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Twisted local wild mapping class groups: configuration spaces, fission trees and complex braids

Philip Boalch, Jean Douçot and Gabriele Rembado*

26 September 2022

Abstract

We continue our investigations of the generalised braid groups appearing in $2d$ gauge theory, as fundamental groups of spaces of admissible deformation parameters (“times”) for the irregular isomonodromy connections. Here we study the local wild mapping class groups in the twisted setting for arbitrary formal structure in type A . General configuration spaces will be defined and shown to admit product decompositions, via a suitable construction of fission trees. Moreover the fission trees will be shown to parameterise admissible deformation classes and used to visualise the configuration spaces. Simple examples give the braid groups of the complex reflection groups known as the generalised symmetric groups, thereby showing how they arise naturally in $2d$ gauge theory (i.e. the theory of meromorphic connections on vector bundles on curves). This enables us to write down the dimensions of the (global) moduli spaces of rank n , trace-free wild Riemann surfaces for any n , a generalisation of “Riemann’s count”.

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1 Introduction

This article is a further step in a series of works [21, 22] investigating the topology of the spaces of times of isomonodromic deformations of connections with irregular singularities, and their fundamental groups, the wild mapping class groups, which are natural generalisations of mapping class groups of surfaces. The motivation for this study comes from the geometric point of view on isomonodromic deformations [6, 10], which views them as natural flat nonlinear symplectic connections, or equivalently as local systems of Poisson varieties

$$\mathbf{M}_{\mathbb{B}} \rightarrow \mathbb{B}$$

over a base space of suitable deformation parameters. The fibres are moduli spaces of connections, possibly with irregular singularities in the de Rham description, or via the Riemann–Hilbert–Birkhoff correspondence, on the Betti side, moduli spaces of generalised monodromy data, i.e. wild character varieties. Such an isomonodromy system depends on the choice of a *wild Riemann surface* [10, 15], that is a triple $(\Sigma, \mathbf{a}, \Theta)$ where Σ is a Riemann surface, $\mathbf{a} = (a_i)$ is the collection of singularities, and $\Theta = (\Theta_i)$ is the collection of *irregular classes* encoding part of the formal normal form of the connection at each singularity. More precisely the base space \mathbb{B} is a space of (admissible) deformations of wild Riemann surfaces, and one gets an isomonodromy system for any such deformation. This gives an intrinsic framework for studying irregular isomonodromy, beyond the generic case studied classically [24, 28], and should generalise the generic twisted case considered in [4].

For connections with regular singularities, the deformation parameters are simply the positions of the singularities (i.e. the irregular classes are all trivial), so that the base space \mathbb{B} is a space of configurations of n points on the Riemann surface Σ . When there are irregular singularities, there are extra “wild” deformation parameters, encoding the irregular class of the connection at each of its singularities, so that the deformation space \mathbb{B} will include both the position of the singularities and the wild moduli. In the regular singular case, the fundamental group of the base is the braid group on n strands, and the monodromy of the isomonodromy connection yields the well-known braid group actions on character varieties. In a similar way, when we allow for irregular singularities, the fundamental group of the base, will act on the wild character variety. This leads to the notion of *wild mapping class group*, the fundamental group of a universal base space of admissible deformations of any wild Riemann surface, as in [11] §8.

As in the previous articles [21, 22], we are here only interested in the local situation: we consider a wild Riemann surface with only one singularity whose position is fixed, and only look at the wild deformation parameters encoding the irregular class of the connection at the singularity. However, in our previous works, we had restricted ourselves to the so-called *untwisted* case, when the irregular classes of our connections do not feature ramification, i.e. do not involve passing to an r -th root of the coordinate z . From the Fabry–Hukuhara–Turrittin–Levelt theorem about the formal classification of irregular connections, tackling the general case requires allowing for the presence of such ramification, so that removing this restriction is needed to have a complete treatment of local wild mapping class groups. We will thus consider twisted irregular classes in this work, but unlike in the previous papers we will this time only consider the type A case of connections on vector bundles, sometimes with a trace-free condition.

A further motivation for allowing twisted irregular classes has to do with comparing different isomonodromy systems, in the spirit of the “Lax project” outlined in [13]. Indeed, it turns out that wild character varieties associated to different wild Riemann surfaces can be isomorphic, so the natural question arises about whether it is possible to go further and relate the corresponding isomonodromy systems, and not just their fibres. This has already been done in the so-called simply laced case in [9]. An important way to obtain such isomorphisms, which was crucial in the study of the simply-laced case, is given by the Fourier–Laplace transform, which changes the rank, the number of singularities and the irregular class of the connection. However, the property of having untwisted irregular class is in general not preserved by the Fourier–Laplace transform. As a consequence if we hope to relate isomonodromy systems obtained from one another by Fourier–Laplace beyond the simply-laced case, it is necessary to understand the twisted wild mapping class groups. Furthermore, it turns out that combining Fourier–Laplace with twists, it is always possible, starting from any genus zero wild

Riemann surface, to obtain a wild Riemann surface with only one singularity at infinity [20]. The position of the singularities of the starting wild Riemann surface become wild moduli for the new wild Riemann surface. We therefore expect that the twisted local situation that we consider here may actually encompass the full *global* situation in genus zero.

Let us try to give some intuition for the main ways in which allowing for twisted irregular classes changes the study of admissible deformations with respect to the untwisted case. Recall that in the untwisted case, an irregular class Θ at a singularity is basically the data of an (unordered) list of exponential factors $q_i \in x\mathbb{C}[x]$, where $z = 1/x$ is a local coordinate vanishing at the singularity, such that formal solutions of the connection involve linear combinations of the essentially singular functions e^{q_i} . In the twisted case, the difference is that now the exponential factors are polynomials in some r -th root of x : we have $q_i \in x^{1/r}\mathbb{C}[x^{1/r}]$, and the list has to be closed under the Galois action. For a deformation of an irregular class to be admissible, in the untwisted case the degree of the differences $q_i - q_j$ has to remain constant. In the twisted case, the condition for a deformation to be admissible will be the same (with the degree replaced by the slope of the difference, which is now a rational number). The main difference is that, among the differences $q_i - q_j$, some are between different Galois conjugates in the same Galois orbit, and requiring the slopes of the differences to be constant creates new types of constraints on the coefficients of the exponential factors which weren't present in the untwisted case. For example let us consider the exponential factor $q_0 := \lambda x^{1/2}$ with $\lambda \in \mathbb{C}$. It has to come together with its Galois conjugate $q_1 = -\lambda x^{1/2}$. If we change the coefficient λ , for this deformation to be admissible the slope of the difference $q_0 - q_1 = 2\lambda x^{1/2}$ has to be constant, which forces λ to remain always non-zero. A key part of the work here consists in investigating systematically these kind of constraints coming from differences between Galois conjugates, and the new types of hyperplane arrangements they lead to. This involves methods similar to those in the theory of curve singularities; the reason such methods are relevant is the bridge from meromorphic connections to meromorphic Higgs fields (and in turn spectral curves) provided by wild nonabelian Hodge theory [5].

Main results. Let us now describe the main results of the article, some of which are in a close parallel to those of [21, 22]. In most previous works discussing connections with twisted irregular singularities only *irregular classes* were considered, that is the exponential factors were not ordered. As in our previous works, to study the admissible deformations it is convenient to also consider *irregular types*, i.e. to fix an order of the exponential factors. We thus start by defining the notion of a (pointed) irregular type Q so that any pointed irregular type Q determines an irregular class $\Theta = [Q]$. We introduce the configuration space $\mathbf{B}(Q)$ of admissible deformations of a pointed irregular type Q . The set of admissible deformations of the corresponding irregular class Θ is in bijection with the quotient $\bar{\mathbf{B}}(\Theta) = \mathbf{B}(Q)/[\cdot]$, where two irregular types Q_1, Q_2 are identified if $[Q_1] = [Q_2]$, i.e. if they have the same irregular class. Our goal is to describe these spaces as explicitly as possible and to determine their respective fundamental group $\Gamma(Q)$ and $\bar{\Gamma}(\Theta)$, the pure and full local (twisted) wild mapping class groups. To this end, we introduce a twisted version of the fission trees.

The first main result is that the admissible deformations of an irregular class are characterised by its fission tree:

Theorem 1.1. *Let Θ and Θ' be arbitrary irregular classes. Then Θ' is an admissible deformation of Θ if and only if their fission trees are isomorphic: $\mathcal{T}(\Theta) \cong \mathcal{T}(\Theta')$.*

This will be deduced from a more precise statement about irregular types (see Cor. 3.33).

We will also give a sharp characterisation of the fission trees (Defn. 3.18, Cor. 3.28), which thus implies a bound (§3.7) on the possible isomorphism classes of the wild character varieties (since the wild character varieties, constructed in general in [10, 15], form a local system of Poisson varieties over any admissible deformation, so they are all isomorphic).

Our next main result is an explicit description of the configuration space of admissible deformations of a pointed irregular type with bounded slope (i.e. bounded Poincaré–Katz rank). This involves a truncation \mathcal{T}^\flat of the fission tree \mathcal{T} just above the maximal slope (see §3.5):

Theorem 1.2. *The configuration space $\mathbf{B}(Q)$ of admissible deformations of the pointed irregular type*

Q is homeomorphic to the following product over the vertices \mathbb{V}^b of its truncated fission tree \mathcal{T}^b :

$$\mathbf{B}(Q) \cong \prod_{v \in \mathbb{V}^b} \mathbf{B}_v(\mathcal{T}).$$

where $\mathbf{B}_v(\mathcal{T})$ is a point if the vertex v has no non-empty children, and if v has n non-empty children, then $\mathbf{B}_v(\mathcal{T})$ is homeomorphic to one of the following spaces:

$$\begin{aligned} X_n &:= \{a_1, \dots, a_n \in \mathbb{C} \mid a_i \neq a_j \text{ for } i \neq j\}, \\ X_{n,N}^* &:= \{a_1, \dots, a_n \in \mathbb{C} \mid a_i \neq 0, a_i \neq \zeta a_j \text{ for } i \neq j, \zeta^N = 1\}. \end{aligned}$$

Compared to the untwisted case, the second factors $X_{n,N}^*$ are new: they are hyperplane complements whose associated hyperplane arrangements are not the complexification of some real hyperplane arrangement. Interestingly, the corresponding braid groups have been studied in [17], and the corresponding Weyl groups are the generalised symmetric groups (see below and Rmk. 4.4).

See Example 3.25 p.20 for a somewhat involved example of how this gives a way to read off the configuration space from the fission tree.

As a consequence, the pure local wild mapping class groups factorise as products of the pure braid groups associated to these hyperplane arrangements:

Theorem 1.3. *Let Q be a pointed irregular type and let \mathcal{T}^b be its truncated fission tree. We have*

$$\Gamma(Q) \cong \prod_{v \in \mathbb{V}^b} \Gamma_v(\mathcal{T}),$$

with $\Gamma_v(\mathcal{T}) := \pi_1(\mathbf{B}_v(\mathcal{T}))$ the pure braid group associated to the hyperplane complement $\mathbf{B}_v(\mathcal{T})$.

In a similar way as in [22], passing from these pure local wild mapping class groups to the full ones (i.e. forgetting the order of the exponential factors) involves considering the automorphisms of the fission tree, leading to a finite group, the *Weyl group* $W(\mathcal{T})$ of the tree, an extension of $\text{Aut}(\mathcal{T})$.

Theorem 1.4. *Let $\Theta = [Q]$ be an irregular class associated to the pointed irregular type Q with fission tree \mathcal{T} . Then the full local wild mapping class group $\bar{\Gamma}(\Theta)$ is an extension of the Weyl group $W(\mathcal{T})$ of the fission tree by the pure wild mapping class group $\Gamma(\Theta)$, i.e. we have the short exact sequence*

$$1 \rightarrow \Gamma(Q) \rightarrow \bar{\Gamma}(\Theta) \rightarrow W(\mathcal{T}) \rightarrow 1.$$

In some simple examples related to the configuration spaces $X_{n,N}^*$ (see Example 4.9) the Weyl group $W(\mathcal{T})$ is isomorphic to the generalised symmetric group $S(N, n)$, which in turn is isomorphic to the complex reflection group $G(N, 1, n)$. Thus we have a modular interpretation (in $2d$ gauge theory) of an infinite family of complex reflection groups and their braid groups. This is parallel to the appearance [7] of the G -braid groups in $2d$ gauge theory for any complex reductive group G .

As an application of these local results we are able to deduce a global statement (see footnote p.30), and write down the dimension of the (global) moduli spaces of rank n , trace-free wild Riemann surfaces for any n , a generalisation of Riemann's count $3g - 3$ of the number of moduli of a compact genus g Riemann surface.

2 General setting

2.1 Twisted irregular types

The exponential local system. The formal data of irregular connections in the twisted case can be formulated geometrically in terms of the so-called exponential local system, which we now briefly recall (see [14] for more details). In summary the idea is to see the exponents q of the exponential factors $\exp(q)$ (occurring in formal solutions of connections, controlling their essentially singular behaviour) as sections of an intrinsic covering space $\pi : \mathcal{I} \rightarrow \partial$ (i.e. a local system of sets) on the circle of directions around the singularity. (We will often abuse language and refer to the exponents q themselves as the

exponential factors.) The basic “extra modular parameters”, the irregular class (that we eventually want to deform), is a finite multiset in the set $\pi_0(\mathcal{I})$ of component circles of \mathcal{I} —this amounts to choosing a finite number of Galois orbits of exponents q_i , each with a multiplicity ≥ 1 .

Let Σ be a Riemann surface and $a \in \Sigma$. Let $\phi : \widehat{\Sigma} \rightarrow \Sigma$ be the real oriented blow-up at a of Σ . The preimage $\partial := \phi^{-1}(a)$ is a circle whose points correspond to directions around a . An open interval $U \subset \partial$ determines a germ of sector Sect_U at a , and if $d \in \partial$ is a direction then Sect_d will denote the germ of an open sector spanning the direction d (where both the opening and the radius may decrease),

Let z be a local coordinate vanishing at a and write $x = z^{-1}$. The exponential local system \mathcal{I} is a local system of sets (that is, a covering space) on ∂ whose sections are germs of holomorphic functions on sectors that are finite sums of the form:

$$q = \sum_i a_i x^{k_i},$$

where $k_i \in \mathbb{Q}_{>0}$, and $a_i \in \mathbb{C}$. More precisely if we fix a direction $d \in \partial$ and choose a branch of $\log(z)$ on Sect_d then the fibre $\mathcal{I}_d = \pi^{-1}(d)$ of \mathcal{I} over d is the set of all such functions on Sect_d , so that

$$\mathcal{I}_d = \left\{ q = \sum_i a_i x^{k_i} \right\} \cong \bigcup_{n \in \mathbb{N}} x^{1/n} \mathbb{C}[x^{1/n}] = \bigcup_{n \in \mathbb{N}} \mathbb{C}((z^{1/n})) / \mathbb{C}[[z^{1/n}]] \quad (2.1)$$

where $x^k := \exp(-k \log(z))$ on the left, and $x = 1/z$ is viewed as a symbol on the right. Thus they are “principal parts of Puiseux series”, but viewed as actual functions on Sect_d , via a choice of logarithm (i.e. the isomorphism from \mathcal{I}_d to Puiseux principal parts depends on this choice). The intrinsic (coordinate free) definition of \mathcal{I} is in [15] Rmk. 3, whence a point $\alpha \in \mathcal{I}_d$ is an *equivalence class* of certain holomorphic functions q_α on Sect_d .

The connected component of \mathcal{I} of such a local section q is a finite order cover of the circle ∂ ; this covering circle “the Stokes circle $\langle q \rangle$ of q ” is essentially the (germ near ∂ of the) Riemann surface where q becomes single valued. A projection of the Stokes circle $\langle x^{3/2} \rangle$ of the Airy equation, making up a trefoil, was drawn in Stokes’ original paper [33], and was reproduced in [15].

More precisely, let $r = \text{Ram}(q)$ be the smallest integer such that the expression $q = \sum_i a_i x^{k_i}$ is a polynomial in $x^{1/r}$, the *ramification order* of q . The corresponding holomorphic function is multivalued, and becomes single-valued when passing to a finite cover $t^r = z$. Therefore, the corresponding connected component, which we denote by $\langle q \rangle$, is an r -sheeted cover of ∂ . As a topological space, it is homeomorphic to a circle, and \mathcal{I} is thus a disjoint union of (an infinite number of) these Stokes circles. Thus $\pi : \mathcal{I} \rightarrow \partial$ is a covering space and if $I = \langle q \rangle \subset \mathcal{I}$ is a connected component then $\pi : I \rightarrow \partial$ is a degree $r = \text{Ram}(q)$ covering map between two circles.

There are several polynomials in $x^{1/r}$ giving rise to the same connected component $I = \langle q \rangle$. They correspond to the Galois orbit of q , under the Galois group of $I \rightarrow \partial$ which is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, and are parameterised by the r points of the fibre $I_d = \pi^{-1}(d)$ for any direction d .

Explicitly, if we write $q = \sum_{j=1}^s a_j x^{j/r}$, $r = \text{Ram}(q)$, $a_s \neq 0$, the polynomials q_i such that $\langle q_i \rangle = \langle q \rangle$ are the Galois conjugates

$$q_i = \sigma^i(q) = \sum_{j=1}^s a_j \omega^{ij} x^{j/r}, \quad i = 0, 1, \dots, r-1,$$

where $\omega = \exp(-2\sqrt{-1}\pi/r)$. The fibre I_d above d of the cover $I \rightarrow \partial$ is equal to the set of germs of functions q_0, \dots, q_{r-1} . The monodromy $\sigma : I_d \rightarrow I_d$ of the cover $I \rightarrow \partial$ is given by $\sigma(q_i) = q_{i+1}$.

The degree s of q as a polynomial in $x^{1/r}$ is called the irregularity of q , which we denote by $\text{Irr}(q) \in \mathbb{N}$. The slope of q , $\text{slope}(q) := \frac{s}{r}$, is the maximal exponent present in q . If $r = 1$ we say that the circle $\langle q \rangle$ is unramified. We will refer to $\langle 0 \rangle$ as the *tame circle*.

Here we view \mathcal{I} as a disjoint union of circles, and the map $\pi : \mathcal{I} \rightarrow \partial$ as a covering space with discrete fibres. Later, below, we will deform the functions q , thus “remembering” the complex vector space structure of the fibres of π .

Irregular classes, finite subcovers and levels. In this language, following [15] Prop. 8, an *irregular class* is a locally constant map $\Theta : \mathcal{I} \rightarrow \mathbb{N}$, assigning an integer to each component of \mathcal{I} , equal to zero for all but a finite number of circles. It is thus constant on each component circle, i.e. corresponds to a map $\pi_0(\mathcal{I}) \rightarrow \mathbb{N}$. An irregular class can be written as a formal sum

$$\Theta = n_1 \langle q_1 \rangle + \cdots + n_m \langle q_m \rangle,$$

where n_1, \dots, n_m are integers. Thus an irregular class Θ is just a finite multiset of Stokes circles, or in concrete terms a finite multiset of Galois orbits of exponential factors. The *rank* of the irregular class Θ is the integer $\text{Rank}(\Theta) := \sum_i n_i \text{Ram}(q_i)$.

The (total) ramification $\text{Ram}(\Theta)$ of an irregular class $\Theta = \sum n_i I_i$ is the lowest common multiple of the ramifications $\text{Ram}(I_i)$ of the active circles I_i in Θ .

From the formal meromorphic classification of meromorphic connections (Fabry, Cope, Hukuhara, Turrittin, Levelt, Jurkat [3, Thm. II], Deligne [29, Thm. IV.2.3]) any connection on a rank n vector bundle on the formal punctured disk determines an irregular class of rank n , taking the Galois orbits of the exponents of the exponential factors e^q that occur, repeated according to their multiplicity. For example a regular singular connection has class $n \langle 0 \rangle$ just involving the tame circle.

A “finite subcover” is a subset $I \subset \mathcal{I}$ such that $I \rightarrow \partial$ is a finite cover, i.e. it is finite set of Stokes circles. An irregular class determines a finite subcover consisting of the active exponents $I = \Theta^{-1}(\mathbb{N}_{>0})$. Explicitly, if $\Theta = \sum_i n_i I_i$, then $I = \bigsqcup_i I_i$. Thus an irregular class corresponds to the data of a finite subcover $I \subset \mathcal{I}$, together with a positive integer n_i for each connected component.

Any rank n irregular class Θ determines another irregular class $\text{End}(\Theta)$ of rank n^2 (as in [14] pp.71-72). The non-zero slopes of the circles in $\text{End}(\Theta)$ are the *levels* of Θ ([19] p.73, [27] p.858, [2] (5.2), [30]). Thus

$$\text{Levels}(\Theta) = \{\text{slope}(q_\alpha - q_\beta) \mid \alpha, \beta \in I_d\} \setminus \{0\} \subset \mathbb{Q}_{>0}$$

where $I \subset \mathcal{I}$ is the finite subcover underlying Θ and I_d is any fibre of I . Note that the existence of irregular classes with multiple levels means there are connections whose formal fundamental solutions are not k -summable for any k (and in particular not Borel summable), and this fact led to the theory of *multisummation* (cf. [2] §5).

Irregular types. In the untwisted case, we have made a distinction between irregular types and irregular classes, the difference being that for irregular classes the exponential factors are unordered, whereas for irregular types the order of the exponential factors matters¹. We will now do the same for the twisted case. As a first step (as in [14] §5.3) we can just choose an ordering of the circles in an irregular class:

Definition 2.1. An “irregular type” of rank n is an ordered list $([n_1, I_1], \dots, [n_m, I_m])$ of distinct Stokes circles $I_i \subset \mathcal{I}$ each with a multiplicity $n_i \geq 1$, such that $\sum n_i \text{Ram}(I_i) = n$.

However here it will more convenient to work with ordered lists of exponential factors. Thus in the rest of the article we fix once and for all a direction $d \in \partial$, local coordinate $z = x^{-1}$ and a choice of logarithm around d , so that a section of the exponential local system \mathcal{I} around the direction d is identified with a Puiseux principal part $q \in x^{1/r} \mathbb{C}[x^{1/r}]$ for some r , as in (2.1).

Definition 2.2. A “full irregular type” of rank n is a Galois closed ordered list $Q = (q_1, \dots, q_n)$ of not necessarily distinct polynomials $q_i \in x^{1/r_i} \mathbb{C}[x^{1/r_i}]$ for some $r_i \in \mathbb{N}_{\geq 1}$.

Asking for the list (q_1, \dots, q_n) to be Galois closed is equivalent to ask for it to be closed under the monodromy σ of the exponential local system \mathcal{I} . Explicitly, the list $Q = (q_1, \dots, q_n)$ is Galois closed if there exists a permutation $\hat{\sigma} \in \text{Sym}_n$ such that $\sigma(q_i) = q_{\hat{\sigma}(i)}$ for $i = 1, \dots, n$.

More intrinsically, given a rank n irregular class Θ and a direction d then Θ determines a length n multiset in \mathcal{I}_d , and the full irregular types determining Θ correspond exactly to the $n!$ possible orderings

¹In the untwisted case an irregular class is the same thing as the “bare irregular type” determined by an irregular type in the sense of [10] Rmk. 10.6, with “irregular type” as defined in [10] Defn. 7.1. (cf. also [6] Defn. 2.4 in the generic case, which gave a coordinate free approach to [24]).

of this multiset. If Q is a full irregular type, let I_1, \dots, I_m be the set of *distinct* Galois orbits of the elements of the list (q_1, \dots, q_n) and $n_i, i = 1, \dots, m$ the multiplicity of I_i , i.e. the number of times that each element of $(I_i)_d$ appears in the list. Then the irregular class associated to Q is $\Theta = n_1 I_1 + \dots + n_m I_m$, and we will write $\Theta = [Q]$ for the class of Q . In particular $n = \text{Rank}(\Theta) = \sum n_i \text{Ram}(I_i)$. In these terms, for any irregular class Θ , the set of full irregular types determining Θ corresponds to all possible orderings of the elements of the list (q_1, \dots, q_n) of one full irregular type determining Θ .

It will also be useful to introduce a variant of these definitions, and consider a specific subset of full irregular types, to account for the fact that the Galois orbits are already naturally cyclically ordered.

Definition 2.3. A “pointed irregular type” is an ordered list

$$Q = [(n_1, q_1), \dots, (n_m, q_m)]$$

where $n_i \in \mathbb{N}_{>0}$, and the q_i are Puiseux principal parts lying in distinct Galois orbits.

We identify a rank n pointed irregular type Q as a full irregular type as follows:

$$Q = \underbrace{(q_1, \dots, q_1)}_{n_1 \text{ times}}, \underbrace{(\sigma(q_1), \dots, \sigma(q_1))}_{n_1 \text{ times}}, \dots, \underbrace{(\sigma^{r_1-1}(q_1), \dots, \sigma^{r_1-1}(q_1))}_{n_1 \text{ times}}, \dots, \dots, \underbrace{(\sigma^{r_m-1}(q_m), \dots, \sigma^{r_m-1}(q_m))}_{n_m \text{ times}}$$

where $r_i = \text{Ram}(q_i)$. Note that (more intrinsically) a rank n pointed irregular type is equivalent to an ordered list

$$[(n_1, p_1, I_1), \dots, (n_m, p_m, I_m)]$$

where $n_i \in \mathbb{N}_{>0}$, the I_i are distinct Stokes circles, $p_i \in (I_i)_d$ is a point of I_i lying over d , and $n = \sum n_i \text{Ram}(I_i)$. The correspondence is given by taking q_i to be the Puiseux principal part determined by the point $p_i \in (I_i)_d$ via the logarithm choice.

Finally observe that the notion of pointed irregular type introduces extra discrete invariants that we don't care about (for example $\text{slope}(q_1 - q_2)$ may vary if q_1 is moved in its Galois orbit). To avoid this we define a notion of “compatibility” between the chosen exponential factors in different orbits.

For any $k \in \mathbb{Q}$ let

$$\tau_k : \mathcal{I}_d \rightarrow \mathcal{I}_d; \quad q = \sum a_i x^{k_i} \mapsto \tau_k(q) = \sum_{k_i \geq k} a_i x^{k_i} \quad (2.2)$$

be the truncation map, discarding all monomials of slope $< k$.

Definition 2.4. A pointed irregular type $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ is *compatible* if for each possible exponent $k \in \mathbb{Q}_{>0}$, and any indices i, j ,

$$\langle \tau_k(q_i) \rangle = \langle \tau_k(q_j) \rangle \implies \tau_k(q_i) = \tau_k(q_j).$$

In other words: if the truncations are in the same Galois orbits, then the truncations are equal.

It is straightforward to see that for any irregular class Θ , a compatible (pointed) irregular type Q exists. Up to isomorphism the configuration space $\mathbf{B}(Q)$ will not depend on the choice of irregular type with irregular class Θ , so we may assume without loss of generality that the pointed irregular types we are considering are compatible.

Pullback to untwisted case. The notions of (twisted) irregular classes and irregular types can easily be related to the corresponding untwisted notions, by passing to a finite cyclic cover. Explicitly, let $Q = (q_1, \dots, q_n)$ be any full irregular type. Let r be an integer multiple of $\text{Ram}(Q)$ so that $\text{Ram}(q_i)$ divides r for all i . Introduce the variable t such that $t^r = x$ (so t^{-1} is a coordinate on a cyclic r -fold ramified cover). Let $\hat{q}_i \in t\mathbb{C}[t]$ be q_i seen as a polynomial in t . Then $\hat{Q} := \text{diag}(\hat{q}_1, \dots, \hat{q}_n)$ is a untwisted irregular type associated to Θ . Its (untwisted) irregular class $\hat{\Theta} = [\hat{Q}]$ only depends on Θ and is simply the pullback. Notice that the irregular class $\hat{\Theta}$ is invariant under the action of $\mathbb{Z}/r\mathbb{Z}$ on the set of untwisted irregular classes obtained by replacing all polynomials $\hat{q}_i(t) \in \mathbb{C}[t]$ by $\hat{q}_i(e^{2\sqrt{-1}k\pi/r}t)$, for any integer k : we say that the untwisted irregular class $\hat{\Theta}$ is r -Galois closed. Conversely, if \hat{Q} is an r -Galois closed untwisted irregular type, it defines a (twisted) irregular type Q such that the ramification orders of all exponential factors divide r . The (twisted) irregular class of Q only depends on the (untwisted) irregular class of \hat{Q} .

2.2 Admissible deformations

The notion of admissible deformations was defined in [10] for arbitrary untwisted meromorphic connections in the context of any reductive group G , extending the generic case in [24, 28] for $\mathrm{GL}_n(\mathbb{C})$ and in [7] for other G . It can be extended to the twisted setting simply by saying that a family of irregular classes is an admissible deformation if and only if some (and hence any) cyclic pullback to the untwisted case is an admissible deformation. In more detail this works out as follows.

Fix \mathcal{I} as above and let \mathbb{B} be a connected complex manifold. Choose a rank n irregular class Θ_b on \mathcal{I} for each $b \in \mathbb{B}$, thus defining a (set theoretic) map

$$\phi : \mathbb{B} \rightarrow \mathrm{IC}_n(\mathcal{I}); \quad b \mapsto \Theta_b$$

to the set $\mathrm{IC}_n(\mathcal{I})$ of rank n irregular classes, i.e. length n multisets in $\pi_0(\mathcal{I})$. We will define when this collection of classes is a (holomorphic) admissible deformation.

Note that since the rank n is fixed the total ramification $\mathrm{Ram}(\Theta_b)$ is uniformly bounded² on \mathbb{B} , for example by $n!$. Thus we can choose an integer N and set $x = t^N$ so that in terms of t Θ_b is a family of untwisted irregular classes, i.e. a multiset in $t\mathbb{C}[t]$ of length n , for each $b \in \mathbb{B}$.

By definition (see [10] Rmk.10.6) this is an admissible deformation if it can locally be represented as $\Theta_b = [Q_b]$ in terms of an admissible family of untwisted irregular types $Q_b = (q_1, \dots, q_n)$ with $q_i \in t\mathbb{C}[t]$ dependent on b . Finally (by [10] Defn. 10.1) this is a (holomorphic) admissible deformation if each q_i varies holomorphically with b and the degree of the polynomial $q_i - q_j \in t\mathbb{C}[t]$ is constant (independent of b) for each $i \neq j$ (the degree is an integer ≥ 0). Similarly one can define smooth admissible deformations etc by allowing the coefficients of the q_i to vary smoothly rather than holomorphically etc. This leads to the following more direct definition, by noting that the slope multiplied by N gives the degree in t when pulled back, upstairs.

Definition 2.5. • A holomorphic family $Q_b = (q_1, \dots, q_n)$ of full irregular types (with $q_i \in \mathcal{I}_d$) is an admissible deformation if

$$\boxed{\text{slope}(q_i - q_j) \text{ is independent of } b \text{ for all } i, j.} \quad (2.3)$$

• The family Θ_b of irregular classes is a holomorphic admissible deformation if it can locally be represented as $\Theta_b = [Q_b]$ for a holomorphic family of full irregular types $Q_b = (q_1, \dots, q_n)$ with $q_i \in \mathcal{I}_d$, varying admissibly, i.e. satisfying (2.3).

If Θ and Θ' are two rank n irregular classes, we say that Θ' is an admissible deformation of Θ if there exists an admissible deformation (Θ_b) indexed by some connected manifold \mathbb{B} equipped with two points $b_1, b_2 \in \mathbb{B}$ such that $\Theta_{b_1} = \Theta$ and $\Theta_{b_2} = \Theta'$. Similarly, if Q and Q' are two rank n full irregular types, we will say that Q' is an admissible deformation of Q , and we write $Q \simeq Q'$ if they are two values of an admissible family of full irregular types.

A continuity argument shows the Galois orbits in an full irregular type do not change under a holomorphic admissible deformation i.e. the same permutation $\hat{\sigma}$ works throughout the deformation (in particular the ramification indices of the Stokes circles are constant).

Example 2.6. Let us consider the holomorphic family of exponential factors $q(b) = x^{1/2} + x^{1/3} + bx^{1/6}$, which has ramification 6 for any $b \in \mathbb{C}$. The first few Galois conjugates of q are:

$$\begin{aligned} q &= q_0(b) = x^{3/6} + x^{2/6} + bx^{1/6} \\ \sigma(q) &= q_1(b) = \varepsilon^3 x^{3/6} + \varepsilon^2 x^{2/6} + b\varepsilon x^{1/6} \\ \sigma^2(q) &= q_2(b) = x^{3/6} + \varepsilon^4 x^{2/6} + b\varepsilon^2 x^{1/6} \end{aligned}$$

where $\varepsilon = \exp(-\pi i/3)$. Considering $\text{slope}(q_i - q_j)$ for $i, j = 0, \dots, 5$ shows that $\Theta_b = \langle q \rangle$ is an admissible deformation over $\mathbb{B} = \mathbb{C}$. Observe that for $b = 0$ we have $\mathrm{Ram}(q_0 - q_2) = 3$, but it is 6 for $b \neq 0$ so not everything behaves continuously.

²In fact since $n = \sum n_i \mathrm{Ram}(I_i)$ it is bounded by the largest possible lowest common multiple of the elements of any integer partition of n ; this is known as Landau's function $g(n)$, whose first 10000 values are listed at <https://oeis.org/A000793>, for example $g(5) = 6 = \mathrm{lcm}(2, 3)$, $g(7) = 12 = \mathrm{lcm}(3, 4)$.

Numerical equivalence of irregular types. We will try to guess a simple numerical criterion for (pointed) irregular types to be admissible deformations of each other. To this end consider the following relation.

Definition 2.7. Let $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ be a pointed irregular type. If $Q' = [(n'_1, q'_1), \dots, (n'_p, q'_p)]$ with each $q'_i \in \mathcal{I}_d$ a Puiseux polar part, then we say that Q, Q' are “numerically equivalent”, and write

$$Q' \sim Q \tag{2.4}$$

if $p = m, n'_i = n_i (i = 1, \dots, m)$ and

$$\text{slope}(\sigma^k(q'_i) - \sigma^l(q'_j)) = \text{slope}(\sigma^k(q_i) - \sigma^l(q_j)) \tag{2.5}$$

for all i, j, k, l with $1 \leq i, j \leq m, 0 \leq k \leq \text{Ram}(q_i), 0 \leq l \leq \text{Ram}(q_j)$ where σ is the Galois action.

Lemma 2.8. *If $Q' \sim Q$ as above, then Q' is a pointed irregular type and moreover $\text{Ram}(q'_i) = \text{Ram}(q_i)$ for all i .*

Proof. Taking $j = i, k = \text{Ram}(q_i), l = 0$ shows that $\text{Ram}(q'_i) \leq \text{Ram}(q_i)$. Thus $\text{Ram}(q'_i) = \text{Ram}(q_i)$ since if $\text{Ram}(q'_i) < \text{Ram}(q_i)$ then there would be some identification amongst the list $\sigma^k(q'_i), k = 1, 2, \dots, \text{Ram}(q'_i)$, but this is not possible as the differences of the slopes matches that of the list $\sigma^k(q_i), k = 1, 2, \dots$. Then the fact that Q' is a pointed irregular type, i.e. its m Galois orbits are distinct, follows from the fact that Q is a pointed irregular type, so none of the slopes between two Galois orbits vanishes. \square

This implies \sim is an equivalence relation when restricted to pairs of pointed irregular types. We will eventually see (Cor. 3.32) that for compatible pointed irregular types it is the same as the relation given by admissible deformation.

Thus it seems we should consider the simple numerical condition (2.5) applied blindly to lists of Puiseux polar parts. This leads to the following configuration spaces.

Remark 2.9. Note that if we just impose that $Q' = [(n_1, q'_1), \dots, (n_m, q'_m)]$ is a rank n pointed irregular type and the apparently weaker condition that (2.5) holds just for $1 \leq i, j \leq m, 0 \leq k \leq \text{Ram}(q_i) - 1, 0 \leq l \leq \text{Ram}(q_j) - 1$, then it follows that $Q' \sim Q$ (because this implies $\text{Ram}(q'_i) \geq \text{Ram}(q_i)$ and then the condition to have rank n implies $\text{Ram}(q'_i) = \text{Ram}(q_i)$ for all i).

Configuration spaces. We will define a *configuration space* for each given (pointed) irregular type $Q = [(n_1, q_1), \dots, (n_m, q_m)]$, and later see it contains all the admissible deformations with bounded slope. Let $r = \text{Ram}(Q) = \text{lcm}\{\text{Ram}(q_i)\}$ be the total ramification of Q and let

$$K = \text{Katz}(Q) := \max(\text{slope}(q_1), \dots, \text{slope}(q_m))$$

be the largest slope, which is essentially the Poincaré–Katz rank of Q (cf. Poincaré [31] p.305, Katz [25] 11.9.7). Thus all the q_i can be expressed as polynomials in $t := x^{1/r}$ of degree at most $s := rK$.

Clearly any pointed irregular type with the same number m of terms, the same multiplicities and ramifications, and that has Katz invariant $\leq K$, will be of the form:

$$Q_{\mathbf{a}} := [(n_1, \sum_{j=1}^s a_{1,j} t^j), \dots, (n_m, \sum_{j=1}^s a_{m,j} t^j)], \quad t = x^{1/r} \tag{2.6}$$

for some unique collection of coefficients $\mathbf{a} = (a_{i,j}) \in \mathbb{C}^{ms}$. This motivates the following definition.

Definition 2.10. Suppose $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ is a pointed irregular type and $r = \text{Ram}(Q), K = \text{Katz}(Q), s = rK$. The configuration space of Q with bounded Katz invariant is the topological space $\mathbf{B}(Q)$ defined by

$$\mathbf{B}(Q) := \{\mathbf{a} = (a_{i,j}) \in \mathbb{C}^{ms} \mid Q_{\mathbf{a}} \sim Q\}, \tag{2.7}$$

with its topology being the one induced from the usual topology of \mathbb{C}^{ms} , where \sim is from (2.4).

We will show below (Cor. 3.34) that $\mathbf{B}(Q)$ is a fine moduli space of all admissible deformations (with Katz invariant $\leq \text{Katz}(Q)$) of the pointed irregular type Q . In the remainder of the article, our goal will then be to explicitly describe $\mathbf{B}(Q)$ and compute its fundamental group. The restriction about having bounded slopes is for the sake of convenience, since it allows us to deal with a finite number of coefficients. As was already the case [21, 22] for the untwisted situation, this entails no loss of generality as far as the topology of $\mathbf{B}(Q)$ is concerned: up to homotopy equivalence $\mathbf{B}(Q)$ does not change if we allow for coefficients associated to higher exponents.

Notice that if Q_1 and Q_2 are two pointed irregular types corresponding to the same irregular class Θ , the spaces $\mathbf{B}(Q_1)$ and $\mathbf{B}(Q_2)$ are homeomorphic, an homeomorphism being given by permuting the active circles and shifting cyclically the distinguished representative of each Galois orbit by the appropriate amount. With a slight abuse of language, we may thus speak of the configuration space $\mathbf{B}(\Theta)$ that is well-defined up to homeomorphism.

Similarly we define a configuration space of trace-free pointed irregular types. First define the *trace* of a full irregular type $Q = (q_1, \dots, q_n)$ to be $\text{Tr}(Q) = \sum_1^n q_i \in \mathcal{I}_d$.

Definition 2.11. Suppose $Q = [(n_1, q_1), \dots, (n_m, q_m)]$ is a pointed irregular type and $r = \text{Ram}(Q)$, $K = \text{Katz}(Q)$, $s = rK$. The traceless (or *special*) configuration space of Q is the topological space $\mathbf{SB}(Q)$ defined by

$$\mathbf{SB}(Q) := \{ \mathbf{a} = (a_{i,j}) \in \mathbb{C}^{ms} \mid Q_{\mathbf{a}} \sim Q, \text{Tr}(Q_{\mathbf{a}}) = 0 \}, \quad (2.8)$$

with its topology being the one induced from the usual topology of \mathbb{C}^{ms} , where \sim is from (2.4).

If $Q = [(n_1, q)]$ just has one Galois orbit then $\text{Tr}(Q) = n_1 \sum_{i=1}^{\text{Ram}(q)} \sigma^i(q)$. In turn, since roots of unity sum to zero, this equals $\text{Tr}(Q) = n_1 \text{Ram}(q) \pi_{un}(q)$ where $\pi_{un} : \mathcal{I}_d \rightarrow x\mathbb{C}[x]$ is the linear map picking out the unramified monomials in q , so that $\pi_{un}(x^k) = x^k$ if $k \in \mathbb{N}$ and $\pi_{un}(x^k) = 0$ otherwise. It follows that the trace of any irregular type lies in the unramified part $x\mathbb{C}[x] \subset \mathcal{I}_d$. Further there is a projection pr:

$$Q = (q_1, \dots, q_n) \mapsto \text{pr}(Q) = Q - \frac{1}{n} \text{Tr}(Q) = (q_1 - \frac{1}{n} \text{Tr}(Q), \dots, q_n - \frac{1}{n} \text{Tr}(Q))$$

mapping any full irregular type to a trace-free irregular type. In particular it makes no difference if we replace Q by its trace-free projection in the definition (2.8), and there are maps

$$\mathbf{B}(Q) \twoheadrightarrow \mathbf{SB}(Q) \hookrightarrow \mathbf{B}(Q) \quad (2.9)$$

where the first map is pr and the second is the natural inclusion. We will see below (Cor. 3.34) that $\mathbf{SB}(Q)$ is a fine moduli space of all trace-free admissible deformations of the pointed irregular type pr(Q). For the moment we just observe the dimensions differ by the integer part of the Katz invariant and they are homotopy equivalent:

Lemma 2.12. *For any pointed irregular type Q , the configuration spaces $\mathbf{SB}(Q), \mathbf{B}(Q)$ are homotopy equivalent, and*

$$\dim(\mathbf{SB}(Q)) = \dim(\mathbf{B}(Q)) - \lfloor \text{Katz}(Q) \rfloor.$$

Proof. Two elements are in the fibre of the map pr : $\mathbf{B}(Q) \twoheadrightarrow \mathbf{SB}(Q)$ if and only if they differ by the operation $(q_1, \dots, q_n) \mapsto (q_1 - q, \dots, q_n - q)$ for some $q \in x\mathbb{C}[x]$ of slope $\leq K$. The dimension of the space of such polynomials q is $\lfloor \text{Katz}(Q) \rfloor$, and this gives a retraction onto $\mathbf{B}(Q)$ \square

Remark 2.13. Isomonodromic deformations of a special class of twisted irregular connections were considered in [4], under a genericity condition (so that the sizes of the Galois orbits are controlled by the Jordan blocks of the leading coefficient). The relation between our general admissibility condition and the Lidskii conditions in [4], specific to their setting, are not immediately clear to us.

3 Classification of admissible deformations

Since an essential difference in the twisted case compared to the untwisted one is that one has to consider differences between different branches of the same exponential factor, it is worth investigating first what the admissible deformations are in the case of an irregular type corresponding to an irregular class with only one active circle.

3.1 A single Stokes circle

Suppose $I = \langle q \rangle \subset \mathcal{I}$ is a single Stokes circle. Recall that the *levels* of I are the non-zero slopes of $\text{End}(I)$, so that, for any $d \in \partial$:

$$\text{Levels}(I) = \{\text{slope}(q_\alpha - q_\beta) \mid \alpha, \beta \in I_d\} \setminus \{0\}$$

where $q_\alpha : \text{Sect}_d \rightarrow \mathbb{C}$ is the function determined by $\alpha \in I_d \subset \mathcal{I}_d$. The set $\text{Levels}(I)$ is a finite, possibly empty, subset of $\mathbb{Q}_{>0}$. Suppose there are m levels and write

$$\text{Levels}(I) = (k_1 > k_2 > \cdots > k_m) \subset \mathbb{Q}_{>0}.$$

The key classification statement is as follows.

Proposition 3.1. *a) Two Stokes circles $I, J \subset \mathcal{I}$ are admissible deformations of each other if and only if $\text{Levels}(I) = \text{Levels}(J) \subset \mathbb{Q}$.*

b) A subset $(k_1 > k_2 > \cdots > k_m) \subset \mathbb{Q}_{>0}$ is the set of levels of some circle $I \subset \mathcal{I}$ if and only if

$$k_1, k_2, \dots, k_m \text{ have strictly increasing common denominators } > 1. \quad (3.1)$$

In other words if d_i is the denominator of k_i (in lowest terms) and

$$r_i \text{ is the lowest common multiple of } d_1, d_2, \dots, d_i \quad (3.2)$$

for each i (so that $r_i \mid r_{i+1}$), then $1 < r_1 < r_2 < \cdots < r_m$.

Proof. Consider $I = \langle q \rangle$ and let $r = \text{Ram}(q)$. Choose a local coordinate z vanishing at 0, set $x = 1/z$ and suppose $x = t^r$. Then $q = \sum_{i=1}^n \alpha_i t^{n_i}$ is a polynomial in t with each α_i non-zero and $n_1 > n_2 > \cdots > n_n \in \mathbb{N}$. Let $r_0 = 1$ and let

$$r_1 < r_2 < \cdots < r_m = r$$

be the set of distinct leading common denominators > 1 that occur, i.e. the distinct numbers > 1 in the set $\{\text{Ram}(\alpha_1 t^{n_1} + \cdots + \alpha_i t^{n_i}) \mid i = 1, 2, \dots, n\}$. Recall that if b_i is the denominator of $n_i/r = \text{slope}(\alpha_i t^{n_i})$, then $\text{Ram}(\alpha_1 t^{n_1} + \cdots + \alpha_i t^{n_i}) = \text{lcm}(b_1, b_2, \dots, b_i)$. Finally, for $i = 1, \dots, m$, let $k_i \in \mathbb{Q}_{>0}$ be the largest exponent such that the ramification is r_i , i.e.

$$k_i = \max\{n_j/r \mid \text{Ram}(\alpha_1 t^{n_1} + \cdots + \alpha_j t^{n_j}) = r_i\}.$$

Then we claim that the levels of I are these numbers $k_1 > k_2 > \cdots > k_m$. This is an exercise, that can be done visibly by drawing the full/naive fission tree, as follows.

For any $k \in \mathbb{R}_{\geq 0}$ define $q_k = \sum \alpha_i t^{n_i}$ where the sum is over the indices i such that $n_i/r > k$. Thus q_k is the leading piece of q whose monomials have slope $> k$. Now for each k consider the finite set

$$N_k := \{q_k(t), q_k(\zeta t), \dots, q_k(\zeta^{r-1}t)\} \subset t\mathbb{C}[t]$$

where $\zeta = \exp(2\pi i/r)$. If $k \in [k_{i+1}, k_i)$ then $|N_k| = r_i$ by definition (of the r_i and k_i). Thus as k varies the sets N_k define a large disjoint union of copies of intervals (i.e. r_i copies of $[k_{i+1}, k_i)$ for each i). Moreover if $k < l$ then truncation gives a map $N_k \twoheadrightarrow N_l$, and this tells us how to glue the intervals into a tree: There is one interval, the trunk, over $[k_1, \infty)$. We glue this to the k_1 -ends of the r_1 intervals over $[k_2, k_1)$. Then over k_2 there are r_1 nodes, and we glue each of them to (r_2/r_1) of the r_2 intervals over $[k_3, k_2)$, etc, ending up with the r leaves of the tree N_0 over 0. Thus the nodes where the tree branches are exactly the points $N_{k_1} \sqcup \cdots \sqcup N_{k_m}$, over $k_1 > \cdots > k_m$.

Now it is easy to see the k_i are the levels: identify I_d with the leaves N_0 ; thus if $i, j \in I_d$ then $\text{slope}(q_i - q_j)$ is the supremum of the unique shortest path in the tree between the leaves i and j . Thus the levels are exactly where the branching occurs. By definition the admissible deformations are thus exactly those which preserve the tree, as an abstract tree lying over $\mathbb{R}_{\geq 0}$ (we can choose an order of N_0 to get an irregular type). Consider the operation of adding to q a monomial of the form αx^k where $k_i > k > k_{i+1}$ and $r_i k \in \mathbb{N}$, as $\alpha \in \mathbb{C}$ varies. This operation clearly gives an admissible deformation of q . Thus we can admissibly deform q to an element of the form $\sum_1^m \beta_i x^{k_i}$ with $\beta_i \neq 0$. In turn we can admissibly deform this so that each $\beta_i = 1$. This implies a). Statement b) is also now clear: the common denominator needs to increase to get into a bigger Galois orbit. \square

Remark 3.2. Note that the statement a) includes the empty set: any two unramified circles $I, J \subset \mathcal{I}$ are admissible deformations of each other. This remark underlies the theory of Baker functions (see [8] §2.2 and the references there).

Remark 3.3. Note that the proof really shows that any two pointed irregular types $[(n_1, q_1)], [(n_2, q_2)]$ (each with just one Galois orbit) are admissible deformations of each other if and only if $\text{Levels}(q_1) = \text{Levels}(q_2)$, and $n_1 = n_2$.

For later use we will formalise the various sets of data in the proof as follows:

Definition 3.4. A “level datum” is a finite, possibly empty, subset $L \subset \mathbb{Q}_{>0}$ satisfying the conditions (3.1) (so they are the possible levels of a single Stokes circle).

Thus, as above, a level datum $L = (k_1 > \cdots > k_m)$ determines ramification indices

$$\text{RI}(L) = (r_1 < \cdots < r_m) \subset \mathbb{N}_{>1}$$

and in particular $\text{Ram}(L) = r_m$. In turn, as in the proof, we can define the *inconsequential exponents*:

$$\text{Inc}(L) = \mathbb{N}_{>0} \cup \left((k_2, k_1) \cap \frac{1}{r_1} \mathbb{N} \right) \cup \cdots \cup \left((k_m, k_{m-1}) \cap \frac{1}{r_{m-1}} \mathbb{N} \right) \cup \left((0, k_m) \cap \frac{1}{r_m} \mathbb{N} \right) \subset \mathbb{Q}_{>0}, \quad (3.3)$$

and the *admissible exponents*:

$$A(L) := L \sqcup \text{Inc}(L) \subset \mathbb{Q}_{>0}. \quad (3.4)$$

If L is empty then $\text{Ram}(L) = 1, \text{RI}(L) = \{1\}$ and $A(L) = \text{Inc}(L) = \mathbb{N}_{>0}$. Note that the admissible exponents can thus also be expressed as:

$$A(L) = \left((0, k_m] \cap \frac{1}{r_m} \mathbb{N} \right) \cup \cdots \cup \left((0, k_2] \cap \frac{1}{r_2} \mathbb{N} \right) \cup \left((0, k_1] \cap \frac{1}{r_1} \mathbb{N} \right) \cup \mathbb{N}_{>0} \subset \mathbb{Q}_{>0}. \quad (3.5)$$

Thus any Stokes circle $I \subset \mathcal{I}$ determines subsets

$$L = L(I) \subset A = A(I) = A(L) \subset \mathbb{R}_{>0}, \quad (3.6)$$

where $L = \text{Levels}(I)$, and A is the set of admissible exponents. The ramification index $\text{Ram}(v) \in \{1, r_1, \dots, r_m\}$ of any $v \in \mathbb{R}_{\geq 0}$ is defined to be the lcm of the denominators of the levels $k_i \geq v$, so that

$$\text{Ram}(v) = r_i \quad \text{if } v \in (k_{i+1}, k_i]. \quad (3.7)$$

Note that (3.3),(3.4) imply the number of non-integral admissible exponents is given by the formula:

$$|A(L) \setminus \mathbb{N}| = \sum_1^m r_i k_i - \lfloor r_{i-1} k_i \rfloor \quad (3.8)$$

where we set $r_0 = 1$.

Example 3.5. Let us look at a few simple examples.

- Suppose q has $\text{Ram}(q) = r > 1$ and $\text{slope}(q) = s/r$ with s and r coprime. Then $q = \sum_1^s a_i x^{i/r}$ with $a_s \neq 0$, the only level of q is its slope $\frac{s}{r}$, $\text{Levels}(q) = (s/r)$ and $A(L) = \mathbb{N}_{>0} \cup \frac{1}{r} \{1, \dots, s\}$.
- Consider $q = x^3 + x^{5/2} + x^{3/2} + x^{1/3}$. It has ramification order $6 = \text{lcm}(2, 3)$. The corresponding list of ramification indices is $\text{RI}(q) = (2 < 6)$, and $\text{Levels}(q) = (5/2 > 1/3)$. In turn $\text{Inc}(q) = \mathbb{N}_{>0} \cup \{3/2, 1/2, 1/6\}$, $A(L) = \mathbb{N}_{>0} \cup \{5/2, 3/2, 1/2, 1/3, 1/6\}$.

Remark 3.6. Note similar (elementary) methods appear in the theory of curve singularities [26]. The reason for such a link to curves is the wild nonabelian Hodge correspondence, between meromorphic connections and meromorphic Higgs bundles, followed by taking the spectral curve of the Higgs field. Indeed the dictionary in [5] p.180 determines the irregular class from the spectral invariants at the singularity of the Higgs field. Note that we only consider *part* of the data of the corresponding curve singularity, and not all of it, i.e. the “principal part” of the singularity, determining the irregular class. This reflects the fact that fission, breaking up the curve at the pole, is about the various growth rates of the essentially singular functions $\exp(q)$ at $a \in \Sigma$, and this only involves the principal part of q .

Single circle configuration spaces. For any pair $q_1, q_2 \in \mathcal{I}_d$ of exponential factors we can consider the pointed irregular types $Q_1 = [(1, q_1)], Q_2 = [(1, q_2)]$ and say that $q_1 \sim q_2$ if $Q_1 \sim Q_2$ in the sense of (2.4), which amounts to the condition:

$$\text{slope}(q_1 - \sigma^i(q_1)) = \text{slope}(q_2 - \sigma^i(q_2))$$

for $i = 0, 1, \dots, \text{Ram}(q_1)$.

Corollary 3.7. *Let q_1 and q_2 be two exponential factors. Then $q_1 \sim q_2$ in the sense of (2.4) if and only if $\text{Levels}(q_1) = \text{Levels}(q_2)$ (i.e. if and only if they are admissible deformations of each other).*

Proof. Clearly if $q_1 \sim q_2$ then $\text{Levels}(q_1) = \text{Levels}(q_2)$. Conversely if they are admissible deformations of each other then they both can be admissibly deformed to $q_0 := \sum x^{k_i}$ (where the k_i are the levels), and so the three lists

$$\text{slope}(q_j - \sigma^i(q_j)), \quad i = 1, 2, 3, 4, \dots$$

of rational numbers are equal, for $j = 0, 1, 2$ (since they remain equal under any small admissible deformation, and under a new choice of initial q). \square

This motivates the definition of the configuration space $\mathbf{B}(q) := \mathbf{B}(Q)$ (from Defn. 2.10, using \sim from (2.4)), for any exponential factor q , where $Q = [(1, q)]$, since we now see $\mathbf{B}(q)$ is the set of all exponential factors that are admissible deformations of q and have Katz invariant (maximal slope) at most $K := \text{Katz}(q)$. Thus we deduce an explicit description of the configuration spaces in the case of one circle:

Proposition 3.8. *Let $q \in \mathcal{I}_d$ be an exponential factor, and let $L = \text{Levels}(q)$ be the levels of q . Then*

$$\mathbf{B}(q) \cong (\mathbb{C}^*)^m \times \mathbb{C}^N, \quad \mathbf{SB}(q) \cong (\mathbb{C}^*)^m \times \mathbb{C}^M$$

where $m = |L|$ is the number of levels, N is the number $|\text{Inc}^b(L)|$ of inconsequential exponents $\leq K$, and $M = |\text{Inc}(L) \setminus \mathbb{N}|$ is the number of non-integral inconsequential exponents. In particular $\dim \mathbf{B}(q)$ is the number $|A^b(L)|$ of admissible exponents $\leq K$ and $\dim \mathbf{SB}(q) = |A(L) \setminus \mathbb{N}|$ is the number of non-integral admissible exponents, as given by the formula (3.8).

Proof. Given $I = \langle q \rangle$ we consider $L(I) \subset A(I) \subset \mathbb{R}_{>0}$ as in (3.6), consisting of the admissible exponents and the subset of levels. We can move the coefficients parameterised by $A(I)$ arbitrarily provided those from L remain non-zero. The descriptions of the configuration spaces then arise a) by not going past K , and b) by only considering trace-free deformations. \square

Example 3.9. Let us come back to our previous examples:

- If $q = a_s x^{s/r} + \dots + a_1 x^{1/r}$, where s and $r = \text{Ram}(q) > 1$ are coprime, $a_s \neq 0$, then its (slope bounded) admissible deformations are of the form

$$q' = \sum_{k=1}^s b_k x^{k/r},$$

with $b_s \in \mathbb{C}^*$ non-zero, and $b_1, \dots, b_{s-1} \in \mathbb{C}$ arbitrary. Removing the integral exponents leaves $s - \lfloor s/r \rfloor$ coefficients, agreeing with formula (3.8) for $\dim \mathbf{SB}(q)$ in this case.

- For $q = x^3 + x^{5/2} + x^{3/2} + x^{1/3}$, the set of levels is $L(q) = \{5/2, 1/3\}$, the set of admissible exponents ≤ 3 is $\{3, 5/2, 2, 3/2, 1, 1/2, 1/3, 1/6\}$, so the (slope bounded) admissible deformations of q are of the form

$$q' = \alpha x^3 + ax^{5/2} + bx^2 + cx^{3/2} + dx + ex^{1/2} + fx^{1/3} + gx^{1/6},$$

with $\alpha, a, b, c, d, e, f, g \in \mathbb{C}$, with a, f non-zero, and the other coefficients arbitrary. The trace-free projection of q is $\text{pr}(q) = q - x^3$ and the trace-free admissible deformations of this are as above, but with $\alpha = b = d = 0$. This deformation space has dimension 5, agreeing with (3.8).

Remark 3.10. Note that if we change coordinates the subsets $L(I) \subset A(I) \subset \mathbb{R}_{>0}$ attached to any Stokes circle I (as in (3.6)) do not change. (Similarly for the fission trees to be defined below.)

3.2 The case of two Stokes circles

As a preparation to tackle the general case, we turn our attention to the case of two distinct active circles. We consider a pointed irregular type of the form

$$Q = [(n, q), (\widehat{n}, \widehat{q})],$$

where q and \widehat{q} have ramification orders r and \widehat{r} . We denote by q_0, \dots, q_{r-1} and $\widehat{q}_0, \dots, \widehat{q}_{\widehat{r}-1}$ the elements of the Galois orbit of the corresponding exponential factors. A holomorphic family defined by $Q_b = [(n, q(b)), (\widehat{n}, \widehat{q}(b))]$ is an admissible deformation if and only if $q(b)$ and $\widehat{q}(b)$ are admissible deformations and if the rational numbers $\text{slope}(q_i - \widehat{q}_j)$ are constant.

We thus have to determine what are the slopes of the differences $q_i - \widehat{q}_j$. This has been studied in [20], the results of which we now briefly recall. If q and \widehat{q} are two distinct exponential factors, we can decompose them into a “common part” and a “different part” in the following way. Recall that $\tau_k : \mathcal{I}_d \rightarrow \mathcal{I}_d$ denotes the truncation map, discarding all monomials of slope $< k$, as in (2.2). (Beware a different truncation was used in the proof of Prop. 3.1, discarding monomials of slope $\leq k$.) Let

$$E(q) \subset \mathbb{Q}_{>0}$$

be the finite set of exponents occurring in q , so $q = \sum_{k \in E(q)} a_k x^k$ with each a_k non-zero. Let $k \in E(q)$ be the smallest number such that

$$\langle \tau_k(q) \rangle = \langle \tau_k(\widehat{q}) \rangle$$

i.e. the Galois orbits of the truncations are equal, if such a number exists. If so then set

$$q_c = \tau_k(q), \quad \widehat{q}_c = \tau_k(\widehat{q}).$$

If there is no such k then set $q_c = \widehat{q}_c = 0$. Then define $q_d = q - q_c, \widehat{q}_d = \widehat{q} - \widehat{q}_c$ so that we get a decomposition

$$q = q_c + q_d, \quad \widehat{q} = \widehat{q}_c + \widehat{q}_d,$$

of q and \widehat{q} as the sum of a *common part* q_c and a *different part* q_d . If $q_c = \widehat{q}_c$ we say that q and \widehat{q} are *compatible*, as in Defn. 2.4. Replacing q or \widehat{q} by another element of their Galois orbit if necessary, we may assume without loss of generality that this is the case. Note that if q and \widehat{q} do not have the same slope then they do not have the same leading term up to Galois conjugate, so $q_c = \widehat{q}_c = 0$. If q and \widehat{q} have a non-zero common part, in particular they have the same slope. We call the rational number

$$f_{q, \widehat{q}} := \max(\text{slope}(q_d), \text{slope}(\widehat{q}_d)) \in \mathbb{Q}_{\geq 0} \tag{3.9}$$

the *fission exponent* of q and \widehat{q} . It is zero if and only if $\langle q \rangle = \langle \widehat{q} \rangle$. Since it only depends on the circles/Galois orbits this defines the fission exponent $f_{I, \widehat{I}}$ for any Stokes circles $I = \langle q \rangle, \widehat{I} = \langle \widehat{q} \rangle$.

We are now in a position to describe the set of slopes we are interested in.

Lemma 3.11. *Let q and \widehat{q} be two exponential factors in distinct Galois orbits. The set of non-zero slopes among the rational numbers $\text{slope}(q_i - \widehat{q}_j)$, for $i = 0, \dots, r-1, j = 0, \dots, \widehat{r}-1$ is equal to*

$$\text{Levels}(q_c) \sqcup \{f_{q, \widehat{q}}\} \subset \mathbb{Q}_{>0},$$

i.e. it consists of the levels of the common part of q and \widehat{q} together with their fission exponent. Furthermore, if q and \widehat{q} are compatible i.e. if $q_c = \widehat{q}_c$ the map $(i, j) \mapsto \text{slope}(q_i - \widehat{q}_j)$ is entirely determined by the data of $\text{Levels}(q_c)$ and $f_{q, \widehat{q}}$.

Proof. This follows directly from the proof of the lemma 4.3 of [20]. The main idea is that the levels of the common part are obtained from the differences between the different Galois conjugates of the common part, while the fission exponent is the slope of all the other differences (for which the Galois conjugates of the common part are the same). \square

As a consequence, the numerical equivalence relation on pointed irregular types can be clarified.

Proposition 3.12. *Let $Q = [(n, q), (\widehat{n}, \widehat{q})]$ be a pointed irregular type with two active circles, such that q and \widehat{q} are compatible. Let $k := f_{q, \widehat{q}}$ be the fission exponent of q and \widehat{q} . Then a pointed irregular type Q' satisfies $Q' \sim Q$ if and only if it is of the form $Q' = [(n, q'), (\widehat{n}, \widehat{q}')]$, with q' and \widehat{q}' compatible and such that*

1. $q \sim q'$ and $\widehat{q} \sim \widehat{q}'$,
2. $q'_c = \widehat{q}'_c$ satisfies $q'_c \sim q_c$,
3. $f_{q', \widehat{q}'} = k$.

These three conditions hold if and only if: $\text{Levels}(q) = \text{Levels}(q')$, $\text{Levels}(\widehat{q}) = \text{Levels}(\widehat{q}')$, and $f_{q', \widehat{q}'} = k$.

Proof. Let us assume that $Q' \sim Q$. Then, considering the differences internal to the two Galois orbits we have $q \sim q'$ and $\widehat{q} \sim \widehat{q}'$. Considering now the set of slopes of the differences between the two distinct Galois orbits for both Q and Q' , we have from lemma 3.11 that $L(q_c) \sqcup \{f_{q, \widehat{q}}\} = L(q'_c) \sqcup \{f_{q', \widehat{q}'}\}$ hence $f_{q', \widehat{q}'} = f_{q, \widehat{q}} = k$, and $L(q_c) = L(q'_c)$, so $q_c \sim q'_c$. Furthermore since q and \widehat{q} are compatible, we have $\text{slope}(q_0 - \widehat{q}_0) = k = \text{slope}(q'_0 - \widehat{q}'_0)$ hence q' and \widehat{q}' are also compatible. For the converse, let us assume that Q' is of the claimed form. Then since $q \sim q'$ and $\widehat{q} \sim \widehat{q}'$ the slopes of the internal differences are the same for Q and Q' . We have $L(q_c) = L(q'_c)$ and $f_{q, \widehat{q}} = f_{q', \widehat{q}'}$. Since q and q' are compatible, as well as \widehat{q} and \widehat{q}' , the second part of lemma 3.11 implies that the slopes of the differences between the distinct Galois orbits are the same for Q and Q' . \square

From the case of a single circle, we know how to make the first two conditions explicit. Let us now investigate the third condition in more detail. Choose $k \in \mathbb{Q}_{>0}$ and let q_c be an exponential factor whose exponents are all strictly greater than k . Write $k = n/m$ with n, m coprime integers. Choose $a, \widehat{a} \in \mathbb{C}$ and consider the exponential factors

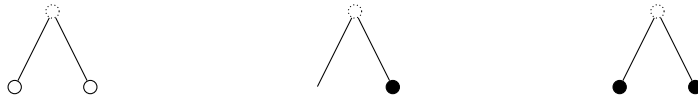
$$q := q_c + az^k + b, \quad \widehat{q} := q_c + \widehat{a}z^k + \widehat{b}$$

where b, \widehat{b} are exponential factors of slope $< k$. The conditions for the fission exponent $f_{q, \widehat{q}}$ to equal k are as follows.

Proposition 3.13. *Let $r = \text{Ram}(q_c), k = n/m$. Then $f_{q, \widehat{q}} = k$ if and only if either 1) or 2) holds:*

1. m divides r and $a \neq \widehat{a}$, or
2. m does not divide r and either:
 - (a) Exactly one of a, \widehat{a} is zero, or
 - (b) Both $a \neq 0$ and $\widehat{a} \neq 0$, and furthermore $a^N \neq \widehat{a}^N$, where $N = \text{lcm}(r, m)/r = m/(r, m)$.

Remark 3.14. Notice that in case 1., one of a and \widehat{a} can be equal to zero. Also note that 2a, 2b can be combined into the single statement that $a^N \neq \widehat{a}^N$. The three cases are distinguished since in 1 the number k is in neither of the sets $\text{Levels}(q), \text{Levels}(\widehat{q})$, for 2a it is in just one, and for 2b it is in both. For later use (cf. Prop 3.26) we encode the three cases 1, 2a, 2b pictorially as follows:



Proof. Clearly we need $a \neq \widehat{a}$, and can set $b = \widehat{b} = 0$ without loss of generality. We then need to see when $\langle q \rangle \neq \langle \widehat{q} \rangle$.

1. Let us first assume that m divides r , so that k is not a level of $\langle q \rangle$ nor $\langle \widehat{q} \rangle$, and $\text{Ram}(q) = \text{Ram}(\widehat{q}) = \text{Ram}(q_c)$. Thus the Galois orbits of q and \widehat{q} are in bijection with that of q_c (via truncation). This implies that the Galois orbits of q and \widehat{q} are distinct if and only if $a \neq \widehat{a}$.

2. 2a) is clear so we consider 2b). Let us set $r' = \text{lcm}(r, m)$ and $N = r'/r$. $\text{Ram}(q)$ is equal to r' if $a \neq 0$, otherwise it is equal to r , and similarly for \hat{q} . If $a \neq 0$, then in the Galois orbit of q there are N elements giving rise, upon truncation, to any given element of the Galois orbit of q_c , and their coefficients of exponent $k = n/m$ differ by an N -th root of unity. The conclusion follows. \square

This enables us to get an explicit description of $\mathbf{B}(Q)$, as we now do for a few examples. In terms of lists of exponents attached to $I = \langle q \rangle, \hat{I} = \langle \hat{q} \rangle$ we have the subsets

$$L(I) \subset A(I) \subset \mathbb{R}_{>0}, \quad L(\hat{I}) \subset A(\hat{I}) \subset \mathbb{R}_{>0}$$

(as in (3.6)) which we should identify just above the fission exponent $f_{I, \hat{I}}$, in order to get the set of exponents whose coefficients we can vary. And these coefficients can be varied arbitrarily provided those from $L(I)$ or $L(\hat{I})$ remain non-zero and those at the fission exponent continue to satisfy the same part of Prop. 3.13.

Example 3.15. Let us look at a few examples illustrating the different cases:

- Consider $Q = [(1, \lambda x^{s/r}), (1, \mu x^{s/r})]$ with $r > 1$ and s coprime and $\lambda \neq \mu e^{2in\pi/r}$ for any integer n (so that the Galois orbits are disjoint). The common part of the two exponential factors is empty, and their fission exponent is the level s/r , so this fits into the case 2(b). The (slope bounded) admissible deformations of Q are of the form $Q' = [(1, q'_1), (1, q'_2)]$ with

$$q'_1 = \sum_{k=1}^s a_k x^{k/r}, \quad q'_2 = \sum_{k=1}^s b_k x^{k/r},$$

with $a_1, \dots, a_s, b_1, \dots, b_s \in \mathbb{C}$, $a_s \neq 0$, $b_s \neq 0$ and $a_s \neq b_s e^{2in\pi/r}$ for any integer n , that is we have

$$\mathbf{B}(Q) \cong \{(a_1, \dots, a_s, b_1, \dots, b_s) \in \mathbb{C}^{2s} \mid a_s \neq 0, b_s \neq 0, \forall n \in \mathbb{N}, a_s \neq b_s e^{2in\pi/r}\}.$$

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/2}$, and $q_2 = x^{3/2} + 2x^{1/2}$. The common part of q_1 and q_2 is $x^{3/2}$, and they are compatible. This fits into the first case, indeed the fission exponent $1/2$ is not a level of q_1 and q_2 . The (slope bounded) admissible deformations of Q are of the form $Q' = [(1, q'_1), (1, q'_2)]$, with

$$q'_1 = ax^{3/2} + bx + cx^{1/2}, \quad q'_2 = ax^{3/2} + bx + dx^{1/2},$$

with $a, b, c, d \in \mathbb{C}$, $a \neq 0$ and $c \neq d$, i.e. we have

$$\mathbf{B}(Q) \cong \{(a, b, c, d) \in \mathbb{C}^4 \mid a \neq 0, c \neq d\}.$$

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/3}$, and $q_2 = x^{3/2}$. The ramification order of $\langle q_1 \rangle$ is 6, the common part of q_1 and q_2 is $x^{3/2}$, and they are compatible. This fits into case 2(a), indeed the fission exponent $1/3$ is a level of q_1 , but does not appear in q_2 . The (slope bounded) admissible deformations of Q are of the form $Q' = [(1, q'_1), (1, q'_2)]$, with

$$q'_1 = ax^{3/2} + bx + cx^{1/2} + dx^{1/3} + ex^{1/6}, \quad q'_2 = ax^{3/2} + bx + cx^{1/2},$$

with $a, b, c, d, e \in \mathbb{C}$, with $a \neq 0$ and $d \neq 0$, i.e. we have

$$\mathbf{B}(Q) \cong \{(a, b, c, d, e) \in \mathbb{C}^5 \mid a \neq 0, d \neq 0\}.$$

3.3 Fission data

As a step towards the general case we will give a first attempt at packaging the relevant data. Recall that a “level datum” is a finite, possibly empty, subset $L \subset \mathbb{Q}_{>0}$ satisfying the conditions (3.1) (so they are the possible levels of a single Stokes circle).

Definition 3.16. A “fission datum” is a pair $\mathcal{F} = (\mathcal{L}, f)$ where \mathcal{L} is a multiset³

$$\mathcal{L} = L_1 + \cdots + L_m$$

of level data and f , the fission exponents, are the choice of a rational number $f_{ij} = f_{ji} \in \mathbb{Q}_{\geq 0}$, for all $i, j \in \{1, \dots, m\}$.

An irregular class determines a fission datum in the obvious way, as follows. If $\Theta = \sum_1^m I_i$ is a rank n irregular class (where the Stokes circles I_i are not necessarily distinct) then define $L_i = L(I_i)$ to be the level datum of I_i for each i , and define $\mathcal{L}(\Theta) := \sum_1^m L_i$ to be the corresponding multiset of level data. Then by taking $f_{ij} = f_{I_i, I_j}$ to be corresponding fission exponents, this determines the fission datum $\mathcal{F}(\Theta) = (\mathcal{L}(\Theta), f)$ of the irregular class Θ . Note that the multiplicity of any given Stokes circle I_j in the class Θ is determined by the fission data by the recipe:

$$\Theta(I_j) = |\{i = 1, 2, \dots, m \mid f_{ij} = 0\}|.$$

In turn a *labelled* fission datum $\widehat{\mathcal{F}} = (\widehat{\mathcal{L}}, f)$ is a pair consisting of an ordered list

$$\widehat{\mathcal{L}} = [(n_1, L_1), \dots, (n_p, L_p)]$$

where the L_i are not necessarily distinct fission data, together with fission exponents $f_{ij} = f_{ji} \in \mathbb{Q}_{\geq 0}$ for $i, j \in \{1, \dots, p\}$ such that $f_{ij} = 0$ if and only if $i = j$. Then a pointed irregular type determines a labelled fission datum in the obvious way (and in turn a labelled fission datum determines a fission datum by forgetting the labelling).

The study in the case of one and two circles then implies one of the main statements:

Theorem 3.17. *Let $Q = [(n_1, q_1), \dots, (n_p, q_p)]$ be a pointed irregular type, which we assume to be compatible, and $\widehat{\mathcal{F}}$ its labelled fission datum. Then a pointed irregular type Q' satisfies $Q' \sim Q$ if and only if it is compatible and its labelled fission datum equals $\widehat{\mathcal{F}}$.*

Proof. This follows from our study of the case of one and two active circles. To see this, notice that the fission data of Q is equivalent to the data of the levels of its active circles, together with the data of the common part and the fission exponent of any pair of distinct active circles. The result now follows from Corollary 3.7 and Proposition 3.12. \square

To go further we will now define fission trees (gluing just above the fission exponents, as above); this will give a way to parameterise the set of coefficients we can vary, and thus to describe the configuration spaces, leading to a proof that two irregular classes are admissible deformations of each other if and only if their fission data are equal. It will also give a way to classify the possible topological data, i.e. the set of possible admissible deformation classes (it seems difficult to write down axioms for the fission data that actually arise from irregular classes, without discussing trees).

3.4 Fission trees in the twisted setting

First we will describe abstractly the exact types of trees we get, and then define how to obtain such trees from irregular types/classes.

³Recall that a multiset is a set with multiplicities. Here this means the L_i are not necessarily distinct level data (but the ordering of the level data, i.e. the labelling by indices $1, \dots, m$, is not part of the data).

Fission trees. Consider a six-tuple $(\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$ where:

- \mathcal{T} is a metrised tree⁴, with vertices $\mathbb{V} \subset \mathcal{T}$
- $\mathbb{A} \subset \mathbb{V}$ is a subset (the admissible vertices),
- $\mathbb{L} \subset \mathbb{A}$ is a finite, possibly empty, subset (the internal levels/mandatory vertices),
- $h : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}$ is a length preserving map, the height map, mapping each edge isomorphically onto an interval, such that $\mathbb{V}_0 := h^{-1}(0) \subset \mathcal{T}$ is a finite set and is the set of leaves of \mathcal{T} ,
- n is a map $\mathbb{V}_0 \rightarrow \mathbb{N}_{>0}$, giving a multiplicity to each leaf.

The edges $E = E(\mathcal{T}) = \pi_0(\mathcal{T} \setminus \mathbb{V})$ of \mathcal{T} are the components of the complement of the vertices. Thus any vertex that is not a leaf is adjacent to ≥ 2 edges, one of which is the “parent” edge, and the others are the descendant edges. The branch vertices $\mathbb{Y} \subset \mathbb{V}$ are those with ≥ 2 descendants (where the branching of the tree occurs). The trunk of the tree is the union of all the edges and vertices above all the branch vertices. The vertices in \mathbb{L} will be called “mandatory”, those in $\mathbb{I} := \mathbb{A} \setminus \mathbb{L}$ will be called “inconsequential”, and the others $(\mathbb{V} \setminus \mathbb{A})$ will be called “empty”.

The “full branch” \mathcal{B}_i of any leaf $i \in \mathbb{V}_0$ is the (minimal) subset of \mathcal{T} all the way from i to the far end of the trunk. Let $\mathbb{L}_i = \mathbb{L} \cap \mathcal{B}_i$ denote the internal levels on the i th full branch, and let $\mathbb{A}_i = \mathbb{A} \cap \mathcal{B}_i$ similarly (the admissible vertices on the i th full branch).

Definition 3.18. Such a tuple $(\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$ is a “fission tree” if:

- 1) $\mathbb{V} = h^{-1}(\{0\} \cup h(\mathbb{A}))$; the vertices are exactly the leaves plus the points that map to $h(\mathbb{A})$,
- 2) h maps each full branch isomorphically onto $\mathbb{R}_{\geq 0}$,
- 3) The internal levels of any full branch map to a set of levels, i.e. $L_i := h(\mathbb{L}_i) \subset \mathbb{Q}_{>0}$ satisfies the conditions (3.1), for any leaf i (so they are the possible levels of a single Stokes circle),
- 4) $A_i := h(\mathbb{A}_i) \subset \mathbb{Q}_{>0}$ is the set $A(L_i)$ of admissible exponents of L_i for each leaf i , as in (3.4),
- 5) The children $\text{Ch}(v) \subset \mathbb{V}$ of each branch vertex $v \in \mathbb{Y}$ satisfy one of the following three conditions:
 1. All the vertices in $\text{Ch}(v)$ are inconsequential,
 - 2a. One vertex in $\text{Ch}(v)$ is empty and the others are mandatory,
 - 2b. All the vertices in $\text{Ch}(v)$ are mandatory.

Note in particular that the leaves of a fission tree are empty. Two fission trees are *isomorphic* if there is an isomorphism between the underlying trees relating all the data $\mathbb{V}, \mathbb{A}, \mathbb{L}, h, n$.

A *labelling* of a fission tree with nodes \mathbb{V}_0 is a total ordering of the set of nodes, i.e. a bijection $\psi : \{1, \dots, |\mathbb{V}_0|\} \cong \mathbb{V}_0$. Two labelled fission trees are isomorphic if there is a label-preserving isomorphism (so there is at most one isomorphism between labelled fission trees).

Remark 3.19. Note that the definition (3.7) extends directly to define the ramification index $\text{Ram}(v)$ of any point $v \in \mathcal{T}$ of a fission tree, taking the lcm of the denominators of the heights of its mandatory ancestors $\geq v$ (i.e. in \mathbb{L} , of height $\geq h(v)$ and on the same full branch). If $p \in \mathbb{V}$ is the parent of some vertex $v \in \mathbb{V}$, by definition the “relative ramification” of p is $\text{Ram}(v)/\text{Ram}(p)$. Observe that the integer $N = \text{lcm}(r, m)/r$ in part 2a of Prop. 3.13 is an example of relative ramification.

Remark 3.20. Axioms 3,4) of Defn. 3.18 imply that axiom 5) could be replaced by the simpler statement “ $\text{Ch}(v)$ contains at most one empty vertex for any $v \in \mathbb{Y}$ ”.

Fission data of a fission tree. Given a fission tree $\mathcal{T} = (\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$ let $A = h(\mathbb{A}) \subset \mathbb{Q}_{>0}$ be the admissible exponents. Given two distinct leaves $i, j \in \mathbb{V}_0$, let $v_{ij} \in \mathbb{Y}$ be their nearest common ancestor (i.e. the branch point where $\mathcal{B}_i, \mathcal{B}_j$ meet). Thus \mathcal{T} determines a number, the fission exponent

$$f_{ij} := \text{prec}(h(v_{ij})) \in A$$

for each pair of leaves, where $\text{prec} : A \rightarrow A \cup \{0\}$ takes $a \in A$ to the preceding element of A , i.e. the largest element $< a$ (or to zero if $a = \min(A)$). Axiom 5) implies $f_{ij} \neq 0$.

Thus a fission tree \mathcal{T} determines a fission datum $\mathcal{F}(\mathcal{T}) = (\mathcal{L}, f)$ where $\mathcal{L} = \sum n_i L_i$ is the set of level data of each full branch (repeated according to their multiplicities) and f encodes the fission exponents between the branches of the tree. The following statement is now an exercise:

⁴See e.g. [1] for metrised graphs, but note that ours are not compact, and recall that a tree is a special type of graph.

Lemma 3.21. *Two fission trees are isomorphic if and only if their fission data are equal:*

$$\mathcal{T}_1 \cong \mathcal{T}_2 \iff \mathcal{F}(\mathcal{T}_1) = \mathcal{F}(\mathcal{T}_2).$$

Similarly two labelled fission trees are isomorphic if and only if their labelled fission data are equal.

Fission tree of an irregular class. We now describe how to define the fission tree of an irregular class $\Theta = \sum n_i I_i$: Firstly there is a full branch \mathcal{B}_i (of multiplicity n_i) for each distinct circle I_i . Thus \mathcal{B}_i is a copy of $\mathbb{R}_{\geq 0}$ equipped with the subsets $\mathbb{L}_i \subset \mathbb{A}_i \subset \mathcal{B}_i$, and an isomorphism $h : \mathcal{B}_i \rightarrow \mathbb{R}_{\geq 0}$, defined so that the subsets $\mathbb{L}_i \subset \mathbb{A}_i$ map onto the sets $L_i = L(I_i) \subset A_i = A(I_i)$ of levels and admissible exponents (defined from I_i as in 3.6).

We then define $A = A(\Theta) := \bigcup A_i \subset \mathbb{Q}_{>0}$ to be the union of all the admissible exponents. This is a discrete subset and for any $k \in \mathbb{Q}_{>0}$ we can define the successor $\text{succ}(k) \in A$ to be the next element of A , i.e. the smallest element of A that is $> k$. If $f_{ij} = f_{I_i, I_j}$ is the fission exponent between I_i, I_j then define the *gluing exponent*

$$g_{ij} := \text{succ}(f_{ij}) \in A$$

to be the next admissible exponent after the fission exponent.

We then glue the full branches $\mathcal{B}_i, \mathcal{B}_j$ over the interval $[g_{ij}, \infty)$ for each i, j , to define the tree \mathcal{T} equipped with the map $h : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}$. The subsets $\mathbb{L}_i \subset \mathbb{A}_i$ fit together to define $\mathbb{L} \subset \mathbb{A} \subset \mathcal{T}$, and we set $\mathbb{V} = h^{-1}(A \cup \{0\}) \subset \mathcal{T}$. Let $\mathbb{V}_k = h^{-1}(k)$ denote the vertices of height k .

This defines the fission tree $\mathcal{T}(\Theta) = (\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$. All the axioms are clear except 5), which will follow from Prop. 3.26 below.

In case we start with a pointed irregular type $Q = [(n_1, q_1), \dots, (n_m, q_m)]$, and not just an irregular class, then we get a labelled fission tree $\hat{\mathcal{T}}(Q)$, by labelling the nodes \mathbb{V}_0 according to the labelling of the exponential factors q_1, \dots, q_m .

The nodes $\mathbb{V} \subset \mathcal{T}$ may be interpreted in terms of truncated circles as follows. Recall that if $k \in \mathbb{Q}$ then $\tau_k(q)$ is the truncation, forgetting monomials of slope $< k$.

Lemma 3.22. *For each $k \in A \cup \{0\}$*

$$\mathbb{V}_k \cong \{ \langle \tau_k(q_1) \rangle, \dots, \langle \tau_k(q_m) \rangle \} \quad (3.10)$$

i.e. the vertices of height k are in bijection with the set of Galois orbits of the exponential factors truncated at k .

Proof. Consider two distinct circles $\langle q_1 \rangle, \langle q_2 \rangle$, corresponding to two leaves. Consider the minimal element k of A such that $\langle \tau_k(q_1) \rangle = \langle \tau_k(q_2) \rangle$. Then, by definition, $k = g_{12} = \text{succ}(f_{12})$. \square

In particular if $k > l$ are two admissible exponents (or zero), we have a surjective map $\phi_{kl} : \mathbb{V}_l \rightarrow \mathbb{V}_k$ defined by $\phi_{kl}(\langle \tau_l(q_i) \rangle) = \langle \tau_k(q_i) \rangle$, and this determines the structure of the tree, by defining the unique parent of each node (if $k, l \in A$ are consecutive). This also proves all the gluings of the full branches can be done consistently.

From this viewpoint, two elements $\langle \tau_l(q_i) \rangle, \langle \tau_l(q_j) \rangle \in \mathbb{V}_l$ are descendants of the same vertex in \mathbb{V}_k , where $l < k$ if they have the same truncation to exponent k , i.e. if $\langle \tau_k(q_i) \rangle = \langle \tau_k(q_j) \rangle$. Furthermore, if $\langle q_i \rangle$ and $\langle q_j \rangle$ are two active circles, corresponding to two leaves, in \mathbb{V}_0 , their closest common ancestor in the tree corresponds to (the Galois orbit of) their common part (this follows immediately from the definition of the common part of two exponential factors).

Remark 3.23. The (seemingly more elegant) alternative construction, gluing $\mathcal{B}_i, \mathcal{B}_j$ at the nearest admissible exponent $g'_{ij} := \min((A_i \cup A_j) \cap (f_{ij}, \infty))$ on either branch, turns out to be much less useful to understand the configuration spaces. Also note that if one glues at f_{ij} instead then the tree will not have enough vertices to parameterise the free coefficients making up the configuration space.

3.5 Truncated fission trees

Since any integer is an admissible exponent of any exponential factor, the set of admissible exponents of any irregular type Q is unbounded from above. For the configuration spaces we are interested in the admissible exponents are bounded by the Katz invariant. Thus we will consider the “truncated fission tree” $\mathcal{T}^\flat = \mathcal{T}^\flat(Q)$ by defining the *root vertex* to be that of height

$$\eta := \lfloor \text{Katz}(Q) + 1 \rfloor \quad (3.11)$$

and then removing all nodes/edges above the root (and marking the root as empty/inadmissible). Note that η is the smallest integer greater than the Katz invariant of Q (the largest slope), so the admissible nodes $\mathbb{A}^\flat \subset \mathcal{T}^\flat$, the subset of \mathbb{A} below the root, are exactly the nodes that will contribute to the configuration space (as these are the admissible nodes of height $\leq \text{Katz}(Q)$). When drawing pictures of trees we will truncate as above, but also we will stop at the smallest admissible exponent, so the leaves will not be drawn. If we are just given a fission tree (and not an irregular type/class) then we will use the “minimal truncation” at $\eta = \lfloor k + 1 \rfloor$ where $k = \max(\{0, h(\mathbb{Y} \cup \mathbb{L})\})$ is the height of the highest branchpoint or internal level. Using this we can attach an integer to any fission tree:

Definition 3.24. The *moduli number* $\mu(\mathcal{T})$ of a fission tree is one plus the number of admissible vertices below the root, minus the number of integers below the root height:

$$\mu(\mathcal{T}) := 1 + |\mathbb{A}^\flat| - (\eta - 1) = |\mathbb{A}^\flat| + 2 - \eta \quad (3.12)$$

$$= 1 + |A_1 \setminus \mathbb{N}| + \sum_2^m |A_i \cap [0, f_i]| \quad (3.13)$$

It is clear that μ just depends on the fission tree and not the irregular class. The dimension of the special configuration space is $\mu - 1$. In practice the second formulation is more convenient (when the tree is labelled by $1, \dots, m$) where $A_i = h(\mathbb{A}_i)$, $f_i = \min_{j < i}(f_{ij})$ and $|A_1 \setminus \mathbb{N}|$ is as in (3.8).

Example 3.25. Consider the rank 10 pointed irregular type $Q = [(1, q_1), (1, q_2), (1, q_3), (1, q_4)]$, with

$$q_1 = x^{3/2} + x, \quad q_2 = x^{3/2} + 2x, \quad q_3 = x^{1/3}, \quad q_4 = 2x^{1/3}.$$

Thus the root height $\eta = 2$ and the set of admissible exponents of Q smaller than η is $\{3/2, 1, 1/2, 1/3\}$. The labelled fission tree is drawn below. We will always draw the mandatory vertices (the internal levels \mathbb{L}) as black circles, the inconsequential vertices ($\mathbb{A} \setminus \mathbb{L}$) as white circles, and the empty vertices without any decoration. The heights are indicated on the left. The root is drawn as a black square.

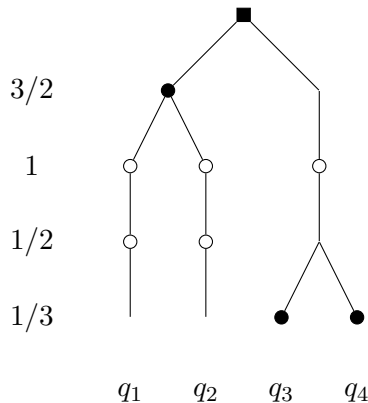


Figure 1: The fission tree \mathcal{T}^\flat associated to Q (not drawn isometrically). The labelling corresponds to the numbering of the q_i . The multiplicities of the leaves are all equal to 1.

We will see below that the configuration space has dimension 8 (the number of nonempty nodes below the root), and further (in Prop. 3.27) that there are three types of conditions on these 8 coefficients: Those from each black vertex \bullet should be non-zero, those from the two inconsequential siblings \circ should be distinct, and those from the two mandatory siblings \bullet should have distinct N th powers (where $N = 3$ in this example). For the special configuration space, the dimension is $7 = 8 - 1$ since there is one positive integer height below the root. The moduli number $\mu(\mathcal{T})$ is eight.

3.6 Realisations of a fission tree

Suppose we are given a fission tree $\mathcal{T} = (\mathcal{T}, \mathbb{V}, \mathbb{A}, \mathbb{L}, h, n)$, and a map $c : \mathbb{A} \rightarrow \mathbb{C}$ with finite support, i.e. $c(v) = 0$ for all but a finite number of the admissible vertices $v \in \mathbb{A} \subset \mathcal{T}$.

Then for each leaf $i \in \mathbb{V}_0$ of the tree we can define an exponential factor

$$q_i = \sum_{v \in \mathbb{A}_i} c(v) x^{h(v)} \quad (3.14)$$

summing over the admissible vertices $\mathbb{A}_i = \mathcal{B}_i \cap \mathbb{A}$ on the i th full branch. Thus c gives the coefficients of a list of exponential factors. Thus if the tree is labelled by some isomorphism $\psi : \{1, \dots, m\} \cong \mathbb{V}_0$ then we get an element

$$Q_c = [(n_1, q_1), \dots, (n_m, q_m)] \quad (3.15)$$

where q_i is determined by c as above, and $n_i = n(i)$ is the multiplicity of the leaf i .

We will say that the coefficient map c is a *realisation* of the tree \mathcal{T} if Q_c is a pointed irregular type and $\mathcal{T}(Q_c) \cong \mathcal{T}$, i.e. if the fission tree of Q_c is isomorphic to \mathcal{T} (the labellings match up by construction).

Thus we wish to make explicit the conditions on c for it to be a realisation.

In effect we just need to check that the tree determined by Q_c has the desired branching and mandatory nodes. Let us focus on a single branch point. Let $l > k$ be two consecutive heights of the tree, and let $v \in \mathbb{V}_l$ be a vertex with n children $\text{Ch}(v) = \{w_1, \dots, w_n\} \subset \mathbb{V}_k$.

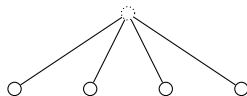
Let q be the exponential factor determined by c at the node v , so that $q = \tau_l(q_j)$ for any leaf j that is a descendant of v . Let $q_i = q + c_i z^k$ (where $c_i = c(w_i)$) be the corresponding exponential factors of the children, $i = 1, \dots, n$. Thus for c to be a realisation, the c_i have to be such that the Galois orbits $\langle q_i \rangle$ are pairwise distinct. Proposition 3.13 allows us to characterise when this is the case. Set $r^+ := \text{Ram}(q)$, and write $k = \frac{s^-}{r^-}$, with s^- and r^- coprime.

Proposition 3.26. *Let q be an exponential factor with all its exponents greater than k , and let $c_1, \dots, c_n \in \mathbb{C}$ and consider $q_i = q + c_i z^k$ for $i = 1, \dots, n$. Now*

1. *If r^- divides r^+ , then k is not a level of any of the q_i . Then the $\langle q_i \rangle$ are distinct if and only if $c_i \neq c_j$ for $1 \leq i < j \leq n$.*
2. *Otherwise, let us assume that r^- does not divide r^+ , and let $N := \frac{\text{lcm}(r^-, r^+)}{r^+}$. Then k is a level of $\langle q_i \rangle$ if and only if $c_i \neq 0$, otherwise if $c_i = 0$, $\langle q_i \rangle \in \mathbb{V}_k$ is an empty vertex. Now the $\langle q_i \rangle$ are distinct if and only if*
 - (a) *Either one of the c_i , $i = 1, \dots, n$, say c_{i_0} , is equal to zero and we have $c_i \neq 0$ for $i \neq i_0$ and $c_i \neq \zeta c_j$ for $i, j \in \{1, \dots, n\} \setminus \{i_0\}$ for any N -th root of unity ζ .*
 - (b) *Or, all of the c_i are non-zero, and we have $c_i \neq \zeta c_j$ for $1 \leq i < j \leq n$ for any N -th root of unity ζ .*

Proof. This follows immediately from the corresponding cases in Proposition 3.13. □

This implies that there are only three possible types of fission at a vertex v in a fission tree, which correspond to the three cases 1, 2(a), and 2(b) of the proposition, and they yield the axiom 5) in the definition of fission tree. In the case 1, since k is not a level of any of the q_i , all the corresponding vertices are inconsequential, which corresponds to the picture below (the parent vertex is dotted on the picture to indicate that it could be mandatory, inconsequential or empty).



Otherwise, in the case 2 the vertices corresponding to the q_i are all mandatory provided they are non-empty. There are two possibilities: either, in the case 2(a), there is one empty vertex, which corresponds to the figure on the left below, or, in case 2(b) there is no empty vertex and all vertices are mandatory, which corresponds to the figure on the right below.



Thus we can write down the exact conditions for c to be a realisation. Recall that two nodes are “siblings” if they have the same parent node. In summary the result is the following:

Proposition 3.27. *The map $c : \mathbb{A} \rightarrow \mathbb{C}$ is a realisation of \mathcal{T} if and only if*

- 1) $c(\mathbb{L}) \subset \mathbb{C}^*$, i.e. $c(v) \neq 0$ for any mandatory node v ,
- 2) $c(u) \neq c(v)$ for any pair u, v of inconsequential siblings,
- 3) $c(u)^N \neq c(v)^N$ for any pair u, v are mandatory siblings where N is the relative ramification of the parent of u, v (defined in Rmk. 3.19).

Proof. The first condition (and the definition of \mathbb{A}) implies each full branch has the correct internal levels. Then 2) and 3) show that the tree of Q_c has the right branching. \square

Note that 2),3) can be combined into the single statement: if u, v are admissible (=non-empty) siblings then $c(u)^N \neq c(v)^N$ where N is the relative ramification of the parent of u or v (since the relative ramification of the parent of an inconsequential vertex is 1).

Corollary 3.28. *Any fission tree admits a realisation.*

Proof. There are just a finite number of Zariski-closed conditions on the coefficients of any set of siblings, so the space of choices of c is non-empty. \square

This immediately gives a clearer description of the configuration spaces. First note that the definition of the fission tree implies:

Lemma 3.29. *Suppose Q is any compatible pointed irregular type with (labelled) fission tree $\mathcal{T} = \mathcal{T}(Q)$, let $r = \text{Ram}(Q), x = t^r$ so that*

$$Q := [(n_1, q_1), \dots, (n_m, q_m)], \quad q_i = \sum_{j=1}^s a_{i,j} t^j \quad (3.16)$$

for some collection of coefficients $\mathbf{a} = (a_{i,j}) \in \mathbb{C}$. Then there is a unique realisation $c = c_Q : \mathbb{A} \rightarrow \mathbb{C}$ of \mathcal{T} with $c(v) = a_{i,k}$ for all i, k where $v = \langle \tau_{k/r}(q_i) \rangle \in \mathbb{A}$ is the vertex of \mathcal{T} determined by the truncation of the exponential factor q_i .

Proof. This amounts to verifying two conditions, which are now straightforward: 1) $a_{i,k} = 0$ if $\langle \tau_{k/r}(q_i) \rangle \in \mathbb{V} \setminus \mathbb{A}$, i.e. if the node of \mathcal{T} determined by the circle $\langle \tau_{k/r}(q_i) \rangle$ is not admissible, and 2) $a_{i,k} = a_{j,k}$ if $\langle \tau_{k/r}(q_i) \rangle = \langle \tau_{k/r}(q_j) \rangle$, i.e. if the truncations determine the same node of \mathcal{T} . \square

It follows that the configuration space $\mathbf{B}(Q)$ of any compatible pointed irregular type Q is isomorphic to the space

$$\begin{aligned} \mathbf{B}(\mathcal{T}^\flat) &:= \{c : \mathbb{A} \rightarrow \mathbb{C} \mid c \text{ is a realisation of } \mathcal{T}, \text{Katz}(c) \leq \text{Katz}(Q)\} \\ &= \{c : \mathbb{A}^\flat \rightarrow \mathbb{C} \mid 1), 2), 3) \text{ of Prop. 3.27 hold for } c\} \end{aligned} \quad (3.17)$$

of realisations of the truncated fission tree \mathcal{T}^\flat . Here \mathbb{A}^\flat is the set of admissible nodes of \mathcal{T}^\flat and $\text{Katz}(c) = \max\{h(a) \mid a \in \mathbb{A}, c(a) \neq 0\}$ is the height of the realisation. In other words we have established the following:

Theorem 3.30. *If Q is a compatible pointed irregular type then $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T}^\flat)$, where \mathcal{T}^\flat is the truncated labelled fission tree of Q .*

Proof. By Thm. 3.17 $\mathbf{B}(Q)$ is the set of compatible pointed irregular type Q' with the same labelled fission data as Q and $\text{Katz}(Q') \leq \text{Katz}(Q)$. Then as in Lem. 3.21 this is the same as saying the labelled fission tree of Q' is isomorphic to that of Q and $\text{Katz}(Q') \leq \text{Katz}(Q)$. Then by Lem. 3.29, any such Q' arises uniquely as a realisation c of \mathcal{T}^\flat . \square

This immediately gives a product decomposition of the configuration space. Given a compatible pointed irregular type Q with fission tree \mathcal{T} , for any vertex $v \in \mathbb{V}$ of \mathcal{T} let $\text{Ch}_{\mathbb{A}}(v) = \mathbb{A} \cap \text{Ch}(v)$ be the set of admissible/nonempty children of v and let $\text{Ch}_{\bullet}(v) = \mathbb{L} \cap \text{Ch}(v)$ be the subset of mandatory children of v (black vertices). Define the local configuration space for the vertex $v \in \mathbb{V}$:

$$\mathbf{B}_v(Q) = \mathbf{B}_v(\mathcal{T}) := \{c : \text{Ch}_{\mathbb{A}}(v) \rightarrow \mathbb{C} \mid c(\text{Ch}_{\bullet}(V)) \subset \mathbb{C}^*, c(u)^N \neq c(w)^N \ \forall u \neq w \in \text{Ch}_{\mathbb{A}}(v)\} \quad (3.18)$$

where N is the relative ramification of v (defined in Rmk. 3.19). $\mathbf{B}_v(\mathcal{T})$ is taken to be a point if v has no non-empty children and otherwise it thus takes the form:

$$\mathbf{B}_v(\mathcal{T}) \cong X_n := \{a_1, \dots, a_n \in \mathbb{C} \mid a_i \neq a_j \text{ for } i \neq j\}, \quad (3.19)$$

if v has n inconsequential children, or

$$\mathbf{B}_v(\mathcal{T}) \cong X_{n,N}^* := \{a_1, \dots, a_n \in \mathbb{C} \mid a_i \neq 0, a_i^N \neq a_j^N \text{ for } i \neq j\} \quad (3.20)$$

if v has n mandatory children and relative ramification N .

Corollary 3.31. *Let Q be a compatible pointed irregular type with fission tree \mathcal{T} and let \mathbb{V}^\flat be the vertices of its truncated fission tree \mathcal{T}^\flat . The configuration space $\mathbf{B}(Q)$ admits a product decomposition:*

$$\mathbf{B}(Q) \cong \prod_{v \in \mathbb{V}^\flat} \mathbf{B}_v(\mathcal{T}).$$

In particular the dimension of $\mathbf{B}(Q)$ is the number of admissible (nonempty) vertices of the fission tree \mathcal{T} of height \leq the Katz invariant of Q .

Proof. Since $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T}^\flat)$ this follows from the characterisation of the realisations $c : \mathbb{A}^\flat \rightarrow \mathbb{C}$ of \mathcal{T}^\flat given in Prop. 3.27. \square

In particular it follows that the configuration spaces are *connected*, since each of the local configuration spaces $\mathbf{B}_v(\mathcal{T}) \cong X_n$ or $X_{n,N}^*$ is connected. This yields a combinatorial/topological characterisation of admissible deformations, as follows.

Corollary 3.32. *Two compatible pointed irregular types are admissible deformations of each other if and only if they have isomorphic labelled fission trees, if and only if they have the same labelled fission data.*

Proof. Given Q, Q' suppose that $\text{Katz}(Q) \geq \text{Katz}(Q')$ and consider $\mathbf{B}(Q)$. If the labelled fission trees are isomorphic then $Q' = Q_c$ for some realisation c of $\mathcal{T}(Q)$, by Theorem 3.30. Thus, by connectedness, Q, Q' are admissible deformations of each other with $\mathbb{B} = \mathbf{B}(Q)$. The converse is clear. The last statement follows as in Lem. 3.21. \square

Corollary 3.33. *Two rank n irregular classes are admissible deformations of each other if and only if they have isomorphic fission trees, if and only if they have the same fission data.*

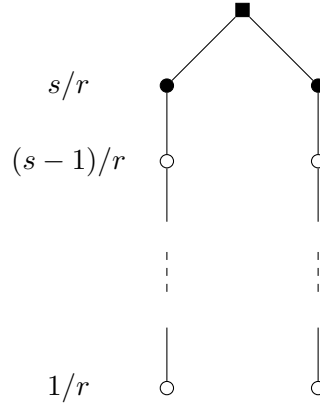
Proof. This follows from Cor. 3.32 by considering local lifts from irregular classes to pointed irregular types. \square

Corollary 3.34. *Let Q be a compatible pointed irregular type. Then the configuration space $\mathbf{B}(Q)$ is a fine moduli space of all pointed irregular types that are admissible deformations of Q , with Katz invariant $\leq \text{Katz}(Q)$. Similarly $\mathbf{SB}(Q)$ is a fine moduli space of all trace-free admissible deformations of any trace-free pointed irregular type Q .*

Proof. The main point is that the product decomposition and the formulae (3.14), (3.15) give a universal family of pointed irregular types over $\mathbf{B}(Q) = \mathbf{B}(\mathcal{T}^b(Q))$. \square

Example 3.35. Let us look at a few examples, starting with the ones with two exponential factors studied previously

- Consider $Q = [(1, \lambda x^{s/r}), (1, \mu x^{s/r})]$ with r and s coprime and $\lambda \neq \mu e^{2in\pi/r}$ for any integer n . The (labelled) fission tree \mathcal{T} is the following:

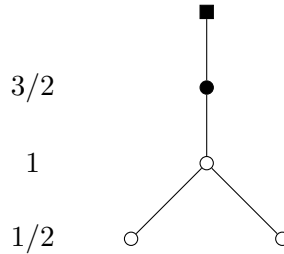


From the tree we read the space of admissible deformations: we have

$$\mathbf{B}(Q) \cong X_{2,r}^* \times \mathbb{C}^{2(s-1)}, \quad X_{2,r}^* = \{a_1, a_2 \in \mathbb{C} \mid a_1 a_2 \neq 0, a_1^r \neq a_2^r\}.$$

Indeed, the factor corresponding to the root vertex is $X_{2,r}^*$, the factors for the leaves are trivial, while for all other vertices v the space $\mathbf{B}_v(\mathcal{T})$ is isomorphic to \mathbb{C} . Similarly $\mathbf{SB}(Q) \cong X_{2,r}^* \times \mathbb{C}^N$ where $N = 2s - 2 - \lfloor s/r \rfloor$, removing one dimension for each integer below the root.

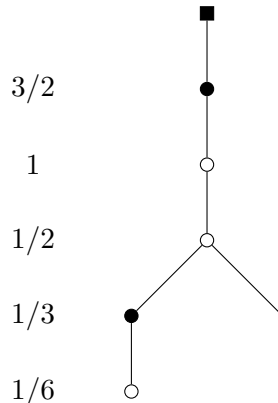
- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/2}$, and $q_2 = x^{3/2} + 2x^{1/2}$. The fission tree is drawn below.



From this we read that the space of admissible deformations satisfies

$$\mathbf{B}(Q) \cong X_{1,2}^* \times X_1 \times X_2 \cong \mathbb{C}^* \times \mathbb{C} \times \{a, b \in \mathbb{C} \mid a \neq b\}.$$

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/3}$, and $q_2 = x^{3/2}$. The fission tree is drawn below.



This yields

$$\mathbf{B}(Q) \cong X_{1,2}^* \times X_{1,3}^* \times X_1^3 \cong (\mathbb{C}^*)^2 \times \mathbb{C}^3.$$

3.7 Topological skeleta

Corollary 3.33 has the following immediate global consequence. Suppose $\Sigma = (\Sigma, \mathbf{a}, \Theta)$ is a rank n wild Riemann surface, with Σ a compact Riemann surface, $\mathbf{a} \subset \Sigma$ a finite subset, and $\Theta = \{\Theta_a \mid a \in \mathbf{a}\}$ the data of a rank n irregular class for each marked point. For each $a \in \mathbf{a}$ let $\mathcal{T}_a = \mathcal{T}(\Theta_a)$ be the fission tree of the irregular class Θ_a . Define the *topological skeleton* of Σ to be the pair

$$\text{Sk}(\Sigma) = (g, \mathbf{F})$$

where $g \geq 0$ is the genus of Σ and $\mathbf{F} = \sum_{a \in \mathbf{a}} [\mathcal{T}_a]$ is the *forest* of Σ , i.e. the multiset of isomorphism classes of fission trees determined by all the \mathcal{T}_a , as a ranges over the marked points $\mathbf{a} \subset \Sigma$. In general it is a multiset rather than a set as some of the fission trees at distinct points may be isomorphic. As explained above the notion of admissible deformations of (twisted) wild Riemann surfaces follows from the untwisted case of [10] (extending the generic case in [24, 28]). Corollary 3.33 then implies:

Corollary 3.36. *Two rank n wild Riemann surfaces are admissible deformations of each other if and only if they have the same topological skeleta.*

Proof. Using the connectedness of the moduli spaces $\mathfrak{M}_{g,[m]}$ of genus g curves with m unordered marked points, this reduces directly to the local statement Cor. 3.33. \square

In particular, since the set of possible topological skeleta is countable, this gives control over the set of possible topological types of the wild character varieties $\mathcal{M}_{\mathbf{B}}$: as in [10] they form a local system of varieties over any admissible deformation, so up to isomorphism there is just one (Poisson) wild character variety for each possible topological skeleton (the key part of the proof in [10] is local on the circle of directions so works equally well for twisted irregular classes).

Remark 3.37. The irregular Deligne–Simpson problem can then be stated as follows: Given a topological skeleton (g, \mathbf{F}) , let \mathbf{L} be the set of all the leaves of all the trees in the forest \mathbf{F} . Choose a conjugacy class $\mathcal{C}_i \subset \text{GL}_{n_i}(\mathbb{C})$ for each leaf $i \in \mathbf{L}$, where $n_i \geq 1$ is the multiplicity of i .

Question: for which choices of skeleton and conjugacy classes is there an irreducible algebraic connection $(V, \nabla) \rightarrow \Sigma^\circ = \Sigma \setminus \mathbf{a}$ with the given topological skeleton and formal monodromy conjugacy classes? Passing to Stokes local systems [14] (by the Stokes version of the irregular Riemann–Hilbert correspondence), this can easily be rewritten as a linear algebra problem (as in [10] §9.4, and the graphical examples in [12] §11).

4 Local wild mapping class groups

4.1 Pure local wild mapping class groups

Let Q be a pointed irregular type. We define the *pure* local (twisted) wild mapping class group of Q as the fundamental group $\Gamma(Q) := \pi_1(\mathbf{B}(Q))$ of the configuration space of admissible deformations (with basepoint \mathbf{a}_Q). From the description of $\mathbf{B}(Q)$, it follows immediately that $\Gamma(Q)$ also factorises as a product of factors associated to each vertex of the fission tree.

Theorem 4.1. *Let \mathcal{T} be the fission tree of Q and let \mathbb{V}^b be the nodes of its truncation. We have*

$$\Gamma(Q) \cong \prod_{v \in \mathbb{V}^b} \Gamma_v(\mathcal{T}),$$

with $\Gamma_v(\mathcal{T}) := \pi_1(\mathbf{B}_v(\mathcal{T}))$.

Since $\mathbf{B}_v(\mathcal{T})$ is isomorphic to a hyperplane complement of the form X_n or $X_{n,N}^*$, setting

$$\Gamma_n := \pi_1(X_n), \quad \Gamma_{n,N}^* := \pi_1(X_{n,N}^*),$$

we get that $\Gamma(Q)$ is always a product of factors of the form Γ_n or $\Gamma_{n,N}^*$. Notice that Γ_n is none other but the pure braid group on n strands PB_n . We recover in this way the untwisted case of [21], since it corresponds to the case where all vertices are inconsequential, hence only the factors Γ_n will appear in the factorisation. In comparison with the untwisted case, new factors $\Gamma_{n,N}^*$ appear as factors of the local wild mapping class group.

Let us now look at a few examples. A simple case to look at is when there is only one exponential factor.

Proposition 4.2. *Let q be an exponential factor. Then $\Gamma(q) := \pi_1(\mathbf{B}(q))$ is isomorphic to $\mathbb{Z}^{|\text{Levels}(q)|}$.*

Proof. This follows immediately from Prop. 3.8. □

Example 4.3. Let us look once again at our previous examples:

- Consider $Q = [(1, \lambda x^{s/r}), (1, \mu x^{s/r})]$ with r and s coprime and $\lambda \neq \mu e^{2in\pi/r}$ for any integer n . The space of realisations is homotopy equivalent to

$$X_{2,r}^* = \{a, b \in \mathbb{C}^* \mid \forall k \in \mathbb{Z}, a \neq be^{2ik\pi/r}\}$$

The fundamental group is thus $\Gamma(Q) \cong \Gamma_{2,r}^*$.

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/2}$, and $q_2 = x^{3/2} + 2x^{1/2}$. The space of realisations is homotopy equivalent to

$$X_{1,2}^* \times X_2 = \mathbb{C}^* \times \{(a, b) \in \mathbb{C}^2 \mid a \neq b\}.$$

Its fundamental group thus satisfies $\Gamma(Q) \cong \Gamma_{1,2}^* \times \Gamma_2 \cong \mathbb{Z}^2$.

- Let us consider $Q = [(1, q_1), (1, q_2)]$ with $q_1 = x^{3/2} + x^{1/3}$, and $q_2 = x^{3/2}$. The space of admissible deformations is homotopy equivalent to $X_{1,2}^* \times X_{1,3}^* \cong (\mathbb{C}^*)^2$, so its fundamental group is isomorphic to \mathbb{Z}^2 .

Remark 4.4. It turns out that the new building blocks $\Gamma_{n,N}^*$ which appear in the twisted case coincide with some braid groups studied in the literature on complex reflections, in particular by Broué–Malle–Rouquier [17]. More precisely, the group $\Gamma_{n,N}^*$ is the same as the group denoted $P(N, 1, n)$ there, and the hyperplane complement $X_{n,N}^*$ is equal to the one denoted by $\mathcal{M}^\#(N, n)$ there (introduced in their Lemma 3.3). Our study of the local wild mapping class groups thus gives a modular interpretation (in 2d gauge theory) for this class of complex braid groups (coming from hyperplane arrangements of the reflecting hyperplanes of these complex/unitary reflection groups). We will see below that the corresponding complex reflection groups, the generalised symmetric groups $S(n, N) = G(N, 1, n)$ appear also in our setting, when passing from irregular types to irregular classes.

4.2 Full local wild mapping class groups

Given an irregular class Θ we have defined a fission tree $\mathcal{T} = \mathcal{T}(\Theta)$ and this determines a configuration space $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T}) \subset \text{Map}(\mathbb{A}^b, \mathbb{C})$ where \mathbb{A}^b is the finite set of admissible nodes of the truncated fission tree. Now we will define a finite group $W(\mathcal{T})$ (the *Weyl group* of the tree) and a free action of $W(\mathcal{T})$ on $\mathbf{B}(\mathcal{T})$ so that two points $Q_1, Q_2 \in \mathbf{B}(\mathcal{T})$ are in the same orbit if and only if $[Q_1] = [Q_2]$, i.e. if they determine the same irregular class. This leads to the *full* local wild mapping class group.

For certain simple examples of fission trees \mathcal{T} we will then find

$$W(\mathcal{T}) \cong \text{Sym}_n \times (\mathbb{Z}/N\mathbb{Z})^n$$

i.e. the Weyl group is a so-called generalised symmetric group $S(N, n)$, isomorphic to the complex reflection group denoted $G(N, 1, n)$ in the Shephard–Todd classification (they are the symmetry groups of the regular complex polytopes called the generalised cubes γ_n^N and the generalised octahedra β_n^N , see e.g. §13.4, p.147 of Coxeter’s book [18] on regular complex polytopes).

The Weyl group of a fission tree. Let \mathcal{T} be a fission tree, let $p = |\mathbb{V}_0|$ be the number of leaves of \mathcal{T} , and choose a labelling $\psi : \{1, \dots, p\} \cong \mathbb{V}_0$. Let $v_i = \psi(i)$ be the i th leaf, and let $v_{ij} \in \mathcal{T}$ be the branchpoint where the full branches $\mathcal{B}_i, \mathcal{B}_j$ meet, i.e. the nearest common ancestor of v_i, v_j . Let $r_i = \text{Ram}(v_i) \in \mathbb{N}$ be the ramification index of the i th leaf, and let $r_{ij} = \text{Ram}(v_{ij})$, for all $i, j = 1, 2, \dots, p$, so that r_{ij} divides both r_i and r_j .

The group $\text{Aut}(\mathcal{T})$ of automorphisms of \mathcal{T} embeds in the symmetric group $\text{Sym}_p = \text{Aut}(\mathbb{V}_0)$ since any automorphism of the tree is determined by its action on the leaves. Thus $\text{Aut}(\mathcal{T})$ acts on the product $(\mathbb{Z}/r_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/r_p\mathbb{Z})$ of cyclic groups, permuting the factors (since if two full branches are isomorphic then they have the same ramification index r_i). Thus we can consider the semi-direct product

$$\text{Aut}(\mathcal{T}) \ltimes ((\mathbb{Z}/r_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/r_p\mathbb{Z})) \quad (4.1)$$

defined via this action. The Weyl group of \mathcal{T} is the following subgroup of this semi-direct product.

Definition 4.5. The Weyl group of the fission tree \mathcal{T} is the subgroup of (4.1) defined by

$$W(\mathcal{T}) := \{(\pi, (d_1, \dots, d_p)) \in \text{Aut}(\mathcal{T}) \ltimes (\mathbb{Z}/r_1\mathbb{Z} \times \dots \times \mathbb{Z}/r_p\mathbb{Z}) \mid d_i \equiv d_j \pmod{r_{ij}}\}.$$

Note that since $r_{ij} \mid r_i$ there is a quotient map $\text{pr}_i : \mathbb{Z}/r_i\mathbb{Z} \rightarrow \mathbb{Z}/r_{ij}\mathbb{Z}$ and the statement that $d_i \equiv d_j \pmod{r_{ij}}$ just means that $\text{pr}_i(d_i) = \text{pr}_j(d_j)$. If $p = 1$ then $W(\mathcal{T}) = \mathbb{Z}/r_1\mathbb{Z}$.

In the rest of this section we will prove the following.

Theorem 4.6. *Let Q be a compatible pointed irregular type with fission tree \mathcal{T} . The Weyl group $W(\mathcal{T})$ acts freely on the configuration space $\mathbf{B}(Q) = \mathbf{B}(\mathcal{T})$ and the quotient*

$$\bar{\mathbf{B}}(\Theta) = \bar{\mathbf{B}}(\mathcal{T}) := \mathbf{B}(Q)/W$$

is the space of all irregular classes that are admissible deformations of $\Theta := [Q]$ with bounded Katz invariant.

It follows that $\bar{\mathbf{B}}(\Theta)$ is a manifold and we can define the *full* local wild mapping class group to be

$$\bar{\Gamma}(\Theta) = \pi_1(\bar{\mathbf{B}}(\Theta)). \quad (4.2)$$

Action on pointed irregular types. The semi-direct product (4.1) is easy to understand via its action on pointed irregular types. Let $Q = ([n_1, q_1], \dots, [n_p, q_p])$ be a pointed irregular type with fission tree \mathcal{T} as above. Let G denote the corresponding group (4.1) defined as a semi-direct product. If $g = (\pi, \mathbf{d}) \in G$ with $\mathbf{d} = (d_1, \dots, d_p)$ then we can obtain another pointed irregular type $g \cdot Q$ with fission tree \mathcal{T} by the formula:

$$\begin{aligned} g \cdot Q &= (\pi, 0) \cdot ([n_1, \sigma^{d_1}(q_1)], \dots, [n_p, \sigma^{d_p}(q_p)]) \\ &= ([n_{\pi(1)}, \sigma^{d_{\pi(1)}}(q_{\pi(1)})], \dots, [n_{\pi(p)}, \sigma^{d_{\pi(p)}}(q_{\pi(p)})]) \end{aligned}$$

so that the cyclic groups rotate the choices of ‘‘pointing’’ and π permutes the exponential factors which have isomorphic full branches.

Note that $g \cdot Q$ will always have the same fission tree as Q but it may not be an admissible deformation of Q , i.e. it may not be a point of the configuration space $\mathbf{B}(Q)$. The Weyl group $W(\mathcal{T})$ is the subgroup characterised by this property:

Lemma 4.7. *Suppose Q is a compatible pointed irregular type, and $g \in G$ is an element of the semi-direct product (4.1). Then $g \cdot Q$ is an admissible deformation of Q (i.e. $g \cdot Q \in \mathbf{B}(Q)$) if and only if $g \in W(\mathcal{T})$.*

Proof. This amounts to characterising the $g \in G$ such that $g \cdot Q$ is still compatible since 1) Any admissible deformation of Q will still be compatible, and 2) by Lemma 3.29, any compatible $g \cdot Q$ will be in $\mathbf{B}(Q)$. Now to see if $g \cdot Q$ is still compatible, we need the exponential factors to ‘‘branch’’ like the circles, and this comes down to requiring

$$\tau_k(\sigma^{d_i}(q_i)) = \tau_k(\sigma^{d_j}(q_j))$$

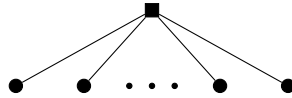
where $k = g_{ij}$ is the height of the nearest common ancestor of the leaves i, j in \mathcal{T} , for all indices $i \neq j$. But this just says that $\sigma^{d_i}(q_c) = \sigma^{d_j}(q_c)$ where $q_c = \tau_k(q_i) = \tau_k(q_j)$ is the common part of q_i, q_j . Now since r_{ij} is the ramification order of q_c this just means that $d_i \equiv d_j$ modulo r_{ij} , as in the definition of $W(\mathcal{T})$. \square

This implies that the finite group $W(\mathcal{T})$ acts on the configuration space $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T})$, and we can now prove the rest of the theorem.

Proof (of Thm 4.6). It remains to show the action is free, and the orbits in $\mathbf{B}(Q)$ are the subsets with the same irregular class. The action is free since 1) the circles corresponding to isomorphic full branches are indeed distinct circles (as else they would be recorded in the multiplicity of the leaf of the full branch), so the permutation is trivial, and 2) the r_i are indeed the exact sizes of the Galois orbits of the q_i , so no smaller cyclic shift will act trivially. Finally note that the pointed irregular types with given irregular class are just related by a choice of ordering of the circles, and the pointings of each circle, are related by the group G . Thus Lemma 4.7 implies the W orbits in $\mathbf{B}(Q) \cong \mathbf{B}(\mathcal{T})$ are exactly the points with the same irregular class. \square

Remark 4.8. The Weyl group $W(\mathcal{T})$ is thus a subgroup of the symmetric group of all permutations of the exponential factors in the corresponding full irregular type (the leaves of the corresponding full/naive fission tree, as in the proof of Prop. 3.1, closely related to the “3d fission tree” in [14]).

Example 4.9. Suppose $Q = ([1, q_1], \dots, [1, q_p])$ with $q_i = a_i x^{s/N}$, $i = 1, \dots, p$ where $(s, N) = 1$, $N > 1$ and the a_i are generic complex numbers (in the sense that $a_i \neq 0$ and $a_i^N \neq a_j^N$ if $i \neq j$). Then the top part of $\mathcal{T}^b(Q)$ looks as follows, with p mandatory nodes branching from the root:



Thus $\text{Aut}(\mathcal{T}) = \text{Sym}_p$, and in turn $W(\mathcal{T}) = \text{Sym}_p \times (\mathbb{Z}/N\mathbb{Z})^p$ is the generalised symmetric group $S(N, p) \cong G(N, 1, p)$, since the ramification indices r_{ij} are all equal to 1.

As in the untwisted case [22], there is an explicit recursive description of the automorphism groups of the fission trees, and in turn the Weyl group. Define a “maximal subtree” of a fission tree \mathcal{T} to be one of the trees obtained by removing the highest branch node of \mathcal{T} (and all the higher edges), so the root of the subtree was the highest branch vertex of \mathcal{T} . The function Ram on \mathcal{T} (from Rmk.3.19) restricts to define a function on any such subtree, so its Weyl group is well-defined.

Theorem 4.10. *Let \mathcal{T} be a fission tree.*

If \mathcal{T} consists only of one full branch whose leaf has ramification order r then $\text{Aut}(\mathcal{T})$ is trivial and $W(\mathcal{T})$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$.

Otherwise, let $\bar{\mathcal{T}}_1, \dots, \bar{\mathcal{T}}_s$ be the distinct isomorphism classes of decorated trees among its maximal subtrees, and for $i = 1, \dots, s$ let $n_i \in \mathbb{N}$ denote the number of such maximal subtrees having the isomorphism class $\bar{\mathcal{T}}_i$. Then $\text{Aut}(\mathcal{T})$ is a product of wreath products:

$$\text{Aut}(\mathcal{T}) \cong \prod_{i=1}^s \text{Sym}_{n_i} \wr \text{Aut}(\bar{\mathcal{T}}_i).$$

In turn if r the ramification order of the root of all the subtrees $\bar{\mathcal{T}}_i$ then

$$W(\mathcal{T}) \cong \left\{ (\pi_i, (g_{i,1}, \dots, g_{i,n_i}))_{i=1, \dots, s} \in \prod_{i=1}^s \text{Sym}_{n_i} \wr W(\bar{\mathcal{T}}_i) \mid \forall i \neq j, \forall k, l, \delta(g_{i,k}) \equiv \delta(g_{j,l}) \pmod{r} \right\},$$

where $\delta(g_{i,k})$ denotes the shift at the root of the subtree induced by the automorphism $g_{i,k}$.

Proof. For the untwisted automorphism group the proof is exactly the same as the one in [22] for the twisted case. For the Weyl group, the proof is similar, the only difference is that the compatibility conditions for the root shifts $\delta(g_{i,k})$ imply that we must restrict to a subgroup of the wreath product $\prod_{i=1}^s \text{Sym}_{n_i} \wr W(\bar{\mathcal{T}}_i)$ that we would get otherwise. \square

The general picture we finally arrive at is the following: the full local wild mapping class group $\bar{\Gamma}(\Theta)$ is an extension of the Weyl group $W(\mathcal{T})$ of the fission tree by the pure local wild mapping class group $\Gamma(\Theta)$, i.e. we have the short exact sequence

$$1 \rightarrow \Gamma(Q) \rightarrow \bar{\Gamma}(\Theta) \rightarrow W(\mathcal{T}) \rightarrow 1 \quad (4.3)$$

The case of one active circle. In general, the exact sequence does not split, so it is not easy to get a fully explicit description of the full local wild mapping class group. In the simple case of only one exponential factor however, we will now see it is possible to be more explicit and to determine completely the full local mapping class group:

Theorem 4.11. *Let $\Theta = \langle q \rangle$ be an irregular class with one circle determined by the exponential factor q . Then the full local wild mapping class group $\bar{\Gamma}(\Theta)$ is isomorphic to $\mathbb{Z}^{|\text{Levels}(q)|}$.*

Proof. Let $r = \text{Ram}(q)$ and $q_i = \sigma^i(q), i = 0, \dots, r-1$ denote the Galois orbit of $q = \sum a_i x^{i/r}$. Let us write $L(q) = \text{Levels}(q) = (k_1 > \dots > k_m)$ and $k_i = n_i/r$. Let us consider the loop $\gamma_i : [0, 1] \rightarrow \mathbf{B}$, $i = 1, \dots, m$ such that γ_i makes the coefficient a_{n_i} of x^{k_i} go once around the origin, and leaves the other coefficient constant. Let us also consider the path ν in \mathbf{B} such that

$$\nu(t) = \sum_j a_j e^{-2\sqrt{-1}\pi j t/r} x^{j/r},$$

so that $\nu(0) = q, \nu(1) = q_1$. Then $\bar{\Gamma}(q)$ is the abelian group generated by the homotopy classes determined by $\gamma_1, \dots, \gamma_m$ and ν , while the subgroup $\Gamma(q)$ is generated by $\gamma_1, \dots, \gamma_m$. There are no relations between the generators $\gamma_1, \dots, \gamma_m$, which recovers the fact that $\Gamma(q) \cong \mathbb{Z}^m = \mathbb{Z}^{|\text{Levels}(q)|}$. On the other hand the family $\gamma_1, \dots, \gamma_m, \nu$ is not free. Indeed, if we follow the loop determined by ν r times, it is the image of a loop upstairs, going from q_0 to itself, with the coefficient a_{n_i} going around the origin a number of times equal to $\text{gcd}(n_i, r) =: d_i$. We thus have the following relation between the $m+1$ generators

$$d_1 \gamma_1 + \dots + d_m \gamma_m = r \nu$$

(we have used an additive notation here since the group is abelian). Using that $\text{gcd}(r, n_1, \dots, n_m) = 1$, the Schmidt algorithm used to classify finitely generated abelian groups transforms the vector $(d_1, \dots, d_m, -r) \in \mathbb{Z}^{m+1}$ corresponding to this relation into $(1, 0, \dots, 0)$, which implies that $\bar{\Gamma}(q) \cong \mathbb{Z}^m = \mathbb{Z}^{|\text{Levels}(q)|}$. \square

In particular, the short exact sequence here reads

$$0 \rightarrow \mathbb{Z}^{|\text{Levels}(q)|} \rightarrow \mathbb{Z}^{|\text{Levels}(q)|} \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow 0,$$

and does not split.

5 Outlook

Several of the directions we plan to pursue are as follows:

1) extend this work beyond type A to any G : the notion of irregular class is already in [15], the analogue of fission trees for any G in the untwisted case is in [21, 22] and the definition of admissible deformations will again follow from that in the untwisted case [10]. Presumably this will lead to other examples of (non-real) complex reflection braid groups.

2) apply the fission trees to the Lax project [13], classifying the (wild) nonabelian Hodge spaces up to isomorphism/deformation. For example how many distinct deformations classes are there in each complex dimension $2, 4, 6, \dots$? Can the fission trees be “combined” with the diagrams of [16, 20] (which

are invariant under Fourier–Laplace) to give a refined invariant? This encompasses the question of classifying isomonodromy systems, and the Painlevé equations are amongst the dimension 2 examples.

3) Study further the full moduli spaces/stacks $\mathfrak{M}_{g,\mathbf{F}}$ of admissible deformations of any wild Riemann surface, whose fundamental groups are the wild mapping class groups, generalising the Riemann moduli spaces $\mathfrak{M}_{g,m}$ and $\mathfrak{M}_{g,[m]}$ in the tame case (where $[m]$ means m unordered marked points), as well as their universal covers (analogues of Teichmüller spaces). This means to bring back in the automorphisms of the underlying curve, and consider arbitrary admissible families of wild Riemann surfaces (as in [10], [11] §8). In general (for each fixed rank n) the pair $g, [m]$ is replaced by the topological skeleton g, \mathbf{F} . The trace-free case is especially nice, when the spaces $\mathfrak{M}_{g,\mathbf{F}}$ have finite type⁵. For example for $g = 0$ and $\mathbf{F} = [\mathcal{T}]$ a single tree, this just amounts to quotienting the configuration space $\bar{\mathbf{B}}(\mathcal{T})$ by the two dimensional group of Möbius transformation fixing one point of the Riemann sphere. All the Painlevé equations (in their standard Lax representations) are especially nice since their (trace-free) moduli spaces $\mathfrak{M}_{g,\mathbf{F}}$ have dimension one, so are *wild modular curves*, reflecting the fact they are ODEs not PDEs (their time variable ranges over a finite cover of this moduli space). This is the basic philosophy of wild Riemann surfaces (that led to [6, 5]); once one realises (as in [6] p.140) that the tame isomonodromy equations (e.g. the Schlesinger equations or Painlevé VI) and the nonabelian Gauss–Manin connections are essentially the same things, and their spaces of times come from maps $\mathbb{B} \rightarrow \mathfrak{M}_{g,[m]}$, then it is natural to *define* the objects $\Sigma = (\Sigma, \mathbf{a}, \Theta)$ whose moduli spaces give the spaces of times for all the irregular isomonodromy equations, i.e. from maps $\mathbb{B} \rightarrow \mathfrak{M}_{g,\mathbf{F}}$.

4) Finally we are interested in quantising the symplectic/Poisson local systems of wild character varieties $\mathbf{M}_{\mathbb{B}} \rightarrow \mathbb{B} \rightarrow \mathfrak{M}_{g,\mathbf{F}}$ (and the corresponding de Rham isomonodromy connections) to get linear representations of the wild mapping class groups (see [7, 32, 23] for some examples).

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⁵Its dimension is $3g - 3 + \sum \mu(\mathcal{T})$ where we sum over all the trees \mathcal{T} in the forest \mathbf{F} , and μ is from Defn. 3.24. We expect it to be Deligne–Mumford if non-empty and $2g - 2 + \sum \mu(\mathcal{T}) > 0$.

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