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Abstract. In this paper, we are dedicated to the global-in-time existence and uniqueness issues of solutions for the inhomogeneous incompressible Hall-MHD system with merely bounded density. In three-dimensional case, assuming that the initial density is a small perturbation of a positive constant in the L^∞ norm, we prove global well-posedness for small initial velocity and magnetic fields in critical Besov spaces. Next, we consider the so-called $2\frac{1}{2}$D flows for the inhomogeneous Hall-MHD system (that is 3D flows independent of the vertical variable), and establish the global existence of strong solutions by assuming only that the initial magnetic field is small in critical spaces and the initial density is bounded and bounded away from zero. Our proofs work for general physical parameters and strongly rely on a new formulation of the system with its Lagrangian formulation. Moreover, some new maximal regularity estimates for parabolic system with just bounded coefficients are developed.

1. Introduction

In this paper, we consider the following inhomogeneous incompressible Hall-magnetohydrodynamics system in the whole spaces $\mathbb{R}^3$:

$$\partial_t \rho + \boldsymbol{u}\cdot\nabla\rho = 0, \tag{1.1}$$

$$\rho\big(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big) + \nabla P = \nu\Delta\boldsymbol{u} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{1.2}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{1.3}$$

$$\partial_t \boldsymbol{B} + h\nabla\times\Big(\frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{\rho}\Big) = \mu\Delta\boldsymbol{B} + \nabla\times(\boldsymbol{u}\times\boldsymbol{B}), \tag{1.4}$$

$$\operatorname{div}\boldsymbol{B} = 0. \tag{1.5}$$

The unknowns are:

- the scalar function $\rho(t,x) : \mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}_+$, that represents the density of the fluid;
- the vector-field $\boldsymbol{u}(t,x) : \mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}^3$, that represents the velocity of the fluid;
- the vector-field $\boldsymbol{B}(t,x) : \mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}^3$, that represents the magnetic field interacting with the fluid;
- the scalar function $P(t,x) : \mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}$, that represents the pressure.

The positive parameters ν and μ are the kinematic viscosity and the magnetic diffusivity, while the number $h>0$ measures the magnitude of the inhomogeneous Hall effect compared to the typical length scale of the fluid.

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The above system is used to model the evolution of electrically conducting fluids such as plasmas or electrolytes (then, $\boldsymbol{u}$ represents the ion velocity), and takes into account the fact that in a moving conductive fluid, the magnetic field can induce currents which, in turn, polarize the fluid and change the magnetic field. That phenomenon which is neglected in the inhomogeneous incompressible magnetohydrodynamics system (corresponding to $h = 0$), is represented by the *inhomogeneous Hall electric field* $\boldsymbol{E}_H := h(\boldsymbol{J}\times\boldsymbol{B})/\rho$, where the current $\boldsymbol{J}$ is defined by $\boldsymbol{J} := \nabla\times\boldsymbol{B}$. Hall term plays an important role in magnetic reconnection [32], as observed in e.g. plasmas [9], star formation [41], solar flares [27].

In the mathematical work of M. Acheritogaray, P. Degond, A. Frouvelle and J.-G. Liu [7], they formally derived the following generalized Ohm's law from the two-fluids Navier-Stokes-Maxwell system under suitable scaling hypotheses

$$\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} = -\nabla(\ln\rho) + h(\boldsymbol{J}\times\boldsymbol{B})/\rho + \boldsymbol{J},$$

where $\boldsymbol{E}$ is the electric field. Later, J. Jang and N. Masmoudi in [34] gave a formal derivation of the Hall effect from the kinetic equations. It is easy to find that generalized Ohm's law combines with the celebrated Maxwell-Faraday equations then gives rise to magnetic equation (1.4). Since the inhomogeneous Hall term is *quasilinear* and *degenerate in vacuum*, it makes the mathematical analysis of the inhomogeneous Hall-MHD system much more complicated than the inhomogeneous MHD system, while the later one has been well-studied in e.g. [4, 6, 13, 31, 43, 44] and the references therein.

To our knowledge, there are few results about the system (1.1)-(1.5). One can find some regularity criteria in [26]. Very recently, the global existence of weak solutions for the Dirichlet problem was established by the first author of the present paper in [40] by an analogy with the classical result due to P.-L. Lions [35] on the inhomogeneous Navier-Stokes equations and later J.-F. Gerbeau and C. Le Bris [28] on the inhomogeneous MHD system. Following the same way as in [40], it is easy to show that system (1.1)-(1.5) formally enjoys the following energy equality

$$\begin{aligned}\|\sqrt{\rho(t)}\boldsymbol{u}(t)\|^2_{L^2(\mathbb{R}^3)} + \|\boldsymbol{B}(t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \big(\nu\|\nabla\boldsymbol{u}(\tau)\|^2_{L^2(\mathbb{R}^3)} + \mu\|\nabla\boldsymbol{B}(\tau)\|^2_{L^2(\mathbb{R}^3)}\big)\,d\tau \\ = \|\sqrt{\rho(0)}\boldsymbol{u}(0)\|^2_{L^2(\mathbb{R}^3)} + \|\boldsymbol{B}(0)\|^2_{L^2(\mathbb{R}^3)}. \quad (1.6)\end{aligned}$$

The primary goal of our paper is to establish the global well-posedness result for the Cauchy problem of the system (1.1)-(1.5) in critical spaces, especially with only bounded initial density. In contrast with the inhomogeneous incompressible Navier–Stokes equations (that corresponds to $\boldsymbol{B} = 0$), however, the system under consideration does not have any scaling invariance owing to the coexistence of the Hall term in (1.4) and of the Lorentz force in (1.2). Recently, R. Danchin and the first author [23] have transformed the incompressible Hall-MHD system (that corresponds to $\rho = 1$) into an extended Hall-MHD system

$$\begin{cases}\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla P = \nu\Delta\boldsymbol{u} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B},\\ \partial_t\boldsymbol{B} - \nabla\times((\boldsymbol{u} - h\boldsymbol{J})\times\boldsymbol{B}) = \mu\Delta\boldsymbol{B},\\ \partial_t\boldsymbol{J} - \nabla\times\big(\nabla\times\big((\boldsymbol{u} - h\boldsymbol{J})\times\operatorname{curl}^{-1}\boldsymbol{J}\big)\big) = \mu\Delta\boldsymbol{J},\\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = \operatorname{div}\boldsymbol{J} = 0,\end{cases} \quad (1.7)$$

which has some scaling invariance (see also [22, 36]). They considered the current function $\boldsymbol{J}$ as an auxiliary vector and found that if $(\boldsymbol{u}(t,x), \boldsymbol{B}(t,x), \boldsymbol{J}(t,x))$ is a

solution of (1.7) on $\mathbb{R}_+ \times \mathbb{R}^d$ subject to the initial data $(\boldsymbol{u}_0(x), \boldsymbol{B}_0(x), \boldsymbol{J}_0(x))$, then for all $\lambda > 0$, the rescaling

$$(\boldsymbol{u}_\lambda, P_\lambda, \boldsymbol{B}_\lambda, \boldsymbol{J}_\lambda) := (\lambda \boldsymbol{u}(\lambda t^2, \lambda x), \lambda^2 P(\lambda t^2, \lambda x), \lambda \boldsymbol{B}(\lambda t^2, \lambda x), \lambda \boldsymbol{J}(\lambda t^2, \lambda x))$$

is a solution of (1.7) with rescaled initial data $(\lambda \boldsymbol{u}_0(\lambda \cdot), \lambda \boldsymbol{B}_0(\lambda \cdot), \lambda \boldsymbol{J}_0(\lambda \cdot))$. Such a scaling invariance is the same as for the incompressible Navier–Stokes equations and thus motivated the definition of critical regularity for (1.7) in [23].

At the moment, let us recall some recent developments on the solvabilities of inhomogeneous incompressible Navier-Stokes equations in the critical regularity framework, i.e. the spaces which have the same invariance with respect to time and space dilation as the system itself, namely

$$\begin{cases} (\rho_0(x), \boldsymbol{u}_0(x)) \rightsquigarrow (\rho_0(\lambda x), \lambda \boldsymbol{u}_0(\lambda x)), \\ (\rho(t,x), \boldsymbol{u}(t,x), P(t,x)) \rightsquigarrow (\rho(\lambda^2 t, \lambda x), \lambda \boldsymbol{u}(\lambda^2 t, \lambda x), \lambda^2 P(\lambda^2 t, \lambda x)). \end{cases}$$

In the critical regularity setting, the local and global existence results were obtained by R. Danchin in [14] for the case of constant viscosity by taking the initial data

$$\rho_0 - 1 \in L^\infty \cap \dot{B}^{\frac{d}{2}}_{2,r} \quad \text{and} \quad \boldsymbol{u}_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}$$

with $r \in [1, +\infty]$ and assuming that $\rho_0 - 1$ is sufficiently small in the space $L^\infty \cap \dot{B}^{\frac{d}{2}}_{2,r}$. After that, H. Abidi in [1] and H. Abidi and M. Paicu in [5] extended these results to the case with variable viscosity for critical Besov spaces of type $\dot{B}^s_{p,1}$ with $p > 1$. Then, many efforts focus on removing the smallness assumptions on the density, see for example R. Danchin [15], H. Abidi, G. Gui, P. Zhang [3] and C. Burtea [11]. Very recently, H. Abidi and G. Gui [2] proved the global well-posedness result at the critical level of regularity, that does not require any smallness condition for two dimensional case. Based on different techniques, H. Xu [42] obtained the global existence and maximal L^1 regularity of solutions provided the initial velocity is small in the Besov space $\dot{B}^{\frac{1}{2}}_{2,1}$.

Concerning rougher density, R. Danchin and P.B. Mucha [17] proved global well-posedness includes discontinuous initial density by a Lagrangian approach, then they proved local well-posedness with $\rho_0 \in L^\infty$ bounded from below and $u_0 \in H^2$ in [18]. Later, M. Paicu, P. Zhang and Z. Zhang [39] established global unique solvability with only bounded initial density and bounded from below. Recently, the lower bound assumption was removed by R. Danchin and P.B. Mucha in [19] in the case where the fluid domain is either bounded or the torus.

Considering initial velocity in critical spaces and initial density only bounded, J. Huang, M. Paicu and P. Zhang [33] proved the global existence of solutions with small variation of density. Recently, P. Zhang [45] removed the smallness assumption of variation of density for u_0 in $\dot{B}^{\frac{1}{2}}_{2,1}$, but lack of uniqueness. A breakthrough on the uniqueness of global solutions obtained in [45] was made by R. Danchin and S. Wang very recently in [24].

Motivated by recent improvements on the inhomogeneous incompressible flows, we are concerned with the global well-posedness of the system (1.1)-(1.5) with only bounded density. Because of the degeneracy of the system in vacuum, we shall only consider the nonvacuum case throughout the paper.

Our results and ideas read as follows:

- In 3D case, to obtain global unique solvability we require that the initial density is a small perturbation of a positive constant (say 1) in the L^∞ norm, and the initial velocity and magnetic fields are small in the critical Besov spaces $\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3) \times (\dot{B}^{-\frac{1}{4}}_{4,1}(\mathbb{R}^3) \cap \dot{B}^{\frac{3}{4}}_{4,1})(\mathbb{R}^3)$. Due to the appearance of the inhomogeneous Hall term $\nabla \times \big((\nabla \times B) \times (B/\rho)\big)$ in (1.3), it seems unavoidable to assume more regularity on the initial density. To overcome this difficulty, we introduce an auxiliary unknown associated with the magnetic field and then use it to reformulate (1.3). More details are presented in the Section 1.1.
- In the $2\frac{1}{2}$D case, we only ask that the initial density is bounded and bounded away from zero and the initial velocity lies in $H^1(\mathbb{R}^2)$, while the initial magnetic field is small in the critical Besov spaces $\dot{B}^{-\frac{1}{3}}_{3,1}(\mathbb{R}^2) \cap \dot{B}^{\frac{2}{3}}_{3,1}(\mathbb{R}^2)$. We use the same strategy as for the 3D case, to enable us consider only bounded density. Meanwhile, one still notice that: the two-dimensional system satisfied by the first two components of the flow is coupled with the equation satisfied by the third component, through the non-linear terms in (1.3), thus hindering any attempt to prove the global well-posedness for large data by means of classical arguments. We thus further introduce a weighted function associated with the previous auxiliary unknown and take advantage of (1.6) and some new maximal estimates for the parabolic system with rough coefficients (see Proposition 2.6), to avoid any smallness assumption on the initial velocity field. To our knowledge, our result partially answers an open problem proposed by D. Chae and J. Lee in the Remark 3 of [12], and improves the result [22] of R. Danchin and the first author to the case $\nu \neq \mu$.

To make our presentation clear, we divide the rest of this section into several parts.

1.1. **New formulation.** In order to prove existence and uniqueness results with only bounded density, our idea is to cancel that first-order derivative on the density in the inhomogeneous Hall term $\nabla\times\big((\nabla \times \boldsymbol{B}) \times (\boldsymbol{B}/\rho)\big)$ in (1.4). Thus we introduce a new unknown named vector potential of magnetic field, that is $\boldsymbol{A} := \operatorname{curl}^{-1}\boldsymbol{B}$, where $\operatorname{curl}^{-1} := -\Delta^{-1}\nabla\times$ is the so-called Biot-Savart operator. Note that $\mathcal{P} = \operatorname{curl}^{-1}\nabla\times$, where $\mathcal{P}$ is the Leray projector over solenoidal vector fields.

To derive the equation of vector $\boldsymbol{A}$, one first find from the following vector identity

$$\Delta + \operatorname{curl}\operatorname{curl} = \nabla\operatorname{div} \tag{1.8}$$

and divergence free condition (1.5) that

$$\operatorname{div}\boldsymbol{A} = 0 \quad \text{and} \quad \nabla \times \boldsymbol{A} = \boldsymbol{B}.$$

Applying the operator curl^{-1} to the equation (1.4) and using identity (1.8) again, the equation of $\boldsymbol{A}$ reads

$$\partial_t \boldsymbol{A} - \mu\Delta\boldsymbol{A} + \nabla Q = \boldsymbol{u} \times (\nabla \times \boldsymbol{A}) + \frac{h}{\rho}\,\Delta\boldsymbol{A} \times (\nabla \times \boldsymbol{A}), \tag{1.9}$$

where ΔQ will be determined thanks to $\operatorname{div}\boldsymbol{A} = 0$.

Notice that the linear part of equation (1.9) is the classical Stokes system that have been widely investigated (see e.g. [16, 18, 29, 38] and the references therein). We will take advantage of the maximal regularity estimates for the heat equation

that was established recently by R. Danchin, P.B. Mucha and P. Tolksdorf in [20], where the time regularity is measured in the Lorentz space. Moreover, to overcome the new difficulties created by the Hall term, we have develop some useful maximal regularity estimates for the heat equation with time-weighted damping (see Proposition 2.4) and the Stokes system with only bounded coefficients (see Proposition 2.6).

In the sequel of the paper, we take $\nu = \mu = h = 1$ for simplicity (our results are valid for general cases). Now, together with the equations of velocity and density, we shall consider the global existence and uniqueness of solutions for the following Cauchy problem:

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0, \\ \rho\big(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\big) + \nabla P = \Delta \boldsymbol{u} - \Delta \boldsymbol{A} \times (\nabla \times \boldsymbol{A}), \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \boldsymbol{A} + \boldsymbol{u} \cdot \nabla \boldsymbol{A} + \nabla Q = \Delta \boldsymbol{A} + (\nabla \boldsymbol{A})^{\mathrm{T}} \boldsymbol{u} + \Delta \boldsymbol{A} \times (\nabla \times \boldsymbol{A})/\rho, \\ \operatorname{div} \boldsymbol{A} = 0, \\ (\rho, \boldsymbol{u}, \boldsymbol{A})|_{t=0} = (\rho_0, \boldsymbol{u}_0, \boldsymbol{A}_0). \end{cases} \tag{1.10}$$

1.2. **Functional spaces.** We briefly recall the definition of the Littlewood–Paley decomposition and define Besov spaces. More details may be found in e.g. [8]. The Littlewood-Paley decomposition is a dyadic localization procedure in the frequency space for tempered distributions over $\mathbb{R}^d$. To define it, fix some nonincreasing smooth radial function χ on $\mathbb{R}^d$, supported in (say) $B(0, 4/3)$ and with value 1 on $B(0, 3/4)$, and set $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$. Then, we have

$$\forall\, \xi \in \mathbb{R}^d,\ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{and} \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\},\ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and low-frequency cut-off operator $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\begin{aligned} \dot{\Delta}_j u &:= \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) u(x-y)\, dy \quad \text{with} \quad h := \mathcal{F}^{-1}\varphi, \\ \dot{S}_j u &:= \chi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y) u(x-y)\, dy \quad \text{with} \quad \tilde{h} := \mathcal{F}^{-1}\chi. \end{aligned}$$

The Littlewood-Paley decomposition of u : $u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u$ holds true modulo polynomials for any tempered distribution u. In order to have an equality in the sense of tempered distributions, we consider only elements of the set $\mathcal{S}'_h(\mathbb{R}^d)$ of tempered distributions u such that $\lim\limits_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$.

We now recall the definition of homogeneous Besov spaces.

Definition 1.1. *Let s be a real number and (p, r) be in $[1, \infty]^2$, we set*

$$\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} := \begin{cases} \|2^{js} \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^d)}\|_{\ell^r(\mathbb{Z})} & \text{for } 1 \leq r < \infty, \\ \sup\limits_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p} & \text{for } r = \infty. \end{cases}$$

The homogeneous Besov space is the set of distributions such that $\|u\|_{\dot{B}^s_{p,r}} < \infty$.

Lorentz spaces can be defined on any measure space (X, μ) via real interpolation between the classical Lebesgue spaces, as follows:

$$L^{p,r}(X, \mu) := (L^\infty, L^1)_{1/p,r} \quad \text{for } p \in (1, \infty) \quad \text{and } r \in [1, \infty].$$

Lorentz spaces may be endowed with the following (quasi)-norm

$$\|f\|_{L^{p,r}} := \begin{cases} p^{\frac{1}{r}} \left(\int_0^\infty (s\,|\{|f| > s\}|^{\frac{1}{p}})^r \frac{ds}{s} \right)^{\frac{1}{r}} & \text{if } \ r < \infty, \\ \sup\limits_{s>0} s\ |\{|f| > s\}|^{\frac{1}{p}} & \text{if } r = \infty. \end{cases}$$

The reason for the pre-factor $p^{\frac{1}{r}}$ is to have $\|f\|_{L^{p,p}} := \|f\|_{L^p}$.

For all $1 \le m \le \infty,\ d = 2, 3$, we define

$$\dot{W}^{2,1}_{p,(m,r)}(\mathbb{R}_+ \times \mathbb{R}^d) := \{\boldsymbol{v} \in C_b(\mathbb{R}_+; \dot{B}^{2-\frac{2}{m}}_{p,r}(\mathbb{R}^d)) :\ \partial_t \boldsymbol{v}, \nabla^2 \boldsymbol{v} \in L^{m,r}(\mathbb{R}_+; L^p(\mathbb{R}^d))\}.$$

1.3. **Main results.** Our first result establish global well-posedness for the 3D inhomogeneous incompressible Hall-MHD system in the critical spaces that allows discontinuous density. For simplicity, in the following, we fix the choice of solution spaces. Other choices might possible, compared to the recent work of R. Danchin and S. Wang [24] for the inhomogeneous incompressible Navier-Stokes equations.

Theorem 1.2. *Let* $\rho_0 \in L^\infty(\mathbb{R}^3)$, $\boldsymbol{u}_0 \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3), \boldsymbol{A}_0 \in (\dot{B}^{\frac{3}{4}}_{4,1} \cap \dot{B}^{\frac{7}{4}}_{4,1})(\mathbb{R}^3)$ *and* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{A}_0 = 0$. *There exists a positive constant* $\varepsilon_1 > 0$ *such that if*

$$\|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)} + \|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} + \|\boldsymbol{A}_0\|_{(\dot{B}^{\frac{3}{4}}_{4,1} \cap \dot{B}^{\frac{7}{4}}_{4,1})(\mathbb{R}^3)} < \varepsilon_1,$$

then the system (1.10) *has a unique global-in-time solution* $(\rho, \boldsymbol{u}, \boldsymbol{A})$ *satisfying*

$$\|\rho - 1\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)} \le \|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)},$$

$$\begin{aligned} \|\boldsymbol{u}\|_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}(\mathbb{R}_+ \times \mathbb{R}^3)} + \|\boldsymbol{A}\|_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}(\mathbb{R}_+ \times \mathbb{R}^3)} + \|\boldsymbol{A}\|_{\dot{W}^{2,1}_{4,(8,1)}(\mathbb{R}_+ \times \mathbb{R}^3)} \\ \le C\Big(\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1}(\mathbb{R}^3)} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{7}{4}}_{4,1}(\mathbb{R}^3)} \Big). \end{aligned} \tag{1.11}$$

Next, we consider the $2\frac{1}{2}$D flows for the System (1.1)-(1.5), that is, 3D flows depending only on the horizontal space variables . This issue is well-known for the incompressible Navier-Stokes equations (see e.g. the book by Bertozzi and Majda [37]). In our case, the corresponding system reads:

$$\begin{aligned} \partial_t \rho + \boldsymbol{u} \cdot \widetilde{\nabla} \rho = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \widetilde{\nabla} \boldsymbol{u} + \widetilde{\nabla} P = \boldsymbol{j} \times \boldsymbol{B} + \widetilde{\Delta} \boldsymbol{u}, \\ \widetilde{\operatorname{div}}\ \boldsymbol{u} = 0, \\ \partial_t \boldsymbol{B} + \widetilde{\nabla} \times \Big(\frac{\boldsymbol{j} \times \boldsymbol{B}}{\rho} \Big) = \widetilde{\Delta} \boldsymbol{B} + \widetilde{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \\ \widetilde{\operatorname{div}}\ \boldsymbol{B} = 0, \end{aligned}$$

where the unknowns $\boldsymbol{u}$ and $\boldsymbol{B}$ are functions from $\mathbb{R}_+ \times \mathbb{R}^2$ to $\mathbb{R}^3$, $\widetilde{\nabla} := (\partial_1, \partial_2, 0)$, $\widetilde{\operatorname{div}} := \widetilde{\nabla} \cdot$, $\widetilde{\Delta} := \partial_1^2 + \partial_2^2$ and $\boldsymbol{j} := \widetilde{\nabla} \times \boldsymbol{B} = (\partial_2 \boldsymbol{B}^3, -\partial_1 \boldsymbol{B}^3, \partial_1 \boldsymbol{B}^2 - \partial_2 \boldsymbol{B}^1)^{\mathrm{T}}$.

Still, in order to prove results with only bounded density, we need to consider the corresponding reformulation of above system in terms of $(\rho, \boldsymbol{u}, \boldsymbol{A})$, with $\boldsymbol{A} :=$

$(-\widetilde{\Delta})^{-1}\widetilde{\nabla}\times\boldsymbol{B}$. It reads as follows:

$$\begin{cases}\partial_t\rho+\boldsymbol{u}\cdot\widetilde{\nabla}\rho=0,\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\widetilde{\nabla}\boldsymbol{u}+\widetilde{\nabla}P=\widetilde{\Delta}\boldsymbol{u}-\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A}),\\ \widetilde{\operatorname{div}}\,\boldsymbol{u}=0,\\ \partial_t\boldsymbol{A}-\widetilde{\Delta}\boldsymbol{A}+\widetilde{\nabla}Q=\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A})+\dfrac{1}{\rho}(\widetilde{\Delta}\boldsymbol{A})\times(\widetilde{\nabla}\times\boldsymbol{A}),\\ \widetilde{\operatorname{div}}\,\boldsymbol{A}=0,\\ (\rho,\boldsymbol{u},\boldsymbol{A})|_{t=0}=(\rho_0,\boldsymbol{u}_0,\boldsymbol{A}_0).\end{cases}\tag{1.12}$$

The global existence and uniqueness result in the $2\frac{1}{2}$ dimensional case reads:

Theorem 1.3. *Let ρ_0 satisfy*

$$c_0\le\rho_0\le C_0,\tag{1.13}$$

for some positive constants c_0, C_0. Let $(\boldsymbol{u}_0,\boldsymbol{A}_0)$ be divergence free vector-fields with $u_0\in H^1(\mathbb{R}^2;\mathbb{R}^3), \boldsymbol{A}_0\in(\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})(\mathbb{R}^2;\mathbb{R}^3)$. There exists a positive constant ε_2 depending only on $c_0, C_0, \|\sqrt{\rho_0}\boldsymbol{u}_0,\widetilde{\nabla}\times\boldsymbol{A}_0\|_{L^2}$ such that, if

$$\|\boldsymbol{A}_0\|_{(\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})(\mathbb{R}^2)}<\varepsilon_2,$$

then the system (1.12) has a unique global-in-time solution $(\rho,\boldsymbol{u},\boldsymbol{A})$ satisfying

$$c_0\le\rho(t,x)\le C_0\tag{1.14}$$

and

$$\|\widetilde{\nabla}\boldsymbol{u}\|_{L^\infty(L^2)}+\|(\sqrt{\rho}\partial_t\boldsymbol{u},\widetilde{\nabla}^2\boldsymbol{u},\widetilde{\nabla}P)\|_{L^2(L^2)}\le M(1+\|\boldsymbol{A}_0\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}}),\tag{1.15}$$

$$\|\boldsymbol{A}\|_{L^\infty(\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})}+\|(\partial_t\boldsymbol{A},\widetilde{\nabla}^2\boldsymbol{A},\widetilde{\nabla}Q)\|_{L^{\frac{3}{2},1}\cap L^{6,1}(L^3)}\le C\|\boldsymbol{A}_0\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}},\tag{1.16}$$

where M depending on $\|\boldsymbol{u}_0\|_{H^1},\|\nabla\times\boldsymbol{A}_0\|_{L^2},c_0,C_0$.

Several comments concerning our Theorem 1.2 and Theorem 1.3 are listed below:

(i) To our knowledge, Theorem 1.2 and Theorem 1.3 are the first results of well-posedness for the inhomogeneous Hall-MHD system (partially) in critical spaces that allow discontinuous density. Meanwhile, Theorem 1.3 is the first to show that the $2\frac{1}{2}$D Hall-MHD system is globally well-posed for any large regular initial velocity and general Magnetic Prandtl number ν/μ.

(ii) Similar to Theorem 1.2, for the $2\frac{1}{2}$D case one may also obtain a small data global well-posedness result in critical spaces, but removing the small variation assumption on the initial density like in Theorem 1.3 is hard. The main problem lies on obtaining L^1-*Lip* estimate for the velocity when density has large variation and is only bounded. To our understanding, because of the existence of Hall term, the time-weighted methods developed in [19, 24] are not likely available for the (quasilinear) system (1.10). Indeed, our idea of proving uniqueness part for the Theorem 1.3 is new, they are different from the recent work [24] of R. Danchin and S. Wang.

(iii) Our proofs of the theorems also work for the inhomogeneous incompressible MHD system without Hall term (that corresponds to $h=0$), thus improve previous results [4, 6, 13, 31, 43, 44].

Structure of the paper. In section 2, we introduce the Lagrangian transform and its properties, and list some useful linear estimates for the heat equation and Stokes system. The next two sections are devoted to the proof of Theorem 1.2 and Theorem 1.3. Finally, for reader's convenience, in the appendix we recall a few properties of Besov and Lorentz spaces and present proofs of some new propositions.

Notation. We end this introductory part presenting a few notations. As usual, we denote by C harmless positive constants that may change from line to line, and $A \lesssim B$ means that $A \leq CB$. For X a Banach space, $p \in [1,\infty]$ and $T > 0$, the notation $L^p(0,T;X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0,T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0,T)$, endowed with the norm $\|\cdot\|_{L^p_T(X)} := \|\|\cdot\|_X\|_{L^p(0,T)}$. We agree that $\mathcal{C}([0,T];X)$ denotes the set of continuous functions from $[0,T]$ to X. Sometimes, we use the notation $L^p(X)$ to designate the space $L^p(\mathbb{R}_+;X)$ and $\|\cdot\|_{L^p(X)}$ for the associated norm. We will keep the same notations for multi-component functions, namely for $f : [0,T] \to X^m$ with $m \in \mathbb{N}$.

2. Preliminary

2.1. **The Lagrangian coordinates.** Since the density is rough, a crucial point in our proofs of uniqueness is the use of Lagrangian coordinates. The following presentation and propositions are borrowed from [17, 18].

Let $\boldsymbol{X}$ be the flow associated to the velocity $\boldsymbol{v}$, that is, the solution to

$$\frac{d}{dt}\boldsymbol{X}_{\boldsymbol{v}}(t,y) = \boldsymbol{v}(t,\boldsymbol{X}_{\boldsymbol{v}}(t,y)), \quad \boldsymbol{X}_{\boldsymbol{v}}(0,y) = y, \quad \forall\, y \in \mathbb{R}^d. \tag{2.1}$$

Equation (2.1) describes the relation between the Eulerian coordinates $x := \boldsymbol{X}_{\boldsymbol{v}}(t,y)$ and the Lagrangian coordinates y.

Let us now list a few basic properties for the Lagrangian change of variables.

Proposition 2.1. *Denote $D := \nabla^{\mathrm{T}}$, $\overline{\boldsymbol{v}}(t,y) := \boldsymbol{v}(t,\boldsymbol{X}_{\boldsymbol{v}}(t,y))$. Suppose that $D_x\boldsymbol{v} \in L^1(0,T;L^\infty(\mathbb{R}^d))$. Then the solution to the system (2.1) exists on the time interval $[0,T]$, and $D_y\boldsymbol{X}_{\boldsymbol{v}} \in L^\infty(0,T;L^\infty(\mathbb{R}^d))$ with, in addition*

$$\|D_y\boldsymbol{X}_{\boldsymbol{v}}(t)\|_{L^\infty(\mathbb{R}^d)} \leq \exp\Big(\int_0^t \|D_x\boldsymbol{v}\|_{L^\infty(\mathbb{R}^d)}\Big).$$

Furthermore,

$$\boldsymbol{X}_{\boldsymbol{v}}(t,y) = y + \int_0^t \overline{\boldsymbol{v}}(\tau,y)\,d\tau,$$

and

$$D_y\boldsymbol{X}(t,y) = \mathrm{Id} + \int_0^t D_y\overline{\boldsymbol{v}}(t,y)\,d\tau.$$

Let $\boldsymbol{Y}_{\boldsymbol{v}}(t,\cdot)$ be the inverse diffeomorphism of $\boldsymbol{X}_{\boldsymbol{v}}(t,\cdot)$. Then

$$D_x\boldsymbol{Y}_{\boldsymbol{v}}(t,x) = (D_y\boldsymbol{X}_{\boldsymbol{v}}(t,y))^{-1}$$

and, if

$$\int_0^t \|D_y\overline{\boldsymbol{v}}(\tau)\|_{L^\infty(\mathbb{R}^d)}\,d\tau \leq \frac{1}{2},$$

then

$$\mathbb{M}_{\overline{\boldsymbol{v}}} := (D_y\boldsymbol{X}_{\boldsymbol{v}}(t,y))^{-1} = \big(\mathrm{Id} + (D_y\boldsymbol{X}_{\boldsymbol{v}} - \mathrm{Id})\big)^{-1} = \sum_{j=0}^{\infty}(-1)^j\Big(\int_0^t D_y\overline{\boldsymbol{v}}\,d\tau\Big)^j,$$

and the following inequalities hold true:

$$|\mathbb{M}_{\overline{\boldsymbol{v}}}(t,y)-\mathrm{Id}|\lesssim\int_0^t|D_y\overline{\boldsymbol{v}}(\tau,y)|\,d\tau,$$
$$|D_y\mathbb{M}_{\overline{\boldsymbol{v}}}(t,y)|\lesssim\int_0^t|D_y^2\overline{\boldsymbol{v}}(\tau,y)|\,d\tau,$$
$$|\partial_t\mathbb{M}_{\overline{\boldsymbol{v}}}(t,y)|\lesssim|D_y\overline{\boldsymbol{v}}(t,y)|.$$

Let us now derive the system (1.10) in the Lagrangian coordinates. We set

$$\begin{aligned}&\overline{\rho}(t,y):=\rho(t,\boldsymbol{X}_{\boldsymbol{u}}(t,y)),\quad \overline{P}(t,y):=P(t,\boldsymbol{X}_{\boldsymbol{u}}(t,y)),\\&\overline{\boldsymbol{u}}(t,y):=\boldsymbol{u}(t,\boldsymbol{X}_{\boldsymbol{u}}(t,y)),\quad \overline{\boldsymbol{B}}(t,y):=\boldsymbol{B}(t,\boldsymbol{X}_{\boldsymbol{u}}(t,y)),\\&\overline{\boldsymbol{A}}(t,y):=\boldsymbol{A}(t,\boldsymbol{X}_{\boldsymbol{u}}(t,y)),\quad \overline{Q}(t,y):=Q(t,\boldsymbol{X}_{\boldsymbol{u}}(t,y)).\end{aligned}\tag{2.2}$$

Using the chain rule, we find that $\overline{\rho}\equiv\rho_0$ and that $(\overline{\boldsymbol{u}},\overline{P},\overline{\boldsymbol{A}},\overline{Q})$ satisfies

$$\begin{cases}\rho_0\partial_t\overline{\boldsymbol{u}}-\Delta_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{u}}+\nabla_{\overline{\boldsymbol{u}}}\overline{P}+\Delta_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{A}}\times(\nabla_{\overline{\boldsymbol{u}}}\times\overline{\boldsymbol{A}})=0,\\ \mathrm{div}_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{u}}=0,\\ \partial_t\overline{\boldsymbol{A}}-\Delta_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{A}}+\nabla_{\overline{\boldsymbol{u}}}\overline{Q}=(\nabla_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{A}})^{\mathrm{T}}\overline{\boldsymbol{u}}+\dfrac{1}{\rho_0}\Delta_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{A}}\times(\nabla_{\overline{\boldsymbol{u}}}\times\overline{\boldsymbol{A}}),\\ \mathrm{div}_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{A}}=0,\\ (\overline{\boldsymbol{u}},\overline{\boldsymbol{A}})|_{t=0}=(\boldsymbol{u}_0,\boldsymbol{A}_0),\end{cases}\tag{2.3}$$

where operators $\Delta_{\overline{\boldsymbol{u}}}$, $\nabla_{\overline{\boldsymbol{u}}}$ and $\mathrm{div}_{\overline{\boldsymbol{u}}}$ correspond to the original operators Δ, ∇ and div , respectively, after performing the change to the Lagrangian coordinates. Index $\overline{\boldsymbol{u}}$ underlines the dependency on $\overline{\boldsymbol{u}}$. Moreover,

$$\nabla_{\overline{\boldsymbol{u}}}=\mathbb{M}_{\overline{\boldsymbol{u}}}^{-\mathrm{T}}\nabla_y,\quad \mathrm{div}_{\overline{\boldsymbol{u}}}=\mathbb{M}_{\overline{\boldsymbol{u}}}^{-\mathrm{T}}:\nabla_y=\mathrm{div}_y(\mathbb{M}_{\overline{\boldsymbol{u}}}\cdot),\quad \Delta_{\overline{\boldsymbol{u}}}=\mathrm{div}_y(\mathbb{M}_{\overline{\boldsymbol{u}}}\mathbb{M}_{\overline{\boldsymbol{u}}}^{-\mathrm{T}}\nabla_y\cdot).\tag{2.4}$$

We point out the equivalence between system (2.3) and system (1.10), whenever, say,

$$\int_0^t\|D_y\overline{\boldsymbol{u}}(\tau)\|_{L^\infty(\mathbb{R}^d)}\,d\tau\le\frac{1}{2}.\tag{2.5}$$

Of course, if that condition is fulfilled, then one may write that

$$\mathbb{M}_{\overline{\boldsymbol{u}}}=\big(\mathrm{Id}+(D_y\boldsymbol{X}_{\boldsymbol{u}}-\mathrm{Id})\big)^{-1}=\sum_{j=0}^{\infty}(-1)^j\Big(\int_0^t D_y\overline{\boldsymbol{u}}\,d\tau\Big)^j.\tag{2.6}$$

In Lagrangian coordinates, we will use repeatedly the fact that

$$\delta\mathbb{M}(t):=\mathbb{M}_{\overline{\boldsymbol{u}}^1}(t)-\mathbb{M}_{\overline{\boldsymbol{u}}^2}(t)=\Big(\int_0^t(D_y\overline{\boldsymbol{u}}^1-D_y\overline{\boldsymbol{u}}^2)\,d\tau\Big)\cdot\Big(\sum_{k\ge1}\sum_{0\le j<k}\mathbb{D}_1^j\mathbb{D}_2^{k-1-j}\Big),$$

with $\mathbb{D}_i(t):=\int_0^t D_y\overline{\boldsymbol{u}}^i\,d\tau$ for $i=1,2$.

Then, we have the proposition below.

Proposition 2.2. *Let $\overline{\boldsymbol{u}}^1$ and $\overline{\boldsymbol{u}}^2$ be two vector fields satisfying* (2.5) *and define $\delta\overline{\boldsymbol{u}}:=\overline{\boldsymbol{u}}^1-\overline{\boldsymbol{u}}^2$. Then*

$$|\delta\mathbb{M}(t)|\lesssim\int_0^t|D_y\delta\overline{\boldsymbol{u}}(\tau,y)|\,d\tau,$$
$$|D\delta\mathbb{M}(t)|\lesssim\int_0^t|D_y^2\delta\overline{\boldsymbol{u}}(\tau,y)|\,d\tau+\int_0^t|D_y\delta\overline{\boldsymbol{u}}(\tau,y)|\,d\tau\int_0^t(|D_y^2\overline{\boldsymbol{u}}^1(\tau,y)|+|D_y^2\overline{\boldsymbol{u}}^2(\tau,y)|)\,d\tau,$$
$$|\partial_t\delta\mathbb{M}(t)|\lesssim|D_y\delta\overline{\boldsymbol{u}}(t,y)|+\int_0^t|D_y\delta\overline{\boldsymbol{u}}(\tau,y)|\,d\tau\ (|D_y\overline{\boldsymbol{u}}^1(\tau,y)|+|D_y\overline{\boldsymbol{u}}^2(\tau,y)|).$$

We still use the same notations as in (2.2) for the $2\frac{1}{2}$D case without confusion. The Lagrangian formulation for system (1.12) reads:

$$\begin{cases} \rho_0 \partial_t \overline{\boldsymbol{u}} - \widetilde{\Delta}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{u}} + \widetilde{\nabla}_{\overline{\boldsymbol{u}}} \overline{P} + \widetilde{\Delta}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{A}} \times (\widetilde{\nabla}_{\overline{\boldsymbol{u}}} \times \overline{\boldsymbol{A}}) = 0, \\ \widetilde{\operatorname{div}}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{u}} = 0, \\ \partial_t \overline{\boldsymbol{A}} - \widetilde{\Delta}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{A}} + \widetilde{\nabla}_{\overline{\boldsymbol{u}}} \overline{Q} = (\widetilde{\nabla}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{A}})^{\mathrm{T}} \overline{\boldsymbol{u}} + \dfrac{1}{\rho_0} \widetilde{\Delta}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{A}} \times (\widetilde{\nabla}_{\overline{\boldsymbol{u}}} \times \overline{\boldsymbol{A}}), \\ \widetilde{\operatorname{div}}_{\overline{\boldsymbol{u}}} \overline{\boldsymbol{A}} = 0, \\ (\overline{\boldsymbol{u}}, \overline{\boldsymbol{A}})|_{t=0} = (\boldsymbol{u}_0, \boldsymbol{A}_0), \end{cases} \tag{2.7}$$

with (similar to (2.4))

$$\widetilde{\nabla}_{\overline{\boldsymbol{u}}} = \mathbb{M}_{\overline{\boldsymbol{u}}}^{\mathrm{T}} \widetilde{\nabla}_y, \quad \widetilde{\operatorname{div}}_{\overline{\boldsymbol{u}}} = \mathbb{M}_{\overline{\boldsymbol{u}}}^{\mathrm{T}} : \widetilde{\nabla}_y = \widetilde{\operatorname{div}}_y (\mathbb{M}_{\overline{\boldsymbol{u}}} \cdot), \quad \widetilde{\Delta}_{\overline{\boldsymbol{u}}} = \widetilde{\operatorname{div}}_y (\mathbb{M}_{\overline{\boldsymbol{u}}} \mathbb{M}_{\overline{\boldsymbol{u}}}^{\mathrm{T}} \widetilde{\nabla}_y \cdot)$$

and $\mathbb{M}_{\overline{\boldsymbol{u}}}$ is now defined by replacing D to $\widetilde{D} := \widetilde{\nabla}^{\mathrm{T}}$ in (2.6).

2.2. **Some linear estimates.** In order to prove existence of solutions in critical spaces that allow discontinuous density for the system (1.10), we need to borrow the following maximal regularity result from [20].

Proposition 2.3. *Let $T, \kappa > 0$, $1 < p, m < \infty$ and $1 \le r \le \infty$. Then, for any $v_0 \in \dot{B}^{2-\frac{2}{m}}_{p,r}(\mathbb{R}^d)$ and $f \in L^{m,r}(0,T; L^p(\mathbb{R}^d))$, the heat equation*

$$\begin{cases} \partial_t v - \kappa \Delta v = f & \text{in } (0,T) \times \mathbb{R}^d, \\ v|_{t=0} = v_0 & \text{in } \mathbb{R}^d \end{cases} \tag{2.8}$$

has a unique solution in the space $\dot{W}^{2,1}_{p,(m,r)}((0,T) \times \mathbb{R}^d)$ and the following inequality holds true:

$$\begin{aligned} &\kappa^{1-\frac{1}{m}} \|v\|_{L^\infty(0,T;\dot{B}^{2-2/m}_{p,r})} + \|v_t, \kappa \nabla^2 v\|_{L^{m,r}(0,T;L^p)} \\ &\qquad \le C\big(\kappa^{1-\frac{1}{m}} \|v_0\|_{\dot{B}^{2-\frac{2}{m}}_{p,r}} + \|f\|_{L^{m,r}(0,T;L^p)}\big). \end{aligned} \tag{2.9}$$

Furthermore, if $\frac{2}{m} + \frac{d}{p} > 2$, then for all $m < s < \infty$ and $p \le \ell$ such that $1 + \frac{d}{2}(\frac{1}{\ell} - \frac{1}{p}) > 0$, interrelated by

$$\frac{d}{2\ell} + \frac{1}{s} = \frac{1}{m} + \frac{d}{2p} - 1,$$

it holds that

$$\kappa^{1+\frac{1}{s}-\frac{1}{m}} \|v\|_{L^{s,r}(0,T;L^\ell)} \le C\big(\kappa^{1-\frac{1}{m}} \|v\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{m}}_{p,r})} + \|v_t, \kappa \nabla^2 v\|_{L^{m,r}(0,T;L^p)}\big). \tag{2.10}$$

In order to prove Theorem 1.3, we introduce a $\lambda-$modified equation for $\boldsymbol{A}$ (that is (4.11)), and we need the following proposition to control $\boldsymbol{A}_\lambda$.

Proposition 2.4. *Let $T, \kappa > 0$, $1 < p, m < \infty$ and $1 \le r \le \infty$. Assume further that $g \in L^1(0,T)$ with $g(t) \ge 0$. Then, for any $v_0 \in \dot{B}^{2-\frac{2}{m}}_{p,r}(\mathbb{R}^d)$ and $f \in L^{m,r}(0,T; L^p(\mathbb{R}^d))$, the heat equation with time-weighted damping*

$$\begin{cases} \partial_t v - \kappa \Delta v + g\, v = f & \text{in } (0,T) \times \mathbb{R}^d, \\ v|_{t=0} = v_0 & \text{in } \mathbb{R}^d \end{cases} \tag{2.11}$$

has a unique solution in the space $\dot{W}^{2,1}_{p,(m,r)}((0,T)\times\mathbb{R}^d)$ *and the following inequality holds true:*

$$\kappa^{1-\frac{1}{m}}\|v\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{m}}_{p,r})}+\|v_t,\kappa\nabla^2 v,g\,v\|_{L^{m,r}(0,T;L^p)}\\ \le C\big(\kappa^{1-\frac{1}{m}}\|v_0\|_{\dot{B}^{2-\frac{2}{m}}_{p,r}}+\|f\|_{L^{m,r}(0,T;L^p)}\big). \quad (2.12)$$

Furthermore, if $\frac{2}{m}+\frac{d}{p}>2$, *then for all* $m<s<\infty$ *and* $p\le\ell$ *such that* $1+\frac{d}{2}(\frac{1}{\ell}-\frac{1}{p})>0$, *interrelated by*

$$\frac{d}{2\ell}+\frac{1}{s}=\frac{1}{m}+\frac{d}{2p}-1,$$

it holds that

$$\kappa^{1+\frac{1}{s}-\frac{1}{m}}\|v\|_{L^{s,r}(0,T;L^l)}\le C\big(\kappa^{1-\frac{1}{m}}\|v\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{m}}_{p,r})}+\|v_t,\kappa\nabla^2 v\|_{L^{m,r}(0,T;L^p)}\big). \quad (2.13)$$

Since the linearization of the system (2.3) is the Stokes system, we need maximal regularity estimates for the Stokes problem. The following one can be found in [18].

Proposition 2.5. *Let* $T,\kappa>0, 1<p,m<\infty$. *Assume that* $\boldsymbol{u}_0\in\dot{B}^{2-\frac{2}{m}}_{p,m}(\mathbb{R}^d)$, $\boldsymbol{f}\in L^m(0,T;L^p(\mathbb{R}^d))$, $\boldsymbol{R}\in\dot{W}^1_m(0,T;L^p(\mathbb{R}^d))$ *with* $\operatorname{div}\boldsymbol{R}\in L^m(0,T;\dot{W}^1_p(\mathbb{R}^d))$. *Suppose that* $\operatorname{div}\boldsymbol{u}_0=\operatorname{div}\boldsymbol{R}|_{t=0}$. *Then there exists a unique solution* $(\boldsymbol{u},\nabla P)$ *to*

$$\begin{cases}\partial_t\boldsymbol{u}-\kappa\Delta\boldsymbol{u}+\nabla P=\boldsymbol{f} & \text{in } (0,T)\times\mathbb{R}^d,\\ \operatorname{div}\boldsymbol{u}=\operatorname{div}\boldsymbol{R} & \text{in } (0,T)\times\mathbb{R}^d,\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0 & \text{in } \mathbb{R}^d\end{cases} \quad (2.14)$$

such that $\boldsymbol{u}\in\dot{W}^{2,1}_{p,m}((0,T)\times\mathbb{R}^d)$, $\nabla P\in L^m(0,T;L^p(\mathbb{R}^d))$, *and the following estimate is valid:*

$$\kappa^{1-\frac{1}{m}}\|\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{m}}_{p,m})}+\|\boldsymbol{u}_t,\kappa\nabla^2\boldsymbol{u},\nabla P\|_{L^m(0,T;L^p)}\\ \le C\Big(\kappa^{1-\frac{1}{m}}\|\boldsymbol{u}_0\|_{\dot{B}^{2-\frac{2}{m}}_{p,m}}+\|\boldsymbol{f},\boldsymbol{R}_t\|_{L^m(0,T;L^p)}+\|\kappa\operatorname{div}\boldsymbol{R}\|_{L^m(0,T;\dot{W}^1_p)}\Big).$$

where C *is independent of* κ *and* T.

In order to prove Theorem 1.3, we develop the following maximal regularity estimates for the Stokes system with only bounded coefficient.

Proposition 2.6. *Let* $T,\kappa>0, 1<p<\infty$. *Assume that* $\boldsymbol{u}_0\in\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^d)$, $\boldsymbol{f}\in L^p((0,T)\times\mathbb{R}^d)$, $\boldsymbol{R}\in\dot{W}^1_p(0,T;L^p(\mathbb{R}^d))$ *with* $\operatorname{div}\boldsymbol{R}\in L^p(0,T;\dot{W}^1_p(\mathbb{R}^d))$. *Suppose that* $\operatorname{div}\boldsymbol{u}_0=\operatorname{div}\boldsymbol{R}|_{t=0}$ *and that*

$$0<\rho_*\le\rho(t,x)\le\rho^*\quad\text{for } (t,x)\in[0,T]\times\mathbb{R}^d.$$

Then there exist positive constants 2^* *and* 2_* *depending only on* ρ_* *and* ρ^*, *with* $2_*<2<2^*$, *such that for all* $p\in(2_*,2^*)$ *the unique solution* $(\boldsymbol{u},\nabla P)$ *of system*

$$\begin{cases}\rho\partial_t\boldsymbol{u}-\kappa\Delta\boldsymbol{u}+\nabla P=\boldsymbol{f} & \text{in } (0,T)\times\mathbb{R}^d,\\ \operatorname{div}\boldsymbol{u}=\operatorname{div}\boldsymbol{R} & \text{in } (0,T)\times\mathbb{R}^d,\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0 & \text{in } \mathbb{R}^d,\end{cases} \quad (2.15)$$

satisfies

$$\kappa^{1-\frac{1}{p}}\|\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p})}+\|\boldsymbol{u}_t,\kappa\nabla^2\boldsymbol{u},\nabla P\|_{L^p((0,T)\times\mathbb{R}^d)}$$
$$\leq C(p,\rho_*,\rho^*)\Big(\|\boldsymbol{u}_0\|_{\dot{B}^{2-\frac{2}{p}}_{p,p}}+\|\boldsymbol{f},\boldsymbol{R}_t\|_{L^p((0,T)\times\mathbb{R}^d)}+\|\kappa\,\mathrm{div}\,\boldsymbol{R}\|_{L^p(0,T;\dot{W}^1_p)}\Big).$$

The proof of the above proposition is given in the appendix.

3. Proof of 3D case

In this section, our aim is to prove Theorem 1.2 for system (1.10). The details of the proof are divided into four steps. In particular, in the last step, we handle the uniqueness in the subsection 3.1, which is the most difficult part.

Step 1: An iterative scheme. Define $a=\rho-1$, we rewrite (1.10) to

$$\begin{cases}\partial_t a+\boldsymbol{u}\cdot\nabla a=0,\\ \partial_t\boldsymbol{u}-\Delta\boldsymbol{u}=\boldsymbol{f}_1(\boldsymbol{u},\boldsymbol{A}),\\ \partial_t\boldsymbol{A}-\Delta\boldsymbol{A}=\boldsymbol{f}_2(\boldsymbol{u},\boldsymbol{A}),\\ (\rho,\boldsymbol{u},\boldsymbol{A})_{t=0}=(\rho_0,\boldsymbol{u}_0,\boldsymbol{A}_0),\end{cases}\tag{3.1}$$

where

$$\boldsymbol{f}_1=-\mathcal{P}\Big(a\partial_t\boldsymbol{u}+(1+a)\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\Delta\boldsymbol{A}\times(\nabla\times\boldsymbol{A})\Big),$$
$$\boldsymbol{f}_2=\mathcal{P}\Big(\boldsymbol{u}\times(\nabla\times\boldsymbol{A})+\frac{1}{1+a}\,\Delta\boldsymbol{A}\times(\nabla\times\boldsymbol{A})\Big).$$

The idea is to construct iteratively a sequence $(a^n,\boldsymbol{u}^n,\boldsymbol{A}^n)_{n\in\mathbb{N}}$ of smooth approximate global solutions by solving only linear transport and heat equations. To this end, we smooth out the initial data: $a_0^n:=\dot{S}_n a_0$, $\boldsymbol{u}_0^n:=\dot{S}_n\boldsymbol{u}_0$ and $\boldsymbol{A}_0^n:=\dot{S}_n\boldsymbol{A}_0$, and define the first term $(a^0,\boldsymbol{u}^0,\boldsymbol{A}^0)$ of our sequence to be

$$a^0:=a_0^0,\quad \boldsymbol{u}^0=e^{t\Delta}\boldsymbol{u}_0^0,\quad \boldsymbol{A}^0=e^{t\Delta}\boldsymbol{A}_0^0.$$

Next, assuming that $(a^n,\boldsymbol{u}^n,\boldsymbol{A}^n)$ has been globally defined on $\mathbb{R}_+\times\mathbb{R}^3$ and smooth, we take for $(a^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{A}^{n+1})$ the solution to the following linear transport-heat equations:

$$\begin{cases}\partial_t a^{n+1}+\boldsymbol{u}^n\cdot\nabla a^{n+1}=0,\\ \partial_t\boldsymbol{u}^{n+1}-\Delta\boldsymbol{u}^{n+1}=\boldsymbol{f}_1^n,\\ \partial_t\boldsymbol{A}^{n+1}-\Delta\boldsymbol{A}^{n+1}=\boldsymbol{f}_2^n,\\ (a^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{A}^{n+1})|_{t=0}=(a_0^{n+1},\boldsymbol{u}_0^{n+1},\boldsymbol{A}_0^{n+1}),\end{cases}\tag{3.2}$$

where

$$\boldsymbol{f}_1^n=-\mathcal{P}\Big(a^n\partial_t\boldsymbol{u}^n+(1+a^n)\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n+\Delta\boldsymbol{A}^n\times(\nabla\times\boldsymbol{A}^n)\Big),$$
$$\boldsymbol{f}_2^n=\mathcal{P}\Big(\boldsymbol{u}^n\times(\nabla\times\boldsymbol{A}^n)+\frac{1}{1+a^n}\,\Delta\boldsymbol{A}^n\times(\nabla\times\boldsymbol{A}^n)\Big).$$

Let us emphasize that as the initial data, transport field and source term are smooth and globally defined, it is also the case for $(a^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{A}^{n+1})$.

Step 2: Uniform Estimates. We are going to show by induction that there exist two generic constants C_1 and C_2 such that for all $n \in \mathbb{N}$,

$$\|a^n\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3)} \leq C_1\|a_0\|_{L^\infty(\mathbb{R}^3)}, \tag{3.3}$$

$$\begin{aligned}\|\boldsymbol{u}^n\|_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}(\mathbb{R}_+\times\mathbb{R}^3)} + \|\boldsymbol{A}^n\|_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}(\mathbb{R}_+\times\mathbb{R}^3)} + \|\boldsymbol{A}^n\|_{\dot{W}^{2,1}_{4,(8,1)}(\mathbb{R}_+\times\mathbb{R}^3)}\\ \leq C_2\Big(\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1}(\mathbb{R}^3)} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{7}{4}}_{4,1}(\mathbb{R}^3)}\Big).\end{aligned} \tag{3.4}$$

In the following, we omit the notation $\mathbb{R}^3$ for simplicity. Noticing that, for initial data we have: $\forall\ n \in \mathbb{N}$, $\|a_0^n\|_{L^\infty} \lesssim \|a_0\|_{L^\infty}$ and

$$\|\boldsymbol{u}_0^n\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \lesssim \|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}}, \quad \|\boldsymbol{A}_0^n\|_{\dot{B}^{\frac{3}{4}}_{4,1}\cap\dot{B}^{\frac{7}{4}}_{4,1}} \lesssim \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1}\cap\dot{B}^{\frac{7}{4}}_{4,1}}.$$

And using classical estimates of heart kernel (or Proposition 2.3), we see that (3.4) is fulfilled for $n = 0$ if C_2 is taken large enough.

Let us now assume that (3.3) and (3.4) are satisfied at rank n, we are going to prove them at rank $n+1$. From $(3.2)_1$, we see that

$$\|a^{n+1}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3)} \leq \|a_0^{n+1}\|_{L^\infty(\mathbb{R}^3)} \leq C_1\|a_0\|_{L^\infty(\mathbb{R}^3)}. \tag{3.5}$$

Regarding $\boldsymbol{u}^{m+1}$, we apply Proposition 2.3 to estimate it in $\dot{B}^{\frac{1}{2}}_{2,1}$ by taking $p = 2, m = \frac{4}{3}, r = 1, s = 8, \ell = 4$. We get

$$\begin{aligned}\|\boldsymbol{u}^{n+1}\|_{L^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\boldsymbol{u}_t^{n+1}, \nabla^2\boldsymbol{u}^{n+1}\|_{L^{\frac{4}{3},1}(L^2)} + \|\boldsymbol{u}^{n+1}\|_{L^{8,1}(L^4)}\\ \leq C\Big(\|\boldsymbol{u}_0^{n+1}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{f}_1^n\|_{L^{\frac{4}{3},1}(L^2)}\Big).\end{aligned} \tag{3.6}$$

Similarly, bounding $\boldsymbol{A}^{n+1}$ also relies on Proposition 2.3: by taking $p = 4, m = \frac{8}{5}, r = 1$ and $p = 4, m = 8, r = 1$, respectively, we get

$$\begin{aligned}\|\boldsymbol{A}^{n+1}\|_{L^\infty(\dot{B}^{\frac{3}{4}}_{4,1})} + \|\boldsymbol{A}_t^{n+1}, \nabla^2\boldsymbol{A}^{n+1}\|_{L^{\frac{8}{5},1}(L^4)}\\ \leq C\left(\|\boldsymbol{A}_0^{n+1}\|_{\dot{B}^{\frac{3}{4}}_{4,1}} + \|\boldsymbol{f}_2^n\|_{L^{\frac{8}{5},1}(L^4)}\right)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\|\boldsymbol{A}^{n+1}\|_{L^\infty(\dot{B}^{\frac{7}{4}}_{4,1})} + \|\boldsymbol{A}_t^{n+1}, \nabla^2\boldsymbol{A}^{n+1}\|_{L^{8,1}(L^4)}\\ \leq C\left(\|\boldsymbol{A}_0^{n+1}\|_{\dot{B}^{\frac{7}{4}}_{4,1}} + \|\boldsymbol{f}_2^n\|_{L^{8,1}(L^4)}\right).\end{aligned} \tag{3.8}$$

Next, we estimate $\boldsymbol{f}_1^n$ and $\boldsymbol{f}_2^n$ term by term.

Bounds for $\boldsymbol{f}_1^n$. Using Hölder's inequality, we have

$$\|a^n\partial_t\boldsymbol{u}^n\|_{L^{\frac{4}{3},1}(L^2)} \lesssim \|a^n\|_{L^\infty}\|\boldsymbol{u}_t^n\|_{L^{\frac{4}{3},1}(L^2)}$$

and

$$\begin{aligned}\|(1+a^n)\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n\|_{L^{\frac{4}{3},1}(L^2)} &\lesssim (1+\|a^n\|_{L^\infty(L^\infty)})\,\big\|\|\boldsymbol{u}^n\|_{L^4}\|\nabla\boldsymbol{u}^n\|_{L^4}\big\|_{L^{\frac{4}{3},1}_t}\\ &\lesssim (1+\|a^n\|_{L^\infty(L^\infty)})\|\boldsymbol{u}^n\|_{L^{8,1}(L^4)}\|\nabla\boldsymbol{u}^n\|_{L^{\frac{8}{5}}(L^4)}\\ &\lesssim (1+C_1\|a_0\|_{L^\infty})\|\boldsymbol{u}^n\|^{\frac{6}{5}}_{L^{8,1}(L^4)}\|\nabla^2\boldsymbol{u}^n\|^{\frac{4}{5}}_{L^{\frac{4}{3}}(L^2)},\end{aligned}$$

where in the last inequality we used (3.3) and the Gagliardo-Nirenberg type inequality

$$\|\nabla \boldsymbol{u}^n\|_{L^{\frac{8}{5}}(L^4)} \lesssim \|\boldsymbol{u}^n\|^{\frac{1}{5}}_{L^8(L^4)} \|\nabla^2 \boldsymbol{u}^n\|^{\frac{4}{5}}_{L^{\frac{4}{3}}(L^2)}.$$

For the last term in $\boldsymbol{f}_1^n$, as a consequence of Hölder's inequality and

$$\|\nabla \boldsymbol{A}^n\|_{L^8(L^4)} \lesssim \Big\| \|\boldsymbol{A}\|^{\frac{4}{5}}_{L^\infty} \|\nabla^2 \boldsymbol{A}\|^{\frac{1}{5}}_{L^4} \Big\|_{L^8_t} \lesssim \|\boldsymbol{A}\|^{\frac{4}{5}}_{L^\infty(L^\infty)} \|\nabla^2 \boldsymbol{A}\|^{\frac{1}{5}}_{L^{\frac{8}{5},1}(L^4)},$$

we obtain

$$\begin{aligned}
\|\Delta \boldsymbol{A}^n \times (\nabla \times \boldsymbol{A}^n)\|_{L^{\frac{4}{3},1}(L^2)} \lesssim & \|\nabla \boldsymbol{A}^n\|_{L^8(L^4)} \|\nabla^2 \boldsymbol{A}^n\|_{L^{\frac{8}{5},1}(L^4)} \\
\lesssim & \|\boldsymbol{A}^n\|^{\frac{4}{5}}_{L^\infty(L^\infty)} \|\nabla^2 \boldsymbol{A}^n\|^{\frac{6}{5}}_{L^{\frac{8}{5},1}(L^4)} \\
\lesssim & \|\boldsymbol{A}^n\|^{\frac{4}{5}}_{L^\infty(\dot{B}^{\frac{3}{4}}_{4,1})} \|\nabla^2 \boldsymbol{A}^n\|^{\frac{6}{5}}_{L^{\frac{8}{5},1}(L^4)},
\end{aligned}$$

where in the last step, we used the embedding $\dot{B}^{\frac{3}{4}}_{4,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$.

Bounds for $\boldsymbol{f}_2^n$. We first estimate $\boldsymbol{f}_2^n$ in $L^{\frac{8}{5}}(\mathbb{R}_+; L^4(\mathbb{R}^3))$. Applying Hölder's inequality, we have

$$\begin{aligned}
\Big\| \frac{1}{1+a^n} \Delta \boldsymbol{A}^n \times (\nabla \times \boldsymbol{A}^n) \Big\|_{L^{\frac{8}{5},1}(L^4)} \lesssim & \frac{1}{1 - \|a^n\|_{L^\infty(L^\infty)}} \|\nabla \boldsymbol{A}^n\|_{L^\infty(L^\infty)} \|\nabla^2 \boldsymbol{A}^n\|_{L^{\frac{8}{5},1}(L^4)} \\
\lesssim & \frac{1}{1 - C_1 \|a_0\|_{L^\infty}} \|\boldsymbol{A}^n\|_{L^\infty(\dot{B}^{\frac{7}{4}}_{4,1})} \|\nabla^2 \boldsymbol{A}^n\|_{L^{\frac{8}{5},1}(L^4)}.
\end{aligned}$$

Using the following Gagliardo-Nirenberg type inequality

$$\|\nabla \boldsymbol{A}^n\|_{L^2(L^\infty)} \lesssim \Big\| \|\boldsymbol{A}^n\|^{\frac{1}{5}}_{L^\infty} \|\nabla^2 \boldsymbol{A}^n\|^{\frac{4}{5}}_{L^4} \Big\|_{L^2_t} \lesssim \|\boldsymbol{A}^n\|^{\frac{1}{5}}_{L^\infty(\dot{B}^{\frac{3}{4}}_{4,1})} \|\nabla^2 \boldsymbol{A}^n\|^{\frac{4}{5}}_{L^{\frac{8}{5}}(L^4)},$$

we get

$$\begin{aligned}
\|\boldsymbol{u}^n \times (\nabla \times \boldsymbol{A}^n)\|_{L^{\frac{8}{5},1}(L^4)} \lesssim & \big\| \|\boldsymbol{u}^n\|_{L^4} \|\nabla \boldsymbol{A}^n\|_{L^\infty} \big\|_{L^{\frac{8}{5},1}_t} \\
\lesssim & \|\boldsymbol{u}^n\|_{L^{8,1}(L^4)} \|\nabla \boldsymbol{A}^n\|_{L^2(L^\infty)} \\
\lesssim & \|\boldsymbol{u}^n\|_{L^{8,1}(L^4)} \|\boldsymbol{A}^n\|^{\frac{1}{5}}_{L^\infty(\dot{B}^{\frac{3}{4}}_{4,1})} \|\nabla^2 \boldsymbol{A}^n\|^{\frac{4}{5}}_{L^{\frac{8}{5},1}(L^4)}. \qquad (3.9)
\end{aligned}$$

Similarly, we estimate $\boldsymbol{f}_2^n$ in $L^{8,1}(\mathbb{R}_+; L^4(\mathbb{R}^3))$. We have

$$\begin{aligned}
\Big\| \frac{1}{1+a^n} \Delta \boldsymbol{A}^n \times (\nabla \times \boldsymbol{A}^n) \Big\|_{L^{8,1}(L^4)} \lesssim & \frac{1}{1 - \|a^n\|_{L^\infty(L^\infty)}} \|\nabla \boldsymbol{A}^n\|_{L^\infty(L^\infty)} \|\nabla^2 \boldsymbol{A}^n\|_{L^{8,1}(L^4)} \\
\lesssim & \frac{1}{1 - C_1 \|a_0\|_{L^\infty}} \|\boldsymbol{A}^n\|_{L^\infty(\dot{B}^{\frac{7}{4}}_{4,1})} \|\nabla^2 \boldsymbol{A}^n\|_{L^{8,1}(L^4)}
\end{aligned}$$

and

$$\|\boldsymbol{u}^n \times (\nabla \times \boldsymbol{A}^n)\|_{L^{8,1}(L^4)} \lesssim \|\boldsymbol{u}^n\|_{L^{8,1}(L^4)} \|\nabla \boldsymbol{A}^n\|_{L^\infty(\dot{B}^{\frac{3}{4}}_{4,1})}.$$

Gathering together all the above bounds for $\boldsymbol{f}_1^n$ and $\boldsymbol{f}_2^n$ with (3.6)-(3.8), we obtain that

$$\|\boldsymbol{u}^{n+1}\|_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}} \leq C \Big(\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|a_0\|_{L^\infty} \|\boldsymbol{u}^n\|_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}}$$

$$+ (1 + C_1\|a_0\|_{L^\infty})\|\boldsymbol{u}^n\|^2_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}} + \|\boldsymbol{A}^n\|^2_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}}\Big)$$

and

$$\|\boldsymbol{A}^{n+1}\|_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}\cap\dot{W}^{2,1}_{4,(8,1)}} \leq C\Big(\|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1}\cap\dot{B}^{\frac{7}{4}}_{4,1}} + \|\boldsymbol{u}^n\|_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}}\|\boldsymbol{A}^n\|_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}\cap\dot{W}^{2,1}_{4,(8,1)}} + \frac{1}{1 - C_1\|a_0\|_{L^\infty}}\|\boldsymbol{A}^n\|^2_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}\cap\dot{W}^{2,1}_{4,(8,1)}}\Big).$$

Define $\varepsilon := \|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1}\cap\dot{B}^{\frac{7}{4}}_{4,1}}$ and using (3.4), then we get by combing above two inequalities that

$$\|\boldsymbol{u}^{n+1}\|_{\dot{W}^{2,1}_{2,(\frac{4}{3},1)}(T)} \leq C\Big(\varepsilon + C_1C_2\ \varepsilon\ \|a_0\|_{L^\infty} + (1 + C_1\|a_0\|_{L^\infty})(C_2\ \varepsilon)^2\Big)$$

and

$$\|\boldsymbol{A}^{n+1}\|_{\dot{W}^{2,1}_{4,(\frac{8}{5},1)}\cap\dot{W}^{2,1}_{4,(8,1)}} \leq C\Big(\varepsilon + (C_2\ \varepsilon)^2 + \frac{1}{1 - C_1\|a_0\|_{L^\infty}}(C_2\ \varepsilon)^2\Big).$$

If the initial data are so small as to satisfy the following smallness condition:

$$C_1C_2\|a_0\|_{L^\infty} + (1 + C_1\|a_0\|_{L^\infty})C_2^2\ \varepsilon \leq 1,$$
$$C^2\varepsilon + \frac{C_2^2\varepsilon}{1 - C_1\|a_0\|_{L^\infty}} \leq 1,$$

then (3.4) are fulfilled by $\boldsymbol{u}^{n+1}$ and $\boldsymbol{A}^{n+1}$ with C_2 being chosen such that $C_2 \geq 4C$.

Step 3: Convergence and checking that the limit is a solution. The bounds (3.3) and (3.4) and Proposition A.3 imply that

$$\|a^n\|_{L^\infty(L^\infty)} + \|\boldsymbol{u}^n\|_{L^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\boldsymbol{u}^n_t, \nabla^2\boldsymbol{u}^n\|_{L^{\frac{4}{3}}(L^2)} + \|\boldsymbol{A}^n\|_{L^\infty(\dot{B}^{\frac{3}{4}}_{4,1}\cap\dot{B}^{\frac{7}{4}}_{4,1})} + \|\boldsymbol{A}^n_t, \nabla^2\boldsymbol{A}^n\|_{(L^{\frac{8}{5}}\cap L^8)(L^4)} < \infty, \quad (3.10)$$

it combined with classical functional analysis arguments already ensure that there exists a subsequence of $(a^n, \boldsymbol{u}^n, \boldsymbol{A}^n)$, which we still denote by $(a^n, \boldsymbol{u}^n, \boldsymbol{A}^n)$ and some $(a, \boldsymbol{u}, \boldsymbol{A})$ with

$$a \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3),\ \boldsymbol{u} \in \dot{W}^{2,1}_{2,(\frac{4}{3},1)}(\mathbb{R}_+ \times \mathbb{R}^3),\ \boldsymbol{A} \in (\dot{W}^{2,1}_{4,(\frac{8}{5},1)} \cap \dot{W}^{2,1}_{4,(8,1)})(\mathbb{R}_+ \times \mathbb{R}^3)$$

such that

- $a^n \rightharpoonup a$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ weak $*$;
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ in $L^\infty(\dot{B}^{\frac{1}{2}}_{2,1})$ weak $*$, $(\boldsymbol{u}^n_t, \nabla^2\boldsymbol{u}^n) \rightharpoonup (\boldsymbol{u}_t, \nabla^2\boldsymbol{u})$ weakly in $L^{\frac{4}{3}}(L^2)$;
- $\boldsymbol{A}^n \rightharpoonup \boldsymbol{A}$ in $L^\infty(\dot{B}^{\frac{3}{4}}_{4,1} \cap \dot{B}^{\frac{3}{4}}_{4,1})$ weak $*$ and $(\boldsymbol{A}^n_t, \nabla^2\boldsymbol{A}^n) \rightharpoonup (\boldsymbol{A}_t, \nabla^2\boldsymbol{A})$ weakly in $(L^{\frac{8}{5}} \cap L^8)(L^4)$.

This implies convergence in the distribution meaning, and one can thus pass to the limit in all the linear terms of (3.2). However, we need to exhibit strong convergence properties to pass to the limit in the non-linear terms.

Notice from (3.10) that $(\boldsymbol{u}^n_t)_{n\in\mathbb{N}}$ is bounded in $L^{\frac{4}{3}}(L^2)$ and $(\boldsymbol{A}^n_t)_{n\in\mathbb{N}}$ is bounded in $(L^{\frac{8}{5}} \cap L^8)(L^4)$, from which (3.10), and the Aubin-Lions Theorem, we infer that for all $0 < \delta < 1, 4 < p < \infty$

- $\boldsymbol{u}^n \to \boldsymbol{u}$ strongly in $L^\infty_{loc}(L^{3-\delta}_{loc})$, $\nabla\boldsymbol{u}^n \to \nabla\boldsymbol{u}$ strongly in $L^{\frac{4}{3}}_{loc}(L^{6-\delta}_{loc})$;
- $\nabla\boldsymbol{A}^n \to \nabla\boldsymbol{A}$ strongly in $L^\infty_{loc}(L^p_{loc})$.

Meanwhile, by Gagliardo-Nirenberg's inequality in (A.1), one has $\nabla \boldsymbol{u} \in L^4(L^2)$, then the Di Perna-Lions theory [25] for mass equation ensures that a is the unique solution for $(3.1)_1$ and

$$a^n \to a \quad \text{strongly in } \mathcal{C}_{loc}(\mathbb{R}_+; L^q_{loc}(\mathbb{R}^3)),\ \forall\ 1 \le q < \infty.$$

Finally, one can pass to the limit in all the terms of $\boldsymbol{f}_1^n, \boldsymbol{f}_2^n$ in the equations $(3.2)_2$ and $(3.2)_3$. To give an example, we only focus on the most complicated one, that is, the inhomogeneous Hall term. We decompose

$$\begin{aligned}
&\frac{\Delta \boldsymbol{A}^n \times (\nabla \times \boldsymbol{A}^n)}{1+a^n} - \frac{\Delta \boldsymbol{A} \times (\nabla \times \boldsymbol{A})}{1+a} \\
&= \frac{a-a^n}{(1+a)(1+a_n)}(\Delta \boldsymbol{A}^n \times (\nabla \times \boldsymbol{A}^n)) + \frac{\Delta \boldsymbol{A}^n \times (\nabla \times (\boldsymbol{A}^n - \boldsymbol{A}))}{1+a} \\
&\quad + \frac{(\Delta \boldsymbol{A}^n - \Delta \boldsymbol{A}) \times (\nabla \times \boldsymbol{A})}{1+a}.
\end{aligned}$$

Thanks to the bounds (3.10) and previous convergence properties we know that the first and second term strongly tends to 0 in $L^1_{loc}(L^2_{loc})$, while the third term weakly tends to 0 in $L^8(L^4)$.

Moreover, it is obvious that $a_0^n = \dot{S}_n a_0$ converges almost everywhere to a_0 in $\mathbb{R}^3$ and $\boldsymbol{u}_0^n \to \boldsymbol{u}_0$ in $\dot{B}^{\frac{1}{2}}_{2,1}$, and $\boldsymbol{A}_0^n \to \boldsymbol{A}_0$ in $\dot{B}^{\frac{3}{4}}_{4,1} \cap \dot{B}^{\frac{7}{4}}_{4,1}$, respectively. In conclusion, we have prove that $(a, \boldsymbol{u}, \boldsymbol{A})$ is a distributional solution of system (3.1), and thus complete the proof of the existence part of Theorems 1.2.

3.1. **Step 4: Uniqueness.**

Time-weighted estimates. To prove the uniqueness, the key point is bounding $\nabla \boldsymbol{u}$ in $L^1(0,T;L^\infty)$. We have the following time-weighted estimates for $\boldsymbol{u}$.

Lemma 3.1. *For all $T > 0$, the estimate below holds:*

$$\begin{aligned}
&\|t\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{\frac{7}{4}}_{4,1})} + \|(t\boldsymbol{u})_t, \nabla^2(t\boldsymbol{u}), t\nabla P\|_{L^{8,1}(0,T;L^4)} \\
&\qquad \le C(1+T)\left(\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1} \cap \dot{B}^{\frac{7}{4}}_{4,1}}\right). \quad (3.11)
\end{aligned}$$

Moreover,

$$\|\boldsymbol{u}\|_{L^2(0,T;L^\infty)} + \|\nabla \boldsymbol{u}\|_{L^1(0,T;L^\infty)} \le C(1+T)\left(\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{3}{4}}_{4,1} \cap \dot{B}^{\frac{7}{4}}_{4,1}}\right). \quad (3.12)$$

Proof. Multiplying $(3.1)_2$ by time t yields

$$\partial_t(t\boldsymbol{u}) - \Delta(t\boldsymbol{u}) = \mathcal{P}\Big((1+a)u - a\partial_t(t\boldsymbol{u}) - (1+a)t\boldsymbol{u}\cdot\nabla\boldsymbol{u} - t\Delta\boldsymbol{A}\times(\nabla\times\boldsymbol{A})\Big).$$

Applying Proposition 2.3 to the above equation by choosing $p = 4, m = 8, r = 1$, we get

$$\begin{aligned}
&\|t\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{\frac{7}{4}}_{4,1})} + \|(t\boldsymbol{u})_t, \nabla^2(t\boldsymbol{u})\|_{L^{8,1}(0,T;L^4)} \\
&\quad \lesssim \|(1+a)\boldsymbol{u}\|_{L^{8,1}_T(L^4)} + \|a\partial_t(t\boldsymbol{u})\|_{L^{8,1}_T(L^4)} + \|(1+a)t\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^{8,1}_T(L^4)} \\
&\qquad + \|t\Delta\boldsymbol{A}\times(\nabla\times\boldsymbol{A})\|_{L^{8,1}_T(L^4)}. \quad (3.13)
\end{aligned}$$

Then Hölder's inequality and Proposition A.2 imply that

$$\|t\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^{8,1}(0,T;L^4)}\lesssim\|\boldsymbol{u}\|_{L^{8,1}(0,T;L^4)}\|t\nabla\boldsymbol{u}\|_{L^\infty(0,T;L^\infty)}\lesssim\|\boldsymbol{u}\|_{L^{8,1}(0,T;L^4)}\|t\boldsymbol{u}\|_{L^\infty(\dot B^{\frac74}_{4,1})}$$

and

$$\|t\Delta\boldsymbol{A}\times(\nabla\times\boldsymbol{A})\|_{L^{8,1}(0,T;L^4)}\lesssim T\|\Delta\boldsymbol{A}\|_{L^{8,1}(0,T;L^4)}\|\nabla\boldsymbol{A}\|_{L^\infty(\dot B^{\frac34}_{4,1})}.$$

Hence, combining above two estimates with the bounds (1.11), we find from (3.13) that

$$\|t\boldsymbol{u}\|_{L^\infty(0,T;\dot B^{\frac74}_{4,1})}+\|(t\boldsymbol{u})_t,\nabla^2(t\boldsymbol{u})\|_{L^{8,1}(0,T;L^4)}\le C(1+T)\left(\|\boldsymbol{u}_0\|_{\dot B^{\frac12}_{2,1}}+\|\boldsymbol{A}_0\|_{\dot B^{\frac34}_{4,1}\cap\dot B^{\frac74}_{4,1}}\right).$$

Next, with (3.11) in hand, we bound $\nabla\boldsymbol{u}$ by using the Hölder inequality, (A.1) and Proposition A.3 as follows:

$$\begin{aligned}\int_0^T\|\nabla\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}dt\le&C\int_0^T t^{-\frac67}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac17}\|\nabla^2(t\boldsymbol{u})\|_{L^4}^{\frac67}dt\\
\le&C\left\|\|\nabla\boldsymbol{u}\|_{L^2}^{\frac17}\right\|_{L_T^{28,7}}\left\|\|\nabla^2(t\boldsymbol{u})\|_{L^4}^{\frac67}\right\|_{L_T^{\frac{28}{3},\frac76}}\|t^{-\frac67}\|_{L_T^{\frac76,\infty}}\\
\le&C\left\|\|\boldsymbol{u}\|_{L^4}^{\frac45}\|\nabla^2\boldsymbol{u}\|_{L^2}^{\frac15}\right\|_{L_T^{4,1}}^{\frac17}\|\nabla^2(t\boldsymbol{u})\|_{L_T^{8,1}(L^4)}^{\frac67}\\
\le&C\left\|\|\boldsymbol{u}\|_{L^4}^{\frac45}\right\|_{L_T^{10,\frac45}}^{\frac17}\left\|\|\nabla^2\boldsymbol{u}\|_{L^2}^{\frac15}\right\|_{L_T^{\frac{20}{3},5}}^{\frac17}\|\nabla^2(t\boldsymbol{u})\|_{L_T^{8,1}(L^4)}^{\frac67}\\
\le&C\|\boldsymbol{u}\|_{L_T^{8,1}(L^4)}^{\frac{4}{35}}\left\|\nabla^2\boldsymbol{u}\right\|_{L_T^{\frac43,1}(L^2)}^{\frac{1}{35}}\|\nabla^2(t\boldsymbol{u})\|_{L_T^{8,1}(L^4)}^{\frac67}\\
\le&C\left(\|\boldsymbol{u}_0\|_{\dot B^{\frac12}_{2,1}}+\|\boldsymbol{A}_0\|_{\dot B^{\frac34}_{4,1}\cap\dot B^{\frac74}_{4,1}}\right).\end{aligned}$$

The proof for $\|\boldsymbol{u}\|_{L^2(0,T;L^\infty)}$ can be performed in a similar way. □

Lagrangian approach. Thanks to (3.12) in Lemma 3.1, we can taking time T small enough so that (2.5) is satisfied. The uniqueness will be shown in low regularity spaces in the Lagrangian coordinates, for which the Proposition 2.5 plays a crucial role in the proof.

Now, we consider two solutions $(\rho^1,\boldsymbol{u}^1,\boldsymbol{A}^1,P^1,Q^1)$ and $(\rho^2,\boldsymbol{u}^2,\boldsymbol{A}^2,P^2,Q^2)$ of (1.10), emanating from the same initial data and denoting by $(\overline{\rho}^1,\overline{\boldsymbol{u}}^1,\overline{\boldsymbol{A}}^1,\overline{P}^1,\overline{Q}^1)$ and $(\overline{\rho}^2,\overline{\boldsymbol{u}}^2,\overline{\boldsymbol{A}}^2,\overline{P}^2,\overline{Q}^2)$ the corresponding ones in Lagrangian coordinates. Note that we have $\overline{\rho}^1=\overline{\rho}^2=\rho_0$, and $(\overline{\rho}^i,\overline{\boldsymbol{u}}^i,\overline{\boldsymbol{A}}^i,\overline{P}^i,\overline{Q}^i)$ $(i=1,2)$ have the regularity stated in Theorem 1.2.

Denoting $\delta\overline{\boldsymbol{u}}=\overline{\boldsymbol{u}}^1-\overline{\boldsymbol{u}}^2$, $\delta\overline{P}=\overline{P}^1-\overline{P}^2$, $\delta\overline{\boldsymbol{A}}=\overline{\boldsymbol{A}}^1-\overline{\boldsymbol{A}}^2$, $\delta\overline{Q}=\overline{Q}^1-\overline{Q}^2$, and $\mathbb{M}^i(t):=\mathbb{M}_{\overline{\boldsymbol{u}}^i}(t)$. From the Lagrangian formulation (2.3), it is easy to find that

$$\begin{cases}\partial_t\delta\overline{\boldsymbol{u}}-\Delta\delta\overline{\boldsymbol{u}}+\nabla\delta\overline{P}=\delta\boldsymbol{f}_1-\delta\boldsymbol{f}_2,\\
\qquad\qquad\operatorname{div}\delta\overline{\boldsymbol{u}}=\operatorname{div}\delta\boldsymbol{R}_1,\\
\partial_t\delta\overline{\boldsymbol{A}}-\Delta\delta\overline{\boldsymbol{A}}+\nabla\delta\overline{Q}=\delta\boldsymbol{f}_2/\rho_0+\delta\boldsymbol{f}_3+\delta\boldsymbol{f}_4,\\
\qquad\qquad\operatorname{div}\delta\overline{\boldsymbol{A}}=\operatorname{div}\delta\boldsymbol{R}_2,\\
\qquad\qquad\delta\overline{\boldsymbol{u}}|_{t=0}=\delta\overline{\boldsymbol{A}}|_{t=0}=0,\end{cases}\tag{3.14}$$

with

$$\begin{aligned}
\delta\boldsymbol{f}_1 :=&(1-\rho_0)\partial_t\delta\overline{\boldsymbol{u}}+[(\Delta_{\overline{\boldsymbol{u}}^1}-\Delta_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{u}}^1-(\Delta-\Delta_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{u}}]\\
&-[(\nabla_{\overline{\boldsymbol{u}}^1}-\nabla_{\overline{\boldsymbol{u}}^2})\overline{P}^1-(\nabla-\nabla_{\overline{\boldsymbol{u}}^2})\delta\overline{P}],\\
\delta\boldsymbol{f}_2 :=&[(\Delta_{\overline{\boldsymbol{u}}^1}-\Delta_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\times(\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)]+[\Delta_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\times(\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)]\\
&+[\Delta_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times((\nabla_{\overline{\boldsymbol{u}}^1}-\nabla_{\overline{\boldsymbol{u}}^2})\times\overline{\boldsymbol{A}}^1)]+[\Delta_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times(\nabla_{\overline{\boldsymbol{u}}^2}\times\delta\overline{\boldsymbol{A}})],\\
\delta\boldsymbol{f}_3 :=&[(\Delta_{\overline{\boldsymbol{u}}^1}-\Delta_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1-(\Delta-\Delta_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{A}}]-[(\nabla_{\overline{\boldsymbol{u}}^1}-\nabla_{\overline{\boldsymbol{u}}^2})\overline{Q}^1-(\nabla-\nabla_{\overline{\boldsymbol{u}}^2})\delta\overline{Q}],\\
\delta\boldsymbol{f}_4 :=&\big((\nabla_{\overline{\boldsymbol{u}}^1}-\nabla_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\big)^{\mathrm{T}}\overline{\boldsymbol{u}}^1+\big(\nabla_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\big)^{\mathrm{T}}\overline{\boldsymbol{u}}^1+\big(\nabla_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\big)^{\mathrm{T}}\delta\overline{\boldsymbol{u}},\\
\delta\boldsymbol{R}_1 :=&(\mathrm{Id}-\mathbb{M}^1)\overline{\boldsymbol{u}}^1-(\mathrm{Id}-\mathbb{M}^2)\overline{\boldsymbol{u}}^2,\\
\delta\boldsymbol{R}_2 :=&(\mathrm{Id}-\mathbb{M}^1)\overline{\boldsymbol{A}}^1-(\mathrm{Id}-\mathbb{M}^2)\overline{\boldsymbol{A}}^2.
\end{aligned}$$

Applying Proposition2.5 to equations of $\delta\overline{\boldsymbol{u}}$ and $\delta\overline{\boldsymbol{A}}$ with $p=2, m=r=\frac{4}{3}$, respectively. One has

$$\begin{aligned}
&\|\delta\overline{\boldsymbol{u}}\|_{L^\infty(0,T;\dot{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(\mathbb{R}^3))}+\|\delta\overline{\boldsymbol{u}}_t,\,\nabla^2\delta\overline{\boldsymbol{u}},\nabla\delta\overline{P}\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))}\\
&\qquad\le C\Big(\|\delta\boldsymbol{f}_1,\delta\boldsymbol{f}_2,(\delta\boldsymbol{R}_1)_t\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))}+\|\mathrm{div}\,\delta\boldsymbol{R}_1\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2(\mathbb{R}^3))}\Big) \quad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&\|\delta\overline{\boldsymbol{A}}\|_{L^\infty(0,T;\dot{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(\mathbb{R}^3))}+\|\delta\overline{\boldsymbol{A}}_t,\nabla^2\delta\overline{\boldsymbol{A}},\nabla\delta\overline{Q}\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))}\\
&\le C\Big(\|\delta\boldsymbol{f}_2/\rho_0,\delta\boldsymbol{f}_3,\delta\boldsymbol{f}_4,(\delta\boldsymbol{R}_2)_t\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))}+\|\mathrm{div}\,\delta\boldsymbol{R}_2\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2(\mathbb{R}^3))}\Big). \quad (3.16)
\end{aligned}$$

In the following, we estimate the right-hand-sides of (3.15) and (3.16) term by term.

Estimates of $(\delta\boldsymbol{R}_1)_t$, $(\delta\boldsymbol{R}_2)_t$, $\mathrm{div}\,\delta\boldsymbol{R}_1$ and $\mathrm{div}\,\delta\boldsymbol{R}_2$. Firstly, we consider the terms $(\delta\boldsymbol{R}_1)_t$ and $(\delta\boldsymbol{R}_2)_t$, and write them as follows:

$$\begin{aligned}
\|(\delta\boldsymbol{R}_1)_t\|_{L^{\frac{4}{3}}_T(L^2)}&\le\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]\|_{L^{\frac{4}{3}}_T(L^2)}+\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)\|_{L^{\frac{4}{3}}_T(L^2)},\\
\|(\delta\boldsymbol{R}_2)_t\|_{L^{\frac{4}{3}}_T(L^2)}&\le\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{A}}]\|_{L^{\frac{4}{3}}_T(L^2)}+\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{A}}^1)\|_{L^{\frac{4}{3}}_T(L^2)}.
\end{aligned}$$

For the term $\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]$, Proposition 2.1 and Proposition 2.2 ensure that

$$\begin{aligned}
\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]\|_{L^{\frac{4}{3}}_T(L^2)}\le&\|\partial_t\mathbb{M}^2\,\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}+\|(\mathrm{Id}-\mathbb{M}^2)\,\partial_t\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}\\
\lesssim&\Big\||D\overline{\boldsymbol{u}}^2|\,|\delta\overline{\boldsymbol{u}}|\Big\|_{L^{\frac{4}{3}}_T(L^2)}+\Big\|\int_0^t|D\overline{\boldsymbol{u}}^2|\,d\tau\,|\partial_t\delta\overline{\boldsymbol{u}}|\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\
\lesssim&\|D\overline{\boldsymbol{u}}^2\|_{L^2_T(L^3)}\|\delta\overline{\boldsymbol{u}}\|_{L^4_T(L^6)}+\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}\|\partial_t\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}.
\end{aligned}$$

Thanks to the Gagliardo-Nirenberg inequality in (A.1), and the embedding $\dot{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, we see that

$$\|D\overline{\boldsymbol{u}}^2\|_{L^2_T(L^3)}\lesssim\Big\|\|\overline{\boldsymbol{u}}^2\|_{L^4}^{\frac{2}{5}}\|D^2\overline{\boldsymbol{u}}^2\|_{L^2}^{\frac{3}{5}}\Big\|_{L^2_T}\lesssim\|\overline{\boldsymbol{u}}^2\|^{\frac{2}{5}}_{L^{8,1}_T(L^4)}\|D^2\overline{\boldsymbol{u}}^2\|^{\frac{3}{5}}_{L^{\frac{4}{3},1}_T(L^2)}\lesssim\varepsilon_1$$

and

$$\|\delta\overline{\boldsymbol{u}}\|_{L^4_T(L^6)}\lesssim\Big\|\|\delta\overline{\boldsymbol{u}}\|_{L^3}^{\frac{2}{3}}\|D^2\delta\overline{\boldsymbol{u}}\|_{L^2}^{\frac{1}{3}}\Big\|_{L^4_T}\lesssim\|\delta\overline{\boldsymbol{u}}\|^{\frac{2}{3}}_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,\frac{4}{3}})}\|D^2\delta\overline{\boldsymbol{u}}\|^{\frac{1}{3}}_{L^{\frac{4}{3}}_T(L^2)}.$$

Therefore, according to Lemma 3.1, one gets

$$\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]\|_{L^{\frac{4}{3}}_T(L^2)}\lesssim\varepsilon_1\ \|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{2,\frac{4}{3}}(T)}. \tag{3.17}$$

Similarly with (3.17), one has

$$\begin{aligned}\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{A}}]\|_{L^{\frac{4}{3}}_T(L^2)}\lesssim&\|D\overline{\boldsymbol{u}}^2\|_{L^2_T(L^3)}\|\delta\overline{\boldsymbol{A}}\|_{L^4_T(L^6)}+\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}\|\partial_t\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^2)}\\\lesssim&\ \varepsilon_1\ \|\delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{2,\frac{4}{3}}(T)}.\end{aligned} \tag{3.18}$$

Next, we consider the term $\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)$. By Proposition 2.2 and Minkowski' inequality, we see that

$$\begin{aligned}\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)\|_{L^{\frac{4}{3}}_T(L^2)}\leq&\|\partial_t\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^2)}+\|\delta\mathbb{M}\,\partial_t\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^2)}\\\lesssim&\Big\||D\delta\overline{\boldsymbol{u}}|\,|\overline{\boldsymbol{u}}^1|\Big\|_{L^{\frac{4}{3}}_T(L^2)}+\Big\|\int_0^t|D\delta\overline{\boldsymbol{u}}|\,d\tau\,|\partial_t\overline{\boldsymbol{u}}^1|\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\&+\Big\|\int_0^t|D\delta\overline{\boldsymbol{u}}|\,d\tau\,(|D\overline{\boldsymbol{u}}^1|+|D\overline{\boldsymbol{u}}^2|)\,|\overline{\boldsymbol{u}}^1|\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\\lesssim&\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)}+\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\|t^{\frac{1}{4}}\partial_t\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^3)}\\&+\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\|t^{\frac{1}{4}}(|D\overline{\boldsymbol{u}}^1|+|D\overline{\boldsymbol{u}}^2|)\|_{L^{\frac{4}{3}}_T(L^\infty)}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)}.\end{aligned}$$

According to the embeddings from Proposition A.2, we have

$$\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\lesssim\|\nabla^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)},\quad\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)}\lesssim\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}\lesssim\varepsilon_1.$$

And (A.1) and A.3 and Lemma 3.1 imply that

$$\begin{aligned}\|t^{\frac{1}{4}}\partial_t\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^3)}\lesssim&\Big\|\|\partial_t\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{1}{3}}\|t\partial_t\overline{\boldsymbol{u}}^1\|_{L^4}^{\frac{2}{3}}t^{-\frac{5}{12}}\Big\|_{L^{\frac{4}{3}}_T}\\\lesssim&\Big\|\|\partial_t\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{1}{3}}\Big\|_{L^{4,3}_T}\Big\|\|t\partial_t\overline{\boldsymbol{u}}^1\|_{L^4}^{\frac{2}{3}}\Big\|_{L^{12,\frac{3}{2}}_T}\|t^{-\frac{5}{12}}\|_{L^{\frac{12}{5},\infty}_T}\lesssim\varepsilon_1\end{aligned}$$

and

$$\begin{aligned}\|t^{\frac{1}{4}}D\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^\infty)}\lesssim&\Big\|\|D^2\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{1}{3}}\|tD^2\overline{\boldsymbol{u}}^1\|_{L^4}^{\frac{2}{3}}t^{-\frac{5}{12}}\Big\|_{L^{\frac{4}{3}}_T}\\\lesssim&\Big\|\|D^2\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{1}{3}}\Big\|_{L^{4,3}_T}\Big\|\|tD^2\overline{\boldsymbol{u}}^1\|_{L^4}^{\frac{2}{3}}\Big\|_{L^{12,\frac{3}{2}}_T}\|t^{-\frac{5}{12}}\|_{L^{\frac{12}{5},\infty}_T}\lesssim\varepsilon_1.\end{aligned}$$

We combine above estimates to obtain that

$$\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)\|_{L^{\frac{4}{3}}_T(L^2)}\lesssim\varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}. \tag{3.19}$$

Similarly, we have

$$\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{A}}^1)\|_{L^{\frac{4}{3}}_T(L^2)}\lesssim\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}. \tag{3.20}$$

Thus, gathering (3.17)-(3.20) together, we readily obtain

$$\|(\delta\boldsymbol{R}_1)_t\|_{L^{\frac{4}{3}}(0,T;L^2)}+\|(\delta\boldsymbol{R}_2)_t\|_{L^{\frac{4}{3}}(0,T;L^2)}$$

$$\lesssim \varepsilon_1 \ (\|\delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{2,\frac{4}{3}}(T)} + \|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{2,\frac{4}{3}}(T)}). \quad (3.21)$$

To estimate $\|\mathrm{div}\,\delta\boldsymbol{R}_1\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}$ and $\|\mathrm{div}\,\delta\boldsymbol{R}_2\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}$, one notice that

$$\|\mathrm{div}\,\delta\boldsymbol{R}_1\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)} \le \|D\delta\overline{\boldsymbol{u}} : (\mathrm{Id}-\mathbb{M}^2)\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)} + \|D\overline{\boldsymbol{u}}^1 : \delta\mathbb{M}\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)},$$
$$\|\mathrm{div}\,\delta\boldsymbol{R}_2\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)} \le \|D\delta\overline{\boldsymbol{A}} : (\mathrm{Id}-\mathbb{M}^2)\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)} + \|D\overline{\boldsymbol{A}}^1 : \delta\mathbb{M}\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}.$$

For the term $D\delta\overline{\boldsymbol{u}} : (\mathrm{Id}-\mathbb{M}^2)$, applying Proposition 2.2, we easily have

$$\begin{aligned}
&\|D\delta\overline{\boldsymbol{u}} : (\mathrm{Id}-\mathbb{M}^2)\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}\\
\lesssim&\Big\||D^2\delta\overline{\boldsymbol{u}}|\int_0^t |D\overline{\boldsymbol{u}}^2|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)} + \Big\||D\delta\overline{\boldsymbol{u}}|\int_0^t |D^2\overline{\boldsymbol{u}}^2|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\
\lesssim&\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)} + \|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)}\\
\lesssim&\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)},
\end{aligned} \quad (3.22)$$

where in the last inequality we used the Gagliardo-Nirenberg inequality of the type:

$$\begin{aligned}
\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)} \lesssim&\Big\|\|D^2\overline{\boldsymbol{u}}^2\|_{L^2}^{\frac{1}{3}}\|tD^2\overline{\boldsymbol{u}}^2\|_{L^4}^{\frac{2}{3}}t^{-\frac{2}{3}}\Big\|_{L^3_T}\\
\lesssim&\|D^2\overline{\boldsymbol{u}}^2\|^{\frac{1}{3}}_{L^{\frac{4}{3},1}_T(L^2)}\|tD^2\overline{\boldsymbol{u}}^2\|^{\frac{2}{3}}_{L^{8,1}_T(L^4)} \lesssim \varepsilon_1.
\end{aligned}$$

In a similar way, we can bound the rest terms in $\mathrm{div}\,\delta\boldsymbol{R}_1$ and $\mathrm{div}\,\delta\boldsymbol{R}_2$ as follows:

$$\begin{aligned}
&\|D\overline{\boldsymbol{u}}^1 : \delta\mathbb{M}\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}\\
\lesssim&\Big\||D^2\overline{\boldsymbol{u}}^1|\int_0^t |D\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)} + \Big\||D\overline{\boldsymbol{u}}^1|\int_0^t |D^2\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\
&+\Big\||D\overline{\boldsymbol{u}}^1|\int_0^t(|D^2\overline{\boldsymbol{u}}^1|+|D^2\overline{\boldsymbol{u}}^2|)\,d\tau\int_0^t |D\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\
\lesssim&\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\|t^{\frac{1}{4}}D^2\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^3)} + \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)}\|t^{\frac{1}{4}}D\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^\infty)}\\
&+\|t^{\frac{1}{4}}D\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^\infty)}(\|D^2\overline{\boldsymbol{u}}^1\|_{L^1_T(L^3)}+\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)})\,\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^6)}\\
\lesssim&\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)},
\end{aligned}$$

$$\begin{aligned}
&\|D\delta\overline{\boldsymbol{A}} : (\mathrm{Id}-\mathbb{M}^2)\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}\\
\lesssim&\|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^2)}\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)} + \|D\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^6)}\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)}\\
\lesssim&\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^2)}
\end{aligned}$$

and

$$\begin{aligned}
&\|D\overline{\boldsymbol{A}}^1 : \delta\mathbb{M}\|_{L^{\frac{4}{3}}(0,T;\dot{W}^1_2)}\\
\lesssim&\Big\||D^2\overline{\boldsymbol{A}}^1|\int_0^t |D\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)} + \Big\||D\overline{\boldsymbol{A}}^1|\int_0^t |D^2\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)}\\
&+\Big\||D\overline{\boldsymbol{A}}^1|\int_0^t(|D^2\overline{\boldsymbol{u}}^1|+|D^2\overline{\boldsymbol{u}}^2|)\,d\tau\int_0^t |D\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac{4}{3}}_T(L^2)}
\end{aligned}$$

$$\begin{aligned}
\lesssim & \|D\delta\overline{\boldsymbol{u}}\|_{L^1_T(L^4)}\|D^2\overline{\boldsymbol{A}}^1\|_{L^{\frac43}_T(L^4)}+\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}\|t^{\frac14}D\overline{\boldsymbol{A}}^1\|_{L^{\frac43}_T(L^\infty)}\\
&+\|t^{\frac14}D\overline{\boldsymbol{A}}^1\|_{L^{\frac43}_T(L^\infty)}(\|D^2\overline{\boldsymbol{u}}^1\|_{L^1_T(L^3)}+\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)})\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^6)}\\
\lesssim & \,\varepsilon_1\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&\|\mathrm{div}\,\delta\boldsymbol{R}_1\|_{L^{\frac43}(0,T;L^2)}+\|\mathrm{div}\,\delta\boldsymbol{R}_2\|_{L^{\frac43}(0,T;L^2)}\\
&\qquad\lesssim\ \varepsilon_1\ (\|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^2)}+\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}).
\end{aligned}\tag{3.23}$$

Estimates of $\delta\boldsymbol{f}_1$, $\delta\boldsymbol{f}_2$, $\delta\boldsymbol{f}_3,\delta\boldsymbol{f}_4$. For the term $\delta\boldsymbol{f}_1$, we see that

$$\begin{aligned}
\|\delta\boldsymbol{f}_1\|_{L^{\frac43}(0,T;L^2)}\le\|1-\rho_0\|_{L^\infty}\|\partial_t\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}&+\|(\Delta_{\overline{\boldsymbol{u}}^2}-\Delta_{\overline{\boldsymbol{u}}^1})\overline{\boldsymbol{u}}^1,(\Delta-\Delta_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}\\
&+\|(\nabla_{\overline{\boldsymbol{u}}^2}-\nabla_{\overline{\boldsymbol{u}}^1})\overline{P}^1,(\nabla-\nabla_{\overline{\boldsymbol{u}}^2})\delta\overline{P}\|_{L^{\frac43}_T(L^2)}.
\end{aligned}$$

The first term at the right-hand side can be easily absorbed by the left-hand-side of (3.15), while we write by analogy with (3.22)

$$\begin{aligned}
\|(\Delta-\Delta_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}&=\|\mathrm{div}\,[(\mathrm{Id}-\mathbb{M}^2(\mathbb{M}^2)^{\mathrm{T}})\nabla\delta\overline{\boldsymbol{u}}]\|_{L^{\frac43}_T(L^2)}\\
&\lesssim\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}+\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)}\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^6)}\\
&\lesssim\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}
\end{aligned}$$

and

$$\begin{aligned}
&\|(\nabla_{\overline{\boldsymbol{u}}^2}-\nabla_{\overline{\boldsymbol{u}}^1})\overline{P}^1,(\nabla-\nabla_{\overline{\boldsymbol{u}}^2})\delta\overline{P}\|_{L^{\frac43}_T(L^2)}\\
\lesssim&\left\|\delta\mathbb{M}\nabla\overline{P}^1\right\|_{L^{\frac43}_T(L^2)}+\left\|\int_0^t|D\overline{\boldsymbol{u}}^2|\,d\tau|\nabla\delta\overline{P}|\right\|_{L^{\frac43}_T(L^2)}\\
\lesssim&\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}\left\|t^{\frac14}\nabla\overline{P}^1\right\|_{L^{\frac43}_T(L^3)}+\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}\left\|\nabla\delta\overline{P}\right\|_{L^{\frac43}_T(L^2)}\\
\lesssim&\,\varepsilon_1\ \left(\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}+\left\|\nabla\delta\overline{P}\right\|_{L^{\frac43}_T(L^2)}\right),
\end{aligned}$$

where we used that

$$\begin{aligned}
\left\|t^{\frac14}\nabla\overline{P}^1\right\|_{L^{\frac43}_T(L^3)}\lesssim&\left\|\|\nabla\overline{P}^1\|_{L^2}^{\frac13}\|t\nabla\overline{P}^1\|_{L^4}^{\frac23}t^{-\frac{5}{12}}\right\|_{L^{\frac43}_T}\\
\lesssim&\left\|\|\nabla\overline{P}^1\|_{L^2}^{\frac13}\right\|_{L^{4,3}_T}\left\|\|t\nabla\overline{P}^1\|_{L^4}^{\frac23}\right\|_{L^{12,\frac32}_T}\|t^{-\frac{5}{12}}\|_{L^{\frac{12}{5},\infty}_T}\lesssim\ \varepsilon_1.
\end{aligned}$$

Noticing that

$$(\Delta_{\overline{\boldsymbol{u}}^2}-\Delta_{\overline{\boldsymbol{u}}^1})\overline{\boldsymbol{u}}^1=\mathrm{div}\,[(\mathbb{M}^2(\mathbb{M}^2)^{\mathrm{T}}-\mathbb{M}^1(\mathbb{M}^1)^{\mathrm{T}})\nabla\overline{\boldsymbol{u}}^1].$$

Then, one can apply the same trick as previous computations to obtain

$$\|(\Delta_{\overline{\boldsymbol{u}}^2}-\Delta_{\overline{\boldsymbol{u}}^1})\overline{\boldsymbol{u}}^1\|_{L^{\frac43}_T(L^2)}\lesssim\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}.$$

To bound $\delta\boldsymbol{f}_2$, according to Proposition A.2, we find that

$$|\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1|=|(\mathbb{M}^1-\mathrm{Id})^{\mathrm{T}}\cdot\nabla\times\overline{\boldsymbol{A}}^1+\nabla\times\overline{\boldsymbol{A}}^1|$$

$$\begin{aligned}&\le\int_0^t|D\overline{\boldsymbol{u}}^1|\,d\tau\,|\nabla\times\overline{\boldsymbol{A}}^1|+|\nabla\times\overline{\boldsymbol{A}}^1|\\&\lesssim\|\overline{\boldsymbol{A}}^1\|_{L^\infty_T(\dot B^{\frac74}_{4,1})}(\|D\overline{\boldsymbol{u}}^1\|_{L^1_T(L^\infty)}+1)\end{aligned}$$

and

$$\begin{aligned}|\Delta_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2|&=|\mathrm{div}\,[(\mathbb{M}^2(\mathbb{M}^2)^{\mathrm T}-\mathrm{Id})\nabla\overline{\boldsymbol{A}}^2]+\Delta\overline{\boldsymbol{A}}^2|\\&\lesssim\int_0^t|D^2\overline{\boldsymbol{u}}^2|\,d\tau\,|D\overline{\boldsymbol{A}}^2|+|D^2\overline{\boldsymbol{A}}^2|.\end{aligned}$$

So that we are able to get

$$\begin{aligned}&\|(\Delta_{\overline{\boldsymbol{u}}^1}-\Delta_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\times(\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)\|_{L^{\frac43}_T(L^2)}\\\le&\|(\Delta_{\overline{\boldsymbol{u}}^1}-\Delta_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\|_{L^{\frac43}_T(L^2)}\|\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1\|_{L^\infty_T(L^\infty)}\lesssim\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}\end{aligned}$$

and

$$\begin{aligned}&\|\Delta_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\times(\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)\|_{L^{\frac43}_T(L^2)}\\\le&\|\Delta_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^2)}\|\nabla_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1\|_{L^\infty_T(L^\infty)}\lesssim\ \varepsilon_1\|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^2)}\end{aligned}$$

and

$$\begin{aligned}&\|\Delta_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times((\nabla_{\overline{\boldsymbol{u}}^1}-\nabla_{\overline{\boldsymbol{u}}^2})\times\overline{\boldsymbol{A}}^1)\|_{L^{\frac43}_T(L^2)}\\=&\|\Delta_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times((\delta\mathbb{M})^{\mathrm T}\cdot\nabla\times\overline{\boldsymbol{A}}^1)\|_{L^{\frac43}_T(L^2)}\\\lesssim&\Big\||\nabla\times\overline{\boldsymbol{A}}^1|\,(\int_0^t|D^2\overline{\boldsymbol{u}}^2|\,d\tau\,|D\overline{\boldsymbol{A}}^2|+|D^2\overline{\boldsymbol{A}}^2|)\int_0^t|D\delta\overline{\boldsymbol{u}}|\,d\tau\Big\|_{L^{\frac43}_T(L^2)}\\\lesssim&\|t^{\frac14}\nabla\times\overline{\boldsymbol{A}}^1\|_{L^{\frac43}_T(L^\infty)}\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)}\|D\overline{\boldsymbol{A}}^2\|_{L^\infty_T(L^\infty)}\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^6)}\\&+\|t^{\frac14}\nabla\times\overline{\boldsymbol{A}}^1\|_{L^2_T(L^6)}\|D^2\overline{\boldsymbol{A}}^2\|_{L^4_T(L^6)}\|D\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^6)}\lesssim\ \varepsilon_1\ \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac43}_T(L^2)}\end{aligned}$$

and

$$\begin{aligned}&\|\Delta_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times(\nabla_{\overline{\boldsymbol{u}}^2}\times\delta\overline{\boldsymbol{A}})\|_{L^{\frac43}_T(L^2)}\\=&\|\Delta^2_{\overline{\boldsymbol{u}}}\overline{\boldsymbol{A}}^2\times((\mathbb{M}^2)^{\mathrm T}\cdot\nabla\times\delta\overline{\boldsymbol{A}})\|_{L^{\frac43}_T(L^2)}\\\lesssim&\Big\||\nabla\times\delta\overline{\boldsymbol{A}}|\,(\int_0^t|D^2\overline{\boldsymbol{u}}^2|\,d\tau\,|D\overline{\boldsymbol{A}}^2|+|D^2\overline{\boldsymbol{A}}^2|)\Big\|_{L^{\frac43}_T(L^2)}\\\lesssim&\|\nabla\times\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^6)}\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)}\|D\overline{\boldsymbol{A}}^2\|_{L^\infty_T(L^\infty)}+\|\nabla\times\delta\overline{\boldsymbol{A}}\|_{L^{\frac83}_T(L^4)}\|D^2\overline{\boldsymbol{A}}^2\|_{L^{\frac83}_T(L^4)}\\\lesssim&\ \varepsilon_1\ \|\delta\overline{A}\|_{\dot W^{2,1}_{2,\frac43}(T)}.\end{aligned}$$

Next, the term $\delta\boldsymbol{f}_3$ may be handled along the same lines. Indeed we have

$$\begin{aligned}&\|(\Delta_{\overline{\boldsymbol{u}}^1}-\Delta_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1,(\Delta-\Delta_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^2)}\\\lesssim&\|\mathrm{div}\,[\delta\mathbb{M}(\mathbb{M}^2)^{\mathrm T}\nabla\overline{\boldsymbol{A}}^1]\|_{L^{\frac43}_T(L^2)}+\|\mathrm{div}\,[\mathbb{M}^1(\delta\mathbb{M})^{\mathrm T}\nabla\overline{\boldsymbol{A}}^1]\|_{L^{\frac43}_T(L^2)}\\&+\|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^2)}\|D\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}+\|D\delta\overline{\boldsymbol{A}}\|_{L^{\frac43}_T(L^6)}\|D^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^3)}\end{aligned}$$

$$\lesssim \varepsilon_1 \ (\|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^2)} + \|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)})$$

and

$$\begin{aligned}
&\|(\nabla_{\overline{\boldsymbol{u}}^1} - \nabla_{\overline{\boldsymbol{u}}^2})\overline{Q}^1, (\nabla - \nabla_{\overline{\boldsymbol{u}}^2})\delta\overline{Q}\|_{L^{\frac{4}{3}}_T(L^2)} \\
\lesssim& \left\|\delta\mathbb{M}\nabla\overline{Q}^1\right\|_{L^{\frac{4}{3}}_T(L^2)} + \left\|\int_0^t |D\overline{\boldsymbol{u}}^2|\,d\tau|\nabla\delta\overline{Q}|\right\|_{L^{\frac{4}{3}}_T(L^2)} \\
\lesssim& \varepsilon_1 \ \left(\|D^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)} + \left\|\nabla\delta\overline{Q}\right\|_{L^{\frac{4}{3}}_T(L^2)}\right).
\end{aligned}$$

Finally, we handle $\delta\boldsymbol{f}_4$ as

$$\begin{aligned}
\|\delta\boldsymbol{f}_4\|_{L^{\frac{4}{3}}_T(L^2)} \leq& \|\delta\mathbb{M}\nabla\overline{\boldsymbol{A}}^1\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^2)} + \|\mathbb{M}^2\nabla\delta\overline{\boldsymbol{A}}\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}_T(L^2)} + \|\mathbb{M}^2\nabla\overline{\boldsymbol{A}}^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{4}{3}}_T(L^2)} \\
\lesssim& \|\nabla\delta\overline{\boldsymbol{u}}\|_{L^1_T(L^4)}\|\nabla\overline{\boldsymbol{A}}^1\|_{L^\infty_T(L^\infty)}\|\overline{\boldsymbol{u}}^1\|_{L^{\frac{4}{3}}(L^4)} \\
&+ \|\nabla\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^6)}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)} + \|\nabla\overline{\boldsymbol{A}}^2\|_{L^{\frac{4}{3}}(L^6)}\|\delta\overline{\boldsymbol{u}}\|_{L^\infty_T(L^3)} \\
\lesssim& \varepsilon_1 \ \left(\|D^2\delta\overline{\boldsymbol{A}}\|_{L^{\frac{4}{3}}_T(L^2)} + \|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{2,\frac{4}{3}}(T)}\right).
\end{aligned}$$

In the end, putting the above estimates of $\delta\boldsymbol{f}_1$ to $\delta\boldsymbol{f}_4$ together with (3.21), (3.23), one gets that

$$\begin{aligned}
&\|\delta\overline{\boldsymbol{u}}\|_{L^\infty(0,T;\dot{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(\mathbb{R}^3))} + \|\delta\overline{\boldsymbol{u}}_t, \nabla^2\delta\overline{\boldsymbol{u}}, \nabla\delta\overline{P}\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|\delta\overline{\boldsymbol{A}}\|_{L^\infty(0,T;\dot{B}^{\frac{1}{2}}_{2,\frac{4}{3}}(\mathbb{R}^3))} + \|\delta\overline{\boldsymbol{A}}_t, \nabla^2\delta\overline{\boldsymbol{A}}, \nabla\delta\overline{Q}\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))} \\
&\qquad \lesssim \ \varepsilon_1 \ \left(\|\delta\overline{\boldsymbol{u}}, \delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{2,\frac{4}{3}}(T)} + \|\nabla\delta\overline{P}, \nabla\delta\overline{Q}\|_{L^{\frac{4}{3}}(0,T;L^2(\mathbb{R}^3))}\right).
\end{aligned}$$

Thus uniqueness fellows on the time interval $[0,T]$, then on the whole interval, thanks to a standard connectivity argument. It completes the proof of Theorem 1.2. □

4. Proof of $2\frac{1}{2}$D case

This section is devoted to the global well-posedness of the $2\frac{1}{2}$D flows of the inhomogeneous Hall-MHD system with large velocity. To this, we use the reformulation (1.12) for convenience. At first, we mention that a small modification of the energy equality (1.6) allows to establish that for any smooth solution $(\rho, \boldsymbol{u}, \boldsymbol{B})$ with initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{B}_0)$ satisfies

$$\begin{aligned}
\|\sqrt{\rho(t)}\boldsymbol{u}(t)\|^2_{L^2(\mathbb{R}^2)} + \|\boldsymbol{B}(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \big(\|\widetilde{\nabla}\boldsymbol{u}\|^2_{L^2(\mathbb{R}^2)} + \|\widetilde{\nabla}\boldsymbol{B}\|^2_{L^2(\mathbb{R}^2)}\big)\,d\tau& \\
= \|\sqrt{\rho_0}\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^2)} + \|\boldsymbol{B}_0\|^2_{L^2(\mathbb{R}^2)}.& \quad (4.1)
\end{aligned}$$

Moreover, thanks to the definition $\boldsymbol{B} = \widetilde{\nabla}\times\boldsymbol{A}$, thus

$$\|\widetilde{\nabla}\times\boldsymbol{A}\|_{\dot{H}^s} = \|\widetilde{\nabla}\boldsymbol{A}\|_{\dot{H}^s} = \|\boldsymbol{B}\|_{\dot{H}^s} \quad \forall\, s\in\mathbb{R},$$

one can further rewrite (4.1) to

$$\|\sqrt{\rho(t)}\boldsymbol{u}(t)\|^2_{L^2(\mathbb{R}^2)}+\|\widetilde{\nabla}\boldsymbol{A}(t)\|^2_{L^2(\mathbb{R}^2)}+2\int_0^t\big(\|\widetilde{\nabla}\boldsymbol{u}\|^2_{L^2(\mathbb{R}^2)}+\|\widetilde{\Delta}\boldsymbol{A}\|^2_{L^2(\mathbb{R}^2)}\big)\,d\tau$$
$$=\|\sqrt{\rho_0}\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^2)}+\|\widetilde{\nabla}\times\boldsymbol{A}_0\|^2_{L^2(\mathbb{R}^2)},\quad t>0. \tag{4.2}$$

In order to prove Theorem 1.3, having (4.2) in hand is not enough, since it is wide open in [35] about the uniqueness of global weak solutions for the 2D inhomogeneous Navier-Stokes equations. Indeed, our proof of Theorem 1.3 essentially relies on the following proposition and Proposition 2.6.

4.1. **The a priori estimates.** We have the following H^1 estimate. Note that we do not assume the smallness of the density.

Proposition 4.1. *Let $(\rho,\boldsymbol{u},\boldsymbol{A})$ be a smooth solution of the system (1.12) on the time interval $[0,T^*)$. Under the assumptions that*

$$0<c_0\le\rho_0\le C_0,\quad \boldsymbol{u}_0\in H^1(\mathbb{R}^2),\quad \boldsymbol{A}_0\in(\dot{B}^{\frac23}_{3,1}\cap\dot{B}^{\frac53}_{3,1})(\mathbb{R}^2). \tag{4.3}$$

There exists a constant ε_2 depending on $M_0:=\|\sqrt{\rho_0}\boldsymbol{u}_0\|_{L^2(\mathbb{R}^2)}+\|\widetilde{\nabla}\times\boldsymbol{A}_0\|_{L^2(\mathbb{R}^2)}$ and c_0,C_0 such that, if

$$\|\boldsymbol{A}_0\|_{\dot{B}^{\frac23}_{3,1}\cap\dot{B}^{\frac53}_{3,1}}\le\varepsilon_2,$$

then we have for all $T<T^*$, (1.14) is satisfied and

$$\|\widetilde{\nabla}\boldsymbol{u}\|^2_{L^\infty(0,T;L^2)}+\int_0^T\|(\sqrt{\rho}\partial_t\boldsymbol{u},\widetilde{\nabla}^2\boldsymbol{u},\widetilde{\nabla}P)(\tau)\|^2_{L^2}\,d\tau\le M_1(1+C_3^2\|\boldsymbol{A}_0\|^2_{\dot{B}^{\frac23}_{3,1}\cap\dot{B}^{\frac53}_{3,1}}), \tag{4.4}$$

$$\|\boldsymbol{A}\|_{L^\infty(0,T;\dot{B}^{\frac23}_{3,1}\cap\dot{B}^{\frac53}_{3,1})}+\|(\partial_t\boldsymbol{A},\widetilde{\nabla}^2\boldsymbol{A},\widetilde{\nabla}Q)\|_{L^{\frac32,1}\cap L^{6,1}(0,T;L^3)}\le C_3\|\boldsymbol{A}_0\|_{\dot{B}^{\frac23}_{3,1}\cap\dot{B}^{\frac53}_{3,1}} \tag{4.5}$$

with $M_1:=(\|\widetilde{\nabla}\boldsymbol{u}_0\|^2_{L^2}+\|(\boldsymbol{u}_0,\widetilde{\nabla}\times\boldsymbol{A}_0)\|^2_{L^2})\ \exp\Big(C(c_0,C_0)\|(\boldsymbol{u}_0,\widetilde{\nabla}\times\boldsymbol{A}_0)\|^4_{L^2}\Big)$.

Proof. Obviously, (1.14) is fulfilled. We then divide the proof into two parts: The first part is concerned with the estimates of the velocity.

Estimate for $\boldsymbol{u}$. Taking $L^2(\mathbb{R}^2)$ inner product of $(1.12)_2$ with $\partial_t\boldsymbol{u}$ gives

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\widetilde{\nabla}\boldsymbol{u}(t)\|^2_{L^2}+\|\sqrt{\rho}\partial_t\boldsymbol{u}\|^2_{L^2}\\
=&-\int_{\mathbb{R}^2}(\rho\boldsymbol{u}\cdot\widetilde{\nabla}\boldsymbol{u})\cdot\partial_t\boldsymbol{u}\,dx-\int_{\mathbb{R}^2}\big(\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A})\big)\cdot\partial_t\boldsymbol{u}\,dx\\
\le&\|\sqrt{\rho}\boldsymbol{u}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^4}\|\sqrt{\rho}\partial_t\boldsymbol{u}\|_{L^2}+\Big\|\frac{1}{\sqrt{\rho}}\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A})\Big\|_{L^2}\|\sqrt{\rho}\partial_t\boldsymbol{u}\|_{L^2}\\
\le&4C_0\|\boldsymbol{u}\|^2_{L^4}\|\widetilde{\nabla}\boldsymbol{u}\|^2_{L^4}+\frac{4}{c_0}\|\widetilde{\Delta}\boldsymbol{A}\|^2_{L^2}\|\widetilde{\nabla}\boldsymbol{A}\|^2_{L^\infty}+\frac12\|\sqrt{\rho}\partial_t\boldsymbol{u}\|_{L^2}.
\end{aligned}\tag{4.6}$$

Noticing that, the equation $(1.12)_2$ can be reformulated as

$$-\widetilde{\Delta}\boldsymbol{u}+\widetilde{\nabla}P=-\rho\partial_t\boldsymbol{u}-\rho\boldsymbol{u}\cdot\widetilde{\nabla}\boldsymbol{u}-\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A}),$$

from which, one has

$$\|\widetilde{\nabla}^2\boldsymbol{u}\|_{L^2}+\|\widetilde{\nabla}P\|_{L^2}\le\sqrt{C_0}\|\sqrt{\rho}\partial_t\boldsymbol{u}\|_{L^2}+C_0\|\boldsymbol{u}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^4}+\|\widetilde{\Delta}\boldsymbol{A}\|_{L^2}\|\widetilde{\nabla}\boldsymbol{A}\|_{L^\infty}.$$

Inserting the above inequality into (4.6) and using the Gagliardo–Nirenberg inequality (A.1) yields

$$\begin{aligned}
&\frac{d}{dt}\|\widetilde{\nabla}\boldsymbol{u}(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_t\boldsymbol{u},\widetilde{\nabla}^2\boldsymbol{u},\widetilde{\nabla}P\|_{L^2}^2\\
\leq& C(c_0,C_0)\left(\|\boldsymbol{u}\|_{L^4}^2\|\widetilde{\nabla}\boldsymbol{u}\|_{L^4}^2+\|\widetilde{\Delta}\boldsymbol{A}\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{A}\|_{L^\infty}^2\right)\\
\leq& C(c_0,C_0)\left(\|\boldsymbol{u}\|_{L^2}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}^2\|\widetilde{\nabla}^2\boldsymbol{u}\|_{L^2}+\|\widetilde{\Delta}\boldsymbol{A}\|_{L^2}^2\|\boldsymbol{A}\|_{\dot{B}^{\frac{5}{3}}_{3,1}}^2\right)\\
\leq& C(c_0,C_0)\left(\|\boldsymbol{u}\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}^4+\|\widetilde{\Delta}\boldsymbol{A}\|_{L^2}^2\|\boldsymbol{A}\|_{\dot{B}^{\frac{5}{3}}_{3,1}}^2\right)+\frac{1}{2}\|\widetilde{\nabla}^2\boldsymbol{u}\|_{L^2}^2. \qquad (4.7)
\end{aligned}$$

In view of (4.2) and (4.7), we get, by applying Gronwall's lemma, that

$$\begin{aligned}
&\|\widetilde{\nabla}\boldsymbol{u}(t)\|_{L^2}^2+\int_0^t\|(\sqrt{\rho}\partial_t\boldsymbol{u},\widetilde{\nabla}^2\boldsymbol{u},\widetilde{\nabla}P)(\tau)\|_{L^2}^2\,d\tau\\
\leq&\big(\|\widetilde{\nabla}\boldsymbol{u}_0\|_{L^2}^2+\int_0^t\|\widetilde{\Delta}\boldsymbol{A}\|_{L^2}^2\|\boldsymbol{A}\|_{\dot{B}^{\frac{5}{3}}_{3,1}}^2\,d\tau\big)\ \exp\Big(C(c_0,C_0)\int_0^t\|\boldsymbol{u}(\tau)\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{u}(\tau)\|_{L^2}^2\,d\tau\Big)\\
\leq&\Big(\|\widetilde{\nabla}\boldsymbol{u}_0\|_{L^2}^2+\|(\boldsymbol{u}_0,\widetilde{\nabla}\times\boldsymbol{A}_0)\|_{L^2}^2\|\boldsymbol{A}\|^2_{L^\infty(0,t;\dot{B}^{\frac{5}{3}}_{3,1})}\Big)\ \exp\Big(C(c_0,C_0)\|(\boldsymbol{u}_0,\widetilde{\nabla}\times\boldsymbol{A}_0)\|_{L^2}^4\Big)\\
\leq&M_1\Big(1+\|\boldsymbol{A}\|^2_{L^\infty(0,t;\dot{B}^{\frac{5}{3}}_{3,1})}\Big). \qquad (4.8)
\end{aligned}$$

Estimate for $\boldsymbol{A}$. Let us first rewrite $(1.12)_4$ to

$$\partial_t\boldsymbol{A}-\widetilde{\Delta}\boldsymbol{A}=\mathcal{P}\Big(\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A})\Big)+\mathcal{P}\left(\frac{\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A})}{\rho}\right),$$

then apply Proposition 2.3 (take $p=3,m=\frac{3}{2},6,r=1$, respectively) to above equation to get

$$\begin{aligned}
&\|\boldsymbol{A}\|_{L^\infty(0,T;\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})}+\|\boldsymbol{A}_t,\widetilde{\nabla}^2\boldsymbol{A}\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)}\\
&\qquad\leq C\left(\|\boldsymbol{A}_0\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}}+\|\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A})\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)}\right.\\
&\qquad\qquad\left.+\frac{1}{c_0}\|\boldsymbol{A}\|_{L^\infty(0,T;\dot{B}^{\frac{5}{3}}_{3,1})}\|\widetilde{\nabla}^2\boldsymbol{A}\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)}\right). \qquad (4.9)
\end{aligned}$$

Due to the coupling $\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A})$, we can not directly close the estimate (4.9) with the help of energy equality (4.2). Since, similar to (3.9), we only have

$$\begin{aligned}
\|\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A})\|_{L^{\frac{3}{2},1}(0,T;L^3)}\leq&C\Big\|\|\boldsymbol{u}\|_{L^3}\|\widetilde{\nabla}\boldsymbol{A}\|_{L^\infty}\Big\|_{L^{\frac{3}{2},1}_T}\\
\leq&C\Big\|\|\boldsymbol{u}\|_{L^3}\|\boldsymbol{A}\|_{L^\infty}^{\frac{1}{4}}\|\widetilde{\nabla}^2\boldsymbol{A}\|_{L^3}^{\frac{3}{4}}\Big\|_{L^{\frac{3}{2},1}_T}\\
\leq&C\|\boldsymbol{u}\|_{L^{6,4}_T(L^3)}\|\boldsymbol{A}\|^{\frac{1}{4}}_{L^\infty(\dot{B}^{\frac{2}{3}}_{3,1})}\Big\|\|\widetilde{\nabla}^2\boldsymbol{A}\|_{L^3}^{\frac{3}{4}}\Big\|_{L^{2,\frac{4}{3}}_T}\\
\leq&C\|\boldsymbol{u}\|_{L^{6,4}_T(L^3)}\|\boldsymbol{A}\|^{\frac{1}{4}}_{L^\infty(\dot{B}^{\frac{2}{3}}_{3,1})}\|\widetilde{\nabla}^2\boldsymbol{A}\|^{\frac{3}{4}}_{L^{\frac{3}{2},1}_T(L^3)},
\end{aligned}$$

and the boundness of $\|\boldsymbol{u}\|_{L_T^{6,6}(L^3)}$ thanks to (4.2). Here, it also explains the importance of considering initial data in critical Besov spaces with the third exponent to be 1.

One may take the advantage of sub-critical controls of velocity in (4.8), however, in this case, inequalities (4.8) and (4.9) are coupled, it seems unavoidable to make smallness assumptions on both $\boldsymbol{u}_0$ and $\boldsymbol{A}_0$. Hence, we use the following time-weighted strategy.

For $\lambda > 0$ (to be determined), let us define an unknown associated with $\boldsymbol{A}$ by

$$\boldsymbol{A}_\lambda(t,x) := e^{-f(t)}\boldsymbol{A}(t,x), \quad \text{with } f(t) = \lambda \int_0^t \|\boldsymbol{u}\|_{L^6}^3 d\tau, \tag{4.10}$$

which satisfies the following $\lambda-$modified equation

$$\partial_t \boldsymbol{A}_\lambda + f'(t)\boldsymbol{A}_\lambda - \widetilde{\Delta}\boldsymbol{A}_\lambda = \mathcal{P}\left(\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A}_\lambda)\right) + \mathcal{P}\left(\frac{\widetilde{\Delta}\boldsymbol{A}_\lambda \times (\widetilde{\nabla}\times\boldsymbol{A}_\lambda)}{\rho}\right)e^{f(t)}. \tag{4.11}$$

Thanks to (4.2) and the Gagliardo-Nirenberg inequality in (A.1), one has

$$\|f(t)\|_{L_T^\infty} \le C\lambda \|\boldsymbol{u}\|_{L^\infty(0,T;L^2)}^{\frac{1}{3}} \|\nabla\boldsymbol{u}\|_{L^2((0,T)\times\mathbb{R}^2)}^{\frac{2}{3}} \le C\lambda M_0, \quad \forall\, T>0. \tag{4.12}$$

Applying Proposition 2.4 to (4.11), for some positive constant C_3 independent of λ, we have

$$\begin{aligned}
&\|\boldsymbol{A}_\lambda\|_{L^\infty(0,T;\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})} + \|\partial_t\boldsymbol{A}_\lambda, \widetilde{\nabla}^2\boldsymbol{A}_\lambda, f'(t)\boldsymbol{A}_\lambda\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)} \\
&\quad\le C_3\Big(\|\boldsymbol{A}_0\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}} + \|\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A}_\lambda)\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)} \\
&\qquad + e^{\|f(t)\|_{L_T^\infty}}\|\boldsymbol{A}_\lambda\|_{L^\infty(0,T;\dot{B}^{\frac{5}{3}}_{3,1})}\|\widetilde{\nabla}^2\boldsymbol{A}_\lambda\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)}\Big).
\end{aligned} \tag{4.13}$$

Applying Hölder's inequality and (A.1), Proposition A.3, we obtain

$$\begin{aligned}
\|\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A}_\lambda)\|_{L^{\frac{3}{2},1}(0,T;L^3)} \le& C\left\|\|\boldsymbol{u}\|_{L^6}\|\widetilde{\nabla}\boldsymbol{A}_\lambda\|_{L^6}\right\|_{L_T^{\frac{3}{2},1}} \\
\le& C\left\|\|\boldsymbol{u}\|_{L^6}\|\boldsymbol{A}_\lambda\|_{L^3}^{\frac{1}{3}}\|\widetilde{\nabla}^2\boldsymbol{A}_\lambda\|_{L^3}^{\frac{2}{3}}\right\|_{L_T^{\frac{3}{2},1}} \\
\le& C\left\|\|\boldsymbol{u}\|_{L^6}\|\boldsymbol{A}_\lambda\|_{L^3}^{\frac{1}{3}}\right\|_{L_T^{\frac{9}{2},3}}\left\|\|\widetilde{\nabla}^2\boldsymbol{A}_\lambda\|_{L^3}^{\frac{2}{3}}\right\|_{L_T^{\frac{9}{4},\frac{3}{2}}} \\
\le& \frac{C}{\lambda}\|f'(t)\boldsymbol{A}_\lambda\|_{L_T^{\frac{3}{2},1}(L^3)}^{\frac{1}{3}}\|\widetilde{\nabla}^2\boldsymbol{A}_\lambda\|_{L_T^{\frac{3}{2},1}(L^3)}^{\frac{2}{3}}
\end{aligned}$$

and

$$\begin{aligned}
\|\boldsymbol{u}\times(\widetilde{\nabla}\times\boldsymbol{A}_\lambda)\|_{L^{6,1}(0,T;L^3)} \le& C\left\|\|\boldsymbol{u}\|_{L^6}\|\boldsymbol{A}_\lambda\|_{L^3}^{\frac{1}{3}}\|\widetilde{\nabla}^2\boldsymbol{A}_\lambda\|_{L^3}^{\frac{2}{3}}\right\|_{L_T^{6,1}} \\
\le& C\left\|\|\boldsymbol{u}\|_{L^6}\|\boldsymbol{A}\|_{L^3}^{\frac{1}{3}}\right\|_{L_T^{18,3}}\left\|\|\widetilde{\nabla}^2\boldsymbol{A}\|_{L^3}^{\frac{2}{3}}\right\|_{L_T^{9,\frac{3}{2}}} \\
\le& \frac{C}{\lambda}\|f'(t)\boldsymbol{A}\|_{L_T^{6,1}(L^3)}^{\frac{1}{3}} + \|\widetilde{\nabla}^2\boldsymbol{A}\|_{L_T^{6,1}(L^3)}^{\frac{2}{3}}.
\end{aligned}$$

For all $T' \in [0,T]$, denote

$$E_\lambda(T') := \|\boldsymbol{A}_\lambda(T')\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}} + \|\partial_t\boldsymbol{A}_\lambda, \widetilde{\nabla}^2\boldsymbol{A}_\lambda, f'(t)\boldsymbol{A}_\lambda\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T';L^3)},$$

we obtain from (4.13) and (4.12) that

$$E_\lambda(T') \le C_3\left(E_\lambda(0) + \frac{C}{\lambda}E_\lambda(T') + \exp(C\lambda M_{00})\ E_\lambda(T')^2\right).$$

Choosing $\lambda = 2C_3C$, one find that as long as

$$16C_3^2\exp(2C_3C^2M_{00})\|\boldsymbol{A}_0\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})} < 1,$$

then $E_\lambda(T') < C_3\left\|\boldsymbol{A}_0\right\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}}$ and thus

$$\|\boldsymbol{A}_\lambda\|_{L^\infty(0,T;\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1})} + \|\partial_t\boldsymbol{A}_\lambda, \widetilde{\nabla}^2\boldsymbol{A}_\lambda, f'(t)\boldsymbol{A}_\lambda\|_{L^{\frac{3}{2},1}\cap L^{6,1}(0,T;L^3)} \le C_3\left\|\boldsymbol{A}_0\right\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}}.$$

Recall the facts that $\boldsymbol{A}(t,x) = e^{f(t)}\boldsymbol{A}_\lambda(t,x)$ and $\partial_t\boldsymbol{A} = e^{f(t)}(f'(t)\boldsymbol{A}_\lambda + \partial_t\boldsymbol{A}_\lambda)$, we get (4.4) and (4.5) from the above inequality and (4.12).

□

4.2. **Existence.** At this stage, apply similar approximation scheme to the one that we used for handling the 3D case allows to conclude to the existence part of Theorem 1.3. We omit details here. We just remind that, since the variation of density is no longer small, and in order to take use of energy inequality (4.2) and (4.4), one may need to slightly change the corresponding approximate system (3.2) to

$$\begin{cases}
\partial_t\rho^{n+1} + \boldsymbol{u}^n\cdot\widetilde{\nabla}\rho^{n+1} = 0,\\
\rho^n\partial_t\boldsymbol{u}^{n+1} + \rho^n\boldsymbol{u}^n\cdot\widetilde{\nabla}\boldsymbol{u}^{n+1} - \widetilde{\Delta}\boldsymbol{u}^{n+1} + \widetilde{\nabla}P^{n+1} = -\widetilde{\Delta}\boldsymbol{A}^n\times(\widetilde{\nabla}\times\boldsymbol{A}^n),\\
\widetilde{\operatorname{div}}\,\boldsymbol{u} = 0,\\
\partial_t\boldsymbol{A}^{n+1} - \widetilde{\Delta}\boldsymbol{A}^{n+1} = \boldsymbol{g}^n,\\
(\rho^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{A}^{n+1})|_{t=0} = (\rho_0^{n+1},\boldsymbol{u}_0^{n+1},\boldsymbol{A}_0^{n+1}),
\end{cases} \tag{4.14}$$

where

$$\boldsymbol{g}^n = \mathcal{P}\Big(\boldsymbol{u}^n\times(\widetilde{\nabla}\times\boldsymbol{A}^n) + \frac{1}{\rho^n}\,\widetilde{\Delta}\boldsymbol{A}^n\times(\widetilde{\nabla}\times\boldsymbol{A}^n)\Big).$$

4.3. **Uniqueness.** Arguing as in the previous section, it is necessary to establish bounds for $\nabla\boldsymbol{u}$ in $L^1(0,T;L^\infty(\mathbb{R}^2))$. Due to the large variation of the density, that argument in Lemma 3.1 is invalid here. We need the following lemma.

Lemma 4.2. *Let $q > 2$ such that $q \to 2$. Define $\alpha := \frac{3}{2} - \frac{2}{q}$. For some suitably small $T > 0$, there exists $c(T)$ going to 0 when T tends to 0 such that the estimate below holds:*

$$\|t^\alpha\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{q}}_{q,q})} + \|(t^\alpha\boldsymbol{u})_t, \widetilde{\nabla}^2(t^\alpha\boldsymbol{u}), t^\alpha\widetilde{\nabla}P\|_{L^q((0,T)\times\mathbb{R}^2)} \le c(T). \tag{4.15}$$

Moreover,

$$\int_0^T \|\nabla\boldsymbol{u}\|_{L^\infty(\mathbb{R}^2)}dt \le c(T). \tag{4.16}$$

Proof. Multiplying $(1.12)_2$ by the weight t^α yields

$$\rho\partial_t(t^\alpha\boldsymbol{u}) - \widetilde{\Delta}(t^\alpha\boldsymbol{u}) + \widetilde{\nabla}(t^\alpha P) = \alpha t^{\alpha-1}\rho\boldsymbol{u} - t^\alpha\Big(\rho\boldsymbol{u}\cdot\widetilde{\nabla}\boldsymbol{u} + \widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A})\Big).$$

Applying the maximal regularity in the Proposition 2.6 (take $p = q$), we have

$$\|t^\alpha\boldsymbol{u}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{q}}_{q,q})} + \|(t^\alpha\boldsymbol{u})_t, \widetilde{\nabla}^2(t^\alpha\boldsymbol{u}), \widetilde{\nabla}(t^\alpha P)\|_{L^q((0,T)\times\mathbb{R}^2)}$$

$$\leq C(q,\rho_*,\rho^*)\|t^{\alpha-1}\boldsymbol{u}, t^\alpha \boldsymbol{u}\cdot\widetilde{\nabla}\boldsymbol{u}, t^\alpha\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A})\|_{L^q((0,T)\times\mathbb{R}^2)}. \quad (4.17)$$

Since q is close to 2, we use Hölder's inequality to get

$$\|t^\alpha\boldsymbol{u}\cdot\widetilde{\nabla}\boldsymbol{u}\|_{L^q((0,T)\times\mathbb{R}^2)} \lesssim \|t^\alpha\boldsymbol{u}\|_{L^{\frac{q}{3-q}}(0,T;L^{\frac{2q}{q-2}})}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^{\frac{q}{q-2}}(0,T;L^{\frac{2q}{4-q}})}$$

and

$$\|t^\alpha\widetilde{\Delta}\boldsymbol{A}\times(\widetilde{\nabla}\times\boldsymbol{A})\|_{L^q((0,T)\times\mathbb{R}^2)} \lesssim T^\alpha\|\widetilde{\Delta}\boldsymbol{A}\|_{L^q((0,T)\times\mathbb{R}^2)}\|\boldsymbol{A}\|_{L^\infty(\dot{B}^{\frac{5}{3}}_{3,1})}.$$

We can control $\boldsymbol{u}$ and $\widetilde{\Delta}\boldsymbol{A}$ by applying (A.1) and the bounds (1.15)-(1.16) in the following way:

$$\|t^{\alpha-1}\boldsymbol{u}\|_{L^q(0,T;L^q)} \leq \|\boldsymbol{u}\|^{\frac{2}{q}}_{L^\infty(0,T;L^2)}\|\widetilde{\nabla}\boldsymbol{u}\|^{1-\frac{2}{q}}_{L^\infty((0,T;L^2)}T^{\frac{1}{2}-\frac{1}{q}} \leq c(T),$$

$$\|t^\alpha\boldsymbol{u}\|_{L^{\frac{q}{3-q}}(0,T;L^{\frac{2q}{q-2}})} \leq \|t^\alpha\|_{L^{\frac{q}{3-q}}_T}\|\boldsymbol{u}\|^{1-\frac{2}{q}}_{L^\infty(0,T;L^2)}\|\widetilde{\nabla}\boldsymbol{u}\|^{\frac{2}{q}}_{L^\infty((0,T;L^2)} \leq c(T),$$

$$\|\widetilde{\nabla}\boldsymbol{u}\|_{L^{\frac{q}{q-2}}(0,T;L^{\frac{2q}{4-q}})} \leq \|\widetilde{\nabla}\boldsymbol{u}\|^{\frac{4-q}{q}}_{L^\infty(0,T;L^2)}\|\widetilde{\nabla}^2\boldsymbol{u}\|^{\frac{2q-4}{q}}_{L^2(0,T;L^2)} < \infty$$

and (use that $\frac{3}{2} \leq \frac{3q-3}{2} \leq 6$)

$$\|\widetilde{\Delta}\boldsymbol{A}\|_{L^q((0,T)\times\mathbb{R}^2)} \leq \|\boldsymbol{A}\|^{\frac{3-q}{2q}}_{L^\infty((0,T)\times\mathbb{R}^2)}\|\widetilde{\Delta}\boldsymbol{A}\|^{\frac{3q-3}{2q}}_{L^{\frac{3q-3}{2}}(0,T;L^3)} \lesssim \|\boldsymbol{A}_0\|_{\dot{B}^{\frac{2}{3}}_{3,1}\cap\dot{B}^{\frac{5}{3}}_{3,1}}.$$

Hence, inserting the above inequalities into (4.17) yields (4.15).

Finally, we can bound $\widetilde{\nabla}\boldsymbol{u}$ by using the Gagliardo-Nirenberg inequality and Hölder's inequality as below

$$\begin{aligned}\int_0^T \|\widetilde{\nabla}\boldsymbol{u}\|_{L^\infty(\mathbb{R}^2)}dt &\lesssim \int_0^T t^{-\frac{q\alpha}{2q-2}}\|\widetilde{\nabla}\boldsymbol{u}\|^{\frac{q-2}{2q-2}}_{L^2}\|\widetilde{\nabla}^2(t^\alpha\boldsymbol{u})\|^{\frac{q}{2q-2}}_{L^q}dt\\ &\lesssim \|t^{-\frac{3q-4}{4q-4}}\|_{L^{\frac{2q-2}{2q-3}}_T}\|\widetilde{\nabla}\boldsymbol{u}\|^{\frac{q-2}{2q-2}}_{L^\infty_T(L^2)}\Big\|\|\widetilde{\nabla}^2(t^\alpha\boldsymbol{u})\|^{\frac{q}{2q-2}}_{L^q}\Big\|_{L^{2q-2}}\\ &\lesssim T^{\frac{q-2}{4q-4}}\|\widetilde{\nabla}\boldsymbol{u}\|^{\frac{q-2}{2q-2}}_{L^\infty_T(L^2)}\|\widetilde{\nabla}^2(t^\alpha\boldsymbol{u})\|^{\frac{q}{2q-2}}_{L^q_T(L^q)} \leq c(T).\end{aligned}$$

□

Now, we are ready to use Lemma 4.2 and the bounds (1.15)-(1.16) to prove uniqueness part in Theorem 1.3, by applying the Lagrangian approach similar to the one in section 3.1 for 3D case. The key ingredient is the proposition 2.6.

Let us assume that (2.5) is satisfied for small enough T, and consider two solutions $(\rho^1,\boldsymbol{u}^1,\boldsymbol{A}^1,P^1,Q^1)$ and $(\rho^2,\boldsymbol{u}^2,\boldsymbol{A}^2,P^2,Q^2)$ of (1.12), emanating from the same initial data and denoting by $(\overline{\rho}^1,\overline{\boldsymbol{u}}^1,\overline{\boldsymbol{A}}^1,\overline{P}^1,\overline{Q}^1)$ and $(\overline{\rho}^2,\overline{\boldsymbol{u}}^2,\overline{\boldsymbol{A}}^2,\overline{P}^2,\overline{Q}^2)$ the corresponding ones in Lagrangian formulation (2.7). Denote $\delta\overline{\boldsymbol{u}} = \overline{\boldsymbol{u}}^1 - \overline{\boldsymbol{u}}^2$, $\delta\overline{P} = \overline{P}^1 - \overline{P}^2$, $\delta\overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}^1 - \overline{\boldsymbol{A}}^2, \delta\overline{Q} = \overline{Q}^1 - \overline{Q}^2$ and $\mathbb{M}^i(t) := \mathbb{M}_{\overline{\boldsymbol{u}}^i}(t), i = 1,2$. We

see that $\overline{\rho}^1=\overline{\rho}^2=\rho_0$ and $(\delta\overline{\boldsymbol{u}},\delta\widetilde{\nabla}\overline{P},\delta\overline{\boldsymbol{A}},\delta\widetilde{\nabla}\overline{Q})$ satisfying

$$\begin{cases}\rho_0\partial_t\delta\overline{\boldsymbol{u}}-\widetilde{\Delta}\delta\overline{\boldsymbol{u}}+\widetilde{\nabla}\delta\overline{P}=\delta\boldsymbol{f}_1-\delta\boldsymbol{f}_2,\\ \qquad\qquad\widetilde{\operatorname{div}}\,\delta\overline{\boldsymbol{u}}=\widetilde{\operatorname{div}}\,\delta\boldsymbol{R}_1,\\ \partial_t\delta\overline{\boldsymbol{A}}-\widetilde{\Delta}\delta\overline{\boldsymbol{A}}+\widetilde{\nabla}\delta\overline{Q}=\delta\boldsymbol{f}_2/\rho_0+\delta\boldsymbol{f}_3+\delta\boldsymbol{f}_4,\\ \qquad\qquad\widetilde{\operatorname{div}}\,\delta\overline{\boldsymbol{A}}=\widetilde{\operatorname{div}}\,\delta\boldsymbol{R}_2,\\ \qquad\qquad\delta\overline{\boldsymbol{u}}|_{t=0}=\delta\overline{\boldsymbol{A}}|_{t=0}=0,\end{cases}\tag{4.18}$$

with

$$\begin{aligned}\delta\boldsymbol{f}_1:=&[(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{u}}^1-(\widetilde{\Delta}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{u}}]-[(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\overline{P}^1-(\widetilde{\nabla}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\delta\overline{P}],\\ \delta\boldsymbol{f}_2:=&[(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\times(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)]+[\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\times(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)]\\ &+[\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times((\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\times\overline{\boldsymbol{A}}^1)]+[\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2}\times\delta\overline{\boldsymbol{A}})],\\ \delta\boldsymbol{f}_3:=&[(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1-(\widetilde{\Delta}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{A}}]-[(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\overline{Q}^1-(\widetilde{\nabla}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\delta\overline{Q}],\\ \delta\boldsymbol{f}_4:=&\big((\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\big)^{\mathrm{T}}\overline{\boldsymbol{u}}^1+\big(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\big)^{\mathrm{T}}\overline{\boldsymbol{u}}^1+\big(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\big)^{\mathrm{T}}\delta\overline{\boldsymbol{u}},\\ \delta\boldsymbol{R}_1:=&(\mathrm{Id}-\mathbb{M}^1)\overline{\boldsymbol{u}}^1-(\mathrm{Id}-\mathbb{M}^2)\overline{\boldsymbol{u}}^2,\\ \delta\boldsymbol{R}_2:=&(\mathrm{Id}-\mathbb{M}^1)\overline{\boldsymbol{A}}^1-(\mathrm{Id}-\mathbb{M}^2)\overline{\boldsymbol{A}}^2.\end{aligned}$$

Applying Proposition 2.6 to equations of $\delta\overline{\boldsymbol{u}}$ and $\delta\overline{\boldsymbol{A}}$ with $p\in(1,2)$ such that $p\to2$, one has

$$\begin{aligned}&\|\delta\overline{\boldsymbol{u}}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^2))}+\|\partial_t\delta\overline{\boldsymbol{u}},\widetilde{\nabla}^2\delta\overline{\boldsymbol{u}},\widetilde{\nabla}\delta\overline{P}\|_{L^p((0,T)\times\mathbb{R}^2)}\\ &\qquad\le C\Big(\|\delta\boldsymbol{f}_1,\delta\boldsymbol{f}_2,(\delta\boldsymbol{R}_1)_t\|_{L^p((0,T)\times\mathbb{R}^2))}+\|\operatorname{div}\delta\boldsymbol{R}_1\|_{L^p(0,T;\dot{W}^1_p(\mathbb{R}^2))}\Big)\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}&\|\delta\overline{\boldsymbol{A}}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^2))}+\|\partial_t\delta\overline{\boldsymbol{A}},\widetilde{\nabla}^2\delta\overline{\boldsymbol{A}},\widetilde{\nabla}\delta\overline{Q}\|_{L^p((0,T)\times\mathbb{R}^2)}\\ &\le C\Big(\|\delta\boldsymbol{f}_2/\rho_0,\delta\boldsymbol{f}_3,\delta\boldsymbol{f}_4,(\delta\boldsymbol{R}_2)_t\|_{L^p((0,T)\times\mathbb{R}^2)}+\|\operatorname{div}\delta\boldsymbol{R}_2\|_{L^p(0,T;\dot{W}^1_p(\mathbb{R}^2))}\Big).\end{aligned}\tag{4.20}$$

Define the functional space

$$\dot{W}^{2,1}_{p,p}(T):=\{\boldsymbol{v}\in L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^2));\boldsymbol{v}_t,\nabla^2\boldsymbol{v}\in L^p((0,T)\times\mathbb{R}^2)\}.$$

We still denote by $c(T)$ harmless positive functions such that $c(T)$ going to 0 when T tends to 0, that may change from line to line. We estimate the right-hand-side terms of (4.19) and (4.20) in the following steps.

Estimates of $(\delta\boldsymbol{R}_1)_t$, $(\delta\boldsymbol{R}_2)_t$, $\operatorname{div}\delta\boldsymbol{R}_1,\operatorname{div}\delta\boldsymbol{R}_2$. Firstly, we consider the terms $(\delta\boldsymbol{R}_1)_t$ and $(\delta\boldsymbol{R}_2)_t$ which contain four terms as follows:

$$\begin{aligned}&\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]\|_{L^p_T((0,T)\times\mathbb{R}^2)},\quad\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)\|_{L^p_T((0,T)\times\mathbb{R}^2)},\\ &\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{A}}]\|_{L^p_T((0,T)\times\mathbb{R}^2)},\quad\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{A}}^1)\|_{L^p_T((0,T)\times\mathbb{R}^2)}.\end{aligned}$$

According to Proposition 2.1 and 2.2, we have

$$\begin{aligned}&\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]\|_{L^p_T((0,T)\times\mathbb{R}^2)}\\ \lesssim&\|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^p_T(L^{\frac{p}{p-1}})}\|\delta\overline{\boldsymbol{u}}\|_{L^\infty_T(L^{\frac{p}{2-p}})}+\|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}\|\partial_t\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}.\end{aligned}$$

Thanks to the Gagliardo-Nirenberg inequality (A.1) and Besov's embedding in Proposition A.2, and the bounds (1.15), we have

$$\begin{aligned}\|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^p_T(L^{\frac{p}{p-1}})} \lesssim& \Big\| \|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^2}^{\frac{2p-2}{p}} \|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^2}^{\frac{2-p}{p}} \Big\|_{L^p_T}\\ \lesssim& \|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^2_T(L^2)}^{\frac{2p-2}{p}} \|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^2)}^{\frac{2-p}{p}} \lesssim c(T)\end{aligned}$$

and (use that $\frac{p}{2-p} > p$)

$$\|\delta\overline{\boldsymbol{u}}\|_{L^\infty_T(L^{\frac{p}{2-p}})} \lesssim \|\delta\overline{\boldsymbol{u}}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p})}.$$

Therefore, one finds that

$$\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{u}}]\|_{L^p((0,T)\times\mathbb{R}^2)} \lesssim c(T)\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)} \tag{4.21}$$

and similarly,

$$\|\partial_t[(\mathrm{Id}-\mathbb{M}^2)\delta\overline{\boldsymbol{A}}]\|_{L^p((0,T)\times\mathbb{R}^2)} \lesssim c(T)\|\delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{p,p}(T)}. \tag{4.22}$$

Next, due to Proposition 2.2 and Hölder's inequality,

$$\begin{aligned}&\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim& \|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{3p}{3-p}})}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)} + \Big\|\int_0^t |\widetilde{D}\delta\overline{\boldsymbol{u}}|\,d\tau\,(|\widetilde{D}\overline{\boldsymbol{u}}^1| + |\widetilde{D}\overline{\boldsymbol{u}}^2|)\Big\|_{L^p_T(L^{\frac{4p}{4-p}})}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^4)}\\ &+ \|\int_0^t |\widetilde{D}\delta\overline{\boldsymbol{u}}|\,d\tau\,\partial_t\overline{\boldsymbol{u}}^1\|_{L^p_T(L^p)}\\ \lesssim& \|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{3p}{3-p}})}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)} + \|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|t^{\frac{p-1}{p}}\partial_t\overline{\boldsymbol{u}}^1\|_{L^p_T(L^2)}\\ &+ \|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|t^{\frac{p-1}{p}}(|\widetilde{D}\overline{\boldsymbol{u}}^1| + |\widetilde{D}\overline{\boldsymbol{u}}^2|)\|_{L^p_T(L^4)}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^4)}.\end{aligned} \tag{4.23}$$

According to Sobolev's embedding and Gagliardo-Nirenberg inequality, we can obtain that for some $p \in (1,2)$,

$$\begin{aligned}\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{3p}{3-p}})} \lesssim& \Big\| \|\delta\overline{\boldsymbol{u}}\|_{L^{\frac{p}{2-p}}}^{\frac{p}{6}} \|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p}^{\frac{6-p}{6}} \Big\|_{L^p_T}\\ \lesssim& \|\delta\overline{\boldsymbol{u}}\|_{L^{\frac{p^2}{6}}_T(\dot{W}^{2-\frac{2}{p}}_p)}^{\frac{p}{6}} \|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^{\frac{(6-p)p}{6}}_T(L^p)}^{\frac{6-p}{6}} \lesssim c(T)\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)},\end{aligned}$$

$$\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})} \lesssim \|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)},$$

and

$$\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^p_T(L^4)} \lesssim \Big\|t^{\frac{p-1}{p}}\|\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{1}{2}}\|\widetilde{D}^2\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{1}{2}}\Big\|_{L^p_T} \lesssim c(T).$$

Then, we find that

$$\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{u}}^1)\|_{L^p((0,T)\times\mathbb{R}^2)} \lesssim c(T)\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)}. \tag{4.24}$$

Similarly,

$$\begin{aligned}&\|\partial_t(\delta\mathbb{M}\,\overline{\boldsymbol{A}}^1)\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim& \|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\|\overline{\boldsymbol{A}}^1\|_{L^\infty((0,T)\times\mathbb{R}^2)} + \|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{3p}{3-p}})}\|t^{\frac{p-1}{p}}\partial_t\overline{\boldsymbol{A}}^1\|_{L^p_T(L^3)}\end{aligned}$$

$$+\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L_T^p(L^{\frac{2p}{2-p}})}\|t^{\frac{p-1}{p}}(|\widetilde{D}\overline{\boldsymbol{u}}^1|+|\widetilde{D}\overline{\boldsymbol{u}}^2|)\|_{L_T^p(L^2)}\|\overline{\boldsymbol{A}}^1\|_{L^\infty(((0,T)\times\mathbb{R}^2))}$$
$$\lesssim c(T)\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)}. \tag{4.25}$$

Gathering (4.21)-(4.25), we get

$$\|(\delta\boldsymbol{R}_1)_t,(\delta\boldsymbol{R}_2)_t\|_{L^p((0,T)\times\mathbb{R}^2)}\lesssim c(T)(\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)}+\|\delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{p,p}(T)}).$$

In the next, we estimate $\|\widetilde{\operatorname{div}}\,\delta\boldsymbol{R}_1\|_{L^p(0,T;\dot{W}^1_p(\mathbb{R}^2))}$ and $\|\widetilde{\operatorname{div}}\,\delta\boldsymbol{R}_2\|_{L^p(0,T;\dot{W}^1_p(\mathbb{R}^2))}$ which can be bounded by the sum of the following four terms:

$$\|\widetilde{D}\delta\overline{\boldsymbol{u}}:(\mathrm{Id}-\mathbb{M}^2)\|_{L^p(0,T;\dot{W}^1_p)},\quad \|\widetilde{D}\overline{\boldsymbol{u}}^1:\delta\mathbb{M}\|_{L^p(0,T;\dot{W}^1_p)},$$
$$\|\widetilde{D}\delta\overline{\boldsymbol{A}}:(\mathrm{Id}-\mathbb{M}^2)\|_{L^p(0,T;\dot{W}^1_p)},\quad \|\widetilde{D}\overline{\boldsymbol{A}}^1:\delta\mathbb{M}\|_{L^p(0,T;\dot{W}^1_p)}.$$

Similar with (4.23), applying Hölder's inequality, Proposition 2.1 and 2.2, we have

$$\begin{aligned}&\|\widetilde{D}\delta\overline{\boldsymbol{u}}:(\mathrm{Id}-\mathbb{M}^2)\|_{L^p(0,T;\dot{W}^1_p)}\\ \lesssim&\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}+\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^1_T(\dot{H}^1)}\\ \lesssim&c(T)\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}&\|\widetilde{D}\overline{\boldsymbol{u}}^1:\delta\mathbb{M}\|_{L^p(0,T;\dot{W}^1_p)}\\ \lesssim&\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^p_T(W^1_2)}+\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^p_T(L^\infty)}\\ &+\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^p_T(L^\infty)}(\|\widetilde{D}^2\overline{\boldsymbol{u}}^1\|_{L^1_T(L^2)}+\|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^2)})\,\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\\ \lesssim&c(T)\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}.\end{aligned} \tag{4.27}$$

In the last step of (4.27), we used Gagliardo-Nirenberg's inequality to write

$$\begin{aligned}\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^p_T(L^\infty)}\lesssim&\left\|t^\beta\|\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^2}^{\frac{q-2}{2q-2}}\|t^\alpha\widetilde{D}^2\overline{\boldsymbol{u}}^1\|_{L^q}^{\frac{q}{2q-2}}\right\|_{L^p_T}\\ \lesssim&\|t^\beta\|_{L_T^{\frac{p(2q-2)}{2q-p-2}}}\|\widetilde{D}\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^2)}^{\frac{q-2}{2q-2}}\|t^\alpha\widetilde{D}^2\overline{\boldsymbol{u}}^1\|_{L^q_T(L^q)}^{\frac{q}{2q-2}}\lesssim c(T),\end{aligned}$$

where $\beta=\frac{p-1}{p}-\frac{q\alpha}{2q-2}$ satisfied $\beta\frac{2q-2}{2q-3}>-1$ whenever p satisfying $4-\frac{6}{q}<p<2$.

The rest two terms can be estimated in a similar way as (4.26)-(4.27). One has

$$\|\widetilde{D}\delta\overline{\boldsymbol{A}}:(\mathrm{Id}-\mathbb{M}^2)\|_{L^p(0,T;\dot{W}^1_p)}\lesssim c(T)\|\widetilde{D}^2\delta\overline{\boldsymbol{A}}\|_{L^p((0,T)\times\mathbb{R}^2)}$$

and

$$\begin{aligned}&\|\widetilde{D}\overline{\boldsymbol{A}}^1:\delta\mathbb{M}\|_{L^p(0,T;\dot{W}^1_p)}\\ \lesssim&\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{3p}{3-p}})}\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{A}}^1\|_{L^p_T(W^1_3)}+\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{A}}^1\|_{L^p_T(L^\infty)}\\ &+\|t^{\frac{p-1}{p}}\widetilde{D}\overline{\boldsymbol{A}}^1\|_{L^p_T(L^\infty)}(\|\widetilde{D}^2\overline{\boldsymbol{u}}^1\|_{L^1_T(L^2)}+\|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^2)})\,\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\\ \lesssim&c(T)\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)}.\end{aligned}$$

Therefore, we obtain that

$$\|\widetilde{\operatorname{div}}\,\delta\boldsymbol{R}_1,\widetilde{\operatorname{div}}\,\delta\boldsymbol{R}_2\|_{L^p(0,T;\dot{W}^1_p)}\lesssim c(T)(\|\delta\overline{\boldsymbol{u}}\|_{\dot{W}^{2,1}_{p,p}(T)}+\|\delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{p,p}(T)}).$$

Estimates of $\delta\boldsymbol{f}_1$, $\delta\boldsymbol{f}_2$, $\delta\boldsymbol{f}_3, \delta\boldsymbol{f}_4$. For the term $\delta\boldsymbol{f}_1$, we see that

$$\begin{aligned}\|\delta\boldsymbol{f}_1\|_{L^p((0,T)\times\mathbb{R}^2)} \leq & \|(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1})\overline{\boldsymbol{u}}^1, (\widetilde{\Delta}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\\ & + \|(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1})\overline{P}^1, (\widetilde{\nabla}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\delta\overline{P}\|_{L^p((0,T)\times\mathbb{R}^2)}.\end{aligned}$$

The first two terms on the righthand side can be bounded similar to (4.26) and (4.27), we have

$$\|(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1})\overline{\boldsymbol{u}}^1, (\widetilde{\Delta}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)} \lesssim c(T)\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}.$$

For other terms in $\delta\boldsymbol{f}_1$, we have

$$\begin{aligned}&\|(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1})\overline{P}^1, (\widetilde{\nabla}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\delta\overline{P}\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & \left\|\widetilde{D}\delta\overline{\boldsymbol{u}}\right\|_{L^p_T(L^{\frac{2p}{2-p}})}\left\|t^{\frac{p-1}{p}}\widetilde{\nabla}\overline{P}^1\right\|_{L^p_T(L^2)} + \left\|\widetilde{D}\overline{\boldsymbol{u}}^2\right\|_{L^1_T(L^\infty)}\left\|\widetilde{\nabla}\delta\overline{P}\right\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & c(T)\left(\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)} + \left\|\widetilde{\nabla}\delta\overline{P}\right\|_{L^p((0,T)\times\mathbb{R}^2)}\right).\end{aligned}$$

Next, we can treat the terms in $\delta\boldsymbol{f}_2$ in the same way as $\delta\boldsymbol{f}_1$, by using (4.5),

$$\begin{aligned}&\|(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1\times(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & \left\||\widetilde{D}^2\overline{\boldsymbol{A}}^1|\int_0^t|\widetilde{D}\delta\overline{\boldsymbol{u}}|\,d\tau\right\|_{L^p((0,T)\times\mathbb{R}^2)} + \left\||D\overline{\boldsymbol{A}}^1|\int_0^t|D^2\delta\overline{\boldsymbol{u}}|\,d\tau\right\|_{L^p((0,T)\times\mathbb{R}^2)}\\ & + \left\||\widetilde{D}\overline{\boldsymbol{A}}^1|\int_0^t(|\widetilde{D}^2\overline{\boldsymbol{u}}^1|+|\widetilde{D}^2\overline{\boldsymbol{u}}^2|)\,d\tau\int_0^t|\widetilde{D}\delta\overline{\boldsymbol{u}}|\,d\tau\right\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & c(T)\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)},\end{aligned}$$

$$\begin{aligned}\|\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\times(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1)\|_{L^p((0,T)\times\mathbb{R}^2)} \leq & \|\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\delta\overline{\boldsymbol{A}}\|_{L^p((0,T)\times\mathbb{R}^2)}\|\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}\times\overline{\boldsymbol{A}}^1\|_{L^\infty_T(L^\infty)}\\ \lesssim & \varepsilon_2\|\widetilde{D}^2\delta\overline{\boldsymbol{A}}\|_{L^p((0,T)\times\mathbb{R}^2)},\end{aligned}$$

$$\begin{aligned}&\|\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times((\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\times\overline{\boldsymbol{A}}^1)\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & \|t^{\frac{p-1}{p}}\widetilde{\nabla}\times\overline{\boldsymbol{A}}^1\|_{L^p_T(L^\infty)}\|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^2)}\|\widetilde{D}\overline{\boldsymbol{A}}^2\|_{L^\infty_T(L^\infty)}\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\\ & + \|t^{\frac{p-1}{p}}\widetilde{\nabla}\times\overline{\boldsymbol{A}}^1\|_{L^\infty_T(L^\infty)}\|\widetilde{D}^2\overline{\boldsymbol{A}}^2\|_{L^p_T(L^3)}\|\widetilde{D}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{3p}{3-p}})}\\ \lesssim & c(T)\|D^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}\end{aligned}$$

and

$$\begin{aligned}&\|\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2}\overline{\boldsymbol{A}}^2\times(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2}\times\delta\overline{\boldsymbol{A}})\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & \|\widetilde{\nabla}\times\delta\overline{\boldsymbol{A}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^2)}\|\widetilde{D}\overline{\boldsymbol{A}}^2\|_{L^\infty_T(L^\infty)}\\ & + \|\widetilde{\nabla}\times\delta\overline{\boldsymbol{A}}\|_{L^{\frac{6p}{6-p}}_T(L^{\frac{3p}{3-p}})}\|\widetilde{D}^2\overline{\boldsymbol{A}}^2\|_{L^6_T(L^3)}\\ \lesssim & c(T)\|\delta\overline{A}\|_{\dot{W}^{2,1}_{p,p}(T)}.\end{aligned}$$

Next, the term $\delta\boldsymbol{f}_3$ may be handled along the same lines. Indeed we have

$$\begin{aligned}&\|(\widetilde{\Delta}_{\overline{\boldsymbol{u}}^1}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\overline{\boldsymbol{A}}^1, (\widetilde{\Delta}-\widetilde{\Delta}_{\overline{\boldsymbol{u}}^2})\delta\overline{\boldsymbol{A}}\|_{L^p((0,T)\times\mathbb{R}^2)}\\ \lesssim & \|\widetilde{\mathrm{div}}\,[\delta\mathbb{M}(\mathbb{M}^2)^{\mathrm{T}}\widetilde{\nabla}\overline{\boldsymbol{A}}^1]\|_{L^p((0,T)\times\mathbb{R}^2)} + \|\widetilde{\mathrm{div}}\,[\mathbb{M}^1(\delta\mathbb{M})^{\mathrm{T}}\widetilde{\nabla}\overline{\boldsymbol{A}}^1]\|_{L^p((0,T)\times\mathbb{R}^2)}\end{aligned}$$

$$
\begin{aligned}
&+\|\widetilde{D}^2\delta\overline{\boldsymbol{A}}\|_{L^p((0,T)\times\mathbb{R}^2)}\|\widetilde{D}\overline{\boldsymbol{u}}^2\|_{L^1_T(L^\infty)}+\|\widetilde{D}\delta\overline{\boldsymbol{A}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|\widetilde{D}^2\overline{\boldsymbol{u}}^2\|_{L^1_T(L^2)}\\
\lesssim& c(T)(\|\widetilde{D}^2\delta\overline{\boldsymbol{A}}\|_{L^p((0,T)\times\mathbb{R}^2)}+\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)})
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\widetilde{\nabla}_{\overline{\boldsymbol{u}}^1}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\overline{Q}^1,(\widetilde{\nabla}-\widetilde{\nabla}_{\overline{\boldsymbol{u}}^2})\delta\overline{Q}\|_{L^p((0,T)\times\mathbb{R}^2)}\\
\lesssim&\left\|\delta\mathbb{M}\widetilde{\nabla}\overline{Q}^1\right\|_{L^p((0,T)\times\mathbb{R}^2)}+\left\|\int_0^t|\widetilde{D}\overline{\boldsymbol{u}}^2|\,d\tau|\nabla\delta\overline{Q}|\right\|_{L^p((0,T)\times\mathbb{R}^2)}\\
\lesssim& c(T)\left(\|\widetilde{D}^2\delta\overline{\boldsymbol{u}}\|_{L^p((0,T)\times\mathbb{R}^2)}+\left\|\widetilde{\nabla}\delta\overline{Q}\right\|_{L^p((0,T)\times\mathbb{R}^2)}\right).
\end{aligned}
$$

Finally, we examine $\delta\boldsymbol{f}_4$ in the following way

$$
\begin{aligned}
&\|\delta\boldsymbol{f}_4\|_{L^p([0,T]\times\mathbb{R}^2)}\\
\lesssim&\|\widetilde{\nabla}\delta\overline{\boldsymbol{u}}\|_{L^p_T(L^{\frac{2p}{2-p}})}\|\widetilde{\nabla}\overline{\boldsymbol{A}}^1\|_{L^\infty_T(L^\infty)}\|\overline{\boldsymbol{u}}^1\|_{L^\infty(L^2)}\\
&+\|\widetilde{\nabla}\delta\overline{\boldsymbol{A}}\|_{L^p_T(L^{\frac{3p}{3-p}})}\|\overline{\boldsymbol{u}}^1\|_{L^\infty_T(L^3)}+\|\widetilde{\nabla}\overline{\boldsymbol{A}}^2\|_{L^p_T(L^{\frac{p}{p-1}})}\|\delta\overline{\boldsymbol{u}}\|_{L^\infty_T(L^{\frac{p}{2-p}})}\\
\lesssim& c(T)\left(\|\delta\overline{\boldsymbol{u}}\|_{W^{2,1}_{p,p}}+\|\delta\overline{\boldsymbol{A}}\|_{W^{2,1}_{p,p}}\right).
\end{aligned}
$$

In the end, one may conclude that

$$
\begin{aligned}
&\|\delta\overline{\boldsymbol{u}}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^2))}+\|\delta\overline{\boldsymbol{u}}_t,\widetilde{\nabla}^2\delta\overline{\boldsymbol{u}},\widetilde{\nabla}\delta\overline{P}\|_{L^p((0,T)\times\mathbb{R}^2)}\\
&\quad+\|\delta\overline{\boldsymbol{A}}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^2))}+\|\delta\overline{\boldsymbol{A}}_t,\widetilde{\nabla}^2\delta\overline{\boldsymbol{A}},\widetilde{\nabla}\delta\overline{Q}\|_{L^p((0,T)\times\mathbb{R}^2)}\\
&\qquad\lesssim c(T)\left(\|\delta\overline{\boldsymbol{u}},\delta\overline{\boldsymbol{A}}\|_{\dot{W}^{2,1}_{p,p}(T)}+\|\widetilde{\nabla}\delta\overline{P},\widetilde{\nabla}\delta\overline{Q}\|_{L^p((0,T)\times\mathbb{R}^2)}\right). \quad (4.28)
\end{aligned}
$$

Uniqueness follows on a sufficiently small time interval $[0,T]$, then on the whole interval, thanks to a standard connectivity argument.

We complete the proof of Theorem 1.3. □

A. Some properties of Besov and Lorentz spaces

We first recall some classical inequalities and then some basic facts on Besov spaces and Lorentz spaces, one may check [8, 30] for more details.

Proposition A.1 ([10]). *The following Gagliardo-Nirenberg inequality holds: If $(q,r)\in(1,\infty]^2$ and $(\sigma,s)\in(0,\infty)^2$ with $\sigma<s$, there exists a constant C such that*

$$
\|u\|_{\dot{W}^\sigma_p(\mathbb{R}^d)}\le C\|u\|^\theta_{\dot{W}^s_r(\mathbb{R}^d)}\|u\|^{1-\theta}_{L^q} \quad (A.1)
$$

with $\frac{1}{p}=\frac{\sigma}{d}+\theta\left(\frac{1}{r}-\frac{s}{d}\right)+\frac{1-\theta}{q}$, $\frac{\sigma}{s}\le\theta<1$.

Proposition A.2 ([8]). *Let $1\le p\le\infty$. Then there hold:*

- *For any $1\le p\le q\le\infty$, we have*
$$
\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1}(\mathbb{R}^d)\hookrightarrow L^q(\mathbb{R}^d).
$$
- *For any $s\in\mathbb{R}$, $1\le p_1\le p_2\le\infty$ and $1\le r_1\le r_2\le\infty$, we have*
$$
\dot{B}^s_{p_1,r_1}(\mathbb{R}^d)\hookrightarrow\dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^d).
$$

- *For any $s \in \mathbb{R}$ and $1 \le r \le \infty$, we have*
$$\|D^k u\|_{\dot{B}^s_{p,r}} \sim \|u\|_{\dot{B}^{s+k}_{p,r}}.$$
- *For any $\theta \in (0,1)$ and $s < \tilde{s}$, we have*
$$\|u\|_{\dot{B}^{\theta s+(1-\theta)\tilde{s}}_{p,1}} \lesssim \|u\|^{\theta}_{\dot{B}^{s}_{p,1}} \|u\|^{1-\theta}_{\dot{B}^{\tilde{s}}_{p,1}}.$$

Proposition A.3 ([30]). *For $1 < p, p_1, p_2 < \infty$ and $1 \le r, r_1, r_2 \le \infty$, we have*

- *The embedding: $L^{p,r_1} \hookrightarrow L^{p,r_2}$ if $r_1 \le r_2$, and $L^{p,p} = L^p$.*
- *The Hölder inequality:*
$$\|fg\|_{L^{p,r}} \lesssim \|f\|_{L^{p_1,r_1}} \|g\|_{L^{p_2,r_2}} \quad \text{with } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and } \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$
This inequality still holds for couples $(1,1)$ and (∞,∞) with convention $L^{1,1} = L^1$ and $L^{\infty,\infty} = L^\infty$.
- *For any $\alpha > 0$ and non-negative measurable function f, we have*
$$\|f^\alpha\|_{L^{p,r}} = \|f\|^\alpha_{L^{p\alpha,r\alpha}}.$$
- *For any $\alpha > 0$, we have $\|t^{-\alpha} 1_{\mathbb{R}_+}\|_{L^{\frac{1}{\alpha},\infty}} = 1$.*

B. Proof of Proposition 2.4 and 2.6

Proof of Proposition 2.4. We first derive the following maximal regularity estimates for the equation (2.11): let $1 < p, m < \infty$, one has

$$\|v\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{m}}_{p,m})} + \|v_t, \kappa\nabla^2 v, g\, v\|_{L^m(0,T;L^p)} \le C\big(\|v_0\|_{\dot{B}^{2-\frac{2}{m}}_{p,m}} + \|f\|_{L^m(0,T;L^p)}\big). \quad \text{(B.1)}$$

According to Duhamel's formula, we have

$$v = e^{\kappa t\Delta} e^{-\int_0^t g(s)ds} v_0 + \int_0^t e^{(t-s)\kappa\Delta} e^{-\int_s^t g(\tau)d\tau} f(s)ds.$$

Noticing that $g(t) \ge 0$ and $g \in L^1(0,T)$ which yields

$$e^{-\int_0^t g(s)ds} \le 1 \quad \text{and} \quad e^{-\int_s^t g(\tau)d\tau} \le 1, \text{ for } s \le t.$$

Then, (B.1) follows from the classical result in [29]. See also the appendix of [16].

As regards the time regularity in Lorentz space, the proof may achieved by (B.1) and interpolations. See for example the appendix of [20]. The details are left to the reader. □

Proof of Proposition 2.6. The proof is motivated by Theorem 4.1 in [21]. First, we transform the system (2.15) into the divergence-free case. Let $g = \operatorname{div} \boldsymbol{R}$, $Q = P - \kappa g$ and $\boldsymbol{v} = \boldsymbol{u} + \nabla(-\Delta)^{-1} g$, then $\boldsymbol{v}$ satisfies

$$\begin{cases} \rho\partial_t \boldsymbol{v} - \kappa\Delta\boldsymbol{v} + \nabla Q = \boldsymbol{f} + \rho\nabla(-\Delta)^{-1}\partial_t g := \boldsymbol{F} & \text{in } (0,T)\times\mathbb{R}^d, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \boldsymbol{v}|_{t=0} = \boldsymbol{u}_0 - \nabla\Delta^{-1} g_0 := \boldsymbol{v}_0 & \text{in } \mathbb{R}^d, \end{cases} \quad \text{(B.2)}$$

Next, we reduce the problem to the one with null initial data by solving

$$\begin{cases} \rho^*\partial_t\overline{\boldsymbol{v}} - \kappa\Delta\overline{\boldsymbol{v}} + \nabla\overline{Q} = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \operatorname{div}\overline{\boldsymbol{v}} = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \overline{\boldsymbol{v}}|_{t=0} = \boldsymbol{v}_0 & \text{in } \mathbb{R}^d. \end{cases}$$

The L^p-estimate of the Stokes system gives us

$$\|\rho^*\partial_t\overline{\boldsymbol{v}}, \kappa\Delta\overline{\boldsymbol{v}}, \nabla\overline{Q}\|_{L^p((0,T)\times\mathbb{R}^d)} \le K_p(\rho^*)^{\frac{1}{p}}\kappa^{1-\frac{1}{p}}\|\boldsymbol{v}_0\|_{\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^d)}, \tag{B.3}$$

for $1 < p < \infty$ and some positive constant K_p depends on p.

Next we look for $\boldsymbol{v}$ in the form $\boldsymbol{v} = \overline{\boldsymbol{v}} + \widetilde{\boldsymbol{v}}$, where $\widetilde{\boldsymbol{v}}$ fulfils

$$\begin{cases} \rho\partial_t\widetilde{\boldsymbol{v}} - \kappa\Delta\widetilde{\boldsymbol{v}} + \nabla\widetilde{Q} = \boldsymbol{F} + (\rho^* - \rho)\partial_t\overline{\boldsymbol{v}} := \boldsymbol{G} & \text{in } (0,T)\times\mathbb{R}^d, \\ \operatorname{div}\widetilde{\boldsymbol{v}} = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \widetilde{\boldsymbol{v}}|_{t=0} = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{B.4}$$

Thanks to the bounds of ρ and (B.3), we have

$$\|\boldsymbol{G}\|_{L^p((0,T)\times\mathbb{R}^d)} \le \|\boldsymbol{F}\|_{L^p((0,T)\times\mathbb{R}^d)} + \|\boldsymbol{v}_0\|_{\dot{B}^{2-\frac{2}{p}}_{p,p}(\mathbb{R}^d)} K_p(\rho^* - \rho_*)(\frac{\kappa}{\rho^*})^{1-\frac{1}{p}}. \tag{B.5}$$

Now, setting $\boldsymbol{H} := \boldsymbol{G} + (\rho^* - \rho)\partial_t\widetilde{\boldsymbol{v}}$, then system (B.4) reduces to

$$\begin{cases} \rho^*\partial_t\widetilde{\boldsymbol{v}} - \kappa\Delta\widetilde{\boldsymbol{v}} + \nabla\widetilde{Q} = \boldsymbol{H} & \text{in } (0,T)\times\mathbb{R}^d, \\ \operatorname{div}\widetilde{\boldsymbol{v}} = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \widetilde{\boldsymbol{v}}|_{t=0} = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{B.6}$$

We claim that for all $p \in (1,\infty)$, there exists a positive constant C_p such that

$$\|\rho^*\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)} \le C_p\|\boldsymbol{H}\|_{L^p((0,T)\times\mathbb{R}^d)},$$

with $C_p \to 1$ for $p \to 2$.

Indeed, we can show that $C_2 \le 1$, we just take the inner product of (B.4) with $\partial_t\widetilde{\boldsymbol{v}}$, which yields

$$\rho^*\|\partial_t\widetilde{\boldsymbol{v}}\|_{L^2}^2 + \frac{\kappa}{2}\frac{d}{dt}\|\nabla\widetilde{\boldsymbol{v}}\|_{L^2}^2 = \int_{\mathbb{R}^d}\boldsymbol{H}\partial_t\widetilde{\boldsymbol{v}}\,dx \le \frac{1}{2}\rho^*\|\partial_t\widetilde{\boldsymbol{v}}\|_{L^2}^2 + \frac{1}{2\rho^*}\|\boldsymbol{H}\|_{L^2}^2.$$

Then for any fixed $p_0 \in (1,\infty)\setminus\{2\}$, the standard maximal regularity estimate is

$$\|\rho^*\partial_t\widetilde{\boldsymbol{v}}\|_{L^{p_0}((0,T)\times\mathbb{R}^d)} \le K_{p_0}\|\boldsymbol{H}\|_{L^{p_0}((0,T)\times\mathbb{R}^d)},$$

and the Hölder inequality gives us for all $\theta \in [0,1]$,

$$\|z\|_{L^r((0,T)\times\mathbb{R}^d)} \le \|z\|^{1-\theta}_{L^2((0,T)\times\mathbb{R}^d)}\|z\|^{\theta}_{L^{p_0}((0,T)\times\mathbb{R}^d)} \quad \text{with } \frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{p_0}.$$

Therefore $C_p \le K_{p_0}^{\theta}$, whence $\limsup C_p \le 1$ for $p \to 2$ (as $\theta \to 0$).

Now, remembering the definition of $\boldsymbol{H}$, we write for all $p \in (1,\infty)$,

$$\begin{aligned} \|\rho^*\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)} \le& C_p\left(\|\boldsymbol{G}\|_{L^p((0,T)\times\mathbb{R}^d)} + \|(\rho^* - \rho)\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)}\right) \\ \le& C_p\|\boldsymbol{G}\|_{L^p((0,T)\times\mathbb{R}^d)} + C_p\left(1 - \frac{\rho_*}{\rho^*}\right)\|\rho^*\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)}. \end{aligned}$$

Therefore, if

$$1 - C_p\left(1 - \frac{\rho_*}{\rho^*}\right) \ge \frac{1}{2}\frac{\rho_*}{\rho^*}, \tag{B.7}$$

then we end up with

$$\|\rho^*\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)} \leq \frac{2\rho^*C_p}{\rho_*}\|\boldsymbol{G}\|_{L^p((0,T)\times\mathbb{R}^d)}. \tag{B.8}$$

Let us emphasize that (B.7) is fulfilled for p close enough to 2, due to $C_p \to 1$ for $p \to 2$.

Now, we rewrite system (B.6) in the form

$$\begin{cases} -\kappa\Delta\widetilde{\boldsymbol{v}} + \nabla\widetilde{Q} = \boldsymbol{G} - \rho\partial_t\widetilde{\boldsymbol{v}} & \text{in } (0,T)\times\mathbb{R}^d, \\ \operatorname{div}\widetilde{\boldsymbol{v}} = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \widetilde{\boldsymbol{v}}|_{t=0} = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{B.9}$$

Applying the standard L^p-estimate to Stokes system (B.9) gives that

$$\begin{aligned} \|\kappa\Delta\widetilde{\boldsymbol{v}}, \nabla\widetilde{Q}\|_{L^p((0,T)\times\mathbb{R}^d)} \leq & K_p\|\boldsymbol{G} - \rho\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)} \\ \leq & K_p\left(\|\boldsymbol{G}\|_{L^p((0,T)\times\mathbb{R}^d)} + \rho^*\|\partial_t\widetilde{\boldsymbol{v}}\|_{L^p((0,T)\times\mathbb{R}^d)}\right). \end{aligned}$$

Then, putting together with (B.8) and assuming that p is close enough to 2 imply

$$\|\partial_t\widetilde{\boldsymbol{v}}, \kappa\Delta\widetilde{\boldsymbol{v}}, \nabla\widetilde{Q}\|_{L^p((0,T)\times\mathbb{R}^d)} \leq C(p,\rho_*,\rho^*)\|\boldsymbol{G}\|_{L^p((0,T)\times\mathbb{R}^d)}.$$

Finally, we combine with estimates (B.5) and (B.3) to obtain that

$$\begin{aligned} \kappa^{1-\frac{1}{p}}\|\boldsymbol{v}\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{p}}_{p,p})} + & \|\partial_t\boldsymbol{v}, \kappa\Delta\boldsymbol{v}, \nabla Q\|_{L^p((0,T)\times\mathbb{R}^d)} \\ & \leq C(p,\rho_*,\rho^*)(\kappa^{1-\frac{1}{p}}\|\boldsymbol{v}_0\|_{\dot{B}^{2-\frac{2}{p}}_{p,p}} + \|\boldsymbol{F}\|_{L^p((0,T)\times\mathbb{R}^d)}). \end{aligned}$$

It is now easy to complete the proof by the fact that $(-\Delta)^{-1}\nabla\operatorname{div}$ is homogeneous of degree 0. □

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References

[1] H. Abidi, *Équation de Navier-Stokes avec densité et viscosité variables dans l'espace critique*, Rev. Mat. Iberoam., 23 (2007), pp. 537–586.

[2] H. Abidi and G. Gui, *Global well-posedness for the 2-D inhomogeneous incompressible Navier-Stokes system with large initial data in critical spaces*, Arch. Ration. Mech. Anal., 242 (2021), pp. 1533–1570.

[3] H. Abidi, G. Gui, and P. Zhang, *On the wellposedness of three-dimensional inhomogeneous Navier–Stokes equations in the critical spaces*, Arch. Rational Mech. Anal., 204 (2012), pp. 189–230.

[4] H. Abidi and T. Hmidi, *Résultats d'existence dans des espaces critiques pour le système de la MHD inhomogène*, Ann. Math. Blaise Pascal, 14 (2007), pp. 103–148.

[5] H. Abidi and M. Paicu, *Existence globale pour un fluide inhomogène*, Ann. Inst. Fourier (Grenoble), 57 (2007), pp. 883–917.

[6] ———, *Global existence for the magnetohydrodynamic system in critical spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 138 (2008), pp. 447–476.
[7] M. Acheritogaray, P. Degond, A. Frouvelle, and J.-G. Liu, *Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system*, Kinet. Relat. Models, 4 (2011), pp. 901–918.
[8] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, vol. 343 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg, 2011.
[9] J. A. Bittencourt, *Fundamentals of Plasma Physics*, Fundamentals of Plasma Physics, Third Edition by J.A. Bittencourt. Published by Springer-Verlag, New York, Inc.; 2004. ISBN 0-387-20975-1., third ed., 2004.
[10] H. Brezis and P. Mironescu, *Gagliardo-Nirenberg inequalities and non-inequalities: The full story*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), pp. 1355–1376.
[11] C. Burtea, *Optimal well-posedness for the inhomogeneous incompressible Navier–Stokes system with general viscosity*, Anal. PDE, 10 (2017), pp. 439–479.
[12] D. Chae and J. Lee, *On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics*, J. Differential Equations, 256 (2014), pp. 3835–3858.
[13] Q. Chen, Z. Tan, and Y. Wang, *Strong solutions to the incompressible magnetohydrodynamic equations*, Math. Methods Appl. Sci., 34 (2011), pp. 94–107.
[14] R. Danchin, *Density-dependent incompressible viscous fluids in critical spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 133 (2003), pp. 1311–1334.
[15] ———, *Local and global well-posedness results for flows of inhomogeneous viscous fluids*, Adv. Differential Equations, 9 (2004), pp. 353–386.
[16] ———, *Density-dependent incompressible fluids in bounded domains*, J. Math. Fluid Mech., 8 (2006), pp. 333–381.
[17] R. Danchin and P. B. Mucha, *A lagrangian approach for the incompressible navier-stokes equations with variable density*, Comm. Pure Appl. Math., 65 (2012), pp. 1458–1480.
[18] ———, *Incompressible flows with piecewise constant density*, Arch. Ration. Mech. Anal., 207 (2013), pp. 991–1023.
[19] ———, *The incompressible Navier-Stokes equations in vacuum*, Comm. Pure Appl. Math., 72 (2019), pp. 1351–1385.
[20] R. Danchin, P. B. Mucha, and P. Tolksdorf, *Lorentz spaces in action on pressureless systems arising from models of collective behavior*, J. Evol. Equ., 21 (2021), pp. 3103–3127.
[21] R. Danchin and P. B. a. Mucha, *From compressible to incompressible inhomogeneous flows in the case of large data*, Tunis. J. Math., 1 (2019), pp. 127–149.
[22] R. Danchin and J. Tan, *The global solvability of the Hall-magnetohydrodynamics system in critical sobolev spaces*, Communications in Contemporary Mathematics, 0 (0), p. 2150099.
[23] ———, *On the well-posedness of the Hall-magnetohydrodynamics system in critical spaces*, Comm. Partial Differential Equations, 46 (2021), pp. 31–65.
[24] R. Danchin and S. Wang, *Global unique solutions for the inhomogeneous Navier-Stokes equation with only bounded density, in critical regularity spaces*, arXiv preprint arXiv:2201.11011, (2022).
[25] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98 (1989), pp. 511–547.
[26] J. Fan, A. Alsaedi, Y. Fukumoto, T. Hayat, and Y. Zhou, *A regularity criterion for the density-dependent Hall-magnetohydrodynamics*, Z. Anal. Anwend., 34 (2015), pp. 277–284.
[27] T. Forbes, *Magnetic reconnection in solar flares*, Geophysical & Astrophysical Fluid Dynamics, 62 (1991), pp. 15–36.
[28] J. F. Gerbeau and C. L. Bris, *Existence of solution for a density-dependant magnetohydrodynamic equation*, Adv. Differ. Equ., 2 (1997), pp. 427–452.
[29] Y. Giga and H. Sohr, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal., 102 (1991), pp. 72–94.
[30] L. Grafakos, *Classical Fourier analysis*, vol. 249 of Graduate Texts in Mathematics, Springer, New York, third ed., 2014.
[31] G. Gui, *Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity*, J. Funct. Anal., 267 (2014), pp. 1488–1539.

[32] H. Homann and R. Grauer, *Bifurcation analysis of magnetic reconnection in Hall-MHD-systems*, Phys. D, 208 (2005), pp. 59–72.
[33] J. Huang, M. Paicu, and P. Zhang, *Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-Lipschitz velocity*, Arch. Ration. Mech. Anal., 209 (2013), pp. 631–682.
[34] J. Jang and N. Masmoudi, *Derivation of Ohm's law from the kinetic equations*, SIAM J. Math. Anal., 44 (2012), pp. 3649–3669.
[35] P.-L. Lions, *Mathematical topics in fluid mechanics*, vol. 1 of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.
[36] L. Liu and J. Tan, *Global well-posedness for the Hall-magnetohydrodynamics system in larger critical Besov spaces*, J. Differential Equations, 274 (2021), pp. 382–413.
[37] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, vol. 27 of Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.
[38] P. Maremonti and V. A. Solonnikov, *On nonstationary Stokes problem in exterior domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24 (1997), pp. 395–449.
[39] M. Paicu, P. Zhang, and Z. Zhang, *Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density*, Comm. Partial Differential Equations, 38 (2013), pp. 1208–1234.
[40] J. Tan, *Global weak solutions to the density-dependent Hall-magnetohydrodynamics system*, J. Math. Fluid Mech., 23 (2021), pp. Paper No. 86, 19.
[41] M. Wardle, *Star formation and the Hall effect*, Astrophysics and Space Science, 292 (2004), pp. 317–323.
[42] H. Xu, *Maximal L^1 regularity for solutions to inhomogeneous incompressible navier-stokes equations*, arXiv preprint arXiv:2103.11513, (2021).
[43] X. Zhai, Y. Li, and W. Yan, *Global well-posedness for the 3D viscous nonhomogeneous incompressible magnetohydrodynamic equations*, Anal. Appl. (Singap.), 16 (2018), pp. 363–405.
[44] X. Zhai and Z. Yin, *Global well-posedness for the 3D incompressible inhomogeneous Navier-Stokes equations and MHD equations*, J. Differential Equations, 262 (2017), pp. 1359–1412.
[45] P. Zhang, *Global Fujita-Kato solution of 3-D inhomogeneous incompressible Navier-Stokes system*, Adv. Math., 363 (2020), pp. 107007, 43.

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