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Forces for the Navier–Stokes equations and the Koch and Tataru theorem.

Pierre Gilles Lemarié-Rieusset*

Abstract

We consider the Cauchy problem for the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^3$, with initial value $\vec{u}_0 \in \mathrm{BMO}^{-1}$ (as in Koch and Tataru's theorem) and with force $\vec{f} = \operatorname{div} \mathbb{F}$ where smallness of $\mathbb{F}$ ensures existence of a mild solution in absence of initial value. We study the interaction of the two solutions and discuss the existence of global solution for the complete problem (i.e. in presence of initial value and forcing term) under smallness assumptions. In particular, we discuss the interaction between Koch and Tataru solutions and Lei-Lin's solutions (in $L^2\mathcal{F}^{-1}L^1$) or solutions in the multiplier space $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$.

Keywords : Navier–Stokes equations, critical spaces, parabolic Sobolev spaces, parabolic Morrey spaces, mild solutions.

AMS classification : 35K55, 35Q30, 76D05.

In this paper, we consider global mild solutions of the Cauchy problem for the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^3$. When looking for assumptions that respect the symmetries of the Navier–Stokes equations (with respect to spatial translation or to dilations), one is lead to consider the initial data to be in BMO^{-1} (this is the famous Koch and Tataru theorem [Koc01]) but there is no natural choice for the forcing term. We are going to consider forces that are known to lead to global mild solutions (if they are small enough) in the absence of initial value, but the interaction between those forces and an initial value in BMO^{-1} or between forces in different functional spaces has not been discussed in the literature.

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1 Navier–Stokes equations with a forcing term and the Koch and Tataru theorem

Let us give a short description of mild solutions in critical spaces for the Navier–Stokes equations. We shall look for minimal regularity assumptions for thhe solution $\vec{u}$. It is therefore better to write the non-linear term $\vec{u}\cdot\vec{\nabla}\vec{u}$ in the Navier–Stokes equations as $\mathrm{div}(\vec{u}\otimes\vec{u})$ (the two vector fields are equal when $\vec{u}$ is a regular divergence free vector field). The Navier–Stokes equations we study are then

$$\begin{cases} \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}p + \mathrm{div}(\mathbb{F} - \vec{u}\otimes\vec{u}) \\ \mathrm{div}\,\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases} \tag{1}$$

Taking the divergence of the first equation, we get

$$\Delta p = (\vec{\nabla}\otimes\vec{\nabla})\cdot(\mathbb{F} - \vec{u}\otimes\vec{u})$$

and thus

$$\Delta\vec{\nabla}p = \vec{\nabla}\left((\vec{\nabla}\otimes\vec{\nabla})\cdot(\mathbb{F} - \vec{u}\otimes\vec{u})\right). \tag{2}$$

Assuming that $\vec{\nabla}p$ is equal to 0 at infinity, equation (2) defines p as a function of $\mathbb{F}$ and $\vec{u}$. More precisely, if $\mathbb{F} - \vec{u}\otimes\vec{u}$ is assumed to belong to $L^1_{\mathrm{loc}}((0,+\infty), L^1(\frac{dx}{1+|x|^4}))$, then [Fer21] shows that the solution $\vec{\nabla}p$ of equation (2) which is equal to 0 at infinity is given by the formula

$$\vec{\nabla}p = \frac{1}{\Delta}\vec{\nabla}\left((\vec{\nabla}\otimes\vec{\nabla})\cdot(\mathbb{F} - \vec{u}\otimes\vec{u})\right)$$

where, writing

$$G(x) = \frac{1}{4\pi|x|}$$

for the fundamental solution of $-\Delta$

$$(-\Delta G = \delta \text{ so that, for } f \in \mathcal{D}(\mathbb{R}^3, f = G * (-\Delta f))$$

and choosing a function $\psi \in \mathcal{D}(\mathbb{R}^3$ which is equal to 1 on a neighbourhood of 0, $\frac{1}{\Delta}\partial_i\partial_j\partial_k$ is defined as

$$\frac{1}{\Delta}\partial_i\partial_j\partial_k f = -\partial_i\partial_j\partial_k\left((\psi G) * f\right) - \left(\partial_i\partial_j\partial_k((1-\psi)G)\right) * f.$$

Thus, we have an equation with one unknown $\vec{u}$ and two data $\vec{u}_0$ and $\mathbb{F}$.

Defining (formally) the Leray projection operator $\mathbb{P}$ as

$$\mathbb{P} = \mathrm{Id} - \frac{1}{\Delta}\vec{\nabla}\,\mathrm{div},$$

the Navier–Stokes equations then become

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} + \mathbb{P}\,\mathrm{div}(\mathbb{F} - \vec{u}\otimes\vec{u}) \\ \vec{u}(0,.) = \vec{u}_0, \quad \mathrm{div}\,\vec{u}_0 = 0 \end{cases} \tag{3}$$

This is viewed as a non-linear heat equation and is transformed into the Duhamel formula

$$\vec{u} = e^{t\Delta}\vec{u}_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(\mathbb{F} - \vec{u}\otimes\vec{u})\,ds \tag{4}$$

where $e^{t\Delta}$ is the convolution operator with the heat kernel: $e^{t\Delta}f = W_t * f$ with

$$W_t = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{x^2}{4t}}.$$

We rewrite (4) as

$$\vec{u} = e^{t\Delta}\vec{u}_0 + \mathcal{L}(\mathbb{F}) - B(\vec{u},\vec{v}) \tag{5}$$

where

$$\mathcal{L}(\mathbb{F}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds$$

and

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(\vec{u}\otimes\vec{v})\,ds.$$

[Remark that we defined $\mathbb{P}\,\mathrm{div}\,\mathbb{F}$ for regular enough tensors $\mathbb{F}$, i.e. $\mathbb{F} \in L^1_{\mathrm{loc}}((0,+\infty), L^1(\frac{dx}{1+|x|^4}))$, but we may also consider more singular data $\mathbb{F}$, as long as we are able to give a sense to $\mathcal{L}(\mathbb{F})$.]

To have lighter computations and notations, it is better to forget the vectorial setting of the problem and to look at the bilinear operator B as a family of scalar operators acting on scalar functions: if $\vec{w} = B(\vec{u},\vec{v})$ then $w_i = \sum_{1\le j,k\le 3} B_{i,j,k}(u_j, v_k)$ with

$$B_{i,j,k}(u,v) = \int_0^t e^{t-s)\Delta}(\delta_{i,k}\partial_j - \frac{1}{\Delta}\partial_i\partial_j\partial_k)(uv)\,ds.$$

More generally, we define $\mathfrak{S}_1$ as the space of smooth functions σ on $\mathbb{R}^3$ which are positively homogeneous of degree 1 ($\sigma(\lambda\xi) = \lambda\sigma(\xi)$ for $\lambda > 0$). To $\sigma \in \mathfrak{S}_1$, we associate the Fourier multiplier $\sigma(D)$ and the operator

$$B_\sigma(u, v) = \int_0^t e^{t-s)\Delta}\sigma(D)(uv)\, ds.$$

The formalism of global mild solutions of the Cauchy problem for the Navier–Stokes equations is then described by the following definition and proposition:

Definition 1.
*An **adapted** Banach space is a Banach space $\mathcal{Y}$ of locally integrable functions on $(0, +\infty) \times \mathbb{R}^3$ such that, for every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is bounded on $\mathcal{Y}$:*

$$\|B_\sigma(u, v))\|_{\mathcal{Y}} \leq C_\sigma \|u\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}.$$

For vector fields $\vec{u}$ with coordinates in $\mathcal{Y}$, we shall write $\vec{u} \in \mathcal{Y}$ instead of $\vec{u} \in \mathcal{Y}^3$. The following theorem is then easy to check (through the Banach contraction principle):

Proposition 1.
Let $\mathcal{Y}$ be an adapted Banach space. If $\vec{u}_0$ is a (divergence free) vector fields of tempered distribution on $\mathbb{R}^3$ such $e^{t\Delta}\vec{u}_0 \in \mathcal{Y}$, $\mathcal{L}(\mathbb{F}) \in \mathcal{Y}$ and if $\vec{u}_0$ and $\mathbb{F}$ are small enough:

$$4C_0(\|e^{t\Delta}\vec{u}_0\|_{\mathcal{Y}} + \|\mathcal{L}(\mathbb{F})\|_{\mathcal{Y}}) < 1,$$

where C_0 is the norm of the bilinear operator B on $\mathcal{Y}^3$, then the Navier–Stokes problem (4) has a global solution $\vec{u}$ with

$$\|\vec{u}\|_{\mathcal{Y}} \leq 2(\|e^{t\Delta}\vec{u}_0\|_{\mathcal{Y}} + \|\mathcal{L}(\mathbb{F})\|_{\mathcal{Y}}).$$

Navier–Stokes equations have symmetries. In particular, we have the two following properties: if $\vec{u}$ is a solution of the Navier–Stokes problem (4) with data $\vec{u}_0$ and $\mathbb{F}$, then

- [space translation] $\vec{u}(t, x-x_0)$ is a solution of the Navier–Stokes problem (4) with data $\vec{u}_0(x - x_0)$ and $\mathbb{F}(t, x - x_0)$,
- [space dilation] if $\lambda > 0$, $\lambda\vec{u}(\lambda^2 t, \lambda x)$ is a solution of the Navier–Stokes problem (4) with data $\lambda\vec{u}_0(\lambda x)$ and $\lambda^2\mathbb{F}(\lambda^2 t, \lambda x)$.

In particular, we shall look for **critical spaces** $\mathcal{Y}$, meaning that we have invariance of the norms under space translations and space dilations: for every $x_0 \in \mathbb{R}^3$ and $\lambda > 0$

$$\|u(t, x - x_0)\|_{\mathcal{Y}} = \|u\|_{\mathcal{Y}}, \qquad \|\lambda u(\lambda^2 t, \lambda x)\|_{\mathcal{Y}} = \|u\|_{\mathcal{Y}}.$$

Finally, in order to give sense to the formula

$$\vec{u}(t, .) = \vec{u}_0 + \Delta \int_0^t \vec{u}(s, .)\, ds + \mathbb{P}\,\mathrm{div}(\int_0^t \mathbb{F} - \vec{u} \otimes \vec{u}\, ds),$$

we require the continuous embedding $\mathcal{Y} \subset \bigcap_{T>0} L^2((0,T), L^2(\frac{dx}{1+|x|^4}))$ [Due to the invariance through space translations or space dilations, it is equivalent to ask that $u \mapsto \mathbb{1}_{(0,1)\times B(0,1)} u$ is bounded from $\mathcal{Y}$ to $L^2((0,1) \times B(0,1))$.]

In particular, we have

$$\mathcal{Y} \subset \mathcal{Y}_2 = \{u \ / \sup_{t>0, x_0 \in \mathbb{R}^3} t^{-3/4} \|u\|_{L^2((0,t)\times B(x_0, \sqrt{t}))} < +\infty\}.$$

Thus, $\mathcal{Y}_2$ is maximal in the class of Banach spaces $\mathcal{Y}$ that satisfy the conditions $\|u(t, x - x_0)\|_{\mathcal{Y}} = \|u\|_{\mathcal{Y}}$, $\|\lambda u(\lambda^2 t, \lambda x)\|_{\mathcal{Y}} = \|u\|_{\mathcal{Y}}$ and

$$\sup_{\|u\|_{\mathcal{Y}} \le 1} \int_0^1 \int_{B(0,1)} |u(s,y)|^2\, ds\, dy < +\infty.$$

Koch and Tataru [Koc01] identified the space $\mathcal{X}$ such that $u_0 \in \mathcal{X}$ implies $S(u_0) \in \mathcal{Y}_2$:

Proposition 2.
For a tempered distribution u_0, the following assertions are equivalent:

(i) $e^{t\Delta} u_0 \in \mathcal{Y}_2$;

(ii) $u_0 \in \mathrm{BMO}^{-1} = \dot{F}^{-1}_{2,\infty}$ *(i.e., there exists* $\vec{v}_0$ *in* BMO^3 *such that* $u_0 = \mathrm{div}\, \vec{v}_0$*).*

However, $\mathcal{Y}_2$ is not an adapted Banach space (see a counter-example in Proposition 5 in the Appendix). The Koch and Tataru theorem deals with a subspace of $\mathcal{Y}_2$. We define the space

$$\mathcal{Z}_0 = \{u \in L^1_{\mathrm{loc}}((0,+\infty) \times \mathbb{R}^3 \ / \ \sup_{t>0} \sqrt{t} \|u(t,.)\|_\infty < +\infty\}.$$

The Koch and Tataru space $\mathcal{Y}_{KT}$ is then defined as:

$$\mathcal{Y}_{KT} = \mathcal{Y}_2 \cap \mathcal{Z}_0.$$

$\mathcal{Y}_{KT}$ is normed with $\|u\|_{\mathcal{Y}_{KT}} = \|u\|_{\mathcal{Y}_2} + \sup_{t>0} \sqrt{t}\|u(t,.)\|_\infty$, where

$$\|u\|_{\mathcal{Y}_2} = \sup_{t>0, x_0 \in \mathbb{R}^3} t^{-\frac{3}{4}} \|u\|_{L^2((0,t)\times B(x_0,\sqrt{t}))}.$$

Koch and Tataru's theorem is then the following one [Koc01, Lem02]:

Theorem 1.
A) For every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\mathcal{Y}_{KT} \times \mathcal{Y}_2$ to $\mathcal{Y}_2$. It is also a bounded bilinear operator from $\mathcal{Y}_{KT} \times \mathcal{Y}_{KT}$ to $\mathcal{Y}_{KT}$.
B) The following assertions are equivalent:

(i) $e^{t\Delta} u_0 \in \mathcal{Y}_{KT}$;

(ii) $u_0 \in \mathrm{BMO}^{-1}$;

C) $\mathcal{Y}_{KT}$ is an adapted Banach space. Thus, there exists a positive constant ϵ_0 such that, if $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{Y}_{KT}} + \|\mathcal{L}(\mathbb{F})\|_{\mathcal{Y}_{KT}} < \epsilon_0$, then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT}$.

Proof. We sketch the proof given by Koch and Tataru in [Koc01], and try to highlight the obstructing term for proving the boundedness of B_σ on $\mathcal{Y}_2$. A simple but key estimate is the following control:

$$|e^{(t-s)\Delta}\sigma(D)(u(s,.)v(s,.))| \leq C_\sigma \int \frac{1}{(\sqrt{t-s}+|x-y|)^4} |u(s,y)|\,|v(s,y)|\,dy.$$

We need to estimate, for every $T > 0$ and $x \in \mathbb{R}^3$, $\|\mathbb{1}_{Q_{T,x}} B_\sigma(u,v)\|_{L^2L^2}$ where

$$Q_{T,x} = \{(t,y) \ / \ 0 < t < T, |x-y| \leq \sqrt{T}\}.$$

Koch and Tataru split $w = B_\sigma(u,v)$ in three parts:

- $w_1 = B_\sigma(u, (1-\mathbb{1}_{Q_{10T,x}})v)$: we easily check that $\mathbb{1}_{Q_{T,x}}|w_1| \leq C\frac{1}{\sqrt{T}}\|u\|_{\mathcal{Y}_2}\|v\|_{\mathcal{Y}_2}$ and thus
$$\|\mathbb{1}_{Q_{T,x}} w_1\|_{L^2L^2} \leq CT^{3/4}\|u\|_{\mathcal{Y}_2}\|v\|_{\mathcal{Y}_2}.$$
- $w_2(t,y) = \sigma(D)e^{t\Delta}\int_0^t \mathbb{1}_{Q_{10T,x}} uv\,ds$. The main lemma in Koch and Tataru's proof states that the operator $Q(u,v) = \sqrt{-\Delta}e^{t\Delta}\int_0^t \mathbb{1}_{Q_{10T,x}} uv\,ds$ maps $\mathcal{Y}_2 \times \mathcal{Y}_2$ to L^2L^2 with a norm of order $T^{3/4}$ and thus
$$\|\mathbb{1}_{Q_{T,x}} w_2\|_{L^2L^2} \leq \|w_2\|_{L^2L^2} \leq CT^{3/4}\|u\|_{\mathcal{Y}_2}\|v\|_{\mathcal{Y}_2}.$$

- $w_3(t,y) = \sigma(D)\int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}(\mathbb{1}_{Q_{10T,x}} uv)\,ds$. They rewrite w_3 as
$$w_3 = \frac{\sigma(D)}{\sqrt{-\Delta}}\int_0^t e^{(t-s)\Delta}\Delta\frac{e^{s\Delta}-\mathrm{Id}}{\sqrt{-\Delta}}(\mathbb{1}_{Q_{10T,x}} uv)\,ds$$
and use the maximal regularity of the heat kernel in L^2L^2 to write
$$\|w_3\|_{L^2L^2} \le C\|\frac{e^{t\Delta}-\mathrm{Id}}{\sqrt{-\Delta}}(\mathbb{1}_{Q_{10T,x}} uv)\|_{L^2L^2} \le C'\|\sqrt{t}\mathbb{1}_{Q_{10T,x}} uv\|_{L^2L^2}.$$
Thus,
$$\|\mathbb{1}_{Q_{T,x}} w_3\|_{L^2L^2} \le \|w_3\|_{L^2L^2} \le C\|\sqrt{t}u\|_\infty \|\mathbb{1}_{Q_{10T,x}} v\|_{L^2L^2} \le CT^{3/4}\|u\|_{\mathcal{Y}_{KT}}\|v\|_{\mathcal{Y}_2}.$$

Thus, the obstruction for the boundedness of B_σ on Y_2 lies in w_3.

To finish the proof, we need to establish the control of B_σ in L^∞ norm. Writing
$$|B_\sigma(u,v)(t,x)| \le C_\sigma \int_0^t \int \frac{1}{(\sqrt{t-s}+|x-y|)^4}|u(s,y)|\,|v(s,y)|\,dy\,ds,$$
we check that
$$\int_0^{t/2}\int \frac{1}{(\sqrt{t-s}+|x-y|)^4}|u(s,y)|\,|v(s,y)|\,dy\,ds \le C\frac{1}{\sqrt{t}}\|u\|_{\mathcal{Y}_2}\|v\|_{\mathcal{Y}_2}$$
and
$$\int_{t/2}^t\int \frac{|u(s,y)|\,|v(s,y)|}{(\sqrt{t-s}+|x-y|)^4}\,dy\,ds \le \frac{C}{\sqrt{t}}\sup_{s>0}\sqrt{s}\|u(s,.)\|_\infty \sup_{s>0}\sqrt{s}\|v(s,.)\|_\infty.\square$$

The proof in Theorem 1 is assumed to satisfy $\mathcal{L}(\mathbb{F}) \in \mathcal{Y}_{KT}$, but one may consider another forcing term; We have the obvious result:

Proposition 3.
Let $\mathcal{Y}$ be an adapted Banach space such that, for every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\mathcal{Y}\times L^{2,\infty}L^\infty$ to $\mathcal{Y}$. Then, there exists a positive constant ϵ_0 such that, if $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathcal{L}(\mathbb{F})\|_{\mathcal{Y}} < \epsilon_0$, then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT} + \mathcal{Y}$.

Let us notice that many adapted spaces studied in the literature satisfy the assumption of Proposition 3. Here are some examples:

a) the Serrin class $\mathcal{Y} = L^p((0,+\infty), L^q(\mathbb{R}^3))$ with $2 < p < +\infty$ and $\frac{2}{p}+\frac{3}{q} = 1$ [this corresponds to the solutions of Fabes, Jones and Rivière [Fab72]);

b) direct generalizations of the Serrin class such that $\mathcal{Y} = L^{p,\rho}((0,+\infty), \mathcal{Y} = L^{q,\sigma}(\mathbb{R}^3))$ or $\mathcal{Y} = L^{p,s}((0,+\infty), \dot{M}^{r,q}(\mathbb{R}^3))$ with $2 < p < +\infty$, $\frac{2}{p} + \frac{3}{q} = 1$, $1 \leq \rho, \sigma \leq +\infty$ and $1 < r \leq q$;

c) the time-weighted Serrin class:

$$\mathcal{Y} = \{u \ / \ t^\alpha u \in L^{p,\rho}((0,+\infty), L^{q,\sigma}(\mathbb{R}^3))\}$$

with $3 < q < +\infty$, $2 < p \leq +\infty$, $0 \leq \alpha$, $1 \leq \rho, \sigma \leq +\infty$ and $2\alpha + \frac{2}{p} + \frac{3}{q} = 1$ [if $p = +\infty$, $L^{p,\rho}$ is to be replaced with L^∞] (this corresponds to the solutions considered by Cannone and Planchon [Can99] or Kozono and Yamazaki [Koz94] and more recently by Farwig, Giga and Shu [Far16] and Kozono and Shimizu [Koz18]);

d) the case $\mathcal{Y} = L^\infty((0,+\infty), L^{3,\infty}(\mathbb{R}^3))$, which is the endpoint of the Serrin class $L^p L^q$ with $p = +\infty$ and corresponds to the solutions of Kozono [Koz96] and Meyer [Mey99];

e) the case of the Lorentz space $L^{5,\rho}_{t,x} = L^{5,\rho}((0,+\infty) \times \mathbb{R}^3)$ with $1 \leq \rho \leq +\infty$ seems to be new but is easy: to check that $L^{5,\rho}_{t,x}$ is an adapted Banach space, just notice that $\frac{1}{(\sqrt{|t|}+|x|)^4} \in L^{5/4,\infty}_{t,x}$ and use convolution inequalities in Lorentz spaces; to check that B_σ is bounded from $L^{5,\rho}_{t,x} \times L^{2,\infty}L^\infty$ to $L^{5,\rho}_{t,x}$, just notice that it is bounded from $L^p_t L^p_x \times L^{2,\infty}L^\infty$ to $L^p_t L^p_x$ for $2 < p < +\infty$ and conclude by interpolation.

All those examples are embedded into larger classes of adapted spaces, namely the parabolic Morrey spaces $\dot{\mathcal{M}}^{p,5}_2$ where $2 < p \leq 5$:

$$\sup_{r>0, t\in\mathbb{R}, x\in\mathbb{R}^3} \frac{1}{r^{\frac{5}{p}-1}} \left(\iint_{(t-r^2,t+r^2)\times B(x,r), s>0} |u(s,y)|^p \, dy \, ds \right)^{1/p} < +\infty.$$

We have $L^p L^q \subset \dot{\mathcal{M}}^{\min(p,q),5}_2$ [case a)], $L^{p,s}((0,+\infty), \dot{M}^{r,q}(\mathbb{R}^3)) \subset \dot{\mathcal{M}}^{\sigma,5}_2$ with $2 < \sigma < \min(p,r)$ [case b)], $t^\alpha u \in L^{p,\rho}((0,+\infty), L^{q,\sigma}(\mathbb{R}^3)) \implies u \in \dot{\mathcal{M}}^{\delta,5}_2$ with $2 < \delta < \min \frac{1}{1+2\alpha p} p, q)$ [case c)], $L^\infty L^{3,\infty} \subset \dot{\mathcal{M}}^{r,5}_2$ for $2 < r < 3$ [case d)], $L^{5,\rho}_{t,x} \subset \dot{\mathcal{M}}^{r,5}_2$ for $2 < r < 5$ [case e)]. We have the easy result on parabolic Morrey spaces:

Proposition 4.
For $2 < p \leq 5$, the parabolic Morrey spaces $\dot{\mathcal{M}}^{p,5}_2$ is an adapted Banach space and, for every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\dot{\mathcal{M}}^{p,5}_2 \times L^{2,\infty}L^\infty$ to $\dot{\mathcal{M}}^{p,5}_2$. Thus, there exists a positive constant ϵ_0 such that, if $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathcal{L}(\mathbb{F})\|_{\dot{\mathcal{M}}^{p,5}_2} < \epsilon_0$, then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT} + \dot{\mathcal{M}}^{p,5}_2$.

Proof. The fact that $\dot{\mathcal{M}}^{p,5}_2$ is an adapted Banach space is proved in [Lem16, Lem18]. One writes that, for $\sigma \in \mathfrak{S}_1$,

$$|B_\sigma(u,v))| \leq C_\sigma \int_0^t \int \frac{1}{(\sqrt{t-s}+|x-y|)^4} |u(s,y)|\,|v(s,y)|\,dy\,ds.$$

Thus, $B_\sigma(u,v)$ is controlled by the parabolic Riesz potential of $|uv|$; as $|uv| \in \dot{\mathcal{M}}^{p/2,5/2}_2$ if u and v belong to $\dot{\mathcal{M}}^{p,5}_2$, we conclude by Hedberg's inequality for Riesz potentals and Morrey spaces that $B_\sigma(u,v)$ is controlled in $\dot{\mathcal{M}}^{p,5}_2$.

Now, let us consider $u \in \dot{\mathcal{M}}^{p,5}_2$ and $v \in L^{2,\infty}L^\infty$. Since $p > 2$, we have $v \in \dot{M}^{\frac{p}{p-1},5}_2$, hence $uv \in \dot{\mathcal{M}}^{1,5/2}_2$. For $r > 0$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$, we want to estimate the L^pL^p norm of $B_\sigma(u,v)$ on $Q_r(t,x) = (t-r^2, t+r^2)\times B(x,r)$. Let $\rho(t-s, x-y) = \sqrt{t-s}+|x-y|$ be the parabolic distance. Let $(s,y) \in Q_r(t,x)$ and (σ,z) be such that $8\ 2^k r \leq \rho(t-\sigma, x-z) \leq 16\ 2^k r$ with $k \in \mathbb{N}$; then

$$\frac{1}{(\sqrt{\sigma-s}+|z-y|)^4} \leq C\frac{1}{(2^k r)^4}$$

and thus

$$\begin{aligned}
\iint_{8\ 2^k r \leq \rho(t-\sigma,x-z) \leq 16\ 2^k r} & \frac{1}{(\sqrt{\sigma-s}+|z-y|)^4} |u(\sigma,z)v(\sigma,z)|\,d\sigma\,dz \\
&\leq C\frac{1}{(2^k r)^4} \iint_{Q_{16\,2^k r}(t,x)} |u(\sigma,z)v(\sigma,z)|\,d\sigma\,dz \\
&\leq C'\frac{1}{2^k r}\|uv\|_{\dot{\mathcal{M}}^{1,5/2}_2}
\end{aligned}$$

so that

$$\begin{aligned}
\iint_{Q_r(t,x)} & \left| \iint_{8r \leq \rho(t-\sigma,x-z))} \frac{1}{(\sqrt{\sigma-s}+|z-y|)^4} |u(\sigma,z)v(\sigma,z)|\,d\sigma\,dz \right|^p ds\,dy \\
&\leq C \iint_{Q_r(t,x)} \left| \sum_{k=0}^{+\infty} \frac{1}{2^k r}\|uv\|_{\dot{\mathcal{M}}^{1,5/2}_2} \right|^p ds\,dy \\
&\leq C' r^{5-p} \|u\|^p_{\dot{\mathcal{M}}^{p,5}_2} \|v\|^p_{L^{2,\infty}L^\infty}.
\end{aligned}$$

On the other hand, we have

$$\mathbb{1}_{Q_{8r}(t,x)} u \in L^pL^p$$

so that

$$w(s,y) = \iint_{\rho(t-\sigma,x-z)<8r} \frac{1}{(\sqrt{\sigma-s}+|z-y|)^4} |u(\sigma,z)v(\sigma,z)|$$

satisfies

$$\|w\|_{L^p(dy)} \leq C \int \frac{1}{\sqrt{|s-\sigma|}} \|\mathbb{1}_{Q_{8r}(t,x)} u(s,z)\|_{L^p(dz)} \|v(s,z)\|_{L^\infty(dz)}\, ds$$

and

$$\|w\|_{L^pL^p} \leq C' \|\mathbb{1}_{Q_{8r}(t,x)} u\|_{L^pL^p} \|v\|_{L^{2,\infty}L^\infty} \leq C'' r^{1-\frac{p}{5}} \|u\|_{\dot{\mathcal{M}}^{p,5}_2} \|v\|_{L^{2,\infty}L^\infty}.$$

Thus, $B_\sigma(u,v)$ belongs to $\dot{\mathcal{M}}^{p,5}_2$. □

However, for some adapted spaces $\mathcal{Y}$, assumptions in Proposition 3 are not satisfied and we cannot use Proposition 4 as they are not included in $\dot{\mathcal{M}}^{p,5}_2$ for any $p > 2$. For instance, let us consider $\mathcal{Y} = L^2((0,+\infty), \mathrm{A}(\mathbb{R}^3))$ where A is the inverse Fourier transform of L^1 (this corresponds to the endpoint $p = 2$ of the Serrin class and to the solutions of Lei and Lin [Lei11] in $L^2\mathrm{A}$). Then, obviously, $L^2\mathrm{A}$ is included in $\dot{\mathcal{M}}^{2,5}_2$ but not in $\dot{\mathcal{M}}^{p,5}_2$ for $p > 2$. Moreover, it is easy to check that, for $\sigma_0(\xi) = |\xi|$, B_{σ_0} is not bounded from $L^2\mathrm{A} \times\ L^{2,\infty}L^\infty$ to $L^2\mathrm{A}$, and even from $L^2\mathrm{A} \times \mathcal{Y}_{KT}$ to $L^2\mathrm{A}$ nor to $\mathcal{Y}_{KT}$ (see counter-examples in Proposition 6 in the Appendix).

Another adapted space for which we don't know whether we may apply Proposition 3 is the space of singular multipliers $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$. This space has been introduced by Lemarié-Rieusset [Lem16, Lem18] and independently by Dao and Nguyen [Dao17]. Notice that, for $2 < p \leq 5$, we have the embeddings

$$\dot{\mathcal{M}}^{p,5}_2 \subset \mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x}) \subset \dot{\mathcal{M}}^{2,5}_2$$

so that $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$ may be viewed as an endpoint of the scale of adapted spaces $\dot{\mathcal{M}}^{p,5}_2$ with $p > 2$. Remark that $L^2\mathrm{A}$ is not included in $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$ (see a counter-example in Proposition 7 in the Appendix).

Thus, we need to find a new adapted space if we want to consider the Cauchy problem with an initial value in BMO^{-1} and a forcing term $\operatorname{div} \mathbb{F}$ leading (in absence of initial value) to a solution in $L^2\mathrm{A}$ or in $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$. This will be done in the next section by modifying the space $\mathcal{Y}_{KT}$ of Koch and Tataru.

2 A variation on the Koch and Tataru theorem

Recall that the Koch and Tataru space $\mathcal{Y}_{KT}$ is defined as

$$\mathcal{Y}_{KT} = \{u \in \mathcal{Y}_2 \ / \ \sup_{t>0} \sqrt{t} \|u(t,.)\|_\infty < +\infty\}.$$

It has been designed to grant that, for every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\mathcal{Y}_{KT} \times \mathcal{Y}_2$ to $\mathcal{Y}_2$. B_σ is also a bounded bilinear operator from $\mathcal{Y}_{KT} \times \mathcal{Y}_{KT}$ to $\mathcal{Y}_{KT}$.

Let us remark that we may easily check that $\mathcal{Y}_{KT} \subset \dot{\mathcal{M}}^{2,5}_2 \subset \mathcal{Y}_2$. We are going to describe new spaces $\mathcal{Y}_{KT,q}$ with $5 < q < +\infty$ so that

$$\mathcal{Y}_{KT} \subset \mathcal{Y}_{KT,q} \subset \dot{\mathcal{M}}^{2,5}_2$$

and, for every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\mathcal{Y}_{KT,q} \times \dot{\mathcal{M}}^{2,5}_2$ to $\mathcal{Y}_{KT,q}$. In particular, $\mathcal{Y}_{KT,q}$ is an adapted space and, for every adapted space $\mathcal{Y}$ such that $\mathcal{Y} \subset \dot{\mathcal{M}}^{2,5}_2$, $\mathcal{Y}_{KT,q} + \mathcal{Y}$ is an adapted space.

Recall that

$$Q_{T,x} = \{(t,y) \ / \ 0 < t < T, |x-y| \leq \sqrt{T}\}$$

and define

$$R_{T,x} = \{(t,y) \ / \ T/2 < t < T, |x-y| \leq \sqrt{T}\}.$$

If $u \in \mathcal{Y}_{KT}$, then $\mathbb{1}_{Q_{T,x}} u \in L^2_{t,x}$ and $\|\mathbb{1}_{Q_{T,x}} u\|_{L^2_{t,x}} \leq \|u\|_{\mathcal{Y}_2} T^{3/4}$. Moreover, $\mathbb{1}_{R_{T,x}} u \in L^\infty_{t,x}$ and $\|\mathbb{1}_{R_{T,x}} u\|_{L^\infty_{t,x}} \leq \|u\|_{\mathcal{Z}_0} \frac{\sqrt{2}}{\sqrt{T}}$. Thus, we have, for $5 \leq q \leq +\infty$,

$$\|\mathbb{1}_{R_{T,x}} u\|_{\dot{\mathcal{M}}^{\frac{2q}{5},q}_2} \leq C \|\mathbb{1}_{R_{T,x}} u\|_{L^q_{t,x}} \leq C' \|u\|_{\mathcal{Y}_{KT}} T^{\frac{5}{2q} - \frac{1}{2}}.$$

Definition 2.
The modified Koch and Tataru space $\mathcal{Y}_{KT,q}$ for $5 < q < +\infty$ is defined as the space of functions u on $(0,+\infty) \times \mathbb{R}^3$ such that

$$\sup_{T>0, x \in \mathbb{R}^3} T^{-3/4} \|\mathbb{1}_{Q_{T,x}} u\|_{L^2_{t,x}} < +\infty$$

and

$$\sup_{T>0, x \in \mathbb{R}^3} T^{-\frac{5}{2q} + \frac{1}{2}} \|\mathbb{1}_{R_{T,x}} u\|_{\dot{\mathcal{M}}^{\frac{2q}{5},q}_2} < +\infty$$

where

$$\|f\|_{\dot{\mathcal{M}}^{\frac{2q}{5},q}_2} = \sup_{r>0, t \in \mathbb{R}, x \in \mathbb{R}^3} r^{-\frac{15}{2q}} \left(\iint_{(t-r^2, t+r^2) \times B(x,r)} |f(s,y)|^{\frac{2q}{5}} \, ds \, dy \right)^{\frac{5}{2q}}.$$

We first remark that $\mathcal{Y}_{KT,q} \subset \dot{\mathcal{M}}^{2,5}_2$: if we want to estimate the $L^2_{t,x}$ norm of u on $(t-r^2, t+r^2) \times B(x,r)$, we may assume that $t \geq r^2$ (otherwise, we control the norm of u on $(t-r^2, t+r^2) \times B(x,r)$ by the norm of u on $(-r^2, r^2) \times B(x,r)$); if $r^2 \leq t \leq 4r^2$, we have $(t-r^2, t+r^2) \times B(x,r) \subset Q_{5r^2,x}$, so we have a control of the $L^2_{t,x}$ norm of u on $(t-r^2, t+r^2) \times B(x,r)$

by $\|u\|_{\mathcal{Y}_2} r^{3/2}$; if $t > 4r^2$, we have $(t-r^2, t+r^2) \times B(x,r) \subset R_{\frac{3}{2}t,x}$, so we have a control of the $L^2_{t,x}$ norm of u on $(t-r^2, t+r^2) \times B(x,r)$ by $r^{5(\frac{1}{2}-\frac{1}{q})} \|\mathbb{1}_{R_{\frac{3}{2}t,x}} u\|_{\dot{\mathcal{M}}_2^{\frac{2q}{5},q}}$, hence in $r^{3/2} \left(\frac{r^2}{t}\right)^{\frac{1}{2}-\frac{5}{2q}} t^{\frac{1}{2}-\frac{5}{2q}} \|\mathbb{1}_{R_{\frac{3}{2}t,x}} u\|_{\dot{\mathcal{M}}_2^{\frac{2q}{5},q}}$.

We may now state our main result:

Theorem 2. *Let $5 < q < +\infty$; then:*
A) For every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\mathcal{Y}_{KT,q} \times \dot{\mathcal{M}}_2^{2,5}$ to $\mathcal{Y}_{KT,q}$.
B) The following assertions are equivalent:

(i) $e^{t\Delta} u_0 \in \mathcal{Y}_2$;

(i) $e^{t\Delta} u_0 \in \mathcal{Y}_{KT,q}$;

(ii) $u_0 \in \mathrm{BMO}^{-1}$.

C) $\mathcal{Y}_{KT,q}$ is an adapted Banach space. Thus, there exists a positive constant ϵ_0 such that, if $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathcal{L}(\mathbb{F})\|_{\mathcal{Y}_{KT,q}} < \epsilon_0$, then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT,q}$.
D) More generally, if $\mathcal{Y}$ is an adapted space such that $\mathcal{Y} \subset \dot{\mathcal{M}}_2^{2,5}$, there exists a positive constant ϵ_1 such that, if $\mathbb{F} = \mathbb{F}_1 + \mathbb{F}_2$ and $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathcal{L}(\mathbb{F}_1)\|_{\mathcal{Y}_{KT,q}} + \|\mathcal{L}(\mathbb{F}_2)\|_{\mathcal{Y}} < \epsilon_1$, then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT,q} + \mathcal{Y}$.

Proof. We only need to prove point A), i.e. to estimate $B_\sigma(u,v)$ in $L^2_{t,x}(Q_{T,x})$ and in $\dot{\mathcal{M}}_2^{\frac{2q}{5},q}(R_{T,x})$ for $u \in \mathcal{Y}_{KT,q}$ and $v \in \dot{\mathcal{M}}_2^{2,5}$.

In order to estimate $B_\sigma(u,v)$ in $L^2_{t,x}(Q_{T,x})$, we follow the proof of Theorem 1 given by Koch and Tataru and we split $w = B_\sigma(u,v)$ in three parts:

- $w_1 = B_\sigma(u, (1 - \mathbb{1}_{Q_{10T,x}})v)$: we saw that
$$\|\mathbb{1}_{Q_{T,x}} w_1\|_{L^2L^2} \le CT^{3/4} \|u\|_{Y_2} \|v\|_{Y_2}.$$
- $w_2(t,y) = \sigma(D)\sqrt{-\Delta} e^{t\Delta} \int_0^t \mathbb{1}_{Q_{10T,x}} uv \, ds$. We saw that
$$\|\mathbb{1}_{Q_{T,x}} w_2\|_{L^2L^2} \le \|w_2\|_{L^2L^2} \le CT^{3/4} \|u\|_{Y_2} \|v\|_{Y_2}.$$
- $w_3(t,y) = \sigma(D) \int_0^t (e^{(t-s)\Delta} - e^{t\Delta}) \sqrt{-\Delta} (\mathbb{1}_{Q_{10T,x}} uv) \, ds$. We are going to prove below (Theorem 3 in next section) that, more generally,
$$\| \int_0^t (e^{(t-s)\Delta} - e^{t\Delta}) \sqrt{-\Delta}(uv) \, ds \|_{L^2L^2} \le C \sup_{T>0, x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2q}} \|\mathbb{1}_{R_{T,x}} u\|_{\dot{\mathcal{M}}_2^{\frac{2q}{5},q}} \|v\|_{L^2L^2}.$$
Thus,
$$\|\mathbb{1}_{Q_{T,x}} w_3\|_{L^2L^2} \le \|w_3\|_{L^2L^2} \le C\|u\|_{Y_{KT,q}} \|\mathbb{1}_{Q_{10T,x}} v\|_{L^2L^2} \le CT^{3/4} \|u\|_{Y_{KT,q}} \|v\|_{Y_2}.$$

Hence, $w \in Y_2$.

In order to estimate $B_\sigma(u,v)$ in $\dot{\mathcal{M}}_2^{\frac{2q}{5},q}$, we write $w = w_4 + w_5$ with $w_4 = B_\sigma((1-\mathbb{1}_{S_{T,x}})u, v)$ and $w_5 = B_\sigma(\mathbb{1}_{S_{T,x}}u, v)$, where

$$S_{T,x} = \{(t,y) \ / \ T/4 < t < T, |x-y| \leq \sqrt{10T}\}.$$

We easily check that $\mathbb{1}_{R_{T,x}}|w_4| \leq C\frac{1}{\sqrt{T}}\|uv\|_{\dot{\mathcal{M}}_2^{1,5/2}}$ (see the proof of Proposition 4) and thus

$$\|\mathbb{1}_{R_{T,x}}w_4\|_{\dot{\mathcal{M}}_2^{\frac{2q}{5},q}} \leq C\|\mathbb{1}_{R_{T,x}}w_4\|_{L^qL^q} \leq C'T^{-\frac{1}{2}+\frac{5}{2q}}\|u\|_{\dot{\mathcal{M}}_2^{2,5}}\|v\|_{\dot{\mathcal{M}}_2^{2,5}}.$$

On the other hand, we have

$$|w_5(t,z)| \leq C_\sigma \int_0^t \int \frac{1}{(\sqrt{t-s}+|z-y|)^4}\mathbb{1}_{S_{T,x}}(s,y)|u(s,y)|\,|v(s,y)|\,dy\,ds.$$

Thus, w_5 is controlled by the parabolic Riesz potential of $|\mathbb{1}_{S_{T,x}}uv|$; as $|\mathbb{1}_{S_{T,x}}uv| \in \dot{\mathcal{M}}_2^{\frac{2}{5}\frac{5q}{5+q},\frac{5q}{5+q}}$ (since $\mathbb{1}_{S_{T,x}}u \in \dot{\mathcal{M}}_2^{\frac{2q}{5},q}$ and $v \in \dot{\mathcal{M}}_2^{2,5}$), we conclude by Hedberg's inequality for Riesz potentals and Morrey spaces that w_5 is controlled in $\dot{\mathcal{M}}_2^{\frac{2q}{5},q}$:

$$\|w_5\|_{\dot{\mathcal{M}}_2^{\frac{2q}{5},q}} \leq C\|\mathbb{1}_{S_{T,x}}u\|_{\dot{\mathcal{M}}_2^{\frac{2q}{5},q}}\|v\|_{\dot{\mathcal{M}}_2^{2,5}} \leq C'T^{\frac{5}{2q}-\frac{1}{2}}\|u\|_{\mathcal{Y}_{KT,q}}\|v\|_{\dot{\mathcal{M}}_2^{2,5}}. \quad \square$$

Corollary 1.
a) There exists a positive constant ϵ_0 such that, if $\mathbb{F} = \mathbb{F}_1+\mathbb{F}_2$ and $\|\vec{u}_0\|_{\mathrm{BMO}^{-1}}+\|\mathcal{L}(\mathbb{F}_1)\|_{\mathcal{Y}_{KT,q}} + \|\mathcal{L}(\mathbb{F}_2)\|_{\mathcal{M}(\dot{H}^{1/2,1}_{t,x}\mapsto L^2_{t,x})} < \epsilon_0$, then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT,q} + \mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$.
b) For $5 < p < +\infty$, there exists a positive constant ϵ_p such that, if $\mathbb{F} = \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3$ and

$$\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathcal{L}(\mathbb{F}_1)\|_{\mathcal{Y}_{KT,q}} + \|\mathcal{L}(\mathbb{F}_2)\|_{\dot{\mathcal{M}}_2^{p,5}} + \|\mathcal{L}(\mathbb{F}_3)\|_{L^2\mathrm{A}} < \epsilon_p,$$

then the Navier–Stokes problem (4) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT,q} + \dot{\mathcal{M}}_2^{p,5} + L^2\mathrm{A}$.

Proof. a) is direct consequence of Theorem 2, as $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$ is an adapted Banach space contained in $\dot{\mathcal{M}}_2^{2,5}$ [Lem16, Lem18, Dao17]. Similarly, $\dot{\mathcal{M}}_2^{p,5}$ is an adapted Banach space contained in $\dot{\mathcal{M}}_2^{2,5}$ [Lem16], $L^2\mathrm{A}$ is an adapted Banach space contained in $\dot{\mathcal{M}}_2^{2,5}$ [Lei11], and, and, for every $\sigma \in \mathfrak{S}_1$, the bilinear operator B_σ is a bounded bilinear operator from $\dot{\mathcal{M}}_2^{p,5} \times L^2\mathrm{A}$ to $\dot{\mathcal{M}}_2^{p,5}$, by Proposition 4 since $L^2\mathrm{A} \subset L^{2,\infty}L^\infty$. $\square$

3 Parabolic dyadic decomposition of the time-space domain

We decompose $(0,+\infty)\times\mathbb{R}^3$ as

$$(0,+\infty)\times\mathbb{R}^3=\bigcup_{j\in\mathbb{Z},k\in\mathbb{Z}^3}\{(t,x)\ /\ 1\le 4^jt<4, 2^jx-k\in[0,1)^3\}=\bigcup_{j\in\mathbb{Z},k\in\mathbb{Z}^3}R_{j,k}$$

and $(0,16\,4^{-j})\times\mathbb{R}^3$ as

$$(0,16\,4^{-j})\times\mathbb{R}^3=\bigcup_{k\in\mathbb{Z}^3}\{(t,x)\ /0<4^jt<16, 2^jx-k\in[0,1)^3\}=\bigcup_{k\in\mathbb{Z}^3}Q_{j,k}.$$

If $v\in L^2L^2$, then v can be decomposed in an orthogonal series

$$v=\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3}\mathbb{1}_{R_{j,k}}v=\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3}v_{j,k}$$

with

$$\|v\|^2_{L^2L^2}=\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3}\|v_{j,k}\|^2_{L^2L^2}.$$

Similarly, if $u\in Y_{KT,q}$, then

$$u=\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3}\mathbb{1}_{R_{j,k}}u=\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3}u_{j,k}$$

with, for every $5/2\le\rho\le q$,

$$\sup_{j\in\mathbb{Z},k\in\mathbb{Z}^3}2^{j(1-\frac{5}{\rho})}\|u_{j,k}\|_{\dot{\mathcal{M}}^{\frac{2}{5}\rho,\rho}_2}<+\infty.$$

Theorem 3.
Let $v\in L^2L^2$ and $u\in Y_{KT,q}$ with $5<q<+\infty$. Write $v_{j,k}=\mathbb{1}_{R_{j,k}}v$, $v_j=\sum_{k\in\mathbb{Z}^3}v_{j,k}$, and $u_{j,k}=\mathbb{1}_{R_{j,k}}u$. Then
A) For $5/2<\rho\le q$ and $\alpha=1-\frac{5}{\rho}$,

$$\|\int_0^t(e^{(t-s)\Delta}-e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds\|_{L^2L^2}\le C\|v_j\|_2\sup_{k\in\mathbb{Z}^3}\|u_{j,k}\|_{\dot{\mathcal{M}}^{\frac{2}{5}\rho,\rho}_2}.$$

B) We have

$$\|\int_0^t(e^{(t-s)\Delta}-e^{t\Delta})\sqrt{-\Delta}(uv)\,ds\|_{L^2L^2}\le C\|v\|_{L^2L^2}\sup_{T>0,x\in\mathbb{R}^3}T^{\frac{1}{2}-\frac{5}{2q}}\|\mathbb{1}_{R_{T,x}}u\|_{\dot{\mathcal{M}}^{\frac{2}{5}q,q}_2}.$$

Proof.

Proof of A).
We first consider $4^j t \leq 16$ and estimate $W = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds$ in $L^2((0, 16\,4^{-j}), L^2)$, then estimate $W^* = \int_0^{16\,4^{-j}} (e^{(16\,4^{-j}-s)\Delta} - e^{16\,4^{-j}\Delta})\sqrt{-\Delta}^{\alpha}(uv_j)\,ds$ in $L^2(\mathbb{R}^3)$ and finally we estimate $W = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds$ in $L^2((16\,4^{-j}, +\infty), L^2)$.

When $t < 16\,4^{-j}$, we write

$$W = \sum_{l\in\mathbb{Z}^3} \mathbb{1}_{Q_{j,l}} \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(\sum_{k\in\mathbb{Z}^3} u_{j,k}v_{,k}j)\,ds$$

which we reorganize as

$$W = \sum_{m\in\mathbb{Z}^3}\sum_{k\in\mathbb{Z}^3} \mathbb{1}_{Q_{j,k+m}} \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(u_{j,k}v_{,k}j)\,ds = \sum_{m\in\mathbb{Z}^3} W_m.$$

We have

$$\|W\|_{L^2((0,16\,4^{-j}),L^2} \leq \sum_{m\in\mathbb{Z}^3} \|W_m\|_{L^2((0,16\,4^{-j}),L^2}$$
$$= \sum_{m\in\mathbb{Z}^3} \left(\sum_{k\in\mathbb{Z}^3} \|\mathbb{1}_{Q_{j,k+m}} \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(u_{j,k}v_{,k}j)\,ds\|^2_{L^2((0,16\,4^{-j}),L^2} \right)^{1/2}$$

We have

$$|\int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(u_{j,k}v_{j,k})\,ds| \leq C Z_\alpha(u_{j,k}v_{j,k}).$$

where

$$Z_\alpha(w) = \int_0^t \int \frac{1}{(\sqrt{t-s} + |x-y|)^{4+\alpha}} |w(s,y)|\,dy\,ds.$$

Z_α is a parabolic Riesz potential and we have the following equivalent of the Fefferman-Phong inequality [Fef83] for the parabolic Riesz potentials and the parabolic Morrey spaces [Lem16]: if $0 < \beta < \frac{5}{2}$ and $2 < p < \frac{5}{\beta}$, then

$$\|\int_0^t \int \frac{1}{(\sqrt{t-s} + |x-y|)^{5-\beta}} |f(s,y)g(s,y)|\,dy\,ds\|_{L^2L^2} \leq C_{p,\beta}\|f\|_{L^2L^2}\|g\|_{\dot{\mathcal{M}}_2^{p,\frac{5}{\beta}}}.$$

As $4 + \alpha = 5 - \frac{5}{\rho}$, we get

$$\|Z_\alpha(u_{j,k}v_{j,k})\|_{L^2_{t,x}} \leq C\|u_{j,k}\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}\rho,\rho}}\|v_{j,k}\|_{L^2L^2}.$$

Thus,

$$\|W_m\|_{L^2L^2} \le C \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{\dot{\mathcal{M}}^{\frac{2}{5}\rho,\rho}_2} \|v_j\|_{L^2L^2}.$$

Moreover, if $|m| \ge 20$ and $|m_0| = 10$, we have, for $0 \le s \le t \le 16\,4^{-j}$, $y \in Q_{j,k}$, $x \in Q_{j,k+m}$ and $z \in Q_{j,k+m_0}$,

$$\frac{1}{(\sqrt{t-s}+|x-y|)^{4+\alpha}} \le \frac{1}{|x-y|^{4+\alpha}} \le C\frac{2^{(4+\alpha)j}}{m^{4+\alpha}} \le C'\frac{1}{m^{4+\alpha}(\sqrt{t-s}+|z-y|)^{4+\alpha}}$$

so that, for $0 < t \le 16\,4^{-j}$,

$$\begin{aligned}&\mathbb{1}_{Q_{j,k+m}}(x) Z_\alpha(u_{j,k}v_{j,k})(t,x)\\ &\qquad \le \frac{C}{m^{4+\alpha}} \mathbb{1}_{Q_{j,k+m_0}}(x-(m-m_0)2^{-j}) Z_\alpha(u_{j,k}v_{j,k})(t, x-(m-m_0)2^{-j}))\end{aligned}$$

and

$$\|W_m\|_{L^2((0,16\,4^{-j}),L^2)} \le C\frac{1}{m^{4+\alpha}} \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{\dot{\mathcal{M}}^{\frac{2}{5}\rho,\rho}_2} \|v_j\|_{L^2L^2}.$$

Thus, we have proved that

$$\|W\|_{L^2((0,16\,4^{-j}),L^2)} \le C \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{\dot{\mathcal{M}}^{\frac{2}{5}\rho,\rho}_2} \|v_j\|_{L^2L^2}.$$

We now estimate $W^* = \int_0^{16\,4^{-j}} (e^{(16\,4^{-j}-s)\Delta} - e^{16\,4^{-j}\Delta})\sqrt{-\Delta}^{\alpha}(uv_j)\,ds$. First, as v_j is supported in $4^{-j} < t < 4\ 4^{-j}$, we write

$$\begin{aligned}W^* &= \int_{4^{-j}}^{4\,4^{-j}} (e^{(16\,4^{-j}-s)\Delta} - e^{16\,4^{-j}\Delta})\sqrt{-\Delta}^{\alpha}(uv_j)\,ds\\ &= \int_{4^{-j}}^{4\,4^{-j}} \int_0^s (e^{(16\,4^{-j}-s+\theta)\Delta})\sqrt{-\Delta}^{2+\alpha}(uv_j)\,d\theta\,ds\end{aligned}$$

so that

$$\begin{aligned}|W^*(x)| &\le C\int_{4^{-j}}^{4\,4^{-j}} \int_0^s \int \frac{1}{(\sqrt{16\,4^{-j}-s+\theta}+|x-y|)^{5+\alpha}} |u(s,y)v_j(s,y)|\,dy\,d\theta\,ds\\ &\le C'\int_{4^{-j}}^{4\,4^{-j}} \int \frac{4^{-j}}{(2^{-j}+|x-y|)^{5+\alpha}} |u(s,y)v_j(s,y)|\,dy\,ds.\end{aligned}$$

If $12\,4^{-j} \le \tau \le 16\,4^{-j}$, we have

$$\begin{aligned}&\int_{4^{-j}}^{4\,4^{-j}} \int \frac{4^{-j}}{(2^{-j}+|x-y|)^{5+\alpha}} |u(s,y)v_j(s,y)|\,dy\,ds\\ &\qquad \le C\int_0^\tau \int \frac{2^{-j}}{(\sqrt{\tau-s}+|x-y|)^{4+\alpha}} |u(s,y)v_j(s,y)|\,dy\,ds = C2^{-j}Z_\alpha(\tau,x).\end{aligned}$$

Thus,

$$\|W^*(x)\|_2 \leq C2^{-j}\frac{1}{4\,4^{-j}}\int_{12\,4^{-j}}^{16\,4^{-j}} \|Z_\alpha(\tau,.)\|_2\,d\tau \leq \frac{C}{2}\|Z_\alpha\|_{L^2((12\,4^{-j},16\,4^{-j}),L^2)}.$$

This gives

$$\|W^*\|_2 \leq C\sup_{k\in\mathbb{Z}^3}\|u_{j,k}\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}\rho,\rho}}\|v_j\|_{L^2L^2}.$$

Finally, we estimate $W=\int_0^t(e^{(t-s)\Delta}-e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds$ in $L^2((16\,4^{-j},+\infty),L^2)$. For $t>16\,4^{-j}$, we have

$$W=\int_0^{16\,4^{-j}}(e^{(t-s)\Delta}-e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds=\sqrt{-\Delta}e^{(t-16\,4^{-j})\Delta}W^*$$

and thus

$$\|W\|_{L^2((16\,4^{-j},+\infty),L^2)}\leq\frac{1}{\sqrt{2}}\|W^*\|_2\leq C\sup_{k\in\mathbb{Z}^3}\|u_{j,k}\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}\rho,\rho}}\|v_j\|_{L^2L^2}.$$

Proof of B).
Let $U=\int_0^t(e^{(t-s)\Delta}-e^{t\Delta})\sqrt{-\Delta}(uv)\,ds$, $U=\sum_{j\in\mathbb{Z}}U_j$ with

$$U_j=\int_0^t(e^{(t-s)\Delta}-e^{t\Delta})\sqrt{-\Delta}(uv_j)\,ds.$$

Let $\gamma=1-\frac{5}{q}$ and $\frac{1}{\rho}=\frac{2}{5}-\frac{1}{q}$. Then $1-\frac{5}{\rho}=-\gamma$. From point A), we know that

$$\|U_j\|_{L^2\dot{H}^\gamma}\leq C\|v_j\|_2\sup_{k\in\mathbb{Z}^3}\|u_{j,k}\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}q,q}}\leq C'2^{j\gamma}\|v_j\|_2\sup_{T>0,x\in\mathbb{R}^3}T^{\frac{1}{2}-\frac{5}{2q}}\|\mathbb{1}_{R_{T,x}}u\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}q,q}}.$$

and

$$\|U_j\|_{L^2\dot{H}^{-\gamma}}\leq C\|v_j\|_2\sup_{k\in\mathbb{Z}^3}\|u_{j,k}\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}\rho,\rho}}\leq C'2^{-j\gamma}\|v_j\|_2\sup_{T>0,x\in\mathbb{R}^3}T^{\frac{1}{2}-\frac{5}{2q}}\|\mathbb{1}_{R_{T,x}}u\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}q,q}}.$$

We then have

$$\begin{aligned}\int_0^{+\infty}\int|U(t,x)|^2\,dx\,dt=&\sum_{j\in\mathbb{Z}}\int_0^{+\infty}\int|U_j(t,x)|^2\,dx\,dt\\&+2\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z},k<j}\int_0^{+\infty}\langle(-\Delta)^{-\gamma}U_j(t,.)|(-\Delta)^\gamma U_k(t,.)\rangle\,dt\\ \leq C(\sup_{T>0,x\in\mathbb{R}^3}T^{\frac{1}{2}-\frac{5}{2q}}\|\mathbb{1}_{R_{T,x}}u\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}q,q}})^2(&\sum_{j\in\mathbb{Z}}\|v_j\|_{L^2L^2}^2+2\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z},k<j}2^{-\gamma(j-k)}\|v_j\|_{L^2L^2}\|v_k\|_{L^2L^2})\\ \leq&C'(\sup_{T>0,x\in\mathbb{R}^3}T^{\frac{1}{2}-\frac{5}{2q}}\|\mathbb{1}_{R_{T,x}}u\|_{\dot{\mathcal{M}}_2^{\frac{2}{5}q,q}})^2\|v\|_{L^2L^2}^2.\end{aligned}$$

The theorem is proved. □

Appendix: counter-examples.

Proposition 5.
Let $\sigma_0(\xi) = |\xi|$. Then $\sigma_0 \in \mathfrak{S}_1$ and B_{σ_0} is not bounded on $\mathcal{Y}_2$. It is not bounded as well from $\mathcal{M}^{2,5}_2 \times \mathcal{M}^{2,5}_2$ to $\mathcal{Y}_2$.

Proof. Due to the invariance of the norms of $\mathcal{Y}_2$ a:nd $\mathcal{M}^{2,5}_2$ through translations and dilations, the operator B_{σ_0} would be bounded from $\mathcal{M}^{2,5}_2 \times \mathcal{M}^{2,5}_2$ to $\mathcal{Y}_2$. if and only if there would exist a constant C_0 such that, for every $u, v \in \mathcal{M}^{2,5}_2$,

$$\int_0^1 \int_{[-1,1]^3} |B_{\sigma_0}(u,v)|^2 \, dt \, dx \leq C_0 \|u\|^2_{\mathcal{M}^{2,5}_2} \|v\|^2_{\mathcal{M}^{2,5}_2}.$$

We then take $u_n(t,x) = v_n(t,x) = \psi_n(x_1,x_2)$ with $\psi_n \in L^2(\mathbb{R}^2)$. We have, for $T > 0$,, $t_0 > 0$ and $x_0 \in \mathbb{R}^3$,

$$\int_{t_0}^{t_0+T} \int_{B(x_0,\sqrt{T})} |u_n(t,x)|^2 \, dt \, dx \leq 2\|\psi_n\|_2^2 T^{3/2}.$$

Thus, $u_n \in \mathcal{M}^{2,5}_2$.

Moreover, for $w_n(x) = u_n(t,x)^2$ and $\phi_n = \psi_n^2$, we have the Fourier transforms

$$\mathcal{F}(e^{(t-s)\Delta}\sigma_0(D)w_n) = e^{-(t-s)|\xi|^2}|\xi|(\hat{\phi_n}(\xi_1,\xi_2) \otimes \delta(\xi_3))$$

hence

$$\mathcal{F}(e^{(t-s)\Delta}\sigma_0(D)w_n) = \left(e^{-(t-s)|(\xi_1,\xi_2)|^2}|(\xi_1,\xi_2)|\hat{\phi}_n(\xi_1,\xi_2)\right) \otimes \delta(\xi_3).$$

Hence, writing Δ_2 for the laplacian operator on $\mathbb{R}^2$, we get

$$B_{\sigma_0}(u_n,u_n) = \int_0^t e^{(t-s)\Delta_2}\sqrt{-\Delta_2}\phi_n \, ds = (\mathrm{Id} - e^{t\Delta_2})\frac{1}{\sqrt{-\Delta_2}}\phi_n.$$

From $\frac{1}{|(y_1,y_2)|} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, we find

$$|e^{t\Delta_2}\frac{1}{\sqrt{-\Delta_2}}\phi_n| \leq C\|\phi_n\|_1(1+\frac{1}{t})$$

and thus

$$\int_{1/2}^1 \int_{[-1,1]^3} |e^{t\Delta_2}\frac{1}{\sqrt{-\Delta_2}}\phi_n|^2 \, dt \, dx \leq 36C^2\|\phi_n\|_1^2.$$

In particular,

$$\int_{1/2}^{1}\int_{[-1,1]^3} |\frac{1}{\sqrt{-\Delta_2}}\phi_n|^2\,dt\,dx \le 8(8C_0+36C^2)\|\phi_n\|_1^2.$$

Thus, if ϕ_n is an approximation of the Dirac mass with $\|\phi_n\|_1 = 1$, we find that the function $\frac{1}{\sqrt{-\Delta_2}}\phi_n$ is bounded in $L^2((-1,1)^2)$; but it converges in $\mathcal{D}'((-1,1)^2)$ to $\frac{\sqrt{2\pi}}{|y|}$ which is not square-integrable on $(-1,1)^2$. Thus, B_{σ_0} is not bounded from $\mathcal{M}^{2,5}_2 \times \mathcal{M}^{2,5}_2$ to $\mathcal{Y}_2$. □

Proposition 6.
Let $\sigma_0(\xi) = |\xi|$. Then $\sigma_0 \in \mathfrak{S}_1$ and B_{σ_0} is not bounded from $\mathcal{Y}_{KT} \times L^2\mathrm{A}$ to $L^2\mathrm{A}$ nor to $\mathcal{Y}_{KT}$.

Proof. Let $u(t,x) = \mathbb{1}_{(0,2)}(t)\frac{x_1}{|x_1|}$ and $v(t,x) = \mathbb{1}_{(0,2)}(t)\phi(x_1)\psi(x_2,x_3)$ where the Fourier transforms of ϕ and ψ are integrable (over $\mathbb{R}$ and over $\mathbb{R}^2$ respectively) and the support of the Fourier transform of ψ is contained in the corona $1 < \xi_2^2+\xi_3^2 < 4$. Then $u \in \mathcal{Y}_{KT}$ and $v \in L^2\mathrm{A}$.

For $t \in (0,2)$, we have

$$\mathcal{F}(B_{\sigma_0}(u,v)) = 2\pi\int_0^t e^{-(t-s)|\xi|^2}|\xi|H(\hat\phi)(\xi_1)\hat\psi(\xi_2,\xi_3)\,ds = 2\pi\frac{1-e^{-t|\xi|^2}}{|\xi|}H(\hat\phi)(\xi_1)\hat\psi(\xi_2,\xi_3)$$

where H is the Hilbert transform. For $1 < t < 2$, we have

$$\int |\mathcal{F}(B_{\sigma_0}(u,v))\,d\xi| \ge \pi(1-e^{-1})\int_{-1}^{1}|H(\hat\phi)(\xi_1)|\,d\xi_1 \iint \frac{|\hat\psi(\xi_2,\xi_3)|}{|(\xi_2,\xi_3)|}\,d\xi_2\,d\xi_3$$

and thus

$$\int_{-1}^{1}|H(\hat\phi)(\xi_1)|\,d\xi_1 \le C\|B_{\sigma_0}(u,v)\|_{L^2\mathrm{A}}$$

whereas

$$\|u\|_{\mathcal{Y}_{KT}}\|v\|_{L^2\mathrm{A}} \le C\|\hat\phi\|_1.$$

Assuming that B_{σ_0} is bounded from $\mathcal{Y}_{KT}\times L^2\mathrm{A}$ to $L^2\mathrm{A}$, we obtain that, for $\phi \in L^1(\mathbb{R})$,

$$\int_{-1}^{1}|H(\hat\phi)(\xi_1)|\,d\xi_1 \le C\|\hat\phi\|_1.$$

Applying this to $\phi_R = \phi(\frac{x}{R})$, we have

$$\int_{-R}^{R}|H(\hat\phi)(\xi_1)|\,d\xi_1 = \int_{-1}^{1}|H(\widehat{\phi_R})(\xi_1)|\,d\xi_1 \le C\|\widehat{\phi_R}\|_1 = C\|\hat\phi\|_1.$$

Letting R go to $+\infty$, we find that the Hilbert transform is bounded on L^1, which is false. Thus, B_{σ_0} cannot be bounded from $\mathcal{Y}_{KT} \times L^2\mathrm{A}$ to $L^2\mathrm{A}$.

Now, consider a function ϕ whose Fourier transform $\hat{\phi} \in \mathcal{D}(\mathbb{R}^3)$ is non-negative, supported in $B(0,1)$ and with $\|\hat{\phi}\|_1 = 1$. Similarly, consider a function ψ whose Fourier transform $\hat{\phi} \in \mathcal{D}(\mathbb{R}^3)$ is non-negative, supported in $B(0,4) \setminus B(0,2)$ and with $\|\hat{\psi}\|_1 = 1$. Writing, for a function f on $\mathbb{R}^3$ and for $\epsilon > 0$, $f_\epsilon(x) = f(\frac{x}{\epsilon})$, we define

$$u(t,x) = \mathbb{1}_{(0,1)}(t)\phi_{\sqrt{1-t}}(x) \text{ and } v(t,x) = \mathbb{1}_{(0,1)}(t)\frac{1}{\sqrt{1-t}|\ln(2-2t)|^{3/4}}\psi_{\sqrt{1-t}}(x).$$

Let us assume that B_{σ_0} is bounded from $\mathcal{Y}_{KT} \times L^2\mathrm{A}$ to $\mathcal{Y}_{KT}$. As $u \in \mathcal{Y}_{KT}$ and $v \in L^2\mathrm{A}$, we should have $\sup_{t>0} \sqrt{t}\|B_{\sigma_0}(u,v)(t,.)\|_\infty < +\infty$, where the supremum bound holds for every $t > 0$ by weak-* continuity of $t > 0 \mapsto B_{\sigma_0}(u,v)(t,.) \in L^\infty$. In particular, $B_{\sigma_0}(u,v)(1,.) \in L^\infty$. The Fourier transform $B_{\sigma_0}(u,v)(1,.)$ is non-negative, as

$$\mathcal{F}(B_{\sigma_0}(u,v)(1,.)) = \int_0^1 e^{-(1-s)|\xi|^2}|\xi|\frac{1}{\sqrt{1-s}|\ln(2-2s)|^{3/4}}\mathcal{F}(\phi_{\sqrt{1-s}}\psi_{\sqrt{1-s}})(\xi)\,ds,$$

and thus

$$\|B_{\sigma_0}(u,v)(1,.)\|_\infty = \frac{1}{(2\pi)^3}\int \mathcal{F}(B_{\sigma_0}(u,v)(1,.))(\xi)\,d\xi.$$

The support of $\mathcal{F}(\phi_{\sqrt{1-s}}\psi_{\sqrt{1-s}})(\xi)$ is contained in $\{\xi \ / \ \frac{1}{\sqrt{1-s}} \leq |\xi| \leq \frac{5}{\sqrt{1-s}}\}$ and thus, for a constant $\gamma > 0$,

$$\int \mathcal{F}(B_{\sigma_0}(u,v)(1,.))(\xi)\,d\xi \geq \gamma \int_0^1 \int \frac{1}{(t-s)|\ln(2-2s)|^{3/4}}\mathcal{F}(\phi_{\sqrt{1-s}}\psi_{\sqrt{1-s}})(\xi)\,ds\,d\xi$$

and thus

$$\int \mathcal{F}(B_{\sigma_0}(u,v)(1,.))(\xi)\,d\xi \geq \gamma\|\hat{\phi}\|_1\|\hat{\psi}\|_1 \int_0^1 \frac{1}{(t-s)|\ln(2-2s)|^{3/4}}\,ds = +\infty.$$

Thus, B_{σ_0} cannot be bounded from $\mathcal{Y}_{KT} \times L^2\mathrm{A}$ to $\mathcal{Y}_{KT}$. □

Proposition 7.
The space $L^2\mathrm{A}$ is not included in $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$.

Proof. If $L^2\mathrm{A} \subset \mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$, then the embedding would be continuous (by Baire's theorem, since convergence of a sequence in $L^2\mathrm{A}$ implies convergence almost everywhere of a subsequence and since $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$

has the Fatou property that an almost everywhere converging sequence of bounded functions in $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$ remains in $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$ with the same bound).

Since the operator T defined by

$$T(u,v)(t,x) = \int_0^t \int \frac{1}{(\sqrt{t-s}+|x-y|)^4} u(s,y)v(s,y)\,ds\,dy$$

is bounded on $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x})$ and since $\mathcal{M}(\dot{H}^{1/2,1}_{t,x} \mapsto L^2_{t,x}) \subset \dot{\mathcal{M}}^{2,5}_2$, we would have the inequality

$$\int_0^1 \int_{B(0,1)} |T(u,u)(t,x)|\,dt\,dx \leq C\|u\|^2_{L^2\mathrm{A}}.$$

Let $u_R(t,x) = \omega(t)e^{-\frac{|x|^2}{R}}$. By monotonous convergence, we have pointwise convergence of $T(u_R,u_R)(t,x)$ to

$$\int_0^t \int \frac{1}{(\sqrt{t-s}+|x-y|)^4} \omega^2(s),ds\,dy = C\int_0^t \frac{1}{\sqrt{t-s}}\omega^2(s)\,ds.$$

Thus, we would have

$$\int_0^1 \left(\int_0^t \frac{1}{\sqrt{t-s}}\omega^2(s)\,ds\right)^2 dt \leq C\|\omega\|_2^4.$$

For $\omega_\epsilon(t) = \frac{1}{\sqrt{\epsilon}}\theta(\frac{x}{\epsilon})$ with $\theta \in \mathcal{D}$ and $\|\theta\|_2 = 1$, we have pointwise convergence (when $\epsilon \to 0$) of $\int_0^t \frac{1}{\sqrt{t-s}}\omega_\epsilon^2(s)\,ds$ to $\frac{1}{\sqrt{t}}$; Fatou's lemma would give $\int_0^1 \frac{dt}{t} \leq C$, which is false. □

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