Forces for the Navier-Stokes equations and the Koch and Tataru theorem

Pierre Gilles Lemarié-Rieusset

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Abstract

We consider the Cauchy problem for the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^3$, with initial value $\vec{u}_0 \in \mathrm{BMO}^{-1}$ (as in Koch and Tataru's theorem) and with force $\vec{f} = \operatorname{div} \mathbb{F}$ with $\mathbb{F} \in L^1\mathcal{F}^{-1}L^1$ (as in Lei and Lin's theorem). If $\vec{u}_0$ and $\mathbb{F}$ are small enough, we show the existence of a global mild solution.

Keywords : Navier–Stokes equations, critical spaces, mild solutions.

AMS classification : 35K55, 35Q30, 76D05.

Introduction

In this paper, we consider global mild solutions of the Cauchy problem for the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^3$. When looking for assumptions that respect the symmetries of the Navier–Stokes equations (with respect to spatial translation or to dilations), one is lead to consider the initial data to be in BMO^{-1} (this is the famous Koch and Tataru theorem [Koc01]) but there is no natural choice for the forcing term. We are going to consider forces that are known to lead to global mild solutions (if they are small enough) in the absence of initial value, but the interaction between those forces and an initial value in BMO^{-1} or between forces in different functional spaces has not been discussed in the literature.

Forces that we shall consider will be written in divergence form ($\vec{f} = \operatorname{div} \mathbb{F}$) where the tensor $\mathbb{F}$ will be a sum of tensors $\mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3 + \mathbb{F}_4$ where:

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- $\mathbb{F}_1$ belongs to a Serrin class $L^{p/2,\infty}((0,+\infty),L^{q/2,\infty}(\mathbb{R}^3))$ with $2<p<+\infty$ and $\frac{2}{p}+\frac{3}{q}=1$ [the case of the Serrin class $L^{p/2}((0,+\infty),L^{q/2}(\mathbb{R}^3))$ corresponds to the solutions of Fabes, Jones and Rivière [Fab72] in L^pL^q]

- $\mathbb{F}_2$ belongs to $L^\infty((0,+\infty),L^{3/2,\infty}(\mathbb{R}^3))$ (this corresponds to the endpoint $p=+\infty$ of the Serrin class and to the solutions of Kozono [Koz96] and Meyer [Mey99] in $L^\infty L^{3,\infty}$)

- $\mathbb{F}_3$ belongs to $L^1((0,+\infty),A(\mathbb{R}^3))$ where A is the inverse Fourier transform of L^1 (this corresponds to the endpoint $p=2$ of the Serrin class and to the solutions of Lei and Lin [Lei11] in L^2A)

- $\mathbb{F}_4$ belongs to a variant of the Koch and Tataru space. The Koch and Tataru space $\mathcal{Z}_{KT}$ is defined by

$$\|\mathbb{F}\|_{\mathcal{Z}_1}=\sup_{t>0,x_0\in\mathbb{R}^3}t^{-\frac{3}{2}}\|\mathbb{F}\|_{L^1((0,t)\times B(x_0,\sqrt{t}))}<+\infty$$

and

$$\sup_{t>0}t\|\mathbb{F}(t,.)\|_\infty<+\infty.$$

We shall define, for $2\leq r\leq+\infty$, the space $\mathcal{Z}_{KT,r}$ by

$$\|\mathbb{F}\|_{\mathcal{Z}_1}=\sup_{t>0,x_0\in\mathbb{R}^3}t^{-\frac{3}{2}}\|\mathbb{F}\|_{L^1((0,t)\times B(x_0,\sqrt{t}))}<+\infty$$

and

$$\sup_{t>0,x_0\in\mathbb{R}^3}t^{1-\frac{5}{r}}\|\mathbb{F}\|_{L^{r/2}((t/2,t)\times B(x_0,\sqrt{t}))}<+\infty.$$

Notice that $\mathcal{Z}_{KT}=\mathcal{Z}_{KT,\infty}\subset\mathcal{Z}_{KT,r}$. We shall assume that $\mathbb{F}_4$ belongs to $\mathcal{Z}_{KT,r}$ for some r such that $5<r<+\infty$.

This allows us to consider many forces, due to the following remarks:

- Assume that $\mathbb{F}$ belongs to a time-weighted Serrin class:

$$t^{2\alpha}\mathbb{F}\in L^{p/2,\rho}((0,+\infty),L^{q/2,\sigma}(\mathbb{R}^3))$$

with $3<q<+\infty$, $2<p\leq+\infty$, $0\leq\alpha$, $1\leq\rho,\sigma\leq+\infty$ and $2\alpha+\frac{2}{p}+\frac{3}{q}=1$ [if $p=+\infty$, $L^{p/2,\rho}$ is to be replaced with L^∞] (this corresponds to the solutions considered by Cannone and Planchon [Can99] or Kozono and Yamazaki [Koz94] and more recently by Farwig, Giga and Shu [Far16] and Kozono and Shimizu [Koz18]). Then $\mathbb{F}$ is of the form $\mathbb{F}=\mathbb{F}_1\in L^{p_0/2,\infty}L^{q/2,\infty}$ with $\frac{1}{p_0}=\alpha+\frac{1}{p}$ and $\frac{2}{p_0}+\frac{3}{q}=1$.

- Assume that $\mathbb{F}$ is a sum $\mathbb{F} = \mathbb{G}_1 + \mathbb{G}_2$ where $\mathbb{G}_i$ belongs to a Serrin class $L^{p_i/2,\infty}((0,+\infty), L^{q_i/2,\infty}(\mathbb{R}^3))$ with $2 < p_i < +\infty$ and $\frac{2}{p_i} + \frac{3}{q_i} = 1$. If $q_1 < q_2$, define

$$\epsilon(t) = \|\mathbb{G}_1(t,.)\|_{L^{q_1/2,\infty}}^{\frac{q_1}{q_1-3}} \|\mathbb{G}_1\|_{L^{p_1/2,\infty}L^{q_1/2,\infty}}^{-\frac{3}{q_1-3}};$$

then define $\mathbb{G}_3 = \mathbb{1}_{|\mathbb{G}_1(t,x)|<\epsilon(t)}\mathbb{G}_1$ and $\mathbb{G}_4 = \mathbb{1}_{|\mathbb{G}_1(t,x)|\geq\epsilon(t)}\mathbb{G}_1$. We have

$$\|\mathbb{G}_3\|_{L^{p_2/2,\infty}L^{q_2/2}} \leq C\|\mathbb{G}_1\|_{L^{p_1/2,\infty}L^{q_1/2,\infty}}$$

and

$$\|\mathbb{G}_4\|_{L^\infty L^{3/2}} \leq C\|\mathbb{G}_1\|_{L^{p_1/2,\infty}L^{q_1/2,\infty}},,$$

thus, $\mathbb{F}$ is of the form $\mathbb{F} = \mathbb{F}_1 + \mathbb{F}_2$ with $\mathbb{F}_1 = \mathbb{G}_2 + \mathbb{G}_3 \in L^{p_2/2,\infty}L^{q_2/2,\infty}$ and $\mathbb{F}_2 = \mathbb{G}_4 \in L^\infty L^{3/2,\infty}$.

- Assume that $\mathbb{F}$ is a sum $\mathbb{F} = \mathbb{G}_1 + \mathbb{G}_2$ where $\mathbb{G}_i$ belongs to $\mathcal{Z}_{KT,r_i}$ for some r_i such that $5 < r_i < +\infty$. If $r_1 < r_2$, then $\mathbb{F}$ is of the form $\mathbb{F} = \mathbb{F}_4 \in \mathcal{Z}_{KT,r_1}$.

Our main result is then the following one:

Theorem 1.
Let $3 < q < +\infty$, $2 < p$ with $\frac{2}{p} + \frac{3}{q} = 1$, $5 < r < +\infty$. Then there exists a positive constant ϵ (which depends on p, q, r) such that if $\vec{u}_0 \in \mathrm{BMO}^{-1}$ is a divergence free vector field, , $\mathbb{F}_1 \in L^{p/2,\infty}L^{q/2,\infty}$, $\mathbb{F}_2 \in L^\infty L^{3/2,\infty}$, $\mathbb{F}_3 \in L^1 A$, $\mathbb{F}_4 \in \mathcal{Z}_{KT,r}$ with

$$\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathbb{F}_1\|_{L^{p/2,\infty}L^{q/2,\infty}} + \|\mathbb{F}_2\|_{L^\infty L^{3/2,\infty}} + \|\mathbb{F}_3\|_{L^1 A} + \|\mathbb{F}_4\|_{\mathcal{Z}_{KT,r}} < \epsilon$$

then the Navier—Stokes problem

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} + \mathbb{P}\,\mathrm{div}(\mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3 + \mathbb{F}_4 - \vec{u}\otimes\vec{u}) \\ \vec{u}(0,.) = \vec{u}_0 \end{cases} \tag{1}$$

has a mild solution $\vec{u}$ such that

$$\vec{u} \in L^{p,\infty}L^{q,\infty} + L^\infty L^{3,\infty} + L^2 A + \mathcal{Y}_{KT,r}$$

where the space $\mathcal{Y}_{KT,r}$ is defined by

$$\|\vec{u}\|_{\mathcal{Y}_2} = \sup_{t>0,x_0\in\mathbb{R}^3} t^{-\frac{3}{4}} \|\vec{u}\|_{L^2((0,t)\times B(x_0,\sqrt{t}))} < +\infty$$

and

$$\sup_{t>0,x_0\in\mathbb{R}^3} t^{\frac{1}{2}-\frac{5}{2r}} \|\vec{u}\|_{L^r((t/2,t)\times B(x_0,\sqrt{t}))} < +\infty.$$

1 Mild solutions of the Navier–Stokes equations

Recall that we consider global mild solutions of the Cauchy problem for the Navier–Stokes equations on the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \Delta \vec{u} - \vec{\nabla} p + \operatorname{div} \mathbb{F} \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0, .) = \vec{u}_0 \end{cases} \tag{2}$$

We shall look for minimal regularity assumptions for $\vec{u}$. It is therefore better to write the non-linear term $\vec{u} \cdot \vec{\nabla} \vec{u}$ as $\operatorname{div}(\vec{u} \otimes \vec{u})$ (the two vector fields are equal when $\vec{u}$ is a regular divergence free vector field):

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla} p + \operatorname{div}(\mathbb{F} - \vec{u} \otimes \vec{u}) \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0, .) = \vec{u}_0 \end{cases} \tag{3}$$

Taking the divergence of the first equation, we get

$$\Delta p = (\vec{\nabla} \otimes \vec{\nabla}) \cdot (\mathbb{F} - \vec{u} \otimes \vec{u})$$

and thus

$$\Delta \vec{\nabla} p = \vec{\nabla} \left((\vec{\nabla} \otimes \vec{\nabla}) \cdot (\mathbb{F} - \vec{u} \otimes \vec{u}) \right). \tag{4}$$

Assuming that $\vec{\nabla} p$ is equal to 0 at infinity, equation (4) defines p as a function of $\mathbb{F}$ and $\vec{u}$. More precisely, if $\mathbb{F} - \vec{u} \otimes \vec{u}$ is assumed to belong to $L^1_{\text{loc}}((0, +\infty), L^1(\frac{dx}{1+|x|^4}))$, then [Fer21] shows that the solution $\vec{\nabla} p$ of equation (4) which is equal to 0 at infinity is given by the formula

$$\vec{\nabla} p = \frac{1}{\Delta} \vec{\nabla} \left((\vec{\nabla} \otimes \vec{\nabla}) \cdot (\mathbb{F} - \vec{u} \otimes \vec{u}) \right)$$

where, writing

$$G(x) = \frac{1}{4\pi |x|}$$

for the fundamental solution of $-\Delta$

$$(-\Delta G = \delta \text{ so that, for } f \in \mathcal{D}(\mathbb{R}^3, f = G * (-\Delta f))$$

and choosing a function $\psi \in \mathcal{D}(\mathbb{R}^3$ which is equal to 1 on a neighbourhood of 0, $\frac{1}{\Delta} \partial_i \partial_j \partial_k$ is defined as

$$\frac{1}{\Delta} \partial_i \partial_j \partial_k f = -\partial_i \partial_j \partial_k \left((\psi G) * f \right) - \left(\partial_i \partial_j \partial_k ((1 - \psi) G) \right) * f.$$

Thus, we have an equation with one unknown $\vec{u}$ and two data $\vec{u}_0$ and $\mathbb{F}$.

Defining (formally) the Leray projection operator $\mathbb{P}$ as

$$\mathbb{P} = \mathrm{Id} - \frac{1}{\Delta}\vec{\nabla}\,\mathrm{div},$$

the Navier–Stokes equations then become

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} + \mathbb{P}\,\mathrm{div}(\mathbb{F} - \vec{u}\otimes\vec{u}) \\ \vec{u}(0,.) = \vec{u}_0, \quad \mathrm{div}\,\vec{u}_0 = 0 \end{cases} \tag{5}$$

This is viewed as a non-linear heat equation and is transformed into the Duhamel formula

$$\vec{u} = e^{t\Delta}\vec{u}_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(\mathbb{F} - \vec{u}\otimes\vec{u})\,ds \tag{6}$$

where $e^{t\Delta}$ is the convolution operator with the heat kernel: $e^{t\Delta}f = W_t * f$ with

$$W_t = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{x^2}{4t}}.$$

The formalism of global mild solutions of the Cauchy problem for the Navier–Stokes equations is then described by the following definition and theorem :

Definition 1.
An adapted triplet of Banach spaces is a triplet of Banach spaces $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $\mathcal{X}$ is a space of tempered vector distributions on $\mathbb{R}^3$, $\mathcal{Y}$ is a space of time-dependent vector fields which are locally integrable on $(0,+\infty)\times\mathbb{R}^3$ and $\mathbb{Z}$ is a space of time-dependent functors which are locally integrable on $(0,+\infty)\times\mathbb{R}^3$ and such that

- *the operator*
$$\vec{u}_0 \mapsto S(\vec{u}_0) = e^{t\Delta}\vec{u}_0$$
is a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$:
$$\|S(\vec{u}_0)\|_{\mathcal{Y}} \leq C_1\|\vec{u}_0\|_{\mathcal{X}};$$

- *the operator*
$$\mathbb{F} \mapsto \mathcal{L}(\mathbb{F}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds$$
is a bounded linear operator from $\mathcal{Z}$ to $\mathcal{Y}$:
$$\|\mathcal{L}(\mathbb{F})\|_{\mathcal{Y}} \leq C_2\|\mathbb{F}\|_{\mathcal{Z}};$$

- *the operator*

$$(\vec{u},\vec{v}) \mapsto B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(\vec{u}\otimes\vec{v})\,ds$$

is a bounded bilinear operator from $\mathcal{Y}\times\mathcal{Y}$ *to* $\mathcal{Y}$*:*

$$\|B(\vec{u},\vec{v})\|_{\mathcal{Y}} \leq C_3\|\vec{u}\|_{\mathcal{Y}}\|\vec{v}\|_{\mathcal{Y}}.$$

Remark: Obviously, if the operator S is a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$, the operator $\mathcal{L}$ is bounded from $\mathcal{Z}$ to $\mathcal{Y}$ and the operator $(\vec{u},\vec{v}) \mapsto \vec{u}\otimes\vec{v}$ is a bounded bilinear operator from $\mathcal{Y}\times\mathcal{Y}$ to $\mathcal{Z}$, then $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is an adapted triplet of Banach spaces.

The following theorem is then easy to check (through the Banach contraction principle):

Theorem 2.
Let $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ *be an adapted triplet of Banach spaces (with associated constants* C_1*,* C_2*,* C_3*). If* $\vec{u}_0$ *and* $\mathbb{F}$ *are small enough:*

$$4C_3(C_1\|\vec{u}_0\|_{\mathcal{X}} + C_2\|\mathbb{F}\|_{\mathcal{Z}}) < 1,$$

then the Navier–Stokes problem (6) has a global solution $\vec{u}$ *with*

$$\|\vec{u}\|_{\mathcal{Y}} \leq 2(C_1\|\vec{u}_0\|_{\mathcal{X}} + C_2\|\mathbb{F}\|_{\mathcal{Z}}).$$

Thus, Theorem 1 will be proved by establishing the following theorem:

Theorem 3.
Let $\mathcal{X} = \mathrm{BMO}^{-1}$*,* $\mathcal{Y} = L^{p,\infty}L^{q,\infty} + L^\infty L^{3,\infty} + L^2 A + \mathcal{Y}_{KT,r}$ *(with* $2 < p < +\infty$*,* $\frac{2}{p}+\frac{3}{q} = 1$ *and* $5 < r < +\infty$*) and* $\mathcal{Z} = L^{p/2,\infty}L^{q/2,\infty} + L^\infty L^{3/2,\infty} + L^1 A + \mathcal{Z}_{KT,r}$*. Then,* $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ *is an adapted triplet of Banach spaces.*

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2 The classical Koch and Tataru theorem.

Navier–Stokes equations have symmetries. In particular, we have the two following properties: if $\vec{u}$ is a solution of the Navier–Stokes problem (6) with data $\vec{u}_0$ and $\mathbb{F}$, then

- [space translation] $\vec{u}(t, x-x_0)$ is a solution of the Navier–Stokes problem (6) with data $\vec{u}_0(x-x_0)$ and $\mathbb{F}(t,x-x_0)$,

- [space dilation] if $\lambda > 0$, $\lambda\vec{u}(\lambda^2 t, \lambda x)$ is a solution of the Navier–Stokes problem (6) with data $\lambda\vec{u}_0(\lambda x)$ and $\lambda^2\mathbb{F}(\lambda^2 t, \lambda x)$.

In particular, we shall look for critical spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, meaning that we have invariance of the norms under space translations and space dilations: for every $x_0 \in \mathbb{R}^3$ and $\lambda > 0$

$$\begin{array}{ll} \|\vec{u}_0(x-x_0)\|_{\mathcal{X}} = \|\vec{u}_0\|_{\mathcal{X}}, & \|\lambda\vec{u}_0(\lambda x)\|_{\mathcal{X}} = \|\vec{u}_0\|_{\mathcal{X}}, \\ \|\vec{u}(t,x-x_0)\|_{\mathcal{Y}} = \|\vec{u}\|_{\mathcal{Y}}, & \|\lambda\vec{u}(\lambda^2 t, \lambda x)\|_{\mathcal{Y}} = \|\vec{u}\|_{\mathcal{Y}}, \\ \|\mathbb{F}(t,x-x_0)\|_{\mathcal{Z}} = \|\mathbb{F}\|_{\mathcal{Z}}, & \|\lambda^2\mathbb{F}(\lambda^2 t, \lambda x)\|_{\mathcal{Z}} = \|\mathbb{F}\|_{\mathcal{Z}}. \end{array}$$

Finally, in order to give sense to the formula

$$\vec{u}(t,.) = \vec{u}_0 + \Delta\int_0^t \vec{u}(s,.)\,ds + \mathbb{P}\,\mathrm{div}(\int_0^t \mathbb{F} - \vec{u}\otimes\vec{u}\,ds),$$

we require the continuous embeddings $\mathcal{Y} \subset \bigcap_{T>0} L^2((0,T), L^2(\frac{dx}{1+|x|^4}))$ and $\mathcal{Z} \subset \bigcap_{T>0} L^1((0,T), L^1(\frac{dx}{1+|x|^4}))$. [Due to the invariance through space translations or space dilations, it is equivalent to ask that $\vec{u} \mapsto \mathbb{1}_{(0,1)\times B(0,1)}\vec{u}$ is bounded from $\mathcal{Y}$ to $L^2((0,1)\times B(0,1))$ and similarly that $\mathbb{F} \mapsto \mathbb{1}_{(0,1)\times B(0,1)}\mathbb{F}$ is bounded from $\mathcal{Z}$ to $L^1((0,1)\times B(0,1))$.]

In particular, we have

$$\mathcal{Y} \subset \mathcal{Y}_2 = \{\vec{u} \,/\, \sup_{t>0, x_0\in\mathbb{R}^3} t^{-3/4}\|\vec{u}\|_{L^2((0,t)\times B(x_0,\sqrt{t}))} < +\infty\}.$$

Thus, $\mathcal{Y}_2$ is maximal in the class of Banach spaces $\mathcal{Y}$ that satisfy the conditions $\|\vec{u}(t,x-x_0)\|_{\mathcal{Y}} = \|\vec{u}\|_{\mathcal{Y}}$, $\|\lambda\vec{u}(\lambda^2 t,\lambda x)\|_{\mathcal{Y}} = \|\vec{u}\|_{\mathcal{Y}}$ and

$$\sup_{\|\vec{u}\|_{\mathcal{Y}}\le 1}\int_0^1\int_{B(0,1)} |\vec{u}(s,y)|^2\,ds\,dy < +\infty.$$

Similarly, let

$$\mathcal{Z}_1 = \{\mathbb{F} \,/\, \sup_{t>0, x_0\in\mathbb{R}^3} t^{-3/2}\|\mathbb{F}\|_{L^1((0,t)\times B(x_0,\sqrt{t}))} < +\infty\}$$

and, for $1 \le p \le +\infty$, Obviously, the bilinear operator $(\vec{u},\vec{v}) \mapsto \vec{u}\otimes\vec{v}$ is bounded from $\mathcal{Y}_2 \times \mathcal{Y}_2$ to $\mathcal{Z}_1$. However, the operator $\mathcal{L}$ fails to be bounded from $\mathcal{Z}_1$ to $\mathcal{Y}_2$:

Proposition 1.
The operator $\mathbb{F} \mapsto \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds$ *is not bounded from* $\mathcal{Z}_1$ *to* $\mathcal{Y}_2$.

Proof. Due to the invariance of the norm of $\mathcal{Z}_1$ through translations and dilations, the operator $\mathcal{L}$ would be bounded from $\mathcal{Z}_1$ to $\mathcal{Y}_2$ if and only if there would exist a constant C_0 such that, for every $\mathbb{F} \in \mathcal{Z}_1$,

$$\int_0^1 \int_{[-1,1]^3} |\mathcal{L}(\mathbb{F})|^2 \, dt \, dx \leq C_0 \|\mathbb{F}\|_{\mathcal{Z}_1}^2.$$

We then take

$$\mathbb{F}_n = \begin{pmatrix} 0 & -\psi_n(x_1, x_2) & 0 \\ \psi_n(x_1, x_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\psi_n \in L^1(\mathbb{R}^2)$. We have, for $T > 0$ and $x_0 \in \mathbb{R}^3$,

$$\int_0^T \int_{B(x_0, \sqrt{T})} |\mathbb{F}_n| \, dt \, dx \leq 2\sqrt{2} \|\psi_n\|_1 T^{3/2}.$$

Thus, $\mathbb{F}_n \in \mathcal{Z}_1$. Moreover, since $\int_{\mathbb{R}} \frac{ds}{(1+s^2)^3} = \frac{19}{16}\pi$,

$$\frac{1}{-\Delta} \mathbb{P} \operatorname{div} \mathbb{F}_n = \frac{1}{-\Delta} \mathbb{P} \operatorname{div} \mathbb{F}_n = \frac{19}{64} \begin{pmatrix} \int_{\mathbb{R}^2} \psi_n(x_1 - y_1, x_2 - y_2) \frac{y_2}{|y|^2} \, dy \\ -\int_{\mathbb{R}^2} \psi_n(x_1 - y_1, x_2 - y_2) \frac{y_1}{|y|^2} \, dy \\ 0 \end{pmatrix}$$

and

$$\mathcal{L}(\mathbb{F}_n) = (\mathrm{Id} - e^{t\Delta}) \frac{1}{-\Delta} \mathbb{P} \operatorname{div} \mathbb{F}_n.$$

From $\frac{y_i}{|y|^2} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, we find

$$|e^{t\Delta} \frac{1}{-\Delta} \mathbb{P} \operatorname{div} \mathbb{F}_n| \leq C \|\psi_n\|_1 (1 + \frac{1}{t})$$

and thus

$$\int_{1/2}^1 \int_{[-1,1]^3} |e^{t\Delta} \frac{1}{-\Delta} \mathbb{P} \operatorname{div} \mathbb{F}_n|^2 \, dt \, dx \leq 9C^2 \|\psi_n\|_1^2.$$

In particular,

$$\int_{1/2}^1 \int_{[-1,1]^3} |\frac{1}{-\Delta} \mathbb{P} \operatorname{div} \mathbb{F}_n|^2 \, dt \, dx \leq (8C_0 + 9C^2) \|\psi_n\|_1^2.$$

Thus, if ψ_n is an approximation of the Dirac mass with $\psi_n = 1$, we find that the vector field

$$\vec{w}_n(x_1, x_2) = (\int_{\mathbb{R}^2} \psi_n(x_1 - y_1, x_2 - y_2) \frac{y_2}{|y|^2} \, dy, -\int_{\mathbb{R}^2} \psi_n(x_1 - y_1, x_2 - y_2) \frac{y_1}{|y|^2} \, dy)$$

is bounded in $L^2((-1,1)^2)$; but it converges in $\mathcal{D}'((-1,1)^2)$ to $(\frac{x_2}{x_1^2 + x_2^2}, -\frac{x_1}{x_1^2 + x_2^2})$ which is not square-integrable on $(-1,1)^2$. Thus, $\mathcal{L}$ is not bounded from $\mathcal{Z}_1$ to $\mathcal{Y}_2$. □

The Koch and Tataru theorem deals with a subspace of $\mathcal{Y}_2$. We define the subspace $\mathcal{Y}_{KT}$ as:

$$\mathcal{Y}_{KT} = \{\vec{u} \in \mathcal{Y}_2 \ / \ \sup_{t>0} \sqrt{t}\|\vec{u}(t,.)\|_\infty < +\infty\}.$$

$\mathcal{Y}_{KT}$ is normed with $\|\vec{u}\|_{\mathcal{Y}_{KT}} = \|\vec{u}\|_{\mathcal{Y}_2} + \sup_{t>0} \sqrt{t}\|\vec{u}(t,.)\|_\infty$, where

$$\|\vec{u}\|_{\mathcal{Y}_2} = \sup_{t>0, x_0 \in \mathbb{R}^3} t^{-\frac{3}{4}} \ \vec{u}\|_{L^2((0,t)\times B(x_0,\sqrt{t}))}.$$

Similarly, let

$$\mathcal{Z}_{KT} = \{\mathbb{F} \in \mathcal{Z}_1 \ / \ \sup_{t>0} t\|\mathbb{F}(t,.)\|_\infty < +\infty\}.$$

$\mathcal{Z}_{KT}$ is normed with $\|\mathbb{F}\|_{\mathcal{Z}_{KT}} = \|\mathbb{F}\|_{\mathcal{Z}_1} + \sup_{t>0} t\|\mathbb{F}(t,.)\|_\infty$, where

$$\|\mathbb{F}\|_{\mathcal{Z}_1} = \sup_{t>0, x_0 \in \mathbb{R}^3} t^{-\frac{3}{2}} \|\mathbb{F}|_{L^1((0,t)\times B(x_0,\sqrt{t}))}.$$

Koch and Tataru's theorem is then the following one [Koc01, Lem02]:

Theorem 4.
A) For $\vec{u}_0$ a divergence-free vector field in $\mathcal{S}'$, the following assertions are equivalent:

(i) $e^{t\Delta}\vec{u}_0 \in \mathcal{Y}_2$;

(ii) $e^{t\Delta}\vec{u}_0 \in \mathcal{Y}_{KT}$;

(iii) $e^{t\Delta}\vec{u}_0 \in \mathrm{BMO}^{-1} = \dot{F}^{-1}_{2,\infty}$ *(i.e., there exists $\vec{v}_0$ in* BMO *such that* $\vec{u}_0 = \vec{\nabla} \wedge \vec{v}_0$*).*

B) The operator $\mathbb{F} \mapsto \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,\mathbb{F}\,ds$ is a bounded linear operator from $\mathcal{Z}_{KT}$ to $\mathcal{Y}_{KT}$. In particular, the operator B defined as

$$(\vec{u},\vec{v}) \mapsto B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(\vec{u}\otimes\vec{v})\,ds$$

is a bounded bilinear operator from $\mathcal{Y}_{KT} \times \mathcal{Y}_{KT}$ to $\mathcal{Y}_{KT}$.
C) $(\mathrm{BMO}^{-1}, \mathcal{Y}_{KT}, \mathcal{Z}_{KT})$ is an adapted triplet of Banach spaces. Thus, there exists a positive constant ϵ_0 such that, if $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{Y}_{KT}} + \|\mathbb{F}\|_{\mathcal{Z}_{KT}} < \epsilon_0$, then the Navier–Stokes problem (6) has a global mild solution $\vec{u} \in \mathcal{Y}_{KT}$.

Proof. We sketch the proof given by Koch and Tataru in [Koc01], and try to highlight the obstructing term for proving the boundedness of B on $\mathcal{Y}_2$. The proof works for more general operators than B. More precisely, let $\sigma(D)$ be a Fourier multiplier associated to a function σ which is smooth on $\mathbb{R}^3 \setminus \{0\}$ and positively homogeneous of order 0 ($\sigma(\lambda\xi) = \sigma(\xi)$ for every $\lambda > 0$) and let $B_\sigma(u,v) = \int_0^t e^{(t-s)\Delta}\sigma(D)\sqrt{-\Delta}(uv)\,ds$. Then Koch and Tataru's theorem states that B_σ is bounded on the scalar version Y_{KT} of $\mathcal{Y}_{KT}$. (Similarly, we write Y_2 for the scalar version of $\mathcal{Y}_2$: $\|u\|_{Y_2} = \sup_{t>0,x_0\in\mathbb{R}^3} t^{-\frac{3}{4}}\|u\|_{L^2((0,t)\times B(x_0,\sqrt{t}))}$.)

The control of B_σ in L^∞ norm is easy: writing

$$|B_\sigma(u,v)(t,x)| \leq C_\sigma \int_0^t \int \frac{1}{(\sqrt{t-s}+|x-y|)^4}|u(s,y)|\,|v(s,y)|\,dy\,ds,$$

we check that

$$\int_0^{t/2} \int \frac{1}{(\sqrt{t-s}+|x-y|)^4}|u(s,y)|\,|v(s,y)|\,dy\,ds \leq C\frac{1}{\sqrt{t}}\|u\|_{Y_2}\|v\|_{Y_2}$$

and

$$\int_{t/2}^t \int \frac{|u(s,y)|\,|v(s,y)|}{(\sqrt{t-s}+|x-y|)^4}\,dy\,ds \leq C\frac{1}{\sqrt{t}}\sup_{s>0}\sqrt{s}\|u(s,.)\|_\infty \sup_{s>0}\sqrt{s}\|u(s,.)\|_\infty.$$

The difficult point is the control of B_σ in Y_2. Koch and Tataru proved more precisely that B_σ is bounded from $Y_{KT}\times Y_2$ to Y_2. We need to estimate, for every $T>0$ and $x\in\mathbb{R}^3$, $\|\mathbb{1}_{Q_{T,x}}B_\sigma(u,v)\|_{L^2L^2}$ where

$$Q_{T,x} = \{(t,y)\ /\ 0<t<T, |x-y|\leq\sqrt{T}\}.$$

Koch and Tataru split $w = B_\sigma(u,v)$ in three parts:

- $w_1 = B_\sigma(u,(1-\mathbb{1}_{Q_{10T,x}})v)$: we easily check that $\mathbb{1}_{Q_{T,x}}|w_1| \leq C\frac{1}{\sqrt{T}}\|u\|_{Y_2}\|v\|_{Y_2}$ and thus
$$\|\mathbb{1}_{Q_{T,x}}w_1\|_{L^2L^2} \leq CT^{3/4}\|u\|_{Y_2}\|v\|_{Y_2}.$$
- $w_2(t,y) = \sigma(D)\sqrt{-\Delta}e^{t\Delta}\int_0^t \mathbb{1}_{Q_{10T,x}}uv\,ds$. The main lemma in Koch and Tataru's proof states that the operator $Q(u,v) = \sqrt{-\Delta}e^{t\Delta}\int_0^t \mathbb{1}_{Q_{10T,x}}uv\,ds$ maps $Y_2\times Y_2$ to L^2L^2 with a norm of order $T^{3/4}$ and thus
$$\|\mathbb{1}_{Q_{T,x}}w_2\|_{L^2L^2} \leq \|w_2\|_{L^2L^2} \leq CT^{3/4}\|u\|_{Y_2}\|v\|_{Y_2}.$$

- $w_3(t,y) = \sigma(D)\int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}(\mathbb{1}_{Q_{10T,x}} uv)\,ds$. They rewrite w_3 as

$$w_3 = \sigma(D)\int_0^t e^{(t-s)\Delta}\Delta \frac{e^{s\Delta} - \mathrm{Id}}{\sqrt{-\Delta}}(\mathbb{1}_{Q_{10T,x}} uv)\,ds$$

and use the maximal regularity of the heat kernel in L^2L^2 to write

$$\|w_3\|_{L^2L^2} \leq C\|\frac{e^{t\Delta} - \mathrm{Id}}{\sqrt{-\Delta}}(\mathbb{1}_{Q_{10T,x}} uv)\|_{L^2L^2} \leq C'\|\sqrt{t}\mathbb{1}_{Q_{10T,x}} uv\|_{L^2L^2}.$$

Thus,

$$\|\mathbb{1}_{Q_{T,x}} w_3\|_{L^2L^2} \leq \|w_3\|_{L^2L^2} \leq C\|\sqrt{t}u\|_\infty \|\mathbb{1}_{Q_{10T,x}} v\|_{L^2L^2} \leq CT^{3/4}\|u\|_{Y_{KT}}\|v\|_{Y_2}.$$

Thus, the obstruction for the boundedness of B_σ on Y_2 lies in w_3. □

An easy consequence of the Koch and Tataru theorem is the following simplified version of Theorem 3:

Proposition 2.
Let $\mathcal{X} = \mathrm{BMO}^{-1}$, $\mathcal{Y} = L^{p,\infty}L^{q,\infty} + L^\infty L^{3,\infty} + \mathcal{Y}_{KT}$ (with $2 < p < +\infty$, $\frac{2}{p} + \frac{3}{q} = 1$) and $\mathcal{Z} = L^{p/2,\infty}L^{q/2,\infty} + L^\infty L^{3/2,\infty} + \mathcal{Z}_{KT}$. Then, $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is an adapted triplet of Banach spaces.

Proof. The boundedness of B on $\mathcal{Y}_{KT}$ is given by the Koch and Tataru theorem and the boundedness of B on $L^\infty L^{3,\infty}$ is given by the results of Kozono and Meyer. Moreover, B is obviously bounded from $\mathcal{Y}_{KT} \times L^\infty L^{3,\infty}$ and $L^\infty L^{3,\infty} \times \mathcal{Y}_{KT}$ to $L^\infty L^{3,\infty}$:

$$\|B(\vec{u},\vec{v})(t,.)\|_{L^{3,\infty}} \leq C\int_0^t \sqrt{s}\|\vec{u}(s,.)\|_\infty \|\vec{v}(s,.)\|_{L^{3,\infty}} \frac{ds}{\sqrt{t-s}\sqrt{s}}$$

(with a similar inequality for $\|B(\vec{v},\vec{u})(t,.)\|_{L^{3,\infty}}$).

Finally, if $1 < r_1 < +\infty$, $1 < r_2 \leq r_3 < +\infty$, we have

$$\|\mathcal{L}(\mathbb{F})\|_{L^{r_3,\infty}} \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2r_2}-\frac{3}{2r_3}}}\|\mathbb{F}(s,.)\|_{L^{r_2,\infty}}\,ds$$

so that $\mathcal{L}$ is bounded from $L^{r_1,\infty}L^{r_2,\infty}$ to $L^{r_4,\infty}L^{r_3,\infty}$ with

$$\frac{1}{r_4} + \frac{3}{2r_3} = \frac{1}{r_1} + \frac{3}{2r_2} - \frac{1}{2}$$

if $\frac{1}{r_2} - \frac{1}{3} < \frac{1}{r_3} < \frac{1}{r_2} - \frac{1}{3} + \frac{2}{3r_1}$. Thus we get the following results:

- B is bounded on $L^{p,\infty}L^{q,\infty}$ (take $r_1 = p/2$, $r_2 = q/2$, $r_3 = q$ and $r_4 = p$)
- B is bounded from $\mathcal{Y}_{KT} \times L^{p,\infty}L^{q,\infty}$ and $L^{p,\infty}L^{q,\infty} \times \mathcal{Y}_{KT}$ to $L^{p,\infty}L^{q,\infty}$ (write $\mathcal{Y}_{KT} \subset L^{2,\infty}L^{\infty}$ and take $r_1 = \frac{2p}{p+2}$, $r_2 = q$, $r_3 = q$ and $r_4 = p$)
- B is bounded from $L^{\infty}L^{3,\infty} \times L^{p,\infty}L^{q,\infty}$ and $L^{p,\infty}L^{q,\infty} \times L^{\infty}L^{3,\infty}$ to $L^{2p,\infty}L^{\frac{6q}{q+3},\infty}$ (take $r_1 = p$, $r_2 = \frac{3q}{q+3}$, $r_3 = \frac{6q}{q+3}$ and $r_4 = 2p$) hence to $L^{\infty}L^{3,\infty} + L^{p,\infty}L^{q,\infty}$.

Thus, B is bounded on $L^{p,\infty}L^{q,\infty} + L^{\infty}L^{3,\infty} + \mathcal{Y}_{KT}$. □

We may notice as well that B is bounded from $L^{\infty}L^{3,\infty} \times L^{2,\infty}L^{\infty}$ and $L^{2,\infty}L^{\infty} \times L^{\infty}L^{3,\infty}$ to $L^{2p,\infty}L^{\frac{6q}{q+3},\infty}$ (take $r_1 = 2$, $r_2 = 3$, $r_3 = \frac{6q}{q+3}$ and $r_4 = 2p$) hence to $L^{\infty}L^{3,\infty} + L^{p,\infty}L^{q,\infty}$.

3 Proof of Theorem 1

To prove Theorem 1, we only need to prove that B is bounded on $\mathcal{Y} = L^{p,\infty}L^{q,\infty} + L^{\infty}L^{3,\infty} + L^2A + \mathcal{Y}_{KT,r}$.

We already know that B is bounded on $L^{p,\infty}L^{q,\infty}$ and that B is bounded from $L^{\infty}L^{3,\infty} \times L^{p,\infty}L^{q,\infty}$ and $L^{p,\infty}L^{q,\infty} \times L^{\infty}L^{3,\infty}$ to $L^{\infty}L^{3,\infty} + L^{p,\infty}L^{q,\infty}$. We have seen as well (since $L^2A \subset L^{2,\infty}L^{\infty}$) that B is bounded from $L^2A \times L^{p,\infty}L^{q,\infty}$ and $L^{p,\infty}L^{q,\infty} \times L^2A$ to $L^{p,\infty}L^{q,\infty}$ and is bounded from $L^{\infty}L^{3,\infty} \times L^2A$ and $L^2A \times L^{\infty}L^{3,\infty}$ to $L^{\infty}L^{3,\infty} + L^{p,\infty}L^{q,\infty}$. Moreover, Lei and Lin's theorem states that B is bounded on L^2A.

Thus, the difficult part lies in the interaction with $\mathcal{Y}_{KT,r}$. We cannot take $\mathcal{Y}_{KT,\infty} = \mathcal{Y}_{KT}$ as we cannot prove that B is bounded from $\mathcal{Y}_{KT} \times L^2A$ to L^2A or $\mathcal{Y}_{KT}$. Replacing $\mathcal{Y}_{KT}$ with the larger space $\mathcal{Y}_{KT,r}$ will solve the problem:

Proposition 3.
A) If $5 < r \leq +\infty$, B is bounded from $\mathcal{Y}_2 \times \mathcal{Y}_{KT,r}$ and from $\mathcal{Y}_{KT,r} \times \mathcal{Y}_2$ to $\mathcal{Y}_2$.
B) Let $\mathcal{Y} = L^{p,\infty}L^{q,\infty} + L^{\infty}L^{3,\infty} + L^2A + \mathcal{Y}_{KT,r}$ (with $2 < p < +\infty$, $\frac{2}{p} + \frac{3}{q} = 1$ and $5 < r < +\infty$). Then B is bounded from $\mathcal{Y} \times \mathcal{Y}_{KT,r}$ and from $\mathcal{Y}_{KT,r} \times \mathcal{Y}$ to $\mathcal{Y}_{KT,r}$.

Proof. Recall that

$$Q_{T,x} = \{(t,y) \ / \ 0 < t < T, |x-y| \leq \sqrt{T}\}$$

and define

$$R_{T,x} = \{(t,y) \ / \ T/2 < t < T, |x-y| \leq \sqrt{T}\}$$

and

$$S_{T,x} = \{(t,y) \ / \ T/4 < t < T, |x-y| \leq \sqrt{10T}\}.$$

$\mathcal{Y}_2$ is defined by

$$\vec{u} \in \mathcal{Y}_2 \Leftrightarrow \sup_{T>0, x\in\mathbb{R}^3} T^{-3/4} \|\mathbb{1}_{Q_{T,x}} \vec{u}\|_{L^2L^2} < +\infty$$

and $\mathcal{Y}_{KT,r}$ is defined by

$$\vec{u} \in \mathcal{Y}_{KT,r} \Leftrightarrow \vec{u} \in \mathcal{Y}_2 \text{ and } \sup_{T>0, x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}} \|\mathbb{1}_{R_{T,x}} \vec{u}\|_{L^rL^r} < +\infty$$

Again, we consider the scalar versions Y_2, $Y_{KT,r}$ and Y of $\mathcal{Y}_2$, $\mathcal{Y}_{KT,r}$ and $\mathcal{Y}$ and replace B with B_σ.

Following Koch and Tataru, we split $w = B_\sigma(u,v)$ in three parts:

- $w_1 = B_\sigma(u, (1 - \mathbb{1}_{Q_{10T,x}})v)$: we saw that

$$\|\mathbb{1}_{Q_{T,x}} w_1\|_{L^2L^2} \leq CT^{3/4} \|u\|_{Y_2} \|v\|_{Y_2}.$$

- $w_2(t,y) = \sigma(D)\sqrt{-\Delta} e^{t\Delta} \int_0^t \mathbb{1}_{Q_{10T,x}} uv \, ds$. We saw that

$$\|\mathbb{1}_{Q_{T,x}} w_2\|_{L^2L^2} \leq \|w_2\|_{L^2L^2} \leq CT^{3/4} \|u\|_{Y_2} \|v\|_{Y_2}.$$

- $w_3(t,y) = \sigma(D) \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}(\mathbb{1}_{Q_{10T,x}} uv) \, ds$. We are going to prove below (Theorem 5 in next section) that

$$\| \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}(uv) \, ds\|_{L^2L^2} \leq C \sup_{T>0, x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}} \|\mathbb{1}_{R_{T,x}} u\|_{L^rL^r} \|v\|_{L^2L^2}.$$

Thus,

$$\|\mathbb{1}_{Q_{T,x}} w_3\|_{L^2L^2} \leq \|w_3\|_{L^2L^2} \leq C\|u\|_{Y_{KT,r}} \|\mathbb{1}_{Q_{10T,x}} v\|_{L^2L^2} \leq CT^{3/4} \|u\|_{Y_{KT,r}} \|v\|_{Y_2}.$$

Hence, $w \in Y_2$.

If $v \in Y$, we write $w = w_4 + w_5$ with $w_4 = B_\sigma(u, (1 - \mathbb{1}_{S_{T,x}})v)$ and $w_5 = B_\sigma(u, \mathbb{1}_{S_{T,x}} v)$. We easily check that $\mathbb{1}_{R_{T,x}}|w_4| \leq C\frac{1}{\sqrt{T}} \|u\|_{Y_2} \|v\|_{Y_2}$ and thus

$$\|\mathbb{1}_{R_{T,x}} w_4\|_{L^rL^r} \leq CT^{-\frac{1}{2}+\frac{5}{2r}} \|u\|_{Y_2} \|v\|_{Y_2}.$$

Now, if $v = v_1 + v_2 + v_3 + v_4 \in L^{p,\infty}L^{q,\infty} + L^\infty L^{3,\infty} + L^2 A + Y_{KT,r}$, we write

$$w_5 = B_\sigma(\mathbb{1}_{S_{T,x}} u, v_1) + B_\sigma(\mathbb{1}_{S_{T,x}} u, v_2) + B_\sigma(\mathbb{1}_{S_{T,x}} u, v_3) + B_\sigma(\mathbb{1}_{S_{T,x}} u, \mathbb{1}_{S_{T,x}} v_4)$$

with $\mathbb{1}_{S_{T,x}} u \in L^r L^r$ and we get:

- $\|B_\sigma(\mathbb{1}_{S_{T,x}}u, v_1)\|_r \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2q}}}\|\mathbb{1}_{S_{T,x}}u\|_r\|v_1\|_{L^{q,\infty}}\,ds$. Since $t^{-(\frac{1}{2}+\frac{3}{2q})} = t^{-1+\frac{1}{p}} \in L^{\frac{p}{p-1},\infty}$ and since $\|\mathbb{1}_{S_{T,x}}u\|_r\|v_1\|_{L^{q,\infty}} \in L^{\frac{rq}{r+q},r}$, we find

  $$\|B_\sigma(\mathbb{1}_{S_{T,x}}u, v_1)\|_{L^rL^r} \leq C\|\mathbb{1}_{S_{T,x}}u\|_{L^rL^r}\|v_1\|_{L^{p,\infty}L^{q,\infty}} \leq C'T^{-\frac{1}{2}+\frac{5}{2r}}\|u\|_{Y_{KT,r}}\|v_1\|_{L^{p,\infty}L^{q,\infty}}$$

- We have $\|\mathbb{1}_{S_{T,x}}u(s,.)v_2(s,.)\|_{L^{\frac{3r}{r+3},r}} \leq C\|\mathbb{1}_{S_{T,x}}u\|_r\|v_2\|_{L^{3,\infty}}$, hence $\frac{1}{\sqrt{-\Delta}}(\mathbb{1}_{S_{T,x}}uv) \in L^rL^r$. We then write

  $$B_\sigma(\mathbb{1}_{S_{T,x}}u, v_2) = -\int_0^t e^{(t-s)\Delta}\Delta\left(\frac{1}{\sqrt{-\Delta}}(\mathbb{1}_{S_{T,x}}uv)\right)\,ds$$

  and use the L^rL^r maximal regularity of the heat kernel to get

  $$\|B_\sigma(\mathbb{1}_{S_{T,x}}u, v_2)\|_{L^rL^r} \leq C\|\mathbb{1}_{S_{T,x}}u\|_{L^rL^r}\|v_2\|_{L^\infty L^{3,\infty}} \leq C'T^{-\frac{1}{2}+\frac{5}{2r}}\|u\|_{Y_{KT,r}}\|v_2\|_{L^\infty L^{3,\infty}}$$

- $\|B_\sigma(\mathbb{1}_{S_{T,x}}u, v_3)\|_r \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|\mathbb{1}_{S_{T,x}}u\|_r\|v_3\|_\infty\,ds$. Since $t^{-(\frac{1}{2}} \in L^{2,\infty}$ and since $\|\mathbb{1}_{S_{T,x}}u\|_r\|v_3\|_\infty \in L^{\frac{2r}{r+2},r}$, we find

  $$\|B_\sigma(\mathbb{1}_{S_{T,x}}u, v_3)\|_{L^rL^r} \leq C\|\mathbb{1}_{S_{T,x}}u\|_{L^rL^r}\|v_3\|_{L^2A} \leq C'T^{-\frac{1}{2}+\frac{5}{2r}}\|u\|_{Y_{KT,r}}\|v_3\|_{L^2A}.$$

- We have $\|B_\sigma(\mathbb{1}_{S_{T,x}}u, \mathbb{1}_{S_{T,x}}v_4)\|_r \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2r}}}\|\mathbb{1}_{S_{T,x}}u\|_r\|\mathbb{1}_{S_{T,x}}v_4\|_r\,ds$. Hence,

  $$\|\mathbb{1}_{R_{T,x}}B_\sigma(\mathbb{1}_{S_{T,x}}u, \mathbb{1}_{S_{T,x}}v_4)\|_{L^rL^r} \leq C(\int_0^T \frac{1}{s^{(\frac{1}{2}+\frac{3}{2r})\frac{r}{r-1}}}ds)^{1-\frac{1}{r}}\|1_{S_{T,x}}u\|_{L^rL^r}\|\mathbb{1}_{S_{T,x}}v_4\|_{L^rL^r}$$

  and thus

  $$\|\mathbb{1}_{R_{T,x}}B_\sigma(\mathbb{1}_{S_{T,x}}u, \mathbb{1}_{S_{T,x}}v_4)\|_{L^rL^r} \leq CT^{1-\frac{1}{r}-\frac{1}{2}-\frac{3}{2r}}\|\mathbb{1}_{S_{T,x}}u\|_{L^rL^r}\|\mathbb{1}_{S_{T,x}}v_4\|_{L^rL^r}$$

  and finally

  $$\|\mathbb{1}_{R_{T,x}}B_\sigma(\mathbb{1}_{S_{T,x}}u, \mathbb{1}_{S_{T,x}}v_4)\|_{L^rL^r} \leq C \leq T^{-\frac{1}{2}+\frac{5}{2r}}\|u\|_{Y_{KT,r}}\|v\|_{Y_{KT,r}}. \qquad \square$$

4 Parabolic dyadic decomposition of the time-space domain

We decompose $(0,+\infty)\times\mathbb{R}^3$ as

$$(0,+\infty)\times\mathbb{R}^3 = \bigcup_{j\in\mathbb{Z},k\in\mathbb{Z}^3}\{(t,x)\ /\ 1 \leq 4^jt < 4, 2^jx-k \in [0,1)^3\} = \bigcup_{j\in\mathbb{Z},k\in\mathbb{Z}^3} R_{j,k}$$

and $(0, 16\,4^{-j}) \times \mathbb{R}^3$ as

$$(0, 16\,4^{-j}) \times \mathbb{R}^3 = \bigcup_{k\in\mathbb{Z}^3} \{(t,x)\ /0 < 4^j t < 16, 2^j x - k \in [0,1)^3\} = \bigcup_{k\in\mathbb{Z}^3} Q_{j,k}.$$

If $v \in L^2L^2$, then v can be decomposed in an orthogonal series

$$v = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3} \mathbb{1}_{R_{j,k}} v = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3} v_{j,k}$$

with

$$\|v\|^2_{L^2L^2} = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3} \|v_{j,k}\|^2_{L^2L^2}.$$

Similarly, if $u \in Y_{KT,r}$, then

$$u = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3} \mathbb{1}_{R_{j,k}} u = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^3} u_{j,k}$$

with, for every $1 \le \rho < r$,

$$\sup_{j\in\mathbb{Z},k\in\mathbb{Z}^3} 2^{j(1-\frac{5}{\rho})} \|u_{j,k}\|_{L^\rho L^\rho} < +\infty.$$

Theorem 5.
Let $v \in L^2L^2$ and $u \in Y_{KT,r}$ with $5 < r < +\infty$. Write $v_{j,k} = \mathbb{1}_{R_{j,k}} v$, $v_j = \sum_{k\in\mathbb{Z}^3} v_{j,k}$, and $u_{j,k} = \mathbb{1}_{R_{j,k}} u$. Then
A) For $5/2 < \rho \le r$ and $\alpha = 1 - \frac{5}{\rho}$,

$$\| \int_0^t (e^{(t-s)\Delta} - e^{t\Delta}) \sqrt{-\Delta}^{1+\alpha} (uv_j)\, ds \|_{L^2L^2} \le C \|v_j\|_2 \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho}.$$

B) We have

$$\| \int_0^t (e^{(t-s)\Delta} - e^{t\Delta}) \sqrt{-\Delta} (uv)\, ds \|_{L^2L^2} \le C \|v\|_{L^2L^2} \sup_{T>0, x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}} \|\mathbb{1}_{R_{T,x}} u\|_{L^r L^r}.$$

Proof.

Proof of A).
We first consider $4^j t \le 16$ and estimate $W = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta}) \sqrt{-\Delta}^{1+\alpha} (uv_j)\, ds$ in $L^2((0, 16\,4^{-j}), L^2)$, then estimate $W^* = \int_0^{16\,4^{-j}} (e^{(16\,4^{-j}-s)\Delta} - e^{16\,4^{-j}\Delta}) \sqrt{-\Delta}^{\alpha} (uv_j)\, ds$ in $L^2(\mathbb{R}^3)$ and finally we estimate $W = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta}) \sqrt{-\Delta}^{1+\alpha} (uv_j)\, ds$ in $L^2((16\,4^{-j}, +\infty), L^2)$.

When $t < 16\,4^{-j}$, we write

$$W = \sum_{l\in\mathbb{Z}^3} \mathbb{1}_{Q_{j,l}} \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(\sum_{k\in\mathbb{Z}^3} u_{j,k}v_{,k}j)\,ds$$

which we reorganize as

$$W = \sum_{m\in\mathbb{Z}^3}\sum_{k\in\mathbb{Z}^3} \mathbb{1}_{Q_{j,k+m}} \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(u_{j,k}v_{,k}j)\,ds = \sum_{m\in\mathbb{Z}^3} W_m.$$

We have

$$\|W\|_{L^2((0,16\,4^{-j}),L^2} \leq \sum_{m\in\mathbb{Z}^3} \|W_m\|_{L^2((0,16\,4^{-j}),L^2}$$
$$= \sum_{m\in\mathbb{Z}^3} \left(\sum_{k\in\mathbb{Z}^3} \|\mathbb{1}_{Q_{j,k+m}} \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(u_{j,k}v_{,k}j)\,ds\|^2_{L^2((0,16\,4^{-j}),L^2} \right)^{1/2}$$

We have

$$|\int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(u_{j,k}v_{j,k})\,ds| \leq C Z_\alpha(u_{j,k}v_{j,k}).$$

where

$$Z_\alpha(w) = \int_0^t \int \frac{1}{(\sqrt{t-s}+|x-y|)^{4+\alpha}} |w(s,y)|\,dy\,ds.$$

As $-1 < \alpha < 1$, $\frac{1}{(\sqrt{t}+|x|)^{4+\alpha}} \in L^{\frac{5}{4+\alpha},\infty}((0,+\infty)\times\mathbb{R}^3)$, hence

$$\|Z_\alpha(u_{j,k}v_{j,k})\|_{L^2_{t,x}} \leq C\|u_{j,k}v_{j,k}\|_{L^{\frac{10}{7-2\alpha},2}_{t,x}} \leq C'\|u_{j,k}\|_{L^{\frac{5}{1-\alpha},\infty}_{t,x}} \|v_{j,k}\|_{L^2_{t,x}}$$

and thus

$$\|(\mathbb{1}_{t>0}\frac{1}{(\sqrt{t}+|x|)^{4+\alpha}}) *_{t,x} (u_{j,k}v_{j,k})\|_{L^2L^2} \leq C\|u_{j,k}\|_{L^\rho L^\rho}\|v_{j,k}\|_{L^2L^2}.$$

Thus,

$$\|W_m\|_{L^2L^2} \leq C \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho}\|v_j\|_{L^2L^2}.$$

Moreover, if $|m| \geq 20$ and $|m_0| = 10$, we have, for $0 \leq s \leq t \leq 16\,4^{-j}$, $y \in Q_{j,k}$, $x \in Q_{j,k+m}$ and $z \in Q_{j,k+m_0}$,

$$\frac{1}{(\sqrt{t-s}+|x-y|)^{4+\alpha}} \leq \frac{1}{|x-y|^{4+\alpha}} \leq C\frac{2^{(4+\alpha)j}}{m^{4+\alpha}} \leq C'\frac{1}{m^{4+\alpha}(\sqrt{t-s}+|z-y|)^{4+\alpha}}$$

so that, for $0 < t \le 16\,4^{-j}$,

$$\begin{aligned}&\mathbb{1}_{Q_{j,k+m}}(x) Z_\alpha(u_{j,k}v_{j,k})(t,x)\\ &\qquad\le \frac{C}{m^{4+\alpha}} \mathbb{1}_{Q_{j,k+m_0}}(x-(m-m_0)2^{-j}) Z_\alpha(u_{j,k}v_{j,k})(t, x-(m-m_0)2^{-j}))\end{aligned}$$

and

$$\|W_m\|_{L^2((0,16\,4^{-j}),L^2)} \le C \frac{1}{m^{4+\alpha}} \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho} \|v_j\|_{L^2L^2}.$$

Thus, we have proved that

$$\|W\|_{L^2((0,16\,4^{-j}),L^2)} \le C \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho} \|v_j\|_{L^2L^2}.$$

We now estimate $W^* = \int_0^{16\,4^{-j}} (e^{(16\,4^{-j}-s)\Delta} - e^{16\,4^{-j}\Delta})\sqrt{-\Delta}^\alpha (uv_j)\,ds$. First, as v_j is supported in $4^{-j} < t < 4\ 4^{-j}$, we write

$$\begin{aligned}W^* &= \int_{4^{-j}}^{4\,4^{-j}} (e^{(16\,4^{-j}-s)\Delta} - e^{16\,4^{-j}\Delta})\sqrt{-\Delta}^\alpha (uv_j)\,ds\\ &= \int_{4^{-j}}^{4\,4^{-j}} \int_0^s (e^{(16\,4^{-j}-s+\theta)\Delta})\sqrt{-\Delta}^{2+\alpha} (uv_j)\,d\theta\,ds\end{aligned}$$

so that

$$\begin{aligned}|W^*(x)| \le& C \int_{4^{-j}}^{4\,4^{-j}} \int_0^s \int \frac{1}{(\sqrt{16\,4^{-j}-s+\theta}+|x-y|)^{5+\alpha}} |u(s,y)v_j(s,y)|\,dy\,d\theta\,ds\\ \le& C' \int_{4^{-j}}^{4\,4^{-j}} \int \frac{4^{-j}}{(2^{-j}+|x-y|)^{5+\alpha}} |u(s,y)v_j(s,y)|\,dy\,ds.\end{aligned}$$

If $12\,4^{-j} \le \tau \le 16\,4^{-j}$, we have

$$\begin{aligned}&\int_{4^{-j}}^{4\,4^{-j}} \int \frac{4^{-j}}{(2^{-j}+|x-y|)^{5+\alpha}} |u(s,y)v_j(s,y)|\,dy\,ds\\ &\qquad\le C \int_0^\tau \int \frac{2^{-j}}{(\sqrt{\tau-s}+|x-y|)^{4+\alpha}} |u(s,y)v_j(s,y)|\,dy\,ds = C2^{-j} Z_\alpha(\tau,x).\end{aligned}$$

Thus,

$$\|W^*(x)\|_2 \le C2^{-j} \frac{1}{4\,4^{-j}} \int_{12\,4^{-j}}^{164^{-j}} \|Z_\alpha(\tau,.)\|_2\,d\tau \le \frac{C}{2} \|Z_\alpha\|_{L^2((12\,4^{-j},16\,4^{-j}),L^2)}.$$

This gives

$$\|W^*\|_2 \le C \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho} \|v_j\|_{L^2L^2}.$$

Finally, we estimate $W = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds$ in $L^2((16\,4^{-j}, +\infty), L^2)$. For $t > 16\,4^{-j}$, we have

$$W = \int_0^{16\,4^{-j}} (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}^{1+\alpha}(uv_j)\,ds = \sqrt{-\Delta}e^{(t-16\,4^{-j})\Delta}W^*$$

and thus

$$\|W\|_{L^2((0,16\,4^{-j}),L^2)} \leq \frac{1}{\sqrt{2}}\|W^*\|_2 \leq C \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho}\|v_j\|_{L^2L^2}.$$

Proof of B).
Let $U = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}(uv)\,ds$, $U = \sum_{j\in\mathbb{Z}} U_j$ with

$$U_j = \int_0^t (e^{(t-s)\Delta} - e^{t\Delta})\sqrt{-\Delta}(uv_j)\,ds.$$

Let $\gamma = 1 - \frac{5}{r}$ and $\frac{1}{\rho} = \frac{2}{5} - \frac{1}{r}$. Then $1 - \frac{5}{\rho} = -\gamma$. From point A), we know that

$$\|U_j\|_{L^2\dot{H}^\gamma} \leq C\|v_j\|_2 \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^rL^r} \leq C'2^{j\gamma}\|v_j\|_2 \sup_{T>0,x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}}\|\mathbb{1}_{R_{T,x}}u\|_{L^rL^r}.$$

and

$$\|U_j\|_{L^2\dot{H}^{-\gamma}} \leq C\|v_j\|_2 \sup_{k\in\mathbb{Z}^3} \|u_{j,k}\|_{L^\rho L^\rho} \leq C'2^{-j\gamma}\|v_j\|_2 \sup_{T>0,x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}}\|\mathbb{1}_{R_{T,x}}u\|_{L^rL^r}.$$

We then have

$$\begin{aligned}\int_0^{+\infty}\int |U(t,x)|^2\,dx\,dt &= \sum_{j\in\mathbb{Z}}\int_0^{+\infty}\int |U_j(t,x)|^2\,dx\,dt \\ &\quad + 2\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z},k<j}\int_0^{+\infty}\langle(-\Delta)^{-\gamma}U_j(t,.)|(-\Delta)^\gamma U_k(t,.)\rangle\,dt \\ \leq C(\sup_{T>0,x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}}\|\mathbb{1}_{R_{T,x}}u\|_{L^rL^r})^2&(\sum_{j\in\mathbb{Z}}\|v_j\|^2_{L^2L^2} + 2\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z},k<j} 2^{-\gamma(j-k)}\|v_j\|_{L^2L^2}\|v_k\|_{L^2L^2}) \\ &\leq C'(\sup_{T>0,x\in\mathbb{R}^3} T^{\frac{1}{2}-\frac{5}{2r}}\|\mathbb{1}_{R_{T,x}}u\|_{L^rL^r})^2\|v\|^2_{L^2L^2}.\end{aligned}$$

The theorem is proved. □

5 Morrey spaces

For $1 < \rho \leq q < +\infty$, the Morrey space $\dot{M}^{\rho,q}$ is defined by

$$u \in \dot{M}^{\rho,q}(\mathbb{R}^3) \Leftrightarrow u \in L^\rho_{\rm loc}(\mathbb{R}^3) \text{ and } \sup_{R>0, x\in\mathbb{R}} R^{\frac{3}{q}-\frac{3}{\rho}} \|\mathbb{1}_{B(x,R)} u\|_\rho < +\infty.$$

We have, if $\rho < q$, $\dot{M}^{q,q} = L^q \subset L^{q,\infty} \subset \dot{M}^{\rho,q}$. In Theorem 1, we can easily replace $L^{q,\infty}$ with the larger space $\dot{M}^{\rho,q}$ with $2 < \rho < q$ (at least, when there is no forcing term $\mathbb{F}_2$ in $L^\infty L^{3/2,\infty}$):

Theorem 6.
Let $3 < q < +\infty$, $2 < p$ with $\frac{2}{p} + \frac{3}{q} = 1$, $2 < \rho < q$, $5 < r < +\infty$. Then there exists a positive constant ϵ (which depends on p, q, r and ρ) such that if $\vec{u}_0 \in \mathrm{BMO}^{-1}$ is a divergence free vector field, , $\mathbb{F}_1 \in L^{p/2,\infty}\dot{M}^{\rho/2,q/2}$, $\mathbb{F}_3 \in L^1 A$, $\mathbb{F}_4 \in \mathcal{Z}_{KT,r}$ with

$$\|\vec{u}_0\|_{\mathrm{BMO}^{-1}} + \|\mathbb{F}_1\|_{L^{p/2,\infty}\dot{M}^{\rho/2,q/2}} + \|\mathbb{F}_3\|_{L^1 A} + \|\mathbb{F}_4\|_{\mathcal{Z}_{KT,r}} < \epsilon$$

then the Navier—Stokes problem

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} + \mathbb{P}\,\mathrm{div}(\mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3 + \mathbb{F}_4 - \vec{u}\otimes\vec{u}) \\ \vec{u}(0,.) = \vec{u}_0 \end{cases} \tag{7}$$

has a mild solution $\vec{u}$ such that

$$\vec{u} \in L^{p,\infty}\dot{M}^{\rho,q} + L^2 A + \mathcal{Y}_{KT,r}.$$

Proof. We need to prove that B is bounded on $\mathcal{Y} = L^{p,\infty}\dot{M}^{\rho,q} + L^\infty L^{3,\infty} + L^2 A + \mathcal{Y}_{KT,r}$. We already know that B is bounded on $L^2 A + \mathcal{Y}_{KT,r}$, on $L^\infty L^{3,\infty}$ and from $L^\infty L^{3,\infty} \times \mathcal{Y}_{KT,r}$ and from $\mathcal{Y}_{KT,r} \times L^\infty L^{3,\infty}$ to $\mathcal{Y}_{KT,r}$. Moreover, since $L^{q,\infty} \subset \dot{M}^{\rho,q}$, B is bounded from $L^\infty L^{3,\infty} \times L^2 A$ and $L^2 A \times L^\infty L^{3,\infty}$ to $L^\infty L^{3,\infty} + L^{p,\infty}\dot{M}^{\rho,q}$. Thus, we only have to study the behavior of B on $\mathcal{Y} \times L^{p,\infty}\dot{M}^{\rho,q}$ and on $L^{p,\infty}\dot{M}^{\rho,q} \times \mathcal{Y}$.

Again, we consider B_σ on $L^{p,\infty}\dot{M}^{\rho,q} \times Y$.

- if $u \in L^{p,\infty}\dot{M}^{\rho,q}$ and $v \in L^{p,\infty}\dot{M}^{\rho,q}$ then $uv \in L^{p,\infty}\dot{M}^{\rho,q}$: we have

  $$\Big|\int_0^t \int \sigma(D) e^{(t-s)\Delta}\sqrt{-\Delta}(uv)\,ds \leq C\int_0^t \frac{1}{(t-s)^{1-\frac{1}{p}}}\frac{1}{|x-y|^{3-\frac{3}{q}}}|uv(s,y)|\,ds\,dy.$$

  As $(-\Delta)^{-\frac{3}{2q}}$ maps $\dot{M}^{\rho/2,q/2}$ to $\dot{M}^{\rho,q}$, we find that

  $$\|B_\sigma(uv)\|_{L^{p,\infty}\dot{M}^{\rho,q}} \leq C\|u\|_{L^{p,\infty}\dot{M}^{\rho,q}}\|v\|_{L^{p,\infty}\dot{M}^{\rho,q}}.$$

- if $u \in L^{p,\infty}\dot{M}^{\rho,q}$ and $v \in L^2 A \subset L^2 L^\infty$ then $uv \in L^{p,\infty}\dot{M}^{\rho,q}$: we have

$$\|\sigma(D)e^{(t-s)\Delta}\sqrt{-\Delta}(uv)\|_{\dot{M}^{p,q}} \leq C\frac{1}{\sqrt{t-s}}\|u\|_{\dot{M}^{\rho,q}}\|v\|_\infty.$$

  As $\|u\|_{\dot{M}^{\rho,q}}\|v\|_\infty$ belongs to $L^{\frac{2p}{2+p},\infty}$ and $\frac{1}{\sqrt{t}}$ belongs to $L^{2,\infty}$ we find that $\int_0^t \frac{1}{\sqrt{t-s}}\|u\|_{\dot{M}^{p,q}}\|v\|_\infty\, ds$ belongs to $L^{p,\infty}$ and thus that

$$\|B_\sigma(uv)\|_{L^{p,\infty}\dot{M}^{\rho,q}} \leq C\|u\|_{L^{p,\infty}\dot{M}^{\rho,q}}\|v\|_{L^2 A}.$$

- if $u \in L^{p,\infty}\dot{M}^{\rho,q}$ and $v \in Y_{KT,r}$ then $uv \in Y_{KT,r}$: we already know that

$$\|B_\sigma(u,v)\|_{Y_2} \leq C\|u\|_{Y_2}\|v\|_{Y_{KT,r}} \leq C'\|u\|_{L^{p,\infty}\dot{M}^{\rho,q}}\|v\|_{Y_{KT,r}}.$$

  Further, we we use Olsen's inequality [Ols95] that states that pointwise multiplication with $\dot{M}^{\rho,q}$ maps $\dot{H}^{\frac{3}{q}}_{r_1}$ to L^{r_1} if $1 < r_1 < \rho$, thus $v \mapsto (-\Delta)^{-\frac{3}{2q}}(uv)$ is bounded from L^{r_2} to L^{r_2} for $r_2 = \frac{r_1}{r_1-1}$, in particular for $r_2 > 2$ as $\rho > 2$. This shows that

$$\|B_\sigma(u, \mathbb{1}_{S_{T,x}}v)\|_r \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2q}}}\|\mathbb{1}_{S_{T,x}}v\|_r\|u\|_{\dot{M}^{\rho,q}}\, ds.$$

  Thus, we get

$$\|B_\sigma(u, \mathbb{1}_{S_{T,x}}v)\|_{L^r L^r} \leq CT^{-\frac{1}{2}+\frac{5}{2r}}\|v\|_{Y_{KT,r}}\|u\|_{L^{p,\infty}\dot{M}^{\rho,q}}. \qquad \square$$

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