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## DETERMINING LIE ALGEBRA OF TRANSVERSE FOLIATED VECTOR FIELD OF THE EXTENSION OF DENSE LEAF LIE FOLIATION ON A COMPACT CONNECTED MANIFOLD

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### Abstract

In [1, 2], we showed that any extension of a Lie  $\mathcal{G}$ -foliation having dense leaves on a compact connected manifold  $M$  corresponds to a Lie subalgebra  $\mathcal{H}$  of  $\mathcal{G}$ . In this paper, we determine the Lie algebra  $\ell(M, \mathcal{F}_{\mathcal{H}})$  of  $\mathcal{F}_{\mathcal{H}}$ -transverse foliated vector fields of an extension

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corresponding to the subalgebra  $\mathcal{H}$ . Noting by  $\tilde{u}$  the  $\mathcal{F}$ -transverse foliated vector field associated to  $u \in \mathcal{G}$ , we prove that

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{\tilde{u}/u \in \mathcal{H}^{\perp} \text{ and } [u, h] = 0, \text{ for every } h \in \mathcal{H}\}.$$

## 1. Introduction

The article aims to determine the Lie algebra  $\ell(M, \mathcal{F}_{\mathcal{H}})$  of  $\mathcal{F}_{\mathcal{H}}$ -transverse foliated vector fields. We recall that for a dense leaf Riemannian foliation  $\mathcal{F}'$  on a compact connected manifold  $M$ , there is a one-to-one correspondence between Lie subalgebras of  $\ell(M, \mathcal{F}')$  and the extensions of  $\mathcal{F}'$  [2]. If  $K$  is a Lie subalgebra of  $\ell(M, \mathcal{F}')$ , then the differential system  $x \rightarrow T_x \mathcal{F}' \oplus ev_x(K)$  defines an extension of  $\mathcal{F}'$ . So the determination of  $\ell(M, \mathcal{F}')$  is necessary for the determination of extension of  $\mathcal{F}'$ .

We recall the following results which are used in the sequel:

(1) Let  $\sigma : \mathcal{G} \rightarrow \ell(M, \mathcal{F})$  be the reciprocal isomorphism of the 1-form of Fédida  $\omega : \ell(M, \mathcal{F}) \rightarrow \mathcal{G}$ , and  $\tilde{\mathcal{H}} = \sigma(\mathcal{H})$  be the Lie subalgebra of  $\mathcal{F}$ -transverse foliated vector fields obtained from  $\mathcal{H} \subset \mathcal{G}$ . Then, we have (cf. [1, 3])  $(T\mathcal{F})^{\perp} \cap T\mathcal{F}_{\mathcal{H}} = \sigma(\mathcal{H}) = \tilde{\mathcal{H}}$  and  $T\mathcal{F}_{\mathcal{H}} = T\mathcal{F} \oplus \tilde{\mathcal{H}}$ .

(2) For every  $x \in M$ ,  $T_x \mathcal{F}_{\mathcal{H}} = T_x \mathcal{F} \oplus ev_x(\tilde{\mathcal{H}})$ , where  $ev_x(X) = X_x$  for any vector field  $X$ .

(3) We have (cf. [1, 3])  $\chi(\mathcal{F}_{\mathcal{H}}) = \chi(\mathcal{F}) \oplus \tilde{\mathcal{H}}$ , where  $\chi(\mathcal{F}_{\mathcal{H}})$  (respectively,  $\chi(\mathcal{F})$ ) is the Lie algebra of vector fields tangent to  $\mathcal{F}_{\mathcal{H}}$  (respectively,  $\mathcal{F}$ ).

(4) Any Lie  $\mathcal{G}$ -foliation  $\mathcal{F}$  having dense leaves on a compact connected manifold  $M$  admits a locally constant sheaf  $\underline{\mathcal{C}}(M, \mathcal{F})$  of germs of local transverse Killing vector fields which has its typical fiber as the Lie algebra  $\mathcal{G}^-$  opposite to  $\mathcal{G}$  (which is identified with the germs defined by the right invariant vector fields obtained from  $\mathcal{G}$ ) [7].

(5) We have (cf. [3])  $\chi(\mathcal{F}_{\mathcal{H}})_{/U} = \chi(\mathcal{F})_{/U} \oplus (\tilde{\mathcal{H}}^-)_{/U}$ , where  $U$  is an open set of local trivialization of  $\underline{\mathcal{C}}(M, \mathcal{F})$  and  $(\tilde{\mathcal{H}}^-)$  is the Lie subalgebra of right invariant vector fields obtained from subalgebra  $\mathcal{H}$  of  $\mathcal{G}$ .

Using the above, we show that all information concerning  $\ell(M, \mathcal{F}_{\mathcal{H}})$  are contained in Lie algebra  $\mathcal{G}$ . Indeed, it is established that

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{\tilde{u}/u \in \mathcal{H}^\perp \text{ and } [u, h] = 0, \text{ for every } h \in \mathcal{H}\},$$

where  $\tilde{u}$  is the  $\mathcal{F}$ -transverse foliated vector field associated to the vector  $u \in \mathcal{G}$ .

This article is organized as follows: Section 2 is devoted to necessary prerequisites on foliations and on extensions of foliations. A property of the orthogonality of a Lie subalgebra of a Lie algebra of Killing vector fields is given in Section 3. The last Section 4 provides the main result.

## 2. Prerequisites

In this section, we recall necessary definitions and results used in the sequel. We refer to [1, 3, 5] for details.

**Definition 2.1.** A  $q$ -codimensional foliation on a manifold  $M$  is the data consisting of open sets  $(U_i)_{i \in I}$  of  $M$ , of submersions  $f_i : U_i \rightarrow T$ , where  $T$  is a  $q$ -dimensional manifold satisfying for  $U_i \cap U_j \neq \emptyset$ , the existence of a diffeomorphism

$$\gamma_{ij} : f_j(U_i \cap U_j) \subset T \rightarrow f_i(U_i \cap U_j) \subset T$$

for which

$$f_i(x) = (\gamma_{ij} \circ f_j)(x), \text{ for all } x \in U_i \cap U_j.$$

It is said that  $(U_i, f_i, T, \gamma_{ij})_{i \in I}$  is a *foliated cocycle* of foliation  $(M, \mathcal{F})$ .

The manifold  $T$  is called the *transverse manifold* of foliation  $(M, \mathcal{F})$ .  
The quotient

$$\mathcal{V}(\mathcal{F}) = \frac{TM}{T\mathcal{F}}$$

is called the *transverse fiber* of foliation  $\mathcal{F}$ .

The set of all fibers of the submersion  $f_i : U_i \rightarrow T$  is the foliation  $\mathcal{F}$  on  $U_i$  which is denoted by  $(U_i, \mathcal{F})$ .

The foliation  $(U_i, \mathcal{F})$  is said to be *simple* and each connected fiber of  $f_i : U_i \rightarrow T$  as a *plate* of  $\mathcal{F}$  in  $U_i$ .

**Definition 2.2.** If  $T = G$  is a connected Lie group and if the  $\gamma_{ij}$ 's are restrictions of translations to left of  $G$ , then the foliation  $\mathcal{F}$  is said to be a *Lie foliation*. It is also called a *Lie  $G$ -foliation* or a *Lie  $\mathcal{G}$ -foliation*, where  $\mathcal{G} = \text{Lie}(G)$ .

**Proposition 2.3.** Let  $\mathcal{F}$  be a Lie  $G$ -foliation on a manifold  $M$ . Then there exist a homomorphism  $\rho : \pi_1(M) \rightarrow G$  and a submersion  $D : \tilde{M} \rightarrow G$  such that

(i) The lifting  $\tilde{\mathcal{F}}$  of  $\mathcal{F}$  on the universal covering  $\tilde{M}$  of  $M$  is simple and is defined by the submersion  $D$ .

(ii)  $D$  is  $\rho$ -equivariant, i.e.,  $D(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot D(\tilde{x})$ , where  $\gamma \in \pi_1(M)$  and  $\tilde{x} \in \tilde{M}$ .

Another proposition from Fédida with differential forms is as follows:

**Proposition 2.4** [5]. A structure of Lie  $G$ -foliation  $\mathcal{F}$  on a manifold  $M$  is equivalent to the data of a differential 1-form  $\omega$  on  $M$  with values in Lie algebra  $\mathcal{G}$  of  $G$  such that

(1)  $\forall x \in M$ , the linear map  $\omega_x : T_x M \rightarrow \mathcal{G}$ .

(2)  $\omega$  satisfies the Maurer-Cartan equation  $d\omega + \frac{1}{2}[\omega, \omega] = 0$ .

(3) Two 1-forms  $\omega$  and  $\omega'$  satisfying these properties are bound by the relationship  $\omega' = ad_g(\omega)$ , for an element  $g \in G$ .

In what follows,  $\mathcal{G}$  is the Lie algebra of a Lie group  $G$  of dimension  $q$ ,  $\mathcal{H}$  is a Lie subalgebra of  $\mathcal{G}$  of codimension  $q'$ ,  $(e_1, \dots, e_q)$  is a base of  $\mathcal{G}$  such that  $(e_{q'+1}, \dots, e_q)$  is a basis of  $\mathcal{H}$ , and  $\omega = \sum_{i=1}^q \omega^i \otimes e_i$  denotes a 1-form on a manifold  $M$  taking values in  $\mathcal{G}$ .

If  $d\omega + \frac{1}{2}[\omega, \omega] = 0$  and  $\omega^1, \dots, \omega^{q'}$  are linearly independent at any point of  $M$ , then the differential system  $\omega^1 = \dots = \omega^{q'} = 0$  defines a foliation  $\mathcal{F}_{\mathcal{H}}$  of codimension  $q'$ .

**Definition 2.5.** The foliation  $\mathcal{F}_{\mathcal{H}}$  is called a  $\frac{\mathcal{G}}{\mathcal{H}}$ -foliation defined by the 1-form  $\omega$ .

Inspired by the proof of the theorem of lifting of a Lie  $G$ -foliation in [4, 5], we establish the following:

**Proposition 2.6.** Let  $\mathcal{F}$  be a  $\frac{\mathcal{G}}{\mathcal{H}}$ -foliation on a manifold  $M$  defined by a 1-form  $\omega$  and let  $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$  be the lifting foliation of  $\mathcal{F}$  on universal covering  $\pi: \tilde{M} \rightarrow M$ . Then there is a differentiable map  $\mathcal{D}: \tilde{M} \rightarrow G$  transverse for  $\mathcal{F}_{G,H}$  (where  $\mathcal{F}_{G,H}$  is the foliation of the translate to the left of  $H$ ) and a homomorphism  $\rho: \pi_1(M) \rightarrow G$  such that

- (i)  $\mathcal{D}$  is equivariant by  $\rho$ , and
- (ii)  $\pi^*\omega = \mathcal{D}^*\theta$ , i.e.,  $\tilde{\mathcal{F}} = \mathcal{D}^*\mathcal{F}_{G,H}$ .

We say that  $\mathcal{D}$  is a *developing map* on  $\tilde{M}$  of  $\frac{\mathcal{G}}{\mathcal{H}}$ -foliation  $\mathcal{F}$ .

It should be noted that if  $\mathcal{F}'$  is a leaf dense Lie  $\mathcal{G}$ -foliation and  $\mathcal{H}$  is a Lie subalgebra, then  $\mathcal{F}'$  and  $\mathcal{F}'_{\mathcal{H}}$  have the same developing map which is a locally trivial fibration [5].

**Definition 2.7.** Let  $\chi(M)$  be the Lie algebra of vector fields over a manifold  $M$ ,  $X$  be an element of  $\chi(M)$  and  $\chi(\mathcal{F})$  be the Lie algebra of vector fields tangent to the foliation  $\mathcal{F}$ . Then  $X$  is foliated if and only if

$$\forall Y \in \chi(\mathcal{F}), \quad [X, Y] \in \chi(\mathcal{F}).$$

The set of foliated fields is denoted by  $\mathcal{L}(M, \mathcal{F})$ . It is a Lie subalgebra of  $\chi(M)$ . Also,  $\chi(\mathcal{F})$  is an ideal of  $\mathcal{L}(M, \mathcal{F})$ .

We have the following exact sequence:

$$0 \rightarrow \chi(\mathcal{F}) \rightarrow \mathcal{L}(M, \mathcal{F}) \rightarrow \frac{\mathcal{L}(M, \mathcal{F})}{\chi(\mathcal{F})} \rightarrow 0.$$

The quotient  $\frac{\mathcal{L}(M, \mathcal{F})}{\chi(\mathcal{F})} = \ell(M, \mathcal{F})$  is called the *Lie algebra* of the transverse foliated vector fields of  $\mathcal{F}$ .

Note that it is a subset of the set of all the sections of the transverse bundle  $\mathcal{V}(\mathcal{F})$ .

**Proposition 2.8** [7]. *For an element  $X$  of  $\chi(M)$ , the following are equivalent:*

- (i)  $X$  is foliated,
- (ii)  $\mathcal{F}$  is invariant by local diffeomorphism  $\phi_t^X$ , where  $(\phi_t^X)_{|_{t|<\varepsilon}}$  is the local group with a parameter associated with  $X$  in the vicinity of an arbitrary point of  $M$ ,
- (iii) in any simple distinguished open set, the local coordinates  $(x_1, \dots, x_p, y_1, \dots, y_q)$  have the  $q$  last components of  $X$  expressed as a function of the only variables  $y_1, \dots, y_q$ .

**Proposition 2.9** [7]. *In any open simple distinguished  $U$ , a vector field  $X$  is foliated if and only if it is projectable into a vector field on the local quotient manifold  $\bar{U}$ .*

**Definition 2.10.** Let  $f \in \mathcal{A}^0(M, \mathcal{F})$  be a differentiable function on  $M$ . Then we say that  $f$  is *basic* for foliation  $\mathcal{F}$  if and only if

$$\forall X \in \chi(\mathcal{F}), Xf \text{ is identically zero.}$$

All the basic functions for foliation  $\mathcal{F}$  is denoted by  $\mathcal{A}_b^0(M, \mathcal{F})$ .

The following are equivalent [7]:

- (i)  $f$  is basic,
- (ii)  $f$  is a constant on each leaf of  $\mathcal{F}$ ,
- (iii) in any simple distinguished open set of local coordinates  $(x_1, \dots, x_p, y_1, \dots, y_q)$ ,  $f$  is expressed as a function of only variables  $y_1, \dots, y_q$ .

**Remark 2.11.** If  $\mathcal{F}$  is an everywhere dense foliation admitting a leaf, then any basic function is constant on  $M$ , i.e.,  $\mathcal{A}_b^0(M, \mathcal{F}) \simeq \mathbb{R}$ .

**Definition 2.12** [1]. An extension of a  $q$ -codimensional foliation  $(M, \mathcal{F})$  is a  $q'$ -codimensional foliation  $(M, \mathcal{F}')$  such that  $0 < q' < q$ .

We show that if  $(M, \mathcal{F}')$  is a simple extension of a simple foliation  $(M, \mathcal{F})$  and if  $(M, \mathcal{F})$  and  $(M, \mathcal{F}')$  are, respectively, defined by submersions  $\pi : M \rightarrow T$  and  $\pi' : M \rightarrow T'$ , then there exists a submersion  $\theta : T \rightarrow T'$  such that  $\pi' = \theta \circ \pi$ .

We say that this submersion  $\theta$  is a *bond* between the foliation  $(M, \mathcal{F})$  and its extension  $(M, \mathcal{F}')$ .

It is shown in [1] that if the foliation  $(M, \mathcal{F})$  and its extension  $(M, \mathcal{F}')$  are, respectively, defined by the cocycles  $(U_i, f_i, T, \gamma_{ij})_{i \in I}$  and



$(U_i, f'_i, T', \gamma'_{ij})_{i \in I}$ , then we have  $f'_i = \theta_i \circ f_i$  and  $\gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij}$ , where  $\theta_s$  is a bond between the foliation  $(U_s, \mathcal{F})$  and its extension  $(U_s, \mathcal{F}')$ .

**Proposition 2.13.** Let  $\mathcal{F}'$  and  $\mathcal{F}$  be two Riemannian foliations on a manifold  $M$ . Then  $\mathcal{F} \subset \mathcal{F}' \Rightarrow \ell(M, \mathcal{F}') \subset \ell(M, \mathcal{F})$ .

**Proposition 2.14** [1]. If  $\mathcal{F}'$  is an extension of a Lie foliation  $(M, \mathcal{F})$  having dense leaves on a connected compact Manifold  $M$ , then any global vector field  $\mathcal{F}$ -foliated transverse tangent to  $\mathcal{F}'$  at a point is tangent to  $\mathcal{F}'$  at any point of  $M$ .

Below, we give the statement of the biunivocal correspondence theorem between the Lie subalgebras of  $\mathcal{G} = \text{Lie}(G)$  and extensions of a Lie  $\mathcal{G}$ -foliation with dense leaves on a compact connected manifold  $M$  [1, 3].

**Theorem 2.15** [1, 3]. Let  $(M, \mathcal{F})$  be a Lie  $\mathcal{G}$ -foliation with dense leaves on a compact connected manifold  $M$  and let  $\mathcal{G} \simeq \ell(M, \mathcal{F})$  be the Lie algebra of  $\mathcal{F}$ -transverse foliated fields which is the structural Lie algebra of  $\mathcal{F}$ . Then

(1) There is a one-to-one correspondence between Lie-subalgebras of  $\ell(M, \mathcal{F}) \simeq \mathcal{G}$  and the extensions of  $\mathcal{F}$ .

(2) An extension of  $\mathcal{F}$  is a  $\frac{\mathcal{G}}{\mathcal{H}}$  transverse Riemannian foliation with trivial normal fiber bundle, defined by a 1-form valued in  $\frac{\mathcal{G}}{\mathcal{H}}$ .

(3) An extension of  $\mathcal{F}$  is transversely homogeneous (resp. of Lie) if and only if the connected Lie subgroup of  $G$  corresponding to it is a closed subgroup (resp. a normal subgroup) of  $G$ .

Note that  $\omega : \ell(M, \mathcal{F}) \rightarrow \mathcal{G}$  is an isomorphism of Lie algebras, where  $\omega$  is the 1-form of Fédida.

We point out before continuing that the following remark is an already established result which is used to prove the main result.

**Remark 2.16** (Cf. [1, 3]). Let  $\mathcal{F}_{\mathcal{H}}$  be an extension of a Lie  $G$ -foliation with dense leaves on a compact manifold corresponding to the Lie-subalgebra  $\mathcal{H}$  of  $\mathcal{G} = \text{Lie}(G)$ .

(i) Let  $U_{x_0}$  be an open set of  $M$  containing  $x_0 \in M$ ,  $\overline{U}_e$  be an open set of  $G$  containing the neutral element  $e$  of  $G$ ,  $f : U_{x_0} \rightarrow \overline{U}_e$  be a submersion defining the Lie foliation  $\mathcal{F}$  on a distinguished  $U_{x_0}$  with  $f(x_0) = e$  and  $\omega$  be a 1-form of Fédida.

We have for every  $X_x \in T_x U_{x_0}$ ,

$$\omega_x(X_x) = (L_{f(x)*})^{-1} \circ f_{*x}(X_x),$$

i.e., the Darboux differential of  $f$  is  $\omega$ , where  $L_{f(x)}$  is the left translation of  $f(x)$ .

(ii) Let  $\omega : \ell(M, \mathcal{F}) \rightarrow \mathcal{G}$  be the 1-form of Fédida of  $\mathcal{F}$ .

Denote the reciprocal isomorphism of  $\omega$  by  $\sigma : \mathcal{G} \rightarrow \ell(M, \mathcal{F})$ . Then  $\tilde{u} = \sigma(u)$  is the  $\mathcal{F}$ -transverse foliated vector field obtained from  $u \in \mathcal{G}$ .

(iii) We have  $\tilde{u}(x) = ((\omega_x)_{/(T_x \mathcal{F})^\perp})^{-1}(u)$ ,  $\forall x \in M$ .

(iv) If  $X \in \ell(M, \mathcal{F})$ , then  $\omega(X)$  is a constant function on  $M$ .

**Definition 2.17.** Let  $M$  be a differential manifold and  $\underline{X}_a$  be the set of  $(U, X)$  such that  $U$  is an open set of  $M$  containing  $a$  and  $X$  is a vector field on  $U$ .

We define an equivalence relation  $\mathcal{R}$  on the set  $\underline{X}_a$  by:

$(U_1, X_1) \mathcal{R} (U_2, X_2) \Leftrightarrow$  there is an open set  $\Omega \subset U_1 \cap U_2$  containing  $a$  such that for every  $x \in \Omega$ ,  $X_1(x) = X_2(x)$ .

The equivalence class of  $(U_1, X_1)$  is noted  $\overline{(U_1, X_1)}$  and it is called the *germ* of  $X_1$  at the point  $a$ .

In what will follow, we give some results on the central transverse sheaf of the foliation  $(M, \mathcal{F})$ .

**Definition 2.18.** Let  $(M, \mathcal{F})$  be a foliation on  $M$ ,  $U$  be an open set  $\mathcal{F}$ -distinguished of  $M$  and  $\bar{Z} \in \ell(U, \mathcal{F})$  be an  $\mathcal{F}$ -transverse local foliated vector field of  $\ell(U, \mathcal{F})$ . Then  $\bar{Z}$  is said to be a *local central transverse vector field* in  $U$  if for every  $\bar{X} \in \ell(M, \mathcal{F})$ , its restriction to  $U$  switches with  $\bar{Z}$ .

The set of germs at one point  $x \in M$  of the local central transverse fields is denoted by  $\underline{\underline{C}}_x(M, \mathcal{F})$ .

The set

$$\underline{\underline{C}}(M, \mathcal{F}) = \bigcup_{x \in M} \underline{\underline{C}}_x(M, \mathcal{F})$$

equipped with a sheaf topology is called the *central transverse sheaf* of the foliation  $(M, \mathcal{F})$ .

It is noted that if  $\mathcal{F}$  is a dense leaf Lie  $\mathcal{G}$ -foliation on a compact connected manifold  $M$ , then  $\underline{\underline{C}}_x(M, \mathcal{F})$  is a Lie algebra with dimension less than or equal to  $\text{codim}(\mathcal{F})$  [7].

It is also noted that according to Molino [7],  $\underline{\underline{C}}_x(M, \mathcal{F})$  is a Lie algebra isomorphic to  $\mathcal{G}^-$ , where  $\mathcal{G}^-$  is the Lie algebra of right invariant vector fields obtained from  $\mathcal{G} = \text{Lie}(G)$ .

Indeed, if  $\tilde{\mathcal{F}}$  is the lifting of  $\mathcal{F}$  on the universal covering  $\tilde{M}$  of  $M$ , then using Darboux lift  $(\tilde{M}, \tilde{\mathcal{F}})$  of  $(M, \mathcal{F})$ , we have the following diagram:

$$\begin{array}{ccc} \mathcal{D} : \tilde{M} & \rightarrow & G \\ & \pi \downarrow & \\ & M & \end{array}$$

where  $\mathcal{D} : \tilde{M} \rightarrow G$  is the lifting of  $\mathcal{F}$  (which is a submersion) and  $\pi : \tilde{M} \rightarrow M$  is the universal covering of  $M$ .

According to [4], the lifting  $\mathcal{D} : \tilde{M} \rightarrow G$  is a locally trivial fibration of  $\tilde{M}$  on  $G$ .

Now, let  $x$  be an element of  $M$ ,  $\tilde{x}$  be an element of  $\pi^{-1}(x)$ , and  $\tilde{U}$  be an open set of  $\tilde{M}$  containing  $\tilde{x}$  such that  $U = \pi(\tilde{U})$  locally trivializing  $\pi$  and  $\bar{U} = \mathcal{D}(\tilde{U})$  locally trivializing  $\mathcal{D}$ . Then we have the following diagram:

$$\begin{array}{ccc} \mathcal{D} : \tilde{U} & \rightarrow & \bar{U} \\ & \pi \downarrow & \\ & U & \end{array}$$

Let  $\mathcal{G}_{\bar{U}}^-$  be the restriction to  $\bar{U}$  of vector field invariant to the right on  $G$ . The previous diagram shows that  $\mathcal{G}_{\bar{U}}^-$  induces on  $U$  a Lie algebra  $\mathcal{G}_U^-$  of local  $\mathcal{F}$ -foliated transverse vector fields isomorphic to  $\mathcal{G}_{\bar{U}}^-$  switches with the  $\mathcal{F}$ -foliated transverse global vector field of  $M$ . Indeed, every vector field of  $\mathcal{G}_{\bar{U}}^-$  switches with any vector field of  $\mathcal{G} \simeq \ell(M, \mathcal{F})$ . As the foliation  $(\tilde{M}, \tilde{\mathcal{F}})$  is simple [4] and is defined by the submersion  $\mathcal{D}$ ,  $\mathcal{G}_{\bar{U}}^-$  induced on  $\tilde{U}$  is a Lie algebra  $(\tilde{\mathcal{G}}^-)_{\tilde{U}}$  of  $\tilde{\mathcal{F}}$ -foliated transverse local vector fields in  $\tilde{U}$  switching with the  $\tilde{\mathcal{F}}$ -foliated transverse global vector field  $\ell(\tilde{M}, \tilde{\mathcal{F}})$  [7]. Thus, as  $\pi_{/\tilde{U}} : \tilde{U} \rightarrow U$  is a diffeomorphism,

$$\pi_*((\tilde{\mathcal{G}}^-)_{|\tilde{U}}) = \mathcal{G}^-_U = (\ell(M, \mathcal{F}))^-_{|U}$$

is a Lie algebra of  $\mathcal{F}$ -foliated transverse local vector field on  $U = \pi(\tilde{U})$  (isomorphic to  $\mathcal{G}^- = (\ell(M, \mathcal{F}))^-$ ) that switches with the  $\mathcal{F}$ -foliated transverse global vector field  $\ell(M, \mathcal{F})$  of  $M$ . As  $\dim(\mathcal{G}^-_U) = \text{codim}(\mathcal{F})$ , it can be said that  $\underline{\underline{\mathcal{C}}}_x(M, \mathcal{F})$  is a Lie algebra isomorphic to  $\mathcal{G}^-$  because the dimension of  $\underline{\underline{\mathcal{C}}}_x(M, \mathcal{F})$  is less than or equal to  $\text{codim}(\mathcal{F})$ .

Thus, the type fiber of  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$  is  $\mathcal{G}^-$  [7].

It is noted that this construction can be done in the vicinity of each point of  $M$ . It is said that the open set  $U$  *trivializes locally*  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$ .

**Theorem 2.19** [7]. *Let  $(M, \mathcal{F})$  be a Lie  $\mathcal{G}$ -foliation with dense leaves on a compact connected manifold  $M$ . Then  $\mathcal{F}$  admits a locally constant sheaf  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$  of germs of central local transverse vector fields which has its typical fiber as the Lie algebra  $\mathcal{G}^-$  opposite to  $\mathcal{G}$  (which is identified with the germs defined by the vector fields right invariant obtained from  $\mathcal{G}$ ). Also,*

- (i) *The orbits of  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$  are the closures of the leaves of  $(M, \mathcal{F})$ .*
- (ii) *The germs of  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$  switch with all global transverse vector fields.*
- (iii) *For any  $\mathcal{F}$ -transverse bundle-like metric on  $M$ ,  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$  is formed by germs of local transverse Killing vector fields.*

Since  $\mathcal{F}$  is a dense leaf Lie foliation, the locally trivial central transverse sheaf  $\underline{\underline{\mathcal{C}}}(M, \mathcal{F})$  identifies germs defined by the opposite structural Lie algebra  $(\ell(M, \mathcal{F}))^- = (\tilde{\mathcal{G}})^-$  constituted by the right-invariant vector fields

$(\tilde{h})^-$ , where  $h \in \mathcal{G}$ . On an open set  $U$  of local trivialization, the restriction of  $\underline{\mathcal{C}}(M, \mathcal{F})$  at  $U$  is

$$\underline{\mathcal{C}}(M, \mathcal{F})|_U = U \times (\ell(M, \mathcal{F})^-)|_U = U \times (\tilde{\mathcal{G}}^-)|_U.$$

Let  $H$  be a subgroup of  $G$ ,  $\mathcal{H}$  be Lie subalgebra of  $\mathcal{G}$  such that  $\mathcal{H} = \text{Lie}(H)$ , and  $\mathcal{F}_{\mathcal{H}}$  be an extension of  $\mathcal{F}$  corresponding to the subalgebra  $\mathcal{H}$ . As  $H$  is a Lie subgroup of  $G$ , we can consider “sub-sheaf”

$$\underline{\mathcal{C}}_H(U, \mathcal{F}) = U \times (\tilde{\mathcal{H}}^-)$$

of  $\underline{\mathcal{C}}(M, \mathcal{F})$  constituted by the right-invariant vector fields tangent to the Lie subgroup  $H$ .

Considering in addition that the open set  $U$  is  $\mathcal{F}_{\mathcal{H}}$ -distinguished, we can look at the leaves of foliation  $(U, \mathcal{F}_{\mathcal{H}}) = \mathcal{F}_{\mathcal{H},U}$  as the orbits of the “sub-sheaf”

$$\underline{\mathcal{C}}_H(U, \mathcal{F}) = U \times (\tilde{\mathcal{H}}^-)|_U$$

which means that transversally to  $\mathcal{F}$ , the leaves of  $(U, \mathcal{F}_{\mathcal{H}})$  are defined by the orbits of the subalgebra  $(\tilde{\mathcal{H}}^-)|_U$ , the restriction to  $U$  of the Lie subalgebra  $(\tilde{\mathcal{H}}^-)$  right-invariant vector fields tangent to the Lie subgroup  $H$  of  $G$ , where  $\mathcal{G} = \text{Lie}(G)$ . Thus,

$$\chi(\mathcal{F}_{\mathcal{H}})|_U = \chi(\mathcal{F})|_U \oplus (\tilde{\mathcal{H}}^-)|_U. \quad (*)$$

### 3. Orthogonality of a Lie Subalgebra of a Lie Algebra of Killing Vector Fields

We establish a result which is useful for the determination of  $\ell(M, \mathcal{F}_{\mathcal{H}})$ .

**Proposition 3.1.** *Let  $(M, g)$  be a Riemannian manifold of metric  $g$ ,  $\tilde{\mathcal{H}}$  be a Lie algebra of Killing vector fields on  $M$  and  $\tilde{\mathcal{H}}^\perp$  be the set of vector fields orthogonal to all vectors in  $\tilde{\mathcal{H}}$ . Then for every vector field  $\tilde{h} \in \tilde{\mathcal{H}}$  and for any vector field  $\tilde{u} \in \tilde{\mathcal{H}}^\perp$ ,  $[\tilde{h}, \tilde{u}] \in \tilde{\mathcal{H}}^\perp$ .*

**Proof.** Let  $(\tilde{h}, \tilde{h}') \in \tilde{\mathcal{H}} \times \tilde{\mathcal{H}}$  be a vector field of  $\tilde{\mathcal{H}} \times \tilde{\mathcal{H}}$  and  $\tilde{u} \in \tilde{\mathcal{H}}^\perp$  be a vector field orthogonal to all vectors of  $\tilde{\mathcal{H}}$ . Then

$$\begin{aligned} 0 &= \tilde{h}' \cdot g(\tilde{u}, \tilde{h}), \text{ because } \tilde{h} \in \tilde{\mathcal{H}} \text{ and } \tilde{u} \in \tilde{\mathcal{H}}^\perp \\ &= g([\tilde{h}', \tilde{u}], \tilde{h}) + g(\tilde{u}, [\tilde{h}', \tilde{h}]), \text{ because } \tilde{h}' \text{ is a Killing vector field} \\ &= g([\tilde{h}', \tilde{u}], \tilde{h}) + 0, \text{ because } [\tilde{h}', \tilde{h}] \in \tilde{\mathcal{H}} \text{ and } \tilde{u} \in \tilde{\mathcal{H}}^\perp \\ &= g([\tilde{h}', \tilde{u}], \tilde{h}). \end{aligned}$$

The equality

$$g([\tilde{h}', \tilde{u}], \tilde{h}) = 0 \text{ for every } (\tilde{h}, \tilde{h}', \tilde{u}) \in \tilde{\mathcal{H}} \times \tilde{\mathcal{H}} \times \tilde{\mathcal{H}}^\perp$$

implies that

$$[\tilde{h}', \tilde{u}] \in \tilde{\mathcal{H}}^\perp \text{ for every } (\tilde{h}', \tilde{u}) \in \tilde{\mathcal{H}} \times \tilde{\mathcal{H}}^\perp. \quad \square$$

#### 4. Main Result

The following theorem is the main result of this article.

**Theorem 4.1.** *Let  $(M, \mathcal{F})$  be a Lie  $\mathcal{G}$ -foliation with dense leaves on a compact connected manifold  $M$ ,  $\mathcal{F}_{\mathcal{H}}$  be an extension of  $\mathcal{F}$  corresponding to the Lie subalgebra  $\mathcal{H}$  of  $\mathcal{G}$ ,  $\omega$  be a 1-form of Fédida of  $(M, \mathcal{F})$ ,  $\sigma : \mathcal{G} \rightarrow \ell(M, \mathcal{F})$  be the reciprocal isomorphism of  $\omega$  and  $\tilde{u} = \sigma(u)$  be the  $\mathcal{F}$ -transverse foliated field associated to the vector  $u \in \mathcal{G}$ . Then*

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{\tilde{u}/u \in \mathcal{H}^\perp \text{ and } [u, h] = 0 \text{ for every } h \in \mathcal{H}\}.$$

**Proof.** (1) Let us first show that

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset \{u \in \mathcal{H}^{\perp} / [u, h] = 0, \text{ for every } h \in \mathcal{H}\}.$$

Since  $\ell(M, \mathcal{F}_{\mathcal{H}}) \subset \ell(M, \mathcal{F})$  because  $\mathcal{F} \subset \mathcal{F}_{\mathcal{H}}$  (cf. Proposition 2.13), the elements of  $\ell(M, \mathcal{F}_{\mathcal{H}})$  are found in  $\ell(M, \mathcal{F})$ .

Further since  $\mathcal{F}$  is a Lie foliation with dense leaves, for all  $(\tilde{u}, \tilde{v}) \in \ell(M, \mathcal{F}) \times \ell(M, \mathcal{F})$ , the function  $\omega([\tilde{u}, \tilde{v}])$  is a constant function because  $[\tilde{u}, \tilde{v}] = [\widetilde{u}, \widetilde{v}] \in \ell(M, \mathcal{F})$  (cf. Remark 2.16(vi)). Hence, the calculation of  $\omega([\tilde{u}, \tilde{v}])$  is equivalent to calculating of  $\omega([\tilde{u}, \tilde{v}]_{/U})$ , where  $U$  is an open set locally trivializing the locally trivialized sheaf  $\underline{\mathcal{C}}(M, \mathcal{F})$  consisting of germs of local central transverse fields of type fiber  $\mathcal{G}^-$ .

Now, we show that

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset \{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for all } h \in \mathcal{H}\}.$$

Based on the equality (\*):

$$\chi(\mathcal{F}_{\mathcal{H}})_{/U} = \chi(U) \oplus (\tilde{\mathcal{H}})_{/U}^-$$

and hence

$$(T(\mathcal{F}_{\mathcal{H}})_{/U})^{\perp} = ((\tilde{\mathcal{H}})_{/U}^-)^{\perp} = ((\tilde{\mathcal{H}})^{\perp})_{/U}^-.$$

Let  $X \in \chi(\mathcal{F}_{\mathcal{H}})$  be a vector field of  $\chi(\mathcal{F}_{\mathcal{H}})$  and  $Y \in \ell(M, \mathcal{F})$  be a vector field of  $\ell(M, \mathcal{F})$ .

As  $Y \in \ell(M, \mathcal{F}_{\mathcal{H}})$ ,  $[X, Y] \in \chi(\mathcal{F}_{\mathcal{H}})$ . Since  $[X, Y]$  is a global vector field on  $M$ , the fact that  $[X, Y] \in \chi(\mathcal{F}_{\mathcal{H}})$  leads to  $[X, Y]_{/U} \in \chi(\mathcal{F}_{\mathcal{H}})_{/U}$ .

So, as  $[X, Y]_{/U} = [X_{/U}, Y_{/U}]$ ,

$$\chi(\mathcal{F}_{\mathcal{H}})_{/U} = \chi(\mathcal{F})_{/U} \oplus (\tilde{\mathcal{H}})_{/U}^-$$



and

$$(T(\mathcal{F}_{\mathcal{H}})_{/U})^\perp = ((\tilde{\mathcal{H}})_{/U}^-)^\perp = ((\tilde{\mathcal{H}})^\perp)_{/U}^-.$$

Thus there exist  $(X_{\mathcal{F}})_{/U} \in \chi(\mathcal{F}_{\mathcal{H}})_{/U}$ ,  $(\tilde{h})_{/U}^- \in (\tilde{\mathcal{H}})_{/U}^-$  and  $(\tilde{k})_{/U}^- \in ((\tilde{\mathcal{H}})^\perp)_{/U}^-$  such that  $X_{/U} = (X_{\mathcal{F}})_{/U} + (\tilde{h})_{/U}^-$ ,  $Y_{/U} = (\tilde{k})_{/U}^-$  and

$$\begin{aligned} [X_{/U}, Y_{/U}] &= [(X_{\mathcal{F}})_{/U}, (\tilde{k})_{/U}^-] + [(\tilde{h})_{/U}^-, (\tilde{k})_{/U}^-] \\ &= [(X_{\mathcal{F}})_{/U}, (\tilde{k})_{/U}^-] + [(\tilde{h})^-, (\tilde{k})^-]_{/U} \\ &= [(X_{\mathcal{F}})_{/U}, (\tilde{k})_{/U}^-] - [\widetilde{[h, k]}]_{/U}^-. \end{aligned} \quad (**)$$

We note before continuing further that

$$[(X_{\mathcal{F}})_{/U}, (\tilde{k})_{/U}^-] \in \chi(\mathcal{F})_{/U} \subset \chi(\mathcal{F}_{\mathcal{H}})$$

because  $(\tilde{k})_{/U}^-$  is  $\mathcal{F}_{/U}$ -foliated. We also notice that vector fields  $(\tilde{h})_{/U}^-$ ,  $(\tilde{k})_{/U}^-$  are local Killing fields from where according to Proposition 3.1, we have  $[(\tilde{h})_{/U}^-, (\tilde{k})_{/U}^-] \in ((\tilde{\mathcal{H}})^\perp)_{/U}^-$ .

Thus, we have the equalities

$$[(\tilde{h})_{/U}^-, (\tilde{k})_{/U}^-] = [(\tilde{h})^-, (\tilde{k})^-]_{/U} = -[\widetilde{[h, k]}]_{/U}^-$$

showing that

$$[\widetilde{[h, k]}]_{/U}^- \in ((\tilde{\mathcal{H}})^\perp)_{/U}^- \text{ because } [(\tilde{h})_{/U}^-, (\tilde{k})_{/U}^-] \in ((\tilde{\mathcal{H}})^\perp)_{/U}^-.$$

That said, in addition, since  $X \in \chi(\mathcal{F}_{\mathcal{H}})$  and  $Y \in \ell(M, \mathcal{F}_{\mathcal{H}})$ , using (\*)

and (\*\*), we have

$$[(\mathcal{X}_{\mathcal{F}})_{/U}, (\tilde{k})_{/U}^-] - [\widetilde{[h, k]}]_{/U}^- \in \mathcal{X}(\mathcal{F})_{/U} \oplus (\tilde{\mathcal{H}})_{/U}^-.$$

It follows that

$$[\widetilde{h, k}]_{/U}^- \in (\widetilde{\mathcal{H}})_{/U}^-. \quad (***)$$

It results from (\*\*) and (\*\*\*) that  $[\widetilde{h, k}]_{/U} \in (\widetilde{\mathcal{H}})_{/U}^\perp$  and  $[\widetilde{h, k}]_{/U} \in (\widetilde{\mathcal{H}})_{/U}$ .

Thus,

$$[\widetilde{h, k}]_{/U} = 0.$$

As

$$[h, k] = \omega([\widetilde{h, k}]) = \omega([\widetilde{h, k}]_{/U}) = \omega([\widetilde{h, k}]_{/U}) = 0,$$

we have

$$[h, k] = 0.$$

The above shows that

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset \{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}.$$

(2) Now, we show that

$$\{u \in \mathcal{H}^\perp / [u, h] = 0, \text{ for every } h \in \mathcal{H}\} \subset \omega(\ell(M, \mathcal{F}_{\mathcal{H}})).$$

Let  $u \in \mathcal{H}^\perp$  be a vector in  $\mathcal{H}^\perp$  such that  $[u, h] = 0$ , for every  $h \in \mathcal{H}$  and let  $X \in \chi(\mathcal{F}_{\mathcal{H}})$  be a vector field of  $\chi(\mathcal{F}_{\mathcal{H}})$ .

First, note that the vector field  $\mathcal{F}$ -foliated transverse  $\tilde{u}$  associated with  $u$  satisfies  $\tilde{u} \in \tilde{\mathcal{H}}^\perp$  and  $\tilde{u} \perp \chi(\mathcal{F})$ , therefore,

$$\tilde{u} \in (\chi(\mathcal{F}) \oplus \tilde{\mathcal{H}})^\perp = (\chi(\mathcal{F}_{\mathcal{H}}))^\perp.$$

Since  $\mathcal{X}(\mathcal{F}_{\mathcal{H}}) = \mathcal{X}(\mathcal{F}) \oplus \mathcal{H}$ , there are  $X_{\mathcal{F}} \in \mathcal{X}(\mathcal{F})$  and  $\tilde{h} \in \tilde{\mathcal{H}}$  such that  $X = X_{\mathcal{F}} + \tilde{h}$ . As a result

$$\begin{aligned}
[\tilde{u}, X] &= [\tilde{u}, X_{\mathcal{F}}] + [\tilde{u}, \tilde{h}] \\
&= [\tilde{u}, X_{\mathcal{F}}] + [\widetilde{u}, \tilde{h}] \\
&= [\tilde{u}, X_{\mathcal{F}}] \text{ because } [u, h] = 0.
\end{aligned}$$

The fact that  $\tilde{u} \in \ell(M, \mathcal{F})$  implies that  $[\tilde{u}, X] \in \chi(\mathcal{F}) \subset \chi(\mathcal{F}_{\mathcal{H}})$ . Thus, equality  $[\tilde{u}, X] = [\tilde{u}, X_{\mathcal{F}}] \in \mathcal{X}(\mathcal{F}) \subset \mathcal{X}(\mathcal{F}_{\mathcal{H}})$  and thus, the equality  $[\tilde{u}, X] = [\tilde{u}, X_{\mathcal{F}}] \in \mathcal{X}(\mathcal{F}_{\mathcal{H}})$ , and the fact that  $\tilde{u} \in (\mathcal{X}(\mathcal{F}) \oplus \tilde{\mathcal{H}})^{\perp} = (\mathcal{X}(\mathcal{F}_{\mathcal{H}}))^{\perp}$  show that  $\tilde{u} \in \ell(M, \mathcal{F}_{\mathcal{H}})$ , which means that  $u \in \omega(\ell(M, \mathcal{F}_{\mathcal{H}}))$ . Thus,

$$\{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for every } h \in \mathcal{H}\} \subset \omega(\ell(M, \mathcal{F}_{\mathcal{H}})).$$

In conclusion of (1) and (2), it can be said that

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) = \{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for all } h \in \mathcal{H}\} \quad (****)$$

which means that

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{\tilde{u} / u \in \mathcal{H}^{\perp} \text{ and } [u, h] = 0 \text{ for every } h \in \mathcal{H}\},$$

where  $\tilde{u}$  is the  $\mathcal{F}$ -foliated transverse vector field associated with the vector  $u \in \mathcal{G}$ . Since  $\sigma : \mathcal{G} \rightarrow \ell(M, \mathcal{F})$  is the reciprocal isomorphism of  $\omega : \ell(M, \mathcal{F}) \rightarrow \mathcal{G}$ , using (\*\*\*\*), we have

$$\begin{aligned}
\ell(M, \mathcal{F}_{\mathcal{H}}) &= \sigma(\omega(\ell(M, \mathcal{F}_{\mathcal{H}}))) \\
&= \sigma(\{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for any } h \in H\}) \\
&= \{\sigma(u) / u \in \mathcal{H}^{\perp} \text{ and } [u, h] = 0 \text{ for every } h \in \mathcal{H}\} \\
&= \{\tilde{u} / u \in \mathcal{H}^{\perp} \text{ and } [u, h] = 0 \text{ for any } h \in \mathcal{H}\}
\end{aligned}$$

because  $\sigma(u) = \tilde{u}$ . □

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