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Hard congestion limit of the dissipative Aw-Rascle system

N. Chaudhuri[*], L. Navoret[†], C. Perrin[‡], E. Zatorska[§]

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Abstract

In this study, we analyse the famous Aw-Rascle system in which the difference between the actual and the desired velocities (the offset function) is a gradient of a singular function of the density. This leads to a dissipation in the momentum equation which vanishes when the density is zero. The resulting system of PDEs can be used to model traffic or suspension flows in one dimension with the maximal packing constraint taken into account. After proving the global existence of smooth solutions, we study the so-called "*hard congestion limit*", and show the convergence of a subsequence of solutions towards a weak solution of a hybrid free-congested system. This is also illustrated numerically using a numerical scheme proposed for the model studied. In the context of suspension flows, this limit can be seen as the transition from a suspension regime, driven by lubrication forces, towards a granular regime, driven by the contacts between the grains.

Keywords: Aw-Rascle system, suspension flows, maximal packing, weak solutions.

MSC: 35Q35, 35B25, 76T20, 90B20.

1 Introduction

The purpose of this work is to study a singular limit $\varepsilon \to 0$ for the following generalization of the Aw-Rascle [3] and Zhang [36] system

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, & \text{(1a)} \\ \partial_t(\rho_\varepsilon w_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon w_\varepsilon) = 0, & \text{(1b)} \end{cases}$$

on one-dimensional periodic domain $\Omega = \mathbb{T}$. The unknowns of the system are the density ρ_ε and the velocity of motion u_ε. The quantity w_ε denotes the desired velocity of motion and it differs from the actual velocity u_ε by the offset function. This function describes the cost of moving in certain direction and it depends on the congestion of the flow. In our case the offset function is equal to the gradient of $p_\varepsilon = p_\varepsilon(\rho_\varepsilon)$, more precisely:

$$w_\varepsilon = u_\varepsilon + \partial_x p_\varepsilon(\rho_\varepsilon), \tag{2}$$

where

$$p_\varepsilon(\rho_\varepsilon) = \varepsilon \frac{F(\rho_\varepsilon)}{(1-\rho_\varepsilon)^\beta}, \quad \text{with } F(s) \underset{s\to 0}{\to} 0, \quad \beta > 1. \tag{3}$$

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This singular function plays formally the role of a barrier by preventing the density to exceed the maximal fixed threshold $\bar{\rho} \equiv 1$. The motivation to study this model and its asymptotic limit $\varepsilon \to 0$ comes mainly from two areas of applications:

The Aw-Rascle model for traffic [3]**.** The system (1) with scalar offset function, i.e. with $w_\varepsilon = w = u + \rho^\gamma$ for $\gamma > 1$, usually known as the Aw-Rascle system (or sometimes ARZ model) has been derived from the Follow the Leader (FTL) microscopic model of one lane vehicular traffic in [2]. The drawback of that model is that the offset function ρ^γ, does not preserve the maximal density constraint, i.e. solutions satisfy the maximal density constraint $\rho^0 \leq \bar{\rho}$ initially but evolve in finite time to a state which violates this constraint. Moreover, the velocity offset should be very small unless the density ρ is very close to the maximal value, $\bar{\rho} = 1$. Indeed, the drivers do not reduce their speed significantly if the traffic is not congested enough. To incorporate these features the authors of [6] proposed to work with the asymptotic limit ($\varepsilon \to 0$) of (1) with $w_\varepsilon = u_\varepsilon + p_\varepsilon(\rho_\varepsilon)$, and a singular scalar offset function p_ε given by (3). The singular Aw-Rascle system and its asymptotic limit $\varepsilon \to 0$ has been studied numerically in [8], and derived from a FTL approximation in [7]. To be able to use this model in the multi-dimensional setting, where velocity and offset function should have the same physical dimension, a possible way would be to take for the offset a gradient rather than a scalar function (see the recent paper [5] for a proposition and analysis of a multi-d extension of the classical Aw-Rascle model). The use of a gradient can be interpreted as ability of the driver to relax their velocity to an average of the speed of the front and the rear vehicles, weighted according to the local density. So, unlike in the classical Aw-Rascle model, both front and rear interactions would have to be incorporated at the level of the particle model. This seems to be a reasonable assumption for interactions between vehicles that can change lanes and overtake each others.

The lubrication model. Equations (1-3) appear also in modeling of suspension flows, i.e. flows of grains suspended in a viscous fluid. To explain this context better, note that system (1) with w_ε given by (2) can be rewritten (formally) as the pressureless compressible Navier-Stokes equations with density dependent viscosity coefficient

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, & \text{(4a)} \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) - \partial_x\left(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\right) = 0, & \text{(4b)} \end{cases}$$

where

$$\lambda_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon^2 p_\varepsilon'(\rho). \tag{5}$$

In system (4) the singular diffusion coefficient $\lambda_\varepsilon(\rho_\varepsilon)$ represents the repulsive lubrication forces and ε is linked to the viscosity of the interstitial fluid. This system has been derived from a microscopic approximation in [23]. The limit $\varepsilon \to 0$ models the transition between the suspension regime, dominated by the lubrication forces, towards the granular regime dictated by the contacts between the solid grains.

Formally, performing the limit $\varepsilon \to 0$ in (1) (or equivalently in (4)), we expect to get the solution (ρ, u) of the compressible pressureless Euler system, at least when $\rho < 1$. In the region where $\rho = 1$ we expect that the singularity of the offset function (3) (equivalently the singularity of the viscosity coefficient (5)) will prevail giving rise to additional forcing term. The limiting equations then read

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{(6a)} \\ \partial_t(\rho u + \partial_x \pi) + \partial_x\big((\rho u + \partial_x \pi)u\big) = 0 & \text{(6b)} \\ 0 \leq \rho \leq 1, \quad (1-\rho)\pi = 0, \quad \pi \geq 0, & \text{(6c)} \end{cases}$$

where π is the additional unknown obtained as a limit of certain singular function of ρ_ε, that will be specified later on. This limiting system has been derived formally before in the papers of Lefebvre-Lepot and Maury [23] and then the Lagrangian solutions based on an explicit formula using the monotone rearrangement associated to the density were constructed by Perrin and

Westdickenberg [32]. As explained in this latter work, system (6) is related to the constrained Euler equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P = 0, \\ \rho \leq 1, \ (1-\rho)P = 0, \ P \geq 0, \end{cases} \tag{7}$$

studied for instance by Berthelin in [4] or Preux and Maury [26], by splitting the momentum equation (6b) as follows:

$$\begin{cases} \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P = 0, \\ \partial_t \pi + u \partial_x \pi = P. \end{cases}$$

Consequently, solutions to system (6) are particular solutions to (7). In [4] the constrained Euler equations are obtained through the *sticky blocks* approximation, while in [1,9,18], the constrained Euler system is approximated by the compressible Euler equations with singular pressure $P_\varepsilon = P_\varepsilon(\rho_\varepsilon)$.

Similar asymptotic limit passage $\varepsilon \to 0$ was analysed in the multi-dimensional setting by Bresch, Necasova and Perrin [14] in the case of heterogeneous fluids flows described by compressible Birkmann equations with singular pressure and bulk viscosity coefficient. The full compressible Navier-Stokes system with exponentially singular viscosity coefficients and pressure was considered by Perrin in [29, 30]. The asymptotic limit when $\varepsilon \to 0$ in the singular pressure term that leads to the two-phase compressible/incompressible Navier-Stokes equations was considered even earlier in the context of crowd dynamics, see Bresch, Perrin and Zatorska [15], Perrin and Zatorska [33], Degond, Minakowski and Zatorska [20], Degond et al. [19]. Moreover, an interested reader can also consult Lions and Masmoudi [25] and Vauchelet and Zatorska [35] for different approximation of the two-phase system. For an overview of results and discussion of models described by the free/congested two-phase flows we refer to [31].

Our paper contains two main results: the existence of strong solutions to (1) at ε fixed , and the convergence as $\varepsilon \to 0$ of the solutions towards a solution of the limit system (6). Let us be a little bit more precise about the framework and the difficulties associated to system (1).

To study (1) at ε fixed, we take advantage of the reformulation (4) as (pressureless) Navier-Stokes equations with a density dependent viscosity. It is now well-known that, in addition to the classical energy estimate which provides a control of $\sqrt{\lambda_\varepsilon}\partial_x u_\varepsilon$, the BD entropy (BD for Bresch and Desjardins [13]) yields a control of a gradient of some density function. We show in this paper, that this estimate is precisely the key ingredient to ensure the maximal density constraint, namely we will show that $\|\rho_\varepsilon\|_{L^\infty_{t,x}} \leq C_\varepsilon$ for some constant $C_\varepsilon < 1$ tending to 1 as $\varepsilon \to 0$.

As $\varepsilon \to 0$, the main issue which is common to the analysis of Navier-Stokes equations with (degenerate close to vacuum) density dependent viscosities, is the fact that, a priori, we do not have any uniform control in L^p of $\partial_x u_\varepsilon$. Therefore, the identification of the limit of the nonlinear convective term $\rho_\varepsilon u_\varepsilon^2$ is not direct. An important difference and difficulty in comparison with [30] or other studies on Navier-Stokes equations with degenerate viscosities, is that the viscosity vanishes, namely the viscosity $\lambda_\varepsilon(\rho)$ goes to 0 as $\varepsilon \to 0$ for any $\rho < 1$. As a result, control of the gradient of ρ_ε provided by the BD entropy is not uniform with respect to ε. This prevents us from using the Mellet-Vasseur type of estimates [28] to pass to the limit in the convective term. An alternative approach was provided by Boudin in his work [12] devoted to the the vanishing viscosity limit for pressureless gases. He studied system (4) in which the singular viscosity term $\partial_x(\lambda_\varepsilon \partial_x u_\varepsilon)$ is replaced by the non-singular term $\varepsilon \partial_x^2 u_\varepsilon$. The key ingredient of [12] is the concept of *duality solutions* for the limit pressureless gas equations. In this framework, introduced by Bouchut and James in [10, 11], it is particularly important to ensure a *one-sided Lipschitz condition*, or the Oleinik entropy condition, on the velocity field. It is related to the *compressive property* of the dynamics which turns out to be useful also in other "compressive systems" such as aggregation equations (see for instance the works of James and Vauchelet [21, 22]). Note that Berthelin in [4] derived the estimate $\partial_x u \leq 1/t$ for solutions of the constrained Euler equations obtained through the sticky blocks approximation. Building upon the recent developments of Constantin et al. [17]

(see also [16]) around the regular solutions for the Navier-Stokes equations, we derive the ε-uniform one-sided Lipschitz condition $\partial_x u_\varepsilon \leq C$ on the approximate solution. This estimate requires no vacuum at the level of fixed ε. The (ε-dependent) lower bound on the density ρ_ε is derived by imposing a specific behavior of ε-dependent viscosity close to vacuum (see (8)-(9) below).

The outline of this paper is to first show that for ε fixed, $(\rho_\varepsilon, u_\varepsilon)$ is a regular solution to (4). We prove the existence of regular solutions to this system in Section 3 following the approach of Constantin et al. [17]. Then, we derive estimates uniform with respect to ε, including the one-sided Lipschitz condition, in Section 4. We justify the limit passage $\varepsilon \to 0$ in Section 5 and show that the limit is solution to the constrained Euler equations. Finally, we propose in Section 6 a numerical scheme to illustrate the behavior of the solutions to (4) as $\varepsilon \to 0$. For the convenience of the reader we included in the Apendix all details of technical estimates of higher regularity of solutions for ε fixed.

2 Main results

Our first main result concerns the existence of strong solutions to system (1) for specific constitutive laws. With a slight abuse of notation, we re-define the functions p_ε, w_ε, and $\lambda_\varepsilon(\rho_\varepsilon)$, from the introduction.

We consider system (1) with w_ε given by

$$w_\varepsilon = u_\varepsilon + \partial_x p_\varepsilon(\rho_\varepsilon) + \partial_x \varphi_\varepsilon(\rho_\varepsilon), \tag{8}$$

where

$$p_\varepsilon(\rho_\varepsilon) = \varepsilon \frac{\rho_\varepsilon^\gamma}{(1-\rho_\varepsilon)^\beta}, \quad \varphi_\varepsilon(\rho_\varepsilon) = \frac{\varepsilon}{\alpha - 1}\rho^{\alpha-1}, \quad \gamma > 0, \quad \beta > 1, \quad \alpha \in \left(0, \frac{1}{2}\right). \tag{9}$$

It means that we consider $F(\rho) = \rho^\gamma + (\alpha-1)^{-1}(1-\rho)^\beta \rho^{\alpha-1}$ in (3). The approximation of (4) can be written in the following form

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, & \text{(10a)} \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) - \partial_x\left(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\right) = 0, & \text{(10b)} \end{cases}$$

with λ_ε re-defined as

$$\lambda_\varepsilon(\rho) = \rho^2 p_\varepsilon'(\rho) + \rho^2 \varphi_\varepsilon'(\rho) = \rho^2 p_\varepsilon'(\rho) + \varepsilon \rho^\alpha.$$

Given the values of the parameters, α, β, γ, we observe that the behavior of λ_ε is dictated by $\rho^2\varphi_\varepsilon'(\rho)$ close to vacuum, and it is dictated by the singularity of $p_\varepsilon'(\rho)$ close to the maximal density constraint $\bar{\rho} \equiv 1$.

We supplement this system with the following set of initial conditions

$$\rho_\varepsilon|_{t=0} = \rho_\varepsilon^0, \quad u_\varepsilon|_{t=0} = u_\varepsilon^0. \tag{11}$$

The existence of unique global smooth solution to the approximate problem (10) at $\varepsilon > 0$ fixed is stated in the theorem below.

Theorem 2.1. *Let $\varepsilon > 0$ be fixed, and let p_ε, φ_ε be given by (9). Assume that the initial data (11) satisfy $\rho_\varepsilon^0, u_\varepsilon^0 \in H^3(\mathbb{T})$, with $0 < \rho_\varepsilon^0 < 1$.*

Then, for all $T > 0$, there exists a unique global solution $(\rho_\varepsilon, u_\varepsilon)$ to system (10) such that $0 < \rho_\varepsilon(t,x) < 1$ for all $t \in [0,T]$, $x \in \mathbb{T}$, and

$$\rho_\varepsilon \in \mathcal{C}([0,T]; H^3(\mathbb{T})), \qquad u_\varepsilon \in \mathcal{C}([0,T]; H^3(\mathbb{T})) \cap L^2(0,T; H^4(\mathbb{T})). \tag{12}$$

We further show that solutions $\rho_\varepsilon, u_\varepsilon$ from the class (12) satisfy some uniform in ε estimates that allow us to justify the asymptotic limit $\varepsilon \to 0$ in the weak sense. To this end, we rewrite system (10) as follows

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, & \text{(13a)} \\ \partial_t \left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon(\rho_\varepsilon)\right) + \partial_x\big(\left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon(\rho_\varepsilon)\right) u_\varepsilon\big) = 0, & \text{(13b)} \end{cases}$$

where we denoted

$$\pi'_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon p'_\varepsilon(\rho_\varepsilon) + \rho_\varepsilon \varphi'_\varepsilon(\rho_\varepsilon). \tag{14}$$

We show that for $\varepsilon \to 0$ the solutions of (13) converge to an entropy weak solution of (6) with the unknowns $\rho,\ u,\ \pi$ in the sense specified in the following result.

Theorem 2.2. *Let assumptions from Theorem 2.1 be satisfied, and moreover let*

$$0 < \rho^0_\varepsilon(x) \le 1 - C_0 \varepsilon^{\frac{1}{\beta}} \quad \forall\ x \in \mathbb{T}, \tag{15}$$

$$\max(\operatorname{ess\,sup}(\lambda_\varepsilon(\rho^0_\varepsilon)\partial_x u^0_\varepsilon), 0) \le C_1 \varepsilon^{\frac{1}{1-2\alpha}}, \tag{16}$$

$$\|\sqrt{\rho^0_\varepsilon} u^0_\varepsilon\|_{L^2_x} \le C, \tag{17}$$

$$\|\partial_x \pi_\varepsilon(\rho^0_\varepsilon)\|_{L^2_x} + \|\sqrt{\rho^0_\varepsilon}\partial_x \varphi_\varepsilon(\rho^0_\varepsilon)\|_{L^2_x} \le C, \tag{18}$$

$$0 < \underline{M}^0 \le M^0_\varepsilon = \int_{\mathbb{T}} \rho^0_\varepsilon \, dx \le \overline{M}^0 < |\mathbb{T}|, \tag{19}$$

for some $C_0, C_1, C, \underline{M}^0, \overline{M}^0 > 0$ independent of ε. Then:

1. *The solution $(\rho_\varepsilon, u_\varepsilon)$ given by Theorem 2.1 satisfies the following uniform estimates*

$$C\varepsilon^{\frac{2}{1-2\alpha}} \le \rho_\varepsilon(t,x) \le 1 - C\varepsilon^{\frac{1}{\beta-1}} \quad \forall\ (t,x) \in [0,T] \times \mathbb{T}, \tag{20}$$

$$\|\pi_\varepsilon\|_{L^\infty_t H^1_x} \le C, \tag{21}$$

and the one-sided Lipschitz condition

$$\partial_x u_\varepsilon(t,x) \le C \quad \forall\ (t,x) \in]0,T] \times \mathbb{T}, \tag{22}$$

for some $C > 0$, independent of ε. Moreover, the following inequality holds for any $S \in \mathcal{C}^1(\mathbb{R})$ convex:

$$\partial_t(\rho_\varepsilon S(u_\varepsilon)) + \partial_x(\rho_\varepsilon u_\varepsilon S(u_\varepsilon)) - \partial_x(\lambda_\varepsilon(\rho_\varepsilon)\partial_x S(u_\varepsilon)) \le 0, \quad \forall\ (t,x) \in (0,T) \times \mathbb{T}.$$

2. *Let in addition*

$$\begin{aligned} &\rho^0_\varepsilon \to \rho^0 \quad \text{weakly in } L^2(\mathbb{T}), \\ &\rho^0_\varepsilon u^0_\varepsilon \to \rho^0 u^0 \quad \text{weakly in } L^2(\mathbb{T}), \\ &\partial_x \pi_\varepsilon(\rho^0_\varepsilon) \to \partial_x \pi^0 \quad \text{weakly in } L^2(\mathbb{T}). \end{aligned} \tag{23}$$

Then there exists a subsequence $(\rho_\varepsilon, u_\varepsilon, \pi_\varepsilon(\rho_\varepsilon))$ of solutions to (13)-(14) with initial datum $(\rho^0_\varepsilon, u^0_\varepsilon, \pi_\varepsilon(\rho^0_\varepsilon))$, which converges to (ρ, u, π) a weak solution of (6) with initial datum (ρ^0, u^0, π^0).
More precisely we have $0 \le \rho \le 1$ a.e. and the following convergences hold:

$$\begin{aligned} &\rho_\varepsilon \to \rho \quad \text{weakly-* in} \quad L^\infty((0,T) \times \mathbb{T}), \\ &\pi_\varepsilon(\rho_\varepsilon) \to \pi \quad \text{weakly-* in} \quad L^\infty(0,T; H^1(\mathbb{T})), \\ &u_\varepsilon \to u \quad \text{weakly-* in} \quad L^\infty((0,T) \times \mathbb{T}). \end{aligned}$$

Moreover, for the limiting system, the following (entropy) conditions hold:

- *one-sided Lipschitz condition*

$$\partial_x u \leq C \qquad in \quad \mathcal{D}', \tag{24}$$

- *entropy inequality:*

$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) - \partial_x \Lambda_S \leq 0 \qquad in \quad \mathcal{D}', \tag{25}$$

for any convex $S \in \mathcal{C}^1(\mathbb{R})$*, and* $\Lambda_S \in \mathcal{M}((0,T)\times\mathbb{T})$ *satisfying* $|\Lambda_S| \leq \mathrm{Lip}_S|\Lambda|$ *where* $-\Lambda = -\overline{\lambda_\varepsilon(\rho)\partial_x u} \in \mathcal{M}^+((0,T)\times\mathbb{T})$.

Remark 2.3. Let us first comment on the choice of initial datum:

- The assumptions for the initial velocity include the upper bound for $(\lambda_\varepsilon(\rho_\varepsilon^0)\partial_x u_\varepsilon^0)_+$ to deduce the one-sided Lipschitz condition discussed in the introduction. This control amounts to take small values of $(\partial_x u_\varepsilon^0)_+$, except in the low density regions.
- The initial condition (19) implies that the limit system (for $\varepsilon = 0$) cannot be fully congested. This condition is required to control the singular potential $\pi_\varepsilon(\rho_\varepsilon)$ in Section 4.1. Analogous constraint has been imposed to study the asymptotic limit of the compressible Navier-Stokes equations with singular pressure, see [33, 34].

Remark 2.4. The choice of the approximate function φ_ε provides uniform lower bound of the density. It was shown in [28] that, for viscosity coefficient proportional to ρ^α with $\alpha \in [0, 1/2)$, the weak solutions to compressible Navier-Stokes equations have density bounded away from zero by a constant. We use this property at the level of ε being fixed in order to derive the one-sided Lipschitz condition, but we also show that in the limit passage $\varepsilon \to 0$, φ_ε converges to zero strongly. We also remark that the lower bound for the density could be obtained without the use of φ_ε, by assuming more regularity on the initial data uniformly w.r.t. ε, see [27].

Remark 2.5. The main difficulty in studying the $\varepsilon \to 0$ limit passage is to justify convergence of the nonlinear terms. In particular, to pass to the limit in the convective term $\rho_\varepsilon u_\varepsilon$ one needs compactness of the velocity sequence w.r.t. space-variable. For compressible Navier-Stokes equations with constant viscosity coefficient this sort of information is deduced directly from the a-priori estimates. When the viscosity coefficient is degenerate, one can compensate lack of the a-priori estimate by higher regularity of the density via the so-called Bresch-Desjardins estimate. Here, the compactness w.r.t. space follows directly from the one-sided Lipschitz condition, which is possible to deduce because the system is pressureless.

Remark 2.6 (More general singular functions p_ε). The specific form of the function p_ε (which blows up close to 1 like a power law) is used in the paper to exhibit the small scales associated to the singular limit $\varepsilon \to 0$ (see in particular Lemma 3.3). Nevertheless, we expect similar results for more general (monotone) hard-sphere potentials. All the estimates will then depend on the specific balance between the parameter ε and the type of the singularity close to 1 encoded in the function p_ε.

3 Proof of Theorem 2.1

The first step of the proof is to construct local in time unique regular solution to (10). Thanks to the presence of the approximation term φ_ε this can be done following the iterative scheme, described for example in Appendix B of Constantin et al. [17]. The extension of this solution to the global in time solution requires some uniform (with respect to time) estimates that are presented below.

3.1 Basic energy estimates

In this section we assume that $\rho_\varepsilon, u_\varepsilon$ are regular solutions to (10) in the time interval $[0,T]$ in the class (12), and that ρ_ε is non-negative. For such solutions we first obtain straight from the continuity equation (10a) that

$$\|\rho_\varepsilon\|_{L^1_x}(t) = \|\rho^0_\varepsilon\|_{L^1_x}, \tag{26}$$

for all $t \in [0,T]$. Multiplying the momentum equation (10b) by u_ε and integrating by parts we obtain the classical energy estimate.

Lemma 3.1. *For a regular solutions of system* (10)*, we have*

$$\|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^\infty_t L^2_x} + \|\sqrt{\lambda_\varepsilon(\rho_\varepsilon)}\partial_x u_\varepsilon\|^2_{L^2_t L^2_x} \le CE_{0,\varepsilon}, \tag{27}$$

with

$$E_{0,\varepsilon} := \|\sqrt{\rho^0_\varepsilon}u^0_\varepsilon\|^2_{L^2_x}.$$

The next energy estimate is an analogue of Bresch-Desjardins entropy for the compressible Navier-Stokes equations. We first introduce the quantity

$$H'_\varepsilon(\rho_\varepsilon) = p_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon). \tag{28}$$

Lemma 3.2. *For a regular solution of system* (10)*, we have*

$$\|\sqrt{\rho_\varepsilon}w_\varepsilon\|^2_{L^\infty_t L^2_x} + \|H_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^1_x} + \|\sqrt{\rho_\varepsilon}\partial_x\big(p_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon)\big)\|^2_{L^2_t L^2_x} \le CE_{1,\varepsilon}(1+T), \tag{29}$$

with

$$E_{1,\varepsilon} := \|\sqrt{\rho^0_\varepsilon}w^0_\varepsilon\|^2_{L^2_x} + \|H_\varepsilon(\rho^0_\varepsilon)\|_{L^1_x}.$$

Proof. Recall that at the level of regular solutions, the system (10) can be reformulated as (1) with w_ε given by (8). Therefore, multiplying the equation (1b) by w_ε and integrating by parts we easily obtain

$$\|\sqrt{\rho_\varepsilon}w_\varepsilon\|_{L^\infty_t L^2_x} \le \|\sqrt{\rho^0_\varepsilon}w^0_\varepsilon\|_{L^2_x}.$$

Using again formula (8) to substitute for u_ε in (10a) we obtain a porous medium equation for ρ_ε, namely:

$$\partial_t\rho_\varepsilon + \partial_x(\rho_\varepsilon w_\varepsilon) - \partial_x\Big(\rho_\varepsilon\partial_x\big(p_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon)\big)\Big) = 0. \tag{30}$$

Multiplying this equation by $H'_\varepsilon(\rho_\varepsilon)$ and integrating over space and time, we get

$$\begin{aligned}\sup_{t\in(0,T)}\int_{\mathbb{T}} H_\varepsilon(\rho_\varepsilon)dx + \|\sqrt{\rho_\varepsilon}\partial_x\big(H'_\varepsilon(\rho_\varepsilon)\big)\|^2_{L^2_t L^2_x} &\le \int_{\mathbb{T}} H_\varepsilon(\rho^0_\varepsilon)dx + C\|\sqrt{\rho_\varepsilon}w_\varepsilon\|^2_{L^2_t L^2_x}\\ &\le \int_{\mathbb{T}} H_\varepsilon(\rho^0_\varepsilon)dx + CT\|\sqrt{\rho^0_\varepsilon}w^0_\varepsilon\|^2_{L^2_x}.\end{aligned}$$

Note that H_ε consists of two components: one positive but singular $H^s_\varepsilon(\rho) = \int_0^\rho p_\varepsilon(r)\,dr$, and a non-singular negative one $H^n_\varepsilon(\rho) = \int_0^\rho \varphi_\varepsilon(r)\,dr$ since $\alpha < 1$ (see (9)). However we can absorb this negative contribution in $H^s_\varepsilon(\rho)$ as follows. Let us first observe that

$$H^s_\varepsilon(\rho) \underset{\rho\to 1^-}{\sim} \frac{\varepsilon}{(1-\rho)^{\beta-1}},$$

and therefore, for any $C_1, C_2 > 0$ (independent of ε) there exist C'_1, C'_2 (also independent of ε) such that

$$\begin{aligned}&H^s_\varepsilon(\rho_\varepsilon)\mathbf{1}_{\{\rho_\varepsilon \ge 1 - C'_1\varepsilon^{\frac{1}{\beta-1}}\}} \ge C_1\rho^\alpha_\varepsilon\mathbf{1}_{\{\rho_\varepsilon \ge 1 - C'_1\varepsilon^{\frac{1}{\beta-1}}\}};\\ &\int_{\mathbb{T}} H^s_\varepsilon(\rho_\varepsilon)\mathbf{1}_{\{1 - C'_2\varepsilon^{\frac{1}{\beta-1}} \le \rho_\varepsilon \le 1 - C'_1\varepsilon^{\frac{1}{\beta-1}}\}}dx \ge C_2\varepsilon.\end{aligned}$$

Hence, splitting the integral of H_ε^n into two parts, we get

$$\begin{aligned}\int_{\mathbb{T}} |H_\varepsilon^n(\rho_\varepsilon)|dx &= \int_{\mathbb{T}} \frac{\varepsilon}{\alpha(1-\alpha)}\rho_\varepsilon^\alpha \mathbf{1}_{\{\rho_\varepsilon \le 1-C_1'\varepsilon^{\frac{1}{\beta-1}}\}}dx + \int_{\mathbb{T}} \frac{\varepsilon}{\alpha(1-\alpha)}\rho_\varepsilon^\alpha \mathbf{1}_{\{\rho_\varepsilon \ge 1-C_1'\varepsilon^{\frac{1}{\beta-1}}\}}dx \\ &\le \frac{\varepsilon}{\alpha(1-\alpha)}|\mathbb{T}| + \int_{\mathbb{T}} |H_\varepsilon^s(\rho_\varepsilon)|\mathbf{1}_{\{\rho_\varepsilon \ge 1-C_1'\varepsilon^{\frac{1}{\beta-1}}\}}dx \\ &\le \int_{\mathbb{T}} |H_\varepsilon^s(\rho_\varepsilon)|\mathbf{1}_{\{1-C_2'\varepsilon^{\frac{1}{\beta-1}} \le \rho_\varepsilon \le 1-C_1'\varepsilon^{\frac{1}{\beta-1}}\}}dx + \int_{\mathbb{T}} |H_\varepsilon^s(\rho_\varepsilon)|\mathbf{1}_{\{\rho_\varepsilon \ge 1-C_1'\varepsilon^{\frac{1}{\beta-1}}\}}dx\end{aligned}$$

for sufficiently large $C_1', C_2' > 0$, independent of ε.

This gives the result of the lemma. □

3.2 The upper and lower bounds on the density

The purpose of this section is to prove that ρ_ε is uniformly (in time) bounded from above by $\overline{\rho}_\varepsilon$ and from below by $\underline{\rho}_\varepsilon$. This will be done in the two lemmas below.

Lemma 3.3 (Upper bound on the density)**.** *Let $T > 0$, and let $(\rho_\varepsilon, u_\varepsilon)$ be a solution to (10) in the class (12) satisfying the energy estimates (27) and (29). Assume moreover that initially $E_{0,\varepsilon}$ and $E_{1,\varepsilon}$, defined in Lemmas 3.1-3.2, are bounded uniformly with respect to ε. Then there exists a positive constant C independent of ε and T such that*

$$\rho_\varepsilon(t,x) \le 1 - C\left(\frac{\varepsilon}{1+\sqrt{T}}\right)^{\frac{1}{\beta-1}} =: \overline{\rho}_\varepsilon \qquad \forall\, t \in [0,T],\ x \in \mathbb{T}. \tag{31}$$

Proof. First of all, the $L_t^\infty L_x^1$ bound (29) on H_ε^s provides the upper bound

$$\|\rho_\varepsilon\|_{L_{t,x}^\infty} \le 1.$$

One can actually derive a more precise upper bound on the density thanks to the previous estimates. First the Nash inequality provides

$$\|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2} \le C\|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^1}^{2/3}\|\partial_x H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2}^{1/3} + C\|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^1},$$

and we have by Sobolev inequality that

$$\begin{aligned}\|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^\infty} &\le \|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2}^{1/2}\|\partial_x H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2}^{1/2} + \|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2} \\ &\le C\Bigg(\|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^1}^{1/3}\|\partial_x H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2}^{2/3} + \|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^1}^{1/2}\|\partial_x H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2}^{1/2} \\ &\quad + \|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^1}^{2/3}\|\partial_x H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^2}^{1/3} + \|H_\varepsilon^s(\rho_\varepsilon(t,\cdot))\|_{L_x^1}\Bigg).\end{aligned} \tag{32}$$

Now, we observe that the control of the energy (27) together with the estimate (29) provide the bound

$$\|\sqrt{\rho_\varepsilon}p_\varepsilon'(\rho_\varepsilon)\partial_x\rho_\varepsilon\|_{L_t^\infty L_x^2}^2 \le C\big(E_{1,\varepsilon}(1+T) + E_{0,\varepsilon}\big), \tag{33}$$

and on the other hand

$$\|\sqrt{\rho_\varepsilon}p_\varepsilon'(\rho_\varepsilon)\partial_x\rho_\varepsilon\|_{L_t^\infty L_x^2}^2 = \left\|\sqrt{\rho_\varepsilon}\frac{p_\varepsilon'(\rho_\varepsilon)}{p_\varepsilon(\rho_\varepsilon)}\partial_x H_\varepsilon^s(\rho_\varepsilon)\right\|_{L_t^\infty L_x^2}^2.$$

Since

$$p_\varepsilon'(\rho_\varepsilon) = \varepsilon\frac{\rho_\varepsilon^{\gamma-1}}{(1-\rho_\varepsilon)^{\beta+1}}\big[\gamma(1-\rho_\varepsilon) + \beta\rho_\varepsilon\big],$$

we deduce that

$$\begin{aligned}\sqrt{\rho_\varepsilon}\frac{p'_\varepsilon(\rho_\varepsilon)}{p_\varepsilon(\rho_\varepsilon)} &= \frac{\big[\gamma(1-\rho_\varepsilon)+\beta\rho_\varepsilon\big]\rho_\varepsilon^{-1/2}}{1-\rho_\varepsilon}\\ &= \left(\gamma+\beta\frac{\rho_\varepsilon}{1-\rho_\varepsilon}\right)\rho_\varepsilon^{-1/2}\\ &\geq \gamma\|\rho_\varepsilon\|_{L^\infty}^{-1/2}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\partial_x H_\varepsilon^s(\rho_\varepsilon)\|_{L_t^\infty L_x^2} &\leq \gamma^{-1}\|\rho_\varepsilon\|_{L_{t,x}^\infty}^{1/2}\|\sqrt{\rho_\varepsilon}p'_\varepsilon(\rho_\varepsilon)\partial_x\rho_\varepsilon\|_{L_t^\infty L_x^2}\\ &\leq C\|\rho_\varepsilon\|_{L_{t,x}^\infty}^{1/2}\sqrt{\big(E_{1,\varepsilon}(1+T)+E_{0,\varepsilon}\big)}\\ &\leq C\sqrt{\big(E_{1,\varepsilon}(1+T)+E_{0,\varepsilon}\big)},\end{aligned}$$

and so, coming back to (32):

$$\|H_\varepsilon^s(\rho_\varepsilon)\|_{L_{t,x}^\infty} \leq C\sqrt{\big(E_{1,\varepsilon}(1+T)+E_{0,\varepsilon}\big)}. \tag{34}$$

Finally, it is easy to show that this latter bound yields

$$\|\rho_\varepsilon\|_{L_{t,x}^\infty} \leq 1 - C\left(\frac{\varepsilon}{1+\sqrt{T}}\right)^{\frac{1}{\beta-1}},$$

for some constant C independent of ε and T, provided that $E_{1,\varepsilon}$ and $E_{0,\varepsilon}$ are bounded uniformly w.r.t. ε. □

Remark 3.4. This upper bound ensures that ρ_ε never reaches 1 at $\varepsilon > 0$ fixed in finite time, and so $p_\varepsilon(\rho_\varepsilon)$ remains bounded by a constant which depends a-priori on ε. Indeed, we have

$$p_\varepsilon(\rho_\varepsilon(t,x)) \leq \frac{\varepsilon}{(1-\|\rho_\varepsilon\|_\infty)^\beta} \leq C(1+\sqrt{T})^{\frac{\beta}{\beta-1}}\varepsilon^{-\frac{1}{\beta-1}} \quad \forall\, t\in[0,T],\ x\in\mathbb{T}. \tag{35}$$

Note that the same type of estimate holds for the potential $\pi_\varepsilon(\rho_\varepsilon)$ defined in (14).

In the next step we progress with the lower bound on the density. This is the only place where the artificial approximation term φ_ε matters.

Lemma 3.5 (Lower bound on the density)**.** *Let the assumptions of Lemma 3.3 be satisfied. Then for a constant $C > 0$ independent of ε and T we have:*

$$\left\|\frac{1}{\rho_\varepsilon}\right\|_{L_{t,x}^\infty} \leq C\varepsilon^{-\frac{2}{1-2\alpha}}\big(1+T\big)^{\frac{1}{1-2\alpha}} =: \frac{1}{\underline{\rho}_\varepsilon}. \tag{36}$$

Proof. The estimate (29) ensures that $\sqrt{\rho_\varepsilon}\partial_x\varphi_\varepsilon(\rho_\varepsilon)$ is controlled uniformly w.r.t. ε in $L_t^\infty L_x^2$, and so, there exists $C > 0$ independent of ε and T such that:

$$\|\partial_x(\varepsilon\rho_\varepsilon^{\alpha-\frac{1}{2}})\|_{L_t^\infty L_x^2}^2 \leq C\big(E_{1,\varepsilon}(1+T)+E_{0,\varepsilon}\big). \tag{37}$$

On the other hand, we have by conservation of mass

$$\int_{\mathbb{T}}\rho_\varepsilon(t,x)dx = \int_{\mathbb{T}}\rho_\varepsilon^0(x)dx = M_\varepsilon^0,$$

and by hypothesis (19) we ensure that

$$0 < \underline{M}^0 \leq M_\varepsilon^0 < |\mathbb{T}|,$$

for some $\underline{M}^0$ independent of ε. Therefore, for any time t, there exists some $\bar{x}(t) = \bar{x}(t,\varepsilon) \in \mathbb{T}$ such that

$$\rho_\varepsilon(t,\bar{x}(t)) \geq \frac{\underline{M}^0}{|\mathbb{T}|}.$$

For all $t \in (0,T]$ and $x \in \mathbb{T}$ we can write

$$\varepsilon\big(\rho_\varepsilon(t,x)\big)^{\alpha-1/2} - \varepsilon\big(\rho_\varepsilon(t,\bar{x}(t))\big)^{\alpha-1/2} = \int_{\bar{x}(t)}^{x} \varepsilon\partial_x\big(\rho_\varepsilon(t,x)\big)^{\alpha-1/2},$$

so that

$$\begin{aligned}\varepsilon\big(\rho_\varepsilon(t,x)\big)^{\alpha-1/2} &\leq \varepsilon\big(\rho_\varepsilon(t,\bar{x}(t))\big)^{\alpha-1/2} + |x-\bar{x}(t)|^{1/2}\|\partial_x(\varepsilon\rho_\varepsilon^{\alpha-\frac{1}{2}})\|_{L^2_x} \\ &\leq \varepsilon\left(\frac{\underline{M}^0}{|\mathbb{T}|}\right)^{\alpha-1/2} + |\mathbb{T}|^{1/2}\|\partial_x(\varepsilon\rho_\varepsilon^{\alpha-\frac{1}{2}})\|_{L^2_x}.\end{aligned}$$

Hence, $\varepsilon\big(\rho_\varepsilon(t,x)\big)^{\alpha-1/2} \leq C\sqrt{1+T}$ and finally

$$\rho_\varepsilon^{-1}(t,x) \leq C\varepsilon^{-\frac{1}{1/2-\alpha}}\big(1+T\big)^{\frac{1}{1-2\alpha}} \quad \forall\, t>0,\ x\in\mathbb{T}.$$

□

Remark 3.6. For the sake of simplicity we sometimes estimate $\underline{\rho}_\varepsilon$ as follows

$$\underline{\rho}_\varepsilon^{-1} = C\varepsilon^{-\frac{1}{1/2-\alpha}}\big(1+T\big)^{\frac{1}{1-2\alpha}} \leq C(T)\varepsilon^{-2}.$$

3.3 Further regularity estimates

Control of the singular diffusion. In the next step we provide the estimates of

$$V_\varepsilon := \lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon, \tag{38}$$

following the reasoning of Constantin et al. [17] (V_ε corresponds to what is called *active potential* in [17]).

Lemma 3.7. *The variable V_ε satisfies*

$$\partial_t V_\varepsilon + \left(u_\varepsilon + \frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\partial_x\rho_\varepsilon\right)\partial_x V_\varepsilon - \frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x^2 V_\varepsilon = -\frac{\big(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\big)}{\big(\lambda_\varepsilon(\rho_\varepsilon)\big)^2}V_\varepsilon^2. \tag{39}$$

Proof. Dividing (10) by $\rho_\varepsilon > 0$ and using the continuity equation we get

$$\partial_t u_\varepsilon + u_\varepsilon\partial_x u_\varepsilon - \frac{1}{\rho_\varepsilon}\partial_x V_\varepsilon = 0.$$

Next, taking the space derivative we obtain

$$\partial_t\partial_x u_\varepsilon + \partial_x(u_\varepsilon\partial_x u_\varepsilon) - \partial_x\left(\frac{1}{\rho_\varepsilon}\partial_x V_\varepsilon\right) = 0.$$

On the other hand, multiplying (10a) by $\lambda_\varepsilon'(\rho_\varepsilon)$, we get

$$\partial_t\lambda_\varepsilon(\rho_\varepsilon) + \partial_x\lambda_\varepsilon(\rho_\varepsilon)u_\varepsilon + \lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon\partial_x u_\varepsilon = 0.$$

Hence

$$
\begin{aligned}
\partial_t V_\varepsilon =& \lambda_\varepsilon(\rho_\varepsilon)\partial_t\partial_x u_\varepsilon + \partial_x u_\varepsilon \partial_t \lambda_\varepsilon(\rho_\varepsilon) \\
=& -\lambda_\varepsilon(\rho_\varepsilon)\partial_x\left(u_\varepsilon\partial_x u_\varepsilon\right) + \lambda_\varepsilon(\rho_\varepsilon)\partial_x\left(\frac{1}{\rho_\varepsilon}\partial_x V_\varepsilon\right) - \partial_x\lambda_\varepsilon(\rho_\varepsilon)u_\varepsilon\partial_x u_\varepsilon - \lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon(\partial_x u_\varepsilon)^2 \\
=& -\partial_x(u_\varepsilon V_\varepsilon) + \partial_x\left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x V_\varepsilon\right) - \lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon\left(\frac{V_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2 - \frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x V_\varepsilon \\
=& -u_\varepsilon\partial_x V_\varepsilon - \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}V_\varepsilon^2 + \partial_x\left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x V_\varepsilon\right) - \lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon\left(\frac{V_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2 - \frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x V_\varepsilon,
\end{aligned}
\tag{40}
$$

from which (39) follows. □

Lemma 3.8. *We have*

$$
\begin{aligned}
&\|V_\varepsilon\|^2_{L^\infty_t L^2_x} + \varepsilon\|\partial_x V_\varepsilon\|^2_{L^2_t L^2_x} \\
&\le C\|V_\varepsilon(0)\|^2_{L^2_x}\exp\left[T\Big(\varepsilon^{-3}\|R\|^4_{L^\infty_t L^2_x} + \frac{\varepsilon^{-1}}{\sqrt{\underline{\rho}_\varepsilon}}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^\infty_t L^2_x}\Big) + \frac{\varepsilon^{-3}}{\underline{\rho}_\varepsilon^{\frac{8}{3}\alpha}(1-\overline{\rho}_\varepsilon)^{\frac{4}{3}(\beta+2)}}\|V_\varepsilon\|^2_{L^2_t L^2_x}\right]
\end{aligned}
\tag{41}
$$

where

$$
\|R\|_{L^\infty_t L^2_x} := \left\|\frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\right\|_{L^\infty_t L^2_x} \le \frac{C}{\sqrt{\underline{\rho}_\varepsilon}(1-\overline{\rho}_\varepsilon)}(1+\sqrt{T}).
$$

Proof. We multiply (39) by V_ε and integrate w.r.t. space to get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}}\frac{V_\varepsilon^2}{2}\,dx + \int_{\mathbb{T}}\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}(\partial_x V_\varepsilon)^2\,dx \\
&= -\int_{\mathbb{T}}\partial_x\left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\right)V_\varepsilon\partial_x V_\varepsilon\,dx - \int_{\mathbb{T}}\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\partial_x\rho_\varepsilon\partial_x V_\varepsilon V_\varepsilon\,dx - \int_{\mathbb{T}}u_\varepsilon\partial_x V_\varepsilon V_\varepsilon\,dx \\
&\quad - \int_{\mathbb{T}}\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}V_\varepsilon^3\,dx \\
&= -\int_{\mathbb{T}}\frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x\rho_\varepsilon\partial_x V_\varepsilon V_\varepsilon\,dx - \int_{\mathbb{T}}u_\varepsilon\partial_x V_\varepsilon V_\varepsilon\,dx - \int_{\mathbb{T}}\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}V_\varepsilon^3\,dx \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Observe that

$$
\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon} = \rho_\varepsilon p_\varepsilon'(\rho_\varepsilon) + \varepsilon\rho_\varepsilon^{\alpha-1} \ge \varepsilon. \tag{42}
$$

We will use this estimate to control each of the terms I_1, I_2, I_3 separately below.
Control of I_1. We have

$$
|I_1| \le \int_{\mathbb{T}}\underbrace{\left|\frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\right|}_{:=R}|\partial_x V_\varepsilon||V_\varepsilon|\,dx \le \|R\|_{L^2_x}\|\partial_x V_\varepsilon\|_{L^2_x}\|V_\varepsilon\|_{L^\infty_x}.
$$

Let us estimate $\|R\|_{L^\infty_t L^2_x}$. To this purpose we split $\lambda_\varepsilon(\rho_\varepsilon)$ to the singular and the non-singular parts denoted by

$$
\lambda_\varepsilon^s = \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon), \qquad \lambda_\varepsilon^n = \varepsilon\rho_\varepsilon^\alpha.
$$

For the singular part we have

$$\begin{aligned}\left\|\frac{\partial_x\lambda_\varepsilon^s}{\rho_\varepsilon}\right\|_{L_t^\infty L_x^2} &= \left\|\frac{2\rho_\varepsilon p_\varepsilon'(\rho_\varepsilon)+\rho_\varepsilon^2 p_\varepsilon''(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x\rho_\varepsilon\right\|_{L_t^\infty L_x^2}\\ &\le \left\|\frac{2\sqrt{\rho_\varepsilon}+\rho_\varepsilon^{3/2}p_\varepsilon''(\rho_\varepsilon)/p_\varepsilon'(\rho_\varepsilon)}{\rho_\varepsilon}\right\|_{L^\infty}\|\sqrt{\rho_\varepsilon}p_\varepsilon'(\rho_\varepsilon)\partial_x\rho_\varepsilon\|_{L_t^\infty L_x^2}\\ &\le C\left(\frac{1}{\sqrt{\underline{\rho}_\varepsilon}}+\frac{1}{\sqrt{\underline{\rho}_\varepsilon}(1-\overline{\rho}_\varepsilon)}\right)\|\sqrt{\rho_\varepsilon}p_\varepsilon'(\rho_\varepsilon)\partial_x\rho_\varepsilon\|_{L_t^\infty L_x^2},\end{aligned}$$

and for the non-singular part

$$\begin{aligned}\left\|\frac{\partial_x\lambda_\varepsilon^n}{\rho_\varepsilon}\right\|_{L_t^\infty L_x^2} &= \left\|\varepsilon\alpha\rho_\varepsilon^{\alpha-2}\partial_x\rho_\varepsilon\right\|_{L_t^\infty L_x^2}\\ &\le C\underline{\rho}_\varepsilon^{-1/2}\left\|\varepsilon\rho_\varepsilon^{\alpha-3/2}\partial_x\rho_\varepsilon\right\|_{L_t^\infty L_x^2}.\end{aligned}$$

Therefore, using estimate (29), we get

$$\|R\|_{L_t^\infty L_x^2}\le C\frac{1+\sqrt{T}}{\sqrt{\underline{\rho}_\varepsilon}(1-\overline{\rho}_\varepsilon)}. \tag{43}$$

Coming back to the control of I_1, we have:

$$\begin{aligned}|I_1| &\le \|V_\varepsilon\|_{L_x^\infty}\|\partial_x V_\varepsilon\|_{L_x^2}\|R\|_{L_x^2}\\ &\le C\|\partial_x V_\varepsilon\|_{L_x^2}\Big(\|\partial_x V_\varepsilon\|^{1/2}\|V_\varepsilon\|_{L_x^2}^{1/2}+\|V_\varepsilon\|_{L_x^2}\Big)\|R\|_{L_x^2}\\ &\le \frac{\bar{C}\varepsilon}{4}\|\partial_x V_\varepsilon\|_{L_x^2}^2+C\Big(\varepsilon^{-3}\|V_\varepsilon\|_{L_x^2}^2\|R\|_{L_x^2}^4+\varepsilon^{-1}\|V_\varepsilon\|_{L_x^2}^2\|R\|_{L_x^2}^2\Big).\end{aligned} \tag{44}$$

Control of I_2. Using the Hölder and the Young inequalities we obtain

$$\begin{aligned}|I_2| &\le \int_{\mathbb{T}}|u_\varepsilon||\partial_x V_\varepsilon||V_\varepsilon|\,dx\\ &\le \|\partial_x V_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{L_x^2}\|u_\varepsilon\|_{L_x^\infty}\\ &\le \frac{\bar{C}\varepsilon}{4}\|\partial_x V_\varepsilon\|_{L_x^2}^2+C\varepsilon^{-1}\|V_\varepsilon\|_{L_x^2}^2\|u_\varepsilon\|_{H_x^1}^2,\end{aligned}$$

with

$$\begin{aligned}\|u_\varepsilon\|_{H_x^1}^2\le\|u_\varepsilon\|_{L_x^2}^2+\left\|\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right\|_{L^\infty}^2\|V_\varepsilon\|_{L_x^2}^2 &\le \underline{\rho}_\varepsilon^{-1/2}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_x^2}^2+\varepsilon^{-2}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L_x^2}^2\\ &\le \underline{\rho}_\varepsilon^{-1/2}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_x^2}^2+\varepsilon^{-2}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L_x^2}^2,\end{aligned}$$

hence

$$|I_2|\le\frac{\bar{C}\varepsilon}{4}\|\partial_x V_\varepsilon\|_{L_x^2}^2+C\varepsilon^{-1}\underline{\rho}_\varepsilon^{-1/2}\|V_\varepsilon\|_{L_x^2}^2\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_x^2}^2+C\varepsilon^{-3}\underline{\rho}_\varepsilon^{-\alpha}\|V_\varepsilon\|_{L_x^2}^4. \tag{45}$$

Control of I_3. We apply the Nash inequality and the Hölder inequality to get

$$
\begin{aligned}
|I_3| &\le \int_{\mathbb{T}} \left| \frac{\big(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\big)}{\big(\lambda_\varepsilon(\rho_\varepsilon)\big)^2} \right| |V_\varepsilon|^3 \, dx \\
&\le \left\| \frac{\big(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\big)}{\big(\lambda_\varepsilon(\rho_\varepsilon)\big)^2} \right\|_{L^\infty} \big(\|\partial_x V_\varepsilon\|_{L^2_x}^{1/2} \|V_\varepsilon\|_{L^2_x}^{5/2} + \|V_\varepsilon\|_{L^2_x}^3 \big) \\
&\le C \frac{\varepsilon^{-1}}{\underline{\rho}_\varepsilon^{2\alpha}(1-\overline{\rho}_\varepsilon)^{\beta+2}} \big(\|\partial_x V_\varepsilon\|_{L^2_x}^{1/2} \|V_\varepsilon\|_{L^2_x}^{5/2} + \|V_\varepsilon\|_{L^2_x}^3 \big) \\
&\le \frac{\bar{C}\varepsilon}{4} \|\partial_x V_\varepsilon\|_{L^2_x}^2 + C \frac{\varepsilon^{-4/3-1/3}}{\big(\underline{\rho}_\varepsilon^{2\alpha}(1-\overline{\rho}_\varepsilon)^{\beta+2}\big)^{4/3}} \|V_\varepsilon\|_{L^2_x}^{5/2\times 4/3} + C \frac{\varepsilon^{-1}}{\underline{\rho}_\varepsilon^{2\alpha}(1-\overline{\rho}_\varepsilon)^{\beta+2}} \|V_\varepsilon\|_{L^2_x}^3 .
\end{aligned}
$$

Putting everything together, we get

$$
\begin{aligned}
&\frac{d}{dt}\|V_\varepsilon\|_{L^2_x}^2 + C\varepsilon \|\partial_x V_\varepsilon\|_{L^2_x}^2 \\
&\le C \Bigg(\varepsilon^{-3} \|V_\varepsilon\|_{L^2_x}^2 \|R\|_{L^2_x}^4 + \varepsilon^{-1} \|V_\varepsilon\|_{L^2_x}^2 \|R\|_{L^2_x}^2 + \varepsilon^{-1} \underline{\rho}_\varepsilon^{-1/2} \|V_\varepsilon\|_{L^2_x}^2 \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^2_x}^2 \\
&\qquad + \varepsilon^{-3} \underline{\rho}_\varepsilon^{-\alpha} \|V_\varepsilon\|_{L^2_x}^4 + \frac{\varepsilon^{-5/3}}{\big(\underline{\rho}_\varepsilon^{2\alpha}(1-\overline{\rho}_\varepsilon)^{\beta+2}\big)^{4/3}} \|V_\varepsilon\|_{L^2_x}^{10/3} + \frac{\varepsilon^{-1}}{\underline{\rho}_\varepsilon^{2\alpha}(1-\overline{\rho}_\varepsilon)^{\beta+2}} \|V_\varepsilon\|_{L^2_x}^3 \Bigg) \\
&\le C \|V_\varepsilon\|_{L^2_x}^2 \left(\varepsilon^{-3} \|R\|_{L^2_x}^4 + \varepsilon^{-1} \underline{\rho}_\varepsilon^{-1/2} \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^2_x}^2 + \frac{\varepsilon^{-3}}{\big(\underline{\rho}_\varepsilon^{2\alpha}(1-\overline{\rho}_\varepsilon)^{\beta+2}\big)^{4/3}} \|V_\varepsilon\|_{L^2_x}^2 \right),
\end{aligned}
$$

and (41) follows by Gronwall's inequality. □

From the above result we infer the estimates on $\partial_x u_\varepsilon$ (which are not uniform with respect to ε).

Lemma 3.9. *We have:*

(i) $\partial_x u_\varepsilon$ is bounded in $L^2_t L^2_x$ with the estimate

$$
\|\partial_x u_\varepsilon\|_{L^2_t L^2_x} \le \left\| \frac{1}{\sqrt{\lambda_\varepsilon(\rho_\varepsilon)}} \right\|_{L^\infty_{t,x}} \|\sqrt{\lambda_\varepsilon(\rho_\varepsilon)} \partial_x u_\varepsilon\|_{L^2_t L^2_x} \le C \varepsilon^{-1/2} \underline{\rho}_\varepsilon^{-\alpha/2} E_{0,\varepsilon}^{1/2}. \tag{46}
$$

(ii) $\partial_x u_\varepsilon$ is bounded in $L^\infty_t L^2_x$ with the estimate

$$
\|\partial_x u_\varepsilon\|_{L^\infty_t L^2_x} \le \left\| \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \right\|_{L^\infty_{t,x}} \|V_\varepsilon\|_{L^\infty_t L^2_x}. \tag{47}
$$

(iii) $\partial_x^2 u_\varepsilon$ is bounded in $L^2_t L^2_x$ with the estimate

$$
\begin{aligned}
\|\partial_x^2 u_\varepsilon\|_{L^2_t L^2_x} \le & C \left\| \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \right\|_{L^\infty_{t,x}} \|\partial_x V_\varepsilon\|_{L^2_t L^2_x} \\
& + C \left\| \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \right\|_{L^\infty_{t,x}}^2 \|\partial_x \lambda_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^2_x} \big(\|\partial_x \lambda_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^2_x} + 1 \big) \|\partial_x u_\varepsilon\|_{L^2_t L^2_x}.
\end{aligned} \tag{48}
$$

Proof. Part (i) follows immediately from (27). Part (ii) results from the definition of V_ε and the control of V_ε in $L^\infty_t L^2_x$ is provided by estimate (41). And finally, part (iii) follows by differentiating V_ε, we get $\partial_x V_\varepsilon = \partial_x \lambda_\varepsilon \partial_x u_\varepsilon + \lambda_\varepsilon \partial_x^2 u_\varepsilon$, so that

$$\begin{aligned}
&\|\partial_x^2 u_\varepsilon\|_{L^2_t L^2_x} \\
&\leq \left\| \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \right\|_{L^\infty_{t,x}} \left(\|\partial_x V_\varepsilon\|_{L^2_t L^2_x} + \|\partial_x \lambda_\varepsilon\|_{L^\infty_t L^2_x} \|\partial_x u_\varepsilon\|_{L^2_t L^\infty_x} \right) \\
&\leq C \left\| \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \right\|_{L^\infty_{t,x}} \left(\|\partial_x V_\varepsilon\|_{L^2_t L^2_x} + \|\partial_x \lambda_\varepsilon\|_{L^\infty_t L^2_x} \left(\|\partial_x u_\varepsilon\|^{1/2}_{L^2_t L^2_x} \|\partial_x^2 u_\varepsilon\|^{1/2}_{L^2_t L^2_x} + \|\partial_x u_\varepsilon\|_{L^2_t L^2_x} \right) \right).
\end{aligned}$$

□

The higher order regularity estimates. We begin with a formal derivation of higher regularity estimates, for which we assume that the functions $\rho_\varepsilon, u_\varepsilon$ are sufficiently smooth.

Lemma 3.10. *Let $m \geq 2$ and let $\rho_\varepsilon \in C^1([0,T]; C^m(\mathbb{T}))$, $u_\varepsilon \in C([0,T]; C^{m+1}(\mathbb{T}))$ satisfy (10a), then we have*

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^m \rho_\varepsilon\|^2_{L^2_x} \leq C \left(\|\partial_x^m u_\varepsilon\|_{L^2_x} \|\partial_x^m \rho_\varepsilon\|^2_{L^2_x} + \|\rho_\varepsilon\|_{L^\infty_x} \|\partial_x^m \rho_\varepsilon\|_{L^2_x} \|\partial_x^{m+1} u_\varepsilon\|_{L^2_x} \right). \tag{49}$$

Moreover, for $l \geq 1$ and $\rho_\varepsilon \in C([0,T]; C^{l+1}(\mathbb{T}^d))$, $u_\varepsilon \in C([0,T]; C^l(\mathbb{T}^d))$, $V_\varepsilon \in C^1([0,T]; C^{l+2}(\mathbb{T}^d))$ satisfying (39), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_x^l V_\varepsilon|^2 \, dx = &- \int_{\mathbb{T}} \partial_x^l \left(\left(u_\varepsilon + \frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2} \partial_x \rho_\varepsilon \right) \partial_x V_\varepsilon \right) \partial_x^l V_\varepsilon \, dx \\
&+ \int_{\mathbb{T}} \partial_x^l \left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon} \partial_x^2 V_\varepsilon \right) \partial_x^l V_\varepsilon \, dx - \int_{\mathbb{T}} \partial_x^l \left(\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2} V_\varepsilon^2 \right) \partial_x^l V_\varepsilon \, dx.
\end{aligned} \tag{50}$$

Proof. First, differentiating (10a) m times with respect to x and multiplying by $\partial_x^m \rho_\varepsilon$, we deduce

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_x^m \rho_\varepsilon|^2 \, dx = - \int_{\mathbb{T}} \partial_x^m \rho_\varepsilon \left(\partial_x^m (u_\varepsilon \partial_x \rho_\varepsilon) + \partial_x^m (\rho_\varepsilon \partial_x u_\varepsilon) \right) \, dx.$$

We introduce the commutator notation $[D, f]g = D(fg) - fDg$, where D is any differential operator and f, g are sufficiently smooth functions. Using this notation we rewrite the above equation as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_x^m \rho_\varepsilon|^2 \, dx = &- \int_{\mathbb{T}} \partial_x^m \rho_\varepsilon \left([\partial_x^m, u_\varepsilon] \partial_x \rho_\varepsilon \right) \, dx - \int_{\mathbb{T}} \partial_x^m \rho_\varepsilon u_\varepsilon \partial_x^{m+1} \rho_\varepsilon \, dx \\
&- \int_{\mathbb{T}} \partial_x^m \rho_\varepsilon \left([\partial_x^m, \rho_\varepsilon] \partial_x u_\varepsilon \right) \, dx - \int_{\mathbb{T}} \partial_x^m \rho_\varepsilon \rho_\varepsilon \partial_x^{m+1} u_\varepsilon \, dx.
\end{aligned} \tag{51}$$

On the periodic domain, for any $g \in W^{n,2}(\mathbb{T})$ with $n \geq 3$, we have

$$\|\partial_x g\|_{L^\infty_x} \leq C \|\partial_x^2 g\|_{L^2_x} \leq C(n) \|\partial_x^n g\|_{L^2_x}, \tag{52}$$

see f.i. [17, Page 12]. Inequality (52) and the Kato-Ponce theory yield

$$\|[\partial_x^m, u_\varepsilon] \partial_x \rho_\varepsilon\|_{L^2_x} \leq C \left(\|\partial_x^m \rho_\varepsilon\|_{L^2_x} \|\partial_x u_\varepsilon\|_{L^\infty_x} + \|\partial_x^m u_\varepsilon\|_{L^2_x} \|\partial_x \rho_\varepsilon\|_{L^\infty_x} \right) \leq C(\|\partial_x^m u_\varepsilon\|_{L^2_x} \|\partial_x^m \rho_\varepsilon\|_{L^2_x}),$$

and similarly

$$\|[\partial_x^m, \rho_\varepsilon] \partial_x u_\varepsilon\|_{L^2_x} \leq C \left(\|\partial_x^m u_\varepsilon\|_{L^2_x} \|\partial_x \rho_\varepsilon\|_{L^\infty_x} + \|\partial_x^m \rho_\varepsilon\|_{L^2_x} \|\partial_x u_\varepsilon\|_{L^\infty_x} \right) \leq C(\|\partial_x^m u_\varepsilon\|_{L^2_x} \|\partial_x^m \rho_\varepsilon\|_{L^2_x}).$$

Going back to (51), we notice that

$$-\int_{\mathbb{T}} \partial_x^m \rho_\varepsilon u_\varepsilon \partial_x^{m+1} \rho_\varepsilon \, dx = \frac{1}{2} \int_{\mathbb{T}} \partial_x u_\varepsilon |\partial_x^m \rho_\varepsilon|^2 \, dx.$$

This implies

$$\int_{\mathbb{T}} \partial_x u_\varepsilon |\partial_x^m \rho_\varepsilon|^2 \, dx \leq C \|\partial_x u_\varepsilon\|_{L_x^\infty} \|\partial_x^m \rho_\varepsilon\|_{L_x^2}^2.$$

Combining all these estimates and plugging into (51) we finally obtain (49). Next, differentiating (39) l times with respect to space and multiplying by $\partial_x^l V_\varepsilon$ we obtain (50). □

We already have the estimate of V_ε in $L_t^\infty L_x^2$ and $\partial_x V_\varepsilon$ in $L_t^2 L_x^2$ from (41). Also, Lemma 3.3 and estimate (37) give

$$\|\partial_x \rho_\varepsilon\|_{L_t^\infty L_x^2} = \left(\alpha - \frac{1}{2}\right)^{-1} \frac{1}{\varepsilon} \left\| \rho_\varepsilon^{\frac{3}{2}-\alpha} \partial_x (\varepsilon \rho_\varepsilon^{\alpha - \frac{1}{2}}) \right\|_{L_t^\infty L_x^2} \leq \frac{1}{\varepsilon} C(\bar{\rho}_\varepsilon, \alpha, T, E_0, E_1). \tag{53}$$

Using the formulas from Lemma 3.10, we want to derive the estimates for two further orders of regularity:

- First, the case corresponding to $m = 2$ and $l = 1$ in (49) and (50), respectively.
- Then, the case corresponding to $m = 3$ and $l = 2$ in (49) and (50), respectively.

The obtained result is summarised in the statement below.

Proposition 3.11. *Let $\varepsilon > 0$ be fixed and $(\rho_\varepsilon, u_\varepsilon)$ be a regular solution of system (10) with*

$$E_{3,\varepsilon} = E_{1,\varepsilon} + \|\partial_x^2 u_\varepsilon^0\|_{L_x^2} + \|\partial_x^2 \rho_\varepsilon^0\|_{L_x^2} + \|\partial_x^3 u_\varepsilon^0\|_{L_x^2} + \|\partial_x^3 \rho_\varepsilon^0\|_{L_x^2}. \tag{54}$$

Then we have

$$\begin{aligned} &\|\partial_x^3 \rho_\varepsilon\|_{L_t^\infty L_x^2} + \|\partial_x^2 V_\varepsilon\|_{L_t^\infty L_x^2} + \|\partial_x^3 V_\varepsilon\|_{L_t^2 L_x^2} + \|\partial_x^3 u_\varepsilon\|_{L_t^\infty L_x^2} + \|\partial_x^4 u_\varepsilon\|_{L_t^2 L_x^2} \\ &\leq C(\varepsilon, E_{3,\varepsilon}, \underline{\rho}_\varepsilon, \overline{\rho}_\varepsilon, T). \end{aligned} \tag{55}$$

Proposition 3.11 is an analogue of [17, Lemma 4.3] and it is based on the Lemma 7.1 which in turns is an analogue of [17, Lemma 4.2]. For completeness we include the proofs of both results, see the Appendix 7.

4 Estimates uniform in ε

In this section we first recall and derive additional uniform w.r.t. ε estimates which eventually will allow us to let $\varepsilon \to 0$. The two key estimates of this section are: the one-sided Lipschitz condition on u_ε, and the control of the singular potential π_ε.

4.1 Estimates based on the energy bounds

First note that the Lemmas 3.1 and 3.2 are already uniform with respect to ε, and so we have the following uniform bounds

Proposition 4.1. *Let the initial data $\rho_\varepsilon^0, u_\varepsilon^0$ be such that*

$$E_{0,\varepsilon} + E_{1,\varepsilon} \leq C \tag{56}$$

for some C independent of ε. Then $\rho_\varepsilon, u_\varepsilon$ enjoy the following bounds:

$$\begin{aligned} &\|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L_t^\infty L_x^2} \leq C, \\ &\|\sqrt{\rho_\varepsilon} w_\varepsilon\|_{L_t^\infty L_x^2} \leq C, \\ &\|\sqrt{\lambda_\varepsilon(\rho_\varepsilon)} \partial_x u_\varepsilon\|_{L_t^2 L_x^2} \leq C, \\ &\|\rho_\varepsilon\|_{L_t^\infty L_x^\infty} \leq 1. \end{aligned} \tag{57}$$

4.2 One-sided Lipschitz condition on $\partial_x u_\varepsilon$

The purpose of this subsection is to prove that u_ε satisfies a one-sided Lipschitz condition, which will yield a control of the full norm of the $\partial_x u_\varepsilon$.

Proposition 4.2. *Let $\rho_\varepsilon, u_\varepsilon$ be solution to system (10), and set*

$$A_\varepsilon := \max(\operatorname{ess\,sup}(\lambda_\varepsilon(\rho_\varepsilon^0)\partial_x u_\varepsilon^0), 0).$$

Then

$$V_\varepsilon = \lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon \leq A_\varepsilon. \tag{58}$$

In particular:

- *If $A_\varepsilon \to 0$ as $\varepsilon \to 0$, then*

$$(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon)_+ \to 0 \quad as \quad \varepsilon \to 0; \tag{59}$$

- *If $A_\varepsilon \leq \lambda_\varepsilon(\underline{\rho}_\varepsilon) \leq \bar{C}\varepsilon^{\frac{1}{1-2\alpha}}$, for some $\bar{C}$ independent of ε, then*

$$\partial_x u_\varepsilon \leq \bar{C}. \tag{60}$$

Proof. The starting point is derivation of equation for

$$\widetilde{V}_\varepsilon = (K_\varepsilon A_\varepsilon t + 1)V_\varepsilon.$$

Similarly to proof of Proposition 3.7 we can show that

$$\begin{aligned}
&\partial_t \widetilde{V}_\varepsilon + \partial_x(u_\varepsilon \widetilde{V}_\varepsilon) - \partial_x\left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x \widetilde{V}_\varepsilon\right) \\
&= \frac{K_\varepsilon A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1}\widetilde{V}_\varepsilon - \frac{\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon}{(K_\varepsilon A_\varepsilon t + 1)}\left(\frac{\widetilde{V}_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2 - \frac{\partial_x \lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x \widetilde{V}_\varepsilon,
\end{aligned} \tag{61}$$

which for ε fixed holds pointwisely. We now derive the renormalised equation for $\widetilde{V}_\varepsilon$. To this purpose we multiply (61) by $S'(\widetilde{V}_\varepsilon)$, where S is smooth, increasing and convex function, we obtain

$$\begin{aligned}
&\partial_t S(\widetilde{V}_\varepsilon) + \partial_x(u_\varepsilon S(\widetilde{V}_\varepsilon)) - \partial_x\left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}S'(\widetilde{V}_\varepsilon)\partial_x \widetilde{V}_\varepsilon\right) \\
&= \left(S(\widetilde{V}_\varepsilon) - S'(\widetilde{V}_\varepsilon)\widetilde{V}_\varepsilon\right)\partial_x u_\varepsilon - S''(\widetilde{V}_\varepsilon)\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\left(\partial_x \widetilde{V}_\varepsilon\right)^2 \\
&+ \frac{K_\varepsilon A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1}S'(\widetilde{V}_\varepsilon)\widetilde{V}_\varepsilon - \frac{S'(\widetilde{V}_\varepsilon)\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon}{(K_\varepsilon A_\varepsilon t + 1)}\left(\frac{\widetilde{V}_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2 - \frac{\partial_x \lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x S(\widetilde{V}_\varepsilon) \\
&= \left(S(\widetilde{V}_\varepsilon) - S'(\widetilde{V}_\varepsilon)(\widetilde{V}_\varepsilon - A_\varepsilon)\right)\partial_x u_\varepsilon - S''(\widetilde{V}_\varepsilon)\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\left(\partial_x \widetilde{V}_\varepsilon\right)^2 \\
&+ \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1}S'(\widetilde{V}_\varepsilon)\widetilde{V}_\varepsilon\left(K_\varepsilon - \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right) - \frac{S'(\widetilde{V}_\varepsilon)\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon}{(K_\varepsilon A_\varepsilon t + 1)}\left(\frac{\widetilde{V}_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2 - \frac{\partial_x \lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x S(\widetilde{V}_\varepsilon)
\end{aligned} \tag{62}$$

We set $S(y) = F_\eta(y)$ where F_η, $\eta > 0$, is a regularization of $(\cdot - A_\varepsilon)_+$:

$$F_\eta(y) = \begin{cases} 0 & \text{if } y - A_\varepsilon \leq \eta \\ \dfrac{y - A_\varepsilon - \eta}{2} + \dfrac{\eta}{2\pi}\sin(\pi\frac{y - A_\varepsilon}{\eta}) & \text{if } \eta \leq y - A_\varepsilon \leq 2\eta \\ y - A_\varepsilon - \dfrac{3\eta}{2} & \text{if } y - A_\varepsilon \geq 2\eta \end{cases} \tag{63}$$

For $\eta > 0$ fixed, $F''_\eta \geq 0$, $0 \leq F'_\eta \leq 1$, and

$$|F_\eta(y) - (y - A_\varepsilon)F'_\eta(y)| \leq \left(\frac{3}{2} + \frac{1}{2\pi}\right)\eta = \kappa\eta. \tag{64}$$

Note that for such choice of S the second and the fourth terms on the r.h.s. of (62) are non-positive. Therefore, integrating (62) in space, we then get:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}} F_\eta(\widetilde{V}_\varepsilon(t))dx \leq \int_{\mathbb{T}} \Big|F_\eta(\widetilde{V}_\varepsilon) - F'_\eta(\widetilde{V}_\varepsilon)(\widetilde{V}_\varepsilon - A_\varepsilon)\Big||\partial_x u_\varepsilon|dx \\ + \int_{\mathbb{T}} \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1} F'_\eta(\widetilde{V}_\varepsilon)\widetilde{V}_\varepsilon \left(K_\varepsilon - \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right) dx + \int_{\mathbb{T}} \left|\partial_x \left(\frac{\partial_x \lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\right)\right| F_\eta(\widetilde{V}_\varepsilon)dx.\end{aligned} \tag{65}$$

The first term on the r.h.s. can be controlled using (64), (57), and the ε-dependent bound from below for ρ_ε (36), we obtain

$$\int_0^T \int_{\mathbb{T}} \Big|F_\eta(\widetilde{V}_\varepsilon) - F'_\eta(\widetilde{V}_\varepsilon)(\widetilde{V}_\varepsilon - A_\varepsilon)\Big||\partial_x u_\varepsilon|dx\,dt \leq \kappa\eta \int_0^T \int_{\mathbb{T}} |\partial_x u_\varepsilon|dx\,dt \leq \kappa C(T)\eta\varepsilon^{-\alpha'} \tag{66}$$

with some $\alpha' > 0$. For the second term of the right-hand side of (65), we notice that

$$\begin{aligned}&\int_{\mathbb{T}} \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1} F'_\eta(\widetilde{V}_\varepsilon)\widetilde{V}_\varepsilon \left(K_\varepsilon - \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right) dx \\ &= \int_{\mathbb{T}} \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1} F'_\eta(\widetilde{V}_\varepsilon)\widetilde{V}_\varepsilon \left(K_\varepsilon - \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right) \mathbf{1}_{\widetilde{V}_\varepsilon \geq A_\varepsilon} dx.\end{aligned} \tag{67}$$

Now, we recall that $\rho_\varepsilon \leq 1 - C_1\varepsilon^{\frac{1}{\beta-1}}$, with $C_1 > 0$ independent of ε, so that $\lambda_\varepsilon(\rho_\varepsilon) \leq C_2\varepsilon^{-\frac{2}{\beta-1}}$. Hence, taking

$$K_\varepsilon \leq C_2^{-1}\varepsilon^{\frac{2}{\beta-1}} \tag{68}$$

we observe that the integral (67) is non-positive so that, by Gronwall's inequality,

$$\int_{\mathbb{T}} F_\eta(\widetilde{V}_\varepsilon(t)) \leq \Big[\int_{\mathbb{T}} F_\eta(\widetilde{V}_\varepsilon(0)) + C\eta \int_{\mathbb{T}} |\partial_x u_\varepsilon|\Big] \exp\left(\left\|\partial_x \left(\frac{\partial_x \lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\right)\right\|_{L^1_t L^\infty_x}\right)$$

Passing to the limit $\eta \to 0$, and observing that initially $(\widetilde{V}_\varepsilon(0) - A_\varepsilon)_+ = (V_\varepsilon(0) - A_\varepsilon)_+ = 0$, we conclude that

$$\int_{\mathbb{T}} (\widetilde{V}_\varepsilon(t) - A_\varepsilon)_+ \, dx \leq 0.$$

This implies that

$$V_\varepsilon(t,x) \leq \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1} \leq A_\varepsilon. \tag{69}$$

This concludes the proof of (58), the two implications (59) and (60) are straightforward. □

Remark 4.3. Note that we have from (69) an Oleinik entropy condition at $\varepsilon > 0$ fixed, namely a decreasing in time: $(\lambda_\varepsilon \partial_x u_\varepsilon)_+ \leq (K_\varepsilon t)^{-1}$. Unfortunately, due to the condition on K_ε (68), this estimate degenerates as $\varepsilon \to 0$.

As a consequence of Proposition 4.2 and due to the periodicity of the domain, we can control the whole norm of the velocity gradient.

Corollary 4.4. *We have*

$$\|\partial_x u_\varepsilon\|_{L^\infty_t L^1_x} \leq C \tag{70}$$

for a constant C independent of ε. As a consequence of it, we deduce

$$\|u_\varepsilon\|_{L^\infty_{t,x}} \leq C. \tag{71}$$

Proof. Let us denote $D_\varepsilon := (\partial_x u_\varepsilon)_+$. We have, for any t

$$\begin{aligned}
&\int_{\mathbb{T}} |\partial_x u_\varepsilon(t,x)| dx \\
&= \int_{\mathbb{T}} \Big(2D_\varepsilon(t,x) - \partial_x u_\varepsilon(t,x)\Big) dx \\
&= \int_{\mathbb{T}} 2D_\varepsilon(t,x)\, dx \\
&\overset{\text{one-sided Lip. (60)}}{\leq} 2\bar{C}.
\end{aligned}$$

Taking the supremum w.r.t. t we finally obtain the uniform control of $\|\partial_x u_\varepsilon\|_{L^\infty_t L^1_x}$. Now as a consequence of the hypothesis on the initial data (57) and generalized Poincaré inequality (See, Proposition 7.2 in Appendix 7) we have

$$\|u_\varepsilon\|_{L^\infty_t L^1_x} \leq C.$$

Finally from the Sobolev embedding, we obtain the $L^\infty_{t,x}$ bound. □

4.3 Improved potential estimate

Note that so far we are lacking the uniform bound on the singular part of the potential $\pi_\varepsilon(\rho_\varepsilon)$ (defined in (14), recall also Remark 3.4 and estimate (35)). This is the purpose of the next lemma.

Lemma 4.5. *Let the conditions of the previous proposition be satisfied and assume furthermore the condition (19). We have then*

$$\|\pi_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^1_x} + \|\partial_x \pi_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^2_x} \leq C, \tag{72}$$

for a positive constant C independent of ε.

Proof. Assume that $\mathbb{T}$ can be identified with $[0,1]$ supplemented with periodic boundary conditions. We first multiply equation (10b) by

$$\psi(t,x) = \int_0^x \big(\rho_\varepsilon(t,y) - <\rho_\varepsilon>\big) dy,$$

where

$$<\rho_\varepsilon> := \frac{M^0_\varepsilon}{|\mathbb{T}|}.$$

Note that ψ is periodic in space. After time and space integration, we get:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p'_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon \big(\rho_\varepsilon - <\rho_\varepsilon>\big) dx\, dt \\
&= -\int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 \varphi'_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon \big(\rho_\varepsilon - <\rho_\varepsilon>\big) dx\, dt + \int_0^t \int_{\mathbb{T}} \rho_\varepsilon u_\varepsilon^2 \big(\rho_\varepsilon - <\rho_\varepsilon>\big) dx\, dt \\
&\quad + \int_0^t \int_{\mathbb{T}} (\rho_\varepsilon u_\varepsilon) \partial_t \psi\, dx\, dt + \int_{\mathbb{T}} (\rho_\varepsilon u_\varepsilon \psi)(0,x) dx - \int_{\mathbb{T}} (\rho_\varepsilon u_\varepsilon \psi)(t,x) dx.
\end{aligned} \tag{73}$$

From the boundednes of the initial conditions and due to (57), we get that the last two terms are uniformly bounded. By the same token, noticing that

$$\partial_t \psi = \int_0^x \partial_t \rho_\varepsilon dy = -\int_0^x \partial_x(\rho_\varepsilon u_\varepsilon) dy = \rho_\varepsilon u_\varepsilon(t,0) - \rho_\varepsilon u_\varepsilon(t,x),$$

and using in addition the uniform bound (71), the third term on the r.h.s. of (73) is bounded as well. The first and second terms of (73) are bounded thanks to (57). Ultimately, we infer that

$$\left| \int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon \big(\rho_\varepsilon - <\rho_\varepsilon>\big) dx\, dt \right| \leq C.$$

Recall that we assumed that there exist $\overline{M}^0, \underline{M}^0$, independent of ε such that

$$0 < \underline{M}^0 \leq M_\varepsilon^0 \leq \overline{M}^0.$$

From the hypotheses (15)-(19), we ensure that

$$<\rho_\varepsilon> \leq \hat{\rho} = \frac{\overline{M}^0}{|\mathbb{T}|} < 1, \tag{74}$$

and we define $\rho_\varepsilon^m = \dfrac{1+<\rho_\varepsilon>}{2} \leq \dfrac{1+\hat{\rho}}{2}$. We then split the previous integral into two parts, depending on the value of ρ_ε. When $\rho_\varepsilon \leq \rho_\varepsilon^m$, the pressure remains far from the singularity uniformly with respect to ε and therefore

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon \big(\rho_\varepsilon - <\rho_\varepsilon>\big) \mathbf{1}_{\{\rho_\varepsilon \leq \rho_\varepsilon^m\}} dx\, dt \right| \\
&\leq C \| \sqrt{p_\varepsilon'(\rho_\varepsilon)} \mathbf{1}_{\{\rho_\varepsilon \leq \rho_\varepsilon^m\}} \|_{L_t^2 L_x^2} \| \rho_\varepsilon \sqrt{p_\varepsilon'(\rho_\varepsilon)} \partial_x u_\varepsilon \|_{L_t^2 L_x^2} \\
&\leq C
\end{aligned}$$

thanks to (57). Hence

$$\begin{aligned}
C &\geq \left| \int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon \big(\rho_\varepsilon - <\rho_\varepsilon>\big) \mathbf{1}_{\{\rho_\varepsilon > \rho_\varepsilon^m\}} dx\, dt \right| \\
&\geq \frac{1-\hat{\rho}}{2} \left| \int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon \mathbf{1}_{\{\rho_\varepsilon > \rho_\varepsilon^m\}} dx\, dt \right|.
\end{aligned}$$

summarizing, we have shown that

$$\left| \int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon dx\, dt \right| \leq C. \tag{75}$$

Next, note that for ε fixed, $s_\varepsilon = \rho_\varepsilon p_\varepsilon(\rho_\varepsilon)$ satisfies the following equation

$$\partial_t s_\varepsilon(\rho_\varepsilon) + \partial_x(s_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -\rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon. \tag{76}$$

From this, the control of the initial density (15) and the estimate (75), we can deduce that

$$\|s_\varepsilon(\rho_\varepsilon)\|_{L_t^\infty L_x^1} \leq C.$$

Now, considering the cases of ρ_ε far away from 1 and close to 1 separately and observing that $s_\varepsilon(\rho)$ and $\pi_\varepsilon(\rho)$ have the same singularity when ρ is close to 1, we obtain

$$\|\pi_\varepsilon(\rho_\varepsilon)\|_{L_t^\infty L_x^1} \leq C.$$

Regarding the control of the gradient $\partial_x \pi_\varepsilon(\rho_\varepsilon)$, we simply use estimate (57) to write

$$\begin{aligned}
\|\partial_x \pi_\varepsilon(\rho_\varepsilon)\|_{L_t^\infty L_x^2} &\leq \|\sqrt{\rho_\varepsilon}\|_{L_{t,x}^\infty} \|\sqrt{\rho_\varepsilon} \partial_x \big(p_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon)\big)\|_{L_t^\infty L_x^2} \\
&\leq \|\sqrt{\rho_\varepsilon}\|_{L_{t,x}^\infty} \Big(\|\sqrt{\rho_\varepsilon} w_\varepsilon\|_{L_t^\infty L_x^2} + \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L_t^\infty L_x^2} \Big) \leq C.
\end{aligned}$$

□

Corollary 4.6. *We have*

$$\|\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\|_{L^1_{t,x}} \leq C \tag{77}$$

for a constant C independent of ε.

Proof. We proceed exactly in the same way as as in the proof of Corollary 4.4 for $V_\varepsilon = \lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon$ by combining the control (75) of $|\int_0^t \int_{\mathbb{T}} V_\varepsilon \, dx \, dt|$ with the control of the positive part given by (58). □

5 Passage to the limit $\varepsilon \to 0$

The purpose of this section is to prove Theorem 2.2. We will show that when $\varepsilon \to 0$ the sequence of solutions $(\rho_\varepsilon, u_\varepsilon, \pi_\varepsilon(\rho_\varepsilon))$ converges to (ρ, u, π), the distributional solution of (6).

Proof of Theorem 2.2. Thanks to the uniform bounds from the previous section, there exist $\rho \in [0,1]$, u, and $\pi \geq 0$ such that

$$\begin{aligned}
&\rho_\varepsilon \rightharpoonup \rho \quad \text{weakly-* in} \quad L^\infty((0,T)\times\mathbb{T}),\\
&u_\varepsilon \rightharpoonup u \quad \text{weakly-* in} \quad L^\infty((0,T)\times\mathbb{T}),\\
&\pi_\varepsilon(\rho_\varepsilon) \to \pi \quad \text{weakly-* in} \quad L^\infty(0,T;H^1(\mathbb{T})),
\end{aligned}$$

up to a selection of a subsequence.
We can immediately justify that

$$(1-\rho_\varepsilon)\pi_\varepsilon(\rho_\varepsilon) \to 0 \quad \text{strongly in} \quad L^q((0,T)\times\mathbb{T}), \ q>1, \tag{78}$$

and that the approximate viscosity term converges to 0 strongly, i.e.

$$\rho_\varepsilon \varphi_\varepsilon(\rho_\varepsilon) \to 0 \quad \text{strongly in } L^\infty((0,T)\times\mathbb{T}).$$

To pass to the limit in the nonlinear terms we first use the continuity and momentum equations of system (10) to deduce that for any $p<\infty$ we have

$$\|\partial_t \rho_\varepsilon\|_{L^\infty_t W^{-1,p}_x} + \|\partial_t(\rho_\varepsilon u_\varepsilon)\|_{L^\infty_t W^{-1,2}_x} \leq C, \tag{79}$$

where to estimate the time derivative of momentum, we use that $\lambda_\varepsilon(\rho_\varepsilon) \approx \pi_\varepsilon(\rho_\varepsilon)^{1+\frac{1}{\beta}}$, along with uniform estimates (57) and (72).

Combining the control of $\partial_t \rho_\varepsilon$ with the control of $\partial_x \pi_\varepsilon(\rho_\varepsilon)$ we can apply the standard compensated compactness argument (see, Lemma 5.1 from [24]) to justify that

$$(1-\rho_\varepsilon)\pi_\varepsilon(\rho_\varepsilon) \rightharpoonup (1-\rho)\pi \quad \text{in } \mathcal{D}',$$

and so, from (78), we deduce that $(1-\rho)\pi = 0$ a.e. in $(0,T)\times\mathbb{T}$ with $1-\rho \geq 0$, $\pi \geq 0$.

Similarly, combining the control of gradient of velocity (77) with the uniform estimates for the time derivatives (79) we can justify that

$$\rho_\varepsilon u_\varepsilon \to \rho u \quad \text{and} \qquad \rho_\varepsilon u_\varepsilon^2 \to \rho u^2 \tag{80}$$

in the sense of distributions.

Finally, we can use the equation for $\partial_t \partial_x \pi_\varepsilon(\rho_\varepsilon)$

$$\partial_t \partial_x \pi_\varepsilon(\rho_\varepsilon) + \partial_x\big(u_\varepsilon \partial_x \pi_\varepsilon\big) = -\partial_x\big(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\big)$$

to deduce that

$$\partial_t \pi_\varepsilon(\rho_\varepsilon) \in L^2_t W^{-1,1}_x, \tag{81}$$

so, repeating the previous argument we can justify that also

$$u_\varepsilon \partial_x \pi_\varepsilon(\rho_\varepsilon) \to u \partial_x \pi, \tag{82}$$

in the sense of the distributions. With this argument, we can justify completely the passage to the limit in the equations leading to system (6).

The last part is to verify the entropy conditions for the limiting system. First, it is clear that the one-sided Lipschitz estimate holds on the limit velocity u:

$$\partial_x u \leq C \qquad \text{in } \mathcal{D}'.$$

Next, we write that for fixed ε, smooth function S:

$$\begin{aligned}\partial_t(\rho_\varepsilon S(u_\varepsilon)) + \partial_x(\rho_\varepsilon u_\varepsilon S(u_\varepsilon)) - \partial_x\big(S'(u_\varepsilon)\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\big) \\ = S''(u_\varepsilon)\lambda_\varepsilon(\rho_\varepsilon)(\partial_x u_\varepsilon)^2,\end{aligned}$$

hence, for convex function S:

$$\partial_t(\rho_\varepsilon S(u_\varepsilon)) + \partial_x(\rho_\varepsilon u_\varepsilon S(u_\varepsilon)) - \partial_x\big(S'(u_\varepsilon)\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\big) \leq 0.$$

As previously, we pass to the limit in the sense of distribution in the first two nonlinear terms thanks to compensated compactness arguments. Next, since $(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon)_\varepsilon$ is bounded in $L^1_{t,x}$ from (77), it converges to some $\Lambda \in \mathcal{M}((0,T)\times\mathbb{T})$. Recall that $(u_\varepsilon)_\varepsilon$ is bounded in $L^\infty_{t,x}$, so $(S'(u_\varepsilon)\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon)_\varepsilon$ is bounded in $L^1_{t,x}$ and converges to some $\Lambda_S \in \mathcal{M}((0,T)\times\mathbb{T})$, where $|\Lambda_S| \leq \mathrm{Lip}_S|\Lambda|$. Finally, we have proven that:

$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) - \partial_x \Lambda_S \leq 0.$$

In addition, from (59) we have for all $t \in [0,T],\ x \in \mathbb{T}$

$$\lambda_\varepsilon(\rho_\varepsilon(t,x))(\partial_x u_\varepsilon(t,x))_+ \to 0 \quad \text{as } \varepsilon \to 0,$$

and therefore

$$\Lambda \leq 0. \tag{83}$$

The proof of the Theorem 2.2 is therefore complete. □

6 Numerical approximations and illustrations

In this Section, we first propose a discretization of model (1)-(2)-(3) (dropping the viscosity φ_ε) and then performs numerical simulations to illustrate the behavior of the solutions in the limit $\varepsilon \to 0$. In particular, we compare with the dynamics the Euler equation with density constraint (7).

6.1 Numerical scheme

We consider the following formulation of the system:

$$\begin{aligned}&\partial_t \rho + \partial_x(\rho w) - \partial_x(\rho\, \partial_x p_\varepsilon(\rho)) = 0,\\ &\partial_t(\rho w) + \partial_x(\rho w^2) - \partial_x(\rho w\, \partial_x p_\varepsilon(\rho)) = 0,\end{aligned}$$

and look for a numerical solution on $[0,T]\times[0,L]$. The density equation involves a non-linear diffusion operator, with unbounded diffusion coefficient $\rho p'_\varepsilon(\rho)$. Therefore implicit schemes are required to avoid too stringent stability condition (in particular with respect to ε).

Time semi-discretization. We denote by (ρ^n, w^n) the quantities at the discrete time $t^n = n\Delta t$, with $n \in \mathbb{N}$ and where $\Delta t > 0$ denotes the time step. We consider the following time semi-discretization scheme:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x((\rho w)^n) - \partial_x(\rho^n \partial_x p_\varepsilon(\rho^{n+1})) = 0, \tag{84}$$

$$\frac{(\rho w)^{n+1} - (\rho w)^n}{\Delta t} + \partial_x((\rho w^2)^n) - \partial_x((\rho w)^n \partial_x p_\varepsilon(\rho^{n+1})) = 0. \tag{85}$$

In order to prevent the density from exceeding the maximal density equal to 1, the first equation in solved in singular pressure variable p^{n+1}:

$$\frac{\rho_\varepsilon(p^{n+1}) - \rho_\varepsilon(p^n)}{\Delta t} + \partial_x(\rho w)^n - \partial_x(\rho^n \partial_x p^{n+1}) = 0, \tag{86}$$

where $\rho_\varepsilon(p)$ is the inverse function of $p_\varepsilon(\rho)$. In practice, at the n-th iteration, we first solve the elliptic equation (86) to get p^{n+1} and deduce $\rho^{n+1} = \rho_\varepsilon(p^{n+1})$ which is less than 1 due to the singularity of the pressure law. Finally w^{n+1} is obtained from (85).

Full discretization. Let us consider the (ρ_j^n, w_j^n) on a uniform mesh $x_j = (j + 1/2)\Delta x$, with $\Delta x > 0$ and $j \in \{0, \dots, J-1\}$, with $J\Delta x = L$. We use finite-volume discretization formula for the advection terms and finite difference discretizations for the elliptic operators:

$$\begin{aligned}\frac{\rho_\varepsilon(p_j^{n+1}) - \rho(p_j^n)}{\Delta t} &+ \frac{F^n_{\rho,j+\frac{1}{2}} - F^n_{\rho,j-\frac{1}{2}}}{\Delta x} \\ &- \frac{(\rho^n_{j+1} + \rho^n_j)\,(p^{n+1}_{j+1} - p^{n+1}_j) - (\rho^n_j + \rho^n_{j-1})\,(p^{n+1}_j - p^{n+1}_{j-1})}{2\Delta x^2} = 0,\end{aligned} \tag{87}$$

$$\begin{aligned}\frac{(\rho w)^{n+1}_j - (\rho w)^n_j}{\Delta t} &+ \frac{F^n_{\rho w,j+\frac{1}{2}} - F^n_{\rho w,j-\frac{1}{2}}}{\Delta x} \\ &- \frac{((\rho w)^n_{j+1} + (\rho w)^n_j)\,(p^{n+1}_{j+1} - p^{n+1}_j) - ((\rho w)^n_j + (\rho w)^n_{j-1})\,(p^{n+1}_j - p^{n+1}_{j-1})}{2\Delta x^2} = 0,\end{aligned} \tag{88}$$

where the fluxes are Lax-Friedrichs fluxes given by:

$$\begin{aligned}F^n_{\rho,j+\frac{1}{2}} &= \frac{(\rho w)^n_j + (\rho w)^n_{j+1}}{2} - \frac{1}{2}C^n\,(\rho^n_{j+1} - \rho^n_j), \\ F^n_{\rho w,j+\frac{1}{2}} &= \frac{(\rho w^2)^n_j + (\rho w^2)^n_{j+1}}{2} - \frac{1}{2}C^n\,((\rho w)^n_{j+1} - (\rho w)^n_j),\end{aligned}$$

where $C^n = \max_j\{|u^n_j|\} = \max_j\{|w^n_j - (p_\varepsilon(\rho^n_{j+1}) - p_\varepsilon(\rho^n_{j-1}))/(2\Delta x)|\}$ is an approximation of the maximal effective velocity. Gathering all the explicit terms on the right-hand side, Equation (87) can be written as follows:

$$\begin{aligned}\rho_\varepsilon(p_j^{n+1}) - \Delta t\,&\frac{(\rho^n_{j+1} + \rho^n_j)\,(p^{n+1}_{j+1} - p^{n+1}_j) - (\rho^n_j + \rho^n_{j-1})\,(p^{n+1}_j - p^{n+1}_{j-1})}{2\Delta x^2} \\ &= \rho_\varepsilon(p^n_j) - \Delta t\,\frac{F^n_{\rho,j+\frac{1}{2}} - F^n_{\rho,j-\frac{1}{2}}}{\Delta x}.\end{aligned} \tag{89}$$

This equation can be complemented by periodic or Dirichlet boundary conditions according to the test case. This is a non-linear equation solved using Newton's algorithm. To summarize, at the n-th iteration, we first compute $(\rho_j^{n+1})_j$ with Eq. (89) and then we compute $(w_j^{n+1})_j$ with Eq. (88). Note that the time step should satisfy the CFL stability condition: $C^n\Delta t \leq \Delta x$, for all $n \in \mathbb{N}$.

6.2 Numerical results

Here, we propose two numerical tests to illustrate the behavior of the numerical solutions as $\varepsilon \to 0$. We compare the results with numerical simulations of the Euler system with maximal density constraints. These latter are obtained with the numerical scheme proposed in [32] based on a Lagrangian formulation.

Test case 1: compressive initial condition. We first consider the following initial condition:

$$\rho^0(x) = 0.5, \quad w^0(x) = 0.5\,\sin(2\pi x),$$

on the domain $[0, 1]$ with periodic boundary conditions. The compressive desired velocity w^0, which is positive for $x < 0.5$ and negative for $x > 0.5$, induces an increase of density in the middle of the domain. The numerical parameters are as follows: $\Delta t = 0.001$ and $J = 150$ and we consider $\gamma = \beta = 2$ in the definition (3) of p_ε. For the congested Euler system, we use 400 discretization points.

Figure 1 shows the solution at time $T = 0.3$ for different values of ε: we plot the density (left), the velocity $u = w - \partial_x p_\varepsilon(\rho)$ and the adhesion potential $-\pi_\varepsilon(\rho)$ defined in (14). Curves labeled by "$\varepsilon = 0$" correspond to the numerical solutions of the constrained Euler system (7). As expected, the approximated solutions tend the solution to the congested Euler system as $\varepsilon \to 0$. In particular, at the limit, the solution solution presents a discontinuity between a congested phase with constant velocity (incompressibility condition $\partial_x u = 0$) and a free phase.

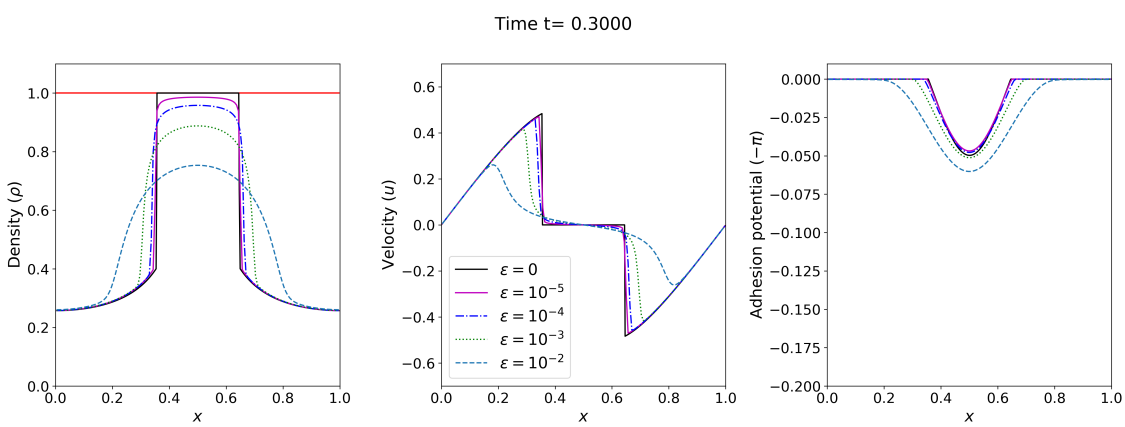

Figure 1: Solution at time $T = 0.3$ with compressive initial condition for different ε. For $\varepsilon > 0$: numerical scheme presented in Section 6.1 with $\Delta t = 0.001$, $J = 150$. For $\varepsilon = 0$: Lagrangian scheme [32] for the congested Euler system with 400 discretization points.

Test case 2: dilatant initial condition. We then perform a dilatant test-case in the congested regime. We consider the following initial data:

$$\rho^0_\varepsilon(x) = 1 - 2\,\varepsilon^{\frac{1}{\beta}}, \quad w^0(x) = \tanh((x - 0.5)/\sigma),$$

with $\sigma = 0.05$, on the domain $\mathbb{R}$. The initial density has been chosen such that $p_\varepsilon(\rho^0_\varepsilon)$ tends to a positive value as $\varepsilon \to 0$: this is the critical scale to approach a congested regime. In practice, we use Neumann boundary conditions for the approximated system with $J = 200$ points and free boundary conditions for the congested Euler model with 200 discretization points on the interval.

On Figure 2, we compare the solutions at time $T = 0.03$ for different ε. We still observe that the solutions tends to the congested Euler dynamics as $\varepsilon \to 0$. We note that the asymptotic adhesion potential equals zero: indeed, due to desired increasing velocity profile, there is no congested force. In consequence, the desired velocity is constant in time.

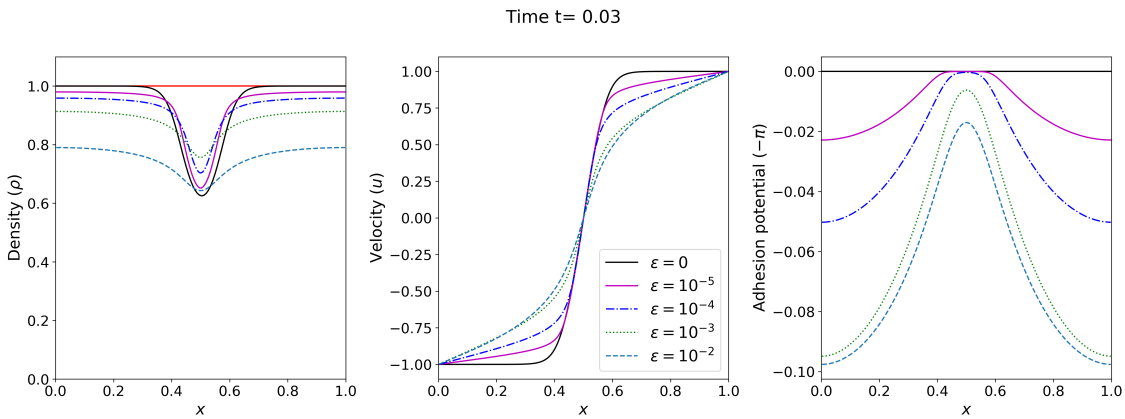

Figure 2: Solution at time $T = 0.03$ with dilatant initial condition for different ε.

7 Appendix

Here our goal is to prove Proposition 3.11. To do so, we first state the following lemma, which provides an estimate of ρ_ε and V_ε for the case $m = 2$ in (49) and $l = 1$ in (50). Then using this lemma we derive the higher order regularity estimates for ρ_ε and V_ε and conclude the proof of Proposition 3.11.

Lemma 7.1. *Let $\varepsilon > 0$ be fixed and $(\rho_\varepsilon, u_\varepsilon)$ be a regular solution of system (10) with*

$$E_{2,\varepsilon} = E_{1,\varepsilon} + \|\partial_x^2 u_\varepsilon^0\|_{L_x^2} + \|\partial_x^2 \rho_\varepsilon^0\|_{L_x^2}. \tag{90}$$

Then we have

$$\begin{aligned}&\|\partial_x^2 \rho_\varepsilon\|_{L_t^\infty L_x^2} + \|\partial_x V_\varepsilon\|_{L_t^\infty L_x^2} + \|\partial_x^2 V_\varepsilon\|_{L_t^2 L_x^2} + \|\partial_x^2 u_\varepsilon\|_{L_t^\infty L_x^2} + \|\partial_x^3 u_\varepsilon\|_{L_t^2 L_x^2} \\ &\le C(\varepsilon, E_{2,\varepsilon}, \underline{\rho}_\varepsilon, \overline{\rho}_\varepsilon, T).\end{aligned} \tag{91}$$

Proof. Here we recall our estimate of $\partial_x \rho_\varepsilon$ in $L_t^\infty L_x^2$:

$$\|\partial_x \rho_\varepsilon\|_{L_t^\infty L_x^2} = \left(\alpha - \frac{1}{2}\right)^{-1} \frac{1}{\varepsilon} \left\| \rho_\varepsilon^{\frac{3}{2}-\alpha} \partial_x (\varepsilon \rho_\varepsilon^{\alpha - \frac{1}{2}}) \right\|_{L_t^\infty L_x^2} \le \frac{1}{\varepsilon} C(\bar{\rho}_\varepsilon, \alpha, T, E_0, E_1).$$

We consider the case that corresponds to $m = 2$ in (49). Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^2 \rho_\varepsilon\|_{L_x^2}^2 \le C\left(\|\partial_x^2 u_\varepsilon\|_{L_x^2} \|\partial_x^2 \rho_\varepsilon\|_{L_x^2}^2 + \|\partial_x^2 \rho_\varepsilon\|_{L_x^2} \|\partial_x^3 u_\varepsilon\|_{L_x^2}\right).$$

On the other hand, integrating by parts with $l = 1$ in (50) gives us

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_x V_\varepsilon|^2 dx + \int_{\mathbb{T}} \frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon} |\partial_x^2 V_\varepsilon|^2 dx \\ &= \int_{\mathbb{T}} \left(u_\varepsilon + \frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2} \partial_x \rho_\varepsilon\right) \partial_x V_\varepsilon \partial_x^2 V_\varepsilon + \int_{\mathbb{T}} \frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon dx + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2} V_\varepsilon^2 \partial_x^2 V_\varepsilon dx.\end{aligned}$$

From the observation

$$\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon} = \rho_\varepsilon p_\varepsilon'(\rho_\varepsilon) + \varepsilon \rho^{\alpha-1} \ge \varepsilon$$

we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x V_\varepsilon|^2dx+\varepsilon\int_{\mathbb{T}}|\partial_x^2 V_\varepsilon|^2dx$$
$$\leq \int_{\mathbb{T}}u_\varepsilon\partial_x V_\varepsilon\partial_x^2 V_\varepsilon dx+\int_{\mathbb{T}}\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\partial_x\rho_\varepsilon\partial_x V_\varepsilon\partial_x^2 V_\varepsilon dx+\int_{\mathbb{T}}\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon dx+\lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}V_\varepsilon^2\partial_x^2 V_\varepsilon dx \quad :=\sum_{i=1}^{3}J_i.$$

Control of J_1. We proceed similarly as in the case of I_2 of Lemma 3.8. We have

$$|J_1|\leq \|u_\varepsilon\|_{L^\infty_x}\|\partial_x V_\varepsilon\|_{L^2_x}\|\partial_x^2 V_\varepsilon\|_{L^2_x}\leq \frac{\varepsilon}{16}\|\partial_x^2 V_\varepsilon\|^2_{L^2_x}+\frac{4}{\varepsilon}\|u_\varepsilon\|^2_{L^\infty_x}\|\partial_x V_\varepsilon\|^2_{L^2_x}.$$

We recall

$$\|u_\varepsilon\|^2_{H^1_x}\leq \underline{\rho}_\varepsilon^{-1/2}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^2_x}+\varepsilon^{-2}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|^2_{L^2_x}$$

to conclude

$$|J_1|\leq \frac{\varepsilon}{16}\|\partial_x^2 V_\varepsilon\|^2_{L^2_x}+4\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^2_x}+\varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|^2_{L^2_x}\right)\|\partial_x V_\varepsilon\|^2_{L^2_x}. \tag{92}$$

Control of J_2. For this term, we observe

$$|J_2|\leq C(\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\|\partial_x\rho_\varepsilon\|_{L^\infty_x}\|\partial_x V_\varepsilon\|_{L^2_x}\|\partial_x^2 V_\varepsilon\|_{L^2_x}\leq \frac{\varepsilon}{16}\|\partial_x^2 V_\varepsilon\|^2_{L^2_x}+C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\|\partial_x\rho_\varepsilon\|^2_{L^\infty_x}\|\partial_x V_\varepsilon\|^2_{L^2_x},$$

from which we deduce

$$|J_2|\leq \frac{\varepsilon}{16}\|\partial_x^2 V_\varepsilon\|^2_{L^2_x}+C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\|\partial_x^2\rho_\varepsilon\|^2_{L^2_x}\|\partial_x V_\varepsilon\|^2_{L^2_x}. \tag{93}$$

Control of J_3. The bound for $\left\|\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon+\lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}\right\|_{L^\infty_x}$ implies

$$|J_3|\leq \frac{\varepsilon}{16}\|\partial_x^2 V_\varepsilon\|^2_{L^2_x}+C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\int_{\mathbb{T}}|V_\varepsilon|^4dx.$$

Moreover, using the Nash inequality we obtain

$$\|V_\varepsilon\|^4_{L^4_x}\leq C\left(\|V_\varepsilon\|^3_{L^2_x}\|\partial_x V_\varepsilon\|_{L^2_x}+\|V_\varepsilon\|^4_{L^2_x}\right).$$

Thus we deduce

$$|J_3|\leq \frac{\varepsilon}{16}\|\partial_x^2 V_\varepsilon\|^2_{L^2_x}+C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|^3_{L^2_x}\|\partial_x V_\varepsilon\|_{L^2_x}+\|V_\varepsilon\|^4_{L^2_x}\right). \tag{94}$$

Finally combining (92), (93) and (94), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x V_\varepsilon|^2dx+\frac{13}{16}\varepsilon\int_{\mathbb{T}}|\partial_x^2 V_\varepsilon|^2dx\\&\leq 4\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^2_x}+\varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|^2_{L^2_x}\right)\|\partial_x V_\varepsilon\|^2_{L^2_x}\\&+C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\|\partial_x^2\rho_\varepsilon\|^2_{L^2_x}\|\partial_x V_\varepsilon\|^2_{L^2_x}+C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|^3_{L^2_x}\|\partial_x V_\varepsilon\|_{L^2_x}+\|V_\varepsilon\|^4_{L^2_x}\right).\end{aligned} \tag{95}$$

Now we want to derive an expression for $\partial_x^3 u_\varepsilon$ in terms of V_ε and its derivatives. A direct calculation gives us

$$\partial_x^2 V_\varepsilon=2\partial_x(\lambda_\varepsilon(\rho_\varepsilon))\partial_x^2 u_\varepsilon+\partial_x^2(\lambda_\varepsilon(\rho_\varepsilon))\partial_x u_\varepsilon+\lambda_\varepsilon(\rho_\varepsilon)\partial_x^3 u_\varepsilon.$$

Using the relation between $\partial_x V_\varepsilon$ and $\partial_x u_\varepsilon$, we have

$$\begin{aligned}\partial_x^3 u_\varepsilon &= \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\partial_x^2 V_\varepsilon - \frac{2\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)}\partial_x V_\varepsilon + 2\left(\frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2 V_\varepsilon - \left(\frac{\lambda_\varepsilon'(\rho_\varepsilon)|\partial_x\rho_\varepsilon|^2 + \lambda_\varepsilon''(\rho_\varepsilon)\partial_x^2\rho_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)^2}\right)V_\varepsilon \\ &:= \sum_{i=1}^4 B_i.\end{aligned}$$

Denoting $\tilde{A} = \dfrac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)}$, we observe that

$$\|\tilde{A}\|_{L_x^\infty} \le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x\rho_\varepsilon\|_{L_x^\infty} \le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}.$$

Using the above estimate, we obtain the following bounds:

$$\begin{aligned}
\|B_1\|_{L_x^2} &\le \left\|\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right\|_{L_x^\infty}\|\partial_x^2 V_\varepsilon\|_{L_x^2} \le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2 V_\varepsilon\|_{L_x^2}; \\
\|B_2\|_{L_x^2} &\le C\|\tilde{A}\|_{L_x^\infty}\|\partial_x V_\varepsilon\|_{L_x^2} \le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}\|\partial_x V_\varepsilon\|_{L_x^2}; \\
\|B_3\|_{L_x^2} &\le C\|\tilde{A}\|_{L_x^\infty}\|\tilde{A}\|_{L_x^2}\|V_\varepsilon\|_{L_x^\infty} \le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x\rho_\varepsilon\|_{L_x^\infty}\|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1} \\
&\le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}\|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1}; \\
\|B_4\|_{L_x^2} &\le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x\rho_\varepsilon\|_{L_x^\infty}\|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1} + C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1} \\
&\le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\partial_x^2\rho_\varepsilon\|_{L_x^2}\|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1} + \|\partial_x^2\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1}\right).
\end{aligned}$$

Therefore, we have

$$\|\partial_x^3 u_\varepsilon\|_{L_x^2} \le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\partial_x^2 V_\varepsilon\|_{L_x^2} + \left(\|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1} + \|V_\varepsilon\|_{H_x^1}\right)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}\right). \tag{96}$$

We recall

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2 \le C\left(\|\partial_x^2 u_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2 + \|\partial_x^2\rho_\varepsilon\|_{L_x^2}\|\partial_x^3 u_\varepsilon\|_{L_x^2}\right)$$

and substitute (96) in the estimate for $\partial_x^2\rho_\varepsilon$, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2 &\le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2 V_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^2} \\
&\quad + C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\partial_x^2 u_\varepsilon\|_{L_x^2} + \|V_\varepsilon\|_{H_x^1} + \|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1}\right)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2.
\end{aligned}$$

Now we add the above estimate with (95) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2 + \frac{1}{2}\frac{d}{dt}\|\partial_x V_\varepsilon\|_{L_x^2}^2 + \frac{13}{16}\varepsilon\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 \\
&\le C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2 V_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^2} \\
&+ C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\partial_x^2 u_\varepsilon\|_{L_x^2} + \|V_\varepsilon\|_{H_x^1} + \|\partial_x\rho_\varepsilon\|_{L_x^2}\|V_\varepsilon\|_{H_x^1} + \|\partial_x V_\varepsilon\|_{L_x^2}^2\right)\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2 \\
&+ 4\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_x^2}^2 + \varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L_x^2}^2\right)\|\partial_x V_\varepsilon\|_{L_x^2}^2 \\
&+ C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|_{L_x^2}^3\|\partial_x V_\varepsilon\|_{L_x^2} + \|V_\varepsilon\|_{L_x^2}^4\right).
\end{aligned}$$

Using the Young's inequality we deduce

$$\|\partial_x^2 V_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^2} \le \frac{\epsilon}{16}\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 + \frac{4}{\epsilon}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2.$$

Similarly, we derive the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^2\rho_\varepsilon\|_{L^2_x}^2+\frac{1}{2}\frac{d}{dt}\|\partial_x V_\varepsilon\|_{L^2_x}^2+\frac{3}{4}\epsilon\|\partial_x^2 V_\varepsilon\|_{L^2_x}^2\leq F_1(t)\|\partial_x^2\rho_\varepsilon\|_{L^2_x}^2+F_2(t)\|\partial_x V_\varepsilon\|_{L^2_x}^2+G(t), \quad (97)$$

where

$$F_1(t)=C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\left(\|\partial_x^2 u_\varepsilon\|_{L^2_x}+\|V_\varepsilon\|_{H^1_x}+\|\partial_x\rho_\varepsilon\|_{L^2_x}\|V_\varepsilon\|_{H^1_x}+\|\partial_x V_\varepsilon\|_{L^2_x}^2+1\right)$$
$$F_2(t)=4\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2_x}^2+\varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L^2_x}^2\right)$$

and

$$G(t)=C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|_{L^2_x}^3\|\partial_x V_\varepsilon\|_{L^2_x}+\|V_\varepsilon\|_{L^2_x}^4\right).$$

From our earlier estimates on the time interval $(0,T)$ described in Lemma 3.8 , we have

$$\|F_1\|_{L^1_t}\leq C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon,T)\left(\|\partial_x^2 u_\varepsilon\|_{L^2_tL^2_x}+\|V_\varepsilon\|_{L^2_tH^1_x}+\|\partial_x\rho_\varepsilon\|_{L^\infty_xL^2_x}\|V_\varepsilon\|_{L^2_tH^1_x}+\|\partial_x V_\varepsilon\|_{L^2_tL^2_x}^2+T\right),$$
$$\|F_2\|_{L^1_t}\leq C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon,T)\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty_tL^2_x}^2+\varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L^\infty_tL^2_x}^2\right),$$
$$\|G\|_{L^1_t}\leq C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon,T)\left(\|V_\varepsilon\|_{L^\infty_tL^2_x}^3\|\partial_x V_\varepsilon\|_{L^2_tL^2_x}+\|V_\varepsilon\|_{L^\infty_tL^2_x}^4\right),$$

and consequently

$$\|F_1\|_{L^1_t}+\|F_2\|_{L^1_t}+\|G\|_{L^1_t}\leq C(\varepsilon,\bar\rho_\varepsilon,\underline{\rho}_\varepsilon,E_{1,\varepsilon},T).$$

Integrating equation (97) with respect to time, and using the additional hypothesis on initial data (90) and Grönwall's inequality, we conclude

$$\|\partial_x^2\rho_\varepsilon\|_{L^\infty_tL^2_x}+\|\partial_x V_\varepsilon\|_{L^\infty_tL^2_x}^2+\frac{\varepsilon}{2}\|\partial_x^2 V_\varepsilon\|_{L^2_tL^2_x}^2\leq C(\varepsilon,E_{2,\varepsilon},\overline{\rho}_\varepsilon,\underline{\rho}_\varepsilon,T).$$

From this we deduce the following estimates:
The $L^\infty_tL^2_x$ estimate for $\partial_x^2 u_\varepsilon$.

$$\begin{aligned}&\|\partial_x^2 u_\varepsilon\|_{L^\infty_tL^2_x}\\&\leq\left\|\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\right\|_{L^\infty_{t,x}}\left(\|\partial_x V_\varepsilon\|_{L^\infty_\infty L^2_x}+\|\lambda_\varepsilon'(\rho_\varepsilon)\|_{L^\infty_{t,x}}(\|\partial_x^2\rho_\varepsilon\|_{L^\infty_tL^2_x}+\|\partial_x\rho_\varepsilon\|_{L^\infty_tL^2_x})\|\partial_x u_\varepsilon\|_{L^\infty_tL^2_x}\right)\\&\leq C(\varepsilon,E_{2,\varepsilon},\underline{\rho}_\varepsilon,T).\end{aligned}$$

The $L^2_tL^2_x$ estimate for $\partial_x^3 u_\varepsilon$. We write

$$\partial_x^3 u_\varepsilon=\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)}\partial_x^2 V_\varepsilon-\frac{2\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)}\partial_x V_\varepsilon+2\left(\frac{\partial_x\lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)}\right)^2V_\varepsilon-\left(\frac{\lambda_\varepsilon'(\rho_\varepsilon)|\partial_x\rho_\varepsilon|^2+\lambda_\varepsilon''(\rho_\varepsilon)\partial_x^2\rho_\varepsilon}{\lambda_\varepsilon(\rho_\varepsilon)^2}\right)V_\varepsilon,$$

and from (96), we get

$$\begin{aligned}\|\partial_x^3 u_\varepsilon\|_{L^2_tL^2_x}\leq&C(\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)\left(\|\partial_x^2 V_\varepsilon\|_{L^2_tL^2_x}+\left(\|\partial_x\rho_\varepsilon\|_{L^\infty_tL^2_x}^{\frac12}\|V_\varepsilon\|_{L^\infty_tH^1_x}\right)^{\frac43}+\|\partial_x^2\rho_\varepsilon\|_{L^2_tL^2_x}^2+\|V_\varepsilon\|_{H^1_x}\|\partial_x^2\rho_\varepsilon\|_{L^2_x}\right)\\&+C(\bar\rho_\varepsilon,\underline{\rho}_\varepsilon)(\|\partial_x\rho_\varepsilon\|_{L^\infty_tL^2_x}+\|\partial_x\rho_\varepsilon\|_{L^\infty_tL^2_x}^2)\|V_\varepsilon\|_{L^\infty_tH^1_x}.\end{aligned}$$

The proof of the lemma is complete. □

Having the above lemma at hand, we next prove the regularity estimates from Proposition 3.11 with the initial data satisfying (54).

Proof of Proposition 3.11. In order to prove the proposition, at first we notice that this corresponds to the case $m = 3$ and $l = 2$ in (51) and (50), respectively.

For $m = 3$ in (49) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2 \leq C\left(\|\partial_x^3 u_\varepsilon\|_{L_x^2}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2 + \|\partial_x^3\rho_\varepsilon\|_{L_x^2}\|\partial_x^4 u_\varepsilon\|_{L_x^2}\right). \tag{98}$$

Similarly, $l = 2$ in (50) gives us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x^2 V_\varepsilon|^2 dx = &-\int_{\mathbb{T}}\partial_x^2\left(\left(u_\varepsilon + \frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\partial_x\rho_\varepsilon\right)\partial_x V_\varepsilon\right)\partial_x^2 V_\varepsilon dx\\ &+\int_{\mathbb{T}}\partial_x^2\left(\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x^2 V_\varepsilon\right)\partial_x^2 V_\varepsilon dx\\ &-\int_{\mathbb{T}}\partial_x^2\left(\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}V_\varepsilon^2\right)\partial_x^2 V_\varepsilon dx.\end{aligned} \tag{99}$$

Applying integration by parts for the terms on the right side of the above equation, followed by an adjustment of the terms, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x^2 V_\varepsilon|^2\,dx + \int_{\mathbb{T}}\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}|\partial_x^3 V_\varepsilon|^2\,dx\\ &= \int_{\mathbb{T}}u_\varepsilon\partial_x^2 V_\varepsilon\partial_x^3 V_\varepsilon + \int_{\mathbb{T}}\partial_x u_\varepsilon\partial_x V_\varepsilon\partial_x^3 V_\varepsilon\,dx + \int_{\mathbb{T}}\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\partial_x^2\rho_\varepsilon\partial_x V_\varepsilon\partial_x^3 V_\varepsilon\,dx\\ &+\int_{\mathbb{T}}\left(\frac{\rho_\varepsilon\lambda_\varepsilon'(\rho_\varepsilon) - \lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon^2}\right)|\partial_x\rho_\varepsilon|^2\partial_x V_\varepsilon\partial_x^3 V_\varepsilon\,dx + \int_{\mathbb{T}}\partial_x\left(\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}\right)V_\varepsilon^2\partial_x^3 V_\varepsilon\,dx\\ &+2\int_{\mathbb{T}}\left(\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2}\right)V_\varepsilon\partial_x V_\varepsilon\partial_x^3 V_\varepsilon\,dx := \sum_{i=1}^{6}K_i.\end{aligned}$$

We use Lemma 7.1 to estimate terms K_i for $i = 1, \cdots, 6$. First, we recall the inequality $\dfrac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon} \geq \varepsilon$ to conclude

$$\int_{\mathbb{T}}\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}|\partial_x^3 V_\varepsilon|^2\,dx \geq \varepsilon\int_{\mathbb{T}}|\partial_x^3 V_\varepsilon|^2\,dx.$$

Control of K_1. Here we have

$$|K_1| \leq \|u_\varepsilon\|_{L_x^\infty}\|\partial_x^2 V_\varepsilon\|_{L_x^2}\|\partial_x^3 V_\varepsilon\|_{L_x^2} \leq \frac{\varepsilon}{16}\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + \frac{4}{\varepsilon}\|u_\varepsilon\|_{L_x^\infty}^2\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2.$$

Proceeding similarly as in (92), we obtain

$$|K_1| \leq \frac{\varepsilon}{16}\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + 4\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_x^2}^2 + \varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L_x^2}^2\right)\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2. \tag{100}$$

Control of K_2. Also, for this term we use Young's inequality and the inequality (52) to get

$$|K_2| \leq \|\partial_x u_\varepsilon\|_{L_x^2}\|\partial_x V_\varepsilon\|_{L_x^\infty}\|\partial_x^3 V_\varepsilon\|_{L_x^2} \leq \frac{\varepsilon}{16}\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + \frac{8}{\varepsilon}\|\partial_x u_\varepsilon\|_{L_x^2}^2\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2.$$

Hence, we have

$$|K_2| \leq \frac{\varepsilon}{16}\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\partial_x u_\varepsilon\|_{L_x^2}^2\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 + \|\partial_x u_\varepsilon\|_{L_x^2}^2\|\partial_x V_\varepsilon\|_{L_x^2}^2\right). \tag{101}$$

Control of K_3. We note that

$$|K_3| \leq \left\|\frac{\lambda_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\right\|_{L_t^\infty}\|\partial_x^2\rho_\varepsilon\|_{L_x^\infty}\|\partial_x V_\varepsilon\|_{L_x^2}\|\partial_x^3 V_\varepsilon\|_{L_x^2} \leq \frac{\varepsilon}{16}\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x V_\varepsilon\|_{L_x^2}^2\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2. \tag{102}$$

Control of K_4. We start with the following estimate:

$$|K_4| \leq \left\| \frac{\rho_\varepsilon \lambda_\varepsilon'(\rho_\varepsilon) - \lambda_\varepsilon(\rho_\varepsilon)}{\lambda_\varepsilon^2} \right\|_{L_x^\infty} \|\partial_x \rho_\varepsilon\|_{L_x^\infty}^2 \|\partial_x V_\varepsilon\|_{L_x^2} \|\partial_x^3 V_\varepsilon\|_{L_x^2}.$$

Additionally, we apply Young's inequality to get

$$|K_4| \leq \frac{\varepsilon}{16} \|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \|\partial_x V_\varepsilon\|_{L_x^2}^2 \|\partial_x^2 \rho_\varepsilon\|_{L_x^2}^4. \tag{103}$$

Control of K_5. A direct calculation gives us the following identity:

$$\partial_x \left(\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2} \right) = \left[\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)^2} \left(\lambda_\varepsilon''(\rho_\varepsilon)\rho_\varepsilon + 2\lambda_\varepsilon'(\rho_\varepsilon)\right) - \frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^3} \left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right) \right] \partial_x \rho_\varepsilon.$$

As a consequence of the above identity, we obtain

$$\left\| \left[\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)^2} \left(\lambda_\varepsilon''(\rho_\varepsilon)\rho_\varepsilon + 2\lambda_\varepsilon'(\rho_\varepsilon)\right) - \frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^3} \left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right) \right] \right\|_{L_x^\infty} \leq C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon),$$

followed by

$$|K_5| \leq C(\bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \|\partial_x \rho_\varepsilon\|_{L_x^2} \|V_\varepsilon\|_{L_x^\infty}^2 \|\partial_x^3 V_\varepsilon\|_{L_x^2}.$$

By using the Young inequality, we can further deduce

$$|K_5| \leq \frac{\varepsilon}{16} \|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \|\partial_x \rho_\varepsilon\|_{L_x^2}^2 \|V_\varepsilon\|_{H_x^1}^4. \tag{104}$$

Control for K_6. Here we observe that

$$|K_6| \leq \left\| \left(\frac{\left(\lambda_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon + \lambda_\varepsilon(\rho_\varepsilon)\right)}{\left(\lambda_\varepsilon(\rho_\varepsilon)\right)^2} \right) \right\|_{L_x^\infty} \|\partial_x V_\varepsilon\|_{L_x^2} \|V_\varepsilon\|_{L_x^\infty} \|\partial_x^3 V_\varepsilon\|_{L_x^2}.$$

This implies

$$|K_6| \leq \frac{\varepsilon}{16} \|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \|V_\varepsilon\|_{H_x^1}^4. \tag{105}$$

Therefore, adding inequalities (100)-(105), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} |\partial_x^2 V_\varepsilon|^2 \, dx + \frac{5}{8}\varepsilon \int_{\mathbb{T}} |\partial_x^3 V_\varepsilon|^2 \, dx \\
&\leq C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \|\partial_x V_\varepsilon\|_{L_x^2}^2 \|\partial_x^3 \rho_\varepsilon\|_{L_x^2}^2 \\
&+ C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \left(\left(\underline{\rho}_\varepsilon^{-1/2} \varepsilon^{-1} \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L_x^2}^2 + \varepsilon^{-3} \underline{\rho}_\varepsilon^{-2\alpha} \|V_\varepsilon\|_{L_x^2}^2 \right) + \|\partial_x u_\varepsilon\|_{L_x^2}^2 \right) \|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 \\
&+ C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon) \left(\|\partial_x V_\varepsilon\|_{L_x^2}^2 \|\partial_x^2 \rho_\varepsilon\|_{L_x^2}^4 + (1 + \|\partial_x \rho_\varepsilon\|_{L_x^2}^2) \|V_\varepsilon\|_{H_x^1}^4 \right).
\end{aligned} \tag{106}$$

Next, we would like to estimate $\|\partial_x^4 u_\varepsilon\|_{L_x^2}$. A direct computation leads to the following identity:

$$\begin{aligned}
\partial_x^4 u_\varepsilon = &\frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \partial_x^3 V_\varepsilon - \left(\frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} + \frac{2\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)} \right) \partial_x \rho_\varepsilon \partial_x^2 V_\varepsilon \\
&- \left(\left(\frac{2\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)} \right)' - \left(\frac{2\lambda_\varepsilon'(\rho_\varepsilon)^2}{\lambda_\varepsilon(\rho_\varepsilon)^2} - \frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} \right) \right) |\partial_x \rho_\varepsilon|^2 \partial_x V_\varepsilon - \left(2\frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)} + \frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} \right) \partial_x^2 \rho_\varepsilon \partial_x V_\varepsilon \\
&+ \left(2\left(\frac{2\lambda_\varepsilon'(\rho_\varepsilon)^2}{\lambda_\varepsilon(\rho_\varepsilon)^2} - \frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} \right) - \left(\frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} \right)' \right) \partial_x \rho_\varepsilon \partial_x^2 \rho_\varepsilon V_\varepsilon \\
&+ \left(\frac{2\lambda_\varepsilon'(\rho_\varepsilon)^2}{\lambda_\varepsilon(\rho_\varepsilon)^2} - \frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} \right)' (\partial_x \rho_\varepsilon)^3 V_\varepsilon - \left(\frac{\lambda_\varepsilon'(\rho_\varepsilon)}{\lambda_\varepsilon(\rho_\varepsilon)^2} \right) \partial_x^3 \rho_\varepsilon V_\varepsilon.
\end{aligned}$$

We rewrite the above expression as

$$\begin{aligned}\partial_x^4 u_\varepsilon &= \mathcal{R}_1(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))\partial_x^3 V_\varepsilon + \mathcal{R}_2(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))\partial_x\rho_\varepsilon\partial_x^2 V_\varepsilon \\ &+ \mathcal{R}_3(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))|\partial_x\rho_\varepsilon|^2\partial_x V_\varepsilon + \mathcal{R}_4(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))\partial_x^2\rho_\varepsilon\partial_x V_\varepsilon \\ &+ \mathcal{R}_5(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))\partial_x\rho_\varepsilon\partial_x^2\rho_\varepsilon V_\varepsilon + \mathcal{R}_6(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))(\partial_x\rho_\varepsilon)^3 V_\varepsilon \\ &+ \mathcal{R}_7(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))\partial_x^3\rho_\varepsilon V_\varepsilon := \sum_{i=1}^{7} D_i,\end{aligned}$$

where for each $i = 1, \cdots, 7$ we have

$$\|\mathcal{R}_i(\lambda_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon'(\rho_\varepsilon), \lambda_\varepsilon''(\rho_\varepsilon))\|_{L_x^\infty} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon).$$

Therefore, we get the following estimates:

$$\begin{aligned}&\|D_1\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^3 V_\varepsilon\|_{L_x^2}; \\ &\|D_2\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2 V_\varepsilon\|_{L_x^2}\|\partial_x\rho_\varepsilon\|_{L_x^\infty} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^2 V_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}; \\ &\|D_3\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x V_\varepsilon\|_{L_x^2}\|\partial_x\rho_\varepsilon\|_{L_x^\infty}^2 \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x V_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2; \\ &\|D_4\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x V_\varepsilon\|_{L_x^2}\|\partial_x^2\rho_\varepsilon\|_{L_x^\infty} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x V_\varepsilon\|_{L_x^2}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}; \\ &\|D_5\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|V_\varepsilon\|_{L_x^\infty}\|\partial_x\rho_\varepsilon\|_{L_x^\infty}\|\partial_x^2\rho_\varepsilon\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|V_\varepsilon\|_{H_x^1}\|\partial_x^2\rho_\varepsilon\|_{L_x^2}^2; \\ &\|D_6\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|V_\varepsilon\|_{L_x^\infty}\|\partial_x\rho_\varepsilon\|_{L_x^\infty}\|\partial_x\rho_\varepsilon\|_{L_x^2}^2 \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|V_\varepsilon\|_{H_x^1}\|\partial_x\rho_\varepsilon\|_{L_x^2}^3; \\ &\|D_7\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|V_\varepsilon\|_{L_x^\infty}\|\partial_x^3\rho_\varepsilon\|_{L_x^2} \le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|V_\varepsilon\|_{H_x^1}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}.\end{aligned}$$

Now, going back to (98) and plugging the above estimate in it, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2 &\le C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^3 V_\varepsilon\|_{L_x^2}\|\partial_x^3\rho_\varepsilon\|_{L_x^2} \\ &+ C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|_{H_x^1} + \|\partial_x^3 u_\varepsilon\|_{L_x^2}\right)\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2 \\ &+ C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\rho_\varepsilon\|_{H_x^2}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}\|\partial_x^2 V_\varepsilon\|_{L_x^2}\right) \\ &+ C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|_{H_x^1}\left(\|\rho_\varepsilon\|_{H_x^2} + \|\rho_\varepsilon\|_{H_x^2}^2 + \|\rho_\varepsilon\|_{H_x^2}^3\right)\right)\|\partial_x^3\rho_\varepsilon\|_{L_x^2}.\end{aligned}$$

Now we add the above inequality with (106) and use the following inequality

$$C(\bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^3 V_\varepsilon\|_{L_x^2}\|\partial_x^3\rho_\varepsilon\|_{L_x^2} \le \frac{\varepsilon}{8}\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 + C(\varepsilon, \bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2$$

to deduce

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2 + \frac{1}{2}\frac{d}{dt}\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 + \frac{1}{2}\varepsilon\|\partial_x^3 V_\varepsilon\|_{L_x^2}^2 \le \tilde F_1(t)\|\partial_x^3\rho_\varepsilon\|_{L_x^2}^2 + \tilde F_2(t)\|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 + \tilde G(t) \quad (107)$$

where

$$\tilde F_1(t) = C(\varepsilon, \bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\left(\|V_\varepsilon\|_{H_x^1} + \|\partial_x^3 u_\varepsilon\|_{L_x^2} + \|\partial_x u_\varepsilon\|_{L_x^2} + \|\partial_x^2 V_\varepsilon\|_{L_x^2}^2 + \|V_\varepsilon\|_{H_x^1}^2 + \|\rho_\varepsilon\|_{H_x^1}^2 + 1\right),$$

$$\tilde F_2(t) = C(\varepsilon, \bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\left(\left(\underline{\rho}_\varepsilon^{-1/2}\varepsilon^{-1}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_x^2}^2 + \varepsilon^{-3}\underline{\rho}_\varepsilon^{-2\alpha}\|V_\varepsilon\|_{L_x^2}^2\right) + \|\partial_x u_\varepsilon\|_{L_x^2}^2 + \|\rho_\varepsilon\|_{H_x^2}^2\right)$$

and

$$\tilde G(t) = C(\varepsilon, \bar\rho_\varepsilon, \underline{\rho}_\varepsilon)\left(\|\partial_x^2 u_\varepsilon\|_{L_x^2}^2 + \|\partial_x^3 u_\varepsilon\|_{L_x^2}^2 + \left(\sum_{k=1}^{6}\|\rho_\varepsilon\|_{H_x^2}^k\right)\left(\sum_{k=1}^{4}\|V_\varepsilon\|_{H_x^1}^k\right)\right).$$

From Lemma 3.8 and Lemma 7.1, we have

$$\|\tilde{F}_1\|_{L^1_t} + \|\tilde{F}_2\|_{L^1_t} + \|\tilde{G}\|_{L^1_t} \leq C(\varepsilon, \bar{\rho}_\varepsilon, \underline{\rho}_\varepsilon, E_2, T).$$

Now, we introduce an additional hypothesis

$$\|\partial_x^3 \rho_\varepsilon^0\|_{L^2_x} + \|\partial_x^2 V_\varepsilon^0\|_{L^2_x} < \infty.$$

Again we use Grönwall's inequality to conclude

$$\begin{aligned}&\|\partial_x^3 \rho_\varepsilon\|^2_{L^\infty_t L^2_x} + \|\partial_x^2 V_\varepsilon\|^2_{L^\infty_t L^2_x} + \frac{\varepsilon}{2}\|\partial_x^3 V_\varepsilon\|^2_{L^2_t L^2_x}\\ &\leq C(\varepsilon, E_{3,\varepsilon}, \overline{\rho}_\varepsilon, \underline{\rho}_\varepsilon, T),\end{aligned}$$

where

$$E_{3,\varepsilon} = E_{2,\varepsilon} + \|\partial_x^3 \rho_\varepsilon^0\|_{L^2_x} + \|\partial_x^2 V_\varepsilon^0\|_{L^2_x}.$$

We proceed analogously as in the proof of Lemma 7.1 to obtain the $L^\infty_t L^2_x$ estimate of $\partial_x^3 u_\varepsilon$ and the $L^2_t L^2_x$ estimate of $\partial_x^4 u_\varepsilon$. □

Next, we state and prove a generalized Poincaré inequality:

Proposition 7.2. *There exists a positive constant C such that the following inequality holds*

$$\|u\|_{L^1_x} \leq C\left(\|\partial_x u\|_{L^1_x} + \int_{\mathbb{T}} r|u|dx\right), \tag{108}$$

for any $u \in W^{1,1}(\mathbb{T})$ and any non-negative function r such that

$$0 < M_0 \leq \int_{\mathbb{T}} r\ dx < \infty, \qquad r \in L^\infty_x(\mathbb{T}). \tag{109}$$

Proof. We prove the statement by methods of contradiction. Suppose (108) is not true, then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ such that

$$\|u_n\|_{L^1_x} = 1, \qquad \|\partial_x u_n\|_{L^1_x} + \int_{\mathbb{T}} r_n|u_n|\ dx \leq \frac{1}{n}$$

and

$$r_n \to r \text{ weakly-* in } L^\infty_x.$$

Therefore, we have

$$\|u_n\|_{W^{1,1}_x} \leq 2.$$

As a consequence of compact embedding of $W^{1,1}_x$ in L^1_x, we obtain

$$u_n \to u \text{ strongly in } L^1_x.$$

Next, the bound $\|\partial_x u_n\|_{L^1_x} \leq \frac{1}{n}$ yields

$$\partial_x u_n \to 0 \text{ strongly in } L^1_x.$$

The above two statements imply

$$u_n \to u \text{ strongly in } W^{1,1}_x \text{ and } \partial_x u = 0 \text{ a.e..}$$

Now, note that the weak-* convergence of r_n in L^∞_x and strong convergence of u_n in L^1_x helps us to deduce

$$\int_{\mathbb{T}} r\ dx = 0,$$

that contradicts the hypothesis (109). □

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