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# Observer Design in LTI Time-Delay Systems using Partial Pole Placement with Applications

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**Abstract:** In order to estimate the state of a dynamical system, we address the problem of designing an observer for linear time-invariant (LTI) dynamical systems including a time-delay. Sufficient conditions for the existence of the proposed observer are given using partial placement of the poles for the error. Namely, we exploit the *multiplicity-induced-dominance* property of the characteristic function corresponding to the system's error. The effectiveness of the proposed observer design is shown in both state lag and input lag respectively through the problems of Mach number control in a wind tunnel and stabilization of the inverted pendulum on a cart by using the delay.

*Keywords:* Time delay, linear system, observer design, frequency-domain, spectral abscissa, partial pole placement

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## 1. INTRODUCTION

The presence of a delay in a dynamical system may alter the latter's performances and may be a source of instability. However, if such a delay is soundly taken into account, it may bring a few advantages in control design (stabilizing oscillations, wait-and-act strategy, etc.), see, for instance, Richard (2003) Sipahi et al. (2011) and the references therein. The main difficulty in the analysis of the asymptotic behavior of systems with delays is that, contrarily to systems governed by ordinary differential equations (ODEs), the corresponding characteristic function admits infinitely many roots. In fact, in all generality, the problem of characterizing the domain in the space of the equation's parameters that guarantee the exponential stability of solutions is a question of ongoing interest, see e.g. Michiels and Niculescu (2007b).

In control theory, to the best of the authors' knowledge, the first pole placement paradigm for continuous-time delay systems, called *finite pole placement* (FPP), was introduced in the late 1970s in Olbrot (1972); Manitius and Olbrot (1979), where a prediction of the state over a delay interval is set to counteract the effect of the delay, hence reducing the closed-loop system to a finite-dimensional plant. However, from a practical point of view, the corresponding controller implementation is subject to a discretization which may induce the loss of stability

of the closed-loop system, see e.g., Mondié and Michiels (2003).

A more recent pole placement analytical paradigm, called *Partial Poles Placement* (PPP), has been introduced in Mazanti et al. (2021a). It derives from two properties called respectively *multiplicity-induced-dominancy* (MID) and *coexistent-real-roots-inducing-dominancy* (CRID), see for instance Boussaada et al. (2020). Indeed, some recent works have shown that, in both retarded and neutral delay systems described by DDEs, a real root of maximal multiplicity is necessarily the rightmost root, which corresponds to the generic MID, see for instance Boussaada et al. (2020, 2022); Mazanti et al. (2021a). In this context, it should be noted that the multiplicity of a root itself is not essential as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. Recently, in Amrane et al. (2018); Bedouhene et al. (2020), it is shown that, under appropriate conditions, the coexistence of distinct negative zeros of a quasipolynomial may guarantee the exponential stability of the zero solution of the corresponding time-delay system.

However, observer design is a major topic in the study of dynamical time-delay systems since it is usually applied in order to have an estimation of the unknown states when they are not accessible.

Early works on the design of observers were devoted to linear systems Luenberger (1966, 1971); Trinh (1999). Since, this research area has become an active field, and the underlying ideas and methods were extended to many

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classes of systems. Several approaches have been developed to ensure the observer convergence.

In the literature, methods developed with the spectral observability assumption or, at least, the  $\nu$ -detectability assumption Ramos and Pearson (1995); Pearson and Figagbedzi (1989) exist. Unfortunately, in some cases, these methods may lead to cumbersome numerical problems. In Fattouh et al. (1999), the authors addressed the problem of designing an observer for linear systems with time-delays using the property that the estimated error converges to zero with a pre-specified decay rate; their work is based on a modification of the criterion proposed in Mori et al. (1982).

The main contribution of the paper is to extend the use of the PPP for observer design in two configurations: delay in the state and in the input. In both cases, the exponential stability of the trivial solution to the error equation is performed using the PPP method based on a control oriented MID property as developed and described in Boussaada et al. (2020); Balogh et al. (2021). To the best of the authors' knowledge, such a construction represents a novelty in the open literature.

The rest of this paper is as follows. We start by the problem formulation and some prerequisites in section 2. In section 3, we give the main results, i.e. conditions of existence and stability of the proposed observer. The feasibility and effectiveness of our approach are illustrated by simulation results in section 4 where two well-known applications are addressed: control of transonic flow in a wind tunnel and the control of an inverted pendulum by delayed feedback. Some remarks conclude the paper.

## 2. PREREQUISITES AND PROBLEM STATEMENT

Consider a generic dynamical system with a single delay described by the DDE:

$$\dot{\xi}(t) = A\xi(t) + A_d\xi(t - \tau), \quad (1)$$

under appropriate initial conditions, where  $\xi \in \mathbb{R}^n$  is the state vector,  $\tau$  is a positive constant delay. Matrices  $A \in \mathbb{R}^{n \times n}$  and  $A_d \in \mathbb{R}^{n \times n}$  are known constant matrices.

It is well-known that the asymptotic behavior of the solutions of (1) is determined from its spectrum (see, e.g. Bellman and Cooke (1963)), that is, the set of the (characteristic) roots of the associated characteristic function (denoted by  $\Delta(s, \tau)$  in the sequel).

The characteristic function  $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$  corresponding to (1) reads as follows

$$\Delta(s, \tau) = \det(sI_n - A - A_d e^{-\tau s}). \quad (2)$$

A generic result on the location of spectral values corresponding to (2) is given by the following proposition.

*Proposition 1.* Michiels and Niculescu (2007a) If  $s$  is a characteristic root of system (1), then it satisfies

$$|s| \leq \|A + A_d e^{-\tau s}\|_2.$$

This result combined with the triangular inequality provides a generic envelope curve around the characteristic roots of system (1).

In the sequel, we are interested by the study of a class of LTI delay systems characterized by the quasipolynomial function of the following form.

$$\Delta(s, \tau) = P_0(s) + P_1(s)e^{-\tau s}, \quad (3)$$

where  $\deg(P_0) > \deg(P_1)$ . Note that the problem of the analytical characterization of its rightmost root will be an essential ingredient in deriving our results.

### 2.1 Partial pole placement in delay systems

One of the most natural and classical way to stabilize a dynamical LTI delay system is to select the free parameters of the controller in order to choose the location of finitely many roots while also guaranteeing that the *dominant root*<sup>1</sup> is among the chosen ones. This has been the subject of several recent works, such as Bedouhene et al. (2020); Boussaada et al. (2020); Mazanti et al. (2021a); Ramirez et al. (2016). Contrarily to the strategy of FPP used, e.g., in Manitius and Olbrot (1979), the controllers designed using these techniques do not render the closed-loop system finite-dimensional, but instead control its rightmost spectral value. These methods also extend to some partial differential equations, as detailed, for example, in Mazanti et al. (2021b). In particular, in this work, we shall use the MID property as described in Balogh et al. (2021).

### 2.2 Problem formulation

Motivated by the control of a transonic flow in a wind tunnel and a friction-free inverted pendulum where the state is not fully accessible, we propose a PPP observer design. The first example includes a constant (pointwise) delay in the state vector and the second one is controlled by a delayed input.

*Delay in the state* Consider the system with delayed state described by the following state space model.

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau) + Bu(t) \quad (4a)$$

$$y(t) = Cx(t) \quad (4b)$$

$$x(t) = \phi(t) \quad t \in [-\tau, 0] \quad (4c)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^p$  are, respectively, the state vector, the control input and the measurements vector. The non-negative scalar  $\tau$  is a known constant delay. Finally,  $\phi(t)$  is a continuous function defined as the initial condition. Matrices  $A \in \mathbb{R}^{n \times n}$ ,  $A_d \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$  are known constant matrices.

For system (4), we propose to design a standard observer, allowing to estimate the states  $x(t)$  and described by:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + A_d \hat{x}(t - \tau) + Bu(t) + L(y(t) - C\hat{x}(t)) \\ &\quad + T(y(t - \tau) - C\hat{x}(t - \tau)) \end{aligned} \quad (5)$$

where  $\hat{x} \in \mathbb{R}^n$  denotes the estimates of  $x$ . Matrices  $L$  and  $T$  are unknown and shall be determined so that the estimation of the states  $\hat{x}$  converges to its real values  $x$ .

*Delay in the input* Now, we consider a system with a known time delay in the input as follows

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \quad (6a)$$

$$y(t) = Cx(t), \quad (6b)$$

<sup>1</sup> the rightmost characteristic root in the complex plane

and for which, we propose the following observer dynamics.

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t - \tau) + L(y(t) - C\hat{x}(t)) \\ &\quad + T(y(t - \tau) - C\hat{x}(t - \tau)) \end{aligned} \quad (7)$$

*The error equation* Let us denote by  $e(t) = x(t) - \hat{x}(t)$  the estimation error vector. In both cases, the dynamic of the estimation error is described by the following system.

$$\dot{e}(t) = (A - LC)e(t) + (A_d - TC)e(t - \tau). \quad (8)$$

Notice that such an error equation applies in both cases, input-delay or state-delay. In the sequel, we shall use the notations  $\bar{A} = A - LC$  and  $\bar{A}_d = A_d - TC$ .

*Proposition 1.* The dynamical system (5) (system (7)) represents an observer for the time-delay system described by (4) (system (6)) if, and only if, the dynamic error system given by (8) is asymptotically stable.

**Proof.** Computing the dynamics of the estimation error  $\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$ , one obtains the system given by (8) (with  $A_d = 0$  for the second case). This system needs to be stable, to ensure the convergence of the estimation error  $e(t)$  to zero.  $\square$

### 3. MAIN RESULTS

Let us consider system (8) written in the following form.

$$\dot{e}(t) - \bar{A}e(t) - \bar{A}_de(t - \tau) = 0. \quad (9)$$

Its characteristic function is  $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$  is given by

$$\Delta(s, \tau) = |sI_n - \bar{A} - \bar{A}_de^{-\tau s}| \quad (10)$$

Hence, the error system is characterised by the quasipolynomial function with the form of equation (3) where the polynomial  $P_0$  is given by:

$$P_0(s) = |sI_n - \bar{A}| = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

The following assumptions are considered:

*Assumption 1.*  $\text{rank}(\bar{A}_d) = 1$ .

*Assumption 2.* The polynomial  $P_0$  is real-rooted.

*Remark 2.* Under the above assumptions we are able to investigate the stability of the estimation error. Notice that Assumption 1 is a sufficient condition to guarantee that the quasipolynomial  $\Delta$  involves a single delay. The assumption 2 is requested since we shall apply Theorem 1 in Balogh et al. (2021).

Let us note  $s_0$  a real root with multiplicity at least  $n$  of the characteristic function (3). It is known that if  $s_0 < 0$  is the corresponding rightmost root then the zero solution of system (9) is asymptotically stable and consequently the estimation error is stable. The following proposition from Balogh et al. (2021) gives explicitly the integral representation of the quasipolynomial.

*Proposition 2.* If the quasipolynomial (3) has a real root  $s_0$  with multiplicity at least  $n$  then it can be written as

$$\Delta(s) = (s - s_0)^n \left( a_n + \int_0^1 e^{-(s-s_0)\tau t} \frac{\tau R_{n-1}(s_0, \tau t)}{(n-1)!} dt \right),$$

where the family of polynomials  $R_k(s, \tau)$  is defined as

$$R_k(s, \tau) = \sum_{i=0}^k \binom{k}{i} P^{(i)}(s) \tau^{k-i}, \quad k \in \mathbb{N}^*. \quad (11)$$

The following proposition, also from Balogh et al. (2021), provides sufficient conditions for the dominance of the multiple spectral value.

*Proposition 3.* Let  $P(s)$  be real-rooted and  $s_a$  be the average of its roots. Then system (3) is  $\gamma$ -stabilizable with  $\gamma \leq s_a$  if, and only if,  $\tau \in ]0, \tau_\gamma[$ , where  $\tau_\gamma$  is the smallest positive root of  $R_n(\gamma, \tau)$  for  $\tau$ .

Now we are able to formulate and prove the following result.

*Theorem 3.* Consider that Assumptions 1 and 2 hold and the quasipolynomial (10) admits a root at some complex number  $s_0$  with multiplicity  $n + 1$ . Then,  $s_0$  is necessarily algebraic and the corresponding characteristic polynomial denoted  $\mathbb{P}$  (called in the sequel *Elimination-produced polynomial*) is real-rooted. Moreover, if  $\mathbb{P}$  is the minimal-degree polynomial such that  $\mathbb{P}(s_0) = 0$  and that  $s_0$  is the spectral abscissa of  $\mathbb{P}$ , then  $s_0$  is the spectral abscissa of (10). Furthermore, if  $\mathbb{P}$  is Hurwitz then the estimation error described by (8) is exponentially stable with  $s_0$  as decay rate and the estimation states vector converges to its real value (in both cases).

**Proof.** The vanishing of the characteristic function  $\Delta$  is equivalent to  $\exp(-s\tau) = -P_0(s)/P_1(s)$ . Then, one considers the algebraic system of  $n$  equations  $\Delta'(s) = 0, \dots, \Delta^{(n)}(s) = 0$ . First, owing to the linearity of the system, one has  $\Delta'(s) = 0, \dots, \Delta^{(n-1)}(s) = 0$  with respect to the coefficients of  $P_1$ , then one proceeds by eliminating them order by order. Next, the values of these coefficients are substituted in the remaining equation  $\Delta^{(n)}(s) = 0$  which gives  $\mathbb{P}$  the so-called elimination-produced polynomial in the variable  $s$ , the delay  $\tau$  and the coefficients of  $P_0$ . Next, by assuming that assumptions 1 and 2 hold, the proof of the theorem is based on proposition 3 since the quasipolynomial associated to the error system (8) is given by equation (3).  $\square$

### 4. ILLUSTRATIVE EXAMPLE

In this section, we provide some illustrative examples in both cases: input-delay and state-delay.

#### 4.1 System with state delay

We revisit in this section the problem of control of a transonic flow in a wind tunnel discussed in Maniatis (1984). The analysis of transonic flows is a challenging problem in compressible fluid dynamics, since a full model of the flow would involve the Navier–Stokes equations in a three-dimensional domain and boundary controls for temperature and pressure regulation. A further simplified model was presented in Armstrong and Tripp (1981) in order to analyze the response of the Mach number of the flow to changes in the guide vane angle. Instead of using partial differential equations, propagation phenomena are modeled in Armstrong and Tripp (1981) through a time-delay, leading to the time-delay system

$$\begin{cases} \kappa m'(t) + m(t) = k\vartheta(t - \tau_0), \\ \vartheta''(t) + 2\zeta\omega\vartheta'(t) + \omega^2\vartheta(t) = \omega^2 u(t), \end{cases} \quad (12)$$

in which  $m$ ,  $\vartheta$ , and  $u$  represent, respectively, perturbations of the Mach number of the flow, the guide vane angle,

and the input of the guide vane actuator, with respect to steady-state values. The parameters  $\kappa$  and  $k$  depend on the steady-state operating point and are assumed to be constant as long as  $m$ ,  $\vartheta$ , and  $u$  remain small, and satisfying  $\kappa > 0$  and  $k < 0$ . The parameters  $\zeta \in (0, 1)$  and  $\omega > 0$  come from the design of the guide vane angle actuator and are thus independent from the operating point. The time-delay  $\tau_0$  is assumed to depend only on the temperature of the flow. In the absence of control ( $u(t) = 0$ ), the open-loop system (12) is exponentially stable.

Equation (12) may be written under a state space model as follows

$$\dot{X}(t) = AX(t) + A_d X(t - \tau) + Bu(t), \quad (13)$$

where  $X(t) = [\dot{\vartheta}(t) \vartheta(t) m(t)]^T$ , matrices  $A$ ,  $A_d$  and  $B$  are given by

$$A = \begin{bmatrix} -2\xi\omega & -\omega^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{\kappa} \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & \frac{k}{\kappa} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \omega^2 \\ 0 \\ 0 \end{bmatrix}.$$

The available measurement vector is described by

$$y(t) = CX(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X(t).$$

As the obtained system fits with system (4), in order to estimate the corresponding state vector  $X(t)$ , we consider an observer in the form (5). In this example, the error dynamics obey to system (8). By choosing the observer matrices in the following fashion

$$L = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ 0 & 0 \\ 1 & \alpha_{3,2} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & \beta_{1,2} \\ 0 & \beta_{2,2} \\ \frac{k}{\kappa} & \beta_{3,2} \end{bmatrix}$$

the rank condition (Assumption 1) is satisfied and the associated characteristic function (3) is defined by

$$\begin{aligned} P_0(s) = & s^3 + \left(2\xi\omega + \frac{\alpha_{3,2}\kappa + 1}{\kappa}\right) s^2 \\ & + \left(\omega^2 + \frac{(2\kappa\xi\alpha_{3,2} + 2\xi)\omega}{\kappa} + \alpha_{1,1}\right) s \\ & + \frac{(\alpha_{3,2}\kappa + 1)\omega^2}{\kappa} + \frac{\kappa\alpha_{1,1}\alpha_{3,2} - \kappa\alpha_{1,2} + \alpha_{1,1}}{\kappa}, \end{aligned} \quad (14a)$$

$$P_1(s) = \beta_{3,2}s^2 + (2\omega\beta_{3,2}\xi - \beta_{2,2})s + \omega^2\beta_{3,2} - 2\omega\beta_{2,2}\xi + \alpha_{1,1}\beta_{3,2} - \beta_{1,2}. \quad (14b)$$

By forcing the multiplicity, one obtains the remaining parameters  $\beta_{i,j}$  and concludes that  $s_0$  is a root with multiplicity  $n + 1 = 4$  if, and only if,  $s_0$  is a root of the *elimination-produced polynomial*

$$\begin{aligned} \mathbb{P}(s) = & \kappa s^3 \tau^3 + ((1 + (2\xi\omega + \alpha_{3,2})\kappa)\tau^3 + 9\kappa\tau^2) s^2 \\ & + (((2\omega\xi\alpha_{3,2} + \omega^2 + \alpha_{1,1})\kappa + 2\xi\omega)\tau^3) s \\ & + ((6 + (12\xi\omega + 6\alpha_{3,2})\kappa)\tau^2 + 18\kappa\tau) s \\ & + ((\alpha_{3,2}\omega^2 + \alpha_{3,2}\alpha_{1,1} - \alpha_{1,2})\kappa + \omega^2 + \alpha_{1,1})\tau^3 \\ & + ((6\omega\xi\alpha_{3,2} + 3\omega^2 + 3\alpha_{1,1})\kappa + 6\xi\omega)\tau^2 \\ & + (6 + (12\xi\omega + 6\alpha_{3,2})\kappa)\tau + 6\kappa. \end{aligned}$$

Now, considering the system (12) with parameter values as the ones proposed in Manitius (1984) where  $\kappa = 2$  s,  $k = -0.67036$  rad<sup>-1</sup>,  $\xi = 1/3$ ,  $\omega = 5$  rad/s and

$\tau_0 = 1/3$ , one obtains  $\alpha_{1,1} = \frac{316}{9}$ ,  $\alpha_{1,2} = \frac{2294}{27}$ ,  $\alpha_{3,2} = -\frac{5}{6}$  guaranteeing the realrootedness of  $P_0$  as well as that of polynomial  $\mathbb{P}$ .

In particular, such a choice of  $\alpha_{i,j}$  leads to the following observer matrices

$$L = \begin{bmatrix} \frac{316}{9} & \frac{2294}{27} \\ \frac{9}{0} & \frac{27}{0} \\ 1 & -\frac{5}{6} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & -\frac{3536 e^{-\frac{5}{3}}}{9} \\ 0 & -\frac{554 e^{-\frac{5}{3}}}{9} \\ \frac{k}{\kappa} & \frac{9}{3} \end{bmatrix},$$

which guarantees that the assignable rightmost multiple root can be achieved at  $s_0 = -5$  as illustrated through Figure 1.

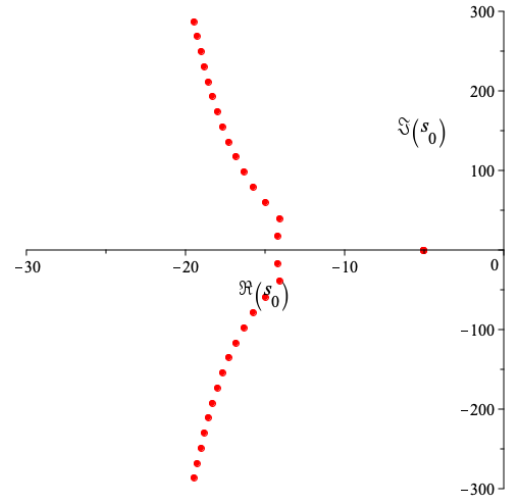


Fig. 1. The spectrum distribution of the quasipolynomial (3) with polynomials  $P_0$  and  $P_1$  defined in (14).

Figure 2 provides the evolution of the Mach number  $m(t)$ , the guide vane position  $\vartheta(t)$ , its derivative and their estimation, in the case where the initial conditions are given by  $m(t) = -0.1$ ,  $\vartheta(0) = \frac{m(0)}{k} = 0.15$ rad, and  $\dot{\vartheta}(t) = 0$ .

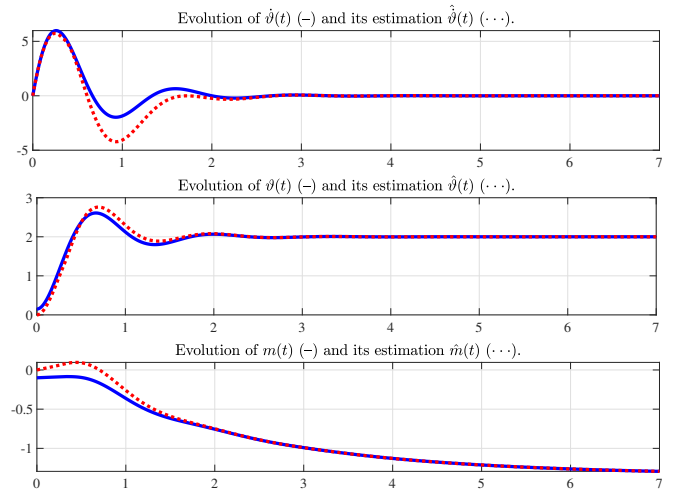
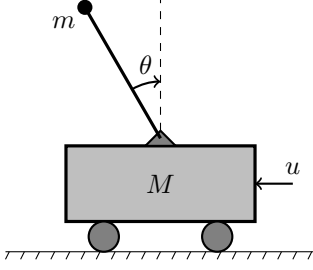


Fig. 2. Evolution of the state vector  $X(t)$  (—) and its estimation  $\hat{X}(t)$  (···).

## 4.2 Systems with input delay

In this section, we shall give an example to show the efficiency and the feasibility of the results obtained in this paper. Indeed, let us consider a dynamical system modeling a friction-free inverted pendulum on a cart.



The model adopted here was discussed in Sieber and Krauskopf (2005). The dynamics of the inverted pendulum on a cart can be modelled by a second-order differential equation for the angular displacement  $\theta$  of the tip of the pendulum

$$\left(1 - \frac{3m}{4(m+M)} \cos^2(\theta)\right) \ddot{\theta} + \frac{3m}{8(m+M)} \dot{\theta}^2 \sin(2\theta) + \sin \theta + u \cos \theta = 0, \quad (15)$$

where  $M$  is the mass of the cart,  $m$  is the mass of the pendulum, and  $u$  represents the control law, which is the horizontal driving force.

We assume that the system is controlled by a delayed input  $u(t - \tau)$ . For small  $\theta$ , we can write  $\sin(\theta) = \theta$  and  $\cos(\theta) = 1$ . Also dropping all non-linear components ( $\dot{\theta}^2 = 0$ ), equation (15) may be rewritten as

$$\left(1 - \frac{3m}{4(m+M)}\right) \ddot{\theta}(t) + \theta(t) + u(t - \tau) = 0. \quad (16)$$

Next, We denote by  $X(t) = [\dot{\theta} \ \theta]^T$ , so system (16) may be rewritten in the following state space model

$$\dot{X}(t) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} X(t) + \begin{bmatrix} -a \\ 0 \end{bmatrix} u(t - \tau) \quad (17)$$

where  $a = \left(1 - \frac{3m}{4(m+M)}\right)^{-1}$ . The available measurement is given by  $y(t) = [0 \ 1]X(t)$ .

In this case the rank condition (Assumption 1) is satisfied and the associated characteristic quasipolynomial is given by:

$$\Delta(s) = (s\beta_{2,1} + \beta_{1,1}) e^{-s\tau} + s^2 + s\alpha_{2,1} + a + \alpha_{1,1}. \quad (18)$$

Next, the realrootedness of the corresponding  $P_0$  (Assumption 2) is guaranteed by the positivity of its discriminant, that is,  $\delta = \alpha_{2,1}^2 - 4a - 4\alpha_{1,1}$ . As shown in Boussaada et al. (2020), if the discriminant  $\delta \geq 0$ , then the multiplicity of any given root of the quasipolynomial function (18) is bounded by 3. For an arbitrary positive delay  $\tau$ , the quasipolynomial (18) admits a real spectral value at  $s = s_{\pm}$  with algebraic multiplicity 3 if, and only if,

$$s_{\pm} = \frac{-\tau \alpha_{2,1} - 4 \pm \sqrt{8 + \tau^2 \delta}}{2\tau},$$

and the system parameters satisfy

$$\begin{cases} \beta_{1,1} = (\alpha_{2,1}s_{\pm} + \frac{\alpha_{2,1}^2}{2} - \frac{\delta}{2} + \frac{6\alpha_{2,1} + 10s_{\pm}}{\tau} + \frac{6}{\tau^2})e^{\tau s_{\pm}}, \\ \beta_{2,1} = \left(2s_0 + \alpha_{2,1} + \frac{2}{\tau}\right)e^{\tau s_{\pm}}. \end{cases} \quad (\star_{\pm})$$

If  $(\star_{\pm})$  is satisfied then  $s = s_{\pm}$  is the spectral abscissa corresponding to (18). Furthermore, the trivial solution is asymptotically stable if, and only if,  $\tau$  satisfies the following conditions

$$\begin{cases} \tau \in ]0, \tau_-[ \text{ when } \alpha_{1,1} < -a, \\ \text{or} \\ \tau \in ]0, \tau_-[ \cup ]\tau_+, \infty[ \text{ when } \alpha_{1,1} > -a \text{ and } \alpha_{2,1} < 0, \end{cases}$$

where  $\tau_{\pm} = \frac{-\alpha_{2,1} \pm \sqrt{\delta + 2\alpha_{1,1} + a}}{\alpha_{1,1} + a}$ .

For instance, choosing  $\alpha_{1,1} = -a - 1$  and  $\alpha_{2,1} = 1$ , we define the rightmost root of (3) as a function of the delay  $\tau$  as illustrated in figure 3.

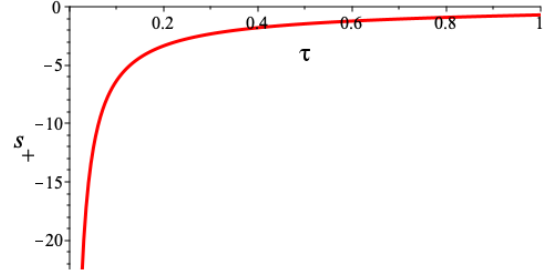


Fig. 3. The spectral abscissa of (3) as a function of the delay  $\tau$  where  $\alpha_{1,1} = -a - 1$  and  $\alpha_{2,1} = 1$ .

For further discussions on the MID property in second-order delay, the reader may consider Boussaada et al. (2020).

As a numerical example, we choose  $m = 0.5$  Kg and  $M = 5$  Kg and a time delay  $\tau = 0.1$ s. By setting the observer matrices to the latter computation, we obtain

$$L = \begin{bmatrix} -2.0732 \\ 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 10.803 \\ 4.453 \end{bmatrix}$$

The initial conditions are given by  $X(0) = [0 \ 0.5]^T$ . Figure 4 shows the convergence of the estimation state vector to its real values.

## 5. CONCLUDING REMARKS

In this paper, a Luenberger-type observer for linear time delay systems has been developed. Two types of delayed systems have been considered, systems with delayed states and systems with delayed inputs. Under sufficient conditions, the stability of the estimation error systems has been proved in both cases using partial placement of the poles for the error, namely the MID property. The practical use of the proposed observers has been demonstrated experimentally for a transonic flow in a wind tunnel which is modeled by a delayed state model and the inverted pendulum system which is described by a delayed input system. A root location of the quasipolynomials was accessed using numerical techniques and the observer matrices were deduced to allow a suitable estimation of the systems states. A generalization of the observer design and an observer based control will be studied in future work.

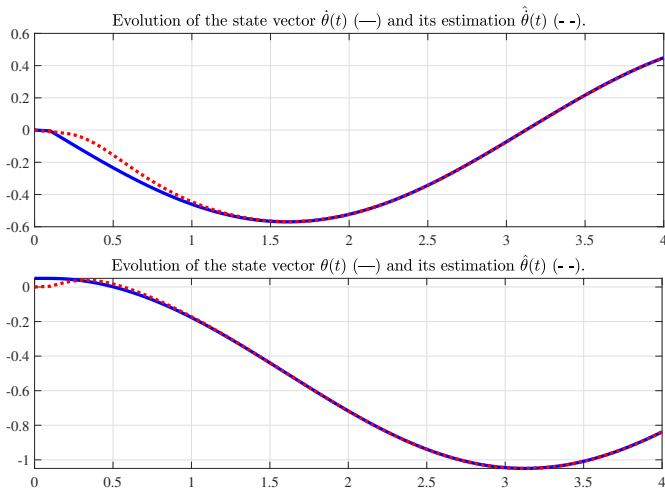


Fig. 4. Evolution of the state vector  $X(t)$  (—) and its estimation  $\hat{X}(t)$  (- -) with  $\tau = 0.1s$ .

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