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Rotating matter-wave beam splitters and consequences for atom gyroscopes

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The effect of a rotation on a matter-wave-laser beam splitter is studied and modeled. This modeling is shown to give important nontrivial corrections to the Sagnac phase shift of matter-wave gyroscopes when the duration of the laser action cannot be neglected with respect to the propagation time between beam splitters. This result is illustrated on a Mach-Zehnder atom gyroscope (rate gyroscope) with running laser beam splitters. A quasi-classical description of the interferometer arms inside the beam splitters is proposed to interpret the corrected Sagnac phase shift in terms of an effective interferometer area.

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I. INTRODUCTION

Two steps are particularly relevant for the modeling of interferometers, whatever the nature of the waves involved (light, electron, neutron, atomic, molecular,...). The first one consists in obtaining a precise analytical expression of the fringes signal (i.e., with the amplitude, contrast factor, and phase shift of the fringes), which can be observed at the output of such devices. The *phase shift* is especially important since it gives the interferometer sensitivity to the effect one wants to detect or avoid. Indeed, at the lowest order, the phase shift expression depends linearly on this effect, and the proportionality coefficient gives the interferometer *intrinsic* sensitivity, i.e., its *scale factor*. The second step deals with the phase shift true origin and the interpretation of the scale factor in terms of geometrical quantities, such as the (space-time) interferometer area for example. These two steps have been regularly reexamined so as to respond to experimental and theoretical advances, and have to be examined afresh in the light of recent works in light as well as matter-wave interferometries.

The first step is all the more important for *atom* interferometers since their precision, sensitivity, and stability have become outstanding [1,2]. Atom gravimeters, gradiometers, gyroscopes,¹ atomic clocks, etc., have now approached—or surpassed—the level of all the other similar devices which do not use atomic waves (such as falling corner cubes, gyrolasers, etc.) [1,3–7]. The superior sensitivity of matter waves over light waves is particularly striking in gyrometry: for equal rotation rates and interferometer areas, an atom gyroscope is about 10^{10} times more sensitive than a gyrolaser

(for visible light and middle-weight atoms). In practice, however, this ratio has to be reduced by several orders of magnitude due to the current differences in enclosed areas and photon and atomic fluxes. The best sensitivity obtained with an atom gyroscope was $6 \times 10^{-10} \text{ rad s}^{-1} \text{ Hz}^{-1/2}$ in 2000 [5] (after less than ten years of expertise in atom interferometry), whereas it was $9 \times 10^{-11} \text{ rad s}^{-1} \text{ Hz}^{-1/2}$ in 2004 for the best laser ring gyro [8]. Moreover, a great improvement of this accuracy level is expected through the use of coherent atomic and molecular sources (to increase the interferometer signal-to-noise ratio) [9,10]; the guiding and trapping of atomic beams (to increase the interferometer area) [11–13]; the implementation on micro- and nanochips (for quantum computing as well as commercial and space applications) [10,13]; the use of particular entangled states (to increase the interferometer sensitivity) [14–17]; the use of microgravity environments (to increase the interaction times) [18,19], etc. Such progress has opened up new possibilities for testing tiny relativistic effects (e.g., Lense-Thirring and geodetic effects [20]), performing tests of the equivalence principle and variability of fundamental constants [21], and contemplating long-term inertial autonomous missions (for positioning and navigation on Earth, under the sea, or in deep space).

This high (expected or already reached) accuracy level requires one to go beyond the former modelings of atom interferometers and, especially, of their main component: the atomic beam splitters. A great effort has already been done to improve both the *global* modeling of atom interferometers [1,22–29] and their *partial* modelings, i.e., the modeling of their constitutive elements: the atomic source, the detection process, and the beam splitters [28,31]. Concerning the global modeling, a very general phase shift formula has recently been obtained with the help of symplectic invariants and Gaussian atom optics tools [27,30]. This formula is valid for any interferometer geometry and accounts for all the (simultaneous) time-dependent effects whose representative potential is at most quadratic in position and momentum. It generalizes the previous analytical expressions [25,26], and, in particular, the usual Sagnac expression of matter-wave gyroscopes [32] and the Wolf-Tourenç formula for atom gradio-gravimeters [33].

Regarding the partial modelings, a unified approach has been developed to describe thermal, cold, and ultracold

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¹The term “gyroscope” is generally used to designate a device which measures both rotation angles and rotation rates. To distinguish between these two different applications, it is preferable to use different words: “gyroscope” for the measurement of rotation angles, for example, and “gyrometer” or “rate gyro” for the measurement of rotation rates. The term “gyrometer” is used in the present paper. A gyroscope can be obtained by integrating over time the signal of a gyrometer.

atomic sources in terms of a statistical distribution of Gaussian wave packets, and corresponding formulas have been obtained for the modeling of the usual detection processes [31]. As for atomic beam splitters, although many of their features have been studied in the past (see [28] and references therein), very simple modelings are still considered sometimes for the global modeling of atom interferometers. In these oversimplified modelings, only the following effects are considered: (a) the splitting of the incident atomic wave packet into several wave packets, (b) among which one is equal to the incident wave packet, up to a change of amplitude, (c) the others differing from the incident wave packet in their central momentum, internal state, amplitude, and phase. Practical though they are, these beam splitter modelings do not account for all the effects which depend explicitly on the duration and/or width of the atomic splitting zones, and, consequently, are sometimes called “infinitely thin modelings.”

In fact, only recently have been modeled several important such effects, such as the effect of additional external potentials during atomic splitting (time-dependent gravito-inertial and trapping potentials for example) or nontrivial dispersive effects (atomic Borrmann effect, for example). The reference [28] gives details of this accurate modeling in the case of a *laser* beam splitter. A particularly interesting variant of this modeling consists in developing the generalized detuning to the first order in the considered inertial or gravitational effect. The case of a uniform acceleration acting on the atoms leads to nontrivial phase shift corrections, as it is explained in [28]. The rotation case is treated in the present paper.

This rotation can act either on the atoms or on the lasers, according to the chosen reference frame where the atomic evolution equation is expressed. Physical results, such as the interferometer phase shift, for example, do not depend on this choice since observables are scalar, i.e., *gauge invariant*. In the following, one will consider the atomic evolution equation in a rotating frame where the lasers are at rest. In this case, the laser wave vectors are constant and a term “ $\vec{r} \cdot (\vec{\Omega} \times \vec{p})$ ” appears in the total Hamiltonian [see Eq. (3)]. The reverse case (where the reference frame is defined as the rest—or inertial—frame of the atoms, with respect to which the lasers frame is rotating) would lead to time-dependent laser wave vectors in the atom-laser interaction Hamiltonian, but without any “ $\vec{r} \cdot (\vec{\Omega} \times \vec{p})$ ” term. These two descriptions are equivalent. Let us underline, however, that this question has to be carefully examined when we consider the addition of several other (unavoidable) inertial and gravitational effects as it is outlined in [27]. Indeed, the evolution equation which is obtained in this case may be simpler in one of these two reference frames, according to the *symmetry* properties of these additional potentials. For example, an atom gyrometer—used as inertial sensor in a vehicle—is sensitive not only to the (artificial) vehicle rotation but also to various accelerations and gradients of acceleration caused by the nonuniform mass distribution around the vehicle (time independent field of gravity of the Earth, for example). In this case, the atomic evolution equation is simpler to solve in the frame which is defined by this constant mass distribu-

tion, with respect to which the vehicle and lasers are rotating.

Actually, regarding the Sagnac effect, i.e., the effect of a rotation on an interferometer fringes pattern, many open questions and problems remain, which highlights the necessity to deepen the “second step” mentioned above. Indeed, this Sagnac effect has always caused controversies, regarding its different derivations (within or without relativistic theories, for example) as well as its interpretations and the question of its true origin (gauge invariance problem, link to Mach principle, topological origin, and geometrical interpretation) [34–39]. Recent works tend to clarify some of these problems, but essentially with light waves [40]. The advent of well controlled slowing, trapping, cooling, and guiding techniques of atoms and molecules as well as promising active matter-wave gyroscopes will certainly give new insights into these questions as well.

The beam splitter modeling is particularly relevant as it has already been outlined in *neutron* interferometry, where the Borrmann effect proved to be crucial to understand discrepancies between phase shift calculations and measurements [41,44]. This effect, discovered by Borrmann in 1941 [42], is well known in dynamical diffraction of x rays [43] and neutron waves [44], and has been recently studied in atom optics [45]. It demonstrates the existence of privileged wave propagation directions (i.e., group velocities) inside the *crystal* where a wave is diffracted. This crystal can be made of matter or light (e.g., with lasers). Historically, this effect was defined for absorbing crystals, but this condition can be relaxed for atom-laser interaction. This effect is valid for any kind of waves and can be stated as follows: “for a certain angle of incidence with respect to the crystal surface, the propagation of an incident wave packet inside the crystal is made (with no attenuation) along a unique trajectory which is orthogonal to the crystal surface.” Conversely, if this condition is not fulfilled, two main wave packets propagate inside the crystal with two different central trajectories, defining what is called the “Borrmann fan.” With neutron and light waves, this particular angle of incidence is the well known “Bragg angle.” Actually, the Borrmann effect can be related in a natural way to the usual “Bragg regime,” which is observed in atomic diffraction when the diffraction zone is thick enough.

The Borrmann effect helps us to answer the question: “Where do the waves propagate inside the diffraction zones?” and if this question is meaningful. This fact is crucial for the definition of the interferometer arms *inside* the laser beam splitters and, consequently, for the definition of both the interferometer area and its geometrical scale factor.

In this paper, I first solve the problem of a rotating atomic laser beam splitter to the first order in $\|\vec{\Omega}\|/\Omega_0$, where $\vec{\Omega}$ is the rotation rate vector and Ω_0 the laser Rabi frequency which characterizes the atom-laser interaction. Then, I express the solution as an effective instantaneous interaction in order to obtain a beam splitter modeling which is particularly well suited for the global modeling of atom interferometers. This beam splitter modeling is illustrated on two important cases: the π and $\pi/2$ beam splitters, which play

respectively the role of mirror and 50:50 beam splitter for atomic waves (Sec. II). Then, after recalling the general interferometer phase shift formula which has been recently obtained [27,30], I apply the $\pi/2$ and π beam splitters modeling to a Mach-Zehnder atom gyrometer and express the phase shift formula to the first order in τ/T , where τ is the beam splitters typical duration and T the propagation time between two beam splitters (Sec. III). Finally, before concluding and giving some prospects, I detail the problems encountered in defining the interferometer arms inside the beam splitters and propose a semiclassical approximation of the interferometer area.

II. EFFECT OF A ROTATION ON A MATTER-WAVE-LASER BEAM SPLITTER

The matter-wave beam splitters considered in this study consist of multilevel atomic systems subject to an interaction potential which couples the levels together. Until now, many different interaction potentials have been used to create atomic beam splitters. A recent review of these techniques is given in [28]. In the present paper I focus on the atomic beam splitters which are based on the quasi-resonant interaction between atoms and laser fields.

The atomic levels are not restricted to internal atomic levels, but more generally refer to energy-momentum states, i.e., eigenstates of both the internal and kinetic Hamiltonians. Atomic transitions can occur between internal states only (spectroscopy without Doppler effect, for example), external states only (diffraction in Kapitza-Dirac and Bragg regimes, for example), or between entangled states, the entanglement being between the internal and external states (stimulated Raman transitions, for example) or between the previous energy-momentum states and the eigenstates of the splitting potential (with the Fock states of the quantized electromagnetic field, for example).

Here, the *two beam approximation* is supposed to be valid: only two energy-momentum eigenstates are effectively coupled by the lasers. This coupling can be direct (for true two-level systems) or indirect (Raman transitions or atom diffraction in the Bragg regime, for example). In fact, it can be shown [46–48] that any N -photon transition (induced by running or standing laser waves) of a multilevel atom is equivalent to an effective 1-photon transition between two atomic levels if the other levels can be adiabatically eliminated. The effective photon may not be real, its frequency being null in the case of atomic Bragg diffraction, for example. Atomic beam splitters using stimulated Raman transitions between two atomic ground state levels belong to this class of matter-wave beam splitters. Indeed, the third (excited) atomic level can be eliminated if the lasers are far enough from resonance, and a single Rabi frequency is needed to characterize the effective laser beam which causes the transition between the two remaining ground states. Eventually, the spatial and temporal structure of the true laser beams appears only in the amplitude of the *effective running laser beam*. Here, for simplicity, is considered a running laser pulse with a square amplitude of duration τ .

Moreover, this laser field is supposed to be classical (no QED effect), and all relativistic effects as well as relaxation processes are neglected (long lifetime of atomic levels, no collision, no absorption, etc.). All these approximations are not necessary but are made to focus on the main features of the triple interaction “laser-matter-rotation” only.

Finally, the usual dipolar electric Hamiltonian $V_{em}(\vec{r}, t)$ which models the atom-laser interaction reads:

$$V_{em}(\vec{r}, t) = -\hbar\Omega_0 \begin{pmatrix} 0 & e^{-i(\omega t - \vec{k} \cdot \vec{r} + \phi)} \\ e^{+i(\omega t - \vec{k} \cdot \vec{r} + \phi)} & 0 \end{pmatrix}, \quad (1)$$

where $\Omega_0 = \Omega_{ba}$ is the Rabi frequency of the atomic transition between the levels a and b . The quantities ω , \vec{k} , and ϕ are the (effective) laser frequency, wave vector, and phase, respectively.

The rotating-wave approximation, which consists in neglecting the off-resonant terms in the Fourier decomposition of the laser pulse, has also been used, and possible diagonal terms in V_{em} have been neglected (no ac Stark shift for simplicity).

This interaction is usually made in the presence of other external potentials: inertial and gravitational fields, trapping potential, van der Waals and Casimir potentials,² etc. In this paper I focus on the effect of a constant and uniform rotation $\vec{\Omega}$. This rotation can have many origins: artificial rotation of the laboratory or vehicle, “natural” rotation of a planet, Lense-Thirring effect near a massive rotating body, etc. As recalled in the introduction, it can act either on the atoms or on the lasers, according to the reference frame which is chosen to express the atomic evolution equation. In the frame where the lasers are at rest, this potential reads

$$\vec{r}_{op} \cdot (\vec{\Omega} \times \vec{p}_{op}) \quad \text{or} \quad -\vec{r}_{op} \cdot \vec{\alpha} \cdot \vec{p}_{op}, \quad (2)$$

where the matrix $\vec{\alpha}$ is defined as $\vec{\alpha} \cdot \vec{u} := -\vec{\Omega} \times \vec{u}$ for any vector \vec{u} . The two operators \vec{r}_{op} and \vec{p}_{op} are the canonical position and momentum operators. Let us underline that there is no (explicit) centrifugal term in the Hamiltonian formalism. This one appears only in the corresponding Lagrangian.

Finally, since the relaxation processes are neglected, the atomic evolution can be described by the following Schrödinger equation:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \left(H_0 + \frac{1}{2m} \vec{p}_{op}^2 - \vec{r}_{op} \cdot \vec{\alpha} \cdot \vec{p}_{op} + V_{em}(\vec{r}_{op}, t) \right) |\Psi(t)\rangle, \quad (3)$$

²The general case of potentials at most quadratic in position and momentum is examined in [28].

where H_0 is the internal Hamiltonian ($E_b > E_a$),

$$H_0 = \begin{pmatrix} E_b & 0 \\ 0 & E_a \end{pmatrix} = \frac{E_a + E_b}{2} 1 + \frac{\hbar\omega_0}{2} \sigma_3, \quad (4)$$

with $\omega_0 = (E_b - E_a)/\hbar$ the atomic transition frequency and σ_3 the usual third Pauli matrix.

The matrix differential equation (3) cannot be directly solved because its right-hand side depends explicitly on time and on the two noncommuting operators \vec{r}_{op} and \vec{p}_{op} , and, consequently, does not commute with itself at two different times. However, it is possible to simplify it with the help of judicious *ket transformations* which progressively eliminate the different (internal and external) sources of atomic evolution. These transformations can be seen as *frame changes* which aim at reaching a more suitable frame (called “least movement frame” in [28]) where the evolution equation is easier to solve.

The first frame change corresponds to the transformation into the interaction picture with respect to H_0 and

$$\frac{1}{2m} \vec{p}_{op}^2 - \vec{r}_{op} \cdot \vec{\alpha} \cdot \vec{p}_{op},$$

$$|\varphi_1(t)\rangle = e^{+i\hbar(H_0 + 1/2m\vec{p}_{op}^2 - \vec{r}_{op} \cdot \vec{\alpha} \cdot \vec{p}_{op})(t-t_1)} |\Psi(t)\rangle, \quad (5)$$

where t_1 is an arbitrary instant different from t by definition.

This instant “ t_1 ” has no physical meaning and can be removed explicitly at each step of the following calculations [31]. However, it can be interesting to keep it and eventually assign it a particular value (the central time of laser interactions, for example) which can be useful to simplify some expressions and describe the atom-laser interaction as an equivalent instantaneous interaction. As underlined in the introduction, this artificial concentration of the beam splitters effects at a single time t_1 is particularly relevant for the modeling of atom interferometers (see also below).

The Schrödinger equation (3) becomes

$$\frac{d}{dt} |\varphi_1(t)\rangle = i\Omega_0 \begin{pmatrix} 0 & e^{-i\Phi_{op}(t,t_1)} \\ e^{+i\Phi_{op}(t,t_1)} & 0 \end{pmatrix} |\varphi_1(t)\rangle, \quad (6)$$

where $\Phi_{op}(t, t_1)$ is defined as

$$\begin{aligned} \Phi_{op}(t, t_1) &= \omega t - \omega_0(t - t_1) - \vec{k} \cdot e^{\vec{\alpha}(t-t_1)} \cdot \left(\vec{r}_{op} + \frac{\vec{p}_{op}}{m}(t - t_1) \right) \\ &+ \phi. \end{aligned} \quad (7)$$

The second (non-unitary) transformation corresponds to the passage into the usual “rotating frame” (δ is the recoil $\hbar k^2/2m$ and t_0 the initial instant of the square laser pulse):³

³This rotating frame is not unique. There is an infinite number of transformations of the type of Eq. (8) as it is detailed in [28].

$$|\varphi_2(t)\rangle = \begin{pmatrix} e^{+i\Phi_{op}(t,t_1)/2} & 0 \\ 0 & e^{-i\Phi_{op}(t,t_1)/2} \end{pmatrix} e^{i\delta(t-t_0)/4} |\varphi_1(t)\rangle, \quad (8)$$

which aims at eliminating the main time dependence of the right-hand side of Eq. (6),

$$\frac{d}{dt} |\varphi_2(t)\rangle = i \begin{pmatrix} \Delta_{op}(t, t_1)/2 & \Omega_0 \\ \Omega_0 & -\Delta_{op}(t, t_1)/2 \end{pmatrix} |\varphi_2(t)\rangle, \quad (9)$$

where $\Delta_{op}(t, t_1)$ is an operator which generalizes the usual detuning $\Delta_0 = \omega - \omega_0 - \vec{k} \cdot \vec{p}_{op}/m$ in the presence of a rotation,

$$\Delta_{op}(t, t_1) = \frac{d}{dt} \Phi_{op}(t, t_1), \quad (10)$$

i.e.,

$$\Delta_{op}(t, t_1) = \Delta_0 + (\vec{k} \times \vec{\Omega}) \cdot \left(\vec{r}_{op} + 2\frac{\vec{p}_{op}}{m}(t - t_1) \right) \quad (11)$$

to the first order in $\|\vec{\Omega}\|/\Omega_0$.

Equation (9) depends explicitly on time and on the two noncommuting operators \vec{r}_{op} and \vec{p}_{op} , and consequently is “non scalar,” i.e., is such that one cannot choose any representation (position or momentum representation, for example) where the previous differential *operator* equation becomes a differential equation (and not a partial derivative equation), which can be solved by applying the common rules of integration. Here, however, the first order solution of Eq. (9) is particularly easy to obtain since all commutators implying \vec{r}_{op} and \vec{p}_{op} vanish to the first order in $\|\vec{\Omega}\|/\Omega_0$,

$$[(\text{function of } \vec{k} \cdot \vec{p}_{op}), (\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op}] = 0. \quad (12)$$

Actually, all happens as if these two canonical operators were commuting, i.e., as if this nonscalar (operator) problem was scalar at this perturbation order. This fact explains why *scalar first-order perturbation methods* can be used to solve this non scalar problem (as in [49] with the “sensitivity function method”).⁴

The first-order solution of Eq. (9) is given by

⁴The sensitivity function method is closely linked to the Dick effect [50] and is usually used in frequency standards to characterize the influence of laser fluctuations on the atomic transition probabilities [51].

$$|\varphi_2(t)\rangle = \left(\begin{array}{c} c(t) + i \frac{y}{\sqrt{1+y^2}} s(t) + \frac{i}{2} (\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op} \left\{ \frac{1}{1+y^2} \frac{s(t)}{\Omega_0} + i \frac{y}{\sqrt{1+y^2}} t s(t) + \frac{y^2}{1+y^2} t c(t) \right\} \\ \frac{i}{\sqrt{1+y^2}} s(t) + \frac{i}{2} (\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op} \frac{y}{1+y^2} \left[t c(t) - \frac{s(t)}{\Omega_0} \right] - (\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_{op}}{2m\Omega_0} \frac{1}{\sqrt{1+y^2}} \left[t c(t) - \frac{s(t)}{\Omega_0} \right] \end{array} \right) \begin{array}{c} \frac{i}{\sqrt{1+y^2}} s(t) + \frac{i}{2} (\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op} \frac{y}{1+y^2} \left[t c(t) - \frac{s(t)}{\Omega_0} \right] + (\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_{op}}{2m\Omega_0} \frac{1}{\sqrt{1+y^2}} \left[t c(t) - \frac{s(t)}{\Omega_0} \right] \\ c(t) - i \frac{y}{\sqrt{1+y^2}} s(t) - \frac{i}{2} (\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op} \left\{ \frac{1}{1+y^2} \frac{s(t)}{\Omega_0} - i \frac{y}{\sqrt{1+y^2}} t s(t) + \frac{y^2}{1+y^2} t c(t) \right\} \end{array} \right) \times |\varphi_2(0)\rangle \quad (13)$$

where, for brevity, the expressions “ $\cos(\sqrt{1+y^2}\Omega_0 t)$ ” and “ $\sin(\sqrt{1+y^2}\Omega_0 t)$ ” have been replaced by “ $c(t)$ ” and “ $s(t)$,” respectively, and where the instants t_0 and t_1 have been chosen equal to 0 and $(t_0+t)/2$, the initial and central times of the interaction.

The y parameter is called “inelasticity parameter” and is defined in analogy to the theory of dynamical diffraction of neutrons in crystals [28,41,44]:

$$y = \frac{\Delta_0}{2\Omega_0} = (\omega - \omega_0 - \vec{k} \cdot \vec{p}_{op}/m)/2\Omega_0. \quad (14)$$

Its scalar version (in momentum representation) tells us how the laser resonance condition is fulfilled. For example, if an incident atomic wave packet (for an atom in the internal state “ a ,” considered at the central time t_1 of the first beam splitter of an atom interferometer) has a mean momentum equal to \vec{p}_1 (i.e., $\vec{p}_1 + \hbar\vec{k}/2$ in the rotating frame), then the corresponding inelasticity parameter reads

$$y_1 = (\omega - \omega_0 - \vec{k} \cdot \vec{p}_1/m - \delta)/2\Omega_0. \quad (15)$$

In the following, we assume that the detuning $\omega - \omega_0 - \vec{k} \cdot \vec{p}_1/m - \delta$ is negligible compared to the bare Rabi frequency Ω_0 (quasi-perfect resonance in the atom-laser interaction) or—as it is explained in details in [28,31]—that the “first Borrmann condition” is fulfilled,

$$y_1 \approx 0. \quad (16)$$

After evolution in an atom gyrometer, the central momentum of this atomic wave packet is changed into $\vec{p}_i = \vec{p}_1 - \vec{\Omega}(t_i - t_1) \times \vec{p}_1$ to the first order in $\|\vec{\Omega}\|(t_i - t_1)$, where $t_i - t_1$ is the evolution duration. For example, within a Mach-Zehnder geometry (see Fig. 1), one obtains $t_2 = t_1 + T + 3\tau/2$ and $t_3 = t_1 + 2T + 3\tau$ for the second and third beam splitters, respectively. Thus, one can write y as

$$y = y_i - \frac{\vec{k} \cdot (\vec{p}_{op} - \vec{p}_i - \hbar\vec{k}/2)}{2m\Omega_0} \quad (17)$$

where the first term $y_i = -\vec{k} \cdot (\vec{p}_i - \vec{p}_1)/2m\Omega_0$ is a first order term in $\|\vec{\Omega}\|/\Omega_0$, and where the second term is assumed to have a small effect on the incident atomic wave packet which is sharply peaked around \vec{p}_1 (“second Borrmann condition” [28]):

$$-\frac{\vec{k} \cdot (\vec{p}_{op} - \vec{p}_i - \hbar\vec{k}/2)}{2m\Omega_0} |\varphi_2(t_i)\rangle \approx -\frac{k\sigma_{p\parallel}}{2m\Omega_0} |\varphi_2(t_i)\rangle \quad (18)$$

where $\sigma_{p\parallel} \ll m\Omega_0/k$ is the momentum width of the atomic wave packet along the \vec{k} direction.

Finally, a Taylor expansion of $|\varphi_2(t)\rangle$ can be carried out up to the first order in y ,

$$|\varphi_2(t)\rangle = \left(\begin{array}{c} \cos(\Omega_0 t) + iy \sin(\Omega_0 t) + i(\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op} \frac{1}{2\Omega_0} \sin(\Omega_0 t) \\ i \sin(\Omega_0 t) - (\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_{op}}{2m\Omega_0} \left(t \cos(\Omega_0 t) - \frac{1}{\Omega_0} \sin(\Omega_0 t) \right) \end{array} \right) \begin{array}{c} i \sin(\Omega_0 t) + (\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_{op}}{2m\Omega_0} \left(t \cos(\Omega_0 t) - \frac{1}{\Omega_0} \sin(\Omega_0 t) \right) \\ \cos(\Omega_0 t) - iy \sin(\Omega_0 t) - i(\vec{k} \times \vec{\Omega}) \cdot \vec{r}_{op} \frac{1}{2\Omega_0} \sin(\Omega_0 t) \end{array} \right) \times |\varphi_2(0)\rangle. \quad (19)$$

Then, one can go back to the initial (laboratory) frame by performing the reverse transformations of Eqs. (8) and (5), and obtain $|\Psi(t)\rangle$ as a function of $|\Psi(0)\rangle$. This result can be reexpressed as a product of three operators,

$$|\Psi(t)\rangle = U_1(t, t_1) S_1(\vec{r}_{op}, \vec{p}_{op}, t, 0, t_1) U_1(t_1, 0) |\Psi(0)\rangle, \quad (20)$$

where $U_1(t_1, 0)$ and $U_1(t, t_1)$ describe the atomic evolution which is due to $H_0 + \frac{1}{2m}\vec{p}_{op}^2 - \vec{r}_{op} \cdot \vec{\alpha} \cdot \vec{p}_{op}$ only (i.e., without any laser beam splitter potential). By this analytic *rearrangement*, one concentrates artificially all the beam splitter effects at a single moment t_1 , and describes the laser interaction as an *effective* instantaneous interaction (generalization of the *ttt* scheme introduced in [52]). Contrary to the former naive modelings of atomic beam splitters, such as the “infinitely thin” one, this modeling does not neglect the effects which depend on the duration of the beam splitter, although it is expressed in the form of an (effective) instantaneous interaction. This “*ttt modeling*” (“*ttt*” for the three instants t_0 , t_1 , and t) is particularly useful not only to make the link with the former oversimplified modelings, but also to provide a practical beam splitter modeling for atom interferometer phase shift calculations.

For illustration, let us consider a Gaussian initial atomic wave packet $|\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle$ (which has the mean position \vec{r}_1 and mean momentum \vec{p}_1 at the time t_1), and let us express the solution $S_1(\vec{r}_{op}, \vec{p}_{op}, t, 0, t_1) |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle$ in two particular cases—

the $\pi/2$ and π beam splitters—for which $\Omega_0 t$ is equal to $\pi/4$, and $\pi/2$, respectively. For example, if the initial “atom” is in the internal state “ a ” (lower atomic state), then the following output is obtained in the case of a $\pi/2$ beam splitter:⁵

$$S_{1,aa}^{\pi/2} |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle = \frac{1}{\sqrt{2}} e^{i(\omega - \omega_0 - \delta)t/2 + i\vec{k} \cdot (\vec{p}_1 + \hbar\vec{k}/2)/2m\Omega_0 - i(\vec{k} \times \vec{\Omega}) \cdot \vec{r}_1 (1 - \Omega_0 t)/2\Omega_0} \left| \Psi_G \left(t_1; \vec{r}_1 - \frac{\hbar\vec{k}}{2m\Omega_0} (1 - \Omega_0 t), \vec{p}_1 - \frac{\hbar\vec{k} \times \vec{\Omega}}{2\Omega_0} (1 - \Omega_0 t) \right) \right\rangle \quad (21)$$

for the atomic part which is associated to the internal state “ a ” after a time t (i.e., at the beam splitter exit), and

$$S_{1,ba}^{\pi/2} |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle = \frac{i}{\sqrt{2}} e^{-i(\omega t_1 - \vec{k} \cdot \vec{r}_1 + \phi)} \left| \Psi_G \left(t_1; \vec{r}_1 + \frac{\hbar\vec{k} \times \vec{\Omega}}{2m\Omega_0^2} \left(\frac{(\Omega_0 t)^2}{2} + \Omega_0 t - 1 \right), \vec{p}_1 + \hbar\vec{k} \right) \right\rangle \quad (22)$$

for the atomic part in the upper state “ b .” The global phase factor $\omega t_1 - \vec{k} \cdot \vec{r}_1 + \phi$ is nothing else than the laser phase in \vec{r}_1 at the time t_1 .

Reversely, if the initial atomic wave packet describes an atom in the upper atomic state “ b ,” one obtains the following output:

$$S_{1,bb}^{\pi/2} |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle = \frac{1}{\sqrt{2}} e^{-i(\omega - \omega_0 + \delta)t/2 - i\vec{k} \cdot (\vec{p}_1 + \hbar\vec{k}/2)/2m\Omega_0 + i(\vec{k} \times \vec{\Omega}) \cdot \vec{r}_1 (1 - \Omega_0 t)/2\Omega_0} \left| \Psi_G \left(t_1; \vec{r}_1 + \frac{\hbar\vec{k}}{2m\Omega_0} (1 - \Omega_0 t), \vec{p}_1 + \frac{\hbar\vec{k} \times \vec{\Omega}}{2\Omega_0} (1 - \Omega_0 t) \right) \right\rangle \quad (23)$$

for the atomic part in the upper state “ b ” and

$$S_{1,ab}^{\pi/2} |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle = \frac{i}{\sqrt{2}} e^{+i(\omega t_1 - \vec{k} \cdot \vec{r}_1 + \phi)} \left| \Psi_G \left(t_1; \vec{r}_1 - \frac{\hbar\vec{k} \times \vec{\Omega}}{2m\Omega_0^2} \left(\frac{(\Omega_0 t)^2}{2} + \Omega_0 t - 1 \right), \vec{p}_1 - \hbar\vec{k} \right) \right\rangle \quad (24)$$

for the atomic part in the lower state “ a .”

Similar results are obtained in the case of a π beam splitter,

$$S_{1,ba}^{\pi} |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle = i e^{-i(\omega t_1 - \vec{k} \cdot \vec{r}_1 + \phi)} \left| \Psi_G \left(t_1; \vec{r}_1 + \frac{\hbar\vec{k} \times \vec{\Omega}}{2m\Omega_0^2} [2(\Omega_0 t)^2 - 1], \vec{p}_1 + \hbar\vec{k} \right) \right\rangle \quad (25)$$

for the atomic transition $a \rightarrow b$ and

$$S_{1,ab}^{\pi} |\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle = i e^{+i(\omega t_1 - \vec{k} \cdot \vec{r}_1 + \phi)} \left| \Psi_G \left(t_1; \vec{r}_1 - \frac{\hbar\vec{k} \times \vec{\Omega}}{2m\Omega_0^2} [2(\Omega_0 t)^2 - 1], \vec{p}_1 - \hbar\vec{k} \right) \right\rangle \quad (26)$$

for the transition $b \rightarrow a$.

⁵The parameters \vec{r}_1 and \vec{p}_1 of the ket $|\Psi_G(t_1; \vec{r}_1, \vec{p}_1)\rangle$ are not to be confused with the variables \vec{r} and \vec{p} which appear in the wave function expression in the corresponding (position $|\vec{r}\rangle$ or momentum $|\vec{p}\rangle$) representation.

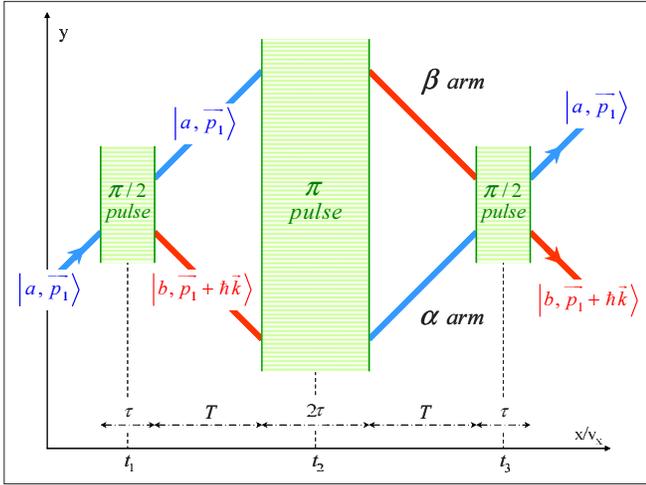


FIG. 1. (Color online) Geometry of a Mach-Zehnder atom interferometer (also called symmetrical *Ramsey-Bordé* or *Chu-Bordé interferometer*). A sequence of $\pi/2$ – π – $\pi/2$ beam splitters is used to coherently split, deflect, and recombine an initial atomic beam. The interferometer phase shift, i.e., the accumulated phase difference between the two main interferometer arms, is encoded in the proportion of atoms in states a and b at the exit. For simplicity, the incidence angle of the atoms (in the first $\pi/2$ beam splitter) has been chosen equal to the Bragg angle, but this condition can be partly relaxed due to the internal atomic structure [see expression (15) and condition (16)].

In addition to the “usual” beam splitter effects (amplitude, phase, and central momentum changes), one observes many other effects which depend on t or $1/\Omega_0$, i.e., on the beam splitter duration, but also on the rotation vector $\vec{\Omega}$. Those which depend only on t account for the usual atomic Borrmann effect (as described in [28]), whereas pure Ω_0 -dependent terms (without $\vec{\Omega}$) give corrections to this effect in the case of a *running* laser pulse. $\vec{\Omega}$ -dependent terms account for the effect of a rotation on such a beam splitter. These modifications concern not only the phase but also the central position and momentum of atomic wave packets, and can induce important changes in the phase shift expression of atom interferometers, as it is shown below [see Eq. (35)].

III. CONSEQUENCES FOR ATOM GYROMETERS

As mentioned in the introduction, the modeling of atom interferometers has been significantly improved during the last 15 years. New theoretical tools have been developed, such as the *ABCD* matrices, for example, to describe the atomic waves evolution in the presence of several and simultaneous inertial and gravitational potentials [25,31], or the use of fundamental invariants to put on light—and explain—some important simplifications in the interferometer phase

shift expression [30,31]. Now is available a full analytical expression of the fringes signal which is very compact and valid for a large variety of atom interferometers, and, in particular, for any interferometer geometry, any kind of atomic source, any kind of atomic beam splitter (for which the two beam approximation is valid⁶), and any kind of detection process. In addition, this general expression accounts for some relativistic effects and all the gravitational and inertial effects whose representative potential is at most quadratic in position and momentum. Rotations, accelerations, gradients of acceleration, gravitational waves, and trapping potentials belong to this class of potentials.

For example, if an atom interferometer possesses two main arms only, called α and β , its (nonrelativistic) phase shift expression reads

$$\Delta\varphi = \sum_{i=1}^N \left\{ \theta_{\beta i} - \theta_{\alpha i} - \frac{\vec{b}_{\beta i} + \vec{b}_{\alpha i}}{2\hbar} \cdot (\vec{r}_{\beta i} - \vec{r}_{\alpha i} + \vec{a}_{\beta i} - \vec{a}_{\alpha i}) - \frac{\vec{p}_{\beta i} + \vec{p}_{\alpha i}}{2\hbar} \cdot (\vec{a}_{\beta i} - \vec{a}_{\alpha i}) \right\}, \quad (27)$$

where the index “ i ” refers to the i th beam splitter. The instant t_i is the central time of this i th beam splitter, and could also be written “ t_i ” since it is the effective instant of interaction t_i of the *ttt* scheme described in Sec. II.

The parameters $\theta_{\alpha i}$, $\vec{a}_{\alpha i}$, $\vec{b}_{\alpha i}$, $\vec{r}_{\alpha i}$, $\vec{p}_{\alpha i}$ are defined as follows (expressed here for the α branch for illustration): $\vec{r}_{\alpha i}$ and $\vec{p}_{\alpha i}$ are the central position and momentum of the (Gaussian) atomic wave packet “ wp_α ” just before the instant t_i , whereas $\theta_{\alpha i}$, $\vec{a}_{\alpha i}$, $\vec{b}_{\alpha i}$ characterize the modification of its phase, central position and momentum due to the i th beam splitter,

$$wp_\alpha(t_i^-; \vec{r}_{\alpha i}, \vec{p}_{\alpha i}) \rightarrow M_{\alpha i} e^{i\theta_{\alpha i}} wp_\alpha(t_i^+; \vec{r}_{\alpha i} + \vec{a}_{\alpha i}, \vec{p}_{\alpha i} + \vec{b}_{\alpha i}), \quad (28)$$

where t_i^- (t_i^+) is the instant just before (after) the instant t_i . All these parameters depend on the considered transition between the internal atomic states ($a \rightarrow b$, for example).

With the modeling of Sec. II, one obtains for a $\pi/2$ beam splitter (of duration τ),

⁶Actually, this condition can be extended to any beam splitting process which preserves the general form (Gaussian, for example) of the wave packet which enters an atomic beam splitter. Only the number of outgoing wave packets and their features (associated internal state, amplitude, phase, central position and momentum, position and momentum widths) can change.

$$a \xrightarrow{\pi/2} a: \begin{cases} \theta_{aa,\alpha i} = (\omega_i - \omega_0 - \delta_i)\pi/2 + \vec{k}_i \cdot (\vec{p}_1 + \hbar\vec{k}_i/2)/2m\Omega_{0i} - (\vec{k}_i \times \vec{\Omega}) \cdot \vec{r}_{\alpha i} \frac{1 - \Omega_{0i}\tau}{2\Omega_{0i}}, \\ \tilde{a}_{aa,\alpha i} = -\frac{\hbar\vec{k}_i}{2m\Omega_{0i}}(1 - \Omega_{0i}\tau), \\ \vec{b}_{aa,\alpha i} = -\frac{\hbar\vec{k}_i \times \vec{\Omega}}{2\Omega_{0i}}(1 - \Omega_{0i}\tau), \end{cases} \quad (29)$$

$$a \xrightarrow{\pi/2} b: \begin{cases} \theta_{ba,\alpha i} = \frac{\pi}{2} - (\omega_i t_i - \vec{k}_i \cdot \vec{r}_{\alpha i} + \phi_i), \\ \tilde{a}_{ba,\alpha i} = +\frac{\hbar\vec{k}_i \times \vec{\Omega}}{2m\Omega_{0i}^2} \left(\frac{(\Omega_{0i}\tau)^2}{2} + \Omega_{0i}\tau - 1 \right), \\ \vec{b}_{ba,\alpha i} = +\hbar\vec{k}_i, \end{cases} \quad (30)$$

$$b \xrightarrow{\pi/2} b: \begin{cases} \theta_{bb,\alpha i} = -(\omega_i - \omega_0 + \delta_i)\pi/2 - \vec{k}_i \cdot (\vec{p}_1 + \hbar\vec{k}_i/2)/2m\Omega_{0i} + (\vec{k}_i \times \vec{\Omega}) \cdot \vec{r}_{\alpha i} \frac{1 - \Omega_{0i}\tau}{2\Omega_{0i}}, \\ \tilde{a}_{bb,\alpha i} = +\frac{\hbar\vec{k}_i}{2m\Omega_{0i}}(1 - \Omega_{0i}\tau), \\ \vec{b}_{bb,\alpha i} = +\frac{\hbar\vec{k}_i \times \vec{\Omega}}{2\Omega_{0i}}(1 - \Omega_{0i}\tau), \end{cases} \quad (31)$$

$$b \xrightarrow{\pi/2} a: \begin{cases} \theta_{ab,\alpha i} = \frac{\pi}{2} + (\omega_i t_i - \vec{k}_i \cdot \vec{r}_{\alpha i} + \phi_i), \\ \tilde{a}_{ab,\alpha i} = -\frac{\hbar\vec{k}_i \times \vec{\Omega}}{2m\Omega_{0i}^2} \left(\frac{(\Omega_{0i}\tau)^2}{2} + \Omega_{0i}\tau - 1 \right), \\ \vec{b}_{ab,\alpha i} = -\hbar\vec{k}_i, \end{cases} \quad (32)$$

where ω_i , \vec{k}_i , ϕ_i , and Ω_{0i} are, respectively, the frequency, wave vector, phase, and Rabi frequency of the i th effective laser pulse.

Similarly, for a π beam splitter (of duration τ'), one obtains

$$a \xrightarrow{\pi} b: \begin{cases} \theta_{ba,\alpha i} = \frac{\pi}{2} - (\omega_i t_i - \vec{k}_i \cdot \vec{r}_{\alpha i} + \phi_i), \\ \tilde{a}_{ba,\alpha i} = +\frac{\hbar\vec{k}_i \times \vec{\Omega}}{2m\Omega_{0i}^2} [2(\Omega_{0i}\tau')^2 - 1], \\ \vec{b}_{ba,\alpha i} = +\hbar\vec{k}_i, \end{cases} \quad (33)$$

$$b \xrightarrow{\pi} a: \begin{cases} \theta_{ab,\alpha i} = \frac{\pi}{2} + (\omega_i t_i - \vec{k}_i \cdot \vec{r}_{\alpha i} + \phi_i), \\ \tilde{a}_{ab,\alpha i} = -\frac{\hbar\vec{k}_i \times \vec{\Omega}}{2m\Omega_{0i}^2} [2(\Omega_{0i}\tau')^2 - 1], \\ \vec{b}_{ab,\alpha i} = -\hbar\vec{k}_i. \end{cases} \quad (34)$$

Let us now apply these expressions to the most commonly used interferometer geometry in atom gyrometry, i.e., the Mach-Zehnder geometry. Such an atom interferometer (also called symmetric *Ramsey-Bordé* or *Chu-Bordé* atom interferometer [1]) has three beam splitters only (with common features, i.e., with the same laser wave vector, intensity, etc.) and the following time sequence (see Fig. 1), a first $\pi/2$ beam splitter centered in t_1 with a duration τ , a first “dark

area” (laser-free region) of duration T , a π beam splitter centered in t_2 with a duration $\tau' = 2\tau$, a second “dark area” of duration T , and a second $\pi/2$ beam splitter centered in t_3 with a duration τ . If the beam splitters (made by Raman laser pulses generally) can be modeled as in Sec. II, then, one can apply the previous phase shift formula, which finally reads:

$$\Delta\varphi_{first} = \Delta\varphi_0 - (\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_1}{2m} \left(T + \frac{1}{\Omega_0} \right) (T + 2\tau), \quad (35)$$

where $\Delta\varphi_0$ accounts for the constant laser phase shift

$$\Delta\varphi_0 = -(\omega t_1 + \phi_1) + 2(\omega t_2 + \phi_2) - (\omega t_3 + \phi_3) \quad (36)$$

and where $1/\Omega_0$ can also be written as $4\tau/\pi$.

The expression (35) is one of the main results of this paper. It gives the Sagnac phase shift of an atom interferometer to the first order in τ/T where τ is the typical beam splitter duration and T the propagation time between two beam splitters.

This result differs greatly from former Sagnac expressions, particularly from the expression obtained with the infinitely thin modeling of beam splitters, i.e., without considering all the effects which depend on the beam splitter duration,

$$\Delta\varphi_{\infty thin} = \Delta\varphi_0 - (\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_1}{2m} T_0^2, \quad (37)$$

where $T_0 = T + \frac{3}{2}\tau$ is the time separating the central instants of two successive beam splitters as illustrated in Fig. 1. Do not be confused: the phase shift $\Delta\varphi_{\infty thin}$ depends on the beam splitter duration τ through T_0 , but does not account for any beam splitter effect depending on τ .

These two phase shifts differ by the following expression:

$$\begin{aligned} \Delta\varphi_{\infty thin} - \Delta\varphi_{first} \\ = -(\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_1}{2m} \left[\left(\frac{4}{\pi} - 1 \right) T\tau + \left(\frac{8}{\pi} - \frac{9}{4} \right) \tau^2 \right]. \end{aligned} \quad (38)$$

To estimate the relevance of this correction, one has to compare it to the experimental sensitivity and accuracy of present (and forthcoming) atom gyrometers. Let us call σ_Ω their sensitivity to a rotation rate $\vec{\Omega}$. For example, the Kasevich’s group at Yale University has obtained the following short-term sensitivity for its thermal (cesium) atom gyrometer [5]:

$$\sigma_{\Omega, Yale} = 6 \times 10^{-10} \text{ rad s}^{-1},$$

for one second averaging time of the signal. An other example is the cold (cesium) atom gyrometer which has been developed in the BNM-SYRTE lab of the Observatoire de Paris where the following sensitivity,

$$\sigma_{\Omega, Paris} = 1.4 \times 10^{-7} \text{ rad s}^{-1}$$

has been demonstrated for 10 min of averaging time, and which is expected to be improved by several orders of magnitude soon [6]. Let us also mention the (one shot) sensitivity which is expected in the cold (rubidium) gyrometer of the

Institute of Quantum Optics in Hannover [53],

$$\sigma_{\Omega, Hannover} = 10^{-9} \text{ rad s}^{-1} \text{ (expected)}$$

as well as the expected accuracy of the ultracold atom gyrometers of the space project HYPER [20],

$$\sigma_{\Omega, HYPER} = 10^{-16} \text{ rad s}^{-1} \text{ (expected)}.$$

These sensitivities have to be compared with the global shift in the $\vec{\Omega}$ measurement due to the leading term of Eq. (38):

$$\left(\frac{4}{\pi} - 1 \right) \frac{\tau}{T} \|\vec{\Omega}\|, \quad (39)$$

which gives a correction of the order of $10^{-4} \|\vec{\Omega}\|$ for the first two atom gyrometers mentioned above (for which $\tau/T \sim 10^{-4} - 10^{-3}$). This correction is larger than the Yale gyrometer sensitivity for rotation rates higher than $6 \times 10^{-6} \text{ rad s}^{-1}$ (for 1 s of signal integration), and larger than the Paris cold atom gyrometer for rotation rates higher than $10^{-4} \text{ rad s}^{-1}$ (for 10 min of signal integration). Let me recall that the Earth’s rotation rate is equal to $7.3 \times 10^{-5} \text{ rad s}^{-1}$ and rotation rates of several rad s^{-1} are currently encountered in navigation.

The expression (35) differs also from the expression which takes into account the usual Borrmann effect in beam splitters [28]:

$$\Delta\varphi_{Borrmann} - \Delta\varphi_{first} = -(\vec{k} \times \vec{\Omega}) \cdot \frac{\vec{p}_1}{2m} \left(\frac{4}{\pi} T\tau + \frac{8}{\pi} \tau^2 \right), \quad (40)$$

which is less relevant here since running laser pulses are considered (and not standing laser pulses where a pure Borrmann effect can take place). In both cases, however, the phase shift correction scales as τ/T and leads to a global shift in the $\vec{\Omega}$ measurement which scales as $\|\vec{\Omega}\| \tau/T$. As explained in [28], the two phase shift expressions $\Delta\varphi_{\infty thin}$ and $\Delta\varphi_{Borrmann}$ have a simple interpretation in terms of interferometer “area,” but it is no more the case for the full first-order expression $\Delta\varphi_{first}$.

IV. PHASE SHIFT AND INTERFEROMETER AREA

The area of an interferometer is generally said to provide its *scale factor*, i.e., to give the interferometer phase shift value up to a proportional factor (linear response in rotation rate).⁷ This area can be spatial or, more generally, spatiotemporal, as for atom gravimeters, for example. However, one can face several problems to define it (with light as well as matter waves), which raises an issue concerning its link to the interferometer phase shift value.

The first problem is due to the fact that an interferometer can have more than two main coherent arms. As explained in [28,31], the nontrivial (dispersive) structuring which occurs

⁷However, let us mention that recent experiments tend to invalidate this assumption for light interferometers [40].

inside beam splitters can lead to a multiplication of the atomic wave packets inside the interferometer, and, consequently, to a multiplication of the coherent interferometer arms. A phase shift expression can still be calculated but the notion of interferometer area (meaning: a *unique* interferometer area) does not make any sense.

A second problem is due to the possible *nonclosure* of the interferometer when it is submitted to external potentials. For example, the external potentials which are quadratic in \vec{r} and \vec{p} (such as rotations, gravity gradients, trapping potentials, effect of gravitational waves, etc.) induce an asymmetric modification of the two main interferometer arms which, consequently, may not cross anymore at the exit of the interferometer. Thus, an interferometer geometry, which would be perfectly closed without any of such external potential, may not be closed when is taken into account the effect of these potentials on the atom interferometer arms. However, it does not mean that one cannot observe interferences at the interferometer exit: the fringes contrast can be sufficient if the matter wave packets overlap enough during the output detection process.

More specifically, for an atom interferometer with a perfect *initial* diamond Mach-Zehnder geometry (see Fig. 1), the following nonclosure formula is obtained (the terms depending on τ have been omitted for simplicity) [31]:

$$\Delta \vec{r}_3^{\pm} = \left(2\alpha T^2 + (3\alpha^2 + \gamma)T^3 + \frac{7}{6}(2\alpha^3 + \alpha\gamma + \gamma\alpha)T^4 + \dots \right) \frac{\hbar \vec{k}}{m}, \quad (41)$$

where α accounts for a rotation and has been defined in Sec. II, and where γ is a 3×3 matrix which accounts for any three-dimensional (3D) quadratic potential $-\vec{m} \cdot \gamma \cdot \vec{r}/2$ (gradient of acceleration, harmonic trap, etc.). Consequently, this atom interferometer cannot be considered as “closed” to the first order in αT and γT^2 . Nevertheless, the two main interferometer arms can sometimes meet before—or after—the final time t_3 , and one can still define an interferometer area in this case. However, in some cases, these two arms never cross and the concept of interferometer area is meaningless (to the perturbation order which is considered).

The notion of “crossing point” of these interferometer arms is *itself* problematic. Indeed, these arms are the support of (light or matter) waves which are not localized in space or space-time. Consequently, what we call an “interferometer arm” is an average position obtained from the evolutionary structure of the main wave packets. These arms are generally defined by the mean position of the two main atomic wave packets which are present during the whole interferometer time sequence. Other definitions can also be considered (with the help of group velocities and probability currents), but they coincide in the present case.

This problem of trajectory or arm definition is particularly relevant inside the beam splitters where dispersive effects change both the structure of atomic wave packets and their dynamical properties [28,31]. For example, a Gaussian wave packet, filtered by a Sec. II kind beam splitter, will not be Gaussian any more during and after the splitting. Several

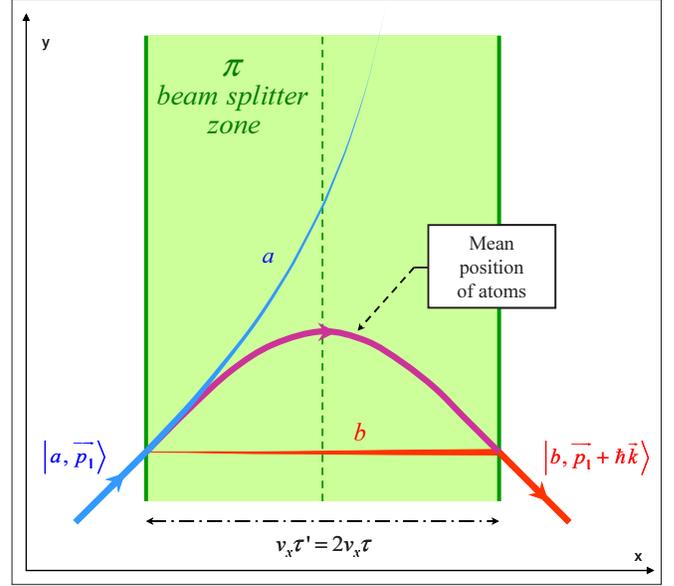


FIG. 2. (Color online) Mean position of the atoms inside a π pulse (running laser beam splitter).

wave packets are created and their spreading and group velocity are nontrivial (e.g., due to the Borrmann effect).

By way of illustration, let us see the case described in Sec. II but without any rotation (usual “free case”). To define the interferometer arms inside this beam splitter, let us consider the mean position of the wave packets which are associated to the internal atomic states a and b . This mean position corresponds to the usual group velocity and probability current, and is defined as follows for a momentum wave packet $\Psi(\vec{p}, t)$ whose modulus is sharply peaked around the mean momentum \vec{p}_1 :

$$\vec{r}_g(t) = -\hbar \left[\frac{\partial}{\partial \vec{p}} \text{Arg}[\Psi(\vec{p}, t)] \right]_{\vec{p}_1}. \quad (42)$$

With this formula and the expressions of Sec. II, one obtains successively (for $\Omega_0 t < \pi/2$)

$$\vec{r}_{g,a \rightarrow b}(t) = \vec{r}_{g,b \rightarrow a}(t) = \left(\frac{\vec{p}_1}{m} + \frac{\hbar \vec{k}}{2m} \right) t, \quad (43)$$

$$\vec{r}_{g,a \rightarrow a}(t) = \left(\frac{\vec{p}_1}{m} + \frac{\hbar \vec{k}}{2m} \right) t - \frac{\hbar \vec{k}}{2m\Omega_0} \tan(\Omega_0 t), \quad (44)$$

$$\vec{r}_{g,b \rightarrow b}(t) = \left(\frac{\vec{p}_1}{m} + \frac{\hbar \vec{k}}{2m} \right) t + \frac{\hbar \vec{k}}{2m\Omega_0} \tan(\Omega_0 t). \quad (45)$$

These trajectories can define interferometer arms inside the beam splitters only if there is a bijection between arms and internal atomic states. It is not the case, for example, inside a π beam splitter, where one has to examine the mean

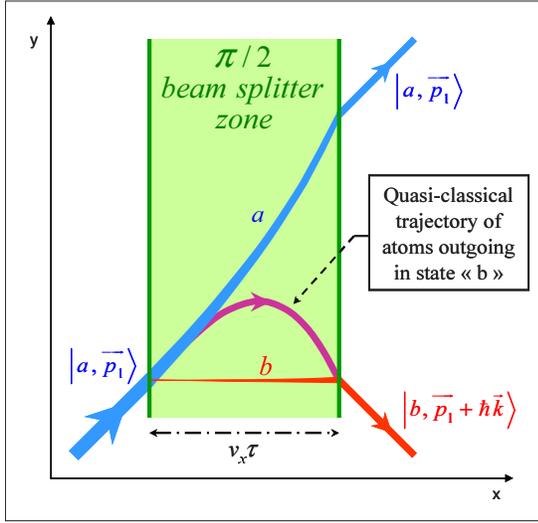


FIG. 3. (Color online) Mean position of the atoms in states a and b inside a $\pi/2$ pulse (running laser beam splitter). The quasi-classical trajectory of the atoms outgoing in state b is also sketched.

position of the whole atomic wave function, and not only of the part associated to the considered transition ($a \rightarrow b$ or $b \rightarrow a$). To be more explicit, let us consider a set of similar atoms, in the internal state a , for example, which enters a perfect π beam splitter. All these atoms will be in state b at the final time τ' (equal to 2τ in our previous example) since a π beam splitter acts as an “internal” mirror for atoms. Consequently, to define an “arm” in this case, one has to consider the mean position or trajectory of the whole packet of atoms and not only of the atoms which *might* be in state b during the splitting. This single arm inside a π beam splitter is consequently defined by the following mean position:

$$\vec{r}_{g,\pi}(t) = \left(\frac{\vec{p}_1}{m} + \frac{\hbar\vec{k}}{2m} \right) t \pm \frac{\hbar\vec{k}}{4m\Omega_0} \sin(2\Omega_0 t), \quad (46)$$

where the sign $+$ ($-$) corresponds to the transition $a \rightarrow b$ ($b \rightarrow a$). This mean trajectory is sketched in Fig. 2 for the transition $a \rightarrow b$.

This problem is all the more fundamental since one cannot know which atoms are in state “ a ” (or “ b ”) during the atomic beam splitting. In particular, one cannot claim that atoms are progressively transferred from one state to the other (so as to obtain an equal internal splitting at the end of a $\pi/2$ beam splitter, for example). Hence, it appears to be impossible to “internally” label the interferometer arms inside the beam splitters.

Nevertheless, in a first approach, one can consider a *quasi-classical* point of view, saying that, inside a $\pi/2$ beam splitter (i.e., a 50:50 beam splitter for matter waves), half of the atoms remain in the initial state, and half are progressively transferred to the other internal state. The two corresponding “classical” arms are thus defined by the following trajectories (written here for an atom initially in state “ a ”):

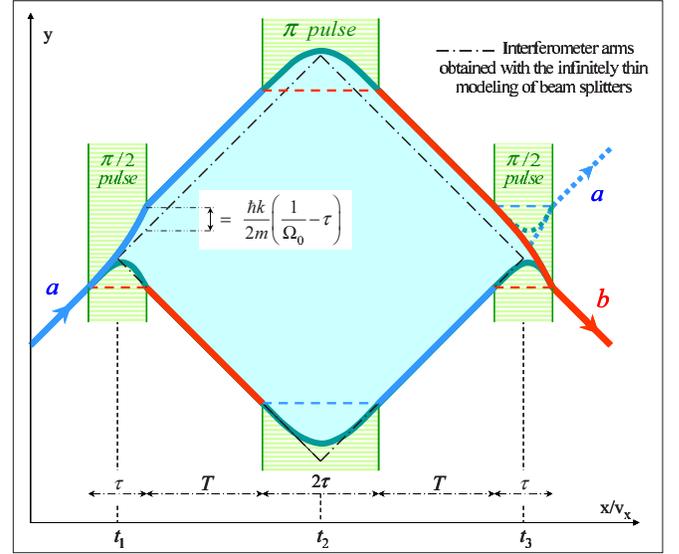


FIG. 4. (Color online) Same figure as Fig. 1, but where the interferometer arms have been defined inside the beam splitters. Inside the π pulse, it is the mean position of the atoms, as it is sketched in Fig. 2. Inside the first (last) $\pi/2$ pulse, one arm is defined by the mean position of the atoms in state a (b), and the other one is the quasiclassical trajectory of the atoms outgoing in state b , as it is sketched in Fig. 3. The Sagnac phase shift obtained in Sec. III is quasi-proportional to the area enclosed by these two arms. This area differs greatly from the area obtained with the former modelings of atomic beam splitters (such as the infinitely thin one, for example).

$$\begin{aligned} \vec{r}_{g,\pi/2,a \rightarrow a}(t) &= \left(\frac{\vec{p}_1}{m} + \frac{\hbar\vec{k}}{2m} \right) t - \frac{\hbar\vec{k}}{2m\Omega_0} \tan(\Omega_0 t), \\ \vec{r}_{g,\pi/2,a \rightarrow b}(t) &= \left(\frac{\vec{p}_1}{m} + \frac{\hbar\vec{k}}{2m} \right) t \\ &\quad - \frac{\hbar\vec{k}}{2m\Omega_0} \tan(\Omega_0 t) \cos(2\Omega_0 t). \end{aligned} \quad (47)$$

This problem arises in the first and third beam splitters of a Mach-Zehnder atom interferometer, as it is sketched in Figs. 3 and 4.

The area \mathcal{A}_{class} which is enclosed by these interferometer arms can be easily calculated:

$$\mathcal{A}_{class} = \mathcal{A}_{first} - \left| \frac{\hbar\vec{k} \cdot \vec{p}_1}{m^2} \right| \frac{\pi/4 - \ln(2)}{2\Omega_0^2}, \quad (48)$$

where \mathcal{A}_{first} is the area corresponding to the full first order phase shift of Sec. III,

$$\mathcal{A}_{first} = \left| \frac{\hbar\vec{k} \cdot \vec{p}_1}{m^2} \right| \left[T^2 + \left(\frac{4}{\pi} + 2 \right) T\tau + \frac{8}{\pi} \tau^2 \right]$$

with

$$|\Delta\varphi_{first} - \Delta\varphi_0| = \frac{2m\|\vec{\Omega}\|}{\hbar} \mathcal{A}_{first}.$$

The quantity $\mathcal{A}_{first}2m/\hbar$ is the gyrometer “scale factor” (linear response in $\vec{\Omega}$).

The difference $\mathcal{A}_{class} - \mathcal{A}_{first}$ is very small,

$$\mathcal{A}_{class} - \mathcal{A}_{first} \sim - \left(\frac{\pi/4 - \ln(2)}{2(\pi/4)^2} \right) \left(\frac{\tau}{T} \right)^2 \mathcal{A}_{first}, \quad (49)$$

and can be estimated to few $10^{-10}\mathcal{A}_{first}$ only (in absolute value) for standard values of T and τ in present atom gyrometers (for which $\tau/T \sim 10^{-4}$).

However, one has to underline the nonperfect proportionality of the phase shift expression and quasiclassical area \mathcal{A}_{class} (it is true only to the first order in τ/T as expected). Other definitions of interferometer arms or area can also be considered, with internally labeled paths, for example, or with intermediate approaches (internally labeled arms for the $\pi/2$ beam splitters only), but they do not lead to a better agreement to \mathcal{A}_{first} .

V. CONCLUSION AND PROSPECTS

Some features of the triple interaction “laser-matter-rotation” have been examined in this paper. It has been shown in particular how to model the effect of a rotation on an atomic laser beam splitter, to the first order in $\|\vec{\Omega}\|/\Omega_0$, where $\vec{\Omega}$ is the rotation rate vector and Ω_0 the laser Rabi frequency. This beam splitter modeling has been applied to the modeling of atom interferometers, and an important nontrivial correction to the phase shift expression of matter wave gyrometers has been put on light [expression (35)]. Finally, thanks to a quasiclassical description of the interferometer arms inside the beam splitters, it has been shown how to interpret the total phase shift expression in terms of an interferometer area (geometrical scale factor).

However, several problems remain, among which an unambiguous definition of these interferometer arms inside the beam splitters, where nontrivial dispersive phenomena lead to changes in the atomic wave packet structure as well as in its dynamical properties. This question is particularly relevant for long-lasting beam splitters, and, consequently, for atomic ones, for which the duration is generally non-negligible compared to the total atomic drift time (the ratio τ/T is equal to 10^{-4} – 10^{-3} in present atom interferometers).

Thus, it would be interesting to deepen this problem by considering more complex interferometer geometries: with a large area, for example (as in the Chu’s experiment [3], where many π beam splitters have been added to increase the area which is enclosed by the two main interferometer arms), with loops (as in the BNM-SYRTE [6] and Kasevich [29] laboratories, where a “butterfly” geometry has recently been considered), with asymmetric geometries (such as asymmetric Ramsey-Bordé interferometers [48]) or geometries which are not closed only because of what happens inside beam splitters (see Sec. IV).

Another direction of research is the study of higher order effects (up to the second order in $\|\vec{\Omega}\|/\Omega_0$, for example) to see if the previous conclusions hold at the considered perturbation order, and especially if the quasi proportionality between phase shift and (quasiclassical) interferometer area remains valid at this order.

It would also be interesting to examine the full effect of other kinds of external potentials on atomic beam splitters and interferometers. The case of a uniform acceleration has already been studied in [28] and leads to a similar result: the phase shift is proportional to the quasiclassical space-time gravimeter area to the first order in $\|\vec{\Omega}\|/\Omega_0$. It is noteworthy that the phase shift correction is identical for atom gyrometers and gravimeters, provided they have the same geometry [compare expression (35) to the last equation of [28]]. This reflects the fact that the sensitivity of an (atom) interferometer is given by its geometrical scale factor only, i.e., the space-time area which is (approximately) enclosed by its two main arms, whatever the effect detected.

One should also easily cope with the effect of simultaneous accelerations, rotations, gradients of acceleration, gravitational waves, trapping potentials, etc., thanks to the recent advances in atom interferometry (see the introduction), especially with the help of the *ABCD* matrices which are well suited to this kind of potentials (at most quadratic in \vec{r} and \vec{p}). The case of a rotation coupled to an acceleration is under progress. This case is particularly important to evaluate the full effect of acceleration fluctuations on the atom gyrometers sensitivity.

As previously underlined, one of the key points of this work is the search for a realistic definition of the interferometer arms inside the atomic beam splitters. To deepen this question, it would be interesting to go beyond the Gaussian approximation of atomic wave packets and contemplate other ways to define these interferometer arms. This point illustrates well the difficulty to link quantum phenomena (matter wave splitting) to classical quantities (interferometer area). Theoretically and experimentally, the use of coherent atomic sources with a large number of atoms should give an interesting perspective to this issue. A different option would be to consider other kinds of beam splitters and, more particularly, the beam splitters made by standing laser waves (with a spatial or temporal working), so as to deal with a pure Borrmann effect [48,54]. More realistic laser beams with a nontrivial amplitude might bring new insight into this problem as well.

Finally, one has also to examine the atom interferometers which have more than two main arms so as to understand the nature of the link between the interferometer phase shift and its area, since the latter cannot be defined generally. A simple case which leads to such multi-arm interferometers is to consider slightly nonresonant beam splitters within a Mach-Zehnder geometry as it is detailed in [31] (presence of a “Borrmann fan” inside all the beam splitters). This case is particularly relevant for present experiments—and future autonomous atom interferometers—where a perfect laser

resonance inside the beam splitters is generally impossible to reach for all the atoms or molecules involved. From this point of view, the advent of new kinds of matter wave gyroscopes, such as guided active gyros and rotating Bose Einstein condensates [12,55–57], is promising to better understand the fundamental origin of the (atomic) Sagnac effect.

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