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Marie-Françoise Bidaut-Véron, Marta Garcia Huidobro

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Elliptic Hamilton-Jacobi systems and Lane-Emden Hardy-Hénon equations

Marie-Françoise Bidaut-Véron Marta Garcia Huidobro

February 18, 2023

Abstract

Here we study the solutions of any sign of the system

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p, \\ -\Delta u_2 = |\nabla u_1|^q, \end{cases}$$

in a domain of \mathbb{R}^N , $N \geq 3$ and $p, q > 0$, $pq > 1$. We show their relation with Lane-Emden Hardy-Hénon equations

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = \varepsilon r^\sigma w^{\mathbf{q}}, \quad \varepsilon = \pm 1,$$

where $u \mapsto \Delta_{\mathbf{p}}^{\mathbf{N}} u$ ($\mathbf{p} > 1$) is the \mathbf{p} -Laplacian in dimension \mathbf{N} , $\mathbf{q} > \mathbf{p} - 1$ and $\sigma \in \mathbb{R}$. This leads us to explore these equations in not often tackled ranges of the parameters $\mathbf{N}, \mathbf{p}, \sigma$. We make a complete description of the radial solutions of the system and of the Hardy-Hénon equations and give nonradial a priori estimates and Liouville type results for the system.

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1 Introduction

Here we study the solutions of any sign of the Hamilton-Jacobi type system

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p, \\ -\Delta u_2 = |\nabla u_1|^q, \end{cases} \quad (1.1)$$

in a domain Ω of \mathbb{R}^N , $N \geq 3$, and

$$p, q > 0, \quad pq > 1, \quad \text{and we can assume } p \geq q,$$

so that $p > 1$. Our purpose is to describe the behaviour and the existence of solutions when $\Omega = B_{r_0} \setminus \{0\}$, $\mathbb{R}^N \setminus \{0\}$, $\mathbb{R}^N \setminus \overline{B_{r_0}}$ or \mathbb{R}^N . Note that if (u_1, u_2) is a solution, then $(u_1 + C_1, u_2 + C_2)$ is a solution, in particular the constants are solutions. The study of the radial solutions is fundamental for the understanding of the system. As shown below, it appears that they are governed by the solutions of Lane-Emden Hardy-Hénon equations

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = \pm 1, \quad (1.2)$$

where $u \mapsto \Delta_{\mathbf{p}}^{\mathbf{N}} u$ ($\mathbf{p} > 1$) is the \mathbf{p} -Laplacian in dimension \mathbf{N} , $\mathbf{q} > \mathbf{p} - 1$ and $\sigma \in \mathbb{R}$. This justifies to make the point on the actual knowledge of these equations, and give a complete study in any range of the parameters $\mathbf{N}, \mathbf{p}, \sigma$.

In the last decades, a great number of elliptic systems deal with positive solutions of semilinear or quasilinear, with source terms in the right hand side, involving powers of the solutions. Our study is motivated by the well known Lane-Emden system

$$\begin{cases} -\Delta u_1 = |x|^a u_2^p, \\ -\Delta u_2 = |x|^b u_1^q, \end{cases} \quad (1.3)$$

which has developed an extremely rich literature, starting from the conjecture of nonexistence of solutions in \mathbb{R}^N for $a = b = 0$ when

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$

first proved in the radial case for any $N \geq 3$ in [30], obtained in of [35] for $N = 3$, extended to $N = 4$ in [37], and still open for $N \geq 5$. The existence of solutions in $\mathbb{R}^N \setminus \{0\}$ of such elliptic systems with a possible singularity at the origin, or in an exterior domain $\mathbb{R}^N \setminus \overline{B_{r_0}}$, and the question of a priori estimates has been developed in many works, with extensions to quasilinear operators, with possible weight functions, or to analogous problems on manifolds, and it is impossible to quote all of them. Let us also mention a significant amount of results of non existence of supersolutions in \mathbb{R}^N , started in [31], [14].

In contrast, only few results are available in the literature for elliptic systems with gradient terms. A general study of nonexistence of positive supersolutions of systems with source terms is given in [22]. We can also mention some works on more specific systems in [23], [25], [6], [18], and [21], [2], [1], [36]. Up to our knowledge the study system (1.1) began very recently, with the publication of [20], where the existence of positive singular solutions of the Dirichlet problem in $\Omega \setminus \{0\}$ was proved when Ω is a small perturbation of a ball of center 0, and the work of [3], where existence and non existence results are given for the system with forcing terms.

In the sequel, we distinguish three types of solutions: the positive ones, solutions of (1.1), with source type gradient terms

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p, \\ -\Delta u_2 = |\nabla u_1|^q, \end{cases} \quad u_1, u_2 \geq 0, \quad (1.4)$$

the negative ones, equivalently, setting $\tilde{u}_1 = -u_1, \tilde{u}_2 = -u_2$, the solutions of a system with absorption terms

$$\begin{cases} -\Delta \tilde{u}_1 + |\nabla \tilde{u}_2|^p = 0, \\ -\Delta \tilde{u}_2 + |\nabla \tilde{u}_1|^q = 0, \end{cases} \quad \tilde{u}_1, \tilde{u}_2 \geq 0, \quad (1.5)$$

and the mixed ones, equivalently setting $\widehat{u}_1 = u_1, \widehat{u}_2 = -u_2$, the solutions of the mixed type system

$$\begin{cases} -\Delta \widehat{u}_1 = |\nabla \widehat{u}_2|^p, \\ \Delta \widehat{u}_2 = |\nabla \widehat{u}_1|^q, \end{cases} \quad \widehat{u}_1, \widehat{u}_2 \geq 0. \quad (1.6)$$

In **Section 2** we study the existence of particular solutions of system (1.1). We briefly mention the main results concerning the scalar case of the well known Hamilton-Jacobi equation

$$-\Delta u = |\nabla u|^q, \quad (1.7)$$

where $q > 1$. The existence of particular positive or negative radial solutions $u^*(r) = A^* r^{\frac{q-2}{q-1}}$, $r = |x|$, for $q \neq 2$, where A^* has the sign of $(2-q)((N-1)q-N)$, puts in evidence two critical values $q = \frac{N}{N-1}$ and $q = 2$. Concerning system (1.1), these two critical values are replaced by

four critical conditions linking the parameters, namely $q = q_i = q_i(p)$, $i = 1, 2, 3, 4$, defined by the relations

$$(N - 1)pq_1 = N + q_1, \quad (N - 1)pq_2 = N + p, \quad q_3(p - 1) = 2, \quad p(q_4 - 1) = 2. \quad (1.8)$$

In the problem, moreover another value is involved, defined by the relation

$$(N - 1)pq^* = N + \frac{p + q^*}{2}, \quad (1.9)$$

corresponding to the Sobolev exponent relative to equation (1.2).

In **Section 3** we study the radial signed solutions of system (1.1). We show that they can be completely described. Indeed the system is invariant by the scaling transformation T_ℓ , $\ell > 0$, defined by

$$T_\ell [(u_1, u_2)](x) = (l^{\frac{p(q-1)-2}{pq-1}} u_1(\ell x), l^{\frac{q(p-1)-2}{pq-1}} u_2(\ell x)),$$

and thus the radial case reduces to an autonomous system of order 4. Due to the particular form of system (1.1), which involves only the gradients of the functions, we can reduce to a system of order 2. The radial study offers a double interest:

- *The radial system reduces to a quadratic Lotka-Volterra type system:*

$$\begin{cases} S_t = S(N - (N - 1)p + S + pZ), \\ Z_t = Z(N - (N - 1)q - qS - Z). \end{cases}$$

where

$$S = r \frac{|u_2'|^p}{u_1'} = -r \frac{(r^{N-1}u_1')'}{r^{N-1}u_1'}, \quad Z = -r \frac{|u_1'|^q}{u_2'} = -r \frac{(r^{N-1}u_2')'}{r^{N-1}u_2'}, \quad t = \ln r.$$

Moreover system (1.1) is governed by the Hardy-Hénon equations (1.2) in the following way:

- *For any radial solution (u_1, u_2) of system (1.1), the function $w = r^{N-1} |u_1'|$ satisfies equation (1.2) with the specific values*

$$\mathbf{N} = 1 + \frac{(N - 1)(p - 1)}{p}, \quad \mathbf{p} = 1 + \frac{1}{p}, \quad \mathbf{q} = q, \quad \sigma = (N - 1) \frac{1 - pq}{p}, \quad \varepsilon = -\text{sign}(u_1' u_2').$$

As an unexpected and remarkable fact, the system (1.4), which presents two *source* terms $|\nabla u_2|^p, |\nabla u_1|^q$, is not linked with a Hardy-Hénon equation with a source term ($\varepsilon = 1$) but an equation with an *absorption* term ($\varepsilon = -1$), see Remark 3.11.

Section 4 is devoted to a complete study of the radial local and global positive solutions of the Hardy-Hénon equations (1.2) for $\varepsilon = 1$. They are well known in the case $\mathbf{N} > \mathbf{p} > -\sigma$, and we cannot quote the immense literature on the subject, starting from the works of [27] and [19]. The main purposes of the articles are the obtention of Liouville type results, symmetry properties and a priori estimates for the equation with source ($\varepsilon = 1$), and the study of large solutions for the equation with absorption ($\varepsilon = -1$). The other ranges of the parameters $\mathbf{N}, \mathbf{p}, \sigma$, such as $\mathbf{p} > \mathbf{N}$ or $\sigma < -\mathbf{p}$, $\sigma < -\mathbf{N}$ seem to be less considered. We first mention the remarkable work of [34] valid for $\sigma = 0$ and any \mathbf{N}, \mathbf{p} . Let us also quote the nonexistence of possibly nonradial solutions, even in

a very weak sense, proved in the pioneer paper [17] for $\varepsilon = 1$, $\mathbf{N} \geq 3$, $\mathbf{p} = 2$, $\sigma \leq -2$; and recent interesting symmetry results of [4] in the case $\mathbf{p} = 2$. see also [24] for an extension to any extension to any $\mathbf{p} > 1$, $\sigma \leq -\max(\mathbf{N}, \mathbf{p})$ under some regularity assumptions.

Here we treat all the cases of distinct $\mathbf{N}, \mathbf{p}, \sigma$ at Theorems 4.9, 4.12, 4.13, 4.14, 4.15, 4.16. The limit cases $\sigma = -\mathbf{p}$, $\sigma = -\mathbf{N}$, $\mathbf{p} = \mathbf{N}$, where logarithmic type solutions appear, are described at Theorems 4.18, 4.19, 4.20, and the critical case $\mathbf{p} = \mathbf{N} = -\sigma$ at Theorem 4.21. Independently of their application to system (1.1), they offer a remarkable diversity of types of behaviour. We hope that they constitute a solid basis for a future nonradial analysis.

Our study is based on the analysis of the phase plane of the associated quadratic Lotka-Volterra system,

$$\begin{cases} s_t = s\left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1}\right), \\ z_t = z(\mathbf{N} + \sigma - \mathbf{q}s - z), \end{cases}$$

where $s = -r\frac{w'}{w}$ and $z = -\varepsilon r^{1+\sigma} w^{\mathbf{q}} |w'|^{-\mathbf{p}} w'$, already started in [12]. We observe that this system is neither competitive, nor cooperative as soon as $(\mathbf{p}-1)\mathbf{q} > 0$. Note also that our study gives local existence results without involving fixed point methods, and global ones without introducing energy functions. Among all the results, let us mention a few striking examples:

- For $\mathbf{p} > \mathbf{N} > -\sigma$ and $\varepsilon = \pm 1$, we get existence of solutions of equation (1.2) in $C^0(B_{r_0}) \cap C^2(B_{r_0} \setminus \{0\})$ of three types : such that $\lim_{r \rightarrow 0} w = c > 0$, and either $\lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = d \neq 0$ or $\lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = d \neq 0$, and solutions such that $\lim_{r \rightarrow 0} w = 0$ and $\lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0$.

- For $-\sigma > \max(\mathbf{p}, \mathbf{N})$, there exist solutions in an exterior domain $\mathbb{R}^N \setminus \overline{B_{r_0}}$ such that $\lim_{r \rightarrow \infty} w = c > 0$, see Theorems 4.12, 4.14, 4.20.

- It is well known that for $\mathbf{N} > \mathbf{p} > -\sigma$ there exist explicit solutions in $C^0(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$, often called ground states, when q is the critical Sobolev exponent $q_S = \frac{\mathbf{N}(\mathbf{p}-1) + \mathbf{p} + \mathbf{p}\sigma}{\mathbf{N}-\mathbf{p}}$, given by $w = c(d + r^{\frac{\mathbf{p}+\sigma}{\mathbf{p}-1}})^{-\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}+\sigma}}$, where $c > 0$ and $d = d(c, \mathbf{N}, \mathbf{p}, \sigma)$. In fact the same phenomena holds when $-\sigma > \mathbf{p} > \mathbf{N}$ (see Theorem 4.12).

- For $\varepsilon = -1$, $\sigma = -\frac{\mathbf{p}(\mathbf{N}-1)}{\mathbf{p}-1}$, in particular when $\mathbf{p} = \mathbf{N} = -\sigma$, there also exists a family of explicit solutions, see Proposition 4.17.

For convenience the proofs of the main results of Section 4, using various techniques of dynamical systems, are given in the Appendix.

In **Section 5** we describe the behaviour of the radial solutions of system (1.1), from the results of Section 4. Because of the great diversity of the possible solutions, we concentrate our study on constant sign solutions in $B_{r_0} \setminus \{0\}$ or $\mathbb{R}^N \setminus \overline{B_{r_0}}$, and above all on global solutions in $\mathbb{R}^N \setminus \{0\}$.

Comparing to the scalar case, the situation is much more intricated: even in the case $p = q$, where we show that the system can be completely integrated, there exists an infinity of solutions such that $u_1 \neq u_2$. In the general case $p \geq q$ we give all the local and global nonconstant solutions of systems (1.4), (1.5) and (1.6). It appears that the components u_1, u_2 of a solution in $(0, \infty)$ can admit different kinds of singularities near 0: either $\lim_{r \rightarrow 0} |u_i| = \infty$, and we say that u_i is ∞ -singular, or if $\lim_{r \rightarrow 0} u_i = c \in \mathbb{R}$ and $\lim_{r \rightarrow 0} |u'_i| = \infty$, and we say that u_i is a *cusp-solution*, see Remark 2.3. Concerning (1.4), we get the following, from Theorems 5.9, 5.12 and 5.15:

Theorem 1.1 *Let $pq > 1, p \geq q$, and let q_1, q_2, q_3, q_4 be defined in (1.8). Consider the system (1.4).*

(1) *Local solutions near 0: There exists radial solutions in $B_{r_0} \setminus \{0\}$ such that*

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 \delta_0, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 \delta_0, \end{cases}$$

with $C_1, C_2 > 0$ when $p < \frac{N}{N-1}$, with $C_1 = 0, C_2 > 0$ when $q < q_1$, with $C_1 > 0, C_2 = 0$ when $q < q_2$.

(2) *Global solutions: up to additive constants,*

• *For $q_2 < q < q_3$, there exists a ∞ -singular solution $(u_1^*, u_2^*) = (A_1^* r^{\frac{p(q-1)-2}{pq-1}}, A_2^* r^{\frac{q(p-1)-2}{pq-1}})$, with $A_1^*, A_2^* > 0$, and also solutions such that*

$$(u_1, u_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \begin{cases} \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \lim_{r \rightarrow \infty} r^{(N-1)q-2} u_2 = c(c_1) > 0, & \text{if } q < \frac{N}{N-1}, \\ \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \lim_{r \rightarrow \infty} r^{N-2} u_2 = c_2 > 0, & \text{if } q > \frac{N}{N-1}. \end{cases}$$

• *For $q > q_4$, there exist cusp-solutions (u_1, u_2) such that*

$$(u_1, u_2) \sim_{r \rightarrow 0} (|A_1^*| r^{\frac{p(q-1)-2}{pq-1}}, |A_2^*| r^{\frac{q(p-1)-2}{pq-1}}), \quad \lim_{r \rightarrow \infty} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} u_2 = c_2 > 0,$$

and there is no other global solution.

We also give the existence of solutions of (1.4) in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ at Theorem 5.10.

The cases of system (1.5) and (1.6), treated at Theorems 5.9, 5.13, 5.14 when $q < q_4$ are even richer. Note the existence of a family of explicit solutions of system (1.6) in the case $q = q^*$, which corresponds to the critical Sobolev exponent for the Hardy-Henon equation, see Theorem 4.9. The case $q > q_4$, treated at Theorem 5.15 is remarkable, since it gives the existence of bounded solutions of any of the three systems (1.4), (1.5) and (1.6).

In **Section 6** we extend the study of system (1.1) to the nonradial case. Our main aim is finding upper estimates, which appears to be a good challenge. Indeed the system is not variational, and does not offer monotony or comparison properties which could be used as it was done in [8], [37]. Moreover the Bernstein technique, developed in [27], [16], [34] for semilinear or quasilinear equations, appears not to be efficient for systems, as in the case of system (1.3), except in special cases, as in [15]. Finally we cannot use the very performant methods of moving planes or moving spheres, see the pioneer results of [19], [33], even with boundedness assumptions on the gradients, because in \mathbb{R}^N there always exist constant nontrivial solutions. Here we give a first estimate, and in the same way a nonexistence result, see Theorem 6.2, Proposition 6.5 and Theorem 6.6:

Theorem 1.2 *Let $pq > 1, p \geq q \geq 1$.*

(i) *If $(N-1)pq < \max(N+p, N+q)$, then any supersolution of system (1.1) in $\Omega = \mathbb{R}^N$, without condition of sign, is constant.*

(ii) *Let (u_1, u_2) be any supersolution of system (1.4) in $B_{r_0} \setminus \{0\}$ (resp. in $\mathbb{R}^N \setminus \overline{B_{r_0}}$). Then there exists $\rho \in (0, r_0)$ and $C > 0$, depending on u_1, u_2 , such that for any $r \in (0, \rho)$,*

$$\begin{aligned} |\overline{u}_1(r)| &\leq C \begin{cases} \max(1, r^{-\frac{2-p(q-1)}{pq-1}}) & \text{if } p(q-1) \neq 2, \\ |\ln r| & \text{if } p(q-1) = 2, \end{cases} \\ |\overline{u}_2(r)| &\leq C \begin{cases} \max(1, r^{-\frac{2-q(p-1)}{pq-1}}) & \text{if } q(p-1) \neq 2, \\ |\ln r| & \text{if } q(p-1) = 2. \end{cases} \end{aligned}$$

(iii) Let $(\widetilde{u}_1, \widetilde{u}_2)$ be any positive subsolution of system (1.5) in $B_{r_0} \setminus \{0\}$ (resp. in $\mathbb{R}^N \setminus \overline{B_{r_0}}$). Then there exists $\rho \in (0, r_0)$ and $C > 0$, depending on $\widetilde{u}_1, \widetilde{u}_2$, such that for $0 < |x| < \rho$,

$$\begin{aligned} \widetilde{u}_1(x) &\leq C \begin{cases} \max(1, |x|^{-\frac{2-p(q-1)}{pq-1}}) & \text{if } p(q-1) \neq 2, \\ |\ln |x|| & \text{if } p(q-1) = 2, \end{cases} \\ \widetilde{u}_2(x) &\leq C \begin{cases} \max(1, |x|^{-\frac{2-q(p-1)}{pq-1}}) & \text{if } q(p-1) \neq 2, \\ |\ln |x|| & \text{if } q(p-1) = 2. \end{cases} \end{aligned}$$

Note that the range $(N-1)pq < \max(N+p, N+q)$ in assertion (i) is optimal for supersolutions. Nevertheless we think that the result is much more general concerning solutions:

Conjecture: for any p, q such that $pq > 1$, all the solutions of system (1.1) in $\Omega = \mathbb{R}^N$, without condition of sign, are constant.

We end this section by the study of the existence of supersolutions of system (1.4) in an exterior domain, see Theorem 6.7, and the existence of solutions in a domain Ω with measure data, see Theorem 6.9.

In **Section 7** we conclude the paper by suggesting some extensions of our results to more general systems.

2 Critical exponents of the system

We first recall some results on the scalar case of equation (1.7), which can be found for example in [9].

2.1 On the scalar case

Clearly a function u is a negative solution of (1.7) if and only if $\widetilde{u} = -u$ is a positive solution of the equation with absorption

$$-\Delta \widetilde{u} + |\nabla \widetilde{u}|^q = 0. \quad (2.1)$$

As we mentioned above, two critical values are involved: $q = \frac{N}{N-1}$, and $q = 2$. When $q = 2$ the equation is equivalent to $\Delta(e^u) = 0$, so all the solutions are described; in particular all the solutions in $\mathbb{R}^N \setminus \{0\}$ are radial, and given by $u = \ln(cr^{2-N} + d)$, $c, d \in \mathbb{R}$.

(1) For $q \neq 1, \frac{N}{N-1}, 2$, there exist particular solutions of (1.7) on $(0, \infty)$, given by

$$u^*(r) = A^* r^{\frac{q-2}{q-1}} + c, \quad r = |x|, \quad c \in \mathbb{R}, \quad (2.2)$$

with $|A^*|^{q-2} A^* = (2-q)((N-1)q-N)|2-q|^{1-q}|q-1|^{2-q}$. This function is ∞ -singular for $q < 2$: positive for $\frac{N}{N-1} < q < 2$, such that $\lim_{r \rightarrow 0} u^* = \infty$, and negative either for $1 < q < \frac{N}{N-1}$ where $\lim_{r \rightarrow 0} u^* = -\infty$; it is a cusp-solution for $q > 2$: $\lim_{r \rightarrow 0} u^* = 0$, $\lim_{r \rightarrow 0} u'^* = -\infty$.

(2) In the radial case, equation (1.7) only depends on the derivative, and it can be completely integrated: it reduces to

$$W' = r^{(N-1)(1-q)} |W|^q, \quad \text{where } W = -r^{N-1} u',$$

hence there exists a union of disjoint intervals (ρ, R) where for $q \neq \frac{N}{N-1}$ setting $b = \frac{q-1}{(N-1)q-N}$,

$$u' = -r^{1-N}(C + br^{N-(N-1)q})^{-\frac{1}{q-1}} < 0, \quad \text{or } u' = r^{1-N}(C - br^{N-(N-1)q})^{-\frac{1}{q-1}} > 0. \quad (2.3)$$

Then we deduce u by integration. The case $C = 0$ corresponds to the particular solutions.

When $q = \frac{N}{N-1}$, there exist negative solutions near 0 with a logarithmic behaviour, given by

$$u^{*}(r) = r^{1-N}((q-1)|\ln r| + C)^{\frac{-1}{q-1}}. \quad (2.4)$$

From (2.3) we get the existence of other radial solutions in $\mathbb{R}^N \setminus \{0\}$:

- for $\frac{N}{N-1} < q < 2$, for any $c \geq 0, k > 0$, there exists positive solutions of (1.7) such that

$$\lim_{r \rightarrow 0} r^{\frac{2-q}{q-1}}(u-c) = A^*, \quad \lim_{r \rightarrow \infty} r^{N-2}(u-c) = k > 0,$$

- for $1 < q < \frac{N}{N-1}$, and for any $c < 0, k > 0$, there exists a negative solution u such that

$$\lim_{r \rightarrow 0} r^{N-2}(u-c) = -k, \quad \lim_{r \rightarrow \infty} r^{\frac{2-q}{q-1}}(u-c) = -|A^*|,$$

- for $q > 2$, and for any $c > d$ there exist decreasing bounded solutions of (1.7), such that

$$\lim_{r \rightarrow 0} u = c, \quad \lim_{r \rightarrow 0} r^{\frac{2-q}{q-1}}(u-c) = -|A^*|, \quad \lim_{r \rightarrow \infty} u = d,$$

and they are either positive for $d > 0$ or negative for $c < 0$.

(3) In the nonradial case, the first upperestimates are due to [29]: if u is any solution (with no condition of sign) in a domain Ω , then for any $q > 1$,

$$|\nabla u(x)| \leq C_{N,q} \text{dist}(x, \partial\Omega)^{-\frac{1}{q-1}}.$$

As a consequence, if $\Omega = \mathbb{R}^N$, then u is constant. Note that the result is false if $q < 1$, since u^* given at (2.2) still exists and $u^* \in C^2(\mathbb{R}^N)$. The estimates have been extended to a quasilinear equation $-\Delta_m u = -\text{div}(|\nabla u|^{m-2} \nabla u) = |\nabla u|^q$, $m > 1$ in [9], where one can find a complete classification of the negative solutions $u \in C^1(\Omega \setminus \{0\})$ near 0, and a partial classification of the positive ones. The negative solutions of the equation (1.7) have been studied in [32]. When $q \geq 2$, the first author gives in [8] a detailed behaviour of the positive solutions in $B_{r_0} \setminus \{0\}$ or in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ in a more general context of equation $-\Delta_m u = u^p |\nabla u|^q$, $p \geq 0$.

2.2 Particular solutions and critical values of the parameters

When searching particular solutions of system (1.1), some critical values of the parameters p, q are involved in the problem, showing the complexity of the system compared to the scalar case :

Definition 2.1 *Let $pq > 1, p \geq q$. Following (1.8) and (1.9), we define*

$$\begin{aligned} q_1 &= q_1(p) = \frac{N}{(N-1)p-1}, & q_2 &= q_2(p) = \frac{N+p}{(N-1)p}, \\ q_3 &= q_3(p) = \frac{2}{p-1}, & q_4 &= q_4(p) = \frac{p+2}{p}, & q^* &= q^*(p) = \frac{2N+p}{2(N-1)p-1}. \end{aligned}$$

corresponding to five curves in the set $\{(p, q) : pq > 1, p \geq q > 0\}$, namely

$$\begin{aligned}\mathcal{L}_1 &= \{q = q_1\} = \{(N-1)pq = N + q\}, & \mathcal{L}_2 &= \{q = q_2\} = \{(N-1)pq = N + p\}, \\ \mathcal{L}_3 &= \{q = q_3\} = \{q(p-1) = 2\}, & \mathcal{L}_4 &= \{q = q_4\} = \{p(q-1) = 2\}, \\ \mathcal{L}^* &= \{q = q^*\} = \left\{ (N-1)pq = N + \frac{p+q}{2} \right\}.\end{aligned}$$

Note that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}^*$ intersect the diagonal $\{p = q\}$ for $p = \frac{N}{N-1}$, and $\mathcal{L}_3, \mathcal{L}_4$ for $p = 2$. And \mathcal{L}_2 intersect \mathcal{L}_3 for $p = 2(N-1)$, \mathcal{L}^* intersects \mathcal{L}_3 for $p = 2(N-1)$, see the figures below in Section 5.

Proposition 2.2 *Let $pq > 1, p \geq q$. Then for $q \neq q_1, q_2$, the system (1.1) admits particular solutions such that*

$$u_1^{*'} = \eta_1 a_1 r^{-\frac{p+1}{pq-1}}, \quad u_2^{*'} = \eta_2 a_2 r^{-\frac{q+1}{pq-1}}, \quad (2.5)$$

where $a_i = a_i(N, p, q) > 0$ for $i = 1, 2$, and

$$\eta_1 = \text{sign}(q_2 - q), \quad \eta_2 = \text{sign}(q_1 - q). \quad (2.6)$$

If moreover $q \neq q_3, q_4$, it has solutions

$$u_1^* = \varepsilon_1 A_1^* r^{\frac{p(q-1)-2}{pq-1}} + c_1 = \varepsilon_1 A_1^* r^{\frac{p(q-q_4)}{pq-1}} + c_1, \quad u_2^* = \varepsilon_2 A_2^* r^{\frac{q(p-1)-2}{pq-1}} + c_2 = \varepsilon_2 A_2^* r^{\frac{(p-1)(q-q_3)}{pq-1}} + c_2, \quad (2.7)$$

where $A_i^* = A_i^*(N, p, q) > 0$, $c_i \in \mathbb{R}$ for $i = 1, 2$, and

$$\varepsilon_1 = \text{sign}(q_2 - q)(q - q_4), \quad \varepsilon_2 = \text{sign}(q_1 - q)(q - q_3).$$

Proof. We search particular solutions of the form $(u_1', u_2') \equiv (C_1 r^{-\lambda_1}, C_2 r^{-\lambda_2})$. Then

$$C_1(\lambda_1 - N + 1)r^{-\lambda_1-1} = r^{-p\lambda_2} |C_2|^p, \quad C_2(\lambda_2 - N + 1)r^{-\lambda_2-1} = r^{-q\lambda_1} |C_1|^p,$$

hence we find $\lambda_1 = \frac{p+1}{pq-1}$, $\lambda_2 = \frac{q+1}{pq-1}$, and setting

$$\beta_1 = \lambda_1 - N + 1 = \frac{N + p - (N-1)pq}{pq-1}, \quad \beta_2 = \lambda_2 - N + 1 = \frac{N + q - (N-1)pq}{pq-1},$$

we get the results with $|C_1| = (|\beta_1| |\beta_2|^p)^{\frac{1}{pq-1}}$, $|C_2| = (|\beta_2| |\beta_1|^q)^{\frac{1}{pq-1}}$. By integration, if the denominator is nonzero, we get in particular the solution (u_1, u_2) , with $|A_i^*| = \left| \frac{C_i(pq-1)}{p(q-1)-2} \right|$, $i = 1, 2$. \blacksquare

Remark 2.3 *Note that u_1^* is ∞ -singular for $q < q_4$ and it is a cusp-solution for $q > q_4$. And u_2^* is ∞ -singular for $q < q_3$ and it is a cusp-solution for $q > q_3$; so there exist solutions (u_1, u_2) of system (1.1) with components of different types when $q_3 < q < q_4$.*

Remark 2.4 *It is clear that the exponents q_1, q_2, q_3, q_4 , are strongly involved in the existence of particular solutions. The exponent q^* corresponds to the Sobolev exponent for equation (1.2), offering other types of explicit solutions, see Theorem 5.6. Note the relations*

$$\begin{aligned}q \leq q_1 &\iff (N-1)pq \leq N + q, & q \leq q_2 &\iff (N-1)pq \leq N + p, \\ q \leq q_3 &\iff q(p-1) \leq 2, & q \leq q_4 &\iff p(q-1) \leq 2, \\ q \leq q^* &\iff (N-1)pq \leq N + \frac{p+q}{2},\end{aligned}$$

$$\begin{aligned}
q_1 &\leq \min(q_2, q_3) \leq \max(q_2, q_3) \leq q_4, \\
q_1 &\leq \frac{N}{N-1} \iff q_2 \leq \frac{N}{N-1} \iff p \geq \frac{N}{N-1}, \\
q_2 &\leq q_3 \iff p \leq N, \\
q_1 &\leq q^* \leq q_2, \quad q^* \leq q_3 \iff p \leq 2(N-1).
\end{aligned}$$

When $q = p$, we find again the two critical values of the scalar case:

$$q_1 = q^* = q_2 = \frac{N}{N-1}, \quad q_3 = q_4 = 2.$$

3 First properties in the radial case

In the radial case, system (1.1) is reduced to

$$\begin{cases} -(u_1'' + \frac{N-1}{r}u_1') = |u_2'|^p, \\ -(u_2'' + \frac{N-1}{r}u_2') = |u_1'|^q, \end{cases} \quad (3.1)$$

so it only involves the derivatives. Moreover, setting

$$w_1 = -r^{N-1}u_1', \quad w_2 = -r^{N-1}u_2', \quad (3.2)$$

system (3.1) is equivalent to a first order system:

$$\begin{cases} w_1' = r^{(N-1)(1-p)} |w_2|^p, \\ w_2' = r^{(N-1)(1-q)} |w_1|^q. \end{cases} \quad (3.3)$$

Consider any solution $(u_1, u_2) \in C^2(0, r_0)$, $r_0 \leq \infty$. We say that u_i is *regular* if u_i' has a limit at 0 and $\lim_{r \rightarrow 0} u_1' = \lim_{r \rightarrow 0} u_2' = 0$; if u_1 and u_2 are regular, then (u_1, u_2) extends as a function in $C^2(B_{r_0} \times B_{r_0})$, from the equations. We say that u_i is *singular* if it is not regular. Among the singular solutions we distinguish the ∞ -*singular* ones, such that $\lim_{r \rightarrow 0} |u_i| = \infty$, and the *cusp-solutions* u_i ($\lim_{r \rightarrow 0} u_i = c \in \mathbb{R}$ and $\lim_{r \rightarrow 0} |u_i'| = \infty$), where the singularity is at the level of the gradient.

3.1 Local existence and uniqueness

Proposition 3.1 *Let $pq > 1, p \geq q$. (i) Let $r_0 \geq 0$ and $c_1, c_2, b_1, b_2 \in \mathbb{R}$ such that $b_2 \neq 0$ if $q < 1$. Then there exists a unique solution (u_1, u_2) of system (3.1) in the neighborhood of r_0 , such that*

$$u_1(r_0) = c_1, \quad u_2(r_0) = c_2, \quad u_1'(r_0) = b_1, \quad u_2'(r_0) = b_2.$$

(ii) *For any $R > 0$, the only radial solutions $(u_1, u_2) \in C^1(B_R \times B_R)$ of system (1.1) are the constants.*

Proof. (i) First suppose $r_0 > 0$. Since the system only depends on the derivatives, it is equivalent to consider system (3.3), with initial data $(w_{01}, w_{02}) \in \mathbb{R}^2$. If $q > 1$, the classical Cauchy-Lipschitz theorem applies. Next suppose $q < 1$; the theorem still applies if $w_{01} \neq 0$. Consider the case $w_{01} = 0, w_{02} \neq 0$. In a neighborhood $[r_0 - \eta, r_0 + \eta]$ of r_0 , there holds $0 < C_1 < |w_2| < C_2$,

$C_3 < w'_1 < C_4$, $0 < |w_1| < C_5$. hence we can take w_1 as a new variable, and consider that r is a function of w_1 . Then we get the system

$$\begin{cases} \frac{dr}{dw_1} = f(w_1, r, w_2) = r^{(N-1)(1-p)} |w_2|^p, \\ \frac{dw_2}{dw_1} = g(w_1, r, w_2) = |w_1|^q r^{(N-1)(2-p-q)} |w_2|^p, \end{cases}$$

for which we can apply the Cauchy-Lipschitz theorem in $[0, C_5) \times (r_0 - \eta, r_0 + \eta) \times (C_1, C_2)$, since f is continuous, and locally Lipschitz with respect to (r, w_2) .

(ii) Next suppose that $b_1 = b_2 = 0$, and $r_0 \geq 0$. Then

$$\begin{aligned} u'_1(r) &= r^{1-N} \int_{r_0}^r s^{N-1} |u'_2|^p ds, & u'_2(r) &= r^{1-N} \int_{r_0}^r s^{N-1} |u'_1|^q ds, \\ |u'_1(r)| &\leq r^{1-N} \left| \int_{r_0}^r s^{N-1} |u'_2|^p ds \right| \leq r^{1-N} \left| \int_{r_0}^r s^{N-1} (s^{1-N} \left| \int_{r_0}^s \theta^{N-1} |u'_1|^q d\theta \right|^p) ds \right|, \end{aligned} \quad (3.4)$$

• First assume $r_0 > 0$. In a neighborhood $[r_0 - \eta, r_0 + \eta]$, there holds

$$|u'_1(r)| \leq C \left| \int_{r_0}^r \left| \int_{r_0}^s |u'_1|^q d\theta \right|^p ds \right| \leq C' \int_{r_0}^r \int_{r_0}^s |u'_1|^{pq} d\theta ds \leq C'' \left| \int_{r_0}^r |u'_1|^{pq} d\theta \right|. \quad (3.5)$$

Let $F(r) = \int_{r_0}^r |u'_1|^{pq} d\theta$. Then $F'(r) \leq C'' F^{pq}$ on $(r_0, r_0 + \eta)$. If u'_1 is not identically 0 on $(r_0, r_0 + \eta)$, then $F > 0$ on $(r_0, r_0 + \rho)$ for some ρ small enough, and $F^{1-pq} + cr$ is nondecreasing, which is contradictory, since $F(r_0) = 0$; thus $u'_1 \equiv 0$ and then $u'_2 \equiv 0$ on $(r_0, r_0 + \rho)$, and similarly on $(r_0 - \rho, r_0)$ for ρ small enough. Then $u_1 \equiv c_1, u_2 \equiv c_2$ in $[r_0 - \eta, r_0 + \eta]$.

• Next assume $r_0 = 0$. Then from (3.4), and since $p > 1$,

$$\begin{aligned} |u'_1(r)| &\leq r^{1-N} \int_0^r s^{(N-1)(1-p)} \left(\int_0^s \theta^{N-1} |u'_1|^q d\theta \right)^p ds \\ &\leq r^{1-N} \int_0^r s^{(N-1)(1-p) + (N-1)p} \left(\int_0^s |u'_1|^q d\theta \right)^p ds \leq \int_0^r \left(\int_0^s |u'_1|^q d\theta \right)^p ds \\ &\leq \int_0^r s^{p-1} \left(\int_0^s |u'_1|^{pq} d\theta \right) ds \leq r^p \int_0^r |u'_1|^{pq} d\theta \leq C \int_0^r |u'_1|^{pq} d\theta. \end{aligned}$$

Defining $F(r) = \int_0^r |u'_1|^{pq} d\theta$ we conclude as above that $u'_1 \equiv 0, u'_2 \equiv 0$ on $(0, \rho)$ for ρ small enough. Finally let $(u_1, u_2) \in C^1(B_R \times B_R)$ be any radial solution. Then the set $\{(0, \rho) : u'_1 \equiv u'_2 \equiv 0 \text{ on } (0, \rho)\}$ is closed and open in $(0, R)$, thus equal to $(0, R)$. ■

Remark 3.2 *The result would be false for $pq < 1$. Indeed in that case there exist solutions of system (1.5):*

$$\widetilde{u}_1^* = A_1^* r^{\frac{2+p(1-q)}{1-pq}} = A_1^* r^{2+\frac{p(q+1)}{1-pq}}, \quad \widetilde{u}_2^* = A_2^* r^{2+\frac{q(p+1)}{1-pq}},$$

belonging to $C^2(\mathbb{R}^N)$ which proves the existence of entire nontrivial solutions, and gives a counterexample to uniqueness.

Remark 3.3 Note a consequence of Proposition 3.1: for any $r_0 > 0$, there exist local radial solutions (u_1, u_2) such that $u_1'(r_0) = 0$, and $u_2'(r_0) = b \neq 0$. This shows the great richness of the solutions of the system: for system (1.4) (resp. (1.5) for example, it means that only one of the functions has a minimum (resp. a maximum) at r_0 .

Remark 3.4 From Proposition 3.1, one can divide the maximal existence interval (ρ, R) of (u_1, u_2) in intervals (a, b) where u_1' and u_2' have a constant sign. Indeed there exists at most one point $\rho < s < R$ where $w_1(s) = 0$; $w_1 < 0$ on (ρ, s) and $w_1 > 0$ on (s, R) ; in the same way, there is at most one point $\rho < \tau < R$ where $w_2(\tau) = 0$; $w_2 < 0$ on (ρ, τ) and $w_2 > 0$ on (τ, R) .

3.2 Upperestimates on the radial supersolutions

Proposition 3.5 Let $pq > 1$, and (u_1, u_2) be any radial supersolution of system (1.1), defined on an interval $(0, r_0)$ (resp. (r_0, ∞)). There exists a constant $C = C(N, p, q) > 0$ such that for $r > 0$ small enough (resp. large enough),

$$|u_1'(r)| \leq Cr^{-\frac{p+1}{pq-1}}, \quad |u_2'(r)| \leq Cr^{-\frac{q+1}{pq-1}}.$$

Proof. Consider any supersolution of system (1.1). Equivalently, the functions w_1, w_2 defined by (3.2) satisfy

$$\begin{cases} w_1' \geq r^{(N-1)(1-p)} |w_2|^p, \\ w_2' \geq r^{(N-1)(1-q)} |w_1|^q. \end{cases}$$

As in Remark 3.4, w_1, w_2 do not vanish for r small enough (resp. large enough), depending on u_1, u_2 . Consider any interval (R_1, R_2) where w_1, w_2 do not vanish. Let $r_0 \in (R_1, R_2)$ and $\varepsilon_0 \in (0, \frac{1}{4}]$ such that $R_1 < r_0(1 - 2\varepsilon_0) < r_0(1 + 2\varepsilon_0) < R_2$. First assume that $w_1 > 0, w_2 > 0$. Since w_i is nondecreasing, for any $\varepsilon \in (0, \varepsilon_0]$, we have $r \in (r_0(1 - \varepsilon), r_0(1 + \varepsilon))$ there holds, since $w_1 > 0$ and $\varepsilon_0 < 1$, with $c_i = c_i(N, p)$

$$\begin{aligned} w_1(r_0) &\geq \int_{r_0(1-\varepsilon)}^{r_0} \theta^{(N-1)(1-p)} w_2^p(\theta) d\theta \\ &\geq w_2^p(r_0(1 - \varepsilon)) \int_{r_0(1-\varepsilon)}^{r_0} \theta^{(N-1)(1-p)} d\theta = c_1 \varepsilon w_2^p(r_0(1 - \varepsilon)) r_0^{N-(N-1)p}, \end{aligned}$$

and symmetrically, since $w_1 > 0$, changing r_0 into $r_0 - \varepsilon$,

$$w_2(r_0 - \varepsilon) \geq c_2 \varepsilon w_1^q(r_0(1 - 2\varepsilon)) (r_0(1 - \varepsilon))^{N-(N-1)q} \geq c_3 \varepsilon w_1^q(r_0 - 2\varepsilon) r_0^{N-(N-1)q}.$$

Then

$$\begin{aligned} w_1(r_0) &\geq c_4 \varepsilon^{p+1} w_1^{pq}(r_0(1 - 2\varepsilon)) r_0^{N+p-(N-1)pq}, \\ w_1(r_0(1 - 2\varepsilon)) &\leq c_5 \varepsilon^{-\frac{p+1}{pq}} (w_1(r_0))^{\frac{1}{pq}} r_0^{-\frac{N+p-(N-1)pq}{pq}}. \end{aligned}$$

Setting $\bar{r}_0 = r_0(1 - 2\varepsilon)$, and using by the bootstrap technique developed at [13, Lemma 2.2], [11, Lemma 2.8] and [?], we obtain, since $d = \frac{1}{pq} < 1$, and

$$w_1(r_0) = r_0^{N-1} |u_1'(r_0)| \leq c_6 r_0^{\frac{N+p-(N-1)pq}{1-d}} = c_6 r_0^{-\frac{N+p-(N-1)pq}{pq-1}},$$

equivalently

$$|u'_1(r_0)| \leq c_7 r_0^{-\frac{p+1}{pq-1}}.$$

If $w_1 < 0, w_2 < 0$, the same conclusion holds by changing $1 + \varepsilon$ into $1 - \varepsilon$. Finally if $w_2 > 0 > w_1$, we get

$$\begin{aligned} w_1(r_0) &\geq c_1 \varepsilon w_2^p(r_0(1 - \varepsilon)) r_0^{N-(N-1)p}, \\ w_2(r_0(1 - \varepsilon)) &\geq c_2 \varepsilon w_1^q(r_0)(r_0(1 - \varepsilon))^{N-(N-1)q} \geq c_3 \varepsilon w_1^q(r_0) r_0^{N-(N-1)q}, \\ w_1(r_0) &\geq c_4 \varepsilon^{p+1} w_1^{pq}(r_0) r_0^{N+p-(N-1)pq}, \end{aligned}$$

hence we obtain the same conclusion without using any bootstrap. \blacksquare

3.3 Reduction to a quadratic system of order 2 and formulation as a Hardy-Hénon equation

In the sequel we show that system (3.1) can be reduced to a polynomial system order 2, and equivalently to an equation of order 2 relative to w_1 :

Proposition 3.6 *Let (u_1, u_2) be any radial solution of system (1.1). At any point r where $u'_1(r) \neq 0, u'_2(r) \neq 0$, we define*

$$S = r \frac{|u'_2|^p}{u'_1}, \quad Z = -r \frac{|u'_1|^q}{u'_2}. \quad (3.6)$$

Then system (3.1) is equivalent to a Lodka-Volterra type system:

$$\begin{cases} S_t = S(N - (N - 1)p + S + pZ), \\ Z_t = Z(N - (N - 1)q - qS - Z). \end{cases} \quad t = \ln r, \quad (3.7)$$

and we recover u'_1 and u'_2 in function of r, S, Z by the formulas

$$u'_1 = (r^{-(p+1)} |S| |Z|^p)^{\frac{1}{pq-1}} \text{sign} S, \quad u'_2 = -(r^{-(q+1)} |S|^q |Z|)^{\frac{1}{pq-1}} \text{sign} Z. \quad (3.8)$$

Proof. Following the ideas of [12], at each point where $u'_1 \neq 0$ and $u'_2 \neq 0$, we set

$$X = r \frac{u''_1}{u'_1}, \quad Y = r \frac{u''_2}{u'_2}, \quad S = r \frac{|u'_2|^p}{u'_1}, \quad Z = -r \frac{|u'_1|^q}{u'_2}, \quad t = \ln r,$$

Then

$$X + N - 1 = -S, \quad Y + N - 1 = Z.$$

Next we differentiate with respect to t . Since $X = \frac{(u'_1)_t}{u'_1}$, $Y = \frac{(u'_2)_t}{u'_2}$, we obtain

$$\frac{S_t}{S} = 1 + p \frac{(u'_2)_t}{u'_2} - \frac{(u'_1)_t}{u'_1} = 1 + pY - X = N - (N - 1)p + S + pZ,$$

$$\frac{Z_t}{Z} = 1 + q \frac{(u'_1)_t}{u'_1} - \frac{(u'_2)_t}{u'_2} = 1 + qX - Y = N - (N - 1)q - qS - Z.$$

So we obtain the quadratic system (3.7) valid in any case of sign of the unknown, and deduce (3.8). \blacksquare

Remark 3.7 *The fixed points of the system are*

$$\begin{aligned} M_0 = (S_0, Z_0) &= \left(\frac{(N-1)p(q_2-q)}{pq-1}, -\frac{((N-1)p-1)(q_1-q)}{pq-1} \right), \\ (0, 0), \quad N_0 &= (0, N - (N-1)q), \quad A_0 = ((N-1)p - N, 0). \end{aligned} \quad (3.9)$$

We easily check that the particular solutions (u'_1, u'_2) of system (3.1) given at (2.5) correspond to the fixed point M_0 .

Next we show the precise link with a Hardy-Hénon equation:

When looking at the system (3.7) for $pq \neq 1$, and comparing to the systems introduced in [12, Section 3], we observe that the system is exactly linked to the positive solutions w of a radial quasilinear equation of Hardy-Hénon type in dimension \mathbf{N} , of the form

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = -(|w'|^{\mathbf{p}-2} w')' - \frac{\mathbf{N}-1}{r} |w'|^{\mathbf{p}-2} w' = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = \pm 1, \quad (3.10)$$

where $q > 0, \sigma \in \mathbb{R}, \mathbf{p} > 1, \mathbf{q} \neq \mathbf{p} - 1$, and \mathbf{N} is not necessarily an integer. Indeed in [12, Section 3], this equation is reduced to a system of order 2, valid for $\varepsilon = \pm 1$, by setting

$$s(t) = -r \frac{w'}{w}, \quad z(t) = -\varepsilon r^{1+\sigma} w^{\mathbf{q}} |w'|^{-\mathbf{p}} w', \quad t = \ln r, \quad (3.11)$$

and obtained the system

$$\begin{cases} s_t = s \left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1} \right), \\ z_t = z (\mathbf{N} + \sigma - \mathbf{q}s - z), \end{cases} \quad (3.12)$$

and one recovers w by the formula

$$w = r^{-\gamma} (|s|^{\mathbf{p}-1} |z|)^{\frac{1}{\mathbf{q}+1-\mathbf{p}}}, \quad \gamma = \frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}. \quad (3.13)$$

It is precisely the case, as we show below.

Proposition 3.8 *Let (u_1, u_2) be any radial solution of system (1.1), and let w_1, w_2 be defined by (3.2). Then the function $w = |w_1|$ satisfies the equation*

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = -\text{sign}(u'_1 u'_2) \quad (3.14)$$

where $\Delta_{\mathbf{p}}^{\mathbf{N}}$ is the \mathbf{p} -Laplace operator in dimension \mathbf{N} , for specific values of \mathbf{p}, σ and \mathbf{N} :

$$\mathbf{p} = 1 + \frac{1}{p} > 1, \quad \mathbf{q} = q, \quad \sigma = (N-1) \frac{1-pq}{p}, \quad \mathbf{N} = 1 + \frac{(N-1)(p-1)}{p}. \quad (3.15)$$

Moreover at each point where $w \neq 0$,

$$S(t) = s(t) = -r \frac{w'_1}{w_1}, \quad Z(t) = z(t) = r \frac{w'_2}{w_2} = -\text{sign}(u'_2) r^{1+\sigma} |w_1|^q |w'_1|^{\frac{1}{p}} w'_1 \quad (3.16)$$

Proof. Computing w_2 from the first equation of (3.3) and reporting in the second one, we get

$$-(r^{\frac{(N-1)(p-1)}{p}}(w'_1)^{\frac{1}{p}})' = \text{sign}(u'_2) r^{(N-1)(1-q)} |w_1|^q,$$

that is in a developed form:

$$-((w'_1)^{\frac{1}{p}})' - \frac{(N-1)(p-1)}{pr} (w'_1)^{\frac{1}{p}} = \text{sign}(u'_2) r^{(N-1)\frac{1-pq}{p}} |w_1|^q$$

So we get an equation with the form (where we recall that $w'_1 > 0$)

$$-(|w'_1|^{\mathbf{p}-2} w'_1)' - \frac{\mathbf{N}-1}{r} |w'_1|^{\mathbf{p}-2} w'_1 = -\Delta_{\mathbf{p}}^{\mathbf{N}} w_1 = \text{sign}(u'_2) r^{\sigma} |w_1|^q$$

where $\mathbf{p}, \sigma, \mathbf{N}$ are defined at (3.15); and the fact that $w_1 = -\text{sign}(u'_1) w$ leads to equation (3.14), and we check easily that $s \equiv S$ and $z \equiv Z$. \blacksquare

Remark 3.9 From the definitions (3.15) and (3.13) we get the relations

$$\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} = N - (N-1)p, \quad \mathbf{N} + \sigma = N - (N-1)q, \quad (3.17)$$

$$\gamma = \frac{\mathbf{p} + \sigma}{q - \mathbf{p} + 1} = \frac{p + N - (N-1)pq}{pq - 1} = \frac{(N-1)p(q_2 - q)}{pq - 1} \quad (3.18)$$

so

$$\mathbf{N} > \mathbf{p} \iff p > \frac{N}{N-1}, \quad \mathbf{N} + \sigma > 0 \iff q < \frac{N}{N-1}, \quad \mathbf{p} + \sigma > 0 \iff q < q_2. \quad (3.19)$$

Remark 3.10 For $pq > 1$, $p \geq q$, there holds $1 < \mathbf{p} < 2$, $\mathbf{q} > \mathbf{p} - 1$, and $\mathbf{N} > 1$. The map $(N, p) \in [1, \infty) \times (1, \infty) \mapsto (\mathbf{N}, \mathbf{p}) \in (1, \infty) \times (1, 2)$ is injective, the reciprocal application is $(\mathbf{N}, \mathbf{p}) \mapsto (\frac{\mathbf{N}+1-\mathbf{p}}{2-\mathbf{p}}, \frac{1}{\mathbf{p}-1})$, and then σ is **fixed** by the relation $\sigma = -\frac{(\mathbf{N}-1)(q+1-\mathbf{p})}{2-\mathbf{p}}$.

Remark 3.11 Equation (3.14) is of source type for $\varepsilon = 1$, of absorption type for $\varepsilon = -1$. One could think that problem (1.4), where u_1, u_2 are positive superharmonic functions, with two **source** terms $|\nabla u_2|^p, |\nabla u_1|^q$ is linked to a Hardy-Hénon equation with source term $\varepsilon = 1$. In fact **it is not the case**: the solutions of system (1.4) on an interval $(0, \rho)$ correspond to a Hardy-Hénon equation of **absorption** type. Indeed consider any positive solution of system (1.4). As it is well known, any positive solution (u_1, u_2) satisfies $u'_1, u'_2 \leq 0$ in $(0, \rho)$. Indeed $r^{N-1}u'_1$ is decreasing; if there is r_0 such that $u'_1(r_0) > 0$, then $r^{N-1}u'_1 \geq C_0 > 0$ on $(0, r_0)$, thus $u_1 + \frac{C_0}{N-2}r^{2-N}$ is increasing, which is impossible. Then $u'_1 \leq 0$ on $(0, \rho)$; moreover if there exists r_1 such that $u'_1(r_1) = 0$, then it is unique and r_1 is a maximum point, so that $u'_1 \geq 0$ on $(0, r_1)$ which is contradictory, unless u_1 is constant. Then the nonconstant solutions satisfy $u'_1, u'_2 < 0$ in $(0, \rho)$.

Remark 3.12 Another way to get an autonomous system is more common in the literature: the change of unknown

$$u'_1 = -r^{-\frac{p+1}{pq-1}}x(t), \quad u'_2 = -r^{-\frac{q+1}{pq-1}}y(t), \quad t = \ln r.$$

leads to the system

$$\begin{cases} x_t = b_1 x - |y|^p, \\ y_t = b_2 y - |x|^q, \end{cases}$$

where $b_1 = \frac{(N-1)pq-(p+N)}{pq-1}$ and $b_2 = \frac{(N-1)pq-(q+N)}{pq-1}$. However this system gives less information on the solutions of system (3.1): it admits at most two fixed points, namely $(0,0)$, and $P_0 = ((|b_1| |b_2|^p)^{\frac{1}{pq-1}} \text{sign}(b_1), (|b_2| |b_1|^q)^{\frac{1}{pq-1}} \text{sign}(b_2))$ which corresponds to the particular solutions. Moreover it is singular at $(0,0)$ whenever $q < 1$. The quadratic system (3.7) gives a great amount of information, because it is obtained by differentiation of the equations of (3.1). It has four fixed points, and each of them corresponds to a type of behaviour near 0 or ∞ .

4 Solutions of the Hardy-Hénon equations

Here we consider the **positive** solutions of the radial equation

$$-\Delta_{\mathbf{P}}^{\mathbf{N}} w = -\frac{d}{dr} \left(\left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} \right) - \frac{\mathbf{N}-1}{r} \left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = \pm 1, \quad (4.1)$$

in dimension \mathbf{N} , where $\mathbf{q} > \mathbf{p} - 1 > 0$. In the sequel, \mathbf{N} is not necessarily an integer.

4.1 General formulation by a quadratic system

On any interval where $w' \neq 0$, we define

$$s(t) = -r \frac{w'}{w}, \quad z(t) = -\varepsilon r^{1+\sigma} w^{\mathbf{q}} |w'|^{-\mathbf{p}} w', \quad t = \ln r, \quad (4.2)$$

and obtain the system, valid for the two equations,

$$\begin{cases} s_t = s \left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1} \right), \\ z_t = z (\mathbf{N} + \sigma - \mathbf{q}s - z). \end{cases} \quad (4.3)$$

We recover w by the formula

$$w = r^{-\gamma} (|s|^{\mathbf{p}-1} |z|)^{\frac{1}{\mathbf{q}+1-\mathbf{p}}}, \quad w' = r^{-(\gamma+1)} (|z| |s|^{\mathbf{q}})^{\frac{1}{\mathbf{q}+1-\mathbf{p}}} \text{sign}(-\varepsilon z), \quad \gamma = \frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}. \quad (4.4)$$

In the plane (s, z) we define the quadrants

$$Q_1 = \{(s, z) \in \mathbb{R}^2 : s > 0, z > 0\}, \quad Q_2 = \{(s, z) \in \mathbb{R}^2 : s < 0, z > 0\}, \quad Q_3 = -Q_1, \quad Q_4 = -Q_2.$$

Remark 4.1 Observe that sz has the sign of ε . If $\varepsilon > 0$ (equation with source), then $(s, z) \in Q_1 \cup Q_3$. If $\varepsilon < 0$ (equation with absorption) then $(s, z) \in Q_2 \cup Q_4$. Moreover, if $\mathbf{p} < \mathbf{N}$, and if w is defined on an interval $(0, \rho)$, then it is always decreasing, hence $(s, z) \in Q_1$. Indeed $(r^{\mathbf{N}-1} |w'|^{\mathbf{p}-2} w')$ is decreasing. If there exists r_0 such that $w'(r_0) > 0$, then for $r < r_0, w'^{\mathbf{p}-1} \geq C_0 r^{1-\mathbf{N}}$, thus w is bounded, which by integration contradicts the assumption $\mathbf{p} < \mathbf{N}$. In case $\mathbf{p} > \mathbf{N}$, it can happen, as we see in the sequel, that $w' > 0$ near 0.

In this paragraph we exclude the limit cases $\mathbf{p} = \mathbf{N}, \sigma = -\mathbf{p}, \sigma = -\mathbf{N}$, which will be studied at paragraph 4.6.

We define two possible critical values

$$\mathbf{q}_c = \frac{(\mathbf{N} + \sigma)(\mathbf{p} - 1)}{\mathbf{N} - \mathbf{p}}, \quad \mathbf{q}_S = \frac{\mathbf{N}(\mathbf{p} - 1) + \mathbf{p} + \mathbf{p}\sigma}{\mathbf{N} - \mathbf{p}}, \quad (4.5)$$

which are the well known Serrin's exponent and Sobolev exponent respectively, in case $\mathbf{N} > \mathbf{p} > -\sigma$. Note that $\mathbf{q}_c < 0$ when $(\mathbf{N} + \sigma)(\mathbf{N} - \mathbf{p}) < 0$.

We first observe that the equation admits particular solutions

$$w^* = a^* r^{-\gamma}, \quad \gamma = \frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}, \quad a^{*\mathbf{q} - \mathbf{p} + 1} = \varepsilon |\gamma|^{\mathbf{p} - 2} \gamma (\mathbf{N} - \mathbf{p} - (\mathbf{p} - 1)\gamma), \quad (4.6)$$

well defined for $\varepsilon \gamma (\mathbf{N} - \mathbf{p} - (\mathbf{p} - 1)\gamma) > 0$ that is $\mathbf{q} - \mathbf{q}_c$ has the sign of $\varepsilon(\mathbf{p} + \sigma)(\mathbf{N} - \mathbf{p})$. Then w^* is ∞ -singular at 0 if $\mathbf{p} + \sigma > 0$, and C^0 -regular if $\mathbf{p} + \sigma < 0$.

Definition 4.2 *In the following we say that a positive solution w of equation (4.1) on an interval $(0, \rho)$ is C^0 -regular if $w \in C^2(0, \rho) \cap C^0[0, \rho)$, that means $\lim_{r \rightarrow 0} w = c \geq 0$. If it exists, such a solution satisfies $(r^{\mathbf{N} - 1} |w'|^{\mathbf{p} - 1})' \sim_{r \rightarrow 0} r^{\mathbf{N} - 1 + \sigma} c^{\mathbf{p}}$, then $r^{\mathbf{N} - 1} |w'|^{\mathbf{p} - 1} \sim_{r \rightarrow 0} Cr^{\mathbf{N} + \sigma} c^{\mathbf{p}}$, $|w'| \sim_{r \rightarrow 0} Cr^{\frac{\sigma + 1}{\mathbf{p} - 1}}$, $|w''| \sim_{r \rightarrow 0} Cr^{\frac{\sigma + 2 - \mathbf{p}}{\mathbf{p} - 1}}$. Then $w \in C^2[0, \rho)$ if $\sigma \geq \mathbf{p} - 2$, $w \in C^1[0, \rho)$ if $\sigma \geq -1$, and w presents a cusp at 0 if $\sigma < -1$. We say that w is a C^0 -ground state if moreover $\rho = \infty$.*

Next we divide the study in 6 regions relative to $(\mathbf{N}, \sigma) \in \mathbb{R}^2$, and we show that we can reduce the study to only 3 regions, for which one of them is well known. This is due to a transformation proved in [12, Remark 2.4]:

Lemma 4.3 *For fixed $\mathbf{p} > 0$, consider the sets defined by*

$$\begin{aligned} \mathbf{A} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{N} > \mathbf{p} > -\sigma\}, & \mathbf{B} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{p} > \mathbf{N} > -\sigma\}, \\ \mathbf{C} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{N} > -\sigma > \mathbf{p}\}, & \mathbf{D} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : -\sigma > \mathbf{N} > \mathbf{p}\}, \\ \mathbf{E} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{p} > -\sigma > \mathbf{N}\}, & \mathbf{F} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : -\sigma > \mathbf{p} > \mathbf{N}\}. \end{aligned} \quad (4.7)$$

Let $(\mathbf{N}, \sigma) \in \mathbb{R}^2$ and $(\widehat{\mathbf{N}}, \widehat{\sigma}) \in \mathbb{R}^2$ such that $(\widehat{\mathbf{N}} - \mathbf{p})(\mathbf{p} + \sigma) = (\mathbf{N} - \mathbf{p})(\mathbf{p} + \widehat{\sigma})$. Set

$$\lambda = \frac{\widehat{\mathbf{N}} - \mathbf{p}}{\mathbf{N} - \mathbf{p}} = \frac{\mathbf{p} + \widehat{\sigma}}{\mathbf{p} + \sigma} = \frac{\widehat{\mathbf{N}} + \widehat{\sigma}}{\mathbf{N} + \sigma}, \quad (4.8)$$

and consider the change of unknown

$$\widehat{w}(\widehat{r}) = Cw(r), \quad r = \widehat{r}^\lambda, \quad C = \lambda^{\frac{\mathbf{p}}{\mathbf{q} + 1 - \mathbf{p}}}. \quad (4.9)$$

Then w satisfies the equation (4.1) if and only if \widehat{w} satisfies the analogous equation with r, \mathbf{N}, σ replaced by $\widehat{r}, \widehat{\mathbf{N}}, \widehat{\sigma}$, and

$$\widehat{r}^{\widehat{\sigma}} \widehat{w}(\widehat{r}) = Cr^\gamma w(r), \quad \widehat{r}^{\frac{\widehat{\mathbf{N}} - \mathbf{p}}{\mathbf{p} - 1}} \widehat{w}(\widehat{r}) = Cr^{\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}} w(r), \quad \widehat{r}^{-\frac{\widehat{\sigma} + 1}{\mathbf{p} - 1}} \widehat{w}'(\widehat{r}) = \lambda Cr^{-\frac{\sigma + 1}{\mathbf{p} - 1}} w'(r). \quad (4.10)$$

Moreover for any $(\mathbf{N}, \sigma) \in \mathbf{F}$ (resp. \mathbf{D} , resp. \mathbf{E}) there exists $(\widehat{\mathbf{N}}, \widehat{\sigma}) \in \mathbf{A}$ (resp. \mathbf{B} , resp. \mathbf{C}), such that (4.8) holds with $\lambda = -1$. As a consequence all the results valid for regions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ apply respectively to $\mathbf{F}, \mathbf{D}, \mathbf{E}$ by changing r into $\frac{1}{r}$.

Proof. Setting $t = \widehat{\lambda}t$ and $(\widehat{s}, \widehat{z}) = \lambda(s, z)$, we obtain a system analogous to (4.3) where \mathbf{N}, σ are replaced by $\widehat{\mathbf{N}}, \widehat{\sigma}$. This corresponds to the change of unknown (4.9). And the regions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are respectively exchanged to $\mathbf{F}, \mathbf{E}, \mathbf{D}$, after choosing suitable reals $\lambda < 0$ such that $\mathbf{N} > \mathbf{0}$ corresponds to $\widehat{\mathbf{N}} > \mathbf{0}$. The relations (4.10) are straightforward, implying the last conclusions. ■

As a consequence we prove that some upperestimates, which are classical when $\mathbf{N} > \mathbf{p} > -\sigma$, and still valid for other ranges of the parameters:

Lemma 4.4 *Let $\mathbf{q} > \mathbf{p} - 1 > 0$, and $\mathbf{N}, \sigma \in \mathbb{R}$. There is a constant $C_{\mathbf{N}, \mathbf{p}, \mathbf{q}} > 0$ such that any positive solution of (4.1) solution with in $(0, r_0)$ (resp in (r_0, ∞)) satisfies*

$$w(r) \leq C_{\mathbf{N}, \mathbf{p}, \mathbf{q}} r^{-\gamma} \quad \text{in } (0, \frac{r_0}{2}) \quad (\text{resp. in } (2r_0, \infty)).$$

Proof. In case $\varepsilon = -1$, this comes the Osserman's property, which is valid in the nonradial case when \mathbf{N} is an integer, see a proof in [39, Proposition 5.2], and for any subsolution. In the case $\varepsilon = 1$, it also extends to integral estimates in the nonradial case, see [14, Theorem 3.1], valid for any $\sigma \in \mathbb{R}$. Both results suppose $\mathbf{N} > \mathbf{1}$, and are given and $\mathbf{N} > \mathbf{p}$, even if this condition does not seem necessary. Next suppose $\mathbf{N} < \mathbf{p}$. We use the transformation (4.8) (4.9) with $\lambda = -1$: it defines $\widehat{\mathbf{N}} = 2\mathbf{p} - \mathbf{N}$ and $\widehat{\sigma} = -2\mathbf{p} - \sigma$, and $w(r) = \widehat{w}(\widehat{r})$, $\widehat{r} = \frac{1}{r}$. From (4.10), $\widehat{r}^{\widehat{\gamma}} \widehat{w}(\widehat{r}) = Cr^{\gamma} w(r)$, and the behaviours near 0 and ∞ are exchanged, hence the conclusion is still valid. ■

From now on in this section, the proofs of the lemmas and theorems are given in the Appendix. We first analyze the nature of the fixed points of the system:

Lemma 4.5 *The fixed points of system (4.3) are*

$$\begin{aligned} M_0 = (s_0, z_0) &= (\gamma, \mathbf{N} - \mathbf{p} - (\mathbf{p} - 1)\gamma) = (\frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}, \frac{(\mathbf{N} - \mathbf{p})(\mathbf{q} - \mathbf{q}_c)}{\mathbf{q} + 1 - \mathbf{p}}), \\ N_0 &= (0, \mathbf{N} + \sigma), \quad A_0 = (\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}, 0), \quad \text{and} \quad (0, 0). \end{aligned} \quad (4.11)$$

(i) *The point M_0 corresponds to the particular solutions defined at (4.6). The eigenvalues λ_1, λ_2 of the linearized system associated to system (4.3) at M_0 are the roots of the equation*

$$T(\lambda) = \lambda^2 - (s_0 - z_0)\lambda + \frac{\mathbf{q} - \mathbf{p} + 1}{\mathbf{p} - 1} s_0 z_0 = \lambda^2 - (\mathbf{p}\gamma + \mathbf{p} - \mathbf{N})\lambda + \frac{(\mathbf{p} + \sigma)(\mathbf{N} - \mathbf{p})(\mathbf{q} - \mathbf{q}_c)}{(\mathbf{p} - 1)(\mathbf{q} - \mathbf{p} + 1)} = 0. \quad (4.12)$$

M_0 is a saddle point when $s_0 z_0 < 0$.

(ii) *The eigenvalues at N_0 are $l_1 = \frac{\mathbf{p} + \sigma}{\mathbf{p} - 1}$ and $l_2 = -(\mathbf{N} + \sigma)$; and $\{s = 0\}$ contains nonadmissible trajectories linked to l_2 . N_0 is a saddle point when $(\mathbf{p} + \sigma)(\mathbf{N} + \sigma) > 0$.*

(iii) *The eigenvalues at A_0 are $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}$ and $\mu_2 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}(\mathbf{q}_c - \mathbf{q}) = \mathbf{N} + \sigma + \mathbf{q} \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}$; and $\{z = 0\}$ contain nonadmissible trajectories linked to μ_1 . A_0 is a saddle point when $\mathbf{q} > \mathbf{q}_c$.*

(iv) *The eigenvalues at $(0, 0)$ are $\rho_1 = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}$ and $\rho_2 = \mathbf{N} + \sigma$, and $\{s = 0\}$ and $\{z = 0\}$ contain nonadmissible trajectories, linked to ρ_1 and ρ_2 . The point $(0, 0)$ is a saddle point when $(\mathbf{N} - \mathbf{p})(\mathbf{N} + \sigma) > 0$.*

Remark 4.6 *Recall that any trajectory in the phase plane distinct from the two axis corresponds to an infinity with one parameter of solutions of equation (4.1). So any saddle-point leads at most to two families of such solutions, and any source or sink leads to an infinity of solutions with two parameters.*

Next we make the link between the convergence to the fixed points and the behaviour of the solutions w near 0 or ∞ .

Lemma 4.7 *Let w be any positive solution of (4.1) defined near 0 (resp near ∞). Then as $r \rightarrow 0, t \rightarrow -\infty$ (resp $r \rightarrow \infty, t \rightarrow \infty$)*

$$\begin{aligned}
(i) \quad \lim(s, z) = M_0 &\implies w \sim w^*, \quad w' \sim w'^*, \\
(ii) \quad \lim(s, z) = N_0 &\implies \lim w = c > 0, \quad \lim r^{-\frac{\sigma+1}{p-1}} w' = \left| \frac{c}{N+\sigma} \right|^{\frac{q}{p-1}} \text{sign}(-\varepsilon(N+\sigma)), \\
(iii) \quad \lim(s, z) = (0, 0) &\implies \lim w = c > 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-1}{p-1}} w' = d \neq 0, \\
(iv) \quad \lim(s, z) = A_0 &\implies \lim r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim r^{\frac{N-1}{p-1}} w' = k^{\frac{p-N}{p-1}} \text{sign}(-\varepsilon z).
\end{aligned}$$

Lemma 4.8 *Let w be any local solution in $(0, r_0)$ (resp. in (r_0, ∞)). Then the associated trajectory in the phase plane is bounded as $t \rightarrow -\infty$ (resp. $t \rightarrow \infty$). It converges to one of the fixed points of the system, or has a limit cycle around M_0 (when $\mathbf{q} = \mathbf{q}_S$).*

4.2 Study of regions A and F

We first consider region **A**, where the behaviour of the system (4.3) has been described in [12], and we deduce the following:

Theorem 4.9 *Let $N > p > -\sigma$ (region **A**).*

There exist local (C^0 -regular) solutions: for $\varepsilon = \pm 1$ and any $w_0 > 0$ there exists a unique solution such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{p-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0.$$

(1) Let $p-1 < \mathbf{q} < \mathbf{q}_c$.

- For $\varepsilon = \pm 1$ and any $k > 0$, there exists an infinity of local solutions near 0 such that

$$\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-1}{p-1}} w' = -c(k) < 0.$$

- For $\varepsilon = 1$, there is no positive solution in (r_0, ∞) , $r_0 > 0$.
- For $\varepsilon = -1$, there exists a global particular solution $w^* = a^* r^{-\frac{p+\sigma}{q+1-p}}$. Moreover there exist solutions such that

$$\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^\gamma w = a^*.$$

(2) Let $\mathbf{q} > \mathbf{q}_c$.

- For $\varepsilon = \pm 1$ and $k > 0$, there exists a local solution near ∞ , unique up to a scaling, such that

$$\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = -c(k) < 0.$$

• For $\varepsilon = -1$, all the local solutions near 0 are C^0 -regular (the singularity is called removable), and not global.

- For $\varepsilon = 1$, there is a particular solution $w^* = a^* r^{-\frac{p+\sigma}{q+1-p}}$. Moreover

(i) either $\mathbf{q} < \mathbf{q}_S$, there exists no C^0 -ground state and there exist solutions such that

$$\lim_{r \rightarrow 0} r^\gamma w = a^*, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}} w = k > 0.$$

(ii) or $\mathbf{q} > \mathbf{q}_S$ and there exist C^0 -ground states, such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow \infty} r^\gamma w = a^*.$$

(iii) or $\mathbf{q} = \mathbf{q}_S$ and there is a family of explicit (well known) C^0 -ground states:

$$w = c(d + r^{\frac{\mathbf{p}+\sigma}{\mathbf{p}-1}})^{-\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}+\sigma}}, \quad d = c^{\mathbf{q}-\mathbf{p}+1}(\mathbf{N} + \sigma)^{-1} \left(\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}\right)^{1-\mathbf{p}}. \quad (4.13)$$

In the case $\mathbf{N} > \mathbf{p} > -\sigma$, $\mathbf{q} = \mathbf{q}_c$, a precise local behaviour of logarithmic type was in great part studied for $\sigma = 0, \varepsilon = 1$ in [5], [28, Theorem 4.1], and for $\varepsilon = 1$ in [38]; the existence of such solutions was not clear. We give below a complete description:

Theorem 4.10 (1) Let $\mathbf{N} > \mathbf{p} > -\sigma$ and $\mathbf{q} = \mathbf{q}_c$. Then

- For $\varepsilon = \pm 1$ and any $w_0 > 0$ there exist a unique solution such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0.$$

- For $\varepsilon = \pm 1$ there exist an infinity of local solutions near 0 such that

$$\lim_{r \rightarrow 0} (r |\ln r|^{\frac{1}{\mathbf{p}+\sigma}})^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = c(\mathbf{N}, \mathbf{p}) > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} |\ln r|^{\frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}+\sigma)(\mathbf{p}-1)}} w' = -\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} c(\mathbf{N}, \mathbf{p}),$$

where $c(\mathbf{N}, \mathbf{p}) = \left(\left(\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}\right)^{\mathbf{p}} \frac{\mathbf{p}-1}{\mathbf{p}+\sigma}\right)^{\frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}-1)(\mathbf{p}+\sigma)}}$.

- For $\varepsilon = 1$, there is no positive solution in (r_0, ∞) , $r_0 > 0$.
- For $\varepsilon = -1$, there exist an infinity of solutions such that

$$\lim_{r \rightarrow \infty} (r |\ln r|^{\frac{1}{\mathbf{p}+\sigma}})^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = c(\mathbf{N}, \mathbf{p}), \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} |\ln r|^{\frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}+\sigma)(\mathbf{p}-1)}} |w'| = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} c(\mathbf{N}, \mathbf{p}).$$

Remark 4.11 As mentioned in [12, Remark 3.2], some energy functions of Pohozaev-type linked to the equation are well known. In fact they are valid for any values of the parameters $\mathbf{p}, \mathbf{q}, \mathbf{N}, \sigma$, and for the two equations ($\varepsilon = \pm 1$):

$$F_\theta(r) = r^{\mathbf{N}} \left(\frac{(\mathbf{p}-1) |w'|^{\mathbf{p}}}{\mathbf{p}} + \varepsilon r^\sigma \frac{w^{\mathbf{q}+1}}{\mathbf{q}+1} + \theta \frac{w |w'|^{\mathbf{p}-2} w'}{r} \right) = r^{\mathbf{N}-\mathbf{p}} w^{\mathbf{p}} |s|^{\mathbf{p}-2} s \left(\frac{(\mathbf{p}-1)s}{\mathbf{p}} + \frac{z}{\mathbf{q}+1} - \theta \right),$$

either with $\theta = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}}$ or $\theta = \frac{\mathbf{N}+\sigma}{\mathbf{q}+1}$, satisfying respectively

$$F'_{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}}}(r) = r^{\mathbf{N}-1+\sigma} \left(\frac{\mathbf{N}+\sigma}{\mathbf{q}+1} - \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}} \right) w^{\mathbf{q}+1}, \quad F'_{\frac{\mathbf{N}+\sigma}{\mathbf{q}+1}}(r) = r^{\mathbf{N}-1} \left(\frac{\mathbf{N}+\sigma}{\mathbf{q}+1} - \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}} \right) |w'|^{\mathbf{p}},$$

Other type of functions can be computed as the ones obtained for $\sigma = 0$ in [7, Proposition 2.2], coinciding with the functions above when $\mathbf{q} = \mathbf{q}_S$. In particular $\mathbf{q} = \mathbf{q}_S$ is the case of constant energy, leading to the existence of the ground states mentioned above at (4.13) when $\varepsilon = 1$, and to explicit local solutions on $[0, r_0)$ and on $[r_0, \infty)$ when $\varepsilon = -1$.

From Lemma 4.3, we deduce the complete behaviour in region **F**. It offers a new striking result in case $\mathbf{q} = \mathbf{q}_S$ of existence of explicit C^0 -ground states, increasing and bounded at ∞ :

Theorem 4.12 *Let $-\sigma > \mathbf{p} > \mathbf{N}$ (region **F**). Then all the conclusions of Theorems 4.9 (for $\mathbf{q} \neq \mathbf{q}_c$) and 4.10 (for $\mathbf{q} = \mathbf{q}_c$) apply after changing r into $\frac{1}{r}$. In particular for $\varepsilon = \pm 1$ and any $c > 0$ there exist a unique local solution near ∞ such that*

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = \varepsilon d(C), \quad d(C) > 0.$$

For $\varepsilon = 1, \mathbf{q} = \mathbf{q}_S$ there are explicit solutions given by

$$w = c(d + r^{\frac{\mathbf{p}+\sigma}{\mathbf{p}-1}})^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}+\sigma}}, \quad d = c^{\mathbf{q}-\mathbf{p}+1} (\mathbf{N} + \sigma)^{-1} \left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}\right)^{1-\mathbf{p}}$$

satisfying $\lim_{r \rightarrow 0} w = 0$, with $w \sim_{r \rightarrow 0} cr^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}}$ and $\lim_{r \rightarrow \infty} w = cd^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}+\sigma}}$.

4.3 Study of Regions **B** and **D**

The case of region **B** is particularly interesting; indeed we prove the following:

Theorem 4.13 *Let $\mathbf{p} > \mathbf{N} > -\sigma$ (region **B**)*

- For $\varepsilon = \pm 1$ there exists local C^0 -**regular** solutions of **three** types:

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0, \quad (4.14)$$

$$\lim_{r \rightarrow 0} w = 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = c(k) > 0 \quad (4.15)$$

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0. \quad (4.16)$$

For $\varepsilon = 1$ they are not global, and there is no solution in (r_0, ∞) . For $\varepsilon = -1$ there exists two types of global solutions in $(0, \infty)$:

- $w^*(r) = a^* r^{-\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}}$,
- solutions such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(w_0) < 0, \quad w \sim_{r \rightarrow \infty} w^*.$$

Note that in the case $\mathbf{N} = 1, \mathbf{p} = 2, \sigma = 0$, Theorem 4.13 can be checked easily, since the equation $-w'' = \varepsilon w^q$ admits a first integral: $w'^2 + \frac{2\varepsilon}{q+1} w^{q+1} = C$.

As a direct consequence we obtain the behaviour in region **D**. Here also we have an very interesting behaviour: we find infinitely many bounded solutions in an exterior domain which do not converge to 0 at ∞ , and global ones when $\varepsilon = -1$:

Theorem 4.14 (region **D**) *Let $\mathbf{p} < \mathbf{N} < -\sigma$. Then*

- For $\varepsilon = \pm 1$ there exists local solutions near ∞ of **three** types:

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = \varepsilon c(C), \quad c(C) > 0,$$

$$\begin{aligned} \lim_{r \rightarrow \infty} w = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = -c(k) < 0, \\ \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = D \neq 0. \end{aligned}$$

For $\varepsilon = 1$ they are not global, and there is no solution in $(0, r_0)$. For $\varepsilon = -1$ there exists two types of global solutions in $(0, \infty)$:

- $w^*(r) = a^* r^{-\frac{p+\sigma}{q+1-p}}$, which is C^0 -regular,
- solutions such that

$$w \sim_{r \rightarrow 0} w^*, \quad \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = c(C) > 0.$$

4.4 Study of regions C and E

In regions C and then in region E from Lemma 4.3, we obtain the following results:

Theorem 4.15 *Let $p < -\sigma < N$ (region C) Then there exists no C^0 -regular solution. For $\varepsilon = \pm 1$ there exist local (nonglobal) solutions near ∞ of two types:*

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{p-1}} w' = -\varepsilon D, \quad D > 0, \quad (4.17)$$

$$\lim_{r \rightarrow \infty} w = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = -c(k) < 0. \quad (4.18)$$

For $\varepsilon = -1$, there exist two types of global solutions in $(0, \infty)$:

- $w^*(r) = a^* r^{-\frac{p+\sigma}{q+1-p}}$, which is a cusp-solution,
- solutions such that

$$w \sim_{r \rightarrow 0} w^*, \quad \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{p-1}} w' = D > 0,$$

and there exist also solutions on (r_0, ∞) such that $u(r_0) = 0$ and others such that $\lim_{r \rightarrow r_0} w = \infty$, and $w \sim_{r \rightarrow \infty} w^*$, and nonglobal solutions on $(0, r_0)$ such that $w \sim_{r \rightarrow 0} w^*$ and $\lim_{r \rightarrow r_0} w = \infty$.

For $\varepsilon = 1$ there is no local solution in $(0, r_0)$.

Theorem 4.16 (region E) *Let $p > -\sigma > N$. For $\varepsilon = \pm 1$ there exist local (nonglobal) solutions near 0 of two types:*

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{p-1}} w' = \varepsilon D, \quad D > 0,$$

$$\lim_{r \rightarrow 0} w = 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-1}{p-1}} w' = d(k) > 0.$$

For $\varepsilon = -1$, there exist two types of global solutions in $(0, \infty)$:

- $w^*(r) = a^* r^{-\frac{p+\sigma}{q+1-p}}$,

- solutions such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = d < 0, \quad w \sim_{r \rightarrow \infty} w^*,$$

and there exist also solutions on $(0, r_0)$ such that $w(r_0) = 0$, and others such that $\lim_{r \rightarrow r_0} w = \infty$, and $w \sim_{r \rightarrow 0} w^*$; and there exist nonglobal solutions on (r_0, ∞) such that $w \sim_{r \rightarrow \infty} w^*$ and $\lim_{r \rightarrow r_0} w = \infty$.

For $\varepsilon = 1$ there is no local solution in (r_0, ∞) .

4.5 Other case of explicit solutions

We have recalled at (4.13) the well-known explicit grounds states obtained for $\mathbf{q} = \mathbf{q}_S$ and $\varepsilon = 1$. Here we give another case where we find global explicit solutions in $\mathbb{R}^N \setminus \{0\}$. We remark that system (1.1) admits the solutions $(u_1, u_2) = (u, u)$ when $p = q$, where u is a solution of the scalar equation (1.7), given explicitly by (2.3) and (2.4); and the corresponding solutions of system (3.7) satisfy the relation $S \equiv -Z$, and $\mathbf{p} = \mathbf{N} = -\sigma$. This suggests that system (4.3) with general $\mathbf{p}, \mathbf{N}, \sigma$ may admit particular explicit solutions for some values of the parameters. We show below that it is true, and this result appears to be new:

Theorem 4.17 *Let $\mathbf{q} > \mathbf{p} - 1 > 0$. When $\sigma = -\mathbf{p} \frac{\mathbf{N}-1}{\mathbf{p}-1}$ there exist explicit radial solutions w of the Hardy-Hénon equation (4.1) with $\varepsilon = -1$, of the form*

$$w = (c \pm d_{\mathbf{p}, \mathbf{q}, \mathbf{N}} r^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}})^{-\frac{\mathbf{p}}{\mathbf{q}-\mathbf{p}+1}}, \quad \text{if } \mathbf{p} \neq \mathbf{N}, \quad (4.19)$$

$$w = (c \pm d_{\mathbf{N}} \ln r)^{-\frac{\mathbf{N}}{\mathbf{q}-\mathbf{N}+1}}, \quad \text{if } \mathbf{p} = \mathbf{N}, \quad (4.20)$$

where $c > 0$ and $d_{\mathbf{p}, \mathbf{q}, \mathbf{N}} = \frac{(\mathbf{p}-1)(\mathbf{q}-\mathbf{p}+1)}{\mathbf{p}(\mathbf{p}-\mathbf{N})} \left(\frac{\mathbf{p}}{(\mathbf{p}-1)(\mathbf{q}+1)} \right)^{\frac{1}{\mathbf{p}}}$, $d_{\mathbf{N}} = \frac{\mathbf{q}-\mathbf{N}+1}{\mathbf{N}} \left(\frac{\mathbf{N}}{(\mathbf{N}-1)(\mathbf{q}+1)} \right)^{\frac{1}{\mathbf{N}}}$.

4.6 Limit cases

Here we give a complete study of all the critical cases $\sigma = -\mathbf{p} \neq -\mathbf{N}$, $\sigma = -\mathbf{N} \neq -\mathbf{p}$, $\mathbf{p} = \mathbf{N} \neq -\sigma$, $\mathbf{p} = \mathbf{N} = -\sigma$.

4.6.1 Case $\sigma = -\mathbf{p} \neq -\mathbf{N}$

Theorem 4.18 (1) *Assume $\mathbf{N} > \mathbf{p} = -\sigma$*

- For $\varepsilon = \pm 1$, there exist local solutions near ∞ such that

$$\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(k) < 0.$$

- For $\varepsilon = 1$, there exists an infinity with 2 parameters of local solutions near ∞ such that

$$\lim_{r \rightarrow \infty} (\ln r)^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}} w(r) = \left(\frac{\mathbf{p}-1}{\mathbf{q}} \right)^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}}. \quad (4.21)$$

- For $\varepsilon = -1$, there exists at least a local solution near 0 such that

$$\lim_{r \rightarrow 0} |\ln r|^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}} w(r) = \left(\frac{\mathbf{p}-1}{\mathbf{q}} \right)^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}}. \quad (4.22)$$

(2) *Assume $\mathbf{p} = -\sigma > \mathbf{N}$. Then the behaviour is deduced from (1) by changing r into $\frac{1}{r}$.*

4.6.2 Case $\sigma = -\mathbf{N} \neq -\mathbf{p}$

Theorem 4.19 (1) Assume $-\sigma = \mathbf{N} > \mathbf{p}$.

- For $\varepsilon = \pm 1$, there exist local solutions near ∞ such that

$$\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(k) < 0. \quad (4.23)$$

- For $\varepsilon = \pm 1$, there exists an infinity with 2 parameters of local solutions near ∞ such that

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} (\ln r)^{-\frac{1}{\mathbf{p}-1}} w' = -\varepsilon c(C). \quad (4.24)$$

- For $\varepsilon = -1$, there exists C^0 -regular solutions $w^* = a^* r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}$. There exist local solutions near 0 such that $w \sim_{r \rightarrow 0} w^*$, and local ones near ∞ such that $w \sim_{r \rightarrow \infty} w^*$. There exists an infinity of solutions such that

$$w \sim_{r \rightarrow 0} w^*, \quad \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} (\ln r)^{-\frac{1}{\mathbf{p}-1}} w' = \left(\frac{\mathbf{q}+1-\mathbf{p}}{\mathbf{N}-\mathbf{p}} \right)^{\frac{\mathbf{q}}{\mathbf{q}+1-\mathbf{p}}}. \quad (4.25)$$

- (2) Assume $\mathbf{p} > \mathbf{N} = -\sigma$. Then the behaviour is deduced from (1) by changing r into $\frac{1}{r}$.

4.6.3 Case $\mathbf{p} = \mathbf{N} \neq -\sigma$

Theorem 4.20 (1) Assume $\mathbf{p} = \mathbf{N} > -\sigma$

- For $\varepsilon = \pm 1$ there exists local (C^0 -regular) solutions: for any $w_0 > 0$ there exists a unique solution such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0.$$

- For $\varepsilon = 1$, there exist an infinity of local solutions near 0 such that

$$\lim_{r \rightarrow 0} |\ln r|^{-1} w = C > 0, \quad \lim_{r \rightarrow 0} r w' = -C.$$

- For $\varepsilon = -1$, there exists a particular solution $w^* = a^* r^{-\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}}$. There exist local solutions near 0 such that $w \sim_{r \rightarrow 0} w^*$, and local ones near ∞ such that $w \sim_{r \rightarrow \infty} w^*$. There exists an infinity of solutions such that

$$\lim_{r \rightarrow 0} |\ln r|^{-1} w = C, \quad \lim_{r \rightarrow 0} r w' = -C, \quad w \sim_{r \rightarrow \infty} w^*.$$

- (2) Assume $\mathbf{p} = \mathbf{N} < -\sigma$. Then the behaviour is deduced from (1) by changing r into $\frac{1}{r}$.

4.6.4 Case $\mathbf{p} = \mathbf{N} = -\sigma$

Theorem 4.21 Assume $\mathbf{p} = \mathbf{N} = -\sigma$. For $\varepsilon = -1$, there exist local explicit solutions near 0 or near ∞ of the form (4.20):

$$w = (C \pm d_{\mathbf{N}} \ln r)^{-\frac{\mathbf{N}}{\mathbf{q}-\mathbf{N}+1}}, \quad C \in \mathbb{R}. \quad (4.26)$$

For $\varepsilon = 1$ there is no local solution near 0 nor ∞ .

5 Description of the radial solutions of system (3.1)

5.1 The case $p = q$

In the case $p = q$, the system (1.1) admits solutions of the form $u_1 \equiv u_2 \equiv u$, where u is any solution of the scalar equation (1.7). However at Proposition 3.1 we have constructed local solutions such that $u_1 \neq u_2$, for example such that $u'_1(r_0) = 0$, $u'_2(r_0) \neq 0$ at some point $r_0 > 0$. A natural question is the existence global solutions in $\mathbb{R}^N \setminus \{0\}$. Here we answer this question:

Proposition 5.1 *Assume that $p = q > 1$. Then all the radial solutions of system (1.1) satisfy the first integral in any interval of definition*

$$|u'_1|^q u'_1 - |u'_2|^q u'_2 \equiv Cr^{(1-N)(q+1)}, \quad C \in \mathbb{R}, \quad (5.1)$$

and can be computed by quadratures. All the radial solutions (u_1, u_2) in $\mathbb{R}^N \setminus \{0\}$ satisfy $u_1 \equiv u_2 \equiv u$, where u is any solution of the scalar Hamilton-Jacobi equation.

Proof. Here system (3.3) reduces to

$$\begin{cases} w'_1 = r^{(N-1)(1-q)} |w_2|^q, \\ w'_2 = r^{(N-1)(1-q)} |w_1|^q. \end{cases}$$

As a consequence,

$$|w_1|^q w'_1 - |w_2|^q w'_2 = r^{(N-1)(1-q)} |w_1|^q |w_2|^q - r^{(N-1)(1-q)} |w_1|^q |w_2|^q = 0.$$

So we get the relation

$$|w_1|^q w_1 - |w_2|^q w_2 \equiv C,$$

equivalent to (5.1). Suppose that $C \neq 0$. By symmetry we can suppose that $C = c^{q+1} > 0$. Then we obtain

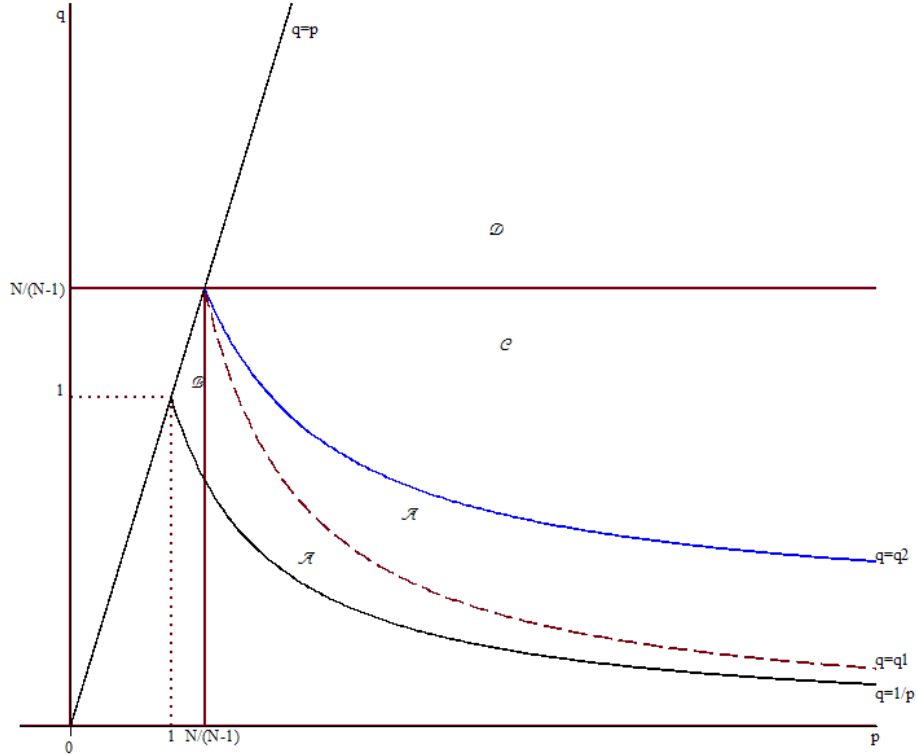
$$\frac{w'_2}{|C + |w_2|^q w_2|^{\frac{q}{q+1}}} = r^{(N-1)(1-q)}.$$

We claim that the solution cannot be defined on $\mathbb{R}^N \setminus \{0\}$. Indeed let

$$F(\theta) = \int_0^\theta \frac{d\theta}{|c^{q+1} + |\theta|^q \theta|^{\frac{q}{q+1}}} = \int_0^{-c} \frac{d\theta}{|c^{q+1} + |\theta|^q \theta|^{\frac{q}{q+1}}} + \int_{-c}^\theta \frac{d\theta}{|c^{q+1} + |\theta|^q \theta|^{\frac{q}{q+1}}}.$$

This function is well defined, since the integrals are convergent at the bound $-c$ since $\frac{q}{q+1} < 1$. And the integrals are convergent at the bounds $\pm\infty$, since $q > 1$, thus F is bounded. If the solution is global, that means r describes $(0, \infty)$, then $F(w_2) = \frac{r^{N-(N-1)q}}{N-(N-1)q} + D$, $D \in \mathbb{R}$, for $q \neq \frac{N}{N-1}$, $F(w_2) = \ln r + D$ if $q = \frac{N}{N-1}$, which is impossible in any case. So the solutions are not global. Hence all the global solutions on $(0, \infty)$ satisfy $w_1 \equiv w_2$, $u_1 = u_2 + c$, $c \in \mathbb{R}$, where u_2 is solution of the scalar equation. The nonglobal solutions can be computed on any interval where w_1, w_2 have a constant sign, by the formulas $F(w_2) = \frac{r^{N-(N-1)q}}{N-(N-1)q} + D$, for $q \neq \frac{N}{N-1}$, $F(w_2) = \ln r + D$ if $q = \frac{N}{N-1}$. ■

Remark 5.2 *When $p = q \neq \frac{N}{N-1}$, the existence of local solutions near 0 or ∞ will be described as a particular case of Theorem 5.7. When $q = \frac{N}{N-1}$, such solutions do not exist, because F is bounded.*



5.2 Constant sign solutions (u'_1, u'_2) of system (3.1)

In this paragraph, we study the existence of radial solutions (u_1, u_2) of system (1.1) in terms of the derivatives u'_1, u'_2 , by applying all the results of Section 4 to system (3.1). It appears that the situation is extremely rich when $p \neq q$. Here we focus our study on solutions defined in $B_{r_0} \setminus \{0\}$, or $\mathbb{R}^N \setminus \overline{B_{r_0}}$, and above all on global solutions in $\mathbb{R}^N \setminus \{0\}$.

We distinguish four regions of study, corresponding respectively to the regions **A**, **B**, **C**, **D** defined at (4.7) for system (4.3), where $\mathbf{p}, \mathbf{N}, \sigma$ are defined at (3.15), (3.17).

Definition 5.3 Let $\mathcal{S} = \left\{ (p, q) \in \mathbb{R}^2 : pq > 1, q \leq p, q \neq q_1, q_2, p, q \neq \frac{N}{N-1} \right\}$ we consider the subsets of \mathcal{S} defined by

$$\begin{aligned} \mathbf{A} &= \left\{ p > \frac{N}{N-1}, q < q_2 \right\}, & \mathbf{B} &= \left\{ p < \frac{N}{N-1} \right\}, \\ \mathbf{C} &= \left\{ q_2 < q < \frac{N}{N-1} \right\}, & \mathbf{D} &= \left\{ q > \frac{N}{N-1} \right\}. \end{aligned}$$

Lemma 5.4 Let (u_1, u_2) be any radial solution of system (1.1) defined near 0 (resp near ∞); let w_1, w_2 be associated by (3.2) and $w = |w_1|$, solution of (3.14) with (3.15). Then

$$u'_1 \text{ has the sign of } -w', \quad \text{and} \quad u'_2 \text{ has the sign of } \varepsilon w'. \quad (5.2)$$

And as $r \rightarrow 0, t \rightarrow -\infty$ (resp $r \rightarrow \infty, t \rightarrow \infty$),

$$\begin{aligned}
(i) \quad & w \sim w^* \quad \Rightarrow \quad (u'_1, u'_2) \sim (u_1^*, u_2^*), \\
(ii) \quad & \begin{cases} \lim w = c > 0, \\ \lim r^{-\frac{\sigma+1}{p-1}} w' = d \neq 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \lim r^{N-1} |u'_1| = c_1 > 0, \\ \lim r^{(N-1)q-1} |u'_2| = c_2 > 0, \end{cases} \\
(iii) \quad & \begin{cases} \lim w = c > 0, \\ \lim r^{\frac{N-1}{p-1}} w' = d \neq 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \lim r^{N-1} |u'_1| = c_1 > 0, \\ \lim r^{N-1} |u'_2| = c_2 > 0, \end{cases} \\
(iv) \quad & \begin{cases} \lim r^{\frac{N-p}{p-1}} w = k > 0, \\ \lim r^{\frac{N-1}{p-1}} w' = d(k) \neq 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \lim r^{(N-1)p-1} |u'_1| = k > 0, \\ \lim r^{N-1} |u'_2| = |d(k)| > 0. \end{cases}
\end{aligned}$$

Proof. From Proposition 3.8, the function $w = |w_1| = r^{N-1} |u'_1|$ satisfies the Hardy-Hénon equation with $\mathbf{p}, \mathbf{q}, \mathbf{N}, \sigma$ given by (3.15),(3.17), and we deduce $|w_2|$ from any of the formulas

$$|w_2| = (r^{(N-1)(p-1)} w'_1)^{\frac{1}{p}}, \quad \text{or } w'_2 = r^{(N-1)(1-q)} w^q, \quad (5.3)$$

and then we obtain the conclusions by computation, from the relations $u'_i = -r^{1-N} w_i, i = 1, 2$. The signs of the derivatives are obtained from (3.8) and (3.11), since $s \equiv S$. \blacksquare

Remark 5.5 (i) From the phase plane analysis in Section 4, all the trajectories relative to singular **global** solutions of the Hardy-Hénon equation (3.14) are located in the quadrant containing the point $M_0 = (S_0, Z_0)$ corresponding to the particular solutions. Moreover from (3.6), u'_1 has the sign of S and u'_2 has the sign of $-Z$. So u'_1 has the sign of S_0 and u'_2 has the sign of $-Z_0$. From (3.9), for $q \neq q_1, q_2$,

$$u'_1 < 0 \iff q > q_2, \quad u'_2 < 0 \iff q > q_1. \quad (5.4)$$

Then, with (5.2) and (5.4) we have a complete knowledge of system (3.1) as soon as we know the absolute value of the derivatives.

(ii) Some results of Section 4 involve the Serrin exponent \mathbf{q}_c and the Sobolev exponent \mathbf{q}_S , defined at (4.5). We easily check that for our system,

$$\mathbf{q} \leq \mathbf{q}_c \iff q \leq q_1 = q_1(p), \quad \mathbf{q} \leq \mathbf{q}_S \iff q \leq q^* = q^*(p).$$

As a consequence of Theorems 4.9, 4.13, 4.14, 4.15, we get the following result on **global** solutions of system (1.1), where we use Remark 5.5:

Theorem 5.6 (1) Let $(p, q) \in \mathcal{A}$, with $q < q_1$. Then

- there exists a particular solution of system (3.1) such that $u_1^{*'} = a_1 r^{-\frac{p+1}{pq-1}} > 0$ and $u_2^{*'} = a_2 r^{-\frac{q+1}{pq-1}} > 0$,
- there exist solutions on $(0, \infty)$ such that $u'_1, u'_2 > 0$ and

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = c(c_1) > 0, \quad (u_1, u_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

(2) Let $(p, q) \in \mathcal{A}$, with $q > q_1$. Then

- there exists a particular solution such that $u_1^{*'} = a_1 r^{-\frac{p+1}{pq-1}} > 0$ and $u_2^{*'} = -a_2 r^{-\frac{q+1}{pq-1}} < 0$,

- if $q < q^*$, there exist solutions such that $u'_1 > 0 > u'_2$, and

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{(N-1)p-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = -c(c_1) < 0,$$

- if $q > q^*$, there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{(N-1)q-1} u'_2 = -c_3 < 0, \quad (u'_1, u'_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*),$$

- if $q = q^*$, there exist **explicit** solutions:

$$u'_1 = cr^{1-N} (d + r^{\frac{p-q^*}{2}})^{\frac{2(N-(N-1)p)}{p-q^*}}, \quad u'_2 = -br^{(1-(N-1)q^*)} (d + r^{\frac{p-q^*}{2}})^{\frac{2((N-1)q^*-N)}{p-q^*}} \quad (5.5)$$

for any $c > 0$, with $d = d(c) = (\frac{c^{pq^*-1}}{(N-1)p-N})^{\frac{1}{p}} \frac{1}{N-(N-1)q^*}$. $b = b(c) = (c((N-1)p - N))^{\frac{1}{p}}$.

(3) Let $(p, q) \in \mathcal{B}$. Then

- there exists a particular solution such that $u_1^* = a_1 r^{-\frac{p+1}{pq-1}} > 0$ and $u_2^* = a_2 r^{-\frac{q+1}{pq-1}} > 0$.
- there exists solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = c_1, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = c_2, \quad (u'_1, u'_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

(4) Let $(p, q) \in \mathcal{C} \cup \mathcal{D}$. Then there exists a particular solution such that $u_1^* = -a_1 r^{-\frac{p+1}{pq-1}} < 0$ and $u_2^* = -a_2 r^{-\frac{q+1}{pq-1}} < 0$.

- If $(p, q) \in \mathcal{C}$, there also exist solutions such that

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-1} u'_1 = -c_1 < 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} u'_2 = -c(c_1) < 0.$$

- If $(p, q) \in \mathcal{D}$, there also exist solutions such that

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-1} u'_1 = -c_1 < 0, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = -c_2 < 0.$$

And all the global solutions are described.

And we also get the existence of **local** but not global solutions near 0 or ∞ with a behaviour of linear type, by using Lemma 5.4:

Theorem 5.7 (1) *Local existence of solutions of system (3.1) near 0:*

- in regions \mathcal{A} or \mathcal{B} , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = \pm c(c_2), \quad \lim_{r \rightarrow 0} r^{(N-1)q-1} u'_2 = -c_2 < 0,$$

- in region \mathcal{A} with $q < q_1$, there exist solutions such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} u'_1 = c(c_2) > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = \pm c_2,$$

- in region \mathcal{B} , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} u'_1 = -c(c_2) < 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = \pm c_2,$$

and solutions such that for any $c_1, c_2 \in \mathbb{R} \setminus \{0\}$,

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = c_1, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = c_2,$$

and no such solution near 0 in regions \mathcal{C}, \mathcal{D} .

(2) Local existence of solutions near ∞ :

- in regions \mathcal{C} and \mathcal{D} , for any $c_1, c_2 \in \mathbb{R} \setminus \{0\}$, there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = c_1, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = c_2,$$

- in region \mathcal{A} with $q > q_1$, and in regions \mathcal{C} and \mathcal{D} , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{(N-1)p-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = \pm c(c_1),$$

- in region \mathcal{C} , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = \pm c(c_2), \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} u'_2 = -c_2 < 0,$$

- in region \mathcal{D} , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = \pm c(c_2), \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} u'_2 = c_2 > 0.$$

Finally we consider the limit cases, where for simplicity we only mention the solutions presenting a logarithmic behaviour:

Theorem 5.8 (1) For $q = q_1 < p$, there exist local solutions of system (1.1) near 0 or ∞ , with respectively

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} |\ln r|^{\frac{p}{pq-1}} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} |\ln r|^{\frac{1}{pq-1}} u'_2 = \pm C_2,$$

$$\lim_{r \rightarrow \infty} r^{(N-1)p-1} |\ln r|^{\frac{p}{pq-1}} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} |\ln r|^{\frac{1}{pq-1}} u'_2 = C_2 > 0,$$

(2) For $q = q_2 < p$, there exist local solutions near 0 or ∞ , with respectively

$$\lim_{r \rightarrow 0} r^{N-1} |\ln r|^{\frac{1}{pq-1}} u'_1(r) = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{(N-1)q-1} |\ln r|^{\frac{q}{pq-1}} u'_2 = C_2 > 0, \quad (5.6)$$

$$\lim_{r \rightarrow \infty} r^{N-1} (\ln r)^{\frac{1}{pq-1}} u'_1(r) = C_1 > 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} (\ln r)^{\frac{q}{pq-1}} u'_2 = -C_2 < 0, \quad (5.7)$$

(3) For $p = \frac{N}{N-1} > q$, there exist local solutions near 0, such that

$$\lim_{r \rightarrow 0} r^{N-1} |\ln r|^{-1} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = \pm C_2,$$

and global solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} |\ln r|^{-1} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = C_2 > 0, \quad (u'_1, u'_2) \sim_{r \rightarrow \infty} (a_1 r^{-\frac{p+1}{pq-1}}, a_2 r^{-\frac{q+1}{pq-1}}),$$

(4) For $q = \frac{N}{N-1} < p$, there exist local solutions near ∞ such that

$$\lim_{r \rightarrow \infty} r^{N-1} u_1' = \pm C_1, \quad \lim_{r \rightarrow \infty} r^{N-1} (\ln r)^{-p} u_2' = -C_2 < 0,$$

and global ones such that,

$$(u_1', u_2') \sim_{r \rightarrow 0} (-a_1 r^{-\frac{p+1}{pq-1}}, -a_2 r^{-\frac{q+1}{pq-1}}), \quad \lim_{r \rightarrow \infty} r^{N-1} u_1' = -C_1 < 0, \quad \lim_{r \rightarrow \infty} r^{N-1} (\ln r)^{-p} u_2' = -C_2 < 0.$$

(5) For $q = p = \frac{N}{N-1}$ (where $\mathbf{p} = \mathbf{N} = -\sigma$) we find again the solutions given at (2.4) and no other local solution.

Proof. We deduce (1) from Theorem 4.10, since $\mathbf{q} = \mathbf{q}_c$, then (2) from Theorem 4.18, since $\mathbf{N} > \mathbf{p} = -\sigma$, in turn (3) from Theorem 4.20 since $\mathbf{p} = \mathbf{N} > -\sigma$, and (4) from Theorem 4.19 where $-\sigma = \mathbf{N} > \mathbf{p}$, and finally (5) 4.21 where $\mathbf{p} = \mathbf{N} = -\sigma$. There exists no other solution near 0 or ∞ , see Remark 5.2. \blacksquare

5.3 Local radial existence results for system (1.1)

Next we deduce local existence results for system (1.1), obtained from Theorem 5.7 by integration. We first give a result of existence of local solutions near 0, with a behaviour of linear type. For simplicity, due to the great number of possibilities, we consider only the positive solutions of the system. One can formulate analogous results for systems (1.5) and (1.6).

Theorem 5.9 *Existence of solutions of system (1.4) in $B_{r_0} \setminus \{0\}$, $r_0 > 0$:*

- in regions \mathcal{A} or \mathcal{B} , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-2} u_2 = 0, \quad \begin{cases} \lim_{r \rightarrow 0} r^{(N-1)q-2} u_2 = c(c_1), & \text{if } q > \frac{2}{N-1}, \\ \lim_{r \rightarrow 0} u_2 = c_2, & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow 0} (|\ln r|^{-1} u_2) = c_2, & \text{if } q = \frac{2}{N-1}; \end{cases}$$

- in region \mathcal{B} , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} u_1 = 0, \quad \lim_{r \rightarrow 0} r^{(N-1)p-2} u_1 = c(c_2), \quad \lim_{r \rightarrow 0} r^{N-2} u_2 = c_2 > 0,$$

and solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-2} u_2 = c_2 > 0.$$

In any case the solutions satisfy the equations in the sense of distributions in B_{r_0} , where δ_0 is the Dirac mass at 0:

$$\begin{cases} -\Delta u_1 = |u_2'|^p + C_1 \delta_0, \\ -\Delta u_2 = |u_1'|^q + C_2 \delta_0, \end{cases} \quad (5.8)$$

with respectively $(C_1, C_2) = (c_N c_1, 0)$, $(C_1, C_2) = (0, c_N c_2)$, $(C_1, C_2) = (c_N c_1, c_N c_2)$.

- There is no radial supersolutions of system (5.8) such that $C_1 > 0$ or $C_2 > 0$ for $p \geq q \geq \frac{2}{N-1}$, and no supersolutions such that $C_2 > 0$ for $p \geq \frac{N}{N-1}$.

Proof. The existence of the solutions is a direct consequence of Theorem 5.7. and the behaviour follows from Brezis-Lions Lemma applied to u_i , $i = 1, 2$. Next consider any radial supersolution (u_1, u_2) and assume $q \geq q_2$. If $C_1 > 0$, then $u_1 \geq cr^{2-N}$ and $|u_1'| \geq cr^{1-N}$ near 0, and $|u_1'|^q \in L_{loc}^1(B_{r_0})$, thus $q < \frac{N}{N-1}$, and $-(r^{N-1}u_2')' \geq cr^{(N-1)(1-q)}$; then by integration we deduce that $\lim_{r \rightarrow 0} r^{N-1}|u_2'| = l \geq 0$, and $|u_2'| \geq \frac{r^{1-(N-1)q}}{N-(N-1)q}$ if $l = 0$. In any case, $|u_2'| \geq cr^{1-(N-1)q}$. Then $-(r^{N-1}u_1')' \geq cr^{(1-(N-1)q)p}$, thus $r^{(1-(N-1)q)p} \in L_{loc}^1(B_{r_0})$, so that $N + p - (N-1)pq > 0$, which is $q < q_2$. Similarly, if $C_2 > 0$, then $q < q_1$; moreover $|u_2'| \geq cr^{1-N}$ near 0, thus $|\overline{u_2'}|^p \geq Cr^{(1-N)p}$, and $|u_1'|^p \in L_{loc}^1(B_{r_0})$, thus $N - (N-1)p > 0$. \blacksquare

In the same way we obtain existence of solutions in an exterior domain with a linear type:

Theorem 5.10 *Existence of solutions of system (1.4) in $\mathbb{R}^N \setminus \overline{B_{r_0}}$:*

- in region \mathcal{C} , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-2}u_1 = c_1 > 0, \quad \begin{cases} \lim_{r \rightarrow \infty} r^{(N-1)q-2}u_2 = \pm c(c_1), & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow \infty} u_2 = c_2 > 0, & \text{if } q > \frac{2}{N-1}, \\ \lim_{r \rightarrow \infty} (|\ln r|^{-1}u_2) = c(c_1), & \text{if } q = \frac{2}{N-1}, \end{cases}$$

- in regions \mathcal{C} and \mathcal{D} , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-2}u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-2}u_2 = c_2 > 0,$$

- in regions \mathcal{A} with $q > q_1$, and \mathcal{D} , there exist solutions such that

$$\lim_{r \rightarrow \infty} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-2}u_2 = c_2 > 0.$$

5.4 Global radial existence and behaviour

Next we study all the *global* constant sign solutions (u_1, u_2) of system (1.1), of any sign. Since they are obtained by integration of the derivatives, we are lead to divide some of the regions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ into subregions, according to the position of q with respect to q_3, q_4 .

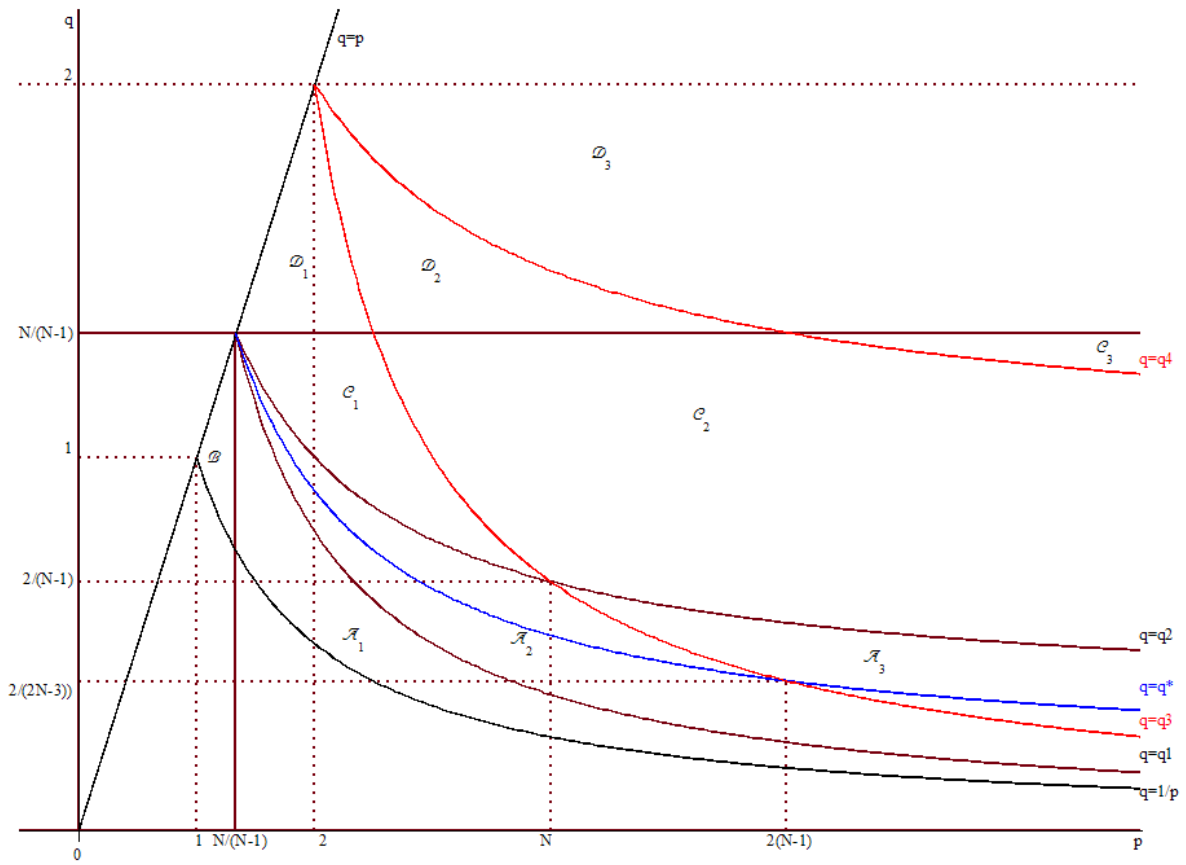
Definition 5.11 *We set $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3, \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3, \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, where*

$$\begin{aligned} \mathcal{A}_1 &= \left\{ p > \frac{N}{N-1}, q < q_1 \right\}, & \mathcal{A}_2 &= \{ q_1 < q < \min(q_2, q_3) \}, & \mathcal{A}_3 &= \{ q_3 < q < q_2 \}, \\ \mathcal{C}_1 &= \left\{ q_2 < q < \min(q_3, \frac{N}{N-1}) \right\}, & \mathcal{C}_2 &= \left\{ \max(q_2, q_3) < q < \min(q_4, \frac{N}{N-1}) \right\}, & \mathcal{C}_3 &= \left\{ q_4 < q < \frac{N}{N-1} \right\}, \\ \mathcal{D}_1 &= \left\{ \frac{N}{N-1} < q < q_3 \right\}, & \mathcal{D}_2 &= \left\{ \max(q_3, \frac{N}{N-1}) < q < q_4 \right\}, & \mathcal{D}_3 &= \left\{ \max(q_4, \frac{N}{N-1}) < q \right\}. \end{aligned}$$

We first study the problem with source terms (1.4) in case $q < q_4$:

Theorem 5.12 *For $(p, q) \in \mathcal{C}_1 \cup \mathcal{D}_1$, up to positive constants, system (1.4) admits two types of global nonconstant solutions:*

- (u_1^*, u_2^*) , with both components ∞ -singular,



- if $(p, q) \in \mathcal{C}_1$, solutions such that

$$(u_1, u_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-2} u_2 = c(c_1) > 0,$$

- if $(p, q) \in \mathcal{D}_1$, solutions such that

$$(u_1, u_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-2} u_2 = c_2 > 0.$$

There is no global nonconstant solution in the other regions such that $q < q_4$.

Proof. Region $\mathcal{C}_1 \cup \mathcal{D}_1$ is contained in $\mathcal{C} \cup \mathcal{D}$, so we can apply Theorem 5.6,(4). In any case $\lim r^{N-1} u_1' = -C_1 < 0$, so u_1' is integrable at ∞ since $N > 2$, and the same holds for u_2' for $q > \frac{N}{N-1}$. If $q < \frac{N}{N-1}$, $\lim r^{(N-1)q-1} u_2' = -C_2 < 0$ also implies that u_2' is integrable; Then the functions

$$u_1(r) = - \int_r^\infty u_1'(\tau) d\tau, \quad u_2(r) = - \int_r^\infty u_2'(\tau) d\tau,$$

satisfy the conclusions. Note that $q > \frac{2}{N-1}$ in $\mathcal{C}_1 \cup \mathcal{D}_1$.

Next we note that there is no global solution of (1.4) for $q < q_2$: indeed if such solution exists then u_1' and u_2' are negative; but all the global solutions are given at Theorems 5.6 and 5.8, and they do not fulfil these conditions. Then we are lead to consider the regions $\mathcal{C}_2, \mathcal{D}_2$. From Theorems 4.14 and 4.15, there is no global solution w on $(0, \infty)$ for $\varepsilon = 1$, and the global solutions relative to $\varepsilon = -1$ do not bring solutions of system (1.4). Indeed near 0, the function w behaves as w^* , and the corresponding solutions (u_1^*, u_2^*) are such that u_2^* is not positive. ■

Next we consider the problem with absorption, for which the situation is very rich. Using Theorem 5.6 and similar arguments of integrability, we obtain the following:

Theorem 5.13 Consider the system (1.5) For $(p, q) \in \mathcal{B} \cup \mathcal{A}_1 \cup \mathcal{A}_3$ it admits a particular solution $(\widetilde{u}_1^*, \widetilde{u}_2^*)$.

- (1) If $(p, q) \in \mathcal{A}_1$, $\widetilde{u}_1^*, \widetilde{u}_2^*$ are ∞ -singular, and there exist solutions such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-2} \widetilde{u}_1 = c(c_2) > 0, \quad \lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_2^* = c_2 > 0, \quad (u_1, u_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

- (2) If $(p, q) \in \mathcal{B}$, $\widetilde{u}_1^*, \widetilde{u}_2^*$ are still ∞ -singular, and there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_2 = c_2 > 0, \quad (u_1, u_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

- (3) If $(p, q) \in \mathcal{A}_3$, \widetilde{u}_1^* is ∞ -singular, and \widetilde{u}_2^* is a cusp-solution; moreover

- if $q < q^*$ there exist solutions such that

$$(\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow 0} (\widetilde{u}_1, \widetilde{u}_2^*), \quad \lim_{r \rightarrow \infty} r^{(N-1)p-2} \widetilde{u}_1 = c(c_2) > 0, \quad \lim_{r \rightarrow \infty} \widetilde{u}_2 = c_2 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_2' = k > 0,$$

so the function \widetilde{u}_2 varies from 0 to C_2 .

- if $q > q^*$ there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{(N-1)q-2} \widetilde{u}_2 = c(c_1) > 0, \quad (\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow \infty} (\widetilde{u}_1, \widetilde{u}_2^*),$$

and there are explicit solutions if $q = q^*$.

There is no global nonconstant solution in the other regions such that $q < q_4$.

Now we study the system (1.6), also of a great richness, in particular involving the Sobolev exponent q^*

Theorem 5.14 *Consider the system (1.6). For $(p, q) \in \mathcal{A}_2 \cup \mathcal{C}_2 \cup \mathcal{D}_2$ it admits a particular solution $(\widehat{u}_1^*, \widehat{u}_2^*)$.*

- (1) *Assume $(p, q) \in \mathcal{A}_2$. Then $\widehat{u}_1^*, \widehat{u}_2^*$ are ∞ -singular; moreover*
- *if $q < q^*$, there exist solutions such that*

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow 0} (\widehat{u}_2^*, \widehat{u}_2^*), \quad \lim_{r \rightarrow \infty} r^{(N-1)p-2} \widehat{u}_1 = c(c_2) > 0, \quad \lim_{r \rightarrow \infty} r^{N-2} \widehat{u}_2 = c_2 > 0;$$

- *if $q > q^*$, there exist solutions such that*

$$\lim_{r \rightarrow 0} r^{N-2} \widehat{u}_1 = c_1 > 0, \quad \begin{cases} \lim_{r \rightarrow 0} \widehat{u}_2 = c_2 > 0, \lim_{r \rightarrow 0} r^{(N-1)q-1} \widehat{u}_2' = -c(c_1) < 0, & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow 0} r^{(N-1)q-2} \widehat{u}_2 = c(c_1) > 0, & \text{if } q > \frac{2}{N-1}, \end{cases}$$

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow \infty} (\widehat{u}_2^*, \widehat{u}_2^*),$$

- *if $q = q^*$, there exist explicit solutions, given by integration of (5.5).*

(2) *Assume $(p, q) \in \mathcal{C}_2$, $q \neq \frac{2}{N-1}$. Then \widehat{u}_1^* is ∞ -singular and \widehat{u}_2^* is a cusp-solution. There exist solutions such that*

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow 0} (\widehat{u}_2^*, \widehat{u}_2^*),$$

$$\lim_{r \rightarrow \infty} r^{N-2} \widehat{u}_1 = c_1 > 0, \quad \begin{cases} \lim_{r \rightarrow \infty} r^{(N-1)q-2} \widehat{u}_2 = c(c_1) > 0, & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow \infty} \widehat{u}_2 = c_2 > 0, \lim_{r \rightarrow \infty} r^{(N-1)q-1} \widehat{u}_2' = -c(c_1) < 0, & \text{if } q > \frac{2}{N-1}. \end{cases}$$

(3) *Assume $(p, q) \in \mathcal{D}_2$. Then again \widehat{u}_1^* is ∞ -singular and \widehat{u}_2^* is a cusp-solution. There exist solutions such that*

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow 0} (\widehat{u}_2^*, \widehat{u}_2^*), \quad \lim_{r \rightarrow \infty} r^{N-2} \widehat{u}_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} \widehat{u}_2 = c(c_2) > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widehat{u}_2' = -c_2 < 0.$$

There is no global nonconstant solution in the other regions such that $q < q_4$.

The region $\{q > q_4\}$, containing $\mathcal{C}_3 \cup \mathcal{D}_3$, plays a particular role, as in the scalar case for $q > 2$, because we can construct solutions such that u_1 and u_2 are **bounded** on $(0, \infty)$, then for any of the three systems (1.4), (1.5) and (1.6) by adding suitable constants.

Theorem 5.15 *Let $(p, q) \in \mathcal{C}_3 \cup \mathcal{D}_3 = \{q > q_4\}$. (i) Then system (1.5) still admits a particular solution $(\widetilde{u}_1^*, \widetilde{u}_2^*)$, where both components are cusp-solutions. Moreover there exist **bounded** global solutions on $(0, \infty)$:*

- *If $(p, q) \in \mathcal{C}_3$, there exist solutions such that*

$$(\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow 0} (\widetilde{u}_1^*, \widetilde{u}_2^*),$$

$$\lim_{r \rightarrow \infty} \widetilde{u}_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} \widetilde{u}_2 = c_2 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_1' = k_1 > 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-2} \widetilde{u}_2' = c(k_1) > 0.$$

- *If $(p, q) \in \mathcal{D}_3$, there exist solutions such that*

$$(\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow 0} (\widetilde{u}_1^*, \widetilde{u}_2^*), \quad \lim_{r \rightarrow \infty} \widetilde{u}_i = c_i > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_1' = k_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_2' = k_2 > 0.$$

(ii) System (1.4) admits global bounded solutions in \mathcal{C}_3 or \mathcal{D}_3 , of the form $(C_1 - \widehat{u}_1, C_2 - \widehat{u}_2)$ for $C_1 > c_1, C_2 > c_2$.

(iii) System (1.6) admits global bounded solutions of the same form for $C_1 > c_1, C_2 < 0$.

Proof of Theorem 1.1. The proofs of (1) (2) are direct consequences of Theorem 5.9, 5.12 and 5.15 respectively, from the definition 5.11 of the regions. \blacksquare

Remark 5.16 The limit cases $p = \frac{N}{N-1}$ and $q = q_1, q_2, \frac{N}{N-1}$ can be deduced from Remark 5.8 by integration. We leave the computations to the reader. In the limit cases $q = q_3 \neq q_2, q_4$, the particular solutions have a logarithmic form: If $p > q = q_3 \neq q_2$, and q we obtain solutions such that $u_1^* = -a_1 r^{1-p}$ and $u_2^* = -a_2 r^{-1}$, so we get non constant sign solutions of the form

$$u_1^* = \pm a_1 r^{2-p} + c_1, \quad u_2^* = -a_2 \ln r + c_2. \quad (5.9)$$

If $q = q_4 < p$ (hence $p > 2$) we get $u_1^* = -a_1 r^{-1}$ and $u_2^* = -a_2 r^{-\frac{2}{p}}$, then we find nonconstant sign solutions of the form

$$u_1^* = a_1 \ln r + c_2, \quad u_2^* = -\frac{p}{p-2} a_2 r^{\frac{p-2}{p}} + c_1.$$

And for $p = q = 2$, we recall the solutions $u_1^* = u_2^* = (2 - N) \ln r + c$. in these limit cases, there is no global nonconstant solution.

Finally when $q = q_2 = q_3$, equivalently $(p, q) = (N, \frac{2}{N-1})$, the behaviours of type (5.6) (5.7) near 0 or ∞ resumes to $\lim r |\ln r|^{\frac{2}{N+1}} u_2' = -c_2 < 0$, which gives by integration solutions such that

$$\lim \ln(|\ln r|^{\frac{2}{N+1}}) u_2 = -C < 0.$$

6 Nonradial case: upper estimates and local behaviour

6.1 Nonexistence results, and upperestimates of mean values

Here we give upperestimates for the supersolutions of system (1.1), implying in particular nonexistence results of entire solutions of the system. the nonexistence results obtained by integral methods were initiated by [31] for the positive supersolutions of the Lane-Emden system (1.3) where $a = b = 0$, and then extended in various directions, to more general operators and second members in [22], involving quasilinear operators and gradient terms, of type

$$\begin{cases} -\Delta_P^N u_1 \geq u_1^a u_2^b |\nabla u_2|^p, \\ -\Delta_Q^N u_2 \geq u_1^c u_2^d |\nabla u_1|^q, \end{cases} \quad (6.1)$$

where $P, Q > 1$, the solutions are positive, and $b, c > 0$. In [14] we also obtained integral estimates of the positive solutions, for example on problems of type (6.1) with $p = q = 0$. In the situation of system (1.1) we adapt the methods of [14], and obtain integral upperestimates of the gradient. A noticeable fact is that the solutions are not supposed to be positive, and not even of constant sign.

Definition 6.1 We say that a couple (u_1, u_2) of C^2 function in a domain $\Omega \subset \mathbb{R}^N$ is a supersolution (resp. a subsolution) of system (1.1) if

$$\begin{cases} -\Delta u_1 \geq |\nabla u_2|^p, \\ -\Delta u_2 \geq |\nabla u_1|^q, \end{cases} \quad \text{in } \Omega, \quad \text{resp.} \quad \begin{cases} -\Delta u_1 \leq |\nabla u_2|^p, \\ -\Delta u_2 \leq |\nabla u_1|^q, \end{cases} \quad \text{in } \Omega. \quad (6.2)$$

Theorem 6.2 Let $pq > 1$, $p \geq q \geq 1$, and (u_1, u_2) be any supersolution of system (1.1) (with no condition of sign) in a domain $\Omega \subset \mathbb{R}^N$.

(i) If $\Omega = B_{r_0} \setminus \{0\}$, resp $\Omega = \mathbb{R}^N \setminus \overline{B_{r_0}}$, then there exist $C = C(N, p, q) > 0$ such that for any $R < \frac{r_0}{2}$ (resp. $R > 2r_0$)

$$\int_{\frac{R}{2} \leq |x| \leq \frac{3R}{2}} |\nabla u_2|^p dx \leq CR^{N - \frac{p(q+1)}{pq-1}}, \quad \int_{\frac{R}{2} \leq |x| \leq \frac{3R}{2}} |\nabla u_1|^q dx \leq CR^{N - \frac{q(p+1)}{pq-1}}. \quad (6.3)$$

(ii) If $\Omega = \mathbb{R}^N$ and

$$(N-1)pq < \max(N+p, N+q) = N+p. \quad (6.4)$$

that means $q < q_2$, then all the **solutions** of system (1.1) are constant.

Proof. (i) Let $R > 0$ and $x_0 \in \Omega$ such that $\overline{B(x_0, 2R)} \subset \Omega$. Let $\zeta \in C_0^\infty(\mathbb{R}^N)$, with values in $[0, 1]$, such that $\zeta = 1$ on $B_{\frac{R}{2}}$, $\zeta = 0$ on B_R^c and $|\nabla \zeta| \leq \frac{2}{R}$. We take as test function $\varphi(x) = \zeta^\lambda(x-x_0)$, $\lambda > 0$, in the first inequality and get by integration, for any $p > 1$ and $\alpha > 0$,

$$\begin{aligned} \int_{B(x_0, R)} |\nabla u_1|^q \zeta^\lambda dx &\leq \lambda \left| \int_{B(x_0, R)} \langle \nabla u_2, \nabla \zeta \rangle \zeta^{\lambda-1} dx \right| = \lambda \left| \int_{B(x_0, R)} |\nabla u_2| \zeta^\alpha |\nabla \zeta| \zeta^{\lambda-1-\alpha} dx \right| \\ &\leq \lambda \left(\int_{B(x_0, R)} |\nabla u_2|^p \zeta^{\alpha p} dx \right)^{\frac{1}{p}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{p'} \zeta^{(\lambda-1-\alpha)p'} dx \right)^{\frac{1}{p'}}; \end{aligned}$$

then for $\lambda \geq 1 + \alpha$,

$$\int_{B(x_0, R)} |\nabla u_1|^q \zeta^\lambda dx \leq \lambda \left(\int_{B(x_0, R)} |\nabla u_2|^p \zeta^{\alpha p} dx \right)^{\frac{1}{p}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{p'}}.$$

Similarly for given $\beta > 0, \mu > 0$ such that $\mu \geq 1 + \beta$,

$$\int_{B(x_0, R)} |\nabla u_2|^p \zeta^\mu dx \leq \mu \left(\int_{B(x_0, R)} |\nabla u_1|^q \zeta^{\beta q} dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}}.$$

Taking $\lambda = \frac{q(p+1)}{pq-1}$, $\alpha = \frac{q+1}{pq-1}$, $\mu = \frac{p(q+1)}{pq-1}$ and $\beta = \frac{p+1}{pq-1}$, we find

$$\int_{B(x_0, R)} |\nabla u_1|^q \zeta^{\frac{q(p+1)}{pq-1}} dx \leq \lambda \left(\int_{B(x_0, R)} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx \right)^{\frac{1}{p}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{p'}},$$

and by symmetry, when $q > 1$,

$$\int_{B(x_0, R)} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx \leq \mu \left(\int_{B(x_0, R)} |\nabla u_1|^q \zeta^{\frac{q(p+1)}{pq-1}} dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}}.$$

As a consequence,

$$\begin{aligned} \int_{B(x_0, R)} |\nabla u_2|^p \xi^{\frac{p(q+1)}{pq-1}} dx &\leq \mu \lambda^{\frac{1}{q}} \left(\int_{B(x_0, R)} |\nabla u_2|^p \xi^{\frac{p(q+1)}{pq-1}} dx \right)^{\frac{1}{pq}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{qp'}}, \\ \left(\int_{B(x_0, \frac{R}{2})} |\nabla u_2|^p dx \right)^{\frac{pq-1}{pq}} &\leq \left(\int_{B(x_0, R)} |\nabla u_2|^p \xi^{\frac{p(q+1)}{pq-1}} dx \right)^{\frac{pq-1}{pq}} \\ &\leq \mu \lambda^{\frac{1}{q}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{qp'}} \leq CR^{\frac{N-q'}{q'} + \frac{N-p'}{qp'}} = CR^{N \frac{pq-1}{pq} - \frac{q+1}{q}}, \end{aligned}$$

for some constant $C = C(N, p, q) > 0$. When $q = 1$, we get directly with $\mu = \frac{2p}{p-1}$

$$\begin{aligned} \int_{B(x_0, R)} |\nabla u_2|^p \zeta^{\frac{2p}{p-1}} dx &\leq \frac{2p}{p-1} \left| \int_{B(x_0, R)} \langle \nabla u_1, \nabla \zeta \rangle \zeta^{\frac{p+1}{p-1}} dx \right| = \frac{2p}{p-1} \left| \int_{B(x_0, R)} |\nabla u_1| \zeta^{\frac{p+1}{p-1}} |\nabla \zeta| dx \right| \\ &\leq \frac{C}{R} \int_{B(x_0, R)} |\nabla u_1| \zeta^{\frac{p+1}{p-1}}, \\ \int_{B(x_0, R)} |\nabla u_1| \zeta^{\frac{p+1}{p-1}} dx &\leq \lambda \left(\int_{B(x_0, R)} |\nabla u_2|^p \zeta^{\frac{2p}{p-1}} dx \right)^{\frac{1}{p}} \left(\int_{B(x_0, R)} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{p'}}, \end{aligned}$$

then we obtain

$$\left(\int_{B(x_0, \frac{R}{2})} |\nabla u_2|^p dx \right)^{\frac{1}{p'}} \leq CR^{\frac{N}{p'} - 2}.$$

In any case the estimates (6.3) follow by considering a finite recovering of $\{\frac{R}{2} \leq |x| \leq \frac{3R}{2}\}$ by balls $B(x_0, R)$.

(ii) If $\Omega = \mathbb{R}^N$ and $N(pq - 1) < p(q + 1)$, equivalently (6.4) holds, we consider any ball $B(0, R)$ and make $R \rightarrow \infty$; we deduce that $\nabla u_2 = 0$, hence $u_2 = C_2$; then $-\Delta u_2 = 0 \geq |\nabla u_1|^q$, thus $u_1 = C_1$. \blacksquare

Remark 6.3 In case $p = q$, and $u_1 \geq 0, u_2 \geq 0$, the nonexistence can be obtained by reducing to the scalar case:

$$-\Delta(u_1 + u_2) = |\nabla u_2|^q + |\nabla u_1|^q \geq c_q^1 (|\nabla u_2| + |\nabla u_1|)^q \geq c_q^2 |\nabla(u_1 + u_2)|^q.$$

If $1 < q < \frac{N}{N-1}$, then the only nonnegative solutions in whole \mathbb{R}^N satisfy $u_1 + u_2 = C$, for example from [10, Proposition 2.1]. Then $-\Delta(u_1 + u_2) = 0 \geq |\nabla u_2|^q + |\nabla u_1|^q$, hence u_1 and u_2 are constant. At Theorem 6.2 we have shown that the positivity is not required. And moreover we obtain integral upper estimates of the gradients when $q \geq \frac{N}{N-1}$.

In the sequel, the mean value of any function u in $\{R \leq |x| \leq R'\}$ on the sphere S^{N-1} is denoted by \bar{u} .

Lemma 6.4 Let $p \geq q \geq 1$ and (u_1, u_2) be a supersolution of system (1.1) in $\{R \leq |x| \leq R'\}$. Then (\bar{u}_1, \bar{u}_2) is also a supersolution. In particular if $(\widetilde{u}_1, \widetilde{u}_2)$ is a positive subsolution of system (1.4), then $(\widetilde{\bar{u}}_1, \widetilde{\bar{u}}_2)$ is also a subsolution.

Proof. For any function $u \in C^1(B(0, R))$, and any $p \geq 1$, from the Jensen inequality,

$$|\overline{\nabla u}|^p \geq |\overline{u'}|^p. \quad (6.5)$$

Indeed

$$\begin{aligned} |\overline{\nabla u}|^p &= \frac{1}{|S^{N-1}|} \int_{S^{N-1}} (u_{2,r}^2 + r^{-2} |\nabla' u_2|^2)^{\frac{p}{2}} d\sigma \geq \frac{1}{|S^{N-1}|} \int_{S^{N-1}} |u_r|^p d\sigma \\ &\geq \left(\frac{1}{|S^{N-1}|} \int_{S^{N-1}} |u_r| d\sigma \right)^p \geq \left| \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u_r d\sigma \right|^p = |\overline{u'}|^p. \end{aligned}$$

Then, since $q \geq 1$,

$$-\Delta \overline{u_1} \geq |\overline{\nabla u_2}|^p \geq |\overline{u_2'}|^p, \quad -\Delta \overline{u_2} \geq |\overline{\nabla u_1}|^q \geq |\overline{u_1'}|^q.$$

■

6.2 Local behaviour near 0 or ∞

As a consequence of Theorem 6.2 and Proposition 3.5, we get

Proposition 6.5 *Let $pq > 1$, $p \geq q \geq 1$, and (u_1, u_2) be any supersolution of system (1.4) in $B_{r_0} \setminus \{0\}$. Then there exist $C = C(N, p, q) > 0$, and $\rho \in (0, r_0)$ depending on u_1, u_2 , such that for $r \in (0, \rho)$,*

$$|\overline{u_1}(r)| \leq \begin{cases} Cr^{-\frac{2-p(q-1)}{pq-1}} & \text{if } q < q_4, \\ C |\ln r| & \text{if } q = q_4, \\ C & \text{if } q > q_4, \end{cases} \quad |\overline{u_2}(r)| \leq \begin{cases} Cr^{-\frac{2-q(p-1)}{pq-1}} & \text{if } q < q_3, \\ C |\ln r| & \text{if } q = q_3, \\ C & \text{if } q > q_3. \end{cases}$$

(ii) (u_1, u_2) be any supersolution of system (1.4) in $\mathbb{R}^N \setminus \overline{B_{r_0}}$. Then there exists $\eta > r_0$ such that for $r > \eta$,

$$|\overline{u_1}(r)| \leq \begin{cases} C |x|^{\frac{p(q-1)-2}{pq-1}} & \text{if } q > q_4, \\ C \ln r & \text{if } q = q_4, \\ C & \text{if } q < q_4, \end{cases} \quad |\overline{u_2}(r)| \leq \begin{cases} Cr^{\frac{q(p-1)-2}{pq-1}} & \text{if } q > q_3, \\ C \ln r & \text{if } q = q_3, \\ C & \text{if } q < q_3. \end{cases}$$

6.2.1 Case of system (1.5)

In this case, we can improve the preceding results. We give Osserman's type estimates for the local solutions near 0 or ∞

Theorem 6.6 *Let $pq > 1$, $p \geq q \geq 1$. (i) Let $(\widetilde{u}_1, \widetilde{u}_2)$ be any positive subsolution of system (1.5) in $B_{r_0} \setminus \{0\}$. Then there exists and $C = C(N, p, q) > 0$, and $\rho \in (0, r_0)$ depending on $\widetilde{u}_1, \widetilde{u}_2$, such that for $0 < |x| < \rho$,*

$$\widetilde{u}_1(x) \leq \begin{cases} C |x|^{-\frac{2-p(q-1)}{pq-1}} & \text{if } q < q_4, \\ C |\ln |x|| & \text{if } q = q_4, \\ C & \text{if } q > q_4, \end{cases} \quad \widetilde{u}_2(x) \leq \begin{cases} C |x|^{-\frac{2-q(p-1)}{pq-1}} & \text{if } q < q_3, \\ C |\ln |x|| & \text{if } q = q_3, \\ C & \text{if } q > q_3. \end{cases}$$

(ii) Let $(\widetilde{u}_1, \widetilde{u}_2)$ be any positive subsolution of system (1.5) in $\mathbb{R}^N \setminus \overline{B_{r_0}}$. Then there exists $\eta > r_0$ such that for $|x| > \eta$,

$$\widetilde{u}_1(x) \leq \begin{cases} C|x|^{\frac{p(q-1)-2}{pq-1}} & \text{if } q > q_4, \\ C|\ln|x|| & \text{if } q = q_4, \\ C & \text{if } q < q_4, \end{cases} \quad \widetilde{u}_2(x) \leq \begin{cases} C|x|^{\frac{q(p-1)-2}{pq-1}} & \text{if } q > q_3, \\ C|\ln|x|| & \text{if } q = q_3, \\ C & \text{if } q < q_3. \end{cases}$$

Proof. (i) Since the functions are subharmonic, if u_i is nonconstant near 0, then \widetilde{u}_i is strictly monotone for small r , and either $r^{N-2}\widetilde{u}_i$ has a positive limit as $r \rightarrow 0$, or \widetilde{u}_i is bounded. And for $|x| < \rho$ small enough, depending on the functions, from Proposition 3.5 applied to the functions \widetilde{u}_i

$$\left| \widetilde{u}_1'(r) \right| \leq C(N, p, q)r^{-\frac{p+1}{pq-1}}, \quad \left| \widetilde{u}_2'(r) \right| \leq Cr^{-\frac{q+1}{pq-1}}.$$

Then by integration, for ρ small enough, there holds $\widetilde{u}_1(r) \leq Cr^{-\frac{2-p(q-1)}{pq-1}}$ if $q < q_4$ and $\widetilde{u}_1(r) \leq C$ if $q > q_4$, similarly for \widetilde{u}_2 by exchanging q_3 and q_4 . Since the functions are subharmonic,

$$u_i(x) \leq C(N) \max\left(\widetilde{u}_i\left(\frac{|x|}{2}\right), \widetilde{u}_i(2|x|)\right),$$

see [13, Lemma 2.1], hence the conclusions hold. Moreover if $q > q_4$, then $\left| \widetilde{u}_1' \right|$ is integrable, hence \widetilde{u}_1 has a finite limit l as $r \rightarrow 0$.

(ii) The proof is similar. ■

6.2.2 Case of system (1.4)

We first consider the exterior problem. We observe that in the scalar case, when $1 < q < \frac{N}{N-1}$, there is no positive radial solution of the equation such that $\lim_{r \rightarrow \infty} u = 0$. But there exist solutions such that u is increasing to some $l > 0$ as $r \rightarrow \infty$, so there exist solutions of system (1.4) for $p = q$. In the general case we prove the following when q is subcritical, namely $q < q_2$:

Theorem 6.7 *Let $pq > 1$, $p \geq q \geq 1$. If $q < q_2$, there is no positive supersolution (u_1, u_2) of system (1.4) such that $\lim_{r \rightarrow \infty} \overline{u}_1 = 0$ in an exterior set $\mathbb{R}^N \setminus \overline{B_{r_0}}$.*

Proof. We have the upper estimates of the derivatives

$$\left| \overline{u}_1'(r) \right| \leq Cr^{-\frac{p+1}{pq-1}}, \quad \left| \overline{u}_2'(r) \right| \leq Cr^{-\frac{q+1}{pq-1}}, \quad (6.6)$$

and

$$\begin{cases} -\Delta \overline{u}_1 \geq |\overline{u}_2'|^p, \\ -\Delta \overline{u}_2 \geq |\overline{u}_1'|^q. \end{cases}$$

The function \overline{u}_1 is superharmonic, and

$$\overline{u}_1(r) \leq \overline{u}_1(r_0) + \int_{r_0}^r \left| \overline{u}_1'(r) \right| dr \leq \overline{u}_1(r_0) + Cr^{-\frac{(2-p(q-1))}{pq-1}},$$

so as soon as $q < q_4$, \overline{u}_1 is bounded. And more precisely $\lim_{r \rightarrow \infty} \overline{u}_1 = l_1$ (and $\lim_{r \rightarrow \infty} \overline{u}_2 = l_2$). Here we suppose that $l_1 = 0$. From the superharmonicity, $r^{N-1}\overline{u}_1'$ is decreasing so either

$\lim_{r \rightarrow \infty} r^{N-1} \bar{u}_1' = l \in (0, \infty)$ or $\lim_{r \rightarrow \infty} r^{N-1} \bar{u}_1' = -\infty$; and $l \in (0, \infty)$ is impossible because then $\bar{u}_1' > 0$ for large r so \bar{u}_1 cannot tend to 0. Thus $\bar{u}_1' < 0$. Then $r^{N-1} \bar{u}_1' < -K$ for large r , so $\bar{u}_1 - \frac{K}{N-2} r^{2-N}$ is decreasing to 0, hence $\bar{u}_1 \geq \frac{K}{N-2} r^{2-N}$. But for large r , there holds $\bar{u}_1'(r) \geq -Cr^{-\frac{p+1}{pq-1}} = \left(\frac{C(pq-1)}{p(q_4-q)} r^{\frac{p(q-q_4)}{pq-1}} \right)'$, then $\bar{u}_1 - \frac{C(pq-1)}{p(q_4-q)} r^{\frac{p(q-q_4)}{pq-1}}$, is increasing to 0, thus $\bar{u}_1 \leq cr^{\frac{p(q-q_4)}{pq-1}} = cr^{\frac{p(q-1)-2}{pq-1}}$. Therefore $r^{2-N} \leq r^{\frac{p(q-1)-2}{pq-1}}$, which implies $q \geq q_2$, leading to a contradiction. \blacksquare

Remark 6.8 *In the case $q < q_2$, from Theorem (5.10) there exist solutions such that $\lim_{r \rightarrow \infty} u_1 = c_1 > 0$. In regions \mathcal{C} and \mathcal{D} , where $q > q_2$, we have proved the existence of solutions in $\mathbb{R}^N \setminus B_{r_0}$ at the same theorem. For $q > q_4$ from Theorem 5.15 there exist solutions on $(0, \infty)$. So our result is optimal.*

Finally we consider the local behaviour of the solutions of system (1.4) near a singularity, and moreover the existence of a system with measure data. We give a nonradial existence result of solutions of a Dirichlet problem when $q < q_2$, extending the radial ones of Theorem 5.9 :

Theorem 6.9 *Let $pq > 1$, $p \geq q \geq 1$. Let Ω be a C^2 bounded domain containing 0.*

(i) *Let $(u_1, u_2) \in W_0^{1,q}(\Omega \setminus \{0\}) \times W_0^{1,p}(\Omega \setminus \{0\})$, such that $\Delta u_1, \Delta u_2 \in L_{loc}^1(\Omega \setminus \{0\})$ be a solution of system (1.4) in $\Omega \setminus \{0\}$, with $u_1 = u_2 = 0$ on $\partial\Omega$. Then $(u_1, u_2) \in W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$ and there exist $C_1 \geq 0, C_2 \geq 0$, such that*

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 \delta_0, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 \delta_0, \end{cases} \quad \text{in } \mathcal{D}'(\Omega). \quad (6.7)$$

Moreover

$$\text{if } C_1 > 0 \text{ then } q < q_2, \text{ and if } C_2 > 0, \text{ then } p < \frac{N}{N-1}. \quad (6.8)$$

(ii) *Reciprocally, let C_1, C_2 satisfying (6.8) with $C_1 > 0$ or $C_2 > 0$, and C_1, C_2 small enough. Then there exists a solution (u_1, u_2) of system (6.7), such that $(u_1, u_2) \in W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$. More generally, for any bounded Radon measures μ, ν in Ω , under the same conditions on C_1, C_2 , there exists a solution of the system*

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 \mu, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 \nu, \end{cases} \quad \text{in } \mathcal{D}'(\Omega).$$

Moreover $u_1, u_2 \in W_0^{1,s}(\Omega)$ for any $s \in \left[1, \frac{N}{N-1}\right)$, and if $C_2 = 0$, then $u_2 \in W_0^{1,s_2}(\Omega)$ for $s_2 < \frac{N}{(N-1)q-1}$.

Proof. (i) The functions u_1, u_2 are superharmonic and nonnegative. Then there exist $C_1 \geq 0, C_2 \geq 0$, such that (6.7) holds, and $|\nabla u_2|^p + |\nabla u_1|^q \in L_{loc}^1(\Omega)$, and moreover $u_1, u_2 \in W_0^{1,s}(\Omega)$ for any $s \in \left[1, \frac{N}{N-1}\right)$, see [?]. From Lemma 6.4 (\bar{u}_1, \bar{u}_2) is a supersolution, then the conditions on C_1, C_2 follow from Theorem 5.9.

(ii) We recall a result of [3, Theorem 3.1]: for any nonnegative $f \in L^m(\Omega)$ and $g \in L^k(\Omega)$, $m, k \in (1, N)$ such that $qk < \frac{mN}{N-m}$ and $pm < \frac{kN}{N-k}$, there exists $\Lambda = \Lambda(p, q, m, k) > 0$ such that the problem

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 f, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 g, \end{cases} \quad \text{in } \Omega,$$

admits a solution such that $(u_1, u_2) \in W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$ under the condition

$$C_1^q \|f\|_{L^m(\Omega)}^q + C_2 \|g\|_{L^k(\Omega)} \leq \Lambda.$$

And $(u_1, u_2) \in W_0^{1,s_1}(\Omega) \times W_0^{1,s_2}(\Omega)$ for any $s_1 \in \left[1, \frac{mN}{N-m}\right)$ and $s_2 \in \left[1, \frac{kN}{N-k}\right)$, with the estimate

$$\|u_1\|_{W_0^{1,s_1}(\Omega)} + \|u_2\|_{W_0^{1,s_2}(\Omega)} \leq C(\Lambda, s_1, s_2, m, k).$$

Let μ, ν be two positive Radon measures in Ω . Let $(\mu_n), (\nu_n)$ be sequences in $C^{\infty,+}(\overline{\Omega})$ converging respectively to μ, ν in the sense of measures, and $\mu_n(\Omega) \leq 2\mu(\Omega), \nu_n(\Omega) \leq 2\nu(\Omega)$. Then there exists a solution of the approximate problem

$$\begin{cases} -\Delta u_{1,n} = |\nabla u_{2,n}|^p + C_1 \mu_n, \\ -\Delta u_{2,n} = |\nabla u_{1,n}|^q + C_2 \nu_n, \end{cases} \quad \text{in } \Omega,$$

First suppose $C_2 > 0$, then $1 \leq q \leq p < \frac{N}{N-1}$. We take $m = k = 1$, and $f = \mu_n, g = \nu_n$. For $2^q C_1^q \mu^q(\Omega) + 2C_2 \nu(\Omega) < \Lambda$ we get the existence of sequences $(u_{1,n}), (u_{2,n})$ bounded in $W_0^{1,s}(\Omega)$ for any $s \in \left[1, \frac{N}{N-1}\right)$. Next suppose $C_2 = 0 < C_1$. We take $f = \mu_n$ and $g = 0$. We need to find some m and k such that $qk < \frac{mN}{N-m}$ and $pm < \frac{kN}{N-k}$. We fix $m = 1$, so we require that that $\frac{Np}{N+p} < k < \frac{N}{(N-1)q}$ (which implies $q < \frac{N}{N-1}$). This is possible because $q < q_2 = \frac{N+p}{(N-1)p} \leq \frac{N}{N-1}$. Then $(u_{1,n})$ is bounded in $W_0^{1,s}(\Omega)$ for any $s \in \left[1, \frac{N}{N-1}\right)$ $u_{2,n}$ bounded in $W_0^{1,s_2}(\Omega)$ for any $s_2 \in \left[1, \frac{N}{(N-1)q-1}\right)$, hence in particular for $s_2 \in \left[1, \frac{N}{N-1}\right)$. Therefore we get the existence for $2^q C_1^q \mu^q(\Omega) \leq \Lambda$. Next we can pass to the limit in $\mathcal{D}'(\Omega)$. We fix s_1 with $q < s_1 < \frac{N}{N-1}$ and s_2 with $p < s_2 < \frac{N}{N-1}$ if $C_2 > 0$ and $p < s_2 < \frac{N}{(N-1)q-1}$ if $C_2 = 0$; up to subsequences, $(u_{1,n})$ converges weakly and *a.e.* in $W_0^{1,s_1}(\Omega)$ and $(u_{2,n})$ converges weakly and *a.e.* in $W_0^{1,s_2}(\Omega)$ to some u_1, u_2 ; in both cases, $(|\nabla u_{2,n}|^p)$ converges strongly in $L^1(\Omega)$, then u satisfies the equation in $\mathcal{D}'(\Omega)$

$$-\Delta u_1 = |\nabla u_2|^p + C_1 \mu,$$

and $(|\nabla u_{1,n}|^q)$ converges strongly in $L^1(\Omega)$, so we also pass to the limit and get in any case

$$-\Delta u_2 = |\nabla u_1|^q + C_2 \nu.$$

■

7 Extensions

(1) First note also that the study of Hardy-Hénon type equations by using of system (4.3) can be adapted to other ranges of the parameters, for example to a "sublinear" case, that is $\mathbf{q} < \mathbf{p} - \mathbf{1}$, corresponding to the case where $pq < 1$ for the system, or also to case $\mathbf{q} < \mathbf{0}$, corresponding to $q < 0$. This allows to extend some results of [26] given for the Hardy-Hénon equation $-\Delta w = |x|^\sigma u^q$ with the Laplacian to the m -Laplacian, in the radial case, in particular their study in dimension 1.

(2) Moreover consider an equation of Hardy-Hénon-type in dimension $\nu \in \mathbb{N}, \nu \geq 1$, with a weight $|x|^\alpha$ ($\alpha \in \mathbb{R}$) inside the operator:

$$-\operatorname{div}(|x|^\alpha |\nabla w|^{\mathbf{p}-2} \nabla w) = \varepsilon |x|^b w^\mathbf{q},$$

In the radial case, it reduces to

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = -\frac{d}{dr} \left(\left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} \right) - \frac{\nu + a - 1}{r} \left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} = \varepsilon r^{b-a} w^\mathbf{q},$$

so it directly falls in the scope of our study, with $N = \nu + a$ and $\sigma = b - a$. It allows for example to find again rapidly some recent results of [40], given for the Laplacian with some conditions on a, b , and extend them to the p -Laplacian, in all the ranges of the parameters.

(3) All our radial study of system (1.1) can be easily directly extended to a system of k -Laplacians,

$$\begin{cases} -\Delta_k u_1 = |\nabla u_2|^p, \\ -\Delta_k u_2 = |\nabla u_1|^q, \end{cases} \quad (7.1)$$

where $k > 1$. Indeed the radial system reduces to

$$\begin{cases} -(|u_1'|^{k_1-2} u_1')' - \frac{N-1}{r} |u_1'|^{k_1-2} u_1' = |u_2'|^p, \\ -(|u_2'|^{k_2-2} u_2')' - \frac{N-1}{r} |u_2'|^{k_2-2} u_2' = |u_1'|^q, \\ \begin{cases} w_1' = r^{(N-1)(1-\frac{p}{k-1})} |w_2|^{\frac{p}{k-1}}, \\ w_2' = r^{(N-1)(1-\frac{q}{k-1})} |w_1|^{\frac{q}{k-1}}, \end{cases} \end{cases}$$

where

$$-r^{N-1} |u_1'|^{k_1-2} u_1' = w_1, \quad -r^{N-1} |u_2'|^{k_2-2} u_2' = w_2.$$

So we are reduced to study functions w_1, w_2 , **with** p, q **replaced by** $\frac{p}{k-1}$ **and** $\frac{q}{k-1}$; and the results apply for $pq > (k-1)^2$. We obtain analogous results as in the case $k = 2$, with new parameters q_i defined by

$$q_1 = \frac{N}{(N-1)p - k + 1}, \quad q_2 = \frac{N + (k-1)p}{(N-1)p}, \quad q_3 = \frac{2(k-1)^2}{p - k + 1}, \quad q_4 = \frac{(k-1)p + 2(k-1)^2}{p}.$$

Moreover one can make an easy adaptation of the results of the nonradial section, relative to the supersolutions of system (1.1), where the estimates of the mean values are replaced by integral estimates, as in [14] in case of source, absorption or mixed terms. The computations are left to the reader.

(4) More generally our study allows to treat systems of the type

$$\begin{cases} -\operatorname{div}(|x|^{a_1} |\nabla u_1|^{k_1-2} \nabla u_1) = |x|^{b_1} |\nabla u_2|^p, \\ -\operatorname{div}(|x|^{a_2} |\nabla u_2|^{k_2-2} \nabla u_2) = |x|^{b_2} |\nabla u_1|^q, \end{cases}$$

which in the radial case reduce to

$$\begin{cases} -(|u_1'|^{k_1-2} u_1')' - \frac{N+a_1-1}{r} |u_1'|^{k_1-2} u_1' = r^{b_1-a_1} |u_2'|^p, \\ -(|u_2'|^{k_2-2} u_2')' - \frac{N+a_2-1}{r} |u_2'|^{k_2-2} u_2' = r^{b_2-a_2} |u_1'|^q, \end{cases}$$

and then to a Hardy-Hénon equation (1.2) with new parameters $\mathbf{p}, \mathbf{q}, \mathbf{N}$ defined by

$$\frac{p}{k_2 - 1} = \frac{1}{\mathbf{p} - 1}, \quad \frac{q}{k_1 - 1} = \mathbf{q}, \quad \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} = N + b_1 - (N + a_2 - 1) \frac{p}{k_2 - 1}, \quad \mathbf{N} + \sigma = N + b_2 - (N + a_1 - 1) \mathbf{q};$$

and $\mathbf{q} > \mathbf{p} - 1$ is equivalent to $pq > (k_1 - 1)(k_2 - 1)$.

(3) Some of our nonradial results can be extended to more general systems, like (7.1), following the methods of [14], for example Theorem 6.2. It would be interesting to extend Theorem 6.9 to such type systems.

8 Appendix

In this section we give the proofs of the main results of Section 4

Proof of Lemma 4.5. (i) Setting $s = s_0 + \bar{s}$, $z = z_0 + \bar{z}$, the linearized problem at M_0 is

$$\begin{cases} \bar{s}_t = s_0 \left(\bar{s} + \frac{\bar{z}}{\mathbf{p} - 1} \right), \\ \bar{z}_t = z_0 (-\mathbf{q}\bar{s} - \bar{z}). \end{cases}$$

It admits the eigenvalues λ_1, λ_2 , roots of the equation (4.12). When the roots are real, that means $s_0 z_0 < 0$ then $\lambda_1 < 0 < \lambda_2$, and $\lambda_1 < s_0 < \lambda_2$ since $T(s_0) < 0$, and the corresponding eigenvectors $\vec{v}_1 = \left(\frac{s_0}{\mathbf{p} - 1}, \lambda_1 - s_0 \right)$ and $\vec{v}_2 = \left(\frac{s_0}{\mathbf{p} - 1}, \lambda_2 - s_0 \right)$ have the slopes $\rho_i = (\mathbf{p} - 1) \left(\frac{\lambda_i}{s_0} - 1 \right) = \frac{\mathbf{q} z_0}{z_0 + \lambda_i}$, $i = 1, 2$.

(ii) Setting $z = \mathbf{N} + \sigma + \bar{z}$, the linearized problem at N_0 is

$$\begin{cases} s_t = \frac{\mathbf{p} + \sigma}{\mathbf{p} - 1} s, \\ \bar{z}_t = (\mathbf{N} + \sigma) (-\mathbf{q}s - \bar{z}). \end{cases}$$

It admits the eigenvalues $l_1 = \frac{\mathbf{p} + \sigma}{\mathbf{p} - 1}$ and $l_2 = -(\mathbf{N} + \sigma)$. When $l_1 \neq l_2$, the corresponding eigenvectors are $\vec{v}_1 = (l_1 - l_2, \mathbf{q}l_2)$ and $\vec{v}_2 = (0, 1)$.

(iii) Setting $s = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} + \bar{s}$, the linearized system at A_0 is

$$\begin{cases} \bar{s}_t = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} \left(\bar{s} + \frac{z}{\mathbf{p} - 1} \right), \\ z_t = z \left(\mathbf{N} + \sigma - \mathbf{q} \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} \right). \end{cases}$$

It admits the eigenvalues $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}$ and $\mu_2 = \mathbf{N} + \sigma - \mathbf{q} \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}$. And $z = 0$ contain particular trajectories linked to μ_1 . When $\mu_1 \neq \mu_2 \neq 0$, corresponding eigenvectors are $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = \left(\frac{\mu_1}{\mathbf{p} - 1}, \mu_2 - \mu_1 \right)$.

(iv) The linearized system at point $(0, 0)$,

$$\begin{cases} s_t = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} z, \\ z_t = (\mathbf{N} + \sigma) z. \end{cases}$$

gives the eigenvalues $\rho_1 = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}$, $\rho_2 = \mathbf{N} + \sigma$, which are distinct for $\sigma \neq -\frac{(N-1)p}{p-1}$, and corresponding eigenvectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (0, 1)$. \blacksquare

Proof of Lemma 4.7. (i) Direct consequence of (4.4).

(ii) Suppose $\lim(s, z) = N_0$. From (4.3) we find $s_t = s(l_1 + s + \bar{z}) = s(l_1 + o(1))$, then $s = o(e^{-\frac{|l_1 t|}{2}})$, and $\bar{z}_t = z - (\mathbf{N} + \sigma) = (-l_2 + \bar{z})(-\mathbf{q}s - \bar{z}) = \bar{z}(l_2 - \mathbf{q}s - \bar{z}) + l_2 \mathbf{q}s = (l_2 + o(1))\bar{z} + o(e^{-2|l_1 t|})$. Then $z = o(e^{-k|t|})$ for some $k > 0$. In turn $s_t = s(l_1 + o(e^{-k|t|}))$ for some $k > 0$, thus $\lim e^{-l_1 t} s = C \neq 0$. The conclusion follows from (4.4) with $\lim e^{-l_1 t} s = C \neq 0$ and $\lim z = N + \sigma$.

(iii) Suppose $\lim(s, z) = (0, 0)$. Then $s_t = s(\rho_1 + s + \bar{z}) = s(\rho_1 + o(1))$, and $z_t = z(\rho_2 + o(1))$. Then $s, z = o(e^{-k|t|})$ for some $k > 0$. Therefore we obtain $s_t = s(\rho_1 + o(e^{kt}))$, $z_t = z(\rho_2 + o(e^{kt}))$. By integration we deduce $\lim e^{-\rho_1 t} s = C \neq 0$ and $\lim e^{-\rho_2 t} z = D \neq 0$, and we still apply (4.4).

(iv) Suppose $\lim(s, z) = A_0$. Then $z_t = z(\mu_2 - \mathbf{q}\bar{s} - z)$ with $\bar{s} = s - \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} = o(1)$, so that $z_t = z(\mu_2 + o(1))$, then $z = o(e^{-\frac{|\mu_2 t|}{2}})$. It follows that $\bar{s}_t = (\mu_1 + \bar{s})(\bar{s} + \frac{z}{\mathbf{p}-1}) = \bar{s}(\mu_1 + \frac{z}{\mathbf{p}-1} + \bar{s}) + \frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}-1)}z = \mu_1 \bar{s} + o(e^{-\frac{|\mu_2 t|}{2}})$. Then $s = o(e^{-k|t|})$ for some $k > 0$; in turn $z_t = z(\mu_2 + o(e^{-k|t|}))$; thus $\lim e^{-\mu_2 t} z = D \neq 0$, and we apply (4.4) where $\lim s = \mu_1$, $\lim e^{-\mu_2 t} z = D$. ■

Proof of Lemma 4.8. Let for example w be any solution in $(0, r_0)$. From Lemma (4.4), and (4.4), $\Phi = |s|^{\mathbf{p}-1} |z|$ is bounded. And

$$\Phi_t = \Phi(\mathbf{p} + \sigma - (\mathbf{q} - \mathbf{p} + 1)s).$$

Suppose that s is unbounded near $-\infty$. Either s is monotone near $-\infty$, then $\lim_{t \rightarrow -\infty} |s| = \infty$, hence $\frac{\Phi_t}{\Phi} \geq -1$ near $-\infty$, $\ln |\Phi| \geq \frac{|t|}{2}$, which is impossible. Or there exists a sequence $t_n \rightarrow -\infty$ such that $|s(t_n)| \rightarrow \infty$ of points of maximum of $|s|$. At these points, and $\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s(t_n) + \frac{z(t_n)}{\mathbf{p}-1} = 0$, hence $s(t_n)z(t_n) < 0$, and

$$s_{tt}(t_n) = \frac{s(t_n)z_t(t_n)}{\mathbf{p}-1} = \frac{s(t_n)z(t_n)}{\mathbf{p}-1}(\mathbf{p} + \sigma - (\mathbf{q} - \mathbf{p} + 1)s(t_n)),$$

thus $s_{tt}(t_n)s(t_n) > 0$, which is contradictory. Then s is bounded. Suppose that z is unbounded. Either z is monotone near $-\infty$, then $\lim_{t \rightarrow -\infty} |z| = \infty$, thus $\lim_{t \rightarrow -\infty} |s_t| = \infty$, which is impossible. Or there exists a sequence $t_n \rightarrow -\infty$ such that $|z(t_n)| \rightarrow \infty$ of points of maximum of $|z|$. At these points there holds $\mathbf{N} + \sigma - \mathbf{q}s(t_n) - z(t_n) = 0$, which is impossible since s is bounded. Then z is bounded. It converges to one of the fixed points of the system, or has a limit cycle around them. Since $N_0, (0, 0)$ or A_0 have real eigenvalues, such cycle can happen only at M_0 , when $\mathbf{q} = \mathbf{q}_S$. ■

Proof of Theorem 4.9. Here we consider region \mathbf{A} , where $\mathbf{N} > \mathbf{p} > -\sigma$. In this case we refer to [12] for the description of the system. Here the behaviour depend on position of \mathbf{q} with respect to \mathbf{q}_c , and also of \mathbf{q}_S when $\varepsilon = 1$, see figures 1, 2, 3, 4. The point M_0 is in Q_4 for $\mathbf{q} < \mathbf{q}_c$, corresponding to solutions of the equation with absorption ($\varepsilon = -1$), and $M_0 \in Q_1$ for $\mathbf{q} > \mathbf{q}_c$. The point N_0 is a saddle point, corresponding to C^0 -regular solutions near 0. The point A_0 is a source for $\mathbf{q} < \mathbf{q}_c$ and a saddle point for $\mathbf{q} > \mathbf{q}_c$. The point $(0, 0)$ is a saddle point, and the associated trajectories are not admissible. The existence of ground states of the equation with source ($\varepsilon = 1$) if and only if $q \geq \mathbf{q}_S$ is well known. The phase plane in the critical Sobolev case $q = \mathbf{q}_S$ is remarkable: there exist particular particular trajectories, located on a straight line

$$(\mathbf{p}-1)s + \frac{\mathbf{p}z}{\mathbf{q}+1} - \mathbf{N} + \mathbf{p} = 0.$$

These trajectories correspond to the ground states given at (4.13) when $\varepsilon = 1$, and to explicit local solutions near 0 or ∞ when $\varepsilon = -1$. ■

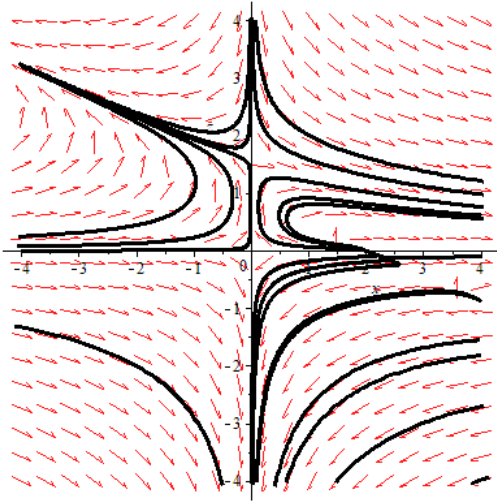


Figure 1, Theorem 4.9:
 $\mathbf{N} = 2.2 > \mathbf{p} = 1.4 > -\sigma = 0.6$, $\mathbf{q} = 0.7 < \mathbf{q}_c = 0.75$

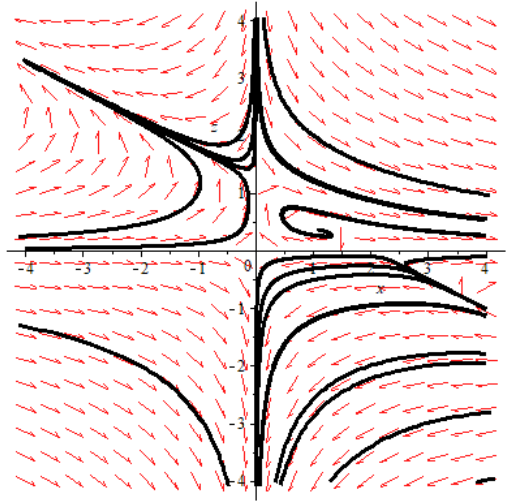


Figure 2, Theorem 4.9:
 $\mathbf{q}_c < \mathbf{q} = 0.85 < \mathbf{q}_S = 85/90$

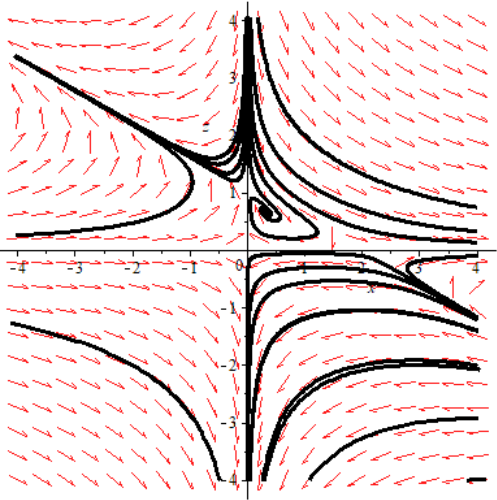


Figure 3, Theorem 4.9:
 $\mathbf{q} = 1 > \mathbf{q}_S = \frac{85}{90}$

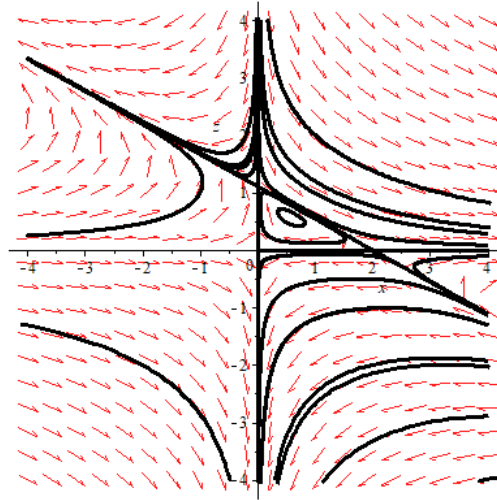


Figure 4, Theorem 4.9:
 $\mathbf{q} = \mathbf{q}_S = \frac{85}{90}$

Proof of Theorem 4.10. • The point N_0 is still as saddle point as in Theorem 4.9, corresponding to C^0 -regular solutions near 0.

• Here the point $M_0 = (\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}, 0)$ coincides with A_0 , see figure 5. The eigenvalues are $\mu_1 = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}$ and $\mu_2 = 0$, with eigenvectors are $\vec{v}_1 = (0, 1)$ and $\vec{v}_2 = (1, -(\mathbf{p} - 1))$. Corresponding to the eigenvalue 0, there exists a central manifold of dimension 1, tangent to \vec{v}_2 , invariant by the flow.

It contains trajectories converging to M_0 as $t \rightarrow \infty$ or $t \rightarrow -\infty$. The eigenspace associated to $\mu_1 > 0$ has dimension 1, so it is precisely the axis $z = 0$, and the trajectories are not admissible.

First assume $\varepsilon = 1$. In fact there exists an infinity of trajectories tangent to \vec{v}_2 , that means an infinity of central manifolds. The idea is the following: we know that there exists a trajectory \mathcal{T}_0 starting from N_0 and corresponding to C^0 -regular solutions; they are not global, w vanishes, which means that $s \rightarrow \infty$ and $z \rightarrow 0$ in finite time. The region \mathcal{R} in Q_1 delimited by \mathcal{T}_0 is invariant. The region $\mathcal{R} \cap \left\{ s \leq \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} \right\}$ is negatively invariant, since the field at $\left(\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}, z \right)$ satisfies $s_t = \frac{sz}{p-1} > 0$. Then any trajectory passing by a point of this region stays in it, and cannot converge to $(0, 0)$, so it converges to M_0 as $t \rightarrow -\infty$.

Setting $s = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} + \bar{s}$, we obtain the system

$$\begin{cases} \bar{s}_t = \left(\bar{s} + \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} \right) \left(\bar{s} + \frac{z}{\mathbf{p}-1} \right), \\ z_t = z(-\mathbf{q}\bar{s} - z). \end{cases}$$

Since the trajectory is tangent to \vec{v}_2 , there holds $z \sim -(\mathbf{p}-1)\bar{s}$, then $z_t \sim \frac{\mathbf{q}+1-\mathbf{p}}{\mathbf{p}-1}z^2$ and by integration $z \sim -\frac{\mathbf{p}-1}{(\mathbf{q}+1-\mathbf{p})t} = -\frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}+\sigma)t}$. Since $s \sim \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}$, we deduce the exact behaviour of w from (4.4).

Then assume $\varepsilon = -1$. There exist trajectories converging to M_0 as $t \rightarrow \infty$. Indeed up to a scaling there exist solutions w with a logarithmic behaviour as $r \rightarrow \infty$, from [38] for $\sigma = 0$, obtained by minimisation. and the construction extends to $\sigma \neq 0$. The exact behaviour follows as before. ■

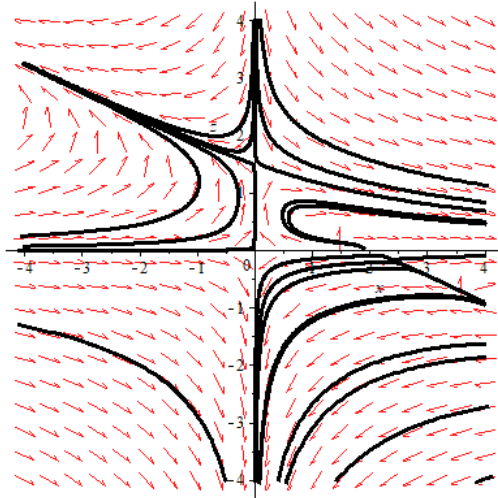


Figure 5, Theorem 4.10:

$$\mathbf{N} = 2.2 > \mathbf{p} = 1.4 > \sigma = -0.6, \mathbf{q} = \mathbf{q}_c = 0.75$$

Proof of Theorem 4.13. Here we consider region \mathbf{B} , where $\mathbf{p} > \mathbf{N} > \sigma$, see figure 6.

- The point M_0 is in Q_4 , since $s_0 = \gamma = \frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}} > 0$, and $z_0 = \mathbf{N} - \mathbf{p} - (\mathbf{p}-1)\gamma < \mathbf{N} - \mathbf{p} < 0$, and M_0 is a saddle point from Lemma 4.5. There exist four trajectories $\mathcal{T}_1^M, \mathcal{T}_2^M, \mathcal{T}_3^M, \mathcal{T}_4^M$, such that $\mathcal{T}_1^M, \mathcal{T}_2^M$ (resp. $\mathcal{T}_3^M, \mathcal{T}_4^M$) converge to M_0 as $t \rightarrow \infty$ (resp. $-\infty$). associated to $\lambda_1 < 0$ (resp.

$\lambda_2 > 0$) with a negative slope $\rho_1 = -(\mathbf{p}-1)(1 + \frac{|\lambda_1|}{s_0}) < -(\mathbf{p}-1)$ (resp. a positive slope). Then \mathcal{T}_1^M lies in the bounded region \mathcal{H} of Q_4 where $s_t < 0, z_t < 0$ for large t . Since this region is negatively invariant, the trajectory \mathcal{T}_1^M stays in \mathcal{H} , so it is bounded, hence defined on $(-\infty, \infty)$, and converges to $(0, 0)$ as $t \rightarrow -\infty$, since it is the only possible fixed point in $\overline{\mathcal{H}}$.

- The point N_0 is located at the boundary of Q_1 and Q_2 . The eigenvalues satisfy $l_2 < 0 < l_1$, then N_0 is a saddle point. There are two trajectories $\mathcal{T}'_1, \mathcal{T}'_2$, starting from N_0 , the first one in Q_1 and one in Q_2 , and the corresponding solutions satisfy (4.14) the solutions are not global, from Lemma 4.8, since in $\overline{Q_1}$ and $\overline{Q_2}$ there is not fixed point attracting at ∞ .

- The point A_0 is located at the boundary of Q_2 and Q_3 . The eigenvalues satisfy $\mu_1 < 0 < \mu_2$, then it is a saddle point. Two trajectories corresponding to μ_1 are ending at A_0 as $t \rightarrow \infty$ are located on the line $z = 0$ not admissible. Two trajectories corresponding to μ_2 are starting from A_0 at $-\infty$, a trajectory \mathcal{T}_2^A in Q_2 and a trajectory \mathcal{T}_3^A in Q_3 , with a negative slope $-(\mathbf{p}-1)\frac{\mu_2+|\mu_1|}{|\mu_1|}$. The corresponding solutions satisfy (4.15). They are not global, from Lemma 4.8, since in $\overline{Q_2}$ and $\overline{Q_3}$ there is no attracting point at ∞ .

- The point $(0, 0)$ is a source, since $\rho_1 > 0$ and $\rho_2 > 0$. So if $\rho_1 \neq \rho_2$, that means $\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} \neq \mathbf{N} + \sigma$, there exists an infinity of trajectories starting from $(0, 0)$. From Lemma 4.7 the corresponding solutions satisfy (4.16). If $\rho_1 = \rho_2$, we prove that there exist particular solutions at Proposition 4.17.

- In $\overline{Q_1}$ and $\overline{Q_3}$ there is no fixed point attracting at ∞ , so for $\varepsilon = 1$ there is no solution in (r_0, ∞) . ■

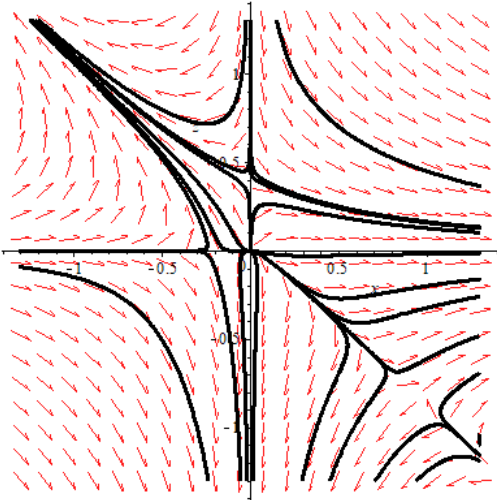


Figure 6: Theorem 4.13
 $\mathbf{p} = \frac{12}{7} > \mathbf{N} = \frac{11}{7} > \sigma = -\frac{1.82}{7}, \mathbf{q} = 1.3$

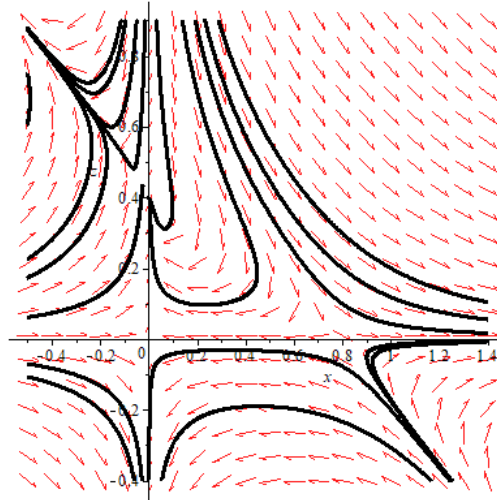


Figure 7: Theorem 4.15
 $\mathbf{p} = \frac{29}{19} < -\sigma = -\frac{29.4}{19} < \mathbf{N} = \frac{37}{19}, \mathbf{q} = 1.3$

Proof of Theorem 4.15. Here we consider region \mathbf{C} , where $\mathbf{p} < -\sigma < \mathbf{N}$, see figure 7.

- The point M_0 is in Q_2 . Indeed $s_0 = \gamma < 0$; and $z_0 = \mathbf{N} - \mathbf{p} + (\mathbf{p}-1)|\gamma| > 0$. It is a saddle point, and the eigenvalues satisfy $\lambda_1 < s_0 < \lambda_2$. So there exist four trajectories $\mathcal{T}_1^M, \mathcal{T}'_1^M, \mathcal{T}_2^M, \mathcal{T}'_2^M$, such that $\mathcal{T}_1^M, \mathcal{T}'_1^M$, (resp. $\mathcal{T}_2^M, \mathcal{T}'_2^M$) converge to M_0 as $t \rightarrow \infty$ (resp. $-\infty$). associated to $\lambda_1 < 0$

(resp. $\lambda_2 > 0$) with the slopes $\rho_i = (\mathbf{p} - 1)(\frac{\lambda_i}{s_0} - 1)$ $i = 1, 2$; hence $\rho_1 > 0 > \rho_2$, and we call $\mathcal{T}_1^M, \mathcal{T}_2^M$, the trajectories such that $s(t) > s_0$ for any $t \in \mathbb{R}$. In particular \mathcal{T}_2^M starts in the region J of Q_2 where $s_t > 0, z_t < 0$. We check easily that the region J is positively invariant. Then the trajectory \mathcal{T}_2^M stays in J , so it is bounded, and since s and z are monotone, we get that \mathcal{T}_2^M converges at ∞ to a fixed point in $\overline{Q_2}$, that is to N_0 if $\mathbf{N} + \sigma > 0$, and to $(0, 0)$ if $\mathbf{N} + \sigma < 0$.

- The point $(0, 0)$ admits the eigenvalues $\rho_1 = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} < 0$ and $\rho_2 = \mathbf{N} + \sigma > 0$, it is a saddle point, the trajectories issued from this point are the two axes, they are not admissible.

- The point N_0 is located at the boundary of Q_1 and Q_2 . The eigenvalues satisfy $l_1 < 0$ and $l_2 < 0$, so N_0 is a sink. In any case, all the corresponding solutions satisfy (4.17), from Lemma 4.7. In any case, we know that the trajectory \mathcal{T}_2^M converges to N_0 .

- The point $A_0 = (\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}, 0)$. is located at the boundary of Q_1 and Q_4 . It has the eigenvalues $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} > 0$ and

$$\mu_2 = \mathbf{N} + \sigma - \mathbf{q} \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} < \mathbf{N} + \sigma - (\mathbf{N} - \mathbf{p}) = \mathbf{p} + \sigma < 0,$$

so it is a saddle point. As we have done before, we check that there exist two trajectories converging to A_0 as $t \rightarrow \infty$, one in Q_1 and one in Q_4 . This show the existence of local solutions of the equations with $\varepsilon = \pm 1$, such that $\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}} w = k > 0$, satisfying (4.18). Clearly those solutions are not global, since there is no fixed point in $\overline{Q_1}$ and $\overline{Q_4}$ attracting at $-\infty$, from Lemma 4.8 So the corresponding functions w are only solutions of the exterior problem. ■

Proof of Theorem 4.17. Setting $s = \Phi z$, we get

$$\Phi_t = \Phi \left(\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} - (\mathbf{N} + \sigma) + \left(\frac{\mathbf{p}}{\mathbf{p} - 1} + (\mathbf{q} + 1)\Phi \right) z \right)$$

Then system admits particular solutions such that $s \equiv -\delta z$ for some $\delta > 0$ if and only if $\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} = \mathbf{N} + \sigma$, that means $\sigma = -\mathbf{p} \frac{\mathbf{N} - 1}{\mathbf{p} - 1}$, and $\delta = \frac{\mathbf{p}}{(\mathbf{p} - 1)(\mathbf{q} + 1)}$. They are given explicitly from the equation

$$s_t = s \left(\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} - \frac{\mathbf{q} - \mathbf{p} + 1}{\mathbf{p} - 1} s \right),$$

and from (4.2), $\delta \varepsilon r^\sigma w^{\mathbf{q} + 1} |w'|^{-\mathbf{p}} = -1$, hence $\varepsilon = -1$ and we get $w^{-\frac{\mathbf{q} + 1}{\mathbf{p}}} w' = \pm \delta^{\frac{1}{\mathbf{p}}} r^{-\frac{\mathbf{N} - 1}{\mathbf{p} - 1}}$, hence (4.19) by integration, with $d_{\mathbf{p}, \mathbf{q}, \mathbf{N}} = \frac{(\mathbf{p} - 1)(\mathbf{q} - \mathbf{p} + 1)}{\mathbf{p}(\mathbf{p} - \mathbf{N})} \left(\frac{\mathbf{p}}{(\mathbf{p} - 1)(\mathbf{q} + 1)} \right)^{\frac{1}{\mathbf{p}}}$. These solutions satisfy $\lim_{r \rightarrow 0} w = C$ when $C > 0$ and when they are global, $w \sim_{r \rightarrow \infty} w^*$.

If $\mathbf{p} = \mathbf{N}$, we get $\sigma = -\mathbf{p} = -\mathbf{N}$, and $w^{-\frac{\mathbf{q} + 1}{\mathbf{N}}} w' = \pm \delta^{\frac{1}{\mathbf{N}}} r^{-1}$, then $w^{1 - \frac{\mathbf{q} + 1}{\mathbf{N}}} = C \pm \delta^{\frac{1}{\mathbf{N}}} \left(\frac{\mathbf{q} + 1}{\mathbf{N}} - 1 \right) \ln r$, where $\delta = \frac{\mathbf{N}}{(\mathbf{N} - 1)(\mathbf{q} + 1)}$, that is $d_{\mathbf{N}} = \frac{\mathbf{q} + 1 - \mathbf{N}}{\mathbf{N}} \left(\frac{\mathbf{N}}{(\mathbf{N} - 1)(\mathbf{q} + 1)} \right)^{\frac{1}{\mathbf{N}}}$. ■

Proof of Theorem 4.18. Using Lemma 4.3, we reduce to the case (1) $\mathbf{N} > \mathbf{p} = -\sigma$, see figure 8.

- Here the point $M_0 = (0, \mathbf{N} - \mathbf{p})$ coincides with N_0 . The eigenvalues are $l_1 = 0$ and $l_2 = \mathbf{p} - \mathbf{N} < 0$. The corresponding eigenvectors are $\vec{v}_1 = (1, -\mathbf{q})$ and $\vec{v}_2 = (0, 1)$. Corresponding to the eigenvalue 0, there exists a central manifold of dimension 1, tangent to \vec{v}_1 , invariant by the flow.

It contains trajectories converging to M_0 as $t \rightarrow \infty$ or $t \rightarrow -\infty$. The eigenspace associated to $l_2 > 0$ has dimension 1, so it is precisely the axis $s = 0$, and the trajectories are not admissible. The system becomes

$$\begin{cases} s_t = s\left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1}\right), \\ z_t = z(\mathbf{N} - \mathbf{p} - \mathbf{q}s - z). \end{cases}$$

- The point $A_0 = \left(\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}, 0\right)$ is located at the boundary of Q_1 and Q_4 . The eigenvalues are $\mu_1 = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} > 0$ and $\mu_2 = (\mathbf{q} - \mathbf{p} + 1)\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} < 0$, with eigenvectors. $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (1, -(\mathbf{p} - 1)(\mathbf{q} - \mathbf{p} + 2))$. Two trajectories associated to A_0 are located on the line $\{z = 0\}$, so they are not admissible. There exist two other trajectories, \mathcal{T}_1 and \mathcal{T}_4 , respectively in Q_1 and Q_4 , converging to A_0 at ∞ . The corresponding solutions w satisfy $\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0$, from Lemma 4.7, for $\varepsilon = \pm 1$.

- First consider the trajectories located in Q_1 , corresponding the case $\varepsilon = 1$. The trajectory \mathcal{T}_1 has a slope less than $-(\mathbf{p} - 1)$, then it arrives as $t \rightarrow \infty$ in the region $\mathcal{R} = \{s_t > 0\}$ in Q_1 delimited by the line N_0A_0 where $s_t = 0$; and \mathcal{R} is negatively invariant, so \mathcal{T}_1 stays in it. The line $\{z_t = 0\}$ is located under the line M_0A_0 , thus $z_t < 0$ in \mathcal{R} . If \mathcal{T}_1 converges to M_0 , then the region \mathcal{R}_1 delimited by the two axis and \mathcal{T}_1 is bounded with no fixed point inside, and this is contradictory. Then \mathcal{T}_1 is asymptotic to the axis $\{s = 0\}$, and the solutions are not global. Consider again the region \mathcal{R}_1 . Then any trajectory passing by a point of \mathcal{R}_1 is bounded, and then converges to M_0 as $t \rightarrow \infty$, so we get a (2 parameters) family of solutions w in (r_0, ∞) such that (4.21) holds. Indeed the trajectory is tangent to line of direction $(1, -\mathbf{q})$ passing by M_0 , which means $\bar{z} \sim -qs$, and we have

$$\begin{cases} s_t = s\left(s + \frac{\bar{z}}{\mathbf{p}-1}\right), \\ z_t = (\mathbf{N} - \mathbf{p} + \bar{z})(-\bar{z} - \mathbf{q}s). \end{cases}$$

Then $s_t \sim -\frac{\mathbf{q}}{\mathbf{p}-1}s^2$, and by integration $s \sim \frac{\mathbf{p}-1}{\mathbf{q}t}$, and $z \sim \mathbf{N} - \mathbf{p}$, thus by (4.4), $w \sim \left(\frac{\mathbf{p}-1}{\mathbf{q}t}\right)^{\frac{\mathbf{p}-1}{\mathbf{q}+1-\mathbf{p}}}$, which is precisely (4.21).

- Next consider the trajectories located in Q_2 , corresponding the case for $\varepsilon = -1$. The line $\{s_t = 0\} = M_0A_0$ is located under the line $\{z_t = 0\}$ and the region \mathcal{G} located between the two lines is negatively invariant. Any trajectory passing by a point above $\{z_t = 0\}$ necessarily crosses this line since z and s are decreasing, and any trajectory passing by a point under $\{s_t = 0\}$ cuts this line, because s and z are increasing. We consider two sets of trajectories: \mathcal{U}_1 is the union of trajectories which cut $\{z_t = 0\}$ at some point, and \mathcal{U}_2 is the union of trajectories which cut $\{s_t = 0\}$ at some point. They are open in Q_2 , so $\mathcal{U}_1 \cup \mathcal{U}_2 \neq Q_2$. Then there exists some point P such that the trajectory passing by P is located in \mathcal{G} , which is negatively invariant. Then it converges to the point M_0 as $t \rightarrow -\infty$. So we get solutions w satisfying (4.22). ■

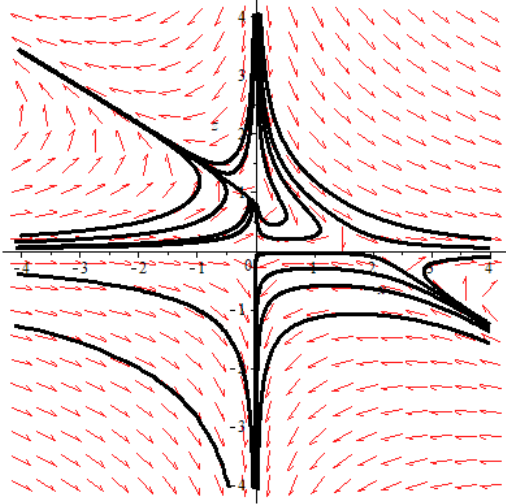


Figure 8: Theorem 4.18
 $\mathbf{N} = 2.2 > \mathbf{p} = -\sigma = 1.4, \mathbf{q} = 1.1$

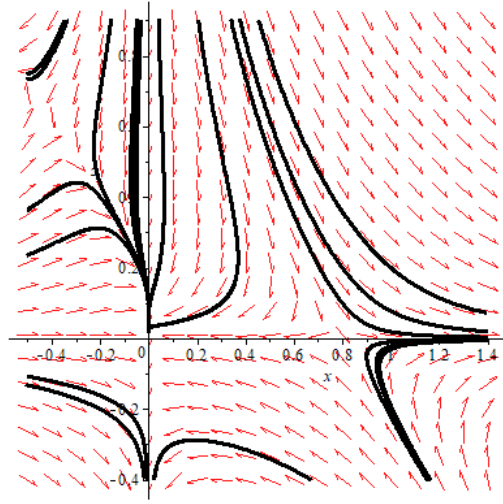


Figure 9: Theorem 4.19
 $-\sigma = \mathbf{N} = \frac{37}{19} > \mathbf{p} = \frac{29}{19}, \mathbf{q} = 1.5$

Proof of Theorem 4.19. We still reduce to the case (1) $-\sigma = \mathbf{N} > p$, see figure 9. Here the system is

$$\begin{cases} s_t = s\left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1}\right), \\ z_t = z(-\mathbf{q}s - z). \end{cases} \quad (8.1)$$

- The point $M_0 = (s_0, z_0) = \left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{q}+1-\mathbf{p}}, \frac{(\mathbf{N}-\mathbf{p})\mathbf{q}}{\mathbf{q}+1-\mathbf{p}}\right)$ is in Q_2 , corresponding to particular solutions of the equation with absorption $\varepsilon = -1$. $w^* = a^* r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}$; note that $\lim_{r \rightarrow 0} w^* = 0$.

- The point N_0 coincides with $(0, 0)$; the eigenvalues are $\rho_1 = \frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}, \rho_2 = 0$, corresponding eigenvectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (0, 1)$. The trajectories associated to ρ_1 are not admissible. There exist at least one central manifold, relative to ρ_2 , tangent to the axis $\{s = 0\}$. We are going to precise it below:

The eigenvalues at point M_0 are given by (4.12), their product is $-(\mathbf{N} - \mathbf{p})^2 \mathbf{q} < 0$, so M_0 is a saddle point. There exist four trajectories $\mathcal{T}_1^M, \mathcal{T}_2^M, \mathcal{T}_3^M, \mathcal{T}_4^M$, such that $\mathcal{T}_1^M, \mathcal{T}_2^M$ (resp. $\mathcal{T}_3^M, \mathcal{T}_4^M$) converge to M_0 as $t \rightarrow \infty$ (resp. $-\infty$). associated to $\lambda_1 < 0$ (resp. $\lambda_2 > 0$) with the slope $\rho_1 = (\mathbf{p} - 1)\left(\frac{\lambda_1}{s_0} - 1\right) > 0$ (resp. $\rho_2 = (\mathbf{p} - 1)\left(-\frac{\lambda_2}{|s_0|} - 1\right) < -(\mathbf{p} - 1)$). Then one of the two last trajectories, denoted by \mathcal{T}_3^M lies in the bounded region \mathcal{H} of Q_2 where $s_t > 0, z_t < 0$ for large t . Since this region is positively invariant, the trajectory \mathcal{T}_3^M stays in \mathcal{H} , so it is bounded, hence converges to $(0, 0)$ which is the only possible fixed point in $\bar{\mathcal{H}}$. Then any trajectory passing by some point of \mathcal{H} converges to $(0, 0)$. So there is an infinity of trajectories converging to this point, staying in Q_2 .

- The point A_0 is located at the boundary of Q_1 and Q_4 . Moreover the eigenvalues at A_0 are $\mu_1 = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} > 0$ and $\mu_2 = \mathbf{q} \frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} < 0$, A_0 is a saddle point. Two trajectories associated to μ_1 are located on the line $z = 0$ are not admissible. There are trajectories $\mathcal{T}_1, \mathcal{T}_4$, respectively in Q_1 and Q_4 converging to A_0 as $t \rightarrow \infty$ corresponding to the eigenvector $\vec{v}_2 = \left(\frac{\mu_1}{\mathbf{p}-1}, \mu_2 - \mu_1\right)$, with the slope $-(\mathbf{p} - 1)(\mathbf{q} + 1)$, the solutions are not global, since there is no fixed point in those quadrants.

Then there exist solutions in (r_0, ∞) for $\varepsilon = \pm 1$, such that $\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0$. Moreover, since $(\mathbf{p}-1)(\mathbf{q}+1) > (\mathbf{p}-1)$, \mathcal{T}_1 is located in the region of Q_1 where $s_t > 0$ and $z_t < 0$, which is negatively invariant, so it stays in it. Next consider the region \mathcal{K} in Q_1 delimited by the two axis and \mathcal{T}_1 (under \mathcal{T}_1). It is invariant, and $z_t < 0$ in \mathcal{K} . Then consider any trajectory \mathcal{T} passing by a point of \mathcal{K} . It cannot converge to A_0 because \mathcal{T}_1 is the only trajectory in Q_1 converging to this point. Then \mathcal{T} necessarily converges to $(0, 0)$. There still exists an infinity of trajectories converging to this point, staying in Q_1 .

Thus in Q_1 (corresponding to $\varepsilon = 1$) as well as in Q_2 , (corresponding to $\varepsilon = -1$) there is an infinity of trajectories converging to $(0, 0)$. From system (8.1), since $\lim_{t \rightarrow \infty} \frac{s}{z} = 0$, we get $s_t \sim s(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1})$, and $z_t \sim -z^2$. Then by integration, $z \sim \frac{1}{t}$. $|s| \leq C e^{-kt}$ for some $k > 0$. From (4.2) we get $w^{\frac{\mathbf{q}}{1-\mathbf{p}}} w' = (-\frac{\mathbf{p}-1}{\mathbf{q}+1-\mathbf{p}} w^{-\frac{\mathbf{q}+1-\mathbf{p}}{\mathbf{p}-1}})' \sim_{r \rightarrow \infty} -\varepsilon r^{\frac{1-\mathbf{N}}{\mathbf{p}-1}} (\ln r)^{\frac{1}{\mathbf{p}-1}}$, then by integration, either

$$w \sim -\varepsilon C (\ln r)^{-\frac{1}{\mathbf{q}+1-\mathbf{p}}} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}, \quad w' \sim -\varepsilon C r^{\frac{\mathbf{q}}{\mathbf{p}-1}} (\ln r)^{-\frac{1}{\mathbf{q}+1-\mathbf{p}}} r^{\frac{\mathbf{N}-1-\mathbf{q}}{\mathbf{q}+1-\mathbf{p}}}, \quad C = \left(\frac{\mathbf{q}+1-\mathbf{p}}{\mathbf{N}-\mathbf{p}} \right)^{\frac{\mathbf{p}-1}{\mathbf{q}+1-\mathbf{p}}},$$

or

$$\lim_{r \rightarrow \infty} w = C > 0, \quad w' \sim -\varepsilon r^{\frac{1-\mathbf{N}}{\mathbf{p}-1}} (\ln r)^{\frac{1}{\mathbf{p}-1}}.$$

The first eventuality is impossible for $\varepsilon = 1$. Let us check if it is possible for $\varepsilon = -1$: it would imply that $s = -r \frac{w'}{w}$ tends to $\mathbf{p} - \mathbf{N} \neq 0$, which is contradictory. Then for $\varepsilon = -1$, for given $r_0 > 0$ there exists an infinity (with 2 parameters) of solutions w in (r_0, ∞) satisfying (??), and an infinity (with one parameter) of solutions satisfying (4.25). \blacksquare

Proof of Theorem 4.20. Here again we reduce to the case (1) $\mathbf{p} = \mathbf{N} > -\sigma$, see figure 10. The system reduces to

$$\begin{cases} s_t = s(s + \frac{z}{\mathbf{p}-1}), \\ z_t = z(\mathbf{p} + \sigma - \mathbf{q}s - z). \end{cases}$$

- The point $M_0 = (s_0, z_0) = (\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}, -\frac{(\mathbf{p}-1)(\mathbf{p}+\sigma)}{\mathbf{q}+1-\mathbf{p}})$ is in Q_4 , corresponding to particular solutions of the equation with absorption $\varepsilon = -1$. $w^* = a^* r^{-\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}}$, which are ∞ -singular.

- The point $N_0 = (\mathbf{p} + \sigma, 0)$ is located at the boundary of Q_1 and Q_2 and admits the eigenvalues $l_1 = \frac{\mathbf{p}+\sigma}{\mathbf{p}-1} > 0$ and $l_2 = -(\mathbf{p} + \sigma) < 0$, with eigenvectors $\vec{v}_1 = (\frac{\mathbf{p}}{\mathbf{p}-1}, -\mathbf{q})$ and $\vec{v}_2 = (0, 1)$. it is a saddle point; the trajectories associated to l_2 are not admissible. There exists two trajectories, \mathcal{T}_1 and \mathcal{T}_2 , respectively in Q_1 and Q_2 starting from N_0 at $-\infty$, corresponding to C^0 -regular solutions w . The trajectory \mathcal{T}_2 starts in the region where $s_t < 0, z_t > 0$, which is positively invariant, so it stays in it. The trajectory \mathcal{T}_1 starts in the region where $s_t > 0, z_t < 0$, with a slope $-\mathbf{q} \frac{\mathbf{p}-1}{\mathbf{p}}$, above the line $N_0 M_0$, and stays in the region of Q_1 where $z_t < 0$, and then it is asymptotic to the axis $\{z = 0\}$.

- Here the point A_0 coincides with $(0, 0)$, and the eigenvalues are $\rho_1 = 0, \rho_2 = \mathbf{N} + \sigma$, with eigenvectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (0, 1)$. The trajectories associated to ρ_2 are not admissible. There exist at least one central manifold, relative to ρ_1 , tangent to the axis $\{z = 0\}$. We are going to precise it below.

The eigenvalues at point M_0 are given by (4.12), their product is $s_0 z_0 < 0$, then $\lambda_1 < 0 < s_0 < \lambda_2$, so M_0 is a saddle point, There exist four trajectories $\mathcal{T}_1^M, \mathcal{T}_2^M, \mathcal{T}_3^M, \mathcal{T}_4^M$, such that $\mathcal{T}_1^M, \mathcal{T}_2^M$

(resp. $\mathcal{T}_3^M, \mathcal{T}_4^M$) converge to M_0 as $t \rightarrow \infty$ (resp. $-\infty$). associated to $\lambda_1 < 0$ (resp. $\lambda_2 > 0$) with the slope $\rho_1 = (\mathbf{p} - 1)(\frac{\lambda_1}{s_0} - 1) < 0$ (resp. $\rho_2 = (\mathbf{p} - 1)(\frac{\lambda_2}{s_0} - 1) > (\mathbf{p} - 1)$). Then one of the two first trajectories, denoted by \mathcal{T}_1^M lies in the bounded region \mathcal{H} of Q_4 where $s_t > 0, z_t < 0$ for large t . Since this region is negatively invariant, the trajectory \mathcal{T}_1^M stays in \mathcal{H} , so it is bounded, hence converges to $(0, 0)$ as $t \rightarrow -\infty$ which is the only possible fixed point in $\overline{\mathcal{H}}$. Moreover any trajectory passing by some point of \mathcal{H} converges to $(0, 0)$. So there is an infinity of trajectories converging to this point as $t \rightarrow \infty$, staying in Q_4 .

Next consider the other quadrants Q_1, Q_2 . There is no trajectory converging to $(0, 0)$ in Q_2 . Indeed suppose that a trajectory converges to $(0, 0)$ in Q_2 as $t \rightarrow \infty$; then it satisfies $s_t > 0, z_t < 0$, near ∞ which is impossible; if it converges as $t \rightarrow -\infty$, then $s_t < 0, z_t > 0$, then it cannot be tangent to the axis $\{z = 0\}$.

Finally consider the quadrant Q_1 . The region \mathcal{K} delimited by the two axis and the line N_0M_0 is negatively invariant. Then any trajectory passing by a point of \mathcal{K} converges to $(0, 0)$ at $-\infty$. Moreover the region \mathcal{L} delimited by the two axis and \mathcal{T}_1 is also invariant. Then any trajectory passing by a point of \mathcal{L} converges to $(0, 0)$ at $-\infty$. Thus there is an infinity of central manifolds.

Those trajectories satisfy $\lim_{t \rightarrow -\infty} \frac{z}{s} = 0$, then $s_t \sim s^2$. By integration, $0 < s = -\frac{w_t}{w} \sim \frac{1}{|t|}$. Thus for given $\varepsilon > 0$, there holds $C_1 |t|^{(1+\varepsilon)} \leq w \leq C_2 |t|^{(1+\varepsilon)}$ near $-\infty$. Since $\mathbf{p} = \mathbf{N}$, equation (4.1) reduces to

$$-(N-1)|w_t|^{N-2} w_{tt} = \varepsilon e^{(\mathbf{N}+\sigma)t} w^q.$$

Then

$$-(\mathbf{N}-1)|w_t|^{N-2-\mathbf{q}} w_{tt} = \left(\frac{\mathbf{N}-1}{\mathbf{q}-\mathbf{N}+1} |w_t|^{N-1-\mathbf{q}}\right)_t \sim \varepsilon e^{(\mathbf{N}+\sigma)t} |t|^q,$$

hence by integration, either $\lim_{t \rightarrow -\infty} w_t = -c < 0$, or $\frac{\mathbf{N}-1}{\mathbf{q}-\mathbf{N}+1} |w_t|^{N-1-\mathbf{q}} \sim -\frac{\varepsilon}{\mathbf{N}+\sigma} e^{(\mathbf{N}+\sigma)t} |t|^q$. The last case is impossible for $\varepsilon = 1$. If $\varepsilon = -1$, since $\frac{|w_t|}{w} \sim \frac{1}{|t|}$, it implies that $w^{-\frac{\mathbf{q}}{\mathbf{N}-1}} |w_t| = \left(\frac{\mathbf{N}-1}{\mathbf{q}-\mathbf{N}+1} w^{-\frac{\mathbf{q}-\mathbf{N}+1}{\mathbf{N}-1}}\right)_t \sim C e^{\frac{\mathbf{N}+\sigma}{\mathbf{N}-1}t}$, which by integration contradicts the estimate $C_1 |t|^{(1+\varepsilon)} \leq w \leq C_2 |t|^{(1+\varepsilon)}$. Then $w \sim_{r \rightarrow 0} C |\ln r|$, and $w' \sim_{r \rightarrow 0} -\frac{C}{r}$. \blacksquare

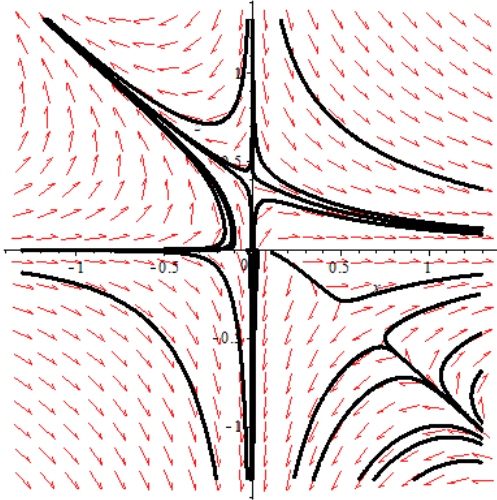


Figure 10: Theorem 4.20
 $\mathbf{p} = \mathbf{N} = \frac{5}{3} > -\sigma = \frac{19}{15}, \mathbf{q} = 1.3$

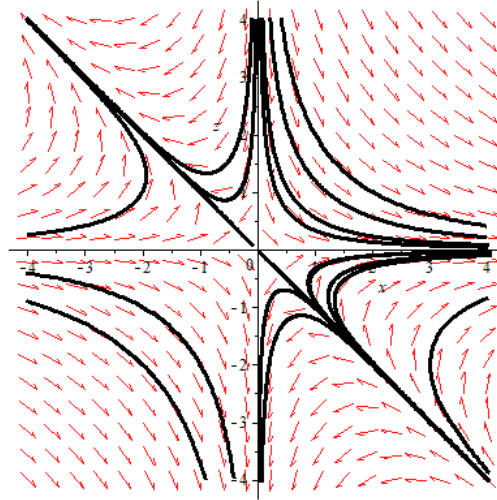


Figure 11: Theorem 4.21
 $\mathbf{p} = \mathbf{N} = -\sigma = \frac{5}{3}, \mathbf{q} = 1.5$

Proof of Theorem 4.21. Here $\mathbf{p} = \mathbf{N} = -\sigma$, see figure 11. The system reduces to

$$\begin{cases} s_t = s(s + \frac{z}{\mathbf{N}-1}), \\ z_t = z(-\mathbf{q}s - z). \end{cases}$$

The unique fixed point is $(0, 0)$, and the two eigenvalues are 0. From Proposition 4.17, there exist solutions such that $s \equiv -\frac{\mathbf{N}}{(\mathbf{N}-1)(\mathbf{q}+1)}z$, corresponding to trajectories in Q_2 and Q_4 . We get a family of solutions w given by (4.20). The sign $+$ (resp. $-$) corresponds to the trajectory in Q_4 (resp. Q_2). It seems that there is no other trajectory converging to $(0, 0)$, thus no other solution w can be defined near 0, or near ∞ .

For $\varepsilon = 1$, corresponding to quadrants Q_1 and Q_3 all the trajectories satisfy $s_t z_t < 0$, hence they cannot converge to the unique fixed point $(0, 0)$ as $t \rightarrow \pm\infty$, from Lemma 4.8. Therefore there is no local solution near 0 or ∞ . ■

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