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# Elliptic Hamilton-Jacobi systems and Lane-Emden Hardy-Hénon equations

Marie-Françoise Bidaut-Véron      Marta Garcia Huidobro

September 22, 2022

## Abstract

Here we study the solutions of any sign of the system

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p, \\ -\Delta u_2 = |\nabla u_1|^q, \end{cases}$$

in a domain of  $\mathbb{R}^N$ ,  $N \geq 3$  and  $p, q > 0$ ,  $pq > 1$ . We show their relation with Lane-Emden Hardy-Hénon equations

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = \varepsilon r^\sigma w^q, \quad \varepsilon = \pm 1,$$

where  $u \mapsto \Delta_{\mathbf{p}}^{\mathbf{N}} u$  ( $\mathbf{p} > 1$ ) is the  $\mathbf{p}$ -Laplacian in dimension  $\mathbf{N}$ ,  $\mathbf{q} > \mathbf{p} - 1$  and  $\sigma \in \mathbb{R}$ . This leads us to explore these equations in not often tackled ranges of the parameters  $\mathbf{N}, \mathbf{p}, \sigma$ . We make a complete description of the radial solutions of the system and of the Hardy-Hénon equations and give nonradial a priori estimates and Liouville type results.

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# 1 Introduction

Here we study the solutions of any sign of the Hamilton-Jacobi type system

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p, \\ -\Delta u_2 = |\nabla u_1|^q, \end{cases} \quad (1.1)$$

in a domain  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 3$ , and

$$p, q > 0, \quad pq > 1, \quad \text{and we can assume } p \geq q,$$

so that  $p > 1$ . Our purpose is to describe the behaviour and the existence of solutions when  $\Omega = B_{r_0} \setminus \{0\}$ ,  $\mathbb{R}^N \setminus \{0\}$ ,  $\mathbb{R}^N \setminus \overline{B_{r_0}}$  or  $\mathbb{R}^N$ . Note that  $(u_1, u_2)$  is a solution, then  $(u_1 + C_1, u_2 + C_2)$  is a solution, in particular the constants are solutions. The study of the radial solutions is fundamental for the understanding of the system. As shown below, it appears that they are governed by the solutions of Lane-Emden Hardy-Hénon equations

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = \pm 1, \quad (1.2)$$

where  $u \mapsto \Delta_{\mathbf{p}}^{\mathbf{N}} u$  ( $\mathbf{p} > 1$ ) is the  $\mathbf{p}$ -Laplacian in dimension  $\mathbf{N}$ ,  $\mathbf{q} > \mathbf{p} - 1$  and  $\sigma \in \mathbb{R}$ . This justifies to make the point on the actual knowledge of these equations, and make a complete study in any range of the parameters  $\mathbf{N}, \mathbf{p}, \sigma$ .

In the last decades, a great number of elliptic systems deal on positive solutions of semilinear or quasilinear, with source terms in the right hand side, involving powers of the solutions. Our study is motivated by the well known Lane-Emden system

$$\begin{cases} -\Delta u_1 = |x|^a u_2^p, \\ -\Delta u_2 = |x|^b u_1^q, \end{cases} \quad (1.3)$$

which has developed an extremely rich literature, starting from the conjecture of nonexistence of solutions in  $\mathbb{R}^N$  for  $a = b = 0$  when

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$

first proved in the radial case for any  $N \geq 3$  in [30], obtained in of [36] for  $N = 3$ , extended to  $N = 4$  in [39], and still open for  $N \geq 5$ . The existence of solutions in  $\mathbb{R}^N \setminus \{0\}$  with a possible singularity at the origin, or in an exterior domain  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ , and the question of a priori estimates has been developed in many works, with extensions to quasilinear operators, with possible weight functions, or to analogous problems on manifolds, and it is impossible to quote all of them. Let us also mention a significant amount of results of non existence of supersolutions in  $\mathbb{R}^N$ , started by [31], [12].

In contrast, only few results are available in the literature for elliptic systems with gradient terms. A general study of nonexistence of positive supersolutions of systems with source terms is given in [22]. We can also mention some works on more specific systems in [23], [25], [4], [18]. Up to our knowledge the study system (1.1) began very recently, with the publication of [20], where the existence of positive singular solutions of the Dirichlet problem in  $\Omega \setminus \{0\}$  was proved when  $\Omega$  is a small perturbation of a ball of center 0, and the work of [3], where existence and non existence results are given for the system with forcing terms.

In the sequel, we distinguish three types of solutions: the positive ones, solutions of (1.1), with source type gradient terms

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p, \\ -\Delta u_2 = |\nabla u_1|^q, \end{cases} \quad u_1, u_2 \geq 0, \quad (1.4)$$

the negative ones, equivalently, setting  $\tilde{u}_1 = -u_1, \tilde{u}_2 = -u_2$ , the solutions of a system with absorption terms

$$\begin{cases} -\Delta \tilde{u}_1 + |\nabla \tilde{u}_2|^p = 0, \\ -\Delta \tilde{u}_2 + |\nabla \tilde{u}_1|^q = 0, \end{cases} \quad \tilde{u}_1, \tilde{u}_2 \geq 0, \quad (1.5)$$

and the mixed ones, equivalently setting  $\widehat{u}_1 = u_1, \widehat{u}_2 = -u_2$ , the solutions of the mixed type system

$$\begin{cases} -\Delta \widehat{u}_1 = |\nabla \widehat{u}_2|^p, \\ \Delta \widehat{u}_2 = |\nabla \widehat{u}_1|^q, \end{cases} \quad \widehat{u}_1, \widehat{u}_2 \geq 0. \quad (1.6)$$

In Section 2 we study the existence of particular solutions of system (1.1). We briefly mention the main results concerning the scalar case of the well known Hamilton-Jacobi equation

$$-\Delta u = |\nabla u|^q, \quad (1.7)$$

where  $q > 1$ . The existence of particular positive or negative radial solutions  $u^*(r) = A^* r^{\frac{q-2}{q-1}}$ ,  $r = |x|$ , for  $q \neq 2$ , where  $A^*$  has the sign of  $(2-q)((N-1)q-N)$ , puts in evidence two critical values  $q = \frac{N}{N-1}$  and  $q = 2$ . Concerning system (1.1), these two critical values are replaced by

four critical conditions linking the parameters, namely  $q = q_i = q_i(p)$ ,  $i = 1, 2, 3, 4$ , defined by the relations

$$(N-1)pq_1 = N + q_1, \quad (N-1)pq_2 = N + p, \quad q_3(p-1) = 2, \quad p(q_4-1) = 2. \quad (1.8)$$

In the problem, moreover another value is involved, defined by the relation

$$(N-1)pq^* = N + \frac{p+q^*}{2}, \quad (1.9)$$

corresponding to the Sobolev exponent relative to equation (1.2).

In Section 3 we study the radial signed solutions of system (1.1). We show that they can be completely described. Indeed it is invariant by the scaling transformation  $T_\ell, \ell > 0$ , defined by

$$T_\ell [(u_1, u_2)](x) = (l^{\frac{p(q-1)-2}{pq-1}} u_1(\ell x), l^{\frac{q(p-1)-2}{pq-1}} u_2(\ell x)),$$

and thus the radial case reduces to an autonomous system of order 4. Due to the particular form of system (1.1), which involves only the gradients of the functions, we can reduce to a system of order 2. The radial study offers a double interest:

- *The radial system reduces to a quadratic Lotka-Volterra type system:*

$$\begin{cases} S_t = S(N - (N-1)p + S + pZ), \\ Z_t = Z(N - (N-1)q - qS - Z). \end{cases}$$

where

$$S = r \frac{|u_2'|^p}{u_1'} = -r \frac{(r^{N-1}u_1')'}{r^{N-1}u_1'}, \quad Z = -r \frac{|u_1'|^q}{u_2'} = -r \frac{(r^{N-1}u_2')'}{r^{N-1}u_2'}, \quad t = \ln r.$$

Moreover system (1.1) is governed by the Hardy-Hénon equations (1.2) in the following way:

- *For any radial solution  $(u_1, u_2)$  of system (1.1), the function  $w = r^{N-1} |u_1'|$  satisfies equation (1.2) with the specific values*

$$\mathbf{N} = 1 + \frac{(N-1)(p-1)}{p}, \quad \mathbf{p} = 1 + \frac{1}{p}, \quad \mathbf{q} = q, \quad \sigma = (N-1) \frac{1-pq}{p}, \quad \varepsilon = -\text{sign}(u_1' u_2').$$

As an unexpected and remarkable fact, the system (1.4), which presents two *source* terms  $|\nabla u_2|^p, |\nabla u_1|^q$ , is not linked with a Hardy-Hénon equation with a source term ( $\varepsilon = 1$ ) but an equation with an *absorption* term ( $\varepsilon = -1$ ), see Remark 3.12.

Section 4 is devoted to a complete study of the radial local and global positive solutions of the Hardy-Hénon equations (1.2). They are well known in the case  $\mathbf{N} > \mathbf{p} > -\sigma$ , and we cannot quote the immense literature on the subject. The main purposes of the articles are the obtention of Liouville type results, symmetry properties and a priori estimates for the equation with source ( $\varepsilon = 1$ ), and the study of large solutions for the equation with absorption ( $\varepsilon = -1$ ). The other ranges of the parameters  $\mathbf{N}, \mathbf{p}, \sigma$ , such as  $\mathbf{p} > \mathbf{N}$  or  $\sigma < -\mathbf{p}$ ,  $\sigma < -\mathbf{N}$  seem to be less considered. Let us mention the nonexistence of possibly nonradial solutions, even in a very weak sense, proved

in the pioneer paper [15] for  $\varepsilon = 1$ ,  $\mathbf{N} \geq 3$ ,  $\mathbf{p} = 2$ ,  $\sigma \leq -2$ ; see also [24] for an extension to any extension to any  $\mathbf{p} > 1$ ,  $\sigma \leq -\max(\mathbf{N}, \mathbf{p})$  under some regularity assumptions.

Here we treat all the cases of distinct  $\mathbf{N}, \mathbf{p}, \sigma$  at Theorems 4.13, 4.12, 4.14, 4.15, 4.16. The limit cases  $\sigma = -\mathbf{p}$ ,  $\sigma = -\mathbf{N}$ ,  $\mathbf{p} = \mathbf{N}$ , where logarithmic type solutions appear, are described at Theorems 4.18, 4.19, 4.20, and the critical case  $\mathbf{p} = \mathbf{N} = -\sigma$  at Theorem 4.21. Independently of their application to system (1.1), they offer a remarkable diversity of types of behaviour. We hope that they constitute a solid basis for a future nonradial analysis.

Our study is based on the analysis of the phase plane of the associated quadratic Lotka-Volterra system,

$$\begin{cases} s_t = s\left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1}\right), \\ z_t = z(\mathbf{N} + \sigma - \mathbf{q}s - z), \end{cases}$$

where  $s = -r\frac{w'}{w}$  and  $z = -\varepsilon r^{1+\sigma} w^{\mathbf{q}} |w'|^{-\mathbf{p}} w'$ , already started in [10]. We observe that this system is neither competitive, nor cooperative as soon as  $(\mathbf{p} - 1)\mathbf{q} > 0$ . Note also that our study gives local existence results without involving fixed point methods, and global ones without introducing energy functions. Among all the results, let us mention a few striking examples:

- For  $\mathbf{p} > \mathbf{N} > -\sigma$  and  $\varepsilon = \pm 1$ , we get existence of solutions in  $C^0(B_{r_0}) \cap C^2(B_{r_0} \setminus \{0\})$  of three types : such that  $\lim_{r \rightarrow 0} w = c > 0$ , and either  $\lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = d \neq 0$  or  $\lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = d \neq 0$ , and solutions such that  $\lim_{r \rightarrow 0} w = 0$  and  $\lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0$ .

- For  $-\sigma > \max(\mathbf{p}, \mathbf{N})$ , there exist solutions in an exterior domain  $\mathbb{R}^N \setminus \overline{B_{r_0}}$  such that  $\lim_{r \rightarrow \infty} w = c > 0$ , see Theorems 4.12, 4.14, 4.20.

- It is well known that for  $\mathbf{N}, \mathbf{p} > -\sigma$  there exist explicit solutions in  $C^0(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ , often called ground states, when  $q$  is the critical Sobolev exponent  $q_S = \frac{\mathbf{N}(\mathbf{p}-1) + \mathbf{p} + \mathbf{p}\sigma}{\mathbf{N}-\mathbf{p}}$ , given by  $w = c(d + r^{\frac{\mathbf{p}+\sigma}{\mathbf{p}-1}})^{-\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}+\sigma}}$ , where  $c > 0$  and  $d = d(c, \mathbf{N}, \mathbf{p}, \sigma)$ . In fact the same phenomena holds when  $-\sigma > \mathbf{p} > \mathbf{N}$  (see Theorem 4.12).

- For  $\varepsilon = -1$ ,  $\sigma = -\frac{\mathbf{p}(\mathbf{N}-1)}{\mathbf{p}-1}$ , in particular when  $\mathbf{p} = \mathbf{N} = -\sigma$ , there also exists a family of explicit solutions, see Proposition 4.17.

For convenience the proofs of the main results of Section 4, using various techniques of dynamical systems, are given in the Appendix.

In Section 5 we describe the behaviour of the radial solutions of system (1.1), from the results of Section 4. Because of the great diversity of the possible solutions, we concentrate our study on constant sign solutions in  $B_{r_0} \setminus \{0\}$  or  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ , and above all on global solutions in  $\mathbb{R}^N \setminus \{0\}$ .

Comparing to the scalar case, the situation is much more intricate: even in the case  $p = q$ , where we show that the system can be completely integrated, there exists an infinity of non symmetrical solutions. In the general case  $p \geq q$  we give all the local and global nonconstant solutions of systems (1.4), (1.5) and (1.6). It appears that each component  $u_1, u_2$  solution in  $(0, \infty)$  can admit different kinds of singularities near 0: either  $\lim_{r \rightarrow 0} |u_i| = \infty$ , and we say that  $u_i$  is  *$\infty$ -singular*, or if  $\lim_{r \rightarrow 0} u_i = c \in \mathbb{R}$  and  $\lim_{r \rightarrow 0} |u'_i| = \infty$ , and we say that  $u_i$  is a *cusps-solution*. Concerning (1.4), we get the following, from Theorems 5.9, 1.4 and 5.15:

**Theorem 1.1** *Let  $pq > 1, p \geq q$ , and let  $q_1, q_2, q_3, q_4$  be defined in (1.8). Consider the system (1.4).*

(1) *Local solutions: There exists radial solutions in  $B_{r_0} \setminus \{0\}$  such that*

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 \delta_0, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 \delta_0, \end{cases}$$

*with  $C_1, C_2 > 0$  when  $p < \frac{N}{N-1}$ , with  $C_1 = 0, C_2 > 0$  when  $q < q_1$ , with  $C_1 > 0, C_2 = 0$  when  $q < q_2$ .*

(2) *Global solutions: up to additive constants,*

• *For  $q_2 < q < q_3$ , there exists a  $\infty$ -singular solution  $(u_1^*, u_2^*) = (A_1^* r^{\frac{p(q-1)-2}{pq-1}}, A_2^* r^{\frac{q(p-1)-2}{pq-1}})$ , with  $A_1^*, A_2^* > 0$ , and also solutions such that*

$$(u_1, u_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \begin{cases} \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \lim_{r \rightarrow \infty} r^{(N-1)q-2} u_2 = c(c_1) > 0, & \text{if } q < \frac{N}{N-1}, \\ \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \lim_{r \rightarrow \infty} r^{N-2} u_2 = c_2 > 0, & \text{if } q > \frac{N}{N-1}. \end{cases}$$

• *For  $q > q_4$ , there exist cusp-solutions  $(u_1, u_2)$  such that*

$$(u_1, u_2) \sim_{r \rightarrow 0} (|A_1^*| r^{\frac{p(q-1)-2}{pq-1}}, |A_2^*| r^{\frac{q(p-1)-2}{pq-1}}), \quad \lim_{r \rightarrow \infty} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} u_2 = c_2 > 0,$$

*and there is no other global solution.*

The cases of system (1.5) and (1.6), treated at Theorems 5.9, 5.13, 5.14 when  $q < q_4$  are even richer. Note the existence of a family of explicit solutions of system (1.6) in the case  $q = q^*$ , which corresponds to the critical Sobolev exponent for the Hardy-Henon equation, see Theorem 4.9. The case  $q > q_4$ , treated at Theorem 5.15 is remarkable, since it gives the existence of bounded solutions of any of the three systems (1.4), (1.5) and (1.6).

In Section 6 we extend the study of system (1.1) to the nonradial case. Our main aim is finding upper estimates, which appears to be a good challenge. Indeed the system is not variational, and does not offer monotony or comparison properties which could be used as it was done in [6], [39]. Moreover the Bernstein technique appears not to be efficient for systems, as in the case of Serrin's system, except in special cases, as developed in [13]. Finally we cannot use the kind of methods of moving planes, even with boundedness assumptions on the gradients, because in  $\mathbb{R}^N$  there always exist constant nontrivial solutions. Here we give a first estimate, and in the same way a nonexistence result, see Proposition 6.5 and Theorem 6.6:

**Theorem 1.2** *Let  $pq > 1, p \geq q \geq 1$ .*

(i) *If  $(N-1)pq < \max(N+p, N+q)$ , then any supersolution of system (1.1) in  $\Omega = \mathbb{R}^N$ , without condition of sign, is constant.*

(ii) *Let  $(u_1, u_2)$  be any supersolution of system (1.5) in  $B_{r_0} \setminus \{0\}$  (resp. in  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ ). Then there exists  $\rho \in (0, r_0)$  and  $C > 0$ , depending on  $u_1, u_2$ , such that for any  $r \in (0, \rho)$ ,*

$$\begin{aligned} |\overline{u}_1(r)| &\leq C \begin{cases} \max(1, r^{-\frac{2-p(q-1)}{pq-1}}) & \text{if } p(q-1) \neq 2, \\ |\ln r| & \text{if } p(q-1) = 2, \end{cases} \\ |\overline{u}_2(r)| &\leq C \begin{cases} \max(1, r^{-\frac{2-q(p-1)}{pq-1}}) & \text{if } q(p-1) \neq 2, \\ |\ln r| & \text{if } q(p-1) = 2. \end{cases} \end{aligned}$$

(iii) Let  $(\widetilde{u}_1, \widetilde{u}_2)$  be any positive subsolution of system (1.5) in  $B_{r_0} \setminus \{0\}$  (resp. in  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ ). Then there exists  $\rho \in (0, r_0)$  and  $C > 0$ , depending on  $\widetilde{u}_1, \widetilde{u}_2$ , such that for  $0 < |x| < \rho$ ,

$$\begin{aligned} \widetilde{u}_1(x) &\leq C \begin{cases} \max(1, |x|^{-\frac{2-p(q-1)}{pq-1}}) & \text{if } p(q-1) \neq 2, \\ |\ln |x|| & \text{if } p(q-1) = 2, \end{cases} \\ \widetilde{u}_2(x) &\leq C \begin{cases} \max(1, |x|^{-\frac{2-q(p-1)}{pq-1}}) & \text{if } q(p-1) \neq 2, \\ |\ln |x|| & \text{if } q(p-1) = 2. \end{cases} \end{aligned}$$

Note that the range  $(N-1)pq < \max(N+p, N+q)$  is optimal for supersolutions. Nevertheless we think that the result is much more general concerning solutions:

**Conjecture:** for any  $p, q$  such that  $pq > 1$ , all the *solutions* of system (1.1) in  $\Omega = \mathbb{R}^N$ , without condition of sign, are constant.

We end this section by the study of the existence of supersolutions of system (1.4) in an exterior domain, see Theorem 6.7, and the existence of solutions in a domain  $\Omega$  with measure data, see Theorem 6.9.

We conclude the paper by suggesting some extensions of our results in Section 7.

## 2 Critical exponents of the system

We first recall some results on the scalar case of equation (1.7), which can be found for example in [7].

### 2.1 On the scalar case

Clearly a function  $u$  is a negative solution of (1.7) if and only if  $\widetilde{u} = -u$  is a positive solution of the equation with absorption

$$-\Delta \widetilde{u} + |\nabla \widetilde{u}|^q = 0. \quad (2.1)$$

As we mentioned above, two critical values are involved:  $q = \frac{N}{N-1}$ , and  $q = 2$ . When  $q = 2$  the equation is equivalent to  $\Delta(e^u) = 0$ , so all the solutions are described; in particular all the solutions in  $\mathbb{R}^N \setminus \{0\}$  are radial, and given by  $u = \ln(cr^{2-N} + d)$ ,  $c, d \in \mathbb{R}$ .

(1) For  $q \neq 1, \frac{N}{N-1}, 2$ , there exist particular solutions of (1.7) on  $(0, \infty)$ , we find

$$u^*(r) = A^* r^{\frac{q-2}{q-1}} + c, \quad r = |x|, \quad c \in \mathbb{R}, \quad (2.2)$$

with  $|A^*|^{q-2} A^* = (2-q)((N-1)q - N) |2-q|^{1-q} |q-1|^{2-q}$ . This function is  $\infty$ -singular for  $q < 2$ , positive for  $\frac{N}{N-1} < q < 2$ , such that  $\lim_{r \rightarrow 0} u^* = \infty$ , and negative either for  $1 < q < \frac{N}{N-1}$  where  $\lim_{r \rightarrow 0} u^* = -\infty$ ; it is a cusp-solution for  $q > 2$ :  $\lim_{r \rightarrow 0} u^* = 0$ ,  $\lim_{r \rightarrow 0} u'^* = -\infty$ .

(2) In the radial case, equation (1.7) only depends on the derivative, and it can be completely integrated: it reduces to

$$W' = r^{(N-1)(1-q)} |W|^q, \quad \text{where } W = -r^{N-1} u',$$



hence there exists a union of disjoint intervals  $(\rho, R)$  where for  $q \neq \frac{N}{N-1}$  setting  $b = \frac{q-1}{(N-1)q-N}$ ,

$$u' = -r^{1-N}(C + br^{N-(N-1)q})^{-\frac{1}{q-1}} < 0, \quad \text{or } u' = r^{1-N}(C - br^{N-(N-1)q})^{-\frac{1}{q-1}} > 0. \quad (2.3)$$

Then we deduce  $u$  by integration. The case  $C = 0$  corresponds to the particular solutions.

When  $q = \frac{N}{N-1}$ , there exist negative solutions near 0 with a logarithmic behaviour, given by

$$u^{*}(r) = r^{1-N}((q-1)|\ln r| + C)^{\frac{-1}{q-1}}. \quad (2.4)$$

From (2.3) we get the existence of other radial solutions in  $\mathbb{R}^N \setminus \{0\}$  :

- for  $\frac{N}{N-1} < q < 2$ , for any  $c \geq 0, k > 0$ , there exists positive solutions of (1.7) such that

$$\lim_{r \rightarrow 0} r^{\frac{2-q}{q-1}}(u-c) = |A^*|, \quad \lim_{r \rightarrow \infty} r^{N-2}(u-c) = k > 0,$$

- for  $1 < q < \frac{N}{N-1}$ , and for any  $c < 0, k > 0$ , there exists a negative solution  $u$  such that

$$\lim_{r \rightarrow 0} r^{N-2}(u-c) = -k, \quad \lim_{r \rightarrow \infty} r^{\frac{2-q}{q-1}}(u-c) = -|A^*|,$$

- for  $q > 2$ , and for any  $c > d$  there exist decreasing bounded solutions of (1.7), such that

$$\lim_{r \rightarrow 0} u = c, \quad \lim_{r \rightarrow 0} r^{\frac{2-q}{q-1}}(u-c) = -|A^*|, \quad \lim_{r \rightarrow \infty} u = d,$$

and they are either positive for  $d > 0$  or negative for  $c < 0$ .

(3) In the nonradial case, the first upper estimates are due to [29]: if  $u$  is any solution (with no condition of sign) in a domain  $\Omega$ , then for any  $q > 1$ ,

$$|\nabla u(x)| \leq C_{N,q} \text{dist}(x, \partial\Omega)^{-\frac{1}{q-1}}.$$

As a consequence, if  $\Omega = \mathbb{R}^N$ , then  $u$  is constant. Note that the result is false if  $q < 1$ , since  $u^*$  given at (2.2) still exists and  $u^* \in C^2(\mathbb{R}^N)$ . The estimates have been extended to a quasilinear equation  $-\Delta_m u = -\text{div}(|\nabla u|^{m-2} \nabla u) = |\nabla u|^q$ ,  $m > 1$  in [7], where one can find a complete classification of the negative solutions  $u \in C^1(\Omega \setminus \{0\})$  near 0, and a partial classification of the positive ones. The negative solutions of the equation (1.7) have been studied in [32, Lemmas 2.3, 2.4, 5.2, Proposition 5.2]. When  $q \geq 2$ , the first author gives in [6] a detailed behaviour of the positive solutions in  $B_{r_0} \setminus \{0\}$  or in  $\mathbb{R}^N \setminus \overline{B_{r_0}}$  in a more general context of equation  $-\Delta_m u = u^p |\nabla u|^q$ ,  $p \geq 0$ .

## 2.2 Particular solutions and critical values of the parameters

When searching particular solutions of system (1.1) critical values of the parameters  $p, q$  are involved in the problem, many more than in the scalar case:

**Definition 2.1** *Let  $pq > 1, p \geq q$ . Following (1.8) and (1.9), we define*

$$\begin{aligned} q_1 &= q_1(p) = \frac{N}{(N-1)p-1}, & q_2 &= q_2(p) = \frac{N+p}{(N-1)p}, \\ q_3 &= q_3(p) = \frac{2}{p-1}, & q_4 &= q_4(p) = \frac{p+2}{p}, & q^* &= q^*(p) = \frac{2N+p}{2(N-1)p-1}. \end{aligned}$$

corresponding to five curves in the set  $\{(p, q) : pq > 1, p \geq q > 0\}$ , namely

$$\begin{aligned}\mathcal{L}_1 &= \{q = q_1\} = \{(N-1)pq = N+q\}, & \mathcal{L}_2 &= \{q = q_2\} = \{(N-1)pq = N+p\}, \\ \mathcal{L}_3 &= \{q = q_3\} = \{q(p-1) = 2\}, & \mathcal{L}_4 &= \{q = q_4\} = \{p(q-1) = 2\}, \\ \mathcal{L}^* &= \{q = q^*\} = \left\{ (N-1)pq = N + \frac{p+q}{2} \right\}.\end{aligned}$$

Note that  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}^*$  intersect the diagonal  $\{p = q\}$  at  $q = \frac{N}{N-1}$ , and  $\mathcal{L}_3, \mathcal{L}_4$  at  $q = 2$ .

**Proposition 2.2** *Let  $pq > 1, p \geq q$ . Then for  $q \neq q_1, q_2$ , the system (1.1) admits particular solutions such that*

$$u_1^{*'} = \eta_1 a_1 r^{-\frac{p+1}{pq-1}}, \quad u_2^{*'} = \eta_2 a_2 r^{-\frac{q+1}{pq-1}}, \quad (2.5)$$

where  $a_i = a_i(N, p, q) > 0$  for  $i = 1, 2$ , and

$$\eta_1 = \text{sign}(q_2 - q), \quad \eta_2 = \text{sign}(q_1 - q). \quad (2.6)$$

If moreover  $q \neq q_3, q_4$ , it has solutions

$$u_1^* = \varepsilon_1 A_1^* r^{\frac{p(q-1)-2}{pq-1}} + c_1 = \varepsilon_1 A_1^* r^{\frac{p(q-q_4)}{pq-1}} + c_1, \quad u_2^* = \varepsilon_2 A_2^* r^{\frac{q(p-1)-2}{pq-1}} + c_2 = \varepsilon_2 A_2^* r^{\frac{(p-1)(q-q_3)}{pq-1}} + c_2, \quad (2.7)$$

where  $A_i^* = A_i^*(N, p, q) > 0$ ,  $c_i \in \mathbb{R}$  for  $i = 1, 2$ , and

$$\varepsilon_1 = \text{sign}(q_2 - q)(q - q_4), \quad \varepsilon_2 = \text{sign}(q_1 - q)(q - q_3).$$

**Proof.** We search particular solutions of the form  $(u_1', u_2') \equiv (C_1 r^{-\lambda_1}, C_2 r^{-\lambda_2})$ . Then

$$C_1(\lambda_1 - N + 1)r^{-\lambda_1-1} = r^{-p\lambda_2} |C_2|^p, \quad C_2(\lambda_2 - N + 1)r^{-\lambda_2-1} = r^{-q\lambda_1} |C_1|^p,$$

hence we find  $\lambda_1 = \frac{p+1}{pq-1}, \lambda_2 = \frac{q+1}{pq-1}$ , and setting

$$\beta_1 = \lambda_1 - N + 1 = \frac{N+p-(N-1)pq}{pq-1}, \quad \beta_2 = \lambda_2 - N + 1 = \frac{N+q-(N-1)pq}{pq-1},$$

we get the results with  $|C_1| = (|\beta_1| |\beta_2|^p)^{\frac{1}{pq-1}}, |C_2| = (|\beta_2| |\beta_1|^q)^{\frac{1}{pq-1}}$ . By integration, if the denominator is nonzero, we get in particular the solution  $(u_1, u_2)$ , with  $|A_i^*| = \left| \frac{C_i(pq-1)}{p(q-1)-2} \right|$ ,  $i = 1, 2$ .  
■

**Remark 2.3** *Note that  $u_1^*$  is  $\infty$ -singular for  $q < q_4$  and it is a cusp-solution for  $q > q_4$ . And  $u_2^*$  is  $\infty$ -singular for  $q < q_3$  and it is a cusp-solution for  $q > q_3$ ; so there exist solutions  $(u_1, u_2)$  of system (1.1) with components of different types when  $q_3 < q < q_4$ .*

**Remark 2.4** *It is clear that the exponents  $q_1, q_2, q_3, q_4$ , are strongly involved in the existence of particular solutions. The exponent  $q^*$  corresponds to the Sobolev exponent for equation (1.2), offering other types of explicit solutions. Note the relations*

$$\begin{aligned}q \leq q_1 &\iff (N-1)pq \leq N+q, & q \leq q_2 &\iff (N-1)pq \leq N+p, \\ q \leq q_3 &\iff q(p-1) \leq 2, & q \leq q_4 &\iff p(q-1) \leq 2, \\ q \leq q^* &\iff (N-1)pq \leq N + \frac{p+q}{2},\end{aligned}$$

$$\begin{aligned}
q_1 &\leq \min(q_2, q_3) \leq \max(q_2, q_3) \leq q_4, \\
q_1 &\leq \frac{N}{N-1} \iff q_2 \leq \frac{N}{N-1} \iff p \geq \frac{N}{N-1}, \\
q_2 &\leq q_3 \iff p \leq N, \\
q_1 &\leq q^* \leq q_2, \quad q^* \leq q_3 \iff p \leq 2(N-1).
\end{aligned}$$

When  $q = p$ , we find again the two critical values of the scalar case:

$$q_1 = q^* = q_2 = \frac{N}{N-1}, \quad q_3 = q_4 = 2.$$

### 3 First properties in the radial case

In the radial case, system (1.1) is reduced to

$$\begin{cases} -(u_1'' + \frac{N-1}{r}u_1') = |u_2'|^p, \\ -(u_2'' + \frac{N-1}{r}u_2') = |u_1'|^q, \end{cases} \quad (3.1)$$

so it only involves the derivatives. Moreover, setting

$$w_1 = -r^{N-1}u_1', \quad w_2 = -r^{N-1}u_2', \quad (3.2)$$

system (3.1) is equivalent to a first order system:

$$\begin{cases} w_1' = r^{(N-1)(1-p)} |w_2|^p, \\ w_2' = r^{(N-1)(1-q)} |w_1|^q. \end{cases} \quad (3.3)$$

Consider any solution  $(u_1, u_2) \in C^2(0, r_0)$ ,  $r_0 \leq \infty$ . We say that  $u_i$  is *regular* if  $u_i'$  has a limit at 0 and  $\lim_{r \rightarrow 0} u_1' = \lim_{r \rightarrow 0} u_2' = 0$ ; if  $u_1$  and  $u_2$  are regular, then  $(u_1, u_2)$  extends as a function in  $C^2(B_{r_0} \times B_{r_0})$ , from the equations. We say that  $u_i$  is *singular* if it is not regular. Among the singular solutions we distinguish the  $\infty$ -*singular* ones, such that  $\lim_{r \rightarrow 0} |u_i| = \infty$ , and the *cusp-solutions*  $u_i$  ( $\lim_{r \rightarrow 0} u_i = c \in \mathbb{R}$  and  $\lim_{r \rightarrow 0} |u_i'| = \infty$ ), where the singularity is at the level of the gradient.

#### 3.1 Local existence and uniqueness

**Proposition 3.1** *Let  $pq > 1, p \geq q$ . (i) Let  $r_0 \geq 0$  and  $c_1, c_2, b_1, b_2 \in \mathbb{R}$  such that  $b_2 \neq 0$  if  $q < 1$ . Then there exists a unique solution  $(u_1, u_2)$  of system (3.1) in the neighborhood of  $r_0$ , such that*

$$u_1(r_0) = c_1, \quad u_2(r_0) = c_2, \quad u_1'(r_0) = b_1, \quad u_2'(r_0) = b_2.$$

(ii) *For any  $R > 0$ , the only radial solutions  $(u_1, u_2) \in C^1(B_R \times B_R)$  of system (1.1) are the constants.*

**Proof.** (i) First suppose  $r_0 > 0$ . Since the system only depends on the derivatives, it is equivalent to consider system (3.3), with initial data  $(w_{01}, w_{02}) \in \mathbb{R}^2$ . If  $q > 1$ , the classical Cauchy-Lipschitz theorem applies. Next suppose  $q < 1$ ; the theorem still applies if  $w_{01} \neq 0$ . Consider the case  $w_{01} = 0, w_{02} \neq 0$ . In a neighborhood  $[r_0 - \eta, r_0 + \eta]$  of  $r_0$ , there holds  $0 < c_1 < |w_2| < c_2$ ,

$c_3 < w'_1 < c_4$ ,  $0 < |w_1| < c_5$ . hence we can take  $w_1$  as a new variable, and consider that  $r$  is a function of  $w_1$ . Then we get the system

$$\begin{cases} \frac{dr}{dw_1} = f(w_1, r, w_2) = r^{(N-1)(1-p)} |w_2|^p, \\ \frac{dw_2}{dw_1} = g(w_1, r, w_2) = |w_1|^q r^{(N-1)(2-p-q)} |w_2|^p, \end{cases}$$

for which we can apply the Cauchy-Lipschitz theorem in  $(0, c_5) \times (r_0 - \eta, r_0 + \eta) \times (c_1, c_2)$ , since  $f$  is continuous, and locally Lipschitz with respect to  $(r, w_2)$ .

(ii) Next suppose that  $b_1 = b_2 = 0$ , and  $r_0 \geq 0$ . Then

$$\begin{aligned} u'_1 &= r^{1-N} \int_{r_0}^r s^{N-1} |u'_2|^p ds, & u'_2 &= r^{1-N} \int_{r_0}^r s^{N-1} |u'_1|^q ds, \\ |u'_1| &\leq r^{1-N} \left| \int_{r_0}^r s^{N-1} |u'_2|^p ds \right| \leq r^{1-N} \left| \int_{r_0}^r s^{N-1} (s^{1-N} \left| \int_{r_0}^s \theta^{N-1} |u'_1|^q)^p d\theta \right| ds \right|, \end{aligned} \quad (3.4)$$

• First assume  $r_0 > 0$ . In a neighborhood  $[r_0 - \eta, r_0 + \eta]$ , there holds

$$|u'_1| \leq C \left| \int_{r_0}^r \left| \int_{r_0}^s |u'_1|^q)^p d\theta \right| ds \right| \leq C' \int_{r_0}^r \int_{r_0}^s |u'_1|^{pq} d\theta ds \leq C'' \left| \int_{r_0}^r |u'_1|^{pq} d\theta \right|. \quad (3.5)$$

Let  $F(r) = \int_{r_0}^r |u'_1|^{pq} d\theta$ . Then  $F'(r) \leq C'' F^{pq}$  on  $(r_0, r_0 + \eta)$ . If  $u'_1$  is not identically 0 on  $(r_0, r_0 + \eta)$ , then  $F > 0$  on  $(r_0, r_0 + \rho)$  for some  $\rho$  small enough, and  $F^{1-pq} + cr$  is nondecreasing, which is contradictory, since  $F(r_0) = 0$ ; thus  $u'_1 \equiv 0$  and then  $u'_2 \equiv 0$  on  $(r_0, r_0 + \rho)$ , and similarly on  $(r_0 - \rho, r_0)$  for  $\rho$  small enough. Then  $u_1 \equiv c_1, u_2 \equiv c_2$  in  $[r_0 - \eta, r_0 + \eta]$ .

• Next assume  $r_0 = 0$ . Then from (3.4),

$$\begin{aligned} |u'_1| &\leq r^{1-N} \int_0^r s^{(N-1)(1-p)} \left( \int_0^s \theta^{N-1} |u'_1|^q)^p d\theta \right) ds \\ &\leq r^{1-N} \int_0^r s^{(N-1)(1-p) + (N-1)p} \left( \int_0^s |u'_1|^q)^p d\theta \right) ds \\ &\leq r^{1-N} \int_0^r s^{(N-1)} \left( \int_0^s |u'_1|^q)^p d\theta \right) ds \leq \int_0^r \left( \int_0^s |u'_1|^q)^p d\theta \right) ds \\ &\leq \int_0^r s^{p-1} \left( \int_0^s |u'_1|^{pq} d\theta \right) ds \leq r^p \int_0^r |u'_1|^{pq} d\theta \leq C \int_0^r |u'_1|^{pq} d\theta. \end{aligned}$$

Defining  $F = \int_0^r |u'_1|^{pq} d\theta$  we conclude as above that  $u'_1 \equiv 0, u'_2 \equiv 0$  on  $(0, \rho)$  for  $\rho$  small enough. Finally let  $(u_1, u_2) \in C^1(B_R \times B_R)$  be any radial solution. Then the set  $\{(0, \rho) : u'_1 \equiv u'_2 \equiv 0 \text{ on } (0, \rho)\}$  is closed and open in  $(0, R)$ , thus equal to  $(0, R)$ . ■

**Remark 3.2** *The result would be false for  $pq < 1$ . Indeed in that case there exist solutions of system (1.5):*

$$\widetilde{u}_1^* = A_1^* r^{\frac{2+p(1-q)}{1-pq}} = A_1^* r^{2+\frac{p(q+1)}{1-pq}}, \quad \widetilde{u}_2^* = A_2^* r^{2+\frac{q(p+1)}{1-pq}},$$

belonging to  $C^2(\mathbb{R}^N)$  which proves the existence of entire nontrivial solutions, and gives a counterexample to uniqueness.

**Remark 3.3** Note a consequence of Proposition 3.1: for any  $r_0 > 0$ , there exist local radial solutions  $(u_1, u_2)$  such that  $u_1'(r_0) = 0$ , and  $u_2'(r_0) = b \neq 0$ . This shows the great richness of the solutions of the system: for system (1.4) (resp. (1.5) for example, it means that only one of the functions has a minimum (resp. a maximum) at  $r_0$ .

**Remark 3.4** From Proposition 3.1, one can divide the maximal existence interval  $(\rho, R)$  of  $(u_1, u_2)$  in intervals  $(a, b)$  where  $u_1'$  and  $u_2'$  have a constant sign. Indeed there exists at most one point  $\rho < s < R$  where  $w_1(s) = 0$ ;  $w_1 < 0$  on  $(\rho, s)$  and  $w_1 > 0$  on  $(s, R)$ ; in the same way, there is at most one point  $\rho < \tau < R$  where  $w_2(\tau) = 0$ ;  $w_2 < 0$  on  $(\rho, \tau)$  and  $w_2 > 0$  on  $(\tau, R)$ .

### 3.2 Upper estimates on the radial supersolutions

**Proposition 3.5** Let  $pq > 1$ , and  $(u_1, u_2)$  be any radial supersolution of system (1.1), defined on an interval  $(\frac{R}{2}, \frac{3R}{2})$  where  $u_1'(r)$  and  $u_2'(r)$  do not vanish. There exists a constant  $C = C(N, p, q) > 0$  such that on  $(\frac{3R}{4}, \frac{5R}{4})$ ,

$$|u_1'| \leq Cr^{-\frac{p+1}{pq-1}}, \quad |u_2'| \leq Cr^{-\frac{q+1}{pq-1}}, \quad \text{on } (\frac{3R}{4}, \frac{5R}{4}).$$

**Proof.** Consider any supersolution of system (1.1). Equivalently, the functions  $w_1, w_2$  defined by (3.2) satisfy

$$\begin{cases} w_1' \geq r^{(N-1)(1-p)} |w_2|^p, \\ w_2' \geq r^{(N-1)(1-q)} |w_1|^q. \end{cases}$$

and consider any interval  $(R_1, R_2)$  where  $w_1, w_2$  do not vanish. Let  $r_0 \in (R_1, R_2)$  and  $\varepsilon_0 \in (0, \frac{1}{4}]$  such that  $R_1 < r_0(1 - 2\varepsilon_0) < r_0(1 + 2\varepsilon_0) < R_2$ . First assume that  $w_1 > 0, w_2 > 0$ . Since  $w_i$  is nondecreasing, for any  $\varepsilon \in (0, \varepsilon_0]$ , we have  $r \in (r_0(1 - \varepsilon), r_0(1 + \varepsilon))$  there holds, since  $w_1 > 0$  and  $\varepsilon_0 < 1$ , with  $c_i = c_i(N, p)$

$$\begin{aligned} w_1(r_0) &\geq \int_{r_0(1-\varepsilon)}^{r_0} \theta^{(N-1)(1-p)} w_2^p(\theta) d\theta \\ &\geq w_2^p(r_0(1 - \varepsilon)) \int_{r_0(1-\varepsilon)}^{r_0} \theta^{(N-1)(1-p)} d\theta = c_1 \varepsilon w_2^p(r_0(1 - \varepsilon)) r_0^{N-(N-1)p}, \end{aligned}$$

and symmetrically, since  $w_1 > 0$ , changing  $r_0$  into  $r_0 - \varepsilon$ ,

$$w_2(r_0 - \varepsilon) \geq c_2 \varepsilon w_1^q(r_0(1 - 2\varepsilon))(r_0(1 - \varepsilon))^{N-(N-1)q} \geq c_3 \varepsilon w_1^q(r_0 - 2\varepsilon) r_0^{N-(N-1)q}.$$

Then

$$\begin{aligned} w_1(r_0) &\geq c_4 \varepsilon^{p+1} w_1^{pq}(r_0(1 - 2\varepsilon)) r_0^{N+p-(N-1)pq}, \\ w_1(r_0(1 - 2\varepsilon)) &\leq c_5 \varepsilon^{-\frac{p+1}{pq}} (w_1(r_0))^{\frac{1}{pq}} r_0^{-\frac{N+p-(N-1)pq}{pq}}. \end{aligned}$$

Setting  $\bar{r}_0 = r_0(1 - 2\varepsilon)$ , and using by the bootstrap technique developed at [11, Lemma 2.2] and [9, Lemma 2.8], we obtain, since  $d = \frac{1}{pq} < 1$ , and

$$w_1(r_0) = r_0^{N-1} |u_1'(r_0)| \leq c_6 r_0^{\frac{N+p-(N-1)pq}{1-d}} = c_6 r_0^{-\frac{N+p-(N-1)pq}{pq-1}},$$

equivalently

$$|u'_1(r_0)| \leq c_7 r_0^{-\frac{p+1}{pq-1}}.$$

If  $w_1 < 0, w_2 < 0$ , the same conclusion holds by changing  $1 + \varepsilon$  into  $1 - \varepsilon$ . Finally if  $w_2 > 0 > w_1$ , we get

$$\begin{aligned} w_1(r_0) &\geq c_1 \varepsilon w_2^p(r_0(1 - \varepsilon)) r_0^{N-(N-1)p}, \\ w_2(r_0(1 - \varepsilon)) &\geq c_2 \varepsilon w_1^q(r_0)(r_0(1 - \varepsilon))^{N-(N-1)q} \geq c_3 \varepsilon w_1^q(r_0) r_0^{N-(N-1)q}, \\ w_1(r_0) &\geq c_4 \varepsilon^{p+1} w_1^{pq}(r_0) r_0^{N+p-(N-1)pq}, \end{aligned}$$

hence we obtain the same conclusion without using any bootstrap.  $\blacksquare$

**Remark 3.6** *In the case  $pq > 1, p \geq q \geq 1$ , we do not need any bootstrap in order to conclude, and our result is valid in the nonradial case, see Theorem 6.2 in Section 6.*

### 3.3 Reduction to a quadratic system of order 2 and formulation as a Hardy-Hénon equation

In the sequel we show that system (3.1) can be reduced to a polynomial system order 2, and equivalently to an equation of order 2 relative to  $w_1$  :

**Proposition 3.7** *Let  $(u_1, u_2)$  be any radial solution of system (1.1). At any point  $r$  where  $u'_1(r) \neq 0, u'_2(r) \neq 0$ , we define*

$$S = r \frac{|u'_2|^p}{u'_1}, \quad Z = -r \frac{|u'_1|^q}{u'_2}, \quad (3.6)$$

system (3.1) is equivalent to a Lotka-Volterra type system:

$$\begin{cases} S_t = S(N - (N - 1)p + S + pZ), \\ Z_t = Z(N - (N - 1)q - qS - Z). \end{cases} \quad t = \ln r, \quad (3.7)$$

and we recover  $u'_1$  and  $u'_2$  in function of  $r, S, Z$  by the formulas

$$u'_1 = (r^{-(p+1)} |S| |Z|^p)^{\frac{1}{pq-1}} \text{sign} S, \quad u'_2 = -(r^{-(q+1)} |S|^q |Z|)^{\frac{1}{pq-1}} \text{sign} Z. \quad (3.8)$$

**Proof.** Following the ideas of [10], at each point where  $u'_1 \neq 0$  and  $u'_2 \neq 0$ , we set

$$X = r \frac{u''_1}{u'_1}, \quad Y = r \frac{u''_2}{u'_2}, \quad S = r \frac{|u'_2|^p}{u'_1}, \quad Z = -r \frac{|u'_1|^q}{u'_2}, \quad t = \ln r,$$

Then

$$X + N - 1 = -S, \quad Y + N - 1 = Z.$$

Next we differentiate with respect to  $t$ . Since  $X = \frac{(u'_1)_t}{u'_1}, Y = \frac{(u'_2)_t}{u'_2}$ , we obtain

$$\begin{aligned} \frac{S_t}{S} &= 1 + p \frac{(u'_2)_t}{u'_2} - \frac{(u'_1)_t}{u'_1} = 1 + pY - X = N - (N - 1)p + S + pZ, \\ \frac{Z_t}{Z} &= 1 + q \frac{(u'_1)_t}{u'_1} - \frac{(u'_2)_t}{u'_2} = 1 + qX - Y = N - (N - 1)q - qS - Z. \end{aligned}$$

So we obtain the quadratic system (3.7) valid in any case of sign of the unknown, and deduce (3.8).  $\blacksquare$

**Remark 3.8** *The fixed points of the system are*

$$\begin{aligned} M_0 = (S_0, Z_0) &= \left( \frac{(N-1)p(q_2-q)}{pq-1}, -\frac{((N-1)p-1)(q_1-q)}{pq-1} \right), \\ (0, 0), \quad N_0 &= (0, N - (N-1)q), \quad A_0 = ((N-1)p - N, 0). \end{aligned} \quad (3.9)$$

We easily check that the particular solutions  $(u'_1, u'_2)$  of system (3.1) correspond to the fixed point  $M_0$ .

Next we show the precise link with a Hardy-Hénon equation:

When looking at the system (3.7) for  $pq \neq 1$ , and comparing to the systems introduced in [10, Section 3], we observe that the system is exactly linked to the positive solutions  $w$  of a radial quasilinear equation of Hardy-Hénon type in dimension  $\mathbf{N}$ , of the form

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = -(|w'|^{\mathbf{p}-2} w')' - \frac{\mathbf{N}-1}{r} |w'|^{\mathbf{p}-2} w' = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = \pm 1, \quad (3.10)$$

where  $q > 0, \sigma \in \mathbb{R}, \mathbf{p} > 1, \mathbf{q} \neq \mathbf{p} - 1$ , and  $\mathbf{N}$  is not necessarily an integer. Indeed in [10, Section 3], this equation was reduced to a system of order 2, by setting

$$s(t) = -r \frac{w'}{w}, \quad z(t) = -\varepsilon r^{1+\sigma} w^{\mathbf{q}} |w'|^{-\mathbf{p}} w', \quad t = \ln r, \quad (3.11)$$

and obtained the system, valid for the two equations,

$$\left\{ \begin{array}{l} s_t = s \left( \frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1} \right), \\ z_t = z (\mathbf{N} + \sigma - \mathbf{q}s - z), \end{array} \right\} \quad (3.12)$$

and one recovers  $w$  by the formula

$$w = r^{-\gamma} (|s^{\mathbf{p}-1} |z|)^{\frac{1}{\mathbf{q}+1-\mathbf{p}}}, \quad \gamma = \frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}. \quad (3.13)$$

It is precisely the case, as we show below.

**Proposition 3.9** *Let  $(u_1, u_2)$  be any radial solution of system (1.1), and let  $w_1, w_2$  be defined by (3.2). Then the function  $w = |w_1|$  satisfies the equation*

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = -\text{sign}(u'_1 u'_2) \quad (3.14)$$

where  $\Delta_{\mathbf{p}}^{\mathbf{N}}$  is the  $\mathbf{p}$ -Laplace operator in dimension  $\mathbf{N}$ , for specific values of  $\mathbf{p}, \sigma$  and  $\mathbf{N}$ :

$$\mathbf{p} = 1 + \frac{1}{p} > 1, \quad \mathbf{q} = q, \quad \sigma = (N-1) \frac{1-pq}{p}, \quad \mathbf{N} = 1 + \frac{(N-1)(p-1)}{p}. \quad (3.15)$$

Moreover at each point where  $w \neq 0$ ,

$$S(t) = s(t) = -r \frac{w'_1}{w_1}, \quad Z(t) = z(t) = r \frac{w'_2}{w_2} = -\text{sign}(u'_2) r^{1+\sigma} |w_1|^q |w'_1|^{\frac{1}{p}} w'_1 \quad (3.16)$$

**Proof.** Computing  $w_2$  from the first equation and reporting in the second one, we get

$$-(r^{\frac{(N-1)(p-1)}{p}}(w_1')^{\frac{1}{p}})' = \text{sign}(u_2') r^{(N-1)(1-q)} |w_1|^q,$$

that is in a developed form:

$$-(w_1')^{\frac{1}{p}}' - \frac{(N-1)(p-1)}{pr} (w_1')^{\frac{1}{p}} = \text{sign}(u_2') r^{(N-1)\frac{1-pq}{p}} |w_1|^q$$

So we get an equation with the form (where we recall that  $w_1' > 0$ )

$$-(|w_1'|^{\mathbf{p}-2} w_1')' - \frac{\mathbf{N}-1}{r} |w_1'|^{\mathbf{p}-2} w_1' = -\Delta_{\mathbf{p}}^{\mathbf{N}} w_1 = \text{sign}(u_2') r^{\sigma} |w_1|^q$$

where  $\mathbf{p}, \sigma, \mathbf{N}$  are defined at (3.15), and the fact that  $w_1 = -\text{sign}(u_1') w$  leads to equation (3.14), and we check easily that  $s \equiv S$  and  $z \equiv Z$ .  $\blacksquare$

**Remark 3.10** From the definition (3.15) and (3.13) we get the relations

$$\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} = N - (N-1)p, \quad \mathbf{N} + \sigma = N - (N-1)q, \quad (3.17)$$

$$\gamma = \frac{\mathbf{p} + \sigma}{q - \mathbf{p} + 1} = \frac{p + N - (N-1)pq}{pq - 1} = \frac{(N-1)p(q_2 - q)}{pq - 1} \quad (3.18)$$

so

$$\mathbf{N} > \mathbf{p} \iff p > \frac{N}{N-1}, \quad \mathbf{N} + \sigma > 0 \iff q < \frac{N}{N-1}, \quad \mathbf{p} + \sigma > 0 \iff q < q_2. \quad (3.19)$$

**Remark 3.11** For  $pq > 1$ ,  $p \geq q$ , there holds  $1 < \mathbf{p} < 2$ ,  $\mathbf{q} > \mathbf{p} - 1$ , and  $\mathbf{N} > 1$ . The map  $(N, p) \in [1, \infty) \times (1, \infty) \mapsto (\mathbf{N}, \mathbf{p}) \in (1, \infty) \times (1, 2)$  is injective, the reciprocal application is  $(\mathbf{N}, \mathbf{p}) \mapsto (\frac{\mathbf{N}+1-\mathbf{p}}{2-\mathbf{p}}, \frac{1}{\mathbf{p}-1})$ , and then  $\sigma$  is **fixed** by the relation  $\sigma = -\frac{(\mathbf{N}-1)(q+1-\mathbf{p})}{2-\mathbf{p}}$ .

**Remark 3.12** Equation (3.14) is of source type for  $\varepsilon = 1$ , of absorption type for  $\varepsilon = -1$ . One could think that problem (1.4), where  $u_1, u_2$  are positive superharmonic functions, with two **source** terms  $|\nabla u_2|^p, |\nabla u_1|^q$  is linked to a Hardy-Hénon equation with source term  $\varepsilon = 1$ . In fact **it is not the case**: the solutions of system (1.4) on an interval  $(0, \rho)$  correspond to a Hardy-Hénon equation of **absorption** type. Indeed consider any positive solution of system (1.4). As it is well known, any positive solution  $(u_1, u_2)$  satisfies  $u_1', u_2' \leq 0$  in  $(0, \rho)$ . Indeed  $r^{N-1}u_1'$  is decreasing; if there is  $r_0$  such that  $u_1'(r_0) > 0$ , then  $r^{N-1}u_1' \geq C_0 > 0$  on  $(0, r_0)$ , thus  $u_1 + \frac{C_0}{N-2}r^{2-N}$  is increasing, which is impossible. Then  $u_1' \leq 0$  on  $(0, \rho)$ ; moreover if there exists  $r_1$  such that  $u_1'(r_1) = 0$ , then it is unique and  $r_1$  is a maximum point, so that  $u_1' \geq 0$  on  $(0, r_1)$  which is contradictory, unless  $u_1$  is constant. Then the nonconstant solutions satisfy  $u_1', u_2' < 0$  in  $(0, \rho)$ .

**Remark 3.13** Another way to get an autonomous system is more common in the literature: the change of unknown

$$u_1' = -r^{-\frac{p+1}{pq-1}}x(t), \quad u_2' = -r^{-\frac{q+1}{pq-1}}y(t), \quad t = \ln r.$$



leads to the system

$$\begin{cases} x_t = b_1 x - |y|^p, \\ y_t = b_2 y - |x|^q, \end{cases}$$

where  $b_1 = \frac{(N-1)pq-(p+N)}{pq-1}$  and  $b_2 = \frac{(N-1)pq-(q+N)}{pq-1}$ . However this system gives less information on the solutions of system (3.1): it admits at most two fixed points, namely  $(0,0)$ , and  $P_0 = ((|b_1| |b_2|^p)^{\frac{1}{pq-1}} \text{sign}(b_1), (|b_2| |b_1|^q)^{\frac{1}{pq-1}} \text{sign}(b_2))$  which corresponds to the particular solutions. Moreover it is singular at  $(0,0)$  whenever  $q < 1$ . The quadratic system (3.7) gives a great amount of information, because it is obtained by differentiation of the equations of (3.1). It has four fixed points, and each of them corresponds to a type of behaviour near 0 or  $\infty$ .

## 4 Solutions of the Hardy-Hénon equations

Here we consider the **positive** solutions of the radial equation

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = -\frac{d}{dr} \left( \left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} \right) - \frac{\mathbf{N}-1}{r} \left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} = \varepsilon r^{\sigma} w^{\mathbf{q}}, \quad \varepsilon = \pm 1, \quad (4.1)$$

in dimension  $\mathbf{N}$ , where  $\mathbf{q} > \mathbf{p} - 1 > 0$ . In the sequel, the results are valid for any real  $\mathbf{N}$  such that  $\mathbf{N} \neq \mathbf{p}$ , and we suppose with  $\sigma \neq -\mathbf{N}, \sigma \neq -\mathbf{p}$ .

We first observe that the equation admits particular solutions

$$w^* = a^* r^{-\gamma}, \quad \gamma = \frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}, \quad a^{*\mathbf{q}-\mathbf{p}+1} = \varepsilon |\gamma|^{\mathbf{p}-2} \gamma (\mathbf{N} - \mathbf{p} - (\mathbf{p} - 1)\gamma), \quad (4.2)$$

well defined for  $\varepsilon \gamma (\mathbf{N} - \mathbf{p} - (\mathbf{p} - 1)\gamma) > 0$  that is  $\mathbf{q} - \mathbf{q}_c$  has the sign of  $\varepsilon (\mathbf{p} + \sigma) (\mathbf{N} - \mathbf{p})$ . Then  $w^*$  is  $\infty$ -singular at 0 if  $\mathbf{p} + \sigma > 0$ , and  $C^0$ -regular if  $\mathbf{p} + \sigma < 0$ .

### 4.1 General formulation by a quadratic system

On any interval where  $w' \neq 0$ , we define

$$s(t) = -r \frac{w'}{w}, \quad z(t) = -\varepsilon r^{1+\sigma} w^{\mathbf{q}} |w'|^{-\mathbf{p}} w', \quad t = \ln r, \quad (4.3)$$

and obtain the system, valid for the two equations,

$$\begin{cases} s_t = s \left( \frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1} \right), \\ z_t = z (\mathbf{N} + \sigma - \mathbf{q}s - z). \end{cases} \quad (4.4)$$

We recover  $w$  by the formula

$$w = r^{-\gamma} (|s|^{\mathbf{p}-1} |z|)^{\frac{1}{\mathbf{q}+1-\mathbf{p}}}, \quad w' = r^{-(\gamma+1)} (|z| |s|^{\mathbf{q}})^{\frac{1}{\mathbf{q}+1-\mathbf{p}}} \text{sign}(-\varepsilon z), \quad \gamma = \frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}. \quad (4.5)$$

In the plane  $(s, z)$  we define the quadrants

$$Q_1 = \{(s, z) \in \mathbb{R}^2 : s > 0, z > 0\}, \quad Q_2 = \{(s, z) \in \mathbb{R}^2 : s < 0, z > 0\}, \quad Q_3 = -Q_1, \quad Q_4 = -Q_2.$$

**Remark 4.1** Observe that  $sz$  has the sign of  $\varepsilon$ . If  $\varepsilon > 0$  (equation with source), then  $(s, z) \in Q_1 \cup Q_3$ . If  $\varepsilon < 0$  (equation with absorption) then  $(s, z) \in Q_2 \cup Q_4$ . Moreover, if  $\mathbf{p} < \mathbf{N}$ , and if  $w$  is defined on an interval  $(0, \rho)$ , then it is always decreasing, hence  $(s, z) \in Q_1$ . Indeed  $(r^{\mathbf{N}-1} |w|^{\mathbf{p}-2} w')$  is decreasing. If there exists  $r_0$  such that  $w'(r_0) > 0$ , then for  $r < r_0, w'^{\mathbf{p}-1} \geq C_0 r^{1-\mathbf{N}}$ , thus  $w$  is bounded, which by integration contradicts the assumption  $\mathbf{p} < \mathbf{N}$ . In case  $\mathbf{p} > \mathbf{N}$ , it can happen, as we see in the sequel, that  $w' > 0$  near 0.

We define two possible critical values

$$\mathbf{q}_c = \frac{(\mathbf{N} + \sigma)(\mathbf{p} - 1)}{\mathbf{N} - \mathbf{p}}, \quad \mathbf{q}_S = \frac{\mathbf{N}(\mathbf{p} - 1) + \mathbf{p} + \mathbf{p}\sigma}{\mathbf{N} - \mathbf{p}}, \quad (4.6)$$

which are the well known Serrin's exponent and Sobolev exponent respectively, in case  $\mathbf{N} < \mathbf{p} > -\sigma$ . Note that  $\mathbf{q}_c < 0$  when  $(\mathbf{N} + \sigma)(\mathbf{N} - \mathbf{p}) < 0$ .

**Definition 4.2** In the following we say that a positive solution  $w$  of equation (4.1) on an interval  $(0, \rho)$  is  $C^0$ -**regular** if  $w \in C^2(0, \rho) \cap C^0[0, \rho)$ , that means  $\lim_{r \rightarrow 0} w = c \geq 0$ . If it exists, such a solution satisfies  $(r^{\mathbf{N}-1} |w|^{\mathbf{p}-1})' \sim r^{\mathbf{N}-1+\sigma} c^{\mathbf{p}}$ , then  $r^{\mathbf{N}-1} |w|^{\mathbf{p}-1} \sim Cr^{\mathbf{N}+\sigma} c^{\mathbf{p}}$ ,  $|w'| \sim Cr^{\frac{\sigma+1}{\mathbf{p}-1}}$ ,  $|w''| \sim Cr^{\frac{\sigma+2-\mathbf{p}}{\mathbf{p}-1}}$ . Then  $w \in C^2[0, \rho)$  if  $\sigma \geq \mathbf{p} - 2$ ,  $w \in C^1[0, \rho)$  if  $\sigma \geq -1$ , and  $w$  presents a cusp at 0 if  $\sigma < -1$ . We say that  $w$  is a  $C^0$ -**ground state** if moreover  $\rho = \infty$ .

Next we divide the study in 6 regions relative to  $(\mathbf{N}, \sigma) \in \mathbb{R}^2$ , and we show that we can reduce the study to only 3 regions, for which one of them is well known. This is due to a transformation proved in [10, Remark 2.4]:

**Lemma 4.3** For fixed  $\mathbf{p} > 0$ , consider the sets defined by

$$\begin{aligned} \mathbf{A} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{N} > \mathbf{p} > -\sigma\}, & \mathbf{B} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{p} > \mathbf{N} > -\sigma\}, \\ \mathbf{C} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{N} > -\sigma > \mathbf{p}\}, & \mathbf{D} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : -\sigma > \mathbf{N} > \mathbf{p}\}, \\ \mathbf{E} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : \mathbf{p} > -\sigma > \mathbf{N}\}, & \mathbf{F} &= \{(\mathbf{N}, \sigma) \in \mathbb{R}^2 : -\sigma > \mathbf{p} > \mathbf{N}\}. \end{aligned} \quad (4.7)$$

Let  $(\mathbf{N}, \sigma) \in \mathbb{R}^2$  and  $(\widehat{\mathbf{N}}, \widehat{\sigma}) \in \mathbb{R}^2$  such that  $(\widehat{\mathbf{N}} - \mathbf{p})(\mathbf{p} + \sigma) = (\mathbf{N} - \mathbf{p})(\mathbf{p} + \widehat{\sigma})$ . Set

$$\lambda = \frac{\widehat{\mathbf{N}} - \mathbf{p}}{\mathbf{N} - \mathbf{p}} = \frac{\mathbf{p} + \widehat{\sigma}}{\mathbf{p} + \sigma} = \frac{\widehat{\mathbf{N}} + \widehat{\sigma}}{\mathbf{N} + \sigma}, \quad (4.8)$$

and consider the change of unknown

$$\widehat{w}(\widehat{r}) = Cw(r), \quad r = \widehat{r}^\lambda, \quad C = \lambda^{\frac{\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}. \quad (4.9)$$

Then  $w$  satisfies the equation (4.1) if and only if  $\widehat{w}$  satisfies the analogous equation with  $r, \mathbf{N}, \sigma$  replaced by  $\widehat{r}, \widehat{\mathbf{N}}, \widehat{\sigma}$ , and

$$\widehat{r}^\gamma \widehat{w}(\widehat{r}) = Cr^\gamma w(r), \quad \widehat{r}^{\frac{\widehat{\mathbf{N}}-\mathbf{p}}{\mathbf{p}-1}} \widehat{w}(\widehat{r}) = Cr^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w(r), \quad \widehat{r}^{-\frac{\widehat{\sigma}+1}{\mathbf{p}-1}} \widehat{w}'(\widehat{r}) = \lambda Cr^{-\frac{\sigma+1}{\mathbf{p}-1}} w'(r). \quad (4.10)$$

Moreover for any  $(\mathbf{N}, \sigma) \in \mathbf{F}$  (resp.  $\mathbf{D}$ , resp.  $\mathbf{E}$ ) there exists  $(\widehat{\mathbf{N}}, \widehat{\sigma}) \in \mathbf{A}$  (resp.  $\mathbf{B}$ , resp.  $\mathbf{C}$ ), such that (4.8) holds with  $\lambda = -1$ . As a consequence all the results valid for regions  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  apply respectively to  $\mathbf{F}, \mathbf{D}, \mathbf{E}$  by changing  $r$  into  $\frac{1}{r}$ .

**Proof.** Setting  $t = \lambda \hat{t}$  and  $(\hat{s}, \hat{z}) = \lambda(s, z)$ , we obtain a same system as (4.4) where  $\mathbf{N}, \sigma$  are replaced by  $\hat{\mathbf{N}}, \hat{\sigma}$ . This corresponds to the change of unknown (4.9). And the regions  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are respectively exchanged to  $\mathbf{F}, \mathbf{E}, \mathbf{D}$ , after choosing suitable reals  $\lambda < 0$  such that  $\mathbf{N} > \mathbf{0}$  corresponds to  $\hat{\mathbf{N}} > 0$ . The relations (4.10) are straightforward, implying the last conclusions. ■

As a consequence we prove that some upper estimates, which are classical when  $\mathbf{N} > \mathbf{p} > -\sigma$ , and still valid for other ranges of the parameters:

**Lemma 4.4** *Let  $\mathbf{q} > \mathbf{p} - 1 > 0$ , and  $\mathbf{N}, \sigma \in \mathbb{R}$ . There is a constant  $C_{\mathbf{N}, \mathbf{p}, \mathbf{q}} > 0$  such that any positive solution of (4.1) solution with in  $(0, r_0)$  (resp in  $(r_0, \infty)$ ) satisfies*

$$w(r) \leq C_{\mathbf{N}, \mathbf{p}, \mathbf{q}} r^{-\gamma} \quad \text{in } (0, \frac{r_0}{2}) \quad (\text{resp. in } (2r_0, \infty)).$$

**Proof.** In case  $\varepsilon = -1$ , this comes the Osserman's property, which is valid in the nonradial case when  $\mathbf{N}$  is an integer, see a proof in [?, Proposition 5.2], and for any subsolution. In the case  $\varepsilon = 1$ , it also extends to integral estimates in the nonradial case, see [12, Proposition 2.2 and Theorem 3.1], valid for any  $\sigma \in \mathbb{R}$ . Both results suppose  $\mathbf{N} > \mathbf{1}$ , for  $\mathbf{p} > \mathbf{1}$ , and are given and  $\mathbf{N} > \mathbf{p}$ , even if this condition does not seem necessary. Next suppose  $\mathbf{N} < \mathbf{p}$ . We use the transformation (4.8) (4.9) with  $\lambda = -1$  : it defines  $\hat{\mathbf{N}} = 2\mathbf{p} - \mathbf{N}$  and  $\hat{\sigma} = -2\mathbf{p} - \sigma$ , and  $w(r) = \hat{w}(\hat{r})$ ,  $\hat{r} = \frac{1}{r}$ . From (4.10),  $\hat{r}^\gamma \hat{w}(\hat{r}) = Cr^\gamma w(r)$ , and the behaviour near 0 or  $\infty$  is exchanged, hence the conclusion is still valid. ■

From now on in this section, the proofs of the lemmas and theorems are given in the Appendix. We first analyze the nature of the fixed points of the system:

**Lemma 4.5** *The fixed points of system (4.4) are*

$$\begin{aligned} M_0 = (s_0, z_0) &= (\gamma, \mathbf{N} - \mathbf{p} - (\mathbf{p} - 1)\gamma) = (\frac{\mathbf{p} + \sigma}{\mathbf{q} + 1 - \mathbf{p}}, \frac{(\mathbf{N} - \mathbf{p})(\mathbf{q} - \mathbf{q}_c)}{\mathbf{q} + 1 - \mathbf{p}}), \\ N_0 = (0, \mathbf{N} + \sigma), \quad A_0 &= (\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}, 0), \quad \text{and} \quad (0, 0). \end{aligned} \quad (4.11)$$

(i) *The point  $M_0$  corresponds to the particular solutions defined at (4.2). The eigenvalues  $\lambda_1, \lambda_2$  of the linearized system associated to system (4.4) at  $M_0$  are the roots of the equation*

$$T(\lambda) = \lambda^2 - (s_0 - z_0)\lambda + \frac{\mathbf{q} - \mathbf{p} + 1}{\mathbf{p} - 1} s_0 z_0 = \lambda^2 - (\mathbf{p}\gamma + \mathbf{p} - \mathbf{N})\lambda + \frac{(\mathbf{p} + \sigma)(\mathbf{N} - \mathbf{p})(\mathbf{q} - \mathbf{q}_c)}{(\mathbf{p} - 1)(\mathbf{q} - \mathbf{p} + 1)} = 0. \quad (4.12)$$

$M_0$  is a saddle point when  $s_0 z_0 < 0$ .

(ii) *The eigenvalues at  $N_0$  are  $l_1 = \frac{\mathbf{p} + \sigma}{\mathbf{p} - 1}$  and  $l_2 = -(\mathbf{N} + \sigma)$ ; and  $\{s = 0\}$  contains nonadmissible trajectories linked to  $l_2$ .  $N_0$  is a saddle point when  $(\mathbf{p} + \sigma)(\mathbf{N} + \sigma) > 0$ .*

(iii) *The eigenvalues at  $A_0$  are  $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}$  and  $\mu_2 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}(\mathbf{q}_c - \mathbf{q}) = \mathbf{N} + \sigma + \mathbf{q} \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}$ ; and  $\{z = 0\}$  contains nonadmissible trajectories linked to  $\mu_1$ .  $A_0$  is a saddle point when  $\mathbf{q} > \mathbf{q}_c$ .*

(iv) *The eigenvalues at  $(0, 0)$  are  $\rho_1 = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}$  and  $\rho_2 = \mathbf{N} + \sigma$ , and  $\{s = 0\}$  and  $\{z = 0\}$  contain nonadmissible trajectories, linked to  $\rho_1$  and  $\rho_2$ . The point  $(0, 0)$  is a saddle point when  $(\mathbf{N} - \mathbf{p})(\mathbf{N} + \sigma) > 0$ .*

**Remark 4.6** *Recall that any trajectory in the phase plane distinct from the two axis corresponds to an infinity with one parameter of solutions of equation (4.1). So any saddle-point leads at most to two families of such solutions, and any source or sink leads to an infinity of solutions with two parameters.*

Next we make the link between the convergence to the fixed points and the behaviour of the solutions  $w$  near 0 or  $\infty$ .

**Lemma 4.7** *Let  $w$  be any positive solution of (4.1) defined near 0 (resp near  $\infty$ ). Then as  $r \rightarrow 0, t \rightarrow -\infty$  (resp  $r \rightarrow \infty, t \rightarrow \infty$ )*

$$\begin{aligned}
(i) \quad \lim(s, z) = M_0 &\implies w \sim w^*, \quad w' \sim w'^*, \\
(ii) \quad \lim(s, z) = N_0 &\implies \lim w = c > 0, \quad \lim r^{-\frac{\sigma+1}{p-1}} w' = \left| \frac{c}{N+\sigma} \right|^{\frac{q}{p-1}} \text{sign}(-\varepsilon(N+\sigma)), \\
(iii) \quad \lim(s, z) = (0, 0) &\implies \lim w = c > 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-1}{p-1}} w' = d \neq 0, \\
(iv) \quad \lim(s, z) = A_0 &\implies \lim r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim r^{\frac{N-1}{p-1}} w' = k \frac{p-N}{p-1} \text{sign}(-\varepsilon z).
\end{aligned}$$

**Lemma 4.8** *Let  $w$  be any local solution in  $(0, r_0)$  (resp. in  $(r_0, \infty)$ ). Then the associated trajectory in the phase plane is bounded as  $t \rightarrow -\infty$  (resp.  $t \rightarrow \infty$ ). It converges to one of the fixed points of the system, or has a limit cycle around  $M_0$  (when  $\mathbf{q} = \mathbf{q}_S$ ).*

## 4.2 Study of regions A and F

We first consider region **A**, where the behaviour of the system (4.4) has been described in [10], and we deduce the following:

**Theorem 4.9** *Let  $N > p > -\sigma$  (region **A**).*

*There exist local ( $C^0$ -regular) solutions: for  $\varepsilon = \pm 1$  and any  $w_0 > 0$  there exists a unique solution such that*

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{p-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0.$$

(1) *Let  $p-1 < \mathbf{q} < \mathbf{q}_c$ .*

- *For  $\varepsilon = \pm 1$  and any  $k > 0$ , there exists an infinity of local solutions near 0 such that*

$$\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-1}{p-1}} w' = -c(k) < 0.$$

- *For  $\varepsilon = 1$ , there is no positive solution in  $(r_0, \infty)$ ,  $r_0 > 0$ .*
- *For  $\varepsilon = -1$ , there exists a global particular solution  $w^* = a^* r^{-\frac{p+\sigma}{q+1-p}}$ . Moreover there exist solutions such that*

$$\lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^\gamma w = a^*.$$

(2) *Let  $\mathbf{q} > \mathbf{q}_c$ .*

- *For  $\varepsilon = \pm 1$  and  $k > 0$ , there exists a local solution near  $\infty$ , unique up to a scaling, such that*

$$\lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = -c(k) < 0.$$

- *For  $\varepsilon = -1$ , all the local solutions near 0 are  $C^0$ -regular (the singularity is called removable), and not global.*

- *For  $\varepsilon = 1$ , there is a particular solution  $w^* = a^* r^{-\frac{p+\sigma}{q+1-p}}$ . Moreover\**

(i) either  $\mathbf{q} < \mathbf{q}_S$ , there exists no ground state and there exist solutions such that

$$\lim_{r \rightarrow 0} r^\gamma w = a^*, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}} w = k > 0.$$

(ii) or  $\mathbf{q} > \mathbf{q}_S$  and there exist  $C^0$ -ground states, such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow \infty} r^\gamma w = a^*.$$

(iii) or  $\mathbf{q} = \mathbf{q}_S$  and there is a family of explicit (well known)  $C^0$ -ground states:

$$w = c(d + r^{\frac{\mathbf{p}+\sigma}{\mathbf{p}-1}})^{-\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}+\sigma}}, \quad d = c^{\mathbf{q}-\mathbf{p}+1}(\mathbf{p} + \sigma)^{-1} \left(\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}\right)^{1-\mathbf{p}}. \quad (4.13)$$

In the case  $\mathbf{N} > \mathbf{p} > -\sigma$ ,  $\mathbf{q} = \mathbf{q}_c$ , a precise local behaviour of logarithmic type was in great part studied for  $\sigma = 0, \varepsilon = 1$  in [?], [28, Theorem 4.1], and for  $\varepsilon = 1$  in [40]; the existence of such solutions were not clear. We give below a complete description:

**Theorem 4.10** (1) Let  $\mathbf{N} > \mathbf{p} > -\sigma$  and  $\mathbf{q} = \mathbf{q}_c$ . Then

- For  $\varepsilon = \pm 1$  there exist an infinity of local solutions near 0 such that

$$\lim_{r \rightarrow 0} (r |\ln r|^{\frac{1}{\mathbf{p}+\sigma}})^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = c(\mathbf{N}, \mathbf{p}) > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} |\ln r|^{\frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}+\sigma)(\mathbf{p}-1)}} |w'| = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} c(\mathbf{N}, \mathbf{p}) > 0.$$

- For  $\varepsilon = 1$ , there is no positive solution in  $(r_0, \infty)$ ,  $r_0 > 0$ .
- For  $\varepsilon = -1$ , there exist an infinity of solutions such that

$$\lim_{r \rightarrow \infty} (r |\ln r|^{\frac{1}{\mathbf{p}+\sigma}})^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = c(\mathbf{N}, \mathbf{p}), \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} |\ln r|^{\frac{\mathbf{N}-\mathbf{p}}{(\mathbf{p}+\sigma)(\mathbf{p}-1)}} |w'| = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} c(\mathbf{N}, \mathbf{p}).$$

**Remark 4.11** As mentioned in [10, Remark 3.2], some energy functions of Pohozaev-type linked to the equation are well known. In fact they are valid for any values of the parameters  $\mathbf{p}, \mathbf{q}, \mathbf{N}, \sigma$ , and for the two equations ( $\varepsilon = \pm 1$ ):

$$F_\theta(r) = r^{\mathbf{N}} \left( \frac{(\mathbf{p}-1) |w'|^{\mathbf{p}}}{\mathbf{p}} + \varepsilon r^\sigma \frac{w^{\mathbf{q}+1}}{\mathbf{q}+1} + \theta \frac{w |w'|^{\mathbf{p}-2} w'}{r} \right) = r^{\mathbf{N}-\mathbf{p}} w^{\mathbf{p}} |s|^{\mathbf{p}-2} s \left( \frac{(\mathbf{p}-1)s}{\mathbf{p}} + \frac{z}{\mathbf{q}+1} - \theta \right),$$

either with  $\theta = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}}$  or  $\theta = \frac{\mathbf{N}+\sigma}{\mathbf{q}+1}$ , satisfying respectively

$$F'_{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}}}(r) = r^{\mathbf{N}-1+\sigma} \left( \frac{\mathbf{N} + \sigma}{\mathbf{q} + 1} - \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p}} \right) w^{\mathbf{q}+1}, \quad F'_{\frac{\mathbf{N}+\sigma}{\mathbf{q}+1}}(r) = r^{\mathbf{N}-1} \left( \frac{\mathbf{N} + \sigma}{\mathbf{q} + 1} - \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p}} \right) |w'|^{\mathbf{p}},$$

Other type of functions can be computed as the ones obtained for  $\sigma = 0$  in [5, Proposition 2.2], coinciding with the functions above when  $\mathbf{q} = \mathbf{q}_S$ . In particular  $\mathbf{q} = \mathbf{q}_S$  is the case of constant energy, leading to the existence of the ground states mentioned above at (4.13) when  $\varepsilon = 1$ , and to explicit local solutions on  $[0, r_0)$  and on  $[r_0, \infty)$  when  $\varepsilon = -1$ .

From Lemma 4.3, we deduce the complete behaviour in region  $\mathbf{F}$ . It offers a new striking result in case  $\mathbf{q} = \mathbf{q}_S$  of existence of explicit  $C^0$ -ground states, increasing and bounded at  $\infty$  :

**Corollary 4.12** *Let  $-\sigma > \mathbf{p} > \mathbf{N}$  (region **F**). Then all the conclusions of Theorems 4.9 (for  $\mathbf{q} \neq \mathbf{q}_c$ ) and 4.10 (for  $\mathbf{q} = \mathbf{q}_c$ ) apply after changing  $r$  into  $\frac{1}{r}$ . In particular for  $\varepsilon = \pm 1$  and any  $c > 0$  there exist a unique local solution near  $\infty$  such that*

$$\lim_{r \rightarrow \infty} w = c > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{\mathbf{p}-1}} |w'| = d(c).$$

For  $\varepsilon = 1, \mathbf{q} = \mathbf{q}_S$  there are explicit solutions given by

$$w = c(d + r^{\frac{\mathbf{p}+\sigma}{\mathbf{p}-1}})^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}+\sigma}}, \quad d = -c^{\mathbf{q}-\mathbf{p}+1}(\mathbf{p} + \sigma)^{-1} \left(\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}\right)^{1-\mathbf{p}}$$

satisfying  $\lim_{r \rightarrow 0} w = 0$ , with  $w \sim_{r \rightarrow 0} cr^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}}$  and  $\lim_{r \rightarrow \infty} w = cd^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}+\sigma}}$ .

### 4.3 Study of Regions **B** and **D**

The case of region **B** is particularly interesting; indeed we prove the following:

**Theorem 4.13** *Let  $\mathbf{p} > \mathbf{N} > -\sigma$  (region **B**)*

- For  $\varepsilon = \pm 1$  there exists local  $C^0$ -**regular** solutions of **three** types:

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = -\varepsilon c(w_0), \quad c(w_0) > 0, \quad (4.14)$$

$$\lim_{r \rightarrow 0} w = 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = f(k) > 0 \quad (4.15)$$

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = \pm c. \quad (4.16)$$

For  $\varepsilon = 1$  they are not global, and there is no solution in  $(r_0, \infty)$ . For  $\varepsilon = -1$  there exists two types of global solutions in  $(0, \infty)$ :

- $w^*(r) = a^* r^{-\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}}$ ,
- solutions such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(w_0) < 0, \quad w \sim_{r \rightarrow \infty} w^*.$$

As a direct consequence we obtain the behaviour in region **D**. Here also we have an very interesting behaviour: we find infinitely many bounded solutions in an exterior domain which do not converge to 0 at  $\infty$ , and global ones when  $\varepsilon = -1$ :

**Corollary 4.14** (region **D**) *Let  $\mathbf{p} < \mathbf{N} < -\sigma$ . Then*

- For  $\varepsilon = \pm 1$  there exists local solutions near  $\infty$  of **three** types:

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{\mathbf{p}-1}} w' = \varepsilon c(C), \quad c(C) > 0,$$

$$\lim_{r \rightarrow \infty} w = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(k) < 0,$$

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = D \neq 0.$$

For  $\varepsilon = 1$  they are not global, and there is no solution in  $(0, r_0)$ . For  $\varepsilon = -1$  there exists two types of global solutions in  $(0, \infty)$  :

- $w^*(r) = a^* r^{-\frac{p+\sigma}{q+1-p}}$ , which is  $C^0$ -regular,
- solutions such that

$$w \sim_{r \rightarrow 0} w^*, \quad \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = c(C) > 0.$$

#### 4.4 Study of regions C and E

We first consider region C:

**Theorem 4.15** *Let  $p < -\sigma < N$  (region C) Then there exists no  $C^0$ -regular solution. For  $\varepsilon = \pm 1$  there exist local (nonglobal) solutions near  $\infty$  of two types:*

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{p-1}} w' = -\varepsilon D, \quad D > 0, \quad (4.17)$$

$$\lim_{r \rightarrow \infty} w = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{p-1}} w' = -c(k) < 0. \quad (4.18)$$

For  $\varepsilon = -1$ , there exist two types of global solutions in  $(0, \infty)$  :

- $w^*(r) = a^* r^{-\frac{p+\sigma}{q+1-p}}$ , which is a cusp-solution,
- solutions such that

$$w \sim_{r \rightarrow 0} w^*, \quad \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{-\frac{\sigma+1}{p-1}} w' = D > 0,$$

and there exist also solutions on  $(r_0, \infty)$  such that  $u(r_0) = 0$  and others such that  $\lim_{r \rightarrow r_0} w = \infty$ , and  $w \sim_{r \rightarrow \infty} w^*$ , and nonglobal solutions on  $(0, r_0)$  such that  $w \sim_{r \rightarrow 0} w^*$  and  $\lim_{r \rightarrow r_0} w = \infty$ .

For  $\varepsilon = 1$  there is no local solution in  $(0, r_0)$ .

**Corollary 4.16** (region E) *Let  $p > -\sigma > N$ . For  $\varepsilon = \pm 1$  there exist local (nonglobal) solutions near 0 of two types:*

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{p-1}} w' = \varepsilon D, \quad D > 0,$$

$$\lim_{r \rightarrow 0} w = 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-p}{p-1}} w = k > 0, \quad \lim_{r \rightarrow 0} r^{\frac{N-1}{p-1}} w' = d(k) > 0.$$

For  $\varepsilon = -1$ , there exist two types of global solutions in  $(0, \infty)$  :

- $w^*(r) = a^* r^{-\frac{p+\sigma}{q+1-p}}$
- solutions such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{p-1}} w' = d < 0, \quad w \sim_{r \rightarrow \infty} w^*,$$

and there exist also solutions on  $(0, r_0)$  such that  $w(r_0) = 0$  and others such that  $\lim_{r \rightarrow r_0} w = \infty$ , and  $w \sim_{r \rightarrow 0} w^*$ , and nonglobal solutions on  $(r_0, \infty)$  such that  $w \sim_{r \rightarrow \infty} w^*$  and  $\lim_{r \rightarrow r_0} w = \infty$ .

For  $\varepsilon = 1$  there is no local solution in  $(r_0, \infty)$ .

## 4.5 Other case of explicit solutions

We have recalled at (4.13) the well-known explicit ground states obtained for  $\mathbf{q} = \mathbf{q}_S$ . Here we give another case where we find global explicit solutions in  $\mathbb{R}^N \setminus \{0\}$ . We remark that system (1.1) admits the solutions  $(u_1, u_2) = (u, u)$  when  $p = q$ , where  $u$  is a solution of the scalar equation (1.7), given explicitly by (2.3) and (2.4); and the corresponding solutions of system (3.7) satisfy the relation  $S \equiv -Z$ , and  $\mathbf{p} = \mathbf{N} = -\sigma$ . This suggests that system (4.4) with general  $\mathbf{p}, \mathbf{N}, \sigma$  may admit particular explicit solutions for some values of the parameters. We show below that it is true, and this result appears to be new:

**Theorem 4.17** *Let  $\mathbf{q} > \mathbf{p} - 1 > 0$ . When  $\sigma = -\mathbf{p} \frac{\mathbf{N}-1}{\mathbf{p}-1}$  there exist explicit radial solutions  $w$  of the Hardy-Hénon equation (4.1) with  $\varepsilon = -1$ , of the form*

$$w = (c \pm d_{\mathbf{p}, \mathbf{q}, \mathbf{N}} r^{\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}})^{-\frac{\mathbf{p}}{\mathbf{q}-\mathbf{p}+1}}, \quad \text{if } \mathbf{p} \neq \mathbf{N}, \quad (4.19)$$

$$w = (c \pm d_{\mathbf{N}} \ln r)^{-\frac{\mathbf{N}}{\mathbf{q}-\mathbf{N}+1}}, \quad \text{if } \mathbf{p} = \mathbf{N}, \quad (4.20)$$

where  $c > 0$  and  $d_{\mathbf{p}, \mathbf{q}, \mathbf{N}} = \frac{(\mathbf{p}-1)(\mathbf{q}-\mathbf{p}+1)}{\mathbf{p}(\mathbf{p}-\mathbf{N})} \left( \frac{\mathbf{p}}{(\mathbf{p}-1)(\mathbf{q}+1)} \right)^{\frac{1}{\mathbf{p}}}$ ,  $d_{\mathbf{N}} = \frac{\mathbf{q}-\mathbf{N}+1}{\mathbf{N}} \left( \frac{\mathbf{N}}{(\mathbf{N}-1)(\mathbf{q}+1)} \right)^{\frac{1}{\mathbf{N}}}$ .

## 4.6 Limit cases

Here we give a complete study of all the critical cases  $\sigma = -\mathbf{p} \neq -\mathbf{N}$ ,  $\sigma = -\mathbf{N} \neq -\mathbf{p}$ ,  $\mathbf{p} = \mathbf{N} \neq -\sigma$ ,  $\mathbf{p} = \mathbf{N} = -\sigma$ .

### 4.6.1 Case $\sigma = -\mathbf{p} \neq -\mathbf{N}$

**Theorem 4.18** (1) *Assume  $\mathbf{N} > \mathbf{p} = -\sigma$*

- *For  $\varepsilon = \pm 1$ , there exist local solutions near  $\infty$  such that*

$$\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(k) < 0.$$

- *For  $\varepsilon = 1$ , there exists an infinity with 2 parameters of local solutions near  $\infty$  such that*

$$\lim_{r \rightarrow \infty} (\ln r)^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}} w(r) = \left( \frac{\mathbf{p}-1}{\mathbf{q}} \right)^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}}. \quad (4.21)$$

- *For  $\varepsilon = -1$ , there exists at least a local solution near 0 such that*

$$\lim_{r \rightarrow 0} |\ln r|^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}} w(r) = \left( \frac{\mathbf{p}-1}{\mathbf{q}} \right)^{\frac{\mathbf{p}-1}{\mathbf{q}-\mathbf{p}+1}}. \quad (4.22)$$

- (2) *Assume  $\mathbf{p} = -\sigma > \mathbf{N}$ . Then the behaviour is deduced from (1) by changing  $r$  into  $\frac{1}{r}$ .*



#### 4.6.2 Case $\sigma = -\mathbf{N} \neq -\mathbf{p}$

**Theorem 4.19** (1) Assume  $-\sigma = \mathbf{N} > \mathbf{p}$ .

- For  $\varepsilon = \pm 1$ , there exist local solutions near  $\infty$  such that

$$\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} w' = -c(k) < 0. \quad (4.23)$$

- For  $\varepsilon = \pm 1$ , there exists an infinity with 2 parameters of local solutions near  $\infty$  such that

$$\lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} (\ln r)^{-\frac{1}{\mathbf{p}-1}} w' = -\varepsilon c(C). \quad (4.24)$$

- For  $\varepsilon = -1$ , there exists  $C^0$ -regular solutions  $w^* = a^* r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}$ . There exist local solutions near 0 such that  $w \sim_{r \rightarrow 0} w^*$ , and local ones near  $\infty$  such that  $w \sim_{r \rightarrow \infty} w^*$ . There exists an infinity of solutions such that

$$w \sim_{r \rightarrow 0} w^*, \quad \lim_{r \rightarrow \infty} w = C > 0, \quad \lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-1}{\mathbf{p}-1}} (\ln r)^{-\frac{1}{\mathbf{p}-1}} w' = C(\mathbf{N}, \mathbf{p}, \mathbf{q}). \quad (4.25)$$

- (2) Assume  $\mathbf{p} > \mathbf{N} = -\sigma$ . Then the behaviour is deduced from (1) by changing  $r$  into  $\frac{1}{r}$ .

#### 4.6.3 Case $\mathbf{p} = \mathbf{N} \neq -\sigma$

**Theorem 4.20** (1) Assume  $\mathbf{p} = \mathbf{N} > -\sigma$

- For  $\varepsilon = \pm 1$  there exists local ( $C^0$ -regular) solutions: for any  $c > 0$  there exists a unique solution such that

$$\lim_{r \rightarrow 0} w = w_0 > 0, \quad \lim_{r \rightarrow 0} r^{-\frac{\sigma+1}{\mathbf{p}-1}} |w'| = c(w_0) > 0.$$

- For  $\varepsilon = 1$ , there exist an infinity of local solutions near 0 such that

$$\lim_{r \rightarrow 0} |\ln r|^{-1} w = C > 0, \quad \lim_{r \rightarrow 0} r w' = -C.$$

- For  $\varepsilon = -1$ , there exists a particular solution  $w^* = a^* r^{-\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}}$ . There exist local solutions near 0 such that  $w \sim_{r \rightarrow 0} w^*$ , and local ones near  $\infty$  such that  $w \sim_{r \rightarrow \infty} w^*$ . There exists an infinity of solutions such that

$$\lim_{r \rightarrow 0} |\ln r|^{-1} w = C, \quad \lim_{r \rightarrow 0} r w' = -C, \quad w \sim_{r \rightarrow \infty} w^*.$$

- (2) Assume  $\mathbf{p} = \mathbf{N} < -\sigma$ . Then the behaviour is deduced from (1) by changing  $r$  into  $\frac{1}{r}$ .

#### 4.6.4 Case $\mathbf{p} = \mathbf{N} = -\sigma$

**Theorem 4.21** Assume  $\mathbf{p} = \mathbf{N} = -\sigma$ . For  $\varepsilon = -1$ , there exist local explicit solutions near 0 or near  $\infty$  of the form (4.20):

$$w = (C \pm d_{\mathbf{N}} \ln r)^{-\frac{\mathbf{N}}{\mathbf{q}-\mathbf{N}+1}}, \quad C \in \mathbb{R}. \quad (4.26)$$

For  $\varepsilon = 1$  there is no local solution near 0 nor  $\infty$ .

## 5 Description of the radial solutions of system (3.1)

### 5.1 The case $p = q$

In the case  $p = q$ , the system (1.1) admits solutions of the form  $u_1 \equiv u_2 \equiv u$ , where  $u$  is any solution of the scalar equation (1.7). However at Proposition 3.1 we have constructed local solutions of non symmetrical type, for example such that  $u'_1(r_0) = 0$ ,  $u'_2(r_0) \neq 0$  at some point  $r_0 > 0$ . A natural question is the existence global solutions in  $\mathbb{R}^N \setminus \{0\}$ . Here we answer this question:

**Proposition 5.1** *Assume that  $p = q > 1$ . Then all the radial solutions of system (1.1) satisfy the first integral in any interval of definition*

$$|u'_1|^q u'_1 - |u'_2|^q u'_2 \equiv Cr^{(1-N)(q+1)}, \quad C \in \mathbb{R}, \quad (5.1)$$

and can be computed by quadrature. All the radial solutions  $(u_1, u_2)$  in  $\mathbb{R}^N \setminus \{0\}$  satisfy  $u_1 \equiv u_2 \equiv u$ , where  $u$  is any solution of the scalar Hamilton-Jacobi equation.

**Proof.** Here system (3.3) reduces to

$$\begin{cases} w'_1 = r^{(N-1)(1-q)} |w_2|^q, \\ w'_2 = r^{(N-1)(1-q)} |w_1|^q. \end{cases}$$

As a consequence,

$$|w_1|^q w'_1 - |w_2|^q w'_2 = r^{(N-1)(1-q)} |w_1|^q |w_2|^q - r^{(N-1)(1-q)} |w_1|^q |w_2|^q = 0.$$

So we get the relation

$$|w_1|^q w_1 - |w_2|^q w_2 \equiv C,$$

equivalent to (5.1). Suppose that  $C \neq 0$ . By symmetry we can suppose that  $C = c^{q+1} > 0$ . Then we obtain

$$\frac{w'_2}{|C + |w_2|^q w_2|^{\frac{q}{q+1}}} = r^{(N-1)(1-q)}.$$

We claim that the solution cannot be defined on  $\mathbb{R}^N \setminus \{0\}$ . Indeed let

$$F(\theta) = \int_0^\theta \frac{d\theta}{|c^{q+1} + |\theta|^q \theta|^{\frac{q}{q+1}}} = \int_0^{-c} \frac{d\theta}{|c^{q+1} + |\theta|^q \theta|^{\frac{q}{q+1}}} + \int_{-c}^\theta \frac{d\theta}{|c^{q+1} + |\theta|^q \theta|^{\frac{q}{q+1}}}.$$

This function is well defined, since the integrals are convergent at the bound  $-c$  since  $\frac{q}{q+1} < 1$ . And the integrals are convergent at the bounds  $\pm\infty$ , since  $q > 1$ , thus  $F$  is bounded. If the solution is global, that means  $r$  describes  $(0, \infty)$ , then  $F(w_2) = \frac{r^{N-(N-1)q}}{N-(N-1)q} + D$ ,  $D \in \mathbb{R}$ , for  $q \neq \frac{N}{N-1}$ ,  $F(w_2) = \ln r + D$  if  $q = \frac{N}{N-1}$ , which is impossible in any case. So the solutions are not global. Hence all the global solutions on  $(0, \infty)$  satisfy  $w_1 \equiv w_2$ ,  $u_1 = u_2 + c$ ,  $c \in \mathbb{R}$ , where  $u_2$  is solution of the scalar equation. The nonglobal solutions can be computed on any interval where  $w_1, w_2$  have a constant sign, by the formulas  $F(w_2) = \frac{r^{N-(N-1)q}}{N-(N-1)q} + D$ , for  $q \neq \frac{N}{N-1}$ ,  $F(w_2) = \ln r + D$  if  $q = \frac{N}{N-1}$ . ■

**Remark 5.2** *When  $q \neq \frac{N}{N-1}$ , the existence of local solutions near 0 or  $\infty$  will be described at Theorem 5.7. When  $q = \frac{N}{N-1}$ , such solutions do not exist, because  $F$  is bounded.*

Note that we will find again these results as a particular case of the general study below.

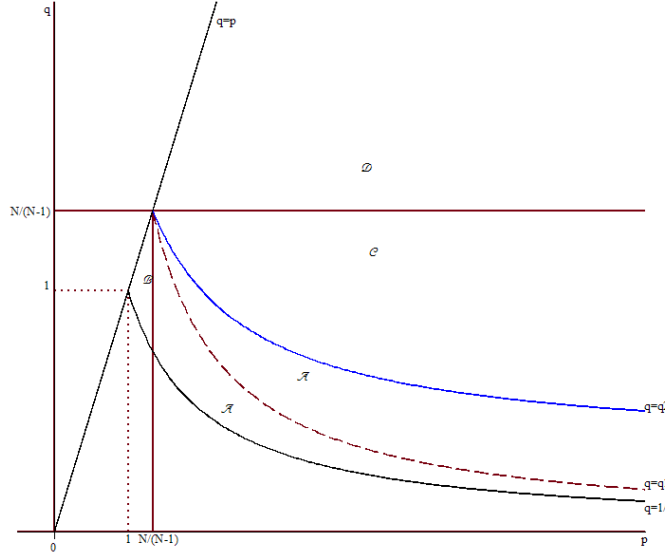
## 5.2 Constant sign solutions $(u'_1, u'_2)$ of system (3.1)

In the sequel we prove that the situation is extremely rich when  $p \neq q$ . In particular we construct many nonsymmetrical solutions of system (3.1) in  $\mathbb{R}^N \setminus \{0\}$ . In this paragraph, we study the existence of solutions in terms of the derivatives, by applying all the results of Section 4 to system (3.1). Note that we can describe all the solutions, defined in a maximal set  $B_{r_0}$  or  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ , where at least one vanishes, explodes or has a singularity at the level of the gradient of  $\partial B_{r_0}$ . Here we focus our study on solutions defined near 0, and above all on global solutions in  $\mathbb{R}^N \setminus \{0\}$ .

We distinguish four regions of study, corresponding respectively to the regions **A**, **B**, **C**, **D** defined at (4.7) for system (4.4), where  $\mathbf{p}, \mathbf{N}, \sigma$  are defined at (3.15), (3.17).

**Definition 5.3** Let  $\mathcal{S} = \left\{ (p, q) \in \mathbb{R}^2 : pq > 1, q \leq p, q \neq q_1, q \neq q_2, p \neq \frac{N}{N-1}, q \neq \frac{N}{N-1} \right\}$  we consider the subsets of  $\mathcal{S}$  defined by

$$\begin{aligned} \mathcal{A} &= \left\{ p > \frac{N}{N-1}, q < q_2 \right\}, & \mathcal{B} &= \left\{ p < \frac{N}{N-1} \right\}, \\ \mathcal{C} &= \left\{ q_2 < q < \frac{N}{N-1} \right\}, & \mathcal{D} &= \left\{ q > \frac{N}{N-1} \right\}. \end{aligned}$$



**Lemma 5.4** Let  $(u_1, u_2)$  be any solution of system (1.1) defined near 0 (resp near  $\infty$ ), and  $w_1, w_2$  be associated by (3.2) and  $w = |w_1|$  be a positive solution of (3.14). Then

$$u'_1 \text{ has the sign of } -w', \quad \text{and} \quad u'_2 \text{ has the sign of } \varepsilon w'. \quad (5.2)$$

And as  $r \rightarrow 0, t \rightarrow -\infty$  (resp  $r \rightarrow \infty, t \rightarrow \infty$ ),

$$\begin{aligned}
(i) \quad & w \sim w^* \quad \Rightarrow \quad (u'_1, u'_2) \sim (u_1^*, u_2^*), \\
(ii) \quad & \begin{cases} \lim w = c > 0, \\ \lim r^{-\frac{\sigma+1}{p-1}} w' = d \neq 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \lim r^{N-1} |u'_1| = c_1 > 0, \\ \lim r^{(N-1)q-1} |u'_2| = c_2 > 0, \end{cases} \\
(iii) \quad & \begin{cases} \lim w = c > 0, \\ \lim r^{\frac{N-1}{p-1}} w' = d \neq 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \lim r^{N-1} |u'_1| = c_1 > 0, \\ \lim r^{N-1} |u'_2| = c_2 > 0, \end{cases} \\
(iv) \quad & \begin{cases} \lim r^{\frac{N-p}{p-1}} w = k > 0, \\ \lim r^{\frac{N-1}{p-1}} w' = d(k) \neq 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \lim r^{(N-1)p-1} |u'_1| = k > 0, \\ \lim r^{N-1} |u'_2| = |d(k)| > 0. \end{cases}
\end{aligned}$$

**Proof.** The function  $w = |w_1| = r^{N-1} |u'_1|$  satisfies the Hardy-Hénon equation with  $\mathbf{p}, \mathbf{q}, \mathbf{N}, \sigma$  satisfy (3.15),(3.17), and we deduce  $|w_2|$  from any of the formulas

$$|w_2| = (r^{(N-1)(p-1)} w_1')^{\frac{1}{p}}, \quad \text{or } w_2' = r^{(N-1)(1-q)} w^q, \quad (5.3)$$

and then we obtain the conclusions by computation, from the relations  $u'_i = -r^{1-N} w_i$ ,  $i = 1, 2$ . The signs of the derivatives are obtained from (3.8) and (3.11), since  $s \equiv S$ .  $\blacksquare$

**Remark 5.5** (i) From the phase plane analysis in Section 4, all the trajectories relative to singular **global** solutions of the Hardy-Hénon equation (4.1) are located in the quadrant containing the point  $M_0 = (S_0, Z_0)$  corresponding to the particular solutions. Moreover from (3.6)  $u'_1$  has the sign of  $S$  and  $u'_2$  has the sign of  $-Z$ . So  $u'_1$  has the sign of  $S_0$  and  $u'_2$  has the sign of  $-Z_0$ . From (3.9),

$$u'_1 < 0 \iff q > q_2, \quad u'_2 < 0 \iff q > q_1. \quad (5.4)$$

Then, with (5.2) and (5.4) we have a complete knowledge as soon as we know the absolute value of the derivatives.

(ii) Some results of Section 4 involve the Serrin exponent  $\mathbf{q}_c$  and the Sobolev exponent  $\mathbf{q}_S$ , defined at (4.6). We easily check that for our system,

$$\mathbf{q} \leq \mathbf{q}_c \iff q \leq q_1 = q_1(p), \quad \mathbf{q} \leq \mathbf{q}_S \iff q \leq q^* = q^*(p).$$

As a consequence of Theorems 4.9, 4.13, 4.15 and Corollary 4.14, we get the following result on **global** solutions of system (1.1), where we use Remark 5.5:

**Theorem 5.6** (1) Let  $(p, q) \in \mathcal{A}$ , with  $q < q_1$ . Then

- there exists a particular solution of system (1.1) such that  $u_1^{*'} = a_1 r^{-\frac{p+1}{pq-1}} > 0$  and  $u_2^{*'} = a_2 r^{-\frac{q+1}{pq-1}} > 0$ ,
- there exist solutions on  $(0, \infty)$  such that  $u'_1, u'_2 > 0$  and

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = c(c_1) > 0, \quad (u_1, u_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

(2) Let  $(p, q) \in \mathcal{A}$ , with  $q > q_1$ . Then

- there exists a particular solution such that  $u_1^{*'} = a_1 r^{-\frac{p+1}{pq-1}} > 0$  and  $u_2^{*'} = -a_2 r^{-\frac{q+1}{pq-1}} < 0$ ,

- if  $q < q^*$ , there exist solutions such that  $u'_1 > 0 > u'_2$ , and

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{(N-1)p-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = -c(c_1) < 0,$$

- if  $q > q^*$ , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{(N-1)q-1} u'_2 = -c_3 < 0, \quad (u'_1, u'_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*),$$

- if  $q = q^*$ , there exist **explicit** solutions:

$$u'_1 = cr^{1-N} (d + r^{\frac{p-q^*}{2}})^{\frac{2(N-(N-1)p)}{p-q^*}}, \quad u'_2 = -br^{(1-(N-1)q^*)} (d + r^{\frac{p-q^*}{2}})^{\frac{2((N-1)q^*-N)}{p-q^*}} \quad (5.5)$$

for any  $c > 0$ , with  $d = d(c) = \left(\frac{c^{pq^*-1}}{(N-1)p-N}\right)^{\frac{1}{p}} \frac{1}{N-(N-1)q^*}$ .  $b = b(c) = (c((N-1)p - N))^{\frac{1}{p}}$ .

(3) Let  $(p, q) \in \mathcal{B}$ . Then

- there exists a particular solution such that  $u_1^* = a_1 r^{-\frac{p+1}{pq-1}} > 0$  and  $u_2^* = a_2 r^{-\frac{q+1}{pq-1}} > 0$ .
- there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = c_1, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = c_2, \quad (u'_1, u'_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

(4) Let  $(p, q) \in \mathcal{C} \cup \mathcal{D}$ . Then there exists a particular solution such that  $u_1^* = -a_1 r^{-\frac{p+1}{pq-1}} < 0$  and  $u_2^* = -a_2 r^{-\frac{q+1}{pq-1}} < 0$ .

- If  $(p, q) \in \mathcal{C}$ , there also exist solutions such that

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-1} u'_1 = -c_1 < 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} u'_2 = -c(c_1) < 0.$$

- If  $(p, q) \in \mathcal{D}$ , there also exist solutions such that

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-1} u'_1 = -c_1 < 0, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = -c_2 < 0.$$

And we also get the existence of **local** but not global solutions near 0 or  $\infty$  with a behaviour of linear type, by using Lemma 5.4:

**Theorem 5.7** (1) *Local existence of solutions of system (1.1) near 0:*

- in regions  $\mathcal{A}$  or  $\mathcal{B}$ , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = \pm c(c_2), \quad \lim_{r \rightarrow 0} r^{(N-1)q-1} u'_2 = -c_2 < 0,$$

- in region  $\mathcal{A}$  with  $q < q_1$ , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} u'_1 = c(c_2) > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = \pm c_2,$$

- in region  $\mathcal{B}$ , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} u'_1 = -c(c_2) < 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = \pm c_2,$$

and solutions such that for any  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ ,

$$\lim_{r \rightarrow 0} r^{N-1} u'_1 = c_1, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = c_2,$$

and no such solution near 0 in regions  $\mathcal{C}, \mathcal{D}$ .

(2) Local existence of solutions near  $\infty$ :

- in regions  $\mathcal{C}$  and  $\mathcal{D}$ , for any  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = c_1, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = c_2,$$

- in region  $\mathcal{A}$  with  $q > q_1$ , and in regions  $\mathcal{C}$  and  $\mathcal{D}$ , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{(N-1)p-1} u'_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} u'_2 = \pm c(c_1),$$

- in region  $\mathcal{C}$ , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = \pm c(c_2), \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} u'_2 = -c_2 < 0,$$

- in region  $\mathcal{D}$ , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = \pm c(c_2), \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} u'_2 = c_2 > 0.$$

Finally we consider the limit cases:

**Theorem 5.8** (1) For  $q = q_1 < p$ , there exist local solutions of system (1.1) near 0 such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-1} |\ln r|^{\frac{p((N-1)p-1)}{p+1}} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} |\ln r|^{\frac{(N-1)p-1}{p+1}} u'_2 = -C_2 < 0,$$

(2) For  $q = q_2 < p$ , there exist local solutions near 0 or  $\infty$ , with respectively

$$\lim_{r \rightarrow 0} r^{N-1} |\ln r|^{\frac{1}{pq-1}} u'_1(r) = -C_1 < 0, \quad \lim_{r \rightarrow 0} r^{(N-1)q-1} |\ln r|^{\frac{q}{pq-1}} u'_2 = -C_2 < 0, \quad (5.6)$$

$$\lim_{r \rightarrow \infty} r^{N-1} (\ln r)^{\frac{1}{pq-1}} u'_1(r) = C_1 > 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-1} (\ln r)^{\frac{q}{pq-1}} u'_2 = -C_2 < 0, \quad (5.7)$$

(3) For  $p = \frac{N}{N-1} > q$ , there exist local solutions near 0, such that

$$\lim_{r \rightarrow 0} r^{N-1} |\ln r|^{-1} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = \pm C_2,$$

and global solutions such that

$$\lim_{r \rightarrow 0} r^{N-1} |\ln r|^{-1} u'_1 = C_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-1} u'_2 = C_2 > 0, \quad (u'_1, u'_2) \sim_{r \rightarrow \infty} (a_1 r^{-\frac{p+1}{pq-1}}, a_2 r^{-\frac{q+1}{pq-1}}),$$

(4) For  $q = \frac{N}{N-1} < p$ , there exist local solutions near  $\infty$  such that

$$\lim_{r \rightarrow \infty} r^{N-1} u'_1 = \pm C_1, \quad \lim_{r \rightarrow \infty} r^{N-1} (\ln r)^{-p} u'_2 = -C_2 < 0,$$

and global ones such that,

$$(u'_1, u'_2) \sim_{r \rightarrow 0} (-a_1 r^{-\frac{p+1}{pq-1}}, -a_2 r^{-\frac{q+1}{pq-1}}), \quad \lim_{r \rightarrow \infty} r^{N-1} u'_1 = -C_1 < 0, \quad \lim_{r \rightarrow \infty} r^{N-1} (\ln r)^{-p} u'_2 = -C_2 < 0.$$

(5) For  $q = p = \frac{N}{N-1}$  (where  $\mathbf{p} = \mathbf{N} = -\sigma$ ) we find again the solutions given at (2.4) and no other local solution.

**Proof.** We deduce (1) from Theorems 4.10, since  $\mathbf{q} = \mathbf{q}_c$ , (2) from 4.18, since  $\mathbf{N} > \mathbf{p} = -\sigma$ , (3) from 4.20  $\mathbf{p} = \mathbf{N} > -\sigma$ , (4) from Theorem 4.19  $-\sigma = \mathbf{N} > \mathbf{p}$ , and (5) 4.21  $\mathbf{p} = \mathbf{N} = -\sigma$ ; there exists no other solution near 0 or  $\infty$ , see Remark 5.2.  $\blacksquare$

### 5.3 Local radial existence results for system (1.1)

Next we deduce existence results for the system obtained by integration of the previous results. We first give a result of existence of local solutions near 0, with a behaviour of linear type. For simplicity, due to the great number of possibilities, we consider only the positive solutions of the system. One can formulate analogous results for systems (1.5) and (1.6).

**Theorem 5.9** *Existence of solutions of system (1.4) in  $B_{r_0} \setminus \{0\}$ ,  $r_0 > 0$  :*

- in regions  $\mathcal{A}$  or  $\mathcal{B}$ , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-2} u_2 = 0, \quad \begin{cases} \lim_{r \rightarrow 0} r^{(N-1)q-2} u_2 = c(c_1), & \text{if } q > \frac{2}{N-1}, \\ \lim_{r \rightarrow 0} u_2 = c_2, & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow 0} (|\ln r|^{-1} u_2) = c_2, & \text{if } q = \frac{2}{N-1}; \end{cases}$$

- in region  $\mathcal{B}$ , there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} u_1 = 0, \quad \lim_{r \rightarrow 0} r^{(N-1)p-2} u_1 = c(c_2), \quad \lim_{r \rightarrow 0} r^{N-2} u_2 = c_2 > 0,$$

and solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-2} u_2 = c_2 > 0.$$

In any case the solutions satisfy the equations in the sense of distributions in  $B_{r_0}$ , where  $\delta_0$  is the Dirac mass at 0:

$$\begin{cases} -\Delta u_1 = |u_2'|^p + C_1 \delta_0, \\ -\Delta u_2 = |u_1'|^q + C_2 \delta_0, \end{cases} \quad (5.8)$$

with respectively  $(C_1, C_2) = (c_N c_1, 0)$ ,  $(C_1, C_2) = (0, c_N c_2)$ ,  $(C_1, C_2) = (c_N c_1, c_N c_2)$ .

- There is no radial supersolutions of system (5.8) such that  $C_1 > 0$  or  $C_2 > 0$  for  $p \geq q \geq q_2$ , and no supersolutions such that  $C_2 > 0$  for  $p \geq \frac{N}{N-1}$ .

**Proof.** The existence of the solutions is a direct consequence of Theorem 5.7). and the behaviour follows from Brezis-Lions Lemma applied to  $u_i$ ,  $i = 1, 2$ . Next consider any radial supersolution  $(u_1, u_2)$  and assume  $q \geq q_2$ . If  $C_1 > 0$ , then  $u_1 \geq cr^{2-N}$  and  $|u_1'| \geq cr^{1-N}$  near 0, and  $|u_1'|^q \in L_{loc}^1(B_{r_0})$ , thus  $q < \frac{N}{N-1}$ , and  $-(r^{N-1}u_2')' \geq cr^{(N-1)(1-q)}$ , then by integration  $\lim_{r \rightarrow 0} r^{N-1}|u_2'| = l \geq 0$ , and  $|u_2'| \geq \frac{r^{1-(N-1)q}}{N-(N-1)q}$  if  $l = 0$ . In any case,  $|u_2'| \geq cr^{1-(N-1)q}$ . Then  $-(r^{N-1}u_1')' \geq cr^{(1-(N-1)q)p}$ , thus  $r^{(1-(N-1)q)p} \in L_{loc}^1(B_{r_0})$ , so that  $N + p - (N-1)pq > 0$ , which is  $q < q_2$ . Similarly, if  $C_2 > 0$ , then  $q < q_1$ ; moreover  $|u_2'| \geq cr^{1-N}$  near 0, thus  $|\bar{u}_2'|^p \geq Cr^{(1-N)p}$ , and  $|u_1'|^p \in L_{loc}^1(B_{r_0})$ , thus  $N - (N-1)p > 0$ . ■

In the same way we obtain existence of solutions in an exterior domain with a linear type:

**Theorem 5.10** *Existence of solutions of system (1.4) in  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ :*

- in region  $\mathcal{C}$ , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \quad \begin{cases} \lim_{r \rightarrow \infty} r^{(N-1)q-2} u_2 = \pm c(c_1), & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow \infty} u_2 = c_2 > 0, & \text{if } q > \frac{2}{N-1}, \\ \lim_{r \rightarrow \infty} (|\ln r|^{-1} u_2) = c(c_1), & \text{if } q = \frac{2}{N-1}, \end{cases}$$

- in regions  $\mathcal{C}$  and  $\mathcal{D}$ , there exist solutions such that

$$\lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-2} u_2 = c_2 > 0,$$

- in regions  $\mathcal{A}$  with  $q > q_1$ , and  $\mathcal{D}$ , there exist solutions such that

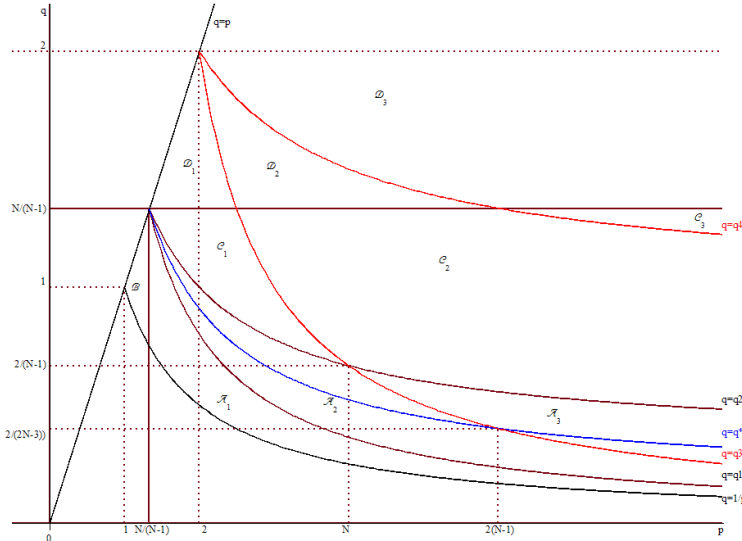
$$\lim_{r \rightarrow \infty} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-2} u_2 = c_2 > 0.$$

## 5.4 Global radial existence and behaviour

Next we study all the *global* constant sign solutions  $(u_1, u_2)$  of system (1.1), of any sign. Since they are obtained by integration of the derivatives, we are led to divide some of the regions  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  into subregions, according to the position of  $q$  with respect to  $q_3, q_4$ .

**Definition 5.11** We set  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3, \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3, \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ , where

$$\begin{aligned} \mathcal{A}_1 &= \left\{ p > \frac{N}{N-1}, q < q_1 \right\}, & \mathcal{A}_2 &= \{ q_1 < q < \min(q_2, q_3) \}, & \mathcal{A}_3 &= \{ q_3 < q < q_2 \}, \\ \mathcal{C}_1 &= \left\{ q_2 < q < \min(q_3, \frac{N}{N-1}) \right\}, & \mathcal{C}_2 &= \left\{ \max(q_2, q_3) < q < \min(q_4, \frac{N}{N-1}) \right\}, & \mathcal{C}_3 &= \left\{ q_4 < q < \frac{N}{N-1} \right\}, \\ \mathcal{D}_1 &= \left\{ \frac{N}{N-1} < q < q_3 \right\}, & \mathcal{D}_2 &= \left\{ \max(q_3, \frac{N}{N-1}) < q < q_4 \right\}, & \mathcal{D}_3 &= \left\{ \max(q_4, \frac{N}{N-1}) < q \right\}. \end{aligned}$$



The region  $\{q > q_4\}$ , containing  $\mathcal{C}_3 \cup \mathcal{D}_3$ , plays a particular role, as in the scalar case for  $q > 2$ , because we can construct solutions such that  $u_1$  and  $u_2$  are bounded on  $(0, \infty)$ , then for any of the three systems by adding suitable constants.

We first study the problem with source terms (1.4) in case  $q < q_4$  :



**Theorem 5.12** For  $(p, q) \in \mathcal{C}_1 \cup \mathcal{D}_1$ , up to positive constants, system (1.4) admits two types of global nonconstant solutions:

- $(u_1^*, u_2^*)$ , with both components  $\infty$ -singular,
- if  $(p, q) \in \mathcal{C}_1$ , solutions such that

$$(u_1, u_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-2} u_2 = c(c_1) > 0,$$

- if  $(p, q) \in \mathcal{D}_1$ , solutions such that

$$(u_1, u_2) \sim_{r \rightarrow 0} (u_1^*, u_2^*), \quad \lim_{r \rightarrow \infty} r^{N-2} u_1 = c_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-2} u_2 = c_2 > 0.$$

There is no global nonconstant solution in the other regions such that  $q < q_4$ .

**Proof.** Region  $\mathcal{C}_1 \cup \mathcal{D}_1$  is contained in  $\mathcal{C} \cup \mathcal{D}$ , so Theorem 5.6,(4) applies. In any case  $\lim r^{N-1} u_1' = -C_1 < 0$ , so  $u_1'$  is integrable at  $\infty$  since  $N > 2$ , and the same holds for  $u_2'$  for  $q > \frac{N}{N-1}$ . If  $q < \frac{N}{N-1}$ ,  $\lim r^{(N-1)q-1} u_2' = -C_2 < 0$  also implies that  $u_2'$  is integrable; Then the functions

$$u_1(r) = - \int_r^\infty u_1'(\tau) d\tau, \quad u_2(r) = - \int_r^\infty u_2'(\tau) d\tau,$$

satisfy the conclusions. Note that  $q > \frac{2}{N-1}$  in  $\mathcal{C}_1 \cup \mathcal{D}_1$ .

We know that there is no global solution in  $\mathcal{A} \cup \mathcal{B}$  from the nonradial study, and we are led to consider the regions  $\mathcal{C}_2, \mathcal{D}_2$ . From Theorems 4.14 and 4.15, there is no global solution  $w$  on  $(0, \infty)$  for  $\varepsilon = 1$ , and the global solutions relative to  $\varepsilon = -1$  do not bring solutions of system (1.4). Indeed near 0 they behave as  $w^*$ , and the corresponding solutions  $u_1^*, u_2^*$  given at (2.7) are not positive. ■

Next we consider the problem with absorption, for which the situation is very rich. Using Theorem 5.6 and similar arguments of integrability, we obtain the following:

**Theorem 5.13** Consider the system (1.5) For  $(p, q) \in \mathcal{B} \cup \mathcal{A}_1 \cup \mathcal{A}_3$  it admits a particular solution  $(\widetilde{u}_1^*, \widetilde{u}_2^*)$ .

- (1) If  $(p, q) \in \mathcal{A}_1$ ,  $\widetilde{u}_1^*, \widetilde{u}_2^*$  are  $\infty$ -singular, and there exist solutions such that

$$\lim_{r \rightarrow 0} r^{(N-1)p-2} \widetilde{u}_1 = c(c_2) > 0, \quad \lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_2^* = c_2 > 0, \quad (u_1, u_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

- (2) If  $(p, q) \in \mathcal{B}$ ,  $\widetilde{u}_1^*, \widetilde{u}_2^*$  are still  $\infty$ -singular, and there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_1 = c_1 > 0, \quad \lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_2^* = c_2 > 0, \quad (u_1, u_2) \sim_{r \rightarrow \infty} (u_1^*, u_2^*).$$

- (3) If  $(p, q) \in \mathcal{A}_3$ ,  $\widetilde{u}_1^*$  is  $\infty$ -singular, and  $\widetilde{u}_2^*$  is a cusp-solution; moreover

- if  $q < q^*$  there exist solutions such that

$$(\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow 0} (\widetilde{u}_1, \widetilde{u}_2^*), \quad \lim_{r \rightarrow \infty} r^{(N-1)p-2} \widetilde{u}_1 = c(c_2), \quad \lim_{r \rightarrow \infty} \widetilde{u}_2 = c_2 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_2' = k,$$

so the function  $\widetilde{u}_2$  varies from 0 to  $C_2$ .

- if  $q > q^*$  there exist solutions such that

$$\lim_{r \rightarrow 0} r^{N-2} \widetilde{u}_1 = c_1, \quad \lim_{r \rightarrow 0} r^{(N-1)q-2} \widetilde{u}_2 = c(c_1), \quad (\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow \infty} (\widetilde{u}_1, \widetilde{u}_2^*),$$

and there are explicit solutions if  $q = q^*$ .

There is no global nonconstant solution in the other regions such that  $q < q_4$ .

Now we study the system (1.6), also of a great richness, in particular involving the Sobolev exponent  $q^*$

**Theorem 5.14** *Consider the system (1.6). For  $(p, q) \in \mathcal{A}_2 \cup \mathcal{C}_2 \cup \mathcal{D}_2$  it admits a particular solution  $(\widehat{u}_1^*, \widehat{u}_2^*)$ .*

(1) *Assume  $(p, q) \in \mathcal{A}_2$ . Then  $\widehat{u}_1^*, \widehat{u}_2^*$  are  $\infty$ -singular; moreover*

- *if  $q < q^*$ , there exist solutions such that*

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow 0} (\widehat{u}_2^*, \widehat{u}_2^*), \quad \lim_{r \rightarrow \infty} r^{(N-1)p-2} \widehat{u}_1 = c(c_2), \quad \lim_{r \rightarrow \infty} r^{N-2} \widehat{u}_2 = c_2,$$

- *if  $q > q^*$ , there exist solutions such that*

$$\lim_{r \rightarrow 0} r^{N-2} \widehat{u}_1 = c_1, \quad \begin{cases} \lim_{r \rightarrow 0} \widehat{u}_2 = c_2, \lim_{r \rightarrow 0} r^{(N-1)q-1} \widehat{u}_2' = -c(c_1), & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow 0} r^{(N-1)q-2} \widehat{u}_2 = c(c_1), & \text{if } q > \frac{2}{N-1}, \end{cases}$$

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow \infty} (\widehat{u}_2^*, \widehat{u}_2^*),$$

- *if  $q = q^*$ , there exist explicit solutions, given by integration of (5.5).*

(2) *Assume  $(p, q) \in \mathcal{C}_2$ ,  $q \neq \frac{2}{N-1}$ . Then  $\widehat{u}_1^*$  is  $\infty$ -singular and  $\widehat{u}_2^*$  is a cusp-solution. There exist solutions such that*

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow 0} (\widehat{u}_2^*, \widehat{u}_2^*),$$

$$\lim_{r \rightarrow \infty} r^{N-2} \widehat{u}_1 = c_1, \quad \begin{cases} \lim_{r \rightarrow \infty} r^{(N-1)q-2} \widehat{u}_2 = c(c_1), & \text{if } q < \frac{2}{N-1}, \\ \lim_{r \rightarrow \infty} \widehat{u}_2 = c_2, \lim_{r \rightarrow \infty} r^{(N-1)q-1} \widehat{u}_2' = -c(c_1), & \text{if } q > \frac{2}{N-1}. \end{cases}$$

(3) *Assume  $(p, q) \in \mathcal{D}_2$ . Then again  $\widehat{u}_1^*$  is  $\infty$ -singular and  $\widehat{u}_2^*$  is a cusp-solution. There exist solutions such that*

$$(\widehat{u}_1, \widehat{u}_2) \sim_{r \rightarrow 0} (\widehat{u}_2^*, \widehat{u}_2^*), \quad \lim_{r \rightarrow \infty} r^{N-2} \widehat{u}_1 = c_1, \quad \lim_{r \rightarrow \infty} \widehat{u}_2 = c_2, \quad \lim_{r \rightarrow \infty} r^{N-1} \widehat{u}_2' = -c_2.$$

*There is no global nonconstant solution in the other regions such that  $q < q_4$ .*

Finally we consider the region  $\{q > q_4\}$  which is very interesting, since it brings a new type of behaviour, valid for all the constant sign solutions of system (1.1):

**Theorem 5.15** *Let  $(p, q) \in \mathcal{C}_3 \cup \mathcal{D}_3 = \{q > q_4\}$ . (i) Then system (1.5) still admits a particular solution  $(\widetilde{u}_1^*, \widetilde{u}_2^*)$ , where both components are cusp-solutions. Moreover there exist **bounded** global solutions on  $(0, \infty)$ :*

- *If  $(p, q) \in \mathcal{C}_3$ , there exist solutions such that*

$$(\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow 0} (\widetilde{u}_1^*, \widetilde{u}_2^*),$$

$$\lim_{r \rightarrow \infty} \widetilde{u}_1 = c_1, \quad \lim_{r \rightarrow \infty} \widetilde{u}_2 = c_2, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_1' = k_1 > 0, \quad \lim_{r \rightarrow \infty} r^{(N-1)q-2} \widetilde{u}_2' = c(k_1) > 0.$$

- *If  $(p, q) \in \mathcal{D}_3$ , there exist solutions such that*

$$(\widetilde{u}_1, \widetilde{u}_2) \sim_{r \rightarrow 0} (\widetilde{u}_1^*, \widetilde{u}_2^*), \quad \lim_{r \rightarrow \infty} \widetilde{u}_i = c_i > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_1' = k_1 > 0, \quad \lim_{r \rightarrow \infty} r^{N-1} \widetilde{u}_2' = k_2 > 0.$$

(ii) *System (1.4) admits global solutions in  $\mathcal{C}_3$  or  $\mathcal{D}_3$ , of the form  $(C_1 - \widetilde{u}_1, C_2 - \widetilde{u}_2)$  for  $C_1 > c_1, C_2 > c_2$ .*

(iii) *System (1.6) admits global solutions of the same form for  $C_1 > c_1, C_2 < 0$ .*

**Proof of Theorem 1.1.** The proofs of (1) (2) and (3) are direct consequences of Theorem 5.9,1.4 and 5.15 respectively, from the definition 5.11 of the regions.  $\blacksquare$

**Remark 5.16** *The limit cases  $p = \frac{N}{N-1}$  and  $q = q_1, q_2, \frac{N}{N-1}$  can be deduced from Remark 5.8 by integration. We leave the computations to the reader. In the limit cases  $q = q_3 \neq q_2, q_4$ , the particular solutions have a logarithmic form: If  $p > q = q_3 \neq q_2$ , and  $q$  we get solutions such that  $u_1^{*'} = -a_1 r^{1-p}$  and  $u_2^{*'} = -a_2 r^{-1}$ , so we get non constant sign solutions of the form*

$$u_1^* = \pm a_1 r^{2-p} + c_1, \quad u_2^* = -a_2 \ln r + c_2. \quad (5.9)$$

If  $q = q_4 < p$  (hence  $p > 2$ ) we get  $u_1^{*'} = -a_1 r^{-1}$  and  $u_2^{*'} = -a_2 r^{-\frac{2}{p}}$ , we get nonconstant sign solutions

$$u_1^* = a_1 \ln r + c_2, \quad u_2^* = -\frac{p}{p-2} a_2 r^{\frac{p-2}{p}} + c_1.$$

And for  $p = q = 2$ , we recall the solutions  $u_1^* = u_2^* = (2 - N) \ln r + c$ . So in these limit cases, we find no global nonconstant solutions.

Finally when  $q = q_2 = q_3$ , equivalently  $(p, q) = (N, \frac{2}{N-1})$ , the behaviours of type (5.6) (5.7) near 0 or  $\infty$  resumes to  $\lim r |\ln r|^{\frac{2}{N+1}} u_2' = -c_2 < 0$ , which gives by integration solutions such that  $\lim \ln(|\ln r|^{\frac{2}{N+1}}) u_2 = -C < 0$ .

## 6 Nonradial case: upper estimates and local behaviour

### 6.1 Nonexistence results, and upper estimates of mean values

Here we give upper estimates for the supersolutions of system (1.1), implying in particular nonexistence results of entire solutions of the system. the nonexistence results obtained by integral methods were initiated by [31] for the positive supersolutions of system Lane-Emden type: case), and then extended in various directions, to more general operators and second members by Filippucci: she has considered much more general systems with gradient terms, in particular of the form

$$\begin{cases} -\Delta_P u_1 \geq u_1^a u_2^b |\nabla u_2|^p, \\ -\Delta_Q u_2 \geq u_1^c u_2^d |\nabla u_1|^q, \end{cases} \quad (6.1)$$

where the solutions are positive and  $b > 0$  and  $c > 0$ . In [12] we also obtained integral estimates of the positive solutions, for example on problems of type (6.1) with  $p = q = 0$ . In the situation of system (1.1) we adapt the methods of [12], and obtain integral upper estimates of the gradient in any case. A noticeable fact is that the solutions are not supposed to be positive, and not even of constant sign.

**Definition 6.1** *We say that a couple  $(u_1, u_2)$  of  $C^2$  function in a domain  $\Omega \subset \mathbb{R}^N$  is a supersolution (resp. a subsolution) of system (1.1) if*

$$\begin{cases} -\Delta u_1 \geq |\nabla u_2|^p, \\ -\Delta u_2 \geq |\nabla u_1|^q, \end{cases} \quad \text{in } \Omega, \quad \text{resp.} \quad \begin{cases} -\Delta u_1 \leq |\nabla u_2|^p, \\ -\Delta u_2 \leq |\nabla u_1|^q, \end{cases} \quad \text{in } \Omega. \quad (6.2)$$

**Theorem 6.2** Let  $pq > 1$ ,  $p \geq q \geq 1$ , and  $(u_1, u_2)$  be any supersolution of system (1.1) (with no condition of sign) in a domain  $\Omega \subset \mathbb{R}^N$ .

(i) If  $\Omega = B_{r_0} \setminus \{0\}$ , resp  $\Omega = \mathbb{R}^N \setminus \overline{B_{r_0}}$ , then for any  $R < \frac{r_0}{2}$  (resp.  $R > 2r_0$ )

$$\int_{\frac{R}{2} \leq |x| \leq \frac{3R}{2}} |\nabla u_2|^p dx \leq CR^{N - \frac{p(q+1)}{pq-1}}, \quad \int_{\frac{R}{2} \leq |x| \leq \frac{3R}{2}} |\nabla u_1|^q dx \leq CR^{N - \frac{q(p+1)}{pq-1}}. \quad (6.3)$$

(ii) If  $\Omega = \mathbb{R}^N$  and

$$(N-1)pq < \max(N+p, N+q) = N+p. \quad (6.4)$$

that means  $q < q_2$ , then the solutions are constant.

**Proof.** (i) Let  $R > 0$  and  $\zeta \in C_0^\infty(\mathbb{R}^N)$ , with values in  $[0, 1]$ , such that  $\zeta = 1$  on  $B_{\frac{R}{2}}$ ,  $\zeta = 0$  on  $B_R^C$  and  $|\nabla \zeta| \leq \frac{2}{R}$ . We take as test function  $\varphi = \zeta^\lambda$ ,  $\lambda > 0$ , in the first inequality and get by integration, for any  $p > 1$ ,

$$\begin{aligned} \int_{B_R} |\nabla u_1|^q \zeta^\lambda dx &\leq \lambda \left| \int_{B_R} \langle \nabla u_2, \nabla \zeta \rangle \zeta^{\lambda-1} dx \right| = \lambda \left| \int_{B_R} |\nabla u_2| \zeta^\alpha |\nabla \zeta| \zeta^{\lambda-1-\alpha} dx \right| \\ &\leq \lambda \left( \int_{B_R} |\nabla u_2|^p \zeta^{\alpha p} dx \right)^{\frac{1}{p}} \left( \int_{B_R} |\nabla \zeta|^{p'} \zeta^{(\lambda-1-\alpha)p'} dx \right)^{\frac{1}{p'}}, \end{aligned}$$

and for  $\lambda \geq 1 + \alpha$ ,

$$\int_{B_R} |\nabla u_1|^q \zeta^\lambda dx \leq \lambda \left( \int_{B_R} |\nabla u_2|^p \zeta^{\alpha p} dx \right)^{\frac{1}{p}} \left( \int_{B_R} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{p'}}.$$

Similarly for given  $\mu > 0$  and  $\mu - 1 - \beta \geq 0$ ,

$$\int_{B_R} |\nabla u_2|^p \zeta^\mu dx \leq \mu \left( \int_{B_R} |\nabla u_1|^q \zeta^{\beta q} dx \right)^{\frac{1}{q}} \left( \int_{B_R} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}},$$

hence taking  $\lambda = \frac{q(p+1)}{pq-1}$ ,  $\alpha = \frac{q+1}{pq-1}$ ,  $\mu = \frac{p(q+1)}{pq-1}$  and  $\beta = \frac{p+1}{pq-1}$ ,

$$\int_{B_R} |\nabla u_1|^q \zeta^{\frac{q(p+1)}{pq-1}} dx \leq \lambda \left( \int_{B_R} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx \right)^{\frac{1}{p}} \left( \int_{B_R} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{p'}},$$

and by symmetry, when  $q > 1$ ,

$$\int_{B_R} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx \leq \mu \left( \int_{B_R} |\nabla u_1|^q \zeta^{\frac{q(p+1)}{pq-1}} dx \right)^{\frac{1}{q}} \left( \int_{B_R} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}}.$$

As a consequence,

$$\begin{aligned} \int_{B_R} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx &\leq \mu \lambda^{\frac{1}{q}} \left( \int_{B_R} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx \right)^{\frac{1}{pq}} \left( \int_{B_R} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}} \left( \int_{B_R} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{qp'}}, \\ \left( \int_{B_{\frac{R}{2}}} |\nabla u_2|^p dx \right)^{\frac{pq-1}{pq}} &\leq \int_{B_R} |\nabla u_2|^p \zeta^{\frac{p(q+1)}{pq-1}} dx)^{\frac{pq-1}{pq}} \leq \mu \lambda^{\frac{1}{q}} \left( \int_{B_R} |\nabla \zeta|^{q'} dx \right)^{\frac{1}{q'}} \left( \int_{B_R} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{qp'}} \\ &\leq CR^{\frac{N-q'}{q} + \frac{N-p'}{qp'}} = CR^{N \frac{pq-1}{pq} - \frac{q+1}{q}}, \end{aligned}$$

for some constant  $C > 0$ , which gives (6.3) after exchanging  $p$  and  $q$ . When  $q = 1$ , we get directly with  $\mu = \frac{2p}{p-1}$

$$\begin{aligned} \int_{B_R} |\nabla u_2|^p \zeta^{\frac{2p}{p-1}} dx &\leq \frac{2p}{p-1} \left| \int_{B_R} \langle \nabla u_1, \nabla \zeta \rangle \zeta^{\frac{p+1}{p-1}} dx \right| = \frac{2p}{p-1} \left| \int_{B_R} |\nabla u_1| \zeta^{\frac{p+1}{p-1}} |\nabla \zeta| dx \right| \\ &\leq \frac{c}{R} \int_{B_R} |\nabla u_1| \zeta^{\frac{p+1}{p-1}}, \\ \int_{B_R} |\nabla u_1| \zeta^{\frac{p+1}{p-1}} dx &\leq \lambda \left( \int_{B_R} |\nabla u_2|^p \zeta^{\frac{2p}{p-1}} dx \right)^{\frac{1}{p}} \left( \int_{B_R} |\nabla \zeta|^{p'} dx \right)^{\frac{1}{p'}}, \end{aligned}$$

then we obtain

$$\left( \int_{B_{\frac{R}{2}}} |\nabla u_2|^p dx \right)^{\frac{1}{p'}} \leq CR^{\frac{N}{p'}-2},$$

hence again (6.3).

(i) If  $\Omega = \mathbb{R}^N$  and  $N(pq - 1) < p(q + 1)$ , equivalently (6.4) holds, making  $R \rightarrow \infty$ , we deduce that  $\nabla u_2 = 0$ , hence  $u_2 = C_2$ ; then  $-\Delta u_2 = 0 \geq |\nabla u_1|^q$ , thus  $u_1 = C_1$ .

(ii) The estimates (6.3) follow by a finite recovering of the annulus  $\{\frac{R}{2} \leq |x| \leq \frac{3R}{2}\}$  by balls of radius  $2R$ .  $\blacksquare$

**Remark 6.3** In case  $p = q$ , and  $u_1 \geq 0, u_2 \geq 0$ , the nonexistence can be obtained by reducing to the scalar case:

$$-\Delta(u_1 + u_2) = |\nabla u_2|^q + |\nabla u_1|^q \geq c_q (|\nabla u_2| + |\nabla u_1|)^q \geq c_q |\nabla(u_1 + u_2)|^q.$$

If  $1 < q < \frac{N}{N-1}$ , then the only nonnegative solutions in whole  $\mathbb{R}^N$  satisfy  $u_1 + u_2 = C$ , for example from [8, Proposition 2.1]. Then  $-\Delta(u_1 + u_2) = 0 \geq |\nabla u_2|^q + |\nabla u_1|^q$ , hence  $u_1$  and  $u_2$  are constant. In fact we have shown that the positivity is not required. And moreover we obtain integral upper estimates of the gradients when  $q \geq \frac{N}{N-1}$ .

In the sequel, the mean value of any function  $u$  in  $\{R \leq |x| \leq R'\}$  on the sphere  $S^{N-1}$  is denoted by  $\bar{u}$ .

**Lemma 6.4** Let  $p \geq q \geq 1$  and  $(u_1, u_2)$  be a supersolution of system (1.1) in  $\{R \leq |x| \leq R'\}$ . Then  $(\bar{u}_1, \bar{u}_2)$  is also a supersolution. In particular if  $(\widetilde{u}_1, \widetilde{u}_2)$  is a positive subsolution of system (1.4), then  $(\widetilde{\bar{u}}_1, \widetilde{\bar{u}}_2)$  is also a subsolution.

**Proof.** For any function  $u \in C^1(B(0, R))$ , and any  $p \geq 1$ , from the Jensen inequality,

$$\overline{|\nabla u|^p} \geq |\bar{u}'|^p. \quad (6.5)$$

Indeed

$$\begin{aligned} \overline{|\nabla u|^p} &= \frac{1}{|S^{N-1}|} \int_{S^{N-1}} (u_{2,r}^2 + r^{-2} |\nabla' u_2|^2)^{\frac{p}{2}} d\sigma \geq \frac{1}{|S^{N-1}|} \int_{S^{N-1}} |u_r|^p d\sigma \\ &\geq \left( \frac{1}{|S^{N-1}|} \int_{S^{N-1}} |u_r| d\sigma \right)^p \geq \left| \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u_r d\sigma \right|^p = |\bar{u}'|^p. \end{aligned}$$

Then, since  $q \geq 1$ ,

$$-\Delta \bar{u}_1 \geq \overline{|\nabla u_2|^p} \geq |\bar{u}_2'|^p, \quad -\Delta \bar{u}_2 \geq \overline{|\nabla u_1|^q} \geq |\bar{u}_1'|^q. \quad \blacksquare$$

## 6.2 Local behaviour near 0 or $\infty$

As a consequence of Theorem 6.2 and Proposition 3.5, we get

**Proposition 6.5** *Let  $pq > 1$ ,  $p \geq q \geq 1$ , and  $(u_1, u_2)$  be any supersolution of system (1.4) in  $B_{r_0} \setminus \{0\}$ . Then there exists  $\rho \in (0, r_0)$  and  $C > 0$ , depending on  $u_1, u_2$ , such that for  $r \in (0, \rho)$ ,*

$$|\overline{u_1}(r)| \leq \begin{cases} Cr^{-\frac{2-p(q-1)}{pq-1}} & \text{if } q < q_4, \\ C |\ln r| & \text{if } q = q_4, \\ C & \text{if } q > q_4, \end{cases} \quad |\overline{u_2}(r)| \leq \begin{cases} Cr^{-\frac{2-q(p-1)}{pq-1}} & \text{if } q < q_3, \\ C |\ln r| & \text{if } q = q_3, \\ C & \text{if } q > q_3. \end{cases}$$

(ii)  $(u_1, u_2)$  be any supersolution of system (1.4) in  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ . Then there exists  $\eta > r_0$  such that for  $r > \eta$ ,

$$|\overline{u_1}(r)| \leq \begin{cases} C |x|^{\frac{p(q-1)-2}{pq-1}} & \text{if } q > q_4, \\ C \ln r & \text{if } q = q_4, \\ C & \text{if } q < q_4, \end{cases} \quad |\overline{u_2}(r)| \leq \begin{cases} Cr^{\frac{q(p-1)-2}{pq-1}} & \text{if } q > q_3, \\ C \ln r & \text{if } q = q_3, \\ C & \text{if } q < q_3. \end{cases}$$

### 6.2.1 Case of system (1.5)

In this case, we can improve the above results. We give Osserman's type estimates for the local solutions near 0 or  $\infty$

**Theorem 6.6** *Let  $pq > 1$ ,  $p \geq q \geq 1$ . (i) Let  $(\widetilde{u}_1, \widetilde{u}_2)$  be any positive subsolution of system (1.5) in  $B_{r_0} \setminus \{0\}$ . Then there exists  $\rho \in (0, r_0)$  and  $C > 0$ , depending on  $\widetilde{u}_1, \widetilde{u}_2$ , such that for  $0 < |x| < \rho$ ,*

$$\widetilde{u_1}(x) \leq \begin{cases} C |x|^{-\frac{2-p(q-1)}{pq-1}} & \text{if } q < q_4, \\ C |\ln |x|| & \text{if } q = q_4, \\ C & \text{if } q > q_4, \end{cases} \quad \widetilde{u_2}(x) \leq \begin{cases} C |x|^{-\frac{2-q(p-1)}{pq-1}} & \text{if } q < q_3, \\ C |\ln |x|| & \text{if } q = q_3, \\ C & \text{if } q > q_3. \end{cases}$$

(ii) Let  $(\widetilde{u}_1, \widetilde{u}_2)$  be any positive subsolution of system (1.5) in  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ . Then there exists  $\eta > r_0$  such that for  $|x| > \eta$ ,

$$\widetilde{u_1}(x) \leq \begin{cases} C |x|^{\frac{p(q-1)-2}{pq-1}} & \text{if } q > q_4, \\ C |\ln |x|| & \text{if } q = q_4, \\ C & \text{if } q < q_4, \end{cases} \quad \widetilde{u_2}(x) \leq \begin{cases} C |x|^{\frac{q(p-1)-2}{pq-1}} & \text{if } q > q_3, \\ C |\ln |x|| & \text{if } q = q_3, \\ C & \text{if } q < q_3. \end{cases}$$

**Proof.** (i) We know that since the functions are subharmonic, if  $u_i$  is nonconstant near 0, then  $\widetilde{u}_i$  are strictly monotone for small  $r$ , and either  $r^{N-2}\widetilde{u}_i$  has a positive limit as  $r \rightarrow 0$ , or  $\widetilde{u}_i$  is bounded. And for  $|x| < 2\rho < c$  with  $c > 0$  small enough, depending on the functions, from Proposition 3.5 applied to the functions  $\widetilde{u}_i$

$$\left| \widetilde{u_1}' \right| \leq C(N, p, q) r^{-\frac{p+1}{pq-1}}, \quad \left| \widetilde{u_2}' \right| \leq Cr^{-\frac{q+1}{pq-1}}, \quad \text{on } \left[ \frac{3\rho}{4}, \frac{5\rho}{4} \right].$$

Then by integration, for  $c$  small enough,  $\widetilde{u_1} \leq Cr^{-\frac{2-p(q-1)}{pq-1}}$  if  $q < q_4$  and  $\widetilde{u_1} \leq C$  if  $q > q_4$ , similarly for  $\widetilde{u_2}$  by exchanging  $q_3$  and  $q_4$ . Since the functions are subharmonic,

$$u_i(x) \leq C(N) \max\left(\widetilde{u}_i\left(\frac{3}{4}|x|\right), \widetilde{u}_i\left(\frac{5}{4}|x|\right)\right),$$

see [11, Lemma 2.1], hence the conclusions hold. Moreover if  $q > q_4$ , then  $\left| \overline{u_1}' \right|$  is integrable, hence  $\overline{u_1}$  has a finite limit  $l$  as  $r \rightarrow 0$ .

(ii) The proof is similar. ■

### 6.2.2 Case of system (1.4)

We first consider the exterior problem. We observe that in the scalar case, when  $1 < q < \frac{N}{N-1}$ , there is no positive radial solution of the equation such that  $\lim_{r \rightarrow \infty} u = 0$ . But there exist solutions such that  $u$  is increasing to some  $l > 0$  as  $r \rightarrow \infty$ , so there exist solutions of system (1.4) for  $p = q$ . In the general case we prove the following when  $q$  is subcritical, namely  $q < q_2$ :

**Theorem 6.7** *Let  $pq > 1$ ,  $p \geq q \geq 1$ . If  $q < q_2$ , there is no positive supersolution  $(u_1, u_2)$  of system (1.4) such that  $\lim_{r \rightarrow \infty} \overline{u_1} = 0$  in an exterior set  $\mathbb{R}^N \setminus \overline{B_{r_0}}$ .*

**Proof.** We have the upper estimates of the derivatives

$$\left| \overline{u_1}'(r) \right| \leq Cr^{-\frac{p+1}{pq-1}}, \quad \left| \overline{u_2}'(r) \right| \leq Cr^{-\frac{q+1}{pq-1}}, \quad (6.6)$$

and

$$\begin{cases} -\Delta \overline{u_1} \geq |\overline{u_2}'|^p, \\ -\Delta \overline{u_2} \geq |\overline{u_1}'|^q. \end{cases}$$

The function  $\overline{u_1}$  is superharmonic, and

$$\overline{u_1}(r) \leq \overline{u_1}(r_0) + \int_{r_0}^r \left| \overline{u_1}'(r) \right| dr \leq \overline{u_1}(r_0) + Cr^{-\frac{(2-p(q-1))}{pq-1}},$$

so as soon as  $q < q_4$ ,  $\overline{u_1}$  is bounded. And more precisely  $\lim_{r \rightarrow \infty} \overline{u_1} = l_1$  (and  $\lim_{r \rightarrow \infty} \overline{u_2} = l_2$ ). Here we suppose that  $l_1 = 0$ . From the superharmonicity,  $r^{N-1} \overline{u_1}'$  is decreasing so either  $\lim_{r \rightarrow \infty} r^{N-1} \overline{u_1}' = l \in (0, \infty)$  or  $\lim_{r \rightarrow \infty} r^{N-1} \overline{u_1}' = -\infty$ ; and  $l \in (0, \infty)$  is impossible because then  $\overline{u_1}' > 0$  for large  $r$  so  $\overline{u_1}$  cannot tend to 0. Then  $\overline{u_1}' < 0$ . Then  $r^{N-1} \overline{u_1}' < -K$  for large  $r$ , so  $\overline{u_1} - \frac{K}{N-2} r^{2-N}$  is decreasing to 0, hence  $\overline{u_1} \geq \frac{K}{N-2} r^{2-N}$ . But for large  $r$ , we have  $\overline{u_1}'(r) \geq -Cr^{-\frac{p+1}{pq-1}} = \left( \frac{C(pq-1)}{p(q_4-q)} r^{\frac{p(q-q_4)}{pq-1}} \right)'$ , then  $\overline{u_1} - \frac{C(pq-1)}{p(q_4-q)} r^{\frac{p(q-q_4)}{pq-1}}$ , is increasing to 0, thus  $\overline{u_1} \leq cr^{\frac{p(q-q_4)}{pq-1}} = cr^{\frac{p(q-1)-2}{pq-1}}$ . Then  $r^{2-N} \leq r^{\frac{p(q-1)-2}{pq-1}}$ , which implies  $q \geq q_2$ , leading to a contradiction. ■

**Remark 6.8** *In the case  $q < q_2$ , from Theorem (5.10) there exist solutions such that  $\lim_{r \rightarrow \infty} u_1 = c_1 > 0$ . In regions  $\mathcal{C}$  and  $\mathcal{D}$ , where  $q > q_2$ , we have proved the existence of solutions in  $\mathbb{R}^N \setminus B_{r_0}$  at the same theorem. For  $q > q_4$  from Theorem 5.15 there exist solutions on  $(0, \infty)$ . So our result is optimal.*

Finally we consider the local behaviour of the solutions of system (1.4) near a singularity, and moreover the existence of a system with measure data. Let  $\Omega$  be a  $C^2$  domain containing 0. We give a nonradial existence result of solutions of a Dirichlet problem in  $\Omega$  when  $q < q_2$ , extending the radial ones of Theorem 5.9 :

**Theorem 6.9** *Let  $pq > 1$ ,  $p \geq q \geq 1$ .*

*(i) Let  $(u_1, u_2) \in W_0^{1,q}(\Omega \setminus \{0\}) \times W_0^{1,p}(\Omega \setminus \{0\})$ , such that  $\Delta u_1, \Delta u_2 \in L_{loc}^1(\Omega \setminus \{0\})$  be a solution of system (1.4) in  $\Omega \setminus \{0\}$ , with  $u_1 = u_2 = 0$  on  $\partial\Omega$ . Then  $(u_1, u_2) \in W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$  and there exist  $C_1 \geq 0, C_2 \geq 0$ , such that*

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 \delta_0, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 \delta_0, \end{cases} \quad \text{in } \mathcal{D}'(\Omega). \quad (6.7)$$

Moreover

$$\text{if } C_1 > 0 \text{ then } q < q_2, \text{ and if } C_2 > 0, \text{ then } p < \frac{N}{N-1}. \quad (6.8)$$

*(ii) Reciprocally, let  $C_1, C_2$  satisfying (6.8) with  $C_1 > 0$  or  $C_2 > 0$ , and  $C_1, C_2$  small enough. Then there exists a solution  $(u_1, u_2)$  of system (1.4) in  $\Omega$ , such that  $(u_1, u_2) \in W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$ . More generally, for any bounded Radon measures  $\mu, \nu$  in  $\Omega$ , under the same conditions on  $C_1, C_2$ , there exists a solution of the system*

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 \mu, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 \nu, \end{cases} \quad \text{in } \mathcal{D}'(\Omega).$$

$C_1, C_2$ . Moreover  $u_1, u_2 \in W_0^{1,s}(\Omega)$  for any  $s \in \left[1, \frac{N}{N-1}\right)$ , and if  $C_2 = 0$ , then  $u_2 \in W_0^{1,s_2}(\Omega)$  for  $s_2 < \frac{N}{(N-1)q-1}$ .

**Proof.** (i) The functions  $u_1, u_2$  are superharmonic and nonnegative. Then there exist  $C_1 \geq 0, C_2 \geq 0$ , such that (6.7) holds, and  $|\nabla u_2|^p + |\nabla u_1|^q \in L_{loc}^1(\Omega)$ , and moreover  $u_1, u_2 \in W_0^{1,s}(\Omega)$  for any  $s \in \left[1, \frac{N}{N-1}\right)$ , see [16]. From Lemma 6.4  $(\bar{u}_1, \bar{u}_2)$  is a supersolution, then the conditions on  $C_1, C_2$  follow from Theorem 5.9.

(ii) We recall a result of [3, Theorem 3.1]: for any nonnegative  $f \in L^m(\Omega)$  and  $g \in L^k(\Omega)$ ,  $m, k \in (1, N)$  such that  $qk < \frac{mN}{N-m}$  and  $pm < \frac{kN}{N-k}$ , there exists  $\Lambda = \Lambda(p, q, m, k) > 0$  such that the problem

$$\begin{cases} -\Delta u_1 = |\nabla u_2|^p + C_1 f, \\ -\Delta u_2 = |\nabla u_1|^q + C_2 g, \end{cases} \quad \text{in } \Omega,$$

admits a solution such that  $(u_1, u_2) \in W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$  under the condition

$$C_1^q \|f\|_{L^m(\Omega)}^q + C_2 \|g\|_{L^k(\Omega)} \leq \Lambda.$$

And  $(u_1, u_2) \in W_0^{1,s_1}(\Omega) \times W_0^{1,s_2}(\Omega)$  for any  $s_1 \in \left[1, \frac{mN}{N-m}\right)$  and  $s_2 \in \left[1, \frac{kN}{N-k}\right)$ , with the estimate

$$\|u_1\|_{W_0^{1,s_1}(\Omega)} + \|u_2\|_{W_0^{1,s_2}(\Omega)} \leq C(\Lambda, s_1, s_2, m, k).$$

Let  $\mu, \nu$  be two positive Radon measures in  $\Omega$ . Let  $(\mu_n), (\nu_n)$  be sequences in  $C^{\infty,+}(\bar{\Omega})$  converging respectively to  $\mu, \nu$  in the sense of measures, and  $\mu_n(\Omega) \leq 2\mu(\Omega), \nu_n(\Omega) \leq 2\nu(\Omega)$ . Then there exists a solution of the approximate problem

$$\begin{cases} -\Delta u_{1,n} = |\nabla u_{2,n}|^p + C_1 \mu_n, \\ -\Delta u_{2,n} = |\nabla u_{1,n}|^q + C_2 \nu_n, \end{cases} \quad \text{in } \Omega,$$



First suppose  $C_2 > 0$ , then  $1 \leq q \leq p < \frac{N}{N-1}$ . We take  $m = k = 1$ , and  $f = \mu_n, g = \nu_n$ . so for  $2^q C_1^q \mu^q(\Omega) + 2C_2 \nu(\Omega) < \Lambda$  we get the existence of sequences  $(u_{1,n}), (u_{2,n})$  bounded in  $W_0^{1,s}(\Omega)$  for any  $s \in \left[1, \frac{N}{N-1}\right)$ . Next suppose  $C_2 = 0 < C_1$ . We take  $f = \mu_n$  and  $g = 0$ . We need to find some  $m$  and  $k$  such that  $qk < \frac{mN}{N-m}$  and  $pm < \frac{kN}{N-k}$ . We fix  $m = 1$ , so we require that that  $\frac{Np}{N+p} < k < \frac{N}{(N-1)q}$  (which implies  $q < \frac{N}{N-1}$ ). This is possible because  $q < q_2 = \frac{N+p}{(N-1)p} \leq \frac{N}{N-1}$ . so  $u_{1,n}$  is bounded in  $W_0^{1,s}(\Omega)$  for any  $s \in \left[1, \frac{N}{N-1}\right)$   $u_{2,n}$  bounded in  $W_0^{1,s_2}(\Omega)$  for any  $s_2 \in \left[1, \frac{N}{(N-1)q-1}\right)$ , hence in particular for  $s_2 \in \left[1, \frac{N}{N-1}\right)$ . So we get the existence for  $2^q C_1^q \mu^q(\Omega) \leq \Lambda$ . Next we can pass to the limit in  $\mathcal{D}'(\Omega)$ . We fix  $s_1$  with  $q < s_1 < \frac{N}{N-1}$  and  $s_2$  with  $p < s_2 < \frac{N}{N-1}$  if  $C_2 > 0$  and  $p < s_2 < \frac{N}{(N-1)q-1}$  if  $C_2 = 0$ ; up to subsequences,  $u_{1,n}$  converges weakly and *a.e.* in  $W_0^{1,s_1}(\Omega)$  and  $u_{2,n}$  converges weakly and *a.e.* in  $W_0^{1,s_2}(\Omega)$  to some  $u_1, u_2$ ; in any of the two cases,  $|\nabla u_{2,n}|^p$  converges strongly in  $L^1(\Omega)$ , then  $u$  satisfies the equation in  $\mathcal{D}'(\Omega)$

$$-\Delta u_1 = |\nabla u_2|^p + C_1 \mu,$$

and  $|\nabla u_{1,n}|^q$  converges strongly in  $L^1(\Omega)$ , so we also pass to the limit and get in any case

$$-\Delta u_2 = |\nabla u_1|^q + C_2 \nu.$$

■

## 7 Extensions

(1) First note also that the study of Hardy-Hénon type equations by the use of system (4.4) can be adapted to other ranges of the parameters, for example to a "sublinear" case, that is  $\mathbf{q} < \mathbf{p} - \mathbf{1}$ , corresponding to the case where  $pq < 1$  for the system, or also to case  $\mathbf{q} < \mathbf{0}$ , corresponding to  $q < 0$ . This allows to extend some results of [26] given for the Hardy-Hénon equation  $-\Delta w = |x|^\sigma w^q$  with the Laplacian to the  $m$ -Laplacian, in the radial case, in particular their study in dimension 1.

(2) Moreover consider an equation of Hardy-Hénon-type in dimension  $\nu \in \mathbb{N}, \nu \geq 1$ , with a weight  $|x|^a$  ( $a \in \mathbb{R}$ ) inside the operator:

$$-div(|x|^a |\nabla w|^{\mathbf{p}-2} \nabla w) = \varepsilon |x|^b w^{\mathbf{q}},$$

In the radial case, it reduces to

$$-\Delta_{\mathbf{p}}^{\mathbf{N}} w = -\frac{d}{dr} \left( \left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} \right) - \frac{\nu+a-1}{r} \left| \frac{dw}{dr} \right|^{\mathbf{p}-2} \frac{dw}{dr} = \varepsilon r^{b-a} w^{\mathbf{q}},$$

so it directly falls in the scope of our study, with  $N = \nu + a$  and  $\sigma = b - a$ . It allows for example to find again rapidly some recent results of [41], given for the Laplacian with some conditions on  $a, b$ , and extend them to the  $p$ -Laplacian, in all the ranges of the parameters.

(1) All our radial study can be easily directly extended to a system of  $k$ -Laplacians,

$$\begin{cases} -\Delta_k u_1 = |\nabla u_2|^p, \\ -\Delta_k u_2 = |\nabla u_1|^q, \end{cases} \quad (7.1)$$

where  $k > 1$ . Indeed the radial system reduces to

$$\begin{cases} -(|u_1'|^{k_1-2} u_1')' - \frac{N-1}{r} |u_1'|^{k_1-2} u_1' = |u_2'|^p, \\ -(|u_2'|^{k_2-2} u_2')' - \frac{N-1}{r} |u_2'|^{k_2-2} u_2' = |u_1'|^q, \end{cases}$$

$$\begin{cases} w_1' = r^{(N-1)(1-\frac{p}{k-1})} |w_2|^{\frac{p}{k-1}}, \\ w_2' = r^{(N-1)(1-\frac{q}{k-1})} |w_1|^{\frac{q}{k-1}}, \end{cases}$$

where

$$-r^{N-1} |u_1'|^{k-2} u_1' = w_1, \quad -r^{N-1} |u_2'|^{k-2} u_2' = w_2.$$

So we are reduced to study functions  $w_1, w_2$ , **with  $p, q$  replaced by  $\frac{p}{k-1}$  and  $\frac{q}{k-1}$** , is valid  $pq > (k-1)^2$ . We obtain analogous results as in the case  $k = 2$ , with new parameters  $q_i$  defined by

$$q_1 = \frac{N}{(N-1)p - k + 1}, \quad q_2 = \frac{N + (k-1)p}{(N-1)p}, \quad q_3 = \frac{2(k-1)^2}{p - k + 1}, \quad q_4 = \frac{(k-1)p + 2(k-1)^2}{p}.$$

And one can make an easy adaptation of the results of the nonradial section, relative to the supersolutions of system (1.1), where the estimates of the mean values are replaced by integral estimates, as in [12] in case of source terms and in case of absorption or mixed terms. The computations are let it to the reader.

(2) More generally our study allows to treat systems of the type

$$\begin{cases} -div(|x|^{a_1} |\nabla u_1|^{k_1-2} \nabla u_1) = |x|^{b_1} |\nabla u_2|^p, \\ -div(|x|^{a_2} |\nabla u_2|^{k_2-2} \nabla u_2) = |x|^{b_2} |\nabla u_1|^q, \end{cases}$$

which in the radial case reduce to

$$\begin{cases} -(|u_1'|^{k_1-2} u_1')' - \frac{N+a_1-1}{r} |u_1'|^{k_1-2} u_1' = r^{b_1-a_1} |u_2'|^p, \\ -(|u_2'|^{k_2-2} u_2')' - \frac{N+a_2-1}{r} |u_2'|^{k_2-2} u_2' = r^{b_2-a_2} |u_1'|^q, \end{cases}$$

that means a system of and then to a Hardy-Hénon equation (1.2) with new parameters  $\mathbf{p}, \mathbf{q}, \mathbf{N}$  defined by  $\frac{p}{k_2-1} = \frac{1}{\mathbf{p}-1}$ ,  $\frac{q}{k_1-1} = \mathbf{q}$ ,  $\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} = N + b_1 - (N + a_2 - 1) \frac{p}{k_2-1}$ ,  $\mathbf{N} + \sigma = N + b_2 - (N + a_1 - 1) \mathbf{q}$ ; and  $\mathbf{q} > \mathbf{p} - 1$  is equivalent to  $pq > (k_1 - 1)(k_2 - 1)$ .

(3) Some of our nonradial results can be extended to more general systems, like (7.1), following the methods of [12], for example Theorem 6.2. It would be interesting to extend Theorem 6.9 to such type systems.

## 8 Appendix

In this section we give the proofs of the main results of Section 4

**Proof of Lemma 4.5.** (i) Setting  $s = s_0 + \bar{s}$ ,  $z = z_0 + \bar{z}$ , the linearized problem at  $M_0$  is

$$\begin{cases} \bar{s}_t = s_0(\bar{s} + \frac{\bar{z}}{\mathbf{p}-1}), \\ \bar{z}_t = z_0(-\mathbf{q}\bar{s} - \bar{z}). \end{cases}$$

It admits the eigenvalues  $\lambda_1, \lambda_2$ , roots of the equation (4.12). When the roots are real, that means  $s_0 z_0 < 0$  then  $\lambda_1 < 0 < \lambda_2$ , and  $\lambda_1 < s_0 < \lambda_2$  since  $T(s_0) < 0$ , and corresponding eigenvectors are  $\vec{v}_1 = (\frac{s_0}{\mathbf{p}-1}, \lambda_1 - s_0)$  and  $\vec{v}_2 = (\frac{s_0}{\mathbf{p}-1}, \lambda_2 - s_0)$  have the slopes  $\rho_i = (\mathbf{p} - 1)(\frac{\lambda_i}{s_0} - 1) = \frac{\mathbf{q}z_0}{z_0 + \lambda_i}$ ,  $i = 1, 2$ .

(ii) Setting  $z = \mathbf{N} + \sigma + \bar{z}$ , the linearized problem at  $N_0$  is

$$\left\{ \begin{array}{l} s_t = \frac{\mathbf{p} + \sigma}{\mathbf{p} - 1} s, \\ \bar{z}_t = (\mathbf{N} + \sigma)(-\mathbf{q}s - \bar{z}). \end{array} \right\}$$

It admits the eigenvalues  $l_1 = \frac{\mathbf{p} + \sigma}{\mathbf{p} - 1}$  and  $l_2 = -(\mathbf{N} + \sigma)$ . When  $l_1 \neq l_2$ , corresponding eigenvectors are  $\vec{v}_1 = (l_1 - l_2, \mathbf{q}l_2)$  and  $\vec{v}_2 = (0, 1)$ .

(iii) Setting  $s = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} + \bar{s}$ , the linearized system at  $A_0$  is

$$\left\{ \begin{array}{l} \bar{s}_t = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} (\bar{s} + \frac{z}{\mathbf{p} - 1}), \\ z_t = z(\mathbf{N} + \sigma - \mathbf{q}\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}). \end{array} \right\}$$

It admits the eigenvalues  $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}$  and  $\mu_2 = \mathbf{N} + \sigma - \mathbf{q}\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}$ , And  $\{z = 0\}$  contains particular trajectories linked to  $\mu_1$ . When  $\mu_1 \neq \mu_2 \neq 0$ , corresponding eigenvectors are  $\vec{v}_1 = (1, 0)$  and  $\vec{v}_2 = (\frac{\mu_1}{\mathbf{p} - 1}, \mu_2 - \mu_1)$ .

(iv) The linearized system at point  $(0, 0)$ ,

$$\left\{ \begin{array}{l} s_t = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} z, \\ z_t = (\mathbf{N} + \sigma)z. \end{array} \right\}$$

gives the eigenvalues  $\rho_1 = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1}$ ,  $\rho_2 = \mathbf{N} + \sigma$ , which are distinct for  $\sigma \neq -\frac{(\mathbf{N} - 1)\mathbf{p}}{\mathbf{p} - 1}$ , and corresponding eigenvectors  $\vec{v}_1 = (1, 0)$  and  $\vec{v}_2 = (0, 1)$ . ■

**Proof of Lemma 4.7.** (i) Direct consequence of (4.3).

(ii) Suppose  $\lim(s, z) = N_0$ . From (4.4) we find  $s_t = s(l_1 + s + \bar{z}) = s(l_1 + o(1))$ , then  $s = o(e^{-\frac{l_1 t}{2}})$ , and  $\bar{z}_t = z - (\mathbf{N} + \sigma) = (-l_2 + \bar{z})(-\mathbf{q}s - \bar{z}) = \bar{z}(l_2 - \mathbf{q}s - \bar{z}) + l_2 \mathbf{q}s = (l_2 + o(1))\bar{z} + o(e^{-2|l_1 t|})$ . Then  $z = o(e^{-k|t|})$  for some  $k > 0$ . In turn  $s_t = s(l_1 + o(e^{-k|t|}))$  for some  $k > 0$ , thus  $\lim e^{-l_1 t} s = C \neq 0$ . The conclusion follows from (4.3) with  $\lim e^{-l_1 t} s = C \neq 0$  and  $\lim z = N + \sigma$ .

(iii) Suppose  $\lim(s, z) = (0, 0)$ . Then  $s_t = s(\rho_1 + s + \bar{z}) = s(\rho_1 + o(1))$ , and  $z_t = z(\rho_2 + o(1))$ . Then  $s, z = o(e^{-k|t|})$  for some  $k > 0$ . Therefore we obtain  $s_t = s(\rho_1 + o(e^{kt}))$ ,  $z_t = z(\rho_2 + o(e^{kt}))$ . By integration we deduce  $\lim e^{-\rho_1 t} s = C \neq 0$  and  $\lim e^{-\rho_2 t} z = D \neq 0$ , and we still apply (4.3).

(iv) Suppose  $\lim(s, z) = A_0$ . Then  $z_t = z(\mu_2 - \mathbf{q}\bar{s} - z)$  with  $\bar{s} = s - \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} = o(1)$ , so that  $z_t = z(\mu_2 + o(1))$ , then  $z = o(e^{-\frac{\mu_2 t}{2}})$ . It follows that  $\bar{s}_t = (\mu_1 + \bar{s})(\bar{s} + \frac{z}{\mathbf{p} - 1}) = \bar{s}(\mu_1 + \frac{z}{\mathbf{p} - 1} + \bar{s}) + \frac{\mathbf{N} - \mathbf{p}}{(\mathbf{p} - 1)} z = \mu_1 \bar{s} + o(e^{-\frac{\mu_2 t}{2}})$ . Then  $s = o(e^{-k|t|})$  for some  $k > 0$ ; in turn  $z_t = z(\mu_2 + o(e^{-k|t|}))$ ; thus  $\lim e^{-\mu_2 t} z = D \neq 0$ , and we apply (4.3) where  $\lim s = \mu_1$ ,  $\lim e^{-\mu_2 t} z = D$ . ■

**Proof of Lemma 4.8.** Let for example  $w$  be any solution in  $(0, r_0)$ . From Lemma (4.4), and (4.5),  $\Phi = |s|^{\mathbf{p}-1} |z|$  is bounded. And

$$\Phi_t = \Phi(\mathbf{p} + \sigma - (\mathbf{q} - \mathbf{p} + 1)s).$$

Suppose that  $s$  is unbounded near  $-\infty$ . Either  $s$  is monotone near  $-\infty$ , then  $\lim_{t \rightarrow -\infty} |s| = \infty$ , hence  $\frac{\Phi_t}{\Phi} \geq -1$  near  $-\infty$ ,  $\ln |\Phi| \geq \frac{|t|}{2}$ , which is impossible. Or there exists a sequence  $t_n \rightarrow -\infty$  such that  $|s(t_n)| \rightarrow \infty$  of points of maximum of  $|s|$ . At these points, and  $\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s(t_n) + \frac{z(t_n)}{\mathbf{p}-1} = 0$ , hence  $s(t_n)z(t_n) < 0$ , and

$$s_{tt}(t_n) = \frac{s(t_n)z_t(t_n)}{\mathbf{p}-1} = \frac{s(t_n)z(t_n)}{\mathbf{p}-1}(\mathbf{p} + \sigma - (\mathbf{q} - \mathbf{p} + 1)s(t_n)),$$

thus  $s_{tt}(t_n)s(t_n) > 0$ , which is contradictory. Then  $s$  is bounded. Suppose that  $z$  is unbounded. Either  $z$  is monotone near  $-\infty$ , then  $\lim_{t \rightarrow -\infty} |z| = \infty$ , thus  $\lim_{t \rightarrow -\infty} |s_t| = \infty$ , which is impossible. Or there exists a sequence  $t_n \rightarrow -\infty$  such that  $|z(t_n)| \rightarrow \infty$  of points of maximum of  $|z|$ . At these points  $\mathbf{N} + \sigma - \mathbf{q}s(t_n) - z(t_n) = 0$ , which is impossible since  $s$  is bounded. Then  $z$  is bounded. It converges to one of the fixed points of the system, or has a limit cycle around them. Since  $N_0, (0, 0)$  or  $A_0$  have real eigenvalues, such cycle can happen only at  $M_0$ , when  $\mathbf{q} = \mathbf{q}_S$ . ■

**Proof of Theorem 4.9.** Here we consider region **A**, where  $\mathbf{N} > \mathbf{p} > -\sigma$ . In this case we refer to [10] for the description of the system. Here the behaviour depend on position of  $\mathbf{q}$  with respect to  $\mathbf{q}_c$ , and also of  $\mathbf{q}_S$  when  $\varepsilon = 1$ , see figures 1, 2, 3, 4. The point  $M_0$  is in  $Q_4$  for  $\mathbf{q} < \mathbf{q}_c$ , corresponding to solutions of the equation with absorption ( $\varepsilon = -1$ ), and  $M_0 \in Q_1$  for  $\mathbf{q} > \mathbf{q}_c$ . The point  $N_0$  is a saddle point, corresponding to  $C^0$ -regular solutions near 0. The point  $A_0$  is a source for  $\mathbf{q} < \mathbf{q}_c$  and a saddle point for  $\mathbf{q} > \mathbf{q}_c$ . The point  $(0, 0)$  is a saddle point, and the associated trajectories are not admissible. The existence of ground states of the equation with source ( $\varepsilon = 1$ ) if and only if  $q \geq \mathbf{q}_S$  is well known. The phase plane in the critical Sobolev case  $q = \mathbf{q}_S$  is remarkable: there exist particular particular trajectories, located on a straight line

$$(\mathbf{p}-1)s + \frac{\mathbf{p}z}{\mathbf{q}+1} - \mathbf{N} + \mathbf{p} = 0.$$

These trajectories correspond to the ground states given at (4.13) when  $\varepsilon = 1$ , and to explicit local solutions near 0 or  $\infty$  when  $\varepsilon = -1$ . ■

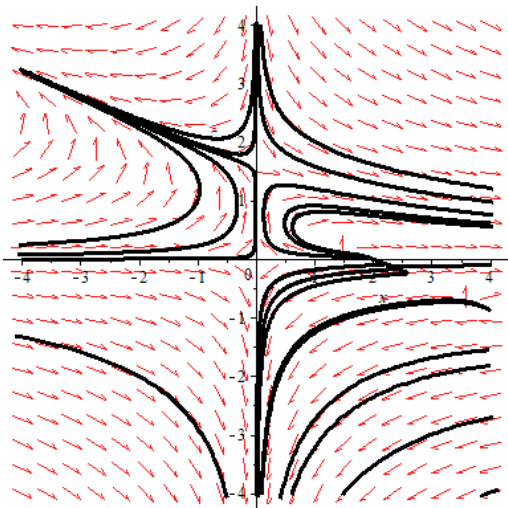


Figure 1. Theorem 4.9  $\mathbf{N} = 2.2 > \mathbf{p} = 1.4 > \sigma = -0.6, \mathbf{q} = 0.7 < \mathbf{q}_c = 0.75$

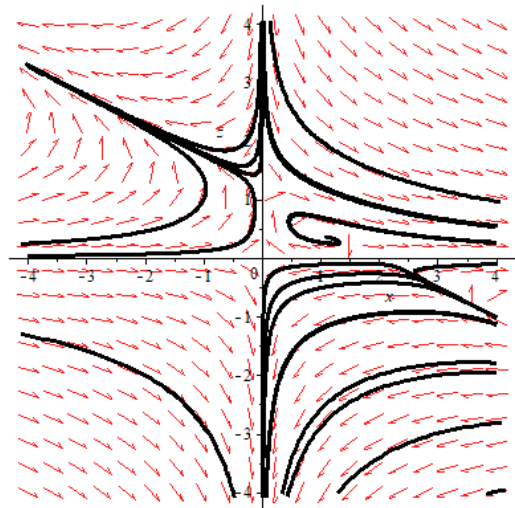


Figure 2. Theorem 4.9  $\mathbf{q}_c < \mathbf{q} = 0.85 < \mathbf{q}_S = \frac{85}{90}$

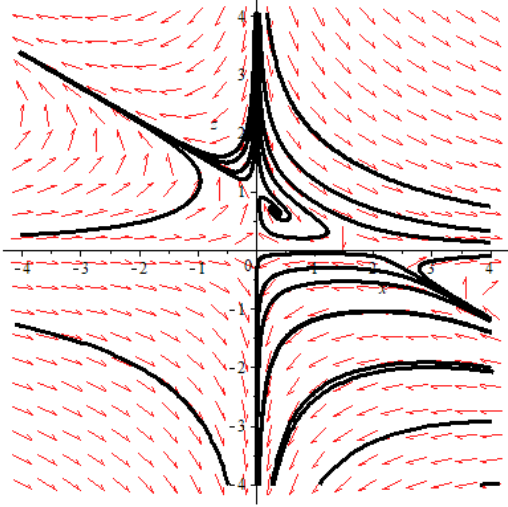


Figure 3. Theorem 4.9  $\mathbf{q} = 1 > \mathbf{q}_S = \frac{85}{90}$

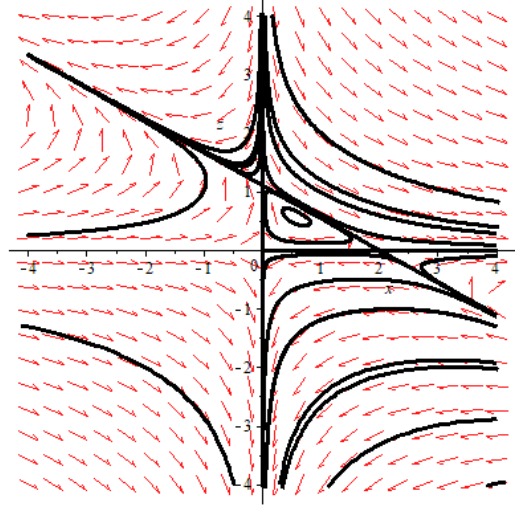


Figure 4. Theorem 4.9  $\mathbf{q} = \mathbf{q}_S = \frac{85}{90}$

**Proof of Theorem 4.10.** In this case,  $M_0 = A_0 = (\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}, 0)$ , see figure 5. The eigenvalues are  $\mu_1 = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}$  and  $\mu_2 = 0$ , with eigenvectors are  $\vec{v}_1 = (0, 1)$  and  $\vec{v}_2 = (1, -(\mathbf{p}-1))$ . Corresponding to the eigenvalue 0, there exists a central manifold of dimension 1, tangent to  $\vec{v}_2$ , invariant by the flow. It contains trajectories converging to  $M_0$  as  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ . The eigenspace associated to  $\mu_1 > 0$  has dimension 1, so it is precisely the axis  $\{z = 0\}$ , and the trajectories are not admissible.

First assume  $\varepsilon = 1$ . In fact there exists an infinity of trajectories tangent to  $\vec{v}_2$ , that means an infinity of central manifolds. The idea is the following: we know that there exists a trajectory  $\mathcal{T}_0$  starting from  $N_0$  and corresponding to  $C^0$ -regular solutions; they are not global,  $w$  vanishes, which means that  $s \rightarrow \infty$  and  $z \rightarrow 0$  in finite time. The region  $\mathcal{R}$  in  $Q_1$  delimited by  $\mathcal{T}_0$  is invariant. The region  $\mathcal{R} \cap \left\{s \leq \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}\right\}$  is negatively invariant, since the field at  $(\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}, z)$  satisfies  $s_t = \frac{sz}{\mathbf{p}-1} > 0$ . Then any trajectory passing by a point of this region stays in it, and cannot converge to  $(0, 0)$ , so it converges to  $M_0$  as  $t \rightarrow -\infty$ .

Then assume  $\varepsilon = -1$ . Then there exist a unique trajectory converging to  $M_0$  as  $t \rightarrow \infty$ . Indeed up to a scaling there exists a unique solution  $w$  with the same logarithmic behaviour as  $r \rightarrow \infty$ , from [40] for  $\sigma = 0$ , and which extends to  $\sigma \neq 0$ , for example by minimisation. ■

**Proof of Theorem 4.13.** Here we consider region  $\mathbf{B}$ , where  $\mathbf{p} > \mathbf{N} > \sigma$ , see figure 6.

- The point  $M_0$  is in  $Q_4$ , since  $s_0 = \gamma = \frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}} > 0$ , and  $z_0 = \mathbf{N}-\mathbf{p}-(\mathbf{p}-1)\gamma < \mathbf{N}-\mathbf{p} < 0$ , see figure .... The eigenvalues satisfy  $\lambda_1 < 0 < \lambda_2$  from (4.12), then  $M_0$  is a saddle point. There exist four trajectories  $\mathcal{T}_1^M, \mathcal{T}_2^M, \mathcal{T}_3^M, \mathcal{T}_4^M$ , such that  $\mathcal{T}_1^M, \mathcal{T}_2^M$  (resp.  $\mathcal{T}_3^M, \mathcal{T}_4^M$ ) converge to  $M_0$  as  $t \rightarrow \infty$  (resp.  $-\infty$ ). associated to  $\lambda_1 < 0$  (resp.  $\lambda_2 > 0$ ) with a negative slope  $\rho_1 = -(\mathbf{p}-1)(1 + \frac{|\lambda_1|}{s_0}) < -(\mathbf{p}-1)$  (resp. a positive slope). Then  $\mathcal{T}_1^M$  lies in the bounded region  $\mathcal{H}$  of  $Q_4$  where  $s_t < 0, z_t < 0$  for large  $t$ . Since this region is negatively invariant, the trajectory  $\mathcal{T}_1^M$  stays in  $\mathcal{H}$ , so it is bounded, hence defined on  $(\infty, \infty)$ , and converges to  $(0, 0)$  which is the only possible fixed point in  $\overline{\mathcal{H}}$ .

- The point  $N_0$  is at the boundary of  $Q_1$  and  $Q_2$ . The eigenvalues satisfy  $l_2 < 0 < l_1$ , then  $N_0$  is a saddle point. There are two trajectories  $\mathcal{T}'_1, \mathcal{T}'_2$ , starting from  $N_0$ , the first one in  $Q_1$  and one in  $Q_2$ , and the corresponding solutions satisfy (4.14) the solutions are not global, from Lemma

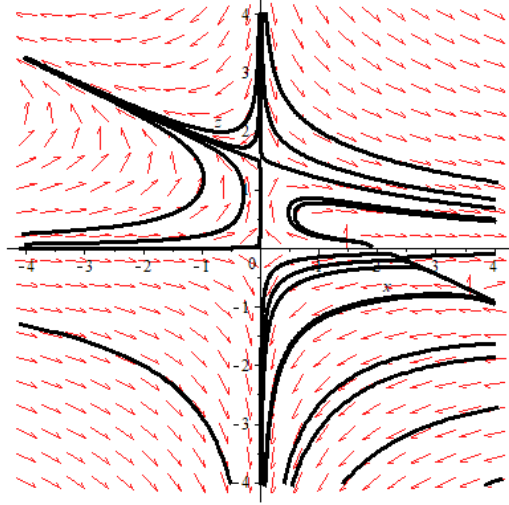


Figure 5, Theorem 4.10:  
 $\mathbf{N} = 2.2 > \mathbf{p} = 1.4 > \sigma = -0.6, \mathbf{q} = \mathbf{q}_c = 0.75,$

4.8, since in  $\overline{Q_1}$  and  $\overline{Q_2}$  there is not fixed point attracting at  $\infty$ . ((Note that  $\mathcal{T}'_1$  corresponds to a solution  $w$  vanishing at finite  $R > 0$ , and  $\mathcal{T}'_1$  corresponds to a large solution)).

- The point  $A_0$  is located at the boundary of  $Q_2$  and  $Q_3$ . The eigenvalues satisfy  $\mu_1 < 0 < \mu_2$ , then it is a saddle point. Two trajectories corresponding to  $\mu_1$  are ending at  $A_0$  as  $t \rightarrow \infty$  are located on the line  $\{z = 0\}$  not admissible. Two trajectories corresponding to  $\mu_2$  are starting from  $A_0$  at  $-\infty$ , a trajectory  $\mathcal{T}_2^A$  in  $Q_2$  and a trajectory  $\mathcal{T}_3^A$  in  $Q_3$ , with a negative slope  $-(\mathbf{p} - 1) \frac{\mu_2 + |\mu_1|}{|\mu_1|}$ . The corresponding solutions satisfy (4.15). They are not global, from Lemma 4.8, since in  $\overline{Q_2}$  and  $\overline{Q_3}$  there is no attracting point at  $\infty$ .

- The point  $(0, 0)$  is a source, since  $\rho_1 > 0$  and  $\rho_2 > 0$ . So if  $\rho_1 \neq \rho_2$ , that means  $\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} \neq \mathbf{N} + \sigma$ , there exists an infinity of trajectories starting from  $(0, 0)$ . From Lemma 4.7 the corresponding solutions satisfy (4.16). If  $\rho_1 = \rho_2$ , we prove that there exist particular solutions at Proposition 4.17.

- In  $\overline{Q_1}$  and  $\overline{Q_3}$  there is no fixed point attracting at  $\infty$ , so for  $\varepsilon = 1$  there is no solution in  $(r_0, \infty)$ . ■

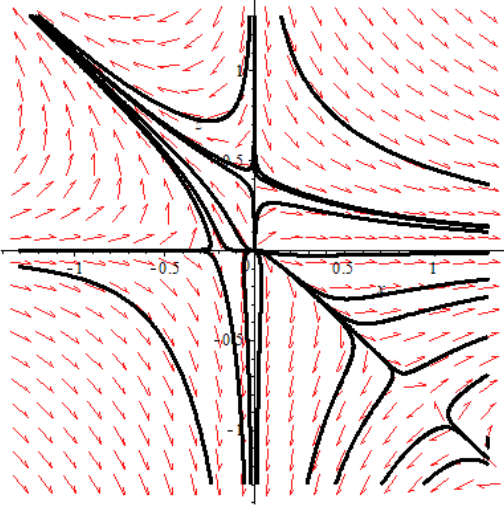


Figure 6. Theorem 4.13  $\mathbf{p} = \frac{12}{7}$   
 $> \mathbf{N} = \frac{11}{7} > \sigma = -\frac{1.82}{7}$ ,  $\mathbf{q} = \mathbf{1.3}$

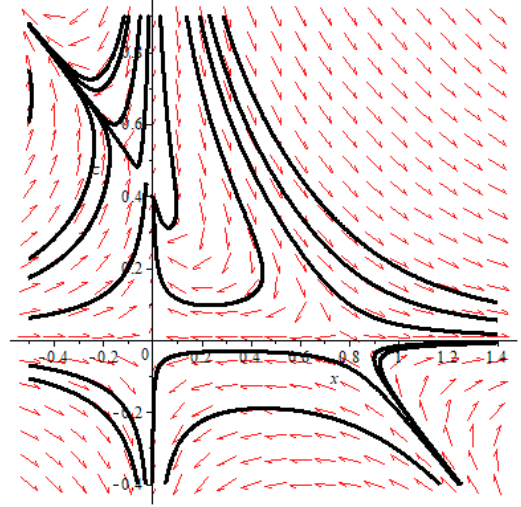


Figure 7. Theorem 4.9 Theorem 4.15:  
 $\mathbf{p} = \frac{29}{19} < -\sigma = -\frac{29.4}{19} < \mathbf{N} = \frac{37}{19}$ ,  $\mathbf{q} = \mathbf{1.3}$

**Proof of Theorem 4.15.** Here we consider region **C**, where  $\mathbf{p} < -\sigma < \mathbf{N}$ , see figure 7.

- The point  $M_0$  is in  $Q_2$ . Indeed  $s_0 = \gamma < 0$ ; and  $z_0 = \mathbf{N} - \mathbf{p} + (\mathbf{p} - 1)|\gamma| > 0$ . It is a saddle point, and the eigenvalues satisfy  $\lambda_1 < s_0 < \lambda_2$ . So there exist four trajectories  $\mathcal{T}_1^M, \mathcal{T}_1'^M, \mathcal{T}_2^M, \mathcal{T}_2'^M$ , such that  $\mathcal{T}_1^M, \mathcal{T}_1'^M$ , (resp.  $\mathcal{T}_2^M, \mathcal{T}_2'^M$ ) converge to  $M_0$  as  $t \rightarrow \infty$  (resp.  $-\infty$ ). associated to  $\lambda_1 < 0$  (resp.  $\lambda_2 > 0$ ) with the slopes  $\rho_i = (\mathbf{p} - 1)(\frac{\lambda_i}{s_0} - 1)$   $i = 1, 2$ , hence  $\rho_1 > 0 > \rho_2$ , and we call  $\mathcal{T}_1^M, \mathcal{T}_2^M$ , the trajectories such that  $s(t) > s_0$  for any  $t \in \mathbb{R}$ . In particular  $\mathcal{T}_2^M$  starts in the region  $J$  of  $Q_2$  where  $s_t > 0, z_t < 0$ . We check easily that the region  $J$  is positively invariant. Then the trajectory  $\mathcal{T}_2^M$  stays in  $J$ , so it is bounded, and since  $s$  and  $z$  are monotone, we get that  $\mathcal{T}_2^M$  converges at  $\infty$  to a fixed point in  $\overline{Q_2}$ , that is to  $N_0$  if  $\mathbf{N} + \sigma > 0$ , and to  $(0, 0)$  if  $\mathbf{N} + \sigma < 0$ .

- The point  $N_0$  is at the boundary of  $Q_1$  and  $Q_2$ . The eigenvalues satisfy  $l_1 < 0$  and  $l_2 < 0$ , so  $N_0$  is a sink. In any case, all the corresponding solutions satisfy (4.17), from Lemma 4.7. In any case, we know that the trajectory  $\mathcal{T}_2^M$  converges to  $N_0$ .

- The point  $(0, 0)$  admits the eigenvalues  $\rho_1 = \frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} < 0$  and  $\rho_2 = \mathbf{N} + \sigma > 0$ , it is a saddle point, the trajectories issued from this point are the two axes, they are not admissible.

- The point  $A_0 = (\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}, 0)$ . is on the boundary common to  $Q_1$  and  $Q_4$ . It has the eigenvalues  $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} > 0$  and

$$\mu_2 = \mathbf{N} + \sigma - \mathbf{q} \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} < \mathbf{N} + \sigma - (\mathbf{N} - \mathbf{p}) = \mathbf{p} + \sigma < 0,$$

so it is a saddle point. As we have done before, we check that there exist two trajectories converging to  $A_0$  as  $t \rightarrow \infty$ , one in  $Q_1$  and one in  $Q_4$ . This show the existence of local solutions of the equations with  $\varepsilon = \pm 1$ , such that  $\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}} w = k > 0$ . And clearly those solutions are not global, since there is no fixed point in  $\overline{Q_1}$  and  $\overline{Q_4}$  attracting at  $-\infty$ , from Lemma 4.8 So the corresponding functions  $w$  are only solutions of the exterior problem. ■

**Proof of Theorem 4.17.** Setting  $s = \Phi z$ , we get

$$\Phi_t = \Phi\left(\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} - (\mathbf{N} + \sigma) + \left(\frac{\mathbf{p}}{\mathbf{p} - 1} + (\mathbf{q} + 1)\Phi\right)z\right)$$

Then system admits particular solutions such that  $s \equiv -\delta z$  for some  $\delta > 0$  if and only if  $\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} = \mathbf{N} + \sigma$ , that means  $\sigma = -\mathbf{p}\frac{\mathbf{N} - 1}{\mathbf{p} - 1}$ , and  $\delta = \frac{\mathbf{p}}{(\mathbf{p} - 1)(\mathbf{q} + 1)}$ . They are given explicitly from the equation

$$s_t = s\left(\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} - \frac{\mathbf{q} - \mathbf{p} + 1}{\mathbf{p} - 1}s\right),$$

and from (4.3),  $\delta \varepsilon r^\sigma w^{\mathbf{q} + 1} |w'|^{-\mathbf{p}} = -1$ , hence  $\varepsilon = -1$  and we get  $w^{-\frac{\mathbf{q} + 1}{\mathbf{p}}} w' = \pm \delta^{\frac{1}{\mathbf{p}}} r^{-\frac{\mathbf{N} - 1}{\mathbf{p} - 1}}$ , hence (4.19) by integration, with  $d_{\mathbf{p}, \mathbf{q}, \mathbf{N}} = \frac{(\mathbf{p} - 1)(\mathbf{q} - \mathbf{p} + 1)}{\mathbf{p}(\mathbf{p} - \mathbf{N})} \left(\frac{\mathbf{p}}{(\mathbf{p} - 1)(\mathbf{q} + 1)}\right)^{\frac{1}{\mathbf{p}}}$ . These solutions satisfy  $\lim_{r \rightarrow 0} w = C$  when  $C > 0$  and when they are global,  $w \sim_{r \rightarrow \infty} w^*$ .

If  $\mathbf{p} = \mathbf{N}$ , we get  $\sigma = -\mathbf{p} = -\mathbf{N}$ , and  $w^{-\frac{\mathbf{q} + 1}{\mathbf{N}}} w' = \pm \delta^{\frac{1}{\mathbf{N}}} r^{-1}$ , then  $w^{1 - \frac{\mathbf{q} + 1}{\mathbf{N}}} = C \pm \delta^{\frac{1}{\mathbf{N}}} \left(\frac{\mathbf{q} + 1}{\mathbf{N}} - 1\right) \ln r$ , where  $\delta = \frac{\mathbf{N}}{(\mathbf{N} - 1)(\mathbf{q} + 1)}$ , that is  $d_{\mathbf{N}} = \frac{\mathbf{q} + 1 - \mathbf{N}}{\mathbf{N}} \left(\frac{\mathbf{N}}{(\mathbf{N} - 1)(\mathbf{q} + 1)}\right)^{\frac{1}{\mathbf{N}}}$ . ■

**Proof of Theorem 4.18.** Using Lemma 4.3, we reduce to the case (1)  $\mathbf{N} > \mathbf{p} = -\sigma$ , see figure 8.

- Here the point  $M_0 = (0, \mathbf{N} - \mathbf{p}) = N_0$ . The eigenvalues are  $l_1 = 0$  and  $l_2 = \mathbf{p} - \mathbf{N} < 0$ . The corresponding eigenvectors are  $\vec{v}_1 = (1, -\mathbf{q})$  and  $\vec{v}_2 = (0, 1)$ . Corresponding to the eigenvalue 0, there exists a central manifold of dimension 1, tangent to  $\vec{v}_1$ , invariant by the flow. It contains trajectories converging to  $M_0$  as  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ . The eigenspace associated to  $l_2 > 0$  has dimension 1, so it is precisely the axis  $\{s = 0\}$ , and the trajectories are not admissible. The system is

$$\begin{cases} s_t = s\left(\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} + s + \frac{z}{\mathbf{p} - 1}\right), \\ z_t = z(\mathbf{N} - \mathbf{p} - \mathbf{q}s - z). \end{cases}$$

- The point  $A_0 = \left(\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}, 0\right)$ . is located at the boundary of  $Q_1$  and  $Q_4$ . The eigenvalues are  $\mu_1 = \frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1} > 0$  and  $\mu_2 = (\mathbf{q} - \mathbf{p} + 1)\frac{\mathbf{p} - \mathbf{N}}{\mathbf{p} - 1} < 0$ , satisfy  $\mu_1 < 0 < \mu_2$ , with eigenvectors.  $\vec{v}_1 = (1, 0)$  and  $\vec{v}_2 = (1, -(\mathbf{p} - 1)(\mathbf{q} - \mathbf{p} + 2))$ . Two trajectories associated to  $A_0$  are located on the line  $\{z = 0\}$ , so they are not admissible. There exist two other trajectories,  $\mathcal{T}_1$  and  $\mathcal{T}_4$ , respectively in  $Q_1$  and  $Q_4$ , converging to  $A_0$  at  $\infty$ . The corresponding solutions  $w$  satisfy  $\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N} - \mathbf{p}}{\mathbf{p} - 1}} w = k > 0$ , from Lemma 4.7.

- First consider the trajectories located in  $Q_1$ , corresponding the case  $\varepsilon = 1$ . The trajectory  $\mathcal{T}_1$  has a slope less than  $-(\mathbf{p} - 1)$ , then it arrives as  $t \rightarrow \infty$  in the region  $\mathcal{R} = \{s_t > 0\}$  in  $Q_1$  delimited by the line  $N_0 A_0$  where  $s_t = 0$ ; and  $\mathcal{R}$  is negatively invariant, so  $\mathcal{T}_1$  stays in it. The line  $\{z_t = 0\}$  is located under the line  $M_0 A_0$ , thus  $z_t < 0$  in  $\mathcal{R}$ . If  $\mathcal{T}_1$  converges to  $M_0$ , then the region  $\mathcal{R}_1$  delimited by the two axis and  $\mathcal{T}_1$  is bounded with any fixed point inside, and this is contradictory. Then  $\mathcal{T}_1$  is asymptotic to the axis  $\{s = 0\}$ , and the solutions are not global. Consider again the region  $\mathcal{R}_1$ . Then any trajectory passing by a point of  $\mathcal{R}_1$  is bounded, and then converges to  $M_0$  as  $t \rightarrow \infty$ , so we get a (2 parameters) family of solutions  $w$  in  $(r_0, \infty)$  such that (4.21) holds. The trajectory is tangent to line of direction  $(1, -\mathbf{q})$  passing by  $M_0$ , which means  $\bar{z} \sim -qs$ , and we have

$$\begin{cases} s_t = s\left(s + \frac{\bar{z}}{\mathbf{p} - 1}\right), \\ z_t = (\mathbf{N} - \mathbf{p} + \bar{z})(-\bar{z} - \mathbf{q}s). \end{cases}$$



Then  $s_t \sim -\frac{\mathbf{q}}{\mathbf{p}-1}s^2$ , and by integration  $s \sim \frac{\mathbf{p}-1}{\mathbf{q}t}$ , and  $z \sim \mathbf{N} - \mathbf{p}$ , thus by (4.5),  $w \sim (\frac{\mathbf{p}-1}{\mathbf{q}t})^{\frac{\mathbf{p}-1}{\mathbf{q}+1-\mathbf{p}}}$ , which is precisely (4.21).

• Next consider the trajectories located in  $Q_2$ , corresponding the case for  $\varepsilon = -1$ . The line  $\{s_t = 0\} = M_0 A_0$  is located under the line  $\{z_t = 0\}$  and the region  $\mathcal{G}$  located between the two lines is negatively invariant. Any trajectory passing by a point above  $\{z_t = 0\}$  necessarily crosses this line since  $z$  and  $s$  are decreasing, and any trajectory passing by a point under  $\{s_t = 0\}$  cuts this line, because  $s$  and  $z$  are increasing. We consider two sets of trajectories:  $\mathcal{U}_1$  is the union of trajectories which cut  $\{z_t = 0\}$  at some point, and  $\mathcal{U}_2$  is the union of trajectories which cut  $\{s_t = 0\}$  at some point. They are open in  $Q_2$ , so  $\mathcal{U}_1 \cup \mathcal{U}_2 \neq Q_2$ . Then there exists some point  $P$  such that the trajectory passing by  $P$  is located in  $\mathcal{G}$ , which is negatively invariant. Then it converges to the point  $M_0$  as  $t \rightarrow -\infty$ . So we get solutions  $w$  satisfying (4.22). ■

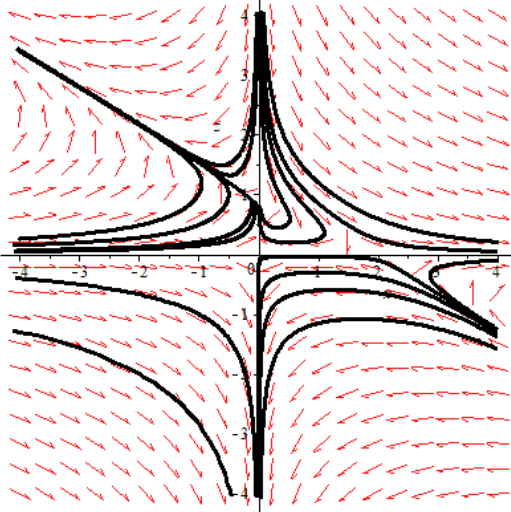


Figure 8. Theorem 4.18  
 $\mathbf{N} = 2.2 > \mathbf{p} = -\sigma = 1.4, \mathbf{q} = 1.1$

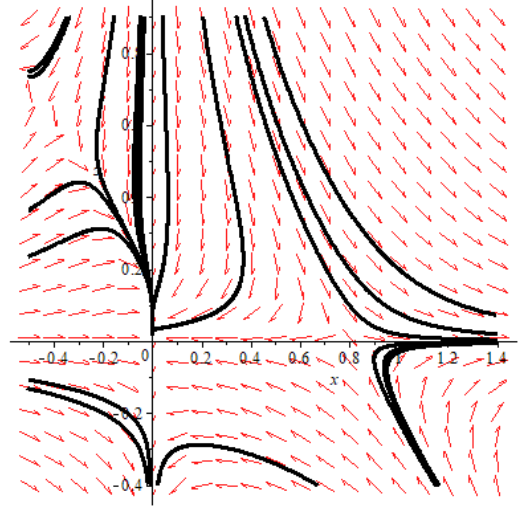


Figure 9. Theorem 4.19  $-\sigma = \mathbf{N} = \frac{37}{19}$   
 $> \mathbf{p} = \frac{29}{19}, \mathbf{q} = 1.5$

**Proof of Theorem 4.19.** We still reduce to the case (1)  $-\sigma = \mathbf{N} > \mathbf{p}$ , see figure 9. Here the system is

$$\begin{cases} s_t = s\left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} + s + \frac{z}{\mathbf{p}-1}\right), \\ z_t = z(-\mathbf{q}s - z). \end{cases} \quad (8.1)$$

The point  $M_0 = (s_0, z_0) = \left(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{q}+1-\mathbf{p}}, \frac{(\mathbf{N}-\mathbf{p})\mathbf{q}}{\mathbf{q}+1-\mathbf{p}}\right)$  is in  $Q_2$ , corresponding to particular solutions of the equation with absorption  $\varepsilon = -1$ .  $w^* = a^* r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}$ ; note that  $\lim_{r \rightarrow 0} w^* = 0$ .

The point  $N_0 = (0, 0)$ , the eigenvalues are  $\rho_1 = \frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1}, \rho_2 = 0$ , corresponding eigenvectors  $\vec{v}_1 = (1, 0)$  and  $\vec{v}_2 = (0, 1)$ . The trajectories associated to  $\rho_1$  are not admissible. There exist at least one central manifold, relative to  $\rho_2$ , tangent to the axis  $\{s = 0\}$ . We are going to precise it below:

The eigenvalues at point  $M_0$  are given by (4.12), their product is  $-(\mathbf{N} - \mathbf{p})^2 \mathbf{q} < 0$ , so  $M_0$  is a saddle point. There exist four trajectories  $\mathcal{T}_1^M, \mathcal{T}_2^M, \mathcal{T}_3^M, \mathcal{T}_4^M$ , such that  $\mathcal{T}_1^M, \mathcal{T}_2^M$  (resp.  $\mathcal{T}_3^M, \mathcal{T}_4^M$ ) converge to  $M_0$  as  $t \rightarrow \infty$  (resp.  $-\infty$ ). associated to  $\lambda_1 < 0$  (resp.  $\lambda_2 > 0$ ) with the slope  $\rho_1 = (\mathbf{p} - 1)\left(\frac{\lambda_1}{s_0} - 1\right) > 0$  (resp.  $\rho_2 = (\mathbf{p} - 1)\left(-\frac{\lambda_2}{|s_0|} - 1\right) < -(\mathbf{p} - 1)$ ). Then one of the two last

trajectories, denoted by  $\mathcal{T}_3^M$  lies in the bounded region  $\mathcal{H}$  of  $Q_2$  where  $s_t > 0, z_t < 0$  for large  $t$ . Since this region is positively invariant, the trajectory  $\mathcal{T}_3^M$  stays in  $\mathcal{H}$ , so it is bounded, hence converges to  $(0,0)$  which is the only possible fixed point in  $\overline{\mathcal{H}}$ . Then any trajectory passing by some point of  $\mathcal{H}$  converges to  $(0,0)$ . So there is an infinity of trajectories converging to this point, staying in  $Q_2$ .

Moreover the eigenvalues at  $A_0$  are  $\mu_1 = \frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1} > 0$  and  $\mu_2 = \mathbf{q} \frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1} < 0$ ,  $A_0$  is a saddle point. Two trajectories associated to  $\mu_1$  are located on the line  $\{z=0\}$  are not admissible. There are trajectories  $\mathcal{T}_1, \mathcal{T}_4$ , respectively in  $Q_1$  and  $Q_4$  converging to  $A_0$  as  $t \rightarrow \infty$  corresponding to the eigenvector  $\vec{v}_2 = (\frac{\mu_1}{\mathbf{p}-1}, \mu_2 - \mu_1)$ , with the slope  $-(\mathbf{p}-1)(\mathbf{q}+1)$ , the solutions are not global, since there is no fixed point in those quadrants. Then there exist solutions in  $(r_0, \infty)$  for  $\varepsilon = \pm 1$ , such that  $\lim_{r \rightarrow \infty} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{p}-1}} w = k > 0$ . Moreover, since  $(\mathbf{p}-1)(\mathbf{q}+1) > (\mathbf{p}-1)$ ,  $\mathcal{T}_1$  is located in the region of  $Q_1$  where  $s_t > 0$  and  $z_t < 0$ , which is negatively invariant, so it stays in it. Next consider the region  $\mathcal{K}$  in  $Q_1$  delimited by the two axis and  $\mathcal{T}_1$  (under  $\mathcal{T}_1$ ). It is invariant, and  $z_t < 0$  in  $\mathcal{K}$ . Then consider any trajectory  $\mathcal{T}$  passing by a point of  $\mathcal{K}$ . It cannot converge to  $A_0$  because  $\mathcal{T}_1$  is the only trajectory in  $Q_1$  converging to this point. Then  $\mathcal{T}$  necessarily converges to  $(0,0)$ . There still exists an infinity of trajectories converging to this point, staying in  $Q_1$ .

So in  $Q_1$  (corresponding to  $\varepsilon = 1$ ) as well as in  $Q_2$ , (corresponding to  $\varepsilon = -1$ ) there is an infinity of trajectories converging to  $(0,0)$ . From system (8.1), since  $\lim_{t \rightarrow \infty} \frac{s}{z} = 0$ , we get  $s_t \sim s(\frac{\mathbf{p}-\mathbf{N}}{\mathbf{p}-1})$ , and  $z_t \sim -z^2$ . Then by integration,  $z \sim \frac{1}{t} \cdot |s| \leq C e^{-kt}$  for some  $k > 0$ . From (4.3) we get  $w^{\frac{\mathbf{q}}{\mathbf{1}-\mathbf{p}}} w' = (-\frac{\mathbf{p}-1}{\mathbf{q}+1-\mathbf{p}} w^{-\frac{\mathbf{q}+1-\mathbf{p}}{\mathbf{p}-1}})' \sim_{r \rightarrow \infty} -\varepsilon r^{\frac{1-\mathbf{N}}{\mathbf{p}-1}} (\ln r)^{\frac{1}{\mathbf{p}-1}}$ , then by integration, either

$$w \sim -\varepsilon C (\ln r)^{-\frac{1}{\mathbf{q}+1-\mathbf{p}} r^{\frac{\mathbf{N}-\mathbf{p}}{\mathbf{q}+1-\mathbf{p}}}}, \quad w' \sim -\varepsilon C^{\frac{\mathbf{q}}{\mathbf{p}-1}} (\ln r)^{-\frac{1}{\mathbf{q}+1-\mathbf{p}} r^{\frac{\mathbf{N}-1-\mathbf{q}}{\mathbf{q}+1-\mathbf{p}}}}, \quad C = \left(\frac{\mathbf{q}+1-\mathbf{p}}{\mathbf{N}-\mathbf{p}}\right)^{\frac{\mathbf{p}-1}{\mathbf{q}+1-\mathbf{p}}},$$

or

$$\lim_{r \rightarrow \infty} w = C > 0, \quad w' \sim -\varepsilon r^{\frac{1-\mathbf{N}}{\mathbf{p}-1}} (\ln r)^{\frac{1}{\mathbf{p}-1}}.$$

The first eventuality is impossible for  $\varepsilon = 1$ . Let us check if it is possible for  $\varepsilon = -1$ : it would imply that  $s = -r \frac{w'}{w}$  tends to  $\mathbf{p} - \mathbf{N} \neq 0$ , which is contradictory. Then for  $\varepsilon = -1$ , for given  $r_0 > 0$  there exists an infinity (with 2 parameters) of solutions  $w$  in  $(r_0, \infty)$  satisfying (??), and an infinity (with one parameter) of solutions satisfying (4.25).  $\blacksquare$

**Proof of Theorem 4.20.** Here again we reduce to the case (1)  $\mathbf{p} = \mathbf{N} > -\sigma$ , see figure 10. The system is

$$\begin{cases} s_t = s(s + \frac{z}{\mathbf{p}-1}), \\ z_t = z(\mathbf{p} + \sigma - \mathbf{q}s - z). \end{cases}$$

The point  $M_0 = (s_0, z_0) = (\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}, -\frac{(\mathbf{p}-1)(\mathbf{p}+\sigma)}{\mathbf{q}+1-\mathbf{p}})$  is in  $Q_4$ , corresponding to particular solutions of the equation with absorption  $\varepsilon = -1$ .  $w^* = a^* r^{-\frac{\mathbf{p}+\sigma}{\mathbf{q}+1-\mathbf{p}}}$ , which are  $\infty$ -singular.

The point  $N_0 = (\mathbf{p} + \sigma, 0)$  is at the boundary of  $Q_1$  and  $Q_2$  admits the eigenvalues  $l_1 = \frac{\mathbf{p}+\sigma}{\mathbf{p}-1} > 0$  and  $l_2 = -(\mathbf{p} + \sigma) < 0$ , with eigenvectors  $\vec{v}_1 = (\frac{\mathbf{p}}{\mathbf{p}-1}, -\mathbf{q})$  and  $\vec{v}_2 = (0, 1)$ . it is a saddle point; the trajectories associated to  $l_2$  are not admissible. There exists two trajectories,  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , respectively in  $Q_1$  and  $Q_2$  starting from  $N_0$  at  $-\infty$ , corresponding to  $C^0$ -regular solutions  $w$ . The trajectory  $\mathcal{T}_2$  starts in the region where  $s_t < 0, z_t > 0$ , which is positively invariant, so it stays in it. The

trajectory  $\mathcal{T}_1$  starts in the region where  $s_t > 0, z_t < 0$ , with a slope  $-\mathbf{q}\frac{\mathbf{p}-1}{\mathbf{p}}$ , above the line  $N_0M_0$ , and stays in the region of  $Q_1$  where  $z_t < 0$ , and then it is asymptotic to the axis  $\{z = 0\}$ .

Here the point  $A_0 = (0, 0)$ , and the eigenvalues are  $\rho_1 = 0, \rho_2 = \mathbf{N} + \sigma$ , with eigenvectors  $\vec{v}_1 = (1, 0)$  and  $\vec{v}_2 = (0, 1)$ . The trajectories associated to  $\rho_2$  are not admissible. There exist at least one central manifold, relative to  $\rho_1$ , tangent to the axis  $\{z = 0\}$ . We are going to precise it below.

The eigenvalues at point  $M_0$  are given by (4.12), their product is  $s_0z_0 < 0$ , then  $\lambda_1 < 0 < s_0 < \lambda_2$ , so  $M_0$  is a saddle point, There exist four trajectories  $\mathcal{T}_1^M, \mathcal{T}_2^M, \mathcal{T}_3^M, \mathcal{T}_4^M$ , such that  $\mathcal{T}_1^M, \mathcal{T}_2^M$  (resp.  $\mathcal{T}_3^M, \mathcal{T}_4^M$ ) converge to  $M_0$  as  $t \rightarrow \infty$  (resp.  $-\infty$ ). associated to  $\lambda_1 < 0$  (resp.  $\lambda_2 > 0$ ) with the slope  $\rho_1 = (\mathbf{p} - 1)(\frac{\lambda_1}{s_0} - 1) < 0$  (resp.  $\rho_2 = (\mathbf{p} - 1)(\frac{\lambda_2}{s_0} - 1) > (\mathbf{p} - 1)$ ). Then one of the two first trajectories, denoted by  $\mathcal{T}_1^M$  lies in the bounded region  $\mathcal{H}$  of  $Q_4$  where  $s_t > 0, z_t < 0$  for large  $t$ . Since this region is negatively invariant, the trajectory  $\mathcal{T}_1^M$  stays in  $\mathcal{H}$ , so it is bounded, hence converges to  $(0, 0)$  as  $t \rightarrow -\infty$  which is the only possible fixed point in  $\overline{\mathcal{H}}$ . Moreover any trajectory passing by some point of  $\mathcal{H}$  converges to  $(0, 0)$ . So there is an infinity of trajectories converging to this point as  $t \rightarrow \infty$ , staying in  $Q_4$ .

Next consider the other quadrants  $Q_1, Q_2$ . There is no trajectory converging to  $(0, 0)$  in  $Q_2$ . Indeed suppose that a trajectory converges to  $(0, 0)$  in  $Q_2$  as  $t \rightarrow \infty$ ; then it satisfies  $s_t > 0, z_t < 0$ , near  $\infty$  which is impossible; if it converges as  $t \rightarrow -\infty$ , then  $s_t < 0, z_t > 0$ , then it cannot be tangent to the axis  $\{z = 0\}$ .

Finally consider the quadrant  $Q_1$ . The region  $\mathcal{K}$  delimited by the two axis and the line  $N_0M_0$  is negatively invariant. Then any trajectory passing by a point of  $\mathcal{K}$  converges to  $(0, 0)$  at  $-\infty$ . Moreover the region  $\mathcal{L}$  delimited by the two axis and  $\mathcal{T}_1$  is also invariant. Then any trajectory passing by a point of  $\mathcal{L}$  converges to  $(0, 0)$  at  $-\infty$ . Thus there is an infinity of central manifolds.

Those trajectories satisfy  $\lim_{t \rightarrow -\infty} \frac{z}{s} = 0$ , then  $s_t \sim s^2$ . By integration,  $0 < s = -\frac{w_t}{w} \sim \frac{1}{|t|}$ . Thus for given  $\varepsilon > 0$ , there holds  $C_1 |t|^{(1+\varepsilon)} \leq w \leq C_2 |t|^{(1+\varepsilon)}$  near  $-\infty$ . Since  $\mathbf{p} = \mathbf{N}$ , equation (4.1) reduces to

$$-(N-1)|w_t|^{N-2} w_{tt} = \varepsilon e^{(\mathbf{N}+\sigma)t} w^q.$$

Then

$$-(\mathbf{N}-1)|w_t|^{N-2-\mathbf{q}} w_{tt} = \left(\frac{\mathbf{N}-1}{\mathbf{q}-\mathbf{N}+1} |w_t|^{N-1-\mathbf{q}}\right)_t \sim \varepsilon e^{(\mathbf{N}+\sigma)t} |t|^q,$$

hence by integration, either  $\lim_{t \rightarrow -\infty} w_t = -c < 0$ , or  $\frac{\mathbf{N}-1}{\mathbf{q}-\mathbf{N}+1} |w_t|^{N-1-\mathbf{q}} \sim -\frac{\varepsilon}{\mathbf{N}+\sigma} e^{(\mathbf{N}+\sigma)t} |t|^q$ . The last case is impossible for  $\varepsilon = 1$ . If  $\varepsilon = -1$ , since  $\frac{|w_t|}{w} \sim \frac{1}{|t|}$ , it implies that  $w^{-\frac{\mathbf{q}}{\mathbf{N}-1}} |w_t| = \left(\frac{\mathbf{N}-1}{\mathbf{q}-\mathbf{N}+1} w^{-\frac{\mathbf{q}-\mathbf{N}+1}{\mathbf{N}-1}}\right)_t \sim C e^{\frac{\mathbf{N}+\sigma}{\mathbf{N}-1}t}$ , which by integration contradicts the estimate  $C_1 |t|^{(1+\varepsilon)} \leq w \leq C_2 |t|^{(1+\varepsilon)}$ . Then  $w \sim_{r \rightarrow 0} C |\ln r|$ , and  $w' \sim_{r \rightarrow 0} -\frac{C}{r}$ . ■

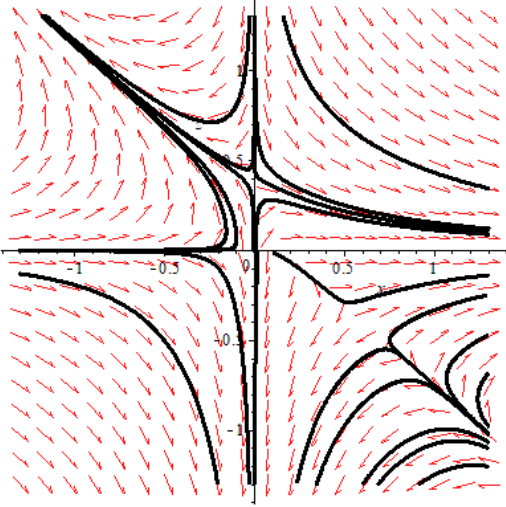


Figure 10. Theorem 4.20:  
 $\mathbf{p} = \mathbf{N} = \frac{5}{3} > -\sigma = \frac{19}{15}, \mathbf{q} = 1.3$

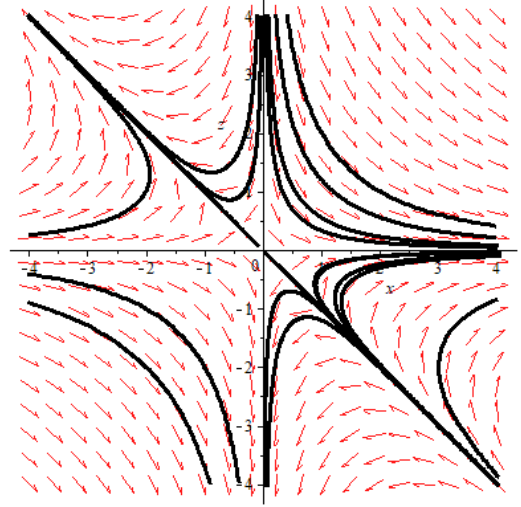


Figure 11. Theorem 4.21  
 $\mathbf{p} = \mathbf{N} = -\sigma = \frac{5}{3}, \mathbf{q} = 1.5$

**Proof of Theorem 4.21.** Here  $\mathbf{p} = \mathbf{N} = -\sigma$ , see figure 11. The system is

$$\begin{cases} s_t = s(s + \frac{z}{\mathbf{N}-1}), \\ z_t = z(-\mathbf{q}s - z). \end{cases}$$

The unique fixed point is  $(0, 0)$ , and the two eigenvalues are 0. From Proposition 4.17, there exist solutions such that  $s \equiv -\frac{\mathbf{N}}{(\mathbf{N}-1)(\mathbf{q}+1)}z$ , corresponding to trajectories in  $Q_2$  and  $Q_4$ . We get a family of solutions  $w$  given by (4.20). The sign  $+$  (resp.  $-$ ) corresponds to the trajectory in  $Q_4$  (resp.  $Q_2$ ). It seems that there is no other trajectory converging to  $(0, 0)$ , thus no other solution  $w$  can be defined near 0, or near  $\infty$ .

For  $\varepsilon = 1$ , corresponding to quadrants  $Q_1$  and  $Q_3$  all the trajectories satisfy  $s_t z_t < 0$ , hence they cannot converge to the unique fixed point  $(0, 0)$  as  $t \rightarrow \pm\infty$ , from Lemma 4.8. So there is no local solution near 0 or  $\infty$ . ■

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