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Superintegrability, symmetry and point particle T-duality

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Abstract

We show that the ideas related to integrability and symmetry play an important role not only in the string T-duality story but also in its point particle counterpart. Applying those ideas, we find that the T-duality seems to be a more widespread phenomenon in the context of the point particle dynamics than it is in the string one; moreover, it concerns physically very relevant point particle dynamical systems and not just somewhat exotic ones fabricated for the purpose. As a source of T-duality examples, we consider maximally superintegrable spherically symmetric electro-gravitational backgrounds in n dimensions. We then describe in detail four such spherically symmetric dynamical systems which are all mutually interconnected by a web of point particle T-dualities. In particular, the dynamics of a charged particle scattered by a repulsive Coulomb potential in a flat space is T-dual to the dynamics of the Coulomb scattering in the space of constant negative curvature, but it is also T-dual to the (conformal) Calogero-Moser inverse square dynamics both in flat and hyperbolic spaces. Thus knowing just the Hamiltonian dynamics of the scattered particle cannot give us an information about the curvature of the space in which the particle moves.

1 Introduction

The motion of a classical string in a gravitational-Kalb-Ramond background is characterized by a dynamical system referred to as a nonlinear σ -model in $1 + 1$ spacetime dimensions. In the case of a topologically trivial Kalb-Ramond field strength, the classical action of this σ -model reads

$$S = \int d\tau \oint d\sigma (g_{ij}(x) + b_{ij}(x)) \partial_+ x^i \partial_- x^j, \quad \partial_{\pm} := \partial_{\tau} \pm \partial_{\sigma}, \quad (1)$$

where τ, σ are respectively time and (circular) space coordinates on the worldsheet, x^i are coordinates on the target space T and $(g_{ij}(x), b_{ij}(x))$ are a metric tensor and a Kalb-Ramond potential in those coordinates.

Consider some other gravitational-Kalb-Ramond background $(\tilde{T}, \tilde{g}_{ij}(\tilde{x}), \tilde{b}_{ij}(\tilde{x}))$ and the σ -model which corresponds to it

$$\tilde{S} = \int d\tau \oint d\sigma (\tilde{g}_{ij}(\tilde{x}) + \tilde{b}_{ij}(\tilde{x})) \partial_+ \tilde{x}^i \partial_- \tilde{x}^j. \quad (2)$$

The phenomenon of stringy T-duality [8, 15, 5, 10, 11, 12] takes place if the backgrounds (T, g, b) and $(\tilde{T}, \tilde{g}, \tilde{b})$ are not *geometrically equivalent* but the σ -models (1) and (2) are *dynamically equivalent*.

The geometrical (non)equivalence of the targets means the (non)existence of a diffeomorphism $D : T \rightarrow \tilde{T}$ such that $(D^* \tilde{g}, D^* \tilde{b}) = (g, b)$. On the other hand, the dynamical equivalence of the σ -models (1) and (2) means the equivalence of their Hamiltonian dynamics [1, 16]. Thus if the σ -models (1),(2) are characterized by their respective phase spaces P, \tilde{P} , symplectic forms $\omega, \tilde{\omega}$ and Hamiltonians h, \tilde{h} they are dynamically equivalent if it exists a symplectomorphism $\Upsilon : P \rightarrow \tilde{P}$ such that $\Upsilon^* \tilde{h} = h$.

In particular, the phase spaces P and \tilde{P} of the σ -models (1) and (2) are parametrized respectively by the functions $x^i(\sigma), p_i(\sigma)$ and $\tilde{x}^i(\sigma), \tilde{p}_i(\sigma)$, the symplectic forms are the canonical ones

$$\omega = \oint d\sigma dp_i(\sigma) \wedge dx^i(\sigma), \quad \tilde{\omega} = \oint d\sigma d\tilde{p}_i(\sigma) \wedge d\tilde{x}^i(\sigma)$$

and the Hamiltonians read

$$h(x, p) = \frac{1}{2} \oint d\sigma g^{ij}(x) (p_i - b_{ik}(x) \partial_\sigma x^k) (p_j - b_{jl}(x) \partial_\sigma x^l) + \frac{1}{2} \oint d\sigma g_{ij}(x) \partial_\sigma x^i \partial_\sigma x^j,$$

$$\tilde{h}(\tilde{x}, \tilde{p}) = \frac{1}{2} \oint d\sigma \tilde{g}^{ij}(\tilde{x}) (\tilde{p}_i - \tilde{b}_{ik}(\tilde{x}) \partial_\sigma \tilde{x}^k) (\tilde{p}_j - \tilde{b}_{jl}(\tilde{x}) \partial_\sigma \tilde{x}^l) + \frac{1}{2} \oint d\sigma \tilde{g}_{ij}(\tilde{x}) \partial_\sigma \tilde{x}^i \partial_\sigma \tilde{x}^j.$$

The symplectomorphism Υ is a canonical transformation $\tilde{x} = \tilde{x}(x, p), \tilde{p} = \tilde{p}(x, p)$ such that

$$h(x, p) = \tilde{h}(\tilde{x}(x, p), \tilde{p}(x, p)).$$

Historically, the dynamical equivalence of strings moving in the geometrically non-equivalent backgrounds came as a surprise and it was often considered to be a distinctive feature of the string dynamics with respect to the point particle one. However, as it was pointed out in [9], the T-duality exists also in the point particle context, where it establishes the dynamical equivalence of geometrically non-equivalent electro-magnetic-gravitational backgrounds.

The motion of a classical point particle in a electro-magnetic-gravitational background is characterized by a classical action of a 0 + 1-dimensional σ model

$$S = \int dt \left(\frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j - A_i(x) \dot{x}^i - V(x) \right), \quad (3)$$

where t is the time, x^i are coordinates on the target space T and $(g_{ij}(x), A_i(x), V(x))$ are respectively the metric tensor as well as the vector and the scalar potentials.

Consider a geometrically non-equivalent background $(\tilde{T}, \tilde{g}_{ij}(\tilde{x}), \tilde{A}_i(\tilde{x}), \tilde{V}(\tilde{x}))$ and the corresponding action

$$\tilde{S} = \int dt \left(\frac{1}{2} \tilde{g}_{ij}(\tilde{x}) \dot{\tilde{x}}^i \dot{\tilde{x}}^j - \tilde{A}_i(\tilde{x}) \dot{\tilde{x}}^i - \tilde{V}(\tilde{x}) \right). \quad (4)$$

The phase spaces P and \tilde{P} of the 0 + 1-dimensional σ -models (3) and (4) are parametrized respectively by the canonically conjugated coordinates x^i, p_i and \tilde{x}^i, \tilde{p}_i , the symplectic forms are the canonical ones

$$\omega = dp_i \wedge dx^i, \quad \tilde{\omega} = d\tilde{p}_i \wedge d\tilde{x}^i$$

and the Hamiltonians read

$$h(x, p) = \frac{1}{2} g^{ij}(x) (p_i + A_i(x)) (p_j + A_j(x)) + V(x), \quad (5)$$

$$\tilde{h}(\tilde{x}, \tilde{p}) = \frac{1}{2} \tilde{g}^{ij}(\tilde{x}) (\tilde{p}_i + \tilde{A}_i(\tilde{x})) (\tilde{p}_j + \tilde{A}_j(\tilde{x})) + \tilde{V}(\tilde{x}). \quad (6)$$

In full analogy with the string case, we declare the point particle models (3) and (4) mutually T-dual if it exists a canonical transformation $\tilde{x} = \tilde{x}(x, p)$, $\tilde{p} = \tilde{p}(x, p)$ such that

$$\tilde{h}(\tilde{x}(x, p), \tilde{p}(x, p)) = h(x, p).$$

The first nontrivial examples of the point particle T-duality obtained in [9] showed that the phenomenon did exist but otherwise they were not particularly physically relevant and they were fabricated for the purpose by essentially a trial and error method. In this paper, we do much better, we show that the point particle T-duality concerns physically very relevant dynamical systems and we give also a method how to obtain many new examples. This method is based on the concepts of integrability and symmetry and was largely inspired by the string T-duality story where, apparently, all known integrable σ -models are (Poisson-Lie) symmetric and T-dualizable. It turns out that the integrability and symmetry help to find the T-duality examples also in the point particle context, moreover, the reason why they help turns out to be much clearer than in the string case where the observed relation between the integrability and T-dualizability remains somewhat mysterious.

Few remarks are perhaps in order about the motivations to study the point particle T-duality. First of all, it is an interesting problem to deal with on its own, since it opens a problem of classification of physical dynamical systems in T-duality equivalence classes. All members of a given class share the same dynamical properties, which maybe manifest or hidden depending on which representative of the class we consider. For example, the T-duality between the Coulomb and the Calogero-Moser scattering, which we establish in the present paper, means that

the manifest conformal symmetry of the Calogero-Moser model is present also in the Coulomb one albeit in a hidden dynamical way.

Another motivation has to do with the problem of zero modes in the string T-duality story. At a first sight it might seem that at least some examples of the point particle T-duality could be obtained by a sort of dimensional reduction of the stringy T-duality, or, said in other words, by restricting the string dynamics to the zero modes. However, this is not the case because (with a notable exception of Abelian T-duality) the T-duality phenomenon in string theory was so far established only for strings deprived of the zero modes. Indeed, the stringy T-duality is in reality *dismembered*, that is, it takes place only if we cut out some zero modes from the string on both original and dual side. This means, in particular, that no examples of point particle T-duality can be obtained by a dimensional reduction of this dismembered string T-duality. However, it might be possible to go in an opposite direction, this is to say, to work out viable examples of the point particle T-duality and to "glue" them to the dismembered stringy T-duality examples to achieve a full-fledged string T-duality.

The plan of the paper is as follows. In Section 2, we construct a particularly simple dynamical system in n dimensions that we call *referential spherically symmetric maximally superintegrable system*. Although this simple system does not have a geometric interpretation as a $0+1$ -dimensional σ -model, still it plays an important role in our analysis because it does naturally represent a T-duality class of $0+1$ -dimensional σ -models which do have the geometric interpretation. Indeed, in Section 3, we show that four physically relevant and geometrically distinct spherically symmetric σ -models are maximally superintegrable and symplectomorphic to the referential system. It follows, that they are all mutually T-dual, or, said differently, they belong all to the same T-duality class represented by the referential model. Those four systems are the (repulsive) Coulomb potential in the flat space and in the space of constant negative curvature, as well as the Calogero-Moser potential in the flat and in the hyperbolic spaces. In Section 4, we provide conclusions and an outlook. Two technical results concerning Section 3 are placed into Appendix.

2 Referential maximally superintegrable system

A dynamical system (P, ω, h) is a smooth manifold P equipped with a symplectic form ω and with a smooth function h , such that all time evolution flows generated by the Hamiltonian h are complete, that is, they can be all smoothly prolonged to both forward and backward infinities $t \rightarrow \pm\infty$.

Let $H > 0$, T be canonically conjugated coordinates on an open symplectic half-plane P_1 equipped with the Darboux symplectic form

$$\omega_1 = dH \wedge dT,$$

or, equivalently, with the Darboux Poisson bracket

$$\{T, H\} = 1. \quad (7)$$

Note that a choice of the Hamiltonian $h_1(H, T) = H$ gives a honest dynamical system $(P_1, \omega_1, h_1 = H)$ with the complete flows $H = \text{const}$, $T = t - t_0$. Indeed, this simple form of the flows follows from the Hamiltonian equations of motion which take the form

$$\dot{T} = \{T, h_1\} = 1, \quad \dot{H} = \{H, h_1\} = 0.$$

Remark 2.1. On the other hand, a choice $h_1(H, T) = T$ does not give a dynamical system because the corresponding flows $T = \text{const}$, $H = -t + t_0$ cannot be prolonged to $t \rightarrow \infty$ (H must remain positive).

Let S^{n-1} be the standard $(n-1)$ -dimensional unit sphere and T^*S^{n-1} its cotangent bundle equipped with its standard symplectic form $\omega_{T^*S^{n-1}}$. We parametrize T^*S^{n-1} by n -vectors \mathbf{B} and \mathbf{k} fulfilling

$$\mathbf{k}\mathbf{k} = 1, \quad \mathbf{B}\mathbf{k} = 0. \quad (8)$$

The vector \mathbf{k} thus represents a point on the sphere S^{n-1} , while \mathbf{B} parametrizes the cotangent space at \mathbf{k} . The symplectic form $\omega_{T^*S^{n-1}}$ then reads

$$\omega_{T^*S^{n-1}} = d\mathbf{B} \wedge d\mathbf{k}.$$

We are now ready to define the *referential spherically symmetric maximally superintegrable dynamical system* (P_n, ω_n, h_n) alluded to in the Introduction. The phase space P_n of this dynamical system is defined as

$$P_n = P_1 \times T^*S^{n-1},$$

its symplectic form ω_n is given by

$$\omega_n = \omega_1 + \omega_{T^*S^{n-1}} = dH \wedge dT + d\mathbf{B} \wedge d\mathbf{k}. \quad (9)$$

and its Hamiltonian h_n is given simply by

$$h_n = H.$$

We now provide an n -dimensional analogue of (7), that is the complete set of Poisson brackets corresponding to (or characterizing) the symplectic form ω_n :

$$\{T, H\} = 1, \quad \{H, \mathbf{k}\} = \mathbf{0}, \quad \{T, \mathbf{k}\} = \mathbf{0}, \quad \{H, \mathbf{B}\} = \mathbf{0}, \quad \{T, \mathbf{B}\} = \mathbf{0}, \quad (10)$$

$$\{B_i, B_j\} = B_i k_j - B_j k_i, \quad \{k_i, B_j\} = \delta_{ij} - k_i k_j, \quad \{k_i, k_j\} = 0, \quad i, j = 1, \dots, n. \quad (11)$$

Note that the brackets (11) are the Dirac ones; they are derived from (9) by taking into account the constraints (8).

The Hamiltonian equations of motion of the referential system (P_n, ω_n, h_n) read

$$\dot{\mathbf{k}} = \{\mathbf{k}, h_n\} = 0, \quad \dot{\mathbf{B}} = \{\mathbf{B}, h_n\} = 0, \quad \dot{T} = \{T, h_n\} = 1, \quad \dot{H} = \{H, h_n\} = 0$$

and this implies the completeness of the flows $\mathbf{k}=\text{const}$, $\mathbf{B}=\text{const}$, $H=\text{const}$, $T = t - t_0$.

The coordinates \mathbf{k}, \mathbf{B} on T^*S^{n-1} Poisson commute with the Hamiltonian $h_n = H$, they are therefore the integrals of motion. Together with the Hamiltonian h_n , those coordinates furnish the $2n-1$ integrals of motion in involution, the referential system (P_n, ω_n, h_n) is therefore *maximally superintegrable*. Note, in particular, that the components of the wedge product $\mathbf{k} \wedge \mathbf{B}$ are conserved generators of n -dimensional rotations (i.e. the angular momenta).

Remark 2.2. We note that the referential dynamical system (P_n, ω_n, h_n) does not lend itself to a geometric interpretation. However, as we shall see in the next section, it is symplectomorphic to at least four dynamical systems which do have the geometrical interpretation as the $0+1$ -dimensional σ -models. We may therefore say that the non-geometric referential system naturally represents a whole T-duality equivalence class of the geometric systems.

We now add some technical stuff which will be useful in the next section. Consider a $2n$ -dimensional manifold

$$M_n = \{(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^n \times \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}\}$$

equipped with the Darboux symplectic form

$$\Omega_n = d\mathbf{p} \wedge d\mathbf{x}.$$

The canonical Poisson brackets corresponding to Ω_n are

$$\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Note that we use the notation $\{.,.\}$ for the Poisson brackets on P_n and the boldface one $\{\mathbf{.}, \mathbf{.}\}$ for the Poisson brackets on M_n .

It turns out that the symplectic manifold M_n is the phase space P_n in a disguise. Indeed, consider a bijection $\mathcal{R}^r : M_n \rightarrow P_n$ defined as

$$\begin{aligned} H = h^r(\mathbf{x}, \mathbf{p}) &:= \frac{1}{2}\mathbf{x}^2, & T = t^r(\mathbf{x}, \mathbf{p}) &:= -\frac{\mathbf{p}\mathbf{x}}{\mathbf{x}^2}, \\ \mathbf{k} = \mathbf{k}^r(\mathbf{x}, \mathbf{p}) &:= \frac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{B} = \mathbf{B}^r(\mathbf{x}, \mathbf{p}) &:= \frac{\mathbf{x}^2\mathbf{p} - (\mathbf{p}\mathbf{x})\mathbf{x}}{|\mathbf{x}|}, \end{aligned} \quad (12)$$

with the inverse map \mathcal{R}_i^r given by

$$\mathbf{x} = \sqrt{2H}\mathbf{k}, \quad \mathbf{p} = -\sqrt{2HT}\mathbf{k} + \frac{\mathbf{B}}{\sqrt{2H}}.$$

The direct calculation of the bold-faced Poisson brackets gives

$$\{t^r, h^r\} = 1, \quad \{h^r, \mathbf{k}^r\} = \mathbf{0}, \quad \{t^r, \mathbf{k}^r\} = \mathbf{0}, \quad \{h^r, \mathbf{B}^r\} = \mathbf{0}, \quad \{t^r, \mathbf{B}^r\} = \mathbf{0}, \quad (13)$$

$$\{B_i^r, B_j^r\} = B_i^r k_j^r - B_j^r k_i^r, \quad \{k_i^r, B_j^r\} = \delta_{ij} - k_i^r k_j^r, \quad \{k_i^r, k_j^r\} = 0, \quad i, j = 1, \dots, n. \quad (14)$$

Comparing (10),(11) with (13),(14), we conclude that the map \mathcal{R}^r is indeed the symplectomorphism.

Remark 2.3. In Section 3, we shall present as the main technical result of this paper an explicit construction of four symplectomorphisms from M_n to P_n denoted respectively as $\mathcal{R}^M, \mathcal{R}^{y^M}, \mathcal{R}^C, \mathcal{R}^y$. Those four symplectomorphisms will have all geometrical interpretation. It is perhaps worth pointing out that there exist also symplectomorphisms which do not have geometric interpretation, like, for example, \mathcal{R}^r where the Hamiltonian $h^r(\mathbf{x}, \mathbf{p})$ does not have a kinetic term. The reason why we have introduced \mathcal{R}^r is the fact that the \mathbf{x}, \mathbf{p} -depending vectors $\mathbf{k}^r(\mathbf{x}, \mathbf{p}), \mathbf{B}^r(\mathbf{x}, \mathbf{p})$ will play an important technical role throughout the paper.

3 Explicit canonical transformations

3.1 Calogero-Moser system in the flat space

In this section, we show that the Calogero-Moser system in the flat space is symplectomorphic to the referential maximally superintegrable system of Section 2.

Spherically symmetric Calogero-Moser dynamical system (M_n, Ω_n, h^M) is defined by the Hamiltonian

$$h^M(\mathbf{x}, \mathbf{p}) := \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\frac{\gamma^2}{\mathbf{x}^2}. \quad (15)$$

The flows generated by (15) are complete due to the conservation of energy and the fact, that the Calogero-Moser Hamiltonian is the sum of two positive terms, therefore neither kinetic nor potential energy may diverge within a given flow characterized by some conserved value of energy. This means that a particle can never reach singularity at $\mathbf{x} = \mathbf{0}$ nor develop an unbounded velocity which would be necessary in order to reach infinity in a finite time.

Following (5), the Calogero-Moser Hamiltonian $h^M(\mathbf{x}, \mathbf{p})$ has the geometric interpretation as the Hamiltonian of the 0 + 1-dimensional σ -model. Indeed, it corresponds to the motion of a charged particle in a flat space \mathbb{R}^n and in a repulsive centrally symmetric electric potential $V(\mathbf{x}) = \frac{1}{2}\gamma^2\mathbf{x}^{-2}$.

Remark 3.1. The Calogero-Moser dynamical system is sometimes referred to as the conformal field theory in 0 + 1-dimensions. The reason for this interpretation is the fact that the conformal group in 0 + 1-dimension is $SL(2, \mathbb{R})$ and it is infinitesimally generated via the Poisson brackets by the Hamiltonian h^M , a dilation charge $D = -\frac{1}{2}\mathbf{p}\mathbf{x}$ and a special conformal transformation

charge $C = \frac{1}{2}\mathbf{x}^2$. It is easy to verify that the Poisson brackets of those generators form the $sl(2, \mathbb{R})$ Lie algebra

$$\{h^M, D\} = h^M, \quad \{C, D\} = -C, \quad \{h^M, C\} = 2D.$$

It is well-known that the flat Calogero-Moser system is superintegrable in three dimensions [13], our goal is now to show that the n -dimensional version (M_n, Ω_n, h^M) is also superintegrable and, moreover, it is symplectomorphic precisely to the superintegrable referential dynamical system¹ (P_n, ω_n, H) . For that, consider a map $\mathcal{R}^M : M_n \rightarrow P_n$ defined as

$$H = h^M(\mathbf{x}, \mathbf{p}) = \frac{1}{2}(\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2}), \quad T = t^M(\mathbf{x}, \mathbf{p}) := \frac{\mathbf{p}\mathbf{x}}{\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2}}, \quad (16)$$

$$\mathbf{k} = \mathbf{k}^M(\mathbf{x}, \mathbf{p}) := \mathbf{k}^r(\mathbf{x}) \cos \Psi^M(\mathbf{x}, \mathbf{p}) - \frac{\mathbf{B}^r(\mathbf{x}, \mathbf{p})}{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|} \sin \Psi^M(\mathbf{x}, \mathbf{p}), \quad (17)$$

$$\mathbf{B} = \mathbf{B}^M(\mathbf{x}, \mathbf{p}) := \mathbf{B}^r(\mathbf{x}, \mathbf{p}) \cos \Psi^M(\mathbf{x}, \mathbf{p}) + |\mathbf{B}^r(\mathbf{x}, \mathbf{p})| \mathbf{k}^r(\mathbf{x}) \sin \Psi^M(\mathbf{x}, \mathbf{p}), \quad (18)$$

where

$$\Psi^M(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|}{\sqrt{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \gamma^2}} \arctan \frac{\mathbf{p}\mathbf{x}}{\sqrt{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \gamma^2}}.$$

We verify easily that it holds

$$(\mathbf{k}^M(\mathbf{x}, \mathbf{p}))^2 = 1, \quad \mathbf{k}^M(\mathbf{x}, \mathbf{p}) \mathbf{B}^M(\mathbf{x}, \mathbf{p}) = 0,$$

we thus observe that the map \mathcal{R}^M is indeed from M_n to P_n . Moreover, the map \mathcal{R}^M is evidently defined on the whole M_n and is smooth everywhere on M_n .

Now consider a map $\mathcal{R}_i^M : P_n \rightarrow M_n$ defined by

$$\mathbf{x} = N^M(H, T, |\mathbf{B}|) \left(\mathbf{k} \cos \Psi^M(H, T, |\mathbf{B}|) + \frac{\mathbf{B}}{|\mathbf{B}|} \sin \Psi^M(H, T, |\mathbf{B}|) \right) \quad (19a)$$

$$\mathbf{p} = \frac{(2HT\mathbf{k} + \mathbf{B}) \cos \Psi^M(H, T, |\mathbf{B}|) + (2HT\frac{\mathbf{B}}{|\mathbf{B}|} - |\mathbf{B}|\mathbf{k}) \sin \Psi^M(H, T, |\mathbf{B}|)}{N^M(H, T, |\mathbf{B}|)}, \quad (19b)$$

where

$$N^M(H, T, |\mathbf{B}|) = \frac{\sqrt{4H^2T^2 + \mathbf{B}^2 + \gamma^2}}{\sqrt{2H}},$$

$$\Psi^M(H, T, |\mathbf{B}|) = \frac{|\mathbf{B}|}{\sqrt{\mathbf{B}^2 + \gamma^2}} \arctan \frac{2HT}{\sqrt{\mathbf{B}^2 + \gamma^2}}.$$

¹It should be noted that two given spherically symmetric maximally superintegrable models need not be necessarily symplectomorphic to each other. In particular, the phase space of one of them may be symplectomorphic to our referential phase space P_n but the phase space of the other may be rather symplectomorphic to a \mathbb{Z} -quotient of P_n (in this case the symplectic half-plane H, T becomes a symplectic half-cylinder with T becoming an angle variable). Other scenarios are also possible.

The map \mathcal{R}_i^M is evidently well defined on the whole P_n and it is everywhere smooth because the apparent singularity at $|\mathbf{B}| = 0$ is smoothly removable due to the multiplication by $\sin \Psi^M$.

We readily verify that

$$\mathcal{R}^M \circ \mathcal{R}_i^M = \text{Id}_{P_n}, \quad \mathcal{R}_i^M \circ \mathcal{R}^M = \text{Id}_{M_n},$$

which means that the both maps $\mathcal{R}^M, \mathcal{R}_i^M$ are diffeomorphisms inverse to each other.

A direct calculation of the bold-faced Poisson brackets then gives

$$\{t^M, h^M\} = 1, \quad \{h^M, \mathbf{k}^M\} = \{t^M, \mathbf{k}^M\} = \{h^M, \mathbf{B}^M\} = \{t^M, \mathbf{B}^M\} = \mathbf{0}, \quad (20)$$

$$\{B_i^M, B_j^M\} = B_i^M k_j^M - B_j^M k_i^M, \quad \{k_i^M, B_j^M\} = \delta_{ij} - k_i^M k_j^M, \quad \{k_i^M, k_j^M\} = 0. \quad (21)$$

Comparing (20),(21) with (10),(11), we conclude that the diffeomorphism \mathcal{R}^M is in fact the symplectomorphism. Said in other words, we have just shown that the flat Calogero-Moser system (M_n, Ω_n, h^M) is symplectomorphic to the referential dynamical system (P_n, ω_n, H) via the symplectomorphism \mathcal{R}^M , in particular, we have

$$H = h^M(\mathbf{x}, \mathbf{p}) = \frac{1}{2}(\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2}).$$

If we interpret the variable T in (19) as time and $H, \mathbf{k}, \mathbf{B}$ as constant quantities, the inverse symplectomorphism (19) can be checked to be the solution of the Calogero-Moser Hamiltonian equations of motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h^M\} = \mathbf{p}, \quad \dot{\mathbf{p}} = \{\mathbf{p}, h^M\} = \frac{\gamma^2 \mathbf{x}}{(\mathbf{x}^2)^2}. \quad (22)$$

In reality, we have used this very fact to find the explicit form (16), (17) and (18) of the symplectomorphism \mathcal{R}^M . We have first found the general solution (19) of the Calogero-Moser Hamiltonian equations of motions (22), we interpreted the time as the variable canonically conjugated to the Hamiltonian and then we expressed $H, T, \mathbf{k}, \mathbf{B}$ as the functions of \mathbf{p}, \mathbf{x} . It was of course not clear from the outset what kind of the bold-faced Poisson brackets would obey those functions, but it turned out eventually that they do obey those of the referential dynamical system (P_n, ω_n, H) . It is this circumstance which makes the spherically symmetric Calogero-Moser model propitious to admit point particle T-duals.

3.2 Calogero-Moser system in the hyperbolic space

In this section, we show that the Calogero-Moser system in the space of constant negative curvature is symplectomorphic to the referential maximally superintegrable system of Section 2.

Equip the space \mathbb{R}^n with a metric

$$g_{jk} = \delta_{jk} - \frac{\alpha^2 x^j x^k}{1 + \alpha^2 \mathbf{x}^2}. \quad (23)$$

The scalar curvature of the metric (23) is constant

$$R = -n(n-1)\alpha^2;$$

the space \mathbb{R}^n equipped with the metric (23) is then called the hyperbolic space or the space of negative constant curvature.

Note that the inverse metric tensor reads

$$g^{jk}(\mathbf{x}) = \delta^{jk} + \alpha^2 x^j x^k,$$

therefore the Hamiltonian (5) of a charged point particle moving in the background (23) and feeling the electric potential $V(\mathbf{x}) = \frac{1}{2} \frac{\gamma^2}{\mathbf{x}^2}$ is

$$h^{yM}(\mathbf{x}, \mathbf{p}) = \frac{1}{2} g^{jk}(\mathbf{x}) p_j p_k + V(\mathbf{x}) = \frac{1}{2} \left(\mathbf{p}^2 + \frac{\gamma^2}{\mathbf{x}^2} + \alpha^2 (\mathbf{p}\mathbf{x})^2 \right). \quad (24)$$

We thus observe, that the dynamical system (M_n, Ω_n, h^{yM}) is an α -deformation of the flat Calogero-Moser dynamics described in the previous section, the deformation which physically corresponds to switching on the negative constant curvature.

Note that the flows generated by (24) are again complete due to a variant of the argument given in the previous section for the case $\alpha = 0$. Indeed, the hyperbolic Calogero-Moser Hamiltonian is the sum of positive terms, therefore neither kinetic nor potential energy may diverge within a given flow characterized by some conserved value of energy.

Our goal is to show that the hyperbolic model (M_n, Ω_n, h^{yM}) is symplectomorphic to the referential dynamical system (P_n, ω_n, H) . For that, consider a map $\mathcal{R}^{yM} : M_n \rightarrow P_n$ defined as

$$H = h^{yM}(\mathbf{x}, \mathbf{p}) = \frac{1}{2} (\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2} + \alpha^2 (\mathbf{p}\mathbf{x})^2), \quad (25)$$

$$T = t^{yM}(\mathbf{p}, \mathbf{x}) = \frac{\operatorname{arctanh} \frac{\alpha \mathbf{p}\mathbf{x}}{\sqrt{\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2} + \alpha^2 (\mathbf{p}\mathbf{x})^2}}}{\alpha \sqrt{\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2} + \alpha^2 (\mathbf{p}\mathbf{x})^2}}, \quad (26)$$

$$\mathbf{k} = \mathbf{k}^{yM}(\mathbf{x}, \mathbf{p}) := \mathbf{k}^r(\mathbf{x}) \cos \Psi^{yM}(\mathbf{x}, \mathbf{p}) - \frac{\mathbf{B}^r(\mathbf{x}, \mathbf{p})}{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|} \sin \Psi^{yM}(\mathbf{x}, \mathbf{p}), \quad (27)$$

$$\mathbf{B} = \mathbf{B}^{yM}(\mathbf{x}, \mathbf{p}) := \mathbf{B}^r(\mathbf{x}, \mathbf{p}) \cos \Psi^{yM}(\mathbf{x}, \mathbf{p}) + |\mathbf{B}^r(\mathbf{x}, \mathbf{p})| \mathbf{k}^r(\mathbf{x}) \sin \Psi^{yM}(\mathbf{x}, \mathbf{p}), \quad (28)$$

where

$$\Psi^{yM}(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|}{\sqrt{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \gamma^2}} \arctan \frac{\mathbf{p}\mathbf{x}}{\sqrt{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \gamma^2}}.$$

We verify easily that it holds

$$(\mathbf{k}^{yM}(\mathbf{x}, \mathbf{p}))^2 = 1, \quad \mathbf{k}^{yM}(\mathbf{x}, \mathbf{p})\mathbf{B}^{yM}(\mathbf{x}, \mathbf{p}) = 0,$$

we thus observe that the map \mathcal{R}^{yM} is indeed from M_n to P_n . Moreover, the map \mathcal{R}^{yM} is evidently defined on the whole M_n and is smooth everywhere on M_n .

Now consider a map $\mathcal{R}_i^{yM} : P_n \rightarrow M_n$ defined by

$$\mathbf{x} = N^{yM}(H, T, |\mathbf{B}|) \left(\mathbf{k} \cos \Psi^{yM}(H, T, |\mathbf{B}|) + \frac{\mathbf{B}}{|\mathbf{B}|} \sin \Psi^{yM}(H, T, |\mathbf{B}|) \right) \quad (29)$$

$$\begin{aligned} \mathbf{p} &= \frac{(\sqrt{2H} \tanh(\alpha\sqrt{2HT})\mathbf{k} + \alpha\mathbf{B}) \cos \Psi^{yM}(H, T, |\mathbf{B}|)}{\alpha N^{yM}(H, T, |\mathbf{B}|)} + \\ &+ \frac{(\sqrt{2H} \tanh(\alpha\sqrt{2HT})\frac{\mathbf{B}}{|\mathbf{B}|} - \alpha|\mathbf{B}|\mathbf{k}) \sin \Psi^{yM}(H, T, |\mathbf{B}|)}{\alpha N^{yM}(H, T, |\mathbf{B}|)}, \end{aligned} \quad (30)$$

where

$$N^{yM}(H, T, |\mathbf{B}|) = \sqrt{\frac{(\mathbf{B}^2 + \gamma^2) \cosh^2(\alpha\sqrt{2HT})}{2H} + \frac{\sinh^2(\alpha\sqrt{2HT})}{\alpha^2}},$$

and

$$\Psi^{yM}(H, T, |\mathbf{B}|) = \frac{|\mathbf{B}|}{\sqrt{\mathbf{B}^2 + \gamma^2}} \arctan \frac{\sqrt{2H} \tanh(\alpha\sqrt{2HT})}{\alpha\sqrt{\mathbf{B}^2 + \gamma^2}}.$$

The map \mathcal{R}_i^{yM} is evidently well defined on the whole P_n and it is everywhere smooth because the apparent singularity at $|\mathbf{B}| = 0$ is smoothly removable due to the multiplication by $\sin \Psi^{yM}$.

We readily verify that

$$\mathcal{R}^{yM} \circ \mathcal{R}_i^{yM} = \text{Id}_{P_n}, \quad \mathcal{R}_i^{yM} \circ \mathcal{R}^{yM} = \text{Id}_{M_n},$$

which means that the both maps $\mathcal{R}^{yM}, \mathcal{R}_i^{yM}$ are diffeomorphisms inverse to each other.

A direct calculation of the bold-faced Poisson brackets then gives

$$\{t^{yM}, h^{yM}\} = 1, \quad \{h^{yM}, \mathbf{k}^{yM}\} = \{t^{yM}, \mathbf{k}^{yM}\} = \{h^{yM}, \mathbf{B}^{yM}\} = \{t^{yM}, \mathbf{B}^{yM}\} = \mathbf{0}, \quad (31)$$

$$\{B_i^{yM}, B_j^{yM}\} = B_i^{yM} k_j^{yM} - B_j^{yM} k_i^{yM}, \quad \{k_i^{yM}, B_j^{yM}\} = \delta_{ij} - k_i^{yM} k_j^{yM}, \quad \{k_i^{yM}, k_j^{yM}\} = 0. \quad (32)$$

Comparing (31),(32) with (10),(11), we conclude that the diffeomorphism \mathcal{R}^{yM} is in fact the symplectomorphism. Said in other words, we have just shown that the hyperbolic Calogero-Moser system (M_n, Ω_n, h^{yM}) is symplectomorphic to the

referential dynamical system (P_n, ω_n, H) via the symplectomorphism \mathcal{R}^{yM} , in particular, we have

$$H = h^{yM}(\mathbf{x}, \mathbf{p}) = \frac{1}{2} (\mathbf{p}^2 + \gamma^2 \mathbf{x}^{-2} + \alpha^2 (\mathbf{p}\mathbf{x})^2).$$

If we interpret the variable T in (29) and (30) as time and $H, \mathbf{k}, \mathbf{B}$ as constant quantities, the inverse symplectomorphism (29) and (30) can be checked to be the solution of the hyperbolic Calogero-Moser Hamiltonian equations of motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h^{yM}\} = \mathbf{p} + \alpha^2 (\mathbf{p}\mathbf{x}) \mathbf{x}, \quad \dot{\mathbf{p}} = \{\mathbf{p}, h^{yM}\} = \frac{\gamma^2 \mathbf{x}}{(\mathbf{x}^2)^2} - \alpha^2 (\mathbf{p}\mathbf{x}) \mathbf{p}. \quad (33)$$

In reality, we have used this very fact to find the explicit form (25),(26),(27) and (28) of the symplectomorphism \mathcal{R}^{yM} . We have first found the general solution (29), (30) of the Calogero-Moser Hamiltonian equations of motions (33), we interpreted the time as the variable canonically conjugated to the Hamiltonian and then we expressed $H, T, \mathbf{k}, \mathbf{B}$ as the functions of \mathbf{p}, \mathbf{x} . It was of course not clear from the outset what kind of the bold-faced Poisson brackets would obey those functions, but it turned out eventually that they do obey those of the referential dynamical system (P_n, ω_n, H) . It is this circumstance which makes the hyperbolic spherically symmetric Calogero-Moser model propitious to admit point particle T-duals.

3.3 Repulsive Coulomb potential in the flat space

In this section, we show that the standard repulsive Coulomb system in the flat space is symplectomorphic to the referential maximally superintegrable system of Section 2.

We consider a Hamiltonian

$$h^C(\mathbf{x}, \mathbf{p}) := \frac{1}{2} \mathbf{p}^2 + \frac{\beta^2}{|\mathbf{x}|}, \quad (34)$$

which has a natural physical interpretation in the dimension $n = 3$ because $\frac{\beta^2}{|\mathbf{x}|}$ is the repulsive Coulomb potential in the flat three-dimensional space.

The flows generated by (34) can be shown to be complete by essentially the same argument as in the Calogero-Moser case.

Our goal is to show that the flat Coulomb system (M_n, Ω_n, h^C) is symplectomorphic to the referential dynamical system (P_n, ω_n, H) . For that, we consider a map $\mathcal{R}^C : M_n \rightarrow P_n$ given by

$$H = h^C(\mathbf{x}, \mathbf{p}) = \frac{1}{2} \mathbf{p}^2 + \frac{\beta^2}{|\mathbf{x}|}, \quad T = t^C(\mathbf{x}, \mathbf{p}) := \tau^C(\mathbf{p}\mathbf{x}, h^C(\mathbf{x}, \mathbf{p}), K^C(\mathbf{x}, \mathbf{p})), \quad (35)$$

$$\mathbf{k} = \mathbf{k}^C(\mathbf{x}, \mathbf{p}) := \frac{(\beta^2 |\mathbf{x}| + |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2) \mathbf{k}^r(\mathbf{x}) - (\mathbf{p}\mathbf{x}) \mathbf{B}^r(\mathbf{x}, \mathbf{p})}{K^C(\mathbf{x}, \mathbf{p}) |\mathbf{x}|} \quad (36)$$

$$\mathbf{B} = \mathbf{B}^C(\mathbf{x}, \mathbf{p}) := \frac{(\beta^2|\mathbf{x}| + |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2)\mathbf{B}^r(\mathbf{x}) + |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2(\mathbf{p}\mathbf{x})\mathbf{k}^r(\mathbf{x})}{K^C(\mathbf{x}, \mathbf{p})|\mathbf{x}|} \quad (37)$$

where $\mathbf{k}^r(\mathbf{x})$, $\mathbf{B}^r(\mathbf{x}, \mathbf{p})$ were defined in (12) and

$$K^C(\mathbf{x}, \mathbf{p}) := \sqrt{2h^C(\mathbf{x}, \mathbf{p})|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \beta^4},$$

$$\tau^C(\mathbf{p}\mathbf{x}, H, K) := \frac{\mathbf{p}\mathbf{x}}{2H} + \frac{\beta^2}{\sqrt{2H}^3} \operatorname{argsinh} \left(\frac{\sqrt{2H}}{K} \mathbf{p}\mathbf{x} \right). \quad (38)$$

We verify easily that it holds

$$(\mathbf{k}^C(\mathbf{x}, \mathbf{p}))^2 = 1, \quad \mathbf{k}^C(\mathbf{x}, \mathbf{p})\mathbf{B}^C(\mathbf{x}, \mathbf{p}) = 0,$$

we thus observe that the map \mathcal{R}^C is indeed from M_n to P_n . Moreover, the map \mathcal{R}^C is evidently defined on the whole M_n and it is smooth everywhere on M_n .

Now consider a map $\mathcal{R}_i^C : P_n \rightarrow M_n$ defined by

$$\mathbf{x} = \frac{(K^2 + \beta^2 \sqrt{K^2 + 2HJ^2(T, H, K)})\mathbf{k} + 2HJ(T, H, K)\mathbf{B}}{2HK}, \quad (39)$$

$$\mathbf{p} = \frac{2H}{K} \left(\mathbf{B} + \frac{\beta^2(J(T, H, K)\mathbf{k} - \mathbf{B})}{\beta^2 + \sqrt{K^2 + 2HJ^2(T, H, K)}} \right), \quad (40)$$

where

$$K \equiv \sqrt{2H|\mathbf{B}|^2 + \beta^4}, \quad \tau^C(J(T, H, K), H, K) = T. \quad (41)$$

Note that the second equation of (41) is the definition of the function $J(T, H, K)$, that is $J(T, H, K)$ is the function inverse to $\tau^C(\mathbf{p}\mathbf{x}, H, K)$ viewed as the function of the first argument. The fact that this inverse function exists follows from taking a partial derivative of τ^C with respect to $\mathbf{p}\mathbf{x}$ for fixed $H > 0$ and $K \geq \beta^2$. Indeed, we find from (38)

$$\frac{\partial \tau^C(\mathbf{p}\mathbf{x}, H, K)}{\partial(\mathbf{p}\mathbf{x})} = \frac{1}{2H} + \frac{\beta^2}{2H\sqrt{K^2 + 2H(\mathbf{p}\mathbf{x})^2}} > 0.$$

Therefore, for H, K fixed, the function $\tau^C(\mathbf{p}\mathbf{x}, H, K)$ is increasing as the function of $\mathbf{p}\mathbf{x}$ and it admits the smooth inverse function $J(T, H, K)$.

The map \mathcal{R}_i^C is evidently well defined on the whole P_n and it is everywhere smooth. Moreover, we readily verify that

$$\mathcal{R}^C \circ \mathcal{R}_i^C = \operatorname{Id}_{P_n}, \quad \mathcal{R}_i^C \circ \mathcal{R}^C = \operatorname{Id}_{M_n},$$

which means that the both maps $\mathcal{R}^C, \mathcal{R}_i^C$ are diffeomorphisms inverse to each other.

A direct calculation of the bold-faced Poisson brackets finally gives

$$\{t^C, h^C\} = 1, \quad \{h^C, \mathbf{k}^C\} = \{t^C, \mathbf{k}^C\} = \{h^C, \mathbf{B}^C\} = \{t^C, \mathbf{B}^C\} = 0, \quad (42)$$

$$\{B_i^C, B_j^C\} = B_i^C k_j^C - B_j^C k_i^C, \{k_i^C, B_j^C\} = \delta_{ij} - k_i^C k_j^C, \{k_i^C, k_j^C\} = 0. \quad (43)$$

Comparing (42),(43) with (10),(11), we conclude that the diffeomorphism \mathcal{R}^C is in fact the symplectomorphism. Said in other words, we have just shown that the flat Coulomb system (M_n, Ω_n, h^C) is symplectomorphic to the referential dynamical system (P_n, ω_n, H) via the symplectomorphism \mathcal{R}^C , in particular, we have

$$H = h^C(\mathbf{x}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^2 + \frac{\beta^2}{|\mathbf{x}|}.$$

If in (39), (40) we interpret the variable T as time and $H, \mathbf{k}, \mathbf{B}$ as constant quantities, the inverse symplectomorphism (39), (40) can be checked to be the solution of the flat Coulomb equations of motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h^C\} = \mathbf{p}, \quad \dot{\mathbf{p}} = \{\mathbf{p}, h^C\} = \frac{\beta^2 \mathbf{x}}{|\mathbf{x}|^3}. \quad (44)$$

In reality, we have used this very fact to find the explicit form (35),(36),(37) of the symplectomorphism \mathcal{R}^C . We have first found the general solution (39), (40) of the flat Coulomb Hamiltonian equations of motions (44), we interpreted the time as the variable canonically conjugated to the Hamiltonian and then we expressed $H, T, \mathbf{k}, \mathbf{B}$ as the functions of \mathbf{p}, \mathbf{x} . It was of course not clear from the outset what kind of the bold-faced Poisson brackets would obey those functions, but it turned out eventually that they do obey those of the referential dynamical system (P_n, ω_n, H) . It is this circumstance which makes the flat Coulomb model propitious to admit point particle T-duals.

Remark 3.2. The reader might not have recognized in (39) and (40) the standard solution of the Coulomb (or Kepler) problem as we have intentionally avoided to employ the spherical coordinates. Indeed, any use of local coordinate systems like the spherical coordinates would obscure our task to find the global symplectomorphism relating the Coulomb model to the referential one.

3.4 Repulsive Coulomb potential in the hyperbolic space

In this section, we show that the repulsive Coulomb model in the space of constant negative curvature is symplectomorphic to the referential maximally super-integrable system of Section 2.

We consider a Hamiltonian of a charged point particle moving in the hyperbolic space background (23) and feeling the electric potential $V(\mathbf{x}) = \frac{\beta^2 \sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|}$:

$$h^y(\mathbf{x}, \mathbf{p}) = \frac{1}{2} (\mathbf{p}^2 + \alpha^2 (\mathbf{p}\mathbf{x})^2) + \frac{\beta^2 \sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} - \alpha \beta^2, \quad \alpha > 0. \quad (45)$$

The flows generated by (45) are complete due to essentially the same argument as in the previous sections.

Note that the Hamiltonian h^y is well defined in any number of dimensions, but in the dimension $n = 3$ it has a natural physical interpretation as the repulsive Coulomb potential in the space of negative constant curvature. Indeed, in three dimensions the Laplace-Beltrami operator in the background (23) acts on the potential $V(\mathbf{x}) = \frac{\beta^2 \sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} - \alpha\beta^2$ with the result

$$\Delta_{LB} \frac{\sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} \equiv \frac{1}{\sqrt{\det g}} \partial_{x_j} \left(\sqrt{\det g} g^{jk} \partial_{x_j} \frac{\sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} \right) = -4\pi\delta(\mathbf{x}).$$

It is well-known that the hyperbolic Coulomb system is superintegrable in three dimensions [6, 4, 3, 14, 2], our goal is to show a little bit more than this, that is to show that its n -dimensional version (M_n, Ω_n, h^y) is symplectomorphic to the referential dynamical system (P_n, ω_n, H) . For that, we consider a map $\mathcal{R}^y : M_n \rightarrow P_n$ given by

$$H = h^y(\mathbf{p}, \mathbf{x}) = \frac{1}{2} (\mathbf{p}^2 + \alpha^2 (\mathbf{p}\mathbf{x})^2) + \frac{\beta^2 \sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} - \alpha\beta^2, \quad (46)$$

$$T = t^y(\mathbf{x}, \mathbf{p}) := \tau^y(\mathbf{p}\mathbf{x}, h^y(\mathbf{x}, \mathbf{p}), |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|), \quad (47)$$

$$\mathbf{k} = \mathbf{k}^y(\mathbf{x}, \mathbf{p}) := \frac{\beta^2 \mathbf{k}^r(\mathbf{x}, \mathbf{p})}{K^y(\mathbf{x}, \mathbf{p})} + \sqrt{1 + \alpha^2 \mathbf{x}^2} \frac{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 \mathbf{k}^r(\mathbf{x}) - (\mathbf{p}\mathbf{x}) \mathbf{B}^r(\mathbf{x}, \mathbf{p})}{K^y(\mathbf{x}, \mathbf{p}) |\mathbf{x}|}, \quad (48)$$

$$\mathbf{B} = \mathbf{B}^y(\mathbf{x}, \mathbf{p}) := \frac{\beta^2 \mathbf{B}^r(\mathbf{x}, \mathbf{p})}{K^y(\mathbf{x}, \mathbf{p})} + |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 \sqrt{1 + \alpha^2 \mathbf{x}^2} \frac{\mathbf{B}^r(\mathbf{x}, \mathbf{p}) + (\mathbf{p}\mathbf{x}) \mathbf{k}^r(\mathbf{x})}{K^y(\mathbf{x}, \mathbf{p}) |\mathbf{x}|} \quad (49)$$

where $\mathbf{k}^r(\mathbf{x})$, $\mathbf{B}^r(\mathbf{x}, \mathbf{p})$ were defined in (12) and

$$K^y(\mathbf{x}, \mathbf{p}) := \sqrt{2h^y(\mathbf{x}, \mathbf{p}) |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + (\beta^2 + \alpha |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2)^2},$$

$$\tau^y(\mathbf{p}\mathbf{x}, H, B) :=$$

$$= \frac{\operatorname{Argtanh} \left(\frac{A^\alpha(B, K)(K - \beta^2) \mathbf{p}\mathbf{x}}{BK + \sqrt{B^2 K^2 + (\mathbf{p}\mathbf{x})^2 (K^2 - \beta^4)}} \right)}{\alpha \sqrt{2H}} - \frac{\operatorname{Argtanh} \left(\frac{A^{-\alpha}(B, K)(K - \beta^2) \mathbf{p}\mathbf{x}}{BK + \sqrt{B^2 K^2 + (\mathbf{p}\mathbf{x})^2 (K^2 - \beta^4)}} \right)}{\alpha \sqrt{2H + 4\alpha\beta^2}} \quad (50)$$

$$A^{\pm\alpha}(B, K) = \sqrt{\frac{K + (\beta^2 \pm \alpha B^2)}{K - (\beta^2 \pm \alpha B^2)}}. \quad (51)$$

We verify that it holds

$$(\mathbf{k}^y(\mathbf{x}, \mathbf{p}))^2 = 1, \quad \mathbf{k}^y(\mathbf{x}, \mathbf{p}) \mathbf{B}^y(\mathbf{x}, \mathbf{p}) = 0,$$

we thus observe that the map \mathcal{R}^y is indeed from M_n to P_n . Moreover, the map \mathcal{R}^y is defined on the whole M_n (see Appendix) and it is smooth everywhere on M_n .

Now consider a map $\mathcal{R}_i^y : P_n \rightarrow M_n$ defined by

$$\mathbf{x} = \frac{|\mathbf{B}|^2 \mathbf{k} \cos \Psi^y(T) + |\mathbf{B}| \mathbf{B} \sin \Psi^y(T)}{\sqrt{(K \cos \Psi^y(T) - \beta^2)^2 - \alpha^2 |\mathbf{B}|^4}}, \quad (52)$$

$$\begin{aligned} \mathbf{p} &= \frac{\sqrt{(K \cos \Psi^y(T) - \beta^2)^2 - \alpha^2 |\mathbf{B}|^4}}{|\mathbf{B}|^2} \times \\ &\times \left((\mathbf{B} + Y(T) \mathbf{k}) \cos \Psi^y(T) + (Y(T) \frac{\mathbf{B}}{|\mathbf{B}|} - |\mathbf{B}| \mathbf{k}) \sin \Psi^y(T) \right), \end{aligned} \quad (53)$$

where

$$\begin{aligned} K &= \sqrt{2H|\mathbf{B}|^2 + (\alpha|\mathbf{B}|^2 + \beta^2)^2}, \\ \sin \Psi^y(T) &:= \frac{(K - \beta^4 K^{-1})Y(T, H, |\mathbf{B}|)}{\sqrt{K^2|\mathbf{B}|^2 + (K^2 - \beta^4)Y^2(T, H, |\mathbf{B}|) + |\mathbf{B}|\beta^2}}, \\ \cos \Psi^y(T) &:= \frac{\beta^2}{K} + \frac{(K - \beta^4 K^{-1})|\mathbf{B}|}{\sqrt{K^2|\mathbf{B}|^2 + (K^2 - \beta^4)Y^2(T, H, |\mathbf{B}|) + |\mathbf{B}|\beta^2}}. \end{aligned}$$

Here the function $Y(T, H, B)$ is inverse to $\tau^y(\mathbf{p}\mathbf{x}, H, B)$ viewed as the function of the first argument (with H, B fixed). The proof that this inverse function exists is presented in the Appendix.

The map \mathcal{R}_i^y is well defined on the whole P_n and it is everywhere smooth. Moreover, we readily verify that

$$\mathcal{R}^y \circ \mathcal{R}_i^y = \text{Id}_{P_n}, \quad \mathcal{R}_i^y \circ \mathcal{R}^y = \text{Id}_{M_n},$$

which means that the both maps $\mathcal{R}^y, \mathcal{R}_i^y$ are diffeomorphisms inverse to each other.

A direct (and tedious) calculation of the bold-faced Poisson brackets finally gives

$$\{t^y, h^y\} = 1, \quad \{h^y, \mathbf{k}^y\} = \{t^y, \mathbf{k}^y\} = \{h^y, \mathbf{B}^y\} = \{t^y, \mathbf{B}^y\} = \mathbf{0}, \quad (54)$$

$$\{B_i^y, B_j^y\} = B_i^y k_j^y - B_j^y k_i^y, \quad \{k_i^y, B_j^y\} = \delta_{ij} - k_i^y k_j^y, \quad \{k_i^y, k_j^y\} = 0. \quad (55)$$

Comparing (54),(55) with (10),(11), we conclude that the diffeomorphism \mathcal{R}^y is in fact the symplectomorphism. Said in other words, we have just shown that the hyperbolic Coulomb system (M_n, Ω_n, h^y) is symplectomorphic to the referential dynamical system (P_n, ω_n, H) via the symplectomorphism \mathcal{R}^y , in particular, we have

$$H = h^y(\mathbf{x}, \mathbf{p}) = \frac{1}{2} (\mathbf{p}^2 + \alpha^2 (\mathbf{p}\mathbf{x})^2) + \frac{\beta^2 \sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} - \alpha \beta^2.$$

If we interpret the variable T in (52) and (53) as time and $H, \mathbf{k}, \mathbf{B}$ as constant quantities, the inverse symplectomorphism (52) and (53) can be checked to be the solution of the hyperbolic Coulomb Hamiltonian equations of motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h^y\} = \mathbf{p} + \alpha^2(\mathbf{p}\mathbf{x})\mathbf{x}, \quad (56a)$$

$$\dot{\mathbf{p}} = \{\mathbf{p}, h^y\} = \frac{\beta^2\mathbf{x}}{|\mathbf{x}|^3\sqrt{1 + \alpha^2\mathbf{x}^2}} - \alpha^2(\mathbf{p}\mathbf{x})\mathbf{p}. \quad (56b)$$

In reality, we have used this very fact to find the explicit form (46),(47), (48),(49) of the symplectomorphism \mathcal{R}^y . We have first found the general solution (52), (53) of the hyperbolic Coulomb Hamiltonian equations of motions (56), we interpreted the time as the variable canonically conjugated to the Hamiltonian and then we expressed $H, T, \mathbf{k}, \mathbf{B}$ as the functions of \mathbf{p}, \mathbf{x} . It was of course not clear from the outset what kind of the bold-faced Poisson brackets would obey those functions, but it turned out eventually that they do obey those of the referential dynamical system (P_n, ω_n, H) . It is this circumstance which makes the hyperbolic Coulomb model propitious to admit point particle T-duals.

4 Discussion, conclusions and outlook

In the preceding Section 3, we have shown that four superintegrable dynamical systems, i.e. flat and hyperbolic Calogero-Moser and flat and hyperbolic Coulomb, are all symplectomorphic to the referential dynamical system (P_n, Ω_n, H) introduced in Section 2. We have constructed explicitly the respective symplectomorphisms $\mathcal{R}^M, \mathcal{R}^{yM}, \mathcal{R}^C, \mathcal{R}^y : M_n \rightarrow P_n$ as well as the inverse symplectomorphisms $\mathcal{R}_i^M, \mathcal{R}_i^{yM}, \mathcal{R}_i^C, \mathcal{R}_i^y : P_n \rightarrow M_n$. The T-duality symplectomorphisms relating those four models are given by the compositions of one original and one inverse symplectomorphism. For example, the hyperbolic Calogero-Moser is related to the flat Coulomb by the composed T-duality symplectomorphism $\mathcal{R}_i^{yM} \circ \mathcal{R}^C$. The explicit formulas for those composed T-duality symplectomorphisms can be worked out straightforwardly but we do not list them because they are cumbersome and, anyway, not very illuminating.

Do some additional 0+1-dimensional σ -models belong to the T-duality equivalence class consisting of the four dynamical systems that we have studied in detail? Very probably yes due to the result of Fradkin [7] which pioneered the quest for superintegrability for general spherically symmetric potentials. However, showing that a given spherically symmetric model is superintegrable is not sufficient, because, as it was already remarked in Footnote 1, not all spherically symmetric superintegrable models must be necessarily symplectomorphic to our referential model (P_n, Ω_n, H) . In particular, the phase spaces of some spherically symmetric superintegrable models may not be symplectomorphic to our referential phase space P_n but they may be rather symplectomorphic to a \mathbb{Z} -quotient of P_n (in this case the symplectic half-plane H, T becomes a symplectic half-cylinder with T becoming an angle variable). Other scenarios are also possible and we expect

that new T-duality equivalence classes can be discovered by following the track of superintegrability, including cases where we abandon the requirement of the spherical symmetry.

A big open issue is a quantum status of the point particle T-duality. Although the problem looks much easier than in the string case, still the high degree of nonlinearity of the explicit symplectomorphisms obtained in Section 3 suggests, that it will not be an effortless task to settle it.

References

- [1] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, “A Canonical approach to duality transformations,” *Phys. Lett. B* **336** (1994), 183-189, [hep-th/9406206],
- [2] A. Ballesteros, A. Blasco and F. J. Herranz, “Curvature as an integrable deformation,” [arXiv:1903.09543 [math-ph]],
- [3] A. Ballesteros, A. Enciso, F. J. Herranz and O. Ragnisco, “Hamiltonian systems admitting a Runge-Lenz vector and an optimal extension of Bertrand’s theorem to curved manifolds”, *Commun. Math. Phys.* **290** (2009), 1033-1049, [arXiv:0810.0999 [math-ph]],
- [4] A. V. Borisov, I. S. Mamaev, ”Superintegrable systems on a sphere”, *Regul. Chaotic Dyn.* **10** (3) (2005), 257–266, [nlin/0504018],
- [5] X. C. de la Ossa and F. Quevedo, ”Duality symmetries from non-Abelian isometries in string theory”, *Nucl. Phys. B* **403** (1993) 377-394, [hep-th/9210021],
- [6] N. W. Evans, ”Superintegrability in classical mechanics”, *Phys. Rev. A* **41** (1990), 5666,
- [7] D. M. Fradkin, ”Existence of the dynamic symmetries O_4 and SU_3 for all classical central potential problems”, *Prog. Theor. Phys.* **37** (1967), 798–812,
- [8] K. Kikkawa and M. Yamasaki, ”Casimir Effects in Superstring Theories”, *Phys. Lett. B* **149** (1984) 357-360,
- [9] C. Klimčík, “T-duality and T-folds for point particles,” *Phys. Lett. B* **812** (2021), 136009 [arXiv:2010.07571 [hep-th]],
- [10] C. Klimčík and P. Ševera, ”Dual non-Abelian duality and the Drinfeld double”, *Phys. Lett. B* **351** (1995) 455-462, hep-th/9502122; C. Klimčík, ”Poisson-Lie T-duality”, *Nucl. Phys. (Proc. Suppl.) B* **46** (1996) 116-121, hep-th/9509095; P. Ševera, ”Minimálne plochy a dualita”, Diploma thesis, Prague University, 1995, in Slovak,

- [11] C. Klimčík and P. Ševera, "Dressing cosets", Phys. Lett. B **381** (1996) 56-61, [hep-th/9602162],
- [12] C. Klimčík and P. Ševera, "Non-Abelian momentum-winding exchange", Phys.Lett. B **383** (1996) 281-286, [hep-th/9605212],
- [13] R. C. O'Connell and K. Jagannathan, "Illustrating Dynamical Symmetries in Classical Mechanics: The Laplace-Runge-Lenz Vector Revisited", American Journal of Physics **71** (2003), 243, [physics/0212043],
- [14] C. Quesne, "Quantum oscillator and Kepler–Coulomb problems in curved spaces: Deformed shape invariance, point canonical transformations, and rational extensions," J. Math. Phys. **57** (2016) no.10, 102101, [arXiv:1604.04136 [math-ph]],
- [15] N. Sakai and I. Senda, "Vacuum Energies of String Compactified on Torus", Prog. Theor. Phys. **75** (1986) 692 [erratum: Prog. Theor. Phys. **77** (1987) 773],
- [16] K. Sfetsos, "Canonical equivalence of nonisometric sigma models and Poisson-Lie T duality," Nucl. Phys. B **517** (1998), 549-566, [arXiv:hep-th/9710163 [hep-th]].

Appendix

First we show that the function (47) is defined everywhere on M_n . For that we rewrite it as

$$\tau^y(\mathbf{x}, \mathbf{p}) = \frac{\text{Argtanh}\left(\frac{\tan\frac{\Psi^y(\mathbf{x}, \mathbf{p})}{2}}{\tan\frac{\Phi^+(\mathbf{x}, \mathbf{p})}{2}}\right)}{\alpha\sqrt{2h^y(\mathbf{x}, \mathbf{p})}} - \frac{\text{Argtanh}\left(\frac{\tan\frac{\Psi^y(\mathbf{x}, \mathbf{p})}{2}}{\tan\frac{\Phi^-(\mathbf{x}, \mathbf{p})}{2}}\right)}{\alpha\sqrt{2h^y(\mathbf{x}, \mathbf{p}) + 4\alpha\beta^2}},$$

where

$$\Psi^y(\mathbf{x}, \mathbf{p}) = \arctan \frac{\sqrt{1 + \alpha^2 \mathbf{x}^2} |\mathbf{B}^r(\mathbf{x}, \mathbf{p})| (\mathbf{p}\mathbf{x})}{\sqrt{1 + \alpha^2 \mathbf{x}^2} |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \beta^2 |\mathbf{x}|}, \quad (57)$$

$$\tan \frac{\Phi^\pm(\mathbf{x}, \mathbf{p})}{2} = \sqrt{\frac{K^y(\mathbf{x}, \mathbf{p}) - (\beta^2 \pm \alpha |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2)}{K^y(\mathbf{x}, \mathbf{p}) + (\beta^2 \pm \alpha |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2)}}. \quad (58)$$

We show without difficulties that $0 < \Phi^+(\mathbf{x}, \mathbf{p}) < \Phi^-(\mathbf{x}, \mathbf{p})$ for all $(\mathbf{x}, \mathbf{p}) \in M_n$, therefore the domain of definition of $t^y(\mathbf{x}, \mathbf{p})$ is given by all $(\mathbf{x}, \mathbf{p}) \in M_n$ which satisfy

$$\left| \tan \frac{\Psi^y(\mathbf{x}, \mathbf{p})}{2} \right| < \tan \frac{\Phi^+(\mathbf{x}, \mathbf{p})}{2} (< 1). \quad (59)$$

Following (57), the image of the map Ψ^y belongs to the interval $]-\frac{1}{2}\pi, \frac{1}{2}\pi[$, whatever $(\mathbf{x}, \mathbf{p}) \in M_n$ we consider. We find also

$$\cos \Psi^y(\mathbf{x}, \mathbf{p}) > \cos \Phi^+(\mathbf{x}, \mathbf{p}), \quad \forall (\mathbf{x}, \mathbf{p}) \in M_n. \quad (60)$$

Indeed, it follows from (57) and (58)

$$\cos \Psi^y(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 \sqrt{1 + \alpha^2 \mathbf{x}^2} + \beta^2 |\mathbf{x}|}{K^y(\mathbf{x}, \mathbf{p}) |\mathbf{x}|}, \quad \cos \Phi^+(\mathbf{x}, \mathbf{p}) = \frac{\alpha |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 + \beta^2}{K^y(\mathbf{x}, \mathbf{p})},$$

therefore

$$\begin{aligned} K^y(\mathbf{x}, \mathbf{p}) \cos \Psi^y(\mathbf{x}, \mathbf{p}) - \beta^2 - \alpha |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 &= |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2 \left(\frac{\sqrt{1 + \alpha^2 \mathbf{x}^2}}{|\mathbf{x}|} - \alpha \right) > \\ &> 0 = K^y(\mathbf{x}, \mathbf{p}) \cos \Phi^+(\mathbf{x}, \mathbf{p}) - \beta^2 - \alpha |\mathbf{B}^r(\mathbf{x}, \mathbf{p})|^2. \end{aligned} \quad (61)$$

Finally, the inequality (61) implies (60), which in turn implies that the relation (59) holds for all $(\mathbf{x}, \mathbf{p}) \in M_n$.

We wish also to show that, for H, B fixed, the function $t^y(\mathbf{p}\mathbf{x}, H, B)$ is invertible as the function of $(\mathbf{p}\mathbf{x})$, which means that the partial derivative $\frac{\partial t^y}{\partial(\mathbf{p}\mathbf{x})}$ must be positive. Set

$$z = \frac{(K - \beta^2)\mathbf{p}\mathbf{x}}{BK + \sqrt{B^2 K^2 + (\mathbf{p}\mathbf{x})^2 (K^2 - \beta^4)}}$$

and find that it holds

$$\frac{\partial z}{\partial(\mathbf{p}\mathbf{x})} = \frac{(K - \beta^2)BK}{(BK + \sqrt{B^2 K^2 + (\mathbf{p}\mathbf{x})^2 (K^2 - \beta^4)}) \sqrt{B^2 K^2 + (\mathbf{p}\mathbf{x})^2 (K^2 - \beta^4)}} > 0.$$

Looking at (50), we therefore see that we have just to show

$$\frac{\partial \tau^y}{\partial z} > 0, \quad (62)$$

where (cf. also (51))

$$\tau^y(z, H, B) := \frac{\text{Argtanh}(A^\alpha z)}{\alpha \sqrt{2H}} - \frac{\text{Argtanh}(A^{-\alpha} z)}{\alpha \sqrt{2H + 4\alpha\beta^2}}.$$

Since $A^\alpha > A^{-\alpha}$, we find

$$\begin{aligned} \frac{\partial \tau^y}{\partial z} &= \frac{A^\alpha}{\alpha \sqrt{2H}(1 - (A^\alpha)^2 z^2)} - \frac{A^{-\alpha}}{\alpha \sqrt{2H + 4\alpha\beta^2}(1 - (A^{-\alpha})^2 z^2)} > \\ &> \frac{A^\alpha \sqrt{2H + 4\alpha\beta^2} - A^{-\alpha} \sqrt{2H}}{\alpha B(1 - (A^{-\alpha})^2 z^2) \sqrt{2H + 4\alpha\beta^2} \sqrt{2H}} = \\ &= \frac{A^\alpha \sqrt{K^2 - (\beta^2 - \alpha B^2)^2} - A^{-\alpha} \sqrt{K^2 - (\beta^2 + \alpha B^2)^2}}{\alpha B(1 - (A^{-\alpha})^2 z^2) \sqrt{2H + 4\alpha\beta^2} \sqrt{2H}} > 0. \end{aligned} \quad (63)$$

Proving the last inequality in (63) therefore boils down to proving the following inequality

$$\sqrt{\frac{K + (\beta^2 + \alpha B^2)}{K - (\beta^2 + \alpha B^2)}} \sqrt{K^2 - (\beta^2 - \alpha B^2)^2} > \sqrt{\frac{K + (\beta^2 - \alpha B^2)}{K - (\beta^2 - \alpha B^2)}} \sqrt{K^2 - (\beta^2 + \alpha B^2)^2},$$

which is equivalent to the evident inequality

$$\frac{1}{K - (\beta^2 + \alpha B^2)} > \frac{1}{K - (\beta^2 - \alpha B^2)}. \quad (64)$$