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A multidimensional spatial model for preference representation in multi-criteria group decision aiding

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Abstract

In this work we propose to use a multidimensional spatial model to represent preferences of a group of decision makers in multi-criteria decision aiding. The decision makers are represented in a shared space with the alternatives so that their positions are consistent with the preferences that they express on pairs of alternatives. We show how the parameters of this preference model can be learnt from holistic preference judgements, and discuss its various consequences and properties.

Keywords: multi-criteria decision aiding, spatial model, preference elicitation, non-monotonic preferences

MSC Classification: 90B50 , 91B06 , 91B08 , 91B10

1 Introduction

1.1 Context and motivations

In the spatial theory of voting, voters and candidates are represented as points in a space defined by a set of attributes or political issues. The position of the voters and the candidates is defined by the way in which they espouse these issues. This space could be single dimensional (left, right), double dimensional

(social and economic policy) or could have many more dimensions. The main purpose of the spatial voting model is to describe and analyse the voters' behaviours and to estimate the outcome of an election using the distance between the voters and the candidates [1]. But other tasks are made possible, such as classifying voters or candidates according to similar characteristics, understanding the preferences that certain voters have for certain candidates, and how candidates should adapt their strategy to meet more voters. Spatial representation of voting situation is an old topic in the field of social choice theory. Since the seminal and early works of Hotelling [2], Black [3] and Downs [1], many developments have been proposed to model voting situations within the spatial model framework. One can refer to Enelow and Hinich [4] or Miller [5] for a good introduction to spatial voting models, to Eguia [6, 7] for a theoretical approach and to Armstrong *et al.* [8] or Negriu and Piatecki [9] for recent applicative approaches.

The objective of multi-criteria decision aiding [10] is to provide a recommendation to a decision maker, confronted with a set of alternatives, described through a set of potentially conflicting criteria, by taking into account his or her preferences. This recommendation usually takes one of three forms: the choice of an alternative considered satisfactory, a sorting of the alternatives into categories, or a ranking of the alternatives from best to worst. Next to that, in many real-world decision problems it may be interesting to understand the behavior of the decision maker when faced with these alternatives or to compare the behavior of several decision makers when faced with these alternatives.

These observations lead us to draw a parallel between multi-criteria decision and spatial voting theory. The decision maker plays the role of the voter, the alternatives are the candidates, and the criteria can be seen as attributes or political issues describing the voters and the candidates.

Therefore, in this work we propose to use an approach similar to spatial voting theory in multi-criteria decision aiding. We represent the decision maker in the criteria space so that his or her position is consistent with the preferences that s/he expresses on pairs of alternatives. The idea is as follows: if s/he says that s/he prefers alternative a to alternative b , then the point representing the decision maker in the criteria space must be closer to a than to b . Similarly, if s/he also says that s/he prefers c to d , then the point representing the decision maker must also be closer to c than to d , while being closer to a than to b . An example of such a spatial representation of multi-criteria preferences is given in Figure 1 for 2 decision makers.

The idea of positioning the decision maker in the same multidimensional space as the alternatives can be used for different purposes. For example, similarly as in the spatial theory of voting, this paradigm allows us to explain to a decision maker in which part of the criteria space s/he is located, and consequently, which alternatives can be the best for him or her. Moreover, it will allow us to position different decision makers in this space in order to evaluate their proximity, or to determine on which alternatives their preferences

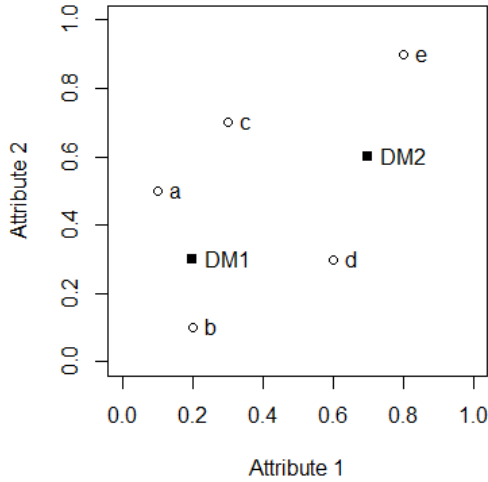


Fig. 1: Example of a spatial representation of 5 alternatives described on 2 attributes and 2 decision makers (DM1 and DM2). Preferences are $a \succ_1 b \succ_1 d \succ_1 c \succ_1 e$ and $d \sim_2 e \succ_2 c \succ_2 a \succ_2 b$.

are similar or dissimilar to facilitate group decision making. And last but not least, the preferences on the criteria do not have to be monotonically increasing or decreasing, as usually supposed in multi-criteria decision aiding, but simply single-peaked. This single-peakedness property of the preferences allows to deal with a larger range of decision problems. Most of all do not require a re-encoding of the underlying criteria scales, which in turn tends to degrade the readability of the results for the decision maker.

1.2 Related work

Just a few related contributions have been identified concerning multi-criteria decision aiding dealing with spatial representations of both alternatives and decision makers. Some works propose to represent the decision maker as a vector of parameters of a particular preference model in the parameter space. For example, Mareschal et al. [11] have introduced a geometrical representation of the decision problem when the preferences are modelled through PROMETHEE [12]. This work has led to various recent improvements and variations [13–15], however it is linked to a very specific preference model, and requires the determination of preference parameters. Pajer et al. [16] propose to perform a visual analysis of weight spaces when the aggregation is performed through a weighted sum. The proposed technique allows to better understand how robust a decision is to plausible weights variations. Again, this work is

situated in the parameters space, and does not allow for any analysis of the alternatives in the criteria space.

Still in multi-criteria decision aiding, several methods make use of an ideal alternative (which could be seen as a representation of the decision maker in the criteria space) and a notion of distance. The TOPSIS method [17–19] evaluates the preferences between the alternatives and their ranking using a distance to an ideal alternative. Chen et al. [20, 21] propose a sorting model using a case-based distance procedure using the weighted Euclidean distance in order to relax the assumption of preference monotonicity. These distance-based approaches are the closest to the model that we develop here, but they are applied for a specific decision problem (ranking or sorting). They suppose that the ideal alternative is known, which is not the case of our work.

As we will see, another advantage of our proposal is that it does not require that the preference directions of the criteria be specified by the decision maker, nor that they be assumed to be monotonic. In multi-criteria decision aiding, several preference models require to identify whether the criteria are to be maximized or to be minimized. Usually it is also supposed that the preferences on the criteria are monotonic. Different contributions propose to handle the case of non-monotonic preferences on criteria. For the Multi-Attribute Value Theory (MAVT) approach, Despotis et al. [22], Kliegr [23], Liu et al. [24], Ghaderi et al. [25], Kadziński et al. [26] or Guo et al. [27] propose various models and elicitation techniques for models with non-monotonic or single-peaked criteria. For the outranking approach, these non-monotonic assumptions are more rarely encountered, except in the work of Minoungou et al. [28, 29].

The works mentioned above are specific to a single decision-maker. The questions of finding a compromise for a group of decision makers and to build a common preference model are not studied in these articles.

In social choice theory, and more specifically the spatial theory of voting, these non-monotonicity and single peaked criteria issues have also been dealt with. We can cite for example the work of Conitzer [30] who deals with eliciting single-peaked preferences using comparison queries. More recently, Escoffier et al. [31] undertake a comparison of popular distance measures for single-peakedness.

1.3 Contribution

In this work, we propose to represent the preferences of decision makers with respect to alternatives through the distance of these decision makers to these alternatives in the attribute space. This model is user-oriented and preferences are fully explainable directly from the available data.

We also propose several algorithms to identify the parameters of the model under various hypotheses. The consequences of this proposal are also discussed, first concerning the ability to provide a position of the decision makers in the attributes space; and second concerning the comparison of several decision makers to each other and to make relevant group decision recommendations.

We also show that the model allows to deal with single-peaked preferences, and we apply the proposal to a real case study.

This article is structured as follows. First, in Section 2 we introduce the spatial decision model and study some of its properties and consequences, especially regarding group decision making. Section 3 details how the parameters of the model can be determined through a holistic approach, where the decision makers provide preferential statements regarding pairs of alternatives. In Section 4 we detail a real-world case study which has been solved using the proposed model, before drawing some conclusions and mentioning some perspectives in Section 5.

2 Multidimensional spatial model

2.1 Spatial decision model

Let us consider a set of alternatives denoted \mathcal{A} . Each alternative $a \in \mathcal{A}$ is evaluated on a set of m attributes denoted $\mathcal{M} = \{1, \dots, m\}$. Let a_j be the quantitative evaluation of a on attribute $j \in \mathcal{M}$, with $a_j \in \mathbb{R}$. a can be identified with its performance vector, i.e., $a \equiv (a_1, \dots, a_m)$. As a consequence, an alternative $a \in \mathcal{A}$ can be represented by a point in a multidimensional space $\mathbf{E} = \mathbb{R}^m$, in which each dimension corresponds to one attribute of \mathcal{M} . We do not suppose in this work any preference direction given by the decision maker on the attribute scales. Following Roy [10], we therefore do not use the word “criteria” here, and rather speak of “attributes”.

The proposed decision model is based on the relative positions of the decision makers and the alternatives in the same multidimensional space. Let $\mathcal{X} = \{1, \dots, n\}$ be the set of n decision makers.

Let $\succsim_i \subseteq \mathcal{A} \times \mathcal{A}$ be the preference relation of i th decision maker $i \in \mathcal{X}$ on the alternatives of \mathcal{A} . The symmetric part of \succsim_i , which models indifferences, is denoted \sim_i , whereas the asymmetric part is denoted \succ_i and models strict preferences.

We define the spatial preference model (SP-model) as follows:

Definition 1. (*Spatial preference model*) Let $i \in \mathcal{X}$. A preference relation \succsim_i on \mathcal{A} follows a spatial preference model if and only if

- there exists a representation $X^i \equiv (x_1^i, \dots, x_m^i)$ of i in \mathbf{E}
- there exists a distance δ on $\mathbf{E} \times \mathbf{E}$ a real function of two variables such that $\forall (a, b) \in \mathcal{A} \times \mathcal{A}$:

$$\begin{cases} a \succ_i b \iff \delta(a, X^i) < \delta(b, X^i) \\ a \sim_i b \iff \delta(a, X^i) = \delta(b, X^i). \end{cases}$$

2.2 Properties

The spatial preference model of Definition 1 is obviously a utility-based model, and it therefore models total pre-orders on the set of alternatives. The hypothesis of single-peakedness on each attribute is the second characteristic of spatial preference models. We develop these properties in the following.

Utility-based models are described as follows, for a given decision maker of \mathcal{X} . To ease the notation, we do not index or superscript each of the elements by $i \in \mathcal{X}$ in Properties 1 and 2 and Proposition 1.

Property 1. *The preference relation \succsim follows an utility-based model if there exists a function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ and m function u_1, \dots, u_m from \mathbb{R} to \mathbb{R} such that $\forall a, b \in \mathcal{A}$,*

$$a \succsim b \iff F(u_1(a_1), \dots, u_m(a_m)) \geq F(u_1(b_1), \dots, u_m(b_m))$$

Moreover the preference relation \succsim is said to follow an additive utility-based model iff

$$a \succsim b \iff \sum_{j \in \mathcal{M}} u_j(a_j) \geq \sum_{j \in \mathcal{M}} u_j(b_j)$$

A complete characterization of preference relations that can be represented by a spatial model using a Minkowski metric as distance has been proposed in [7] in the framework of political issues. This characterization is based on both utility functions and single-peakedness property. Following [7], and denoting (a_j, b_{-j}) the alternative that takes the value a_j on attribute j and b_i on attributes $i \in \mathcal{M}, i \neq j$, we define multi-attribute single-peakedness as follows.

Property 2. *(Multi-attribute single-peakedness) There exists an ideal alternative $x^* = (x_1^*, \dots, x_m^*) \in \mathbf{E}$ such that for each attribute $j \in \mathcal{M}$ and any values $a_k, b_k, c_k, d_k \in \mathbb{R}$,*

$$a_k < b_k < x_k^* < c_k < d_k \Rightarrow (b_k, x_{-k}^*) \succ (a_k, x_{-k}^*) \text{ and } (c_k, x_{-k}^*) \succ (d_k, x_{-k}^*)$$

Eguia [7] therefore proposes a characterization theorem, which we adapt in our framework with the following proposition:

Proposition 1. *For any positive real number α , there exists a spatial representation $f^\alpha = (f_1^\alpha, \dots, f_m^\alpha)$ such that the utility function $u(a)_x = -\sum_{j=1}^m |f_j^\alpha(a_j) - f_j^\alpha(x_j)|^\alpha$ represents the preference order \succsim if and only if \succsim follows an additive utility model satisfying the multiple single-peakedness properties.*

Note that the utility function $u(a)_x$ is negative to ensure that the closer a is to x (and the smaller the absolute value of $u(a)_x$), the most a is preferred by the decision maker. Proposition 1 is not so informative as, emphasized by

Nehring and Puppe in [32], *obviously, every given preference ordering is single-peaked with respect to some appropriate betweenness relation*. Therefore, using both utility functions and a specific α distance, Proposition 1 leads to over-parametric models facing an observed preference situation. We propose in our proposal to restrict the number of parameters in two different ways. First, we consider that the model should be user-oriented, and fully explainable directly from the available data. Therefore, we do not encourage the use of utility functions as all the attributes are described with values on \mathbb{R} . The issue of using different scales for the different attributes can be fixed by two other means: standardization of the attributes (for example using the same 0-1 or 0-100 scale for each attribute) and / or a weighted distance, which is developed below.

Proposition 1 proposes a large (infinite) possibility of distances to model spatial preferences. Keeping in mind the aim that the model should be easily understandable by the decision-makers, we focus on the classical Euclidean distance, i.e. $\alpha = 2$.

However, with a fixed distance and without any utility functions, the model could sometimes be under-parameterized. Therefore, in order to capture the importance given by different decision makers to the same attributes, we propose to use an extension of the Euclidean distance, which weighs the various attributes differently, and gives further flexibility to the model. The weighted Euclidean distance is defined as follows for an alternative $a \in \mathcal{A}$ and $i \in \mathcal{X}$:

$$\delta(a, X^i) = \sqrt{\sum_{j \in \mathcal{M}} w_j^i \cdot (x_j^i - a_j)^2}, \quad (1)$$

where $w_j^i \geq 0$ and $\sum_{j=1}^m w_j^i = 1$.

We use this weighted Euclidean distance in the model of Definition 1 to specify our spatial preference model.

Definition 2. (*Spatial weighted Euclidean preference model*)

Let $i \in \mathcal{X}$. A preference relation \succsim_i on \mathcal{A} follows a spatial weighted Euclidean preference model if and only if

- there exist a representation $X^i \equiv (x_1^i, \dots, x_m^i)$ of i in \mathbf{E}
- a set of weights $w_j^i \geq 0$, $j = 1, \dots, m$ and $\sum_{j=1}^m w_j^i = 1$

such that $\forall (a, b) \in \mathcal{A} \times \mathcal{A}$:

$$a \succsim_i b \iff \sqrt{\sum_{j \in \mathcal{M}} w_j^i \cdot (x_j^i - a_j)^2} \leq \sqrt{\sum_{j \in \mathcal{M}} w_j^i \cdot (x_j^i - b_j)^2}$$

The advantage of this weighted Euclidean distance compared to the classical Euclidean distance, is that it leads to a larger flexibility of the model, by allowing for differences on the attributes to be more or less important in the distance calculation for each decision maker. We therefore keep the main interest of the SP-model, that is to capture in a single explainable model preferences that are different for different decision makers.

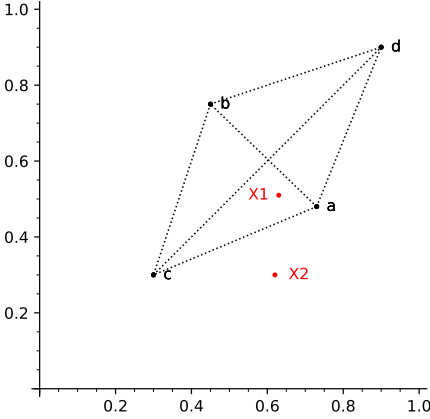


Fig. 2: Two decision makers positioned in $X^1 \equiv (0.63, 0.51)$ and $X^2 \equiv (0.63, 0.30)$.

| | | \mathcal{M} | |
|---------------|-----|---------------|------|
| | | 1 | 2 |
| \mathcal{A} | a | 0.73 | 0.48 |
| | b | 0.45 | 0.75 |
| | c | 0.30 | 0.30 |
| | d | 0.90 | 0.90 |

Table 1: 4 alternatives on two quantitative attributes

2.3 Use cases of the model

Let us now discuss a few intuitive consequences of the preference model of Definition 2. Remember that the parameters of a spatial weighted Euclidean preference model are 1) the positions of each of the decision makers X^i in the space \mathbf{E} , 2) the weights $w_j^i, \forall j \in \mathcal{M}$, for each decision maker.

Influence of the weights

For a given position of a decision maker, different weights in the weighted Euclidean distance can lead to different preference orders of the alternatives.

To illustrate this, consider the two-dimensional example of Figure 2. A decision maker is represented as point X^1 which is positioned in $(0.63, 0.51)$. We consider four alternatives a, b, c and d whose performances can be found in Table 1.

If we consider for this decision maker equal weights in the distance ($w_1^1 = w_2^1 = 0.5$) the preference order on a, b, c and d is $a \succ_1 b \succ_1 c \succ_1 d$. If we consider non-equal weights for this same decision maker ($w_1^1 = 0.9, w_2^1 = 0.1$) the preference order is however $a \succ_1 b \succ_1 d \succ_1 c$.

This shows that these weights allow to modify how differences on the attributes influence the distance calculation in the preference model, and thus the final pre-order. To illustrate this, when non-equal weights are considered, very small difference on the first attribute have a high impact on the distance calculation, whereas the differences on the second attribute must be very large in order to influence the distance significantly. This allows to modify the preference order between c and d , without changing the position of the decision maker.

Influence of the position

We also observe that the weighted Euclidean distance allows to obtain the same preference orders on the alternatives for very different decision maker positions and their associated weights, i.e. for very different decision maker profiles.

To illustrate this, reconsider the same example as in Table 1. The first decision maker considered here is again positioned in $X^1 \equiv (0.63, 0.51)$ and we consider equal weights. As before this leads to $a \succ_1 b \succ_1 c \succ_1 d$. A second decision maker whose optimal alternative is in $X^2 = (0.63, 0.30)$ with $(w_1^2 = 0.75, w_2^2 = 0.25)$ leads to the same preference order.

Positions of the decision maker compatible with a given preference order

If we fix the weights for a decision maker, it is quite straightforward to determine his or her positions which are compatible with a given preference order of the alternatives.

As it can be deduced from Definition 1, these positions are located in a polyhedron defined by a set of inequalities, which can be deduced from the preference model. The way of obtaining this set of possible positions will be further explored in Section 3, where we show how to determine the parameters of the model.

Preference directions

As already said, the preference directions on the attributes are not specified beforehand in the proposed decision model. This removes a burden of the decision maker, as s/he does not have to specify whether the attributes are to be maximized or minimized. In the examples we can also see that the representation of the decision maker in the attributes space does not necessarily have to be in an extreme position, i.e., in one of the “corners”. This means that his or her “ideal” alternative can have intermediate evaluations on the attributes. This is directly linked to the single-peakedness of the attributes, as mentioned in Section 2.2.

In the classical situation, with known criteria preference directions and monotone preferences, the proposal still holds. In this situation the decision maker will be located at the position of the classically called “zenith” alternative, which is the ideal solution (maximizing or minimizing the performances on all the criteria) [17].

Compromise alternative

As already said above, depending on the positions of the decision makers and the weights used in the weighted Euclidean distance, the pre-orders of the alternatives may potentially be very different. Therefore, determining a compromise alternative (for the choice problem) for the group of decision makers is a hard task, if it is solely based on these rankings. The proposed model allows

to determine a compromise alternative by searching for the alternative which is “closest” to all the decision makers in \mathbf{E} .

However, this notion of proximity can be interpreted and defined in various ways. Here we propose to use the “average” distance between the decision makers and the alternatives, and select the alternative, which on average, is the closest to the positions of the decision makers in \mathbf{E} . More formally this comes down to searching for alternative $c \in \mathcal{A}$ such that:

$$c = \arg \min_{a \in \mathcal{A}} \sum_{i \in \mathcal{X}} \delta(a, X^i)$$

As mentioned in the introduction, if we apply this rule to the problem of Figure 1, this leads to recommending c to the two decision makers.

Clusters of similar decision makers

Another interesting use of the proposed model is to group together decision makers who are close to each other in \mathbf{E} , in order to be able to describe them in a synthetic way.

However, this supposes that they share the same definition of the distance, i.e., in our case the same weights in the Euclidean distance. In that case, any distance-based clustering method can be used, as for example k-means or hierarchical clustering [33].

3 Inference of parameters

Usually in the spatial voting models, the unknown parameters are both the positions of the candidates (the decision makers for us) and the voters (the alternatives). The methods used in this case, as shown by Armstrong *et al.* [8], are not suitable for our paradigm where the positions of the alternatives are known, and where mainly two types of parameters have to be determined: the positions of the decision makers in \mathbf{E} and the weights used in the weighted Euclidean distance.

As classically done in Multi-Criteria Decision Aiding, they could be elicited in a direct way [34], by questioning the decision makers. However, as one or both of these elements are not necessarily known by the decision makers, we propose to use an indirect elicitation approach to learn the parameters of the model through a holistic approach [35]. Each of the decision makers will express his or her preferences on couples of alternatives (either a strict preference or an indifference).

We will consider two situations that characterize the relationships among the decision makers in the group. First, we will consider a situation where the decision makers in the group do not share any elements of the preference model among themselves. This means that each individual has their own position and weights. The second situation considers that a part of the model is shared among the decision makers, either the weights or the positions. We will refer to the decision makers as “unconnected” in the first situation, or “connected” in the second situation.

In addition to this, it is possible that the preferences expressed by some decision makers in the group are incompatible with the proposed model. However, in this study, we will attempt to find model parameters that minimize errors in the expressed holistic preferences. We will refer to decision makers as "compatible" or "incompatible" with the proposed model.

In this section we first present various mathematical models to learn the preference parameters of the proposed model from preference judgments, for the various situations discussed above. Then we detail an experimental study on artificial data to estimate the calculation times for this parameter identification and the generalization power of the model on unseen data.

3.1 Mathematical modeling and resolution

Let us now present how the preference parameters of the proposed model can be determined by using mathematical programming techniques. The unknowns (or the variables) to be determined are the values of the preference parameters of the proposed model, i.e.:

1. the positions of the decision makers in the multidimensional space \mathbf{E} , $X^i \equiv (x_1^i, x_2^i, \dots, x_m^i)$, $\forall i \in \mathcal{X}$,
2. the weights of the different attributes w_j^i , for $j \in \mathcal{M}$ used in the distance, $\forall i \in \mathcal{X}$ (see Equation 1).

3.1.1 Unconnected decision makers compatible with the model

In this first formulation we consider that all the decision makers are unconnected and thus do not share any preferential parameters (neither the weights nor the positions). This means they all have independent preference models. Next to that, we assume here that the preference statements expressed by these decision makers are all compatible with the proposed model.

To determine the values of these preference parameters in this first case, we model the problem as a non-linear mathematical program. The formulation of this program is presented in Table 4 and its parameters and variables are respectively given in Tables 2 and 3.

| | |
|-------------------|---|
| \mathcal{A} | the set of alternatives |
| \mathcal{M} | the set of m attributes |
| \mathcal{L}_P^i | a set of pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ where a is preferred to b by i , $\forall i \in \mathcal{X}$ |
| \mathcal{L}_I^i | a set of pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ where a and b are considered as indifferent by i , $\forall i \in \mathcal{X}$ |
| γ | a small constant used to model strict inequalities |

Table 2: Input parameters of the first mathematical model

| | | | |
|--------------|-------------------|---|---|
| w_j^i | <i>continuous</i> | : | the weights of the attributes, $\forall j \in \mathcal{M}$ for $i, \forall i \in \mathcal{X}$ |
| x_j^i | <i>continuous</i> | : | the position of i on dimension j , $\forall j \in \mathcal{M}, \forall i \in \mathcal{X}$ |
| δ_a^i | <i>continuous</i> | : | the distance between X^i and a , $\forall a \in \mathcal{A}, \forall i \in \mathcal{X}$ (auxiliary variables) |

Table 3: Variables of the first mathematical model

$$\begin{array}{ll}
\min 0 & \\
\text{s.t.} & \\
w_j^i \geq \gamma & \forall j \in \mathcal{M}, \forall i \in \mathcal{X} \\
\sum_{j=1}^m w_j^i = 1 & \\
0 \leq x_j^i \leq 1 & \forall j \in \mathcal{M}, \forall i \in \mathcal{X} \\
\delta_a^i = \sqrt{\sum_{j=1}^m w_j^i (a_j - x_j^i)^2} & \forall a \in \mathcal{A}, \forall i \in \mathcal{X} \\
\delta_a^i + \gamma \leq \delta_b^i & \forall (a, b) \in \mathcal{L}_P^i, \forall i \in \mathcal{X} \\
\delta_b^i = \delta_a^i & \forall (a, b) \in \mathcal{L}_I, \forall i \in \mathcal{X}
\end{array} \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right| \begin{array}{l} (i) \\ \\ \\ (ii) \\ (iii) \\ (iv) \end{array}$$

Table 4: First mathematical model

In the model of Table 4, no objective function is considered, which comes down to simply looking for a feasible solution. However, it could be possible to avoid extreme values for the weights by minimizing the standard deviation of the different criteria weights. The constraints (i) specify that all the weights have to be strictly positive and sum up to 1 and that the positions of the decision makers are bounded in the unit interval, while constraints (ii) model the distance between an alternative $a \in \mathcal{A}$ and the i th decision maker's position $X^i \equiv (x_1^i, x_2^i, \dots, x_m^i)$. Constraints (iii) (resp. (iv)) are used to represent strict preference (resp. indifference) judgments of each decision maker.

Let us illustrate the use of this mathematical program on a small example in which we consider a problem with 2 attributes and 6 alternatives a, b, c, d, e and f of \mathcal{A} and a single decision maker, i.e. $\mathcal{X} = \{1\}$. The performances of the 6 alternatives on the 2 attributes are given in Table 5. This decision maker considers that $a \succ_1 b$, $c \succ_1 d$ and $e \succ_1 f$. Solving the first mathematical model with these preferential statements leads to a feasible solution

given by $X^1 = (0.2, 0.1)$, $w_1^1 = 0.75$ and $w_2^1 = 0.25$. It is possible to repre-

| $\mathcal{A} \setminus \mathcal{M}$ | 1 | 2 |
|-------------------------------------|-----|-----|
| a | 0.1 | 0.1 |
| b | 0.3 | 0.3 |
| c | 0.5 | 0.2 |
| d | 0.6 | 0.5 |
| e | 0.3 | 0.4 |
| f | 0.2 | 0.5 |

Table 5: Performances of the alternatives for the preference elicitation

sent this problem in a plane, which is shown on Figure 3. The axes represent the attributes and the alternatives are represented as points. When two alternatives are linked by a dotted segment, the decision maker has expressed a preferential judgement on this pair. The line passing through the center of this segment delimits two half-planes. In each of these half-planes, either one of the two considered alternatives is strictly preferred to the other one, or vice versa. On the line delimiting these two half-planes, both alternatives are considered as indifferent. Each gray half-plane correspond to the area in which the decision maker has to be located to respect the corresponding strict preference constraint. The intersection of the 3 half-planes corresponding to the 3 strict preferential statements is the dark grey triangle-shaped area on the bottom left (the darker an area, the more half-planes are intersecting in this area). All the points in this triangle area are closer to a than they are to b , while being closer to c than to d and closer to e than to f . X^1 , determined by the previous mathematical program, is obviously located in this triangle. The angles of the indifference lines with the dotted segment depend on the values of the weights in the weighted Euclidean distance.

As already said, using this first model assumes that all the preferences expressed by all the decision makers are compatible with the proposed distance model. If this is the case, this program will have a feasible solution. However, in real-world cases this might not always be the case, and some of the strict preference and indifference judgements expressed by the decision makers might not be compatible with the proposed model. To solve this issue, we propose an extension of this initial mathematical program hereafter.

3.1.2 Unconnected decision makers partially incompatible with the model

The idea behind this first extension is that if there is no feasible solution compatible with the decision maker i 's preferences, then we try to position X^i in such a way as to minimize an error measure. Intuitively, for each strict preference and indifference statement, we allow the distance condition to be violated by a certain amount. The goal of the mathematical program is then

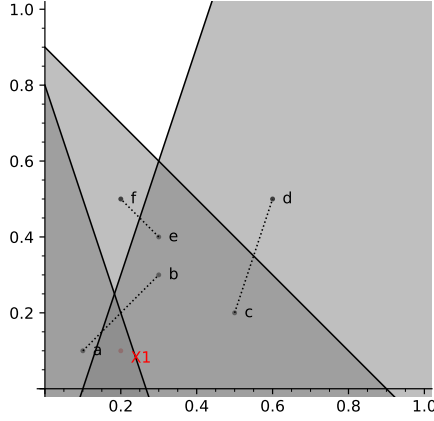


Fig. 3: 3 pairs: $a \succ_1 b$, $c \succ_1 d$, $e \succ_1 f$

to minimize the total error, throughout all the decision makers. In case all the preference and indifference statements of all the decision makers are compatible with the proposed model, then there are no such errors, and the program comes down to that of the initial mathematical program. More precisely, for a strict preference constraint (iii), a positive continuous auxiliary variable $\sigma_{(a,b)}^{P,i}$ is added to the right hand side, so that in case δ_a^i cannot be strictly smaller than δ_b^i , then $\sigma_{(a,b)}^{P,i}$ will be chosen positive by the solver. For an indifference constraint (iv), a continuous (positive or negative) auxiliary variable $\sigma_{(a,b)}^{I,i}$ is added to the right hand side in order to allow for the distance from X^i to a to be larger or smaller than that from X^i to b . The objective function is the sum over all decision makers of the sum of two terms: first the sum of the (positive) errors from the preference constraints, and second the sum of the absolute values of the errors from the indifference constraints. To model the absolute values $\tau_{(a,b)}^{I,i}$ of $\sigma_{(a,b)}^{I,i}$, we use constraints (v), where $\rho_{(a,b)}^{I,i}$ is a binary variable, and N a positive parameter, larger than the largest possible distance which may be encountered. The extra variables and parameters needed for this program are given in Tables 6 and 7.

This elicitation model supposes again that all the decision makers are unconnected and therefore do not share any common preference parameters.

Figure 4 shows the example from Table 1 in which the decision maker has expressed the following holistic preferences : $a \succ_1 b$, $c \succ_1 d$, $e \succ_1 f$, $f \sim_1 d$ and $e \sim_1 d$. The 2 dashed lines correspond to the 2 indifference statements. The first mathematical program is not able to find a feasible solution for this problem. However the second program finds a solution in which the decision maker is positioned in $X^1 \equiv (0.40, 14)$, with $w_1^1 = 0.2$ and $w_2^1 = 0.8$). This position violates the indifference statement between e and d , but the other statements are respected. The weighted distance between X^1 and e is lower

| | | |
|------------------------|-------------------|---|
| $\sigma_{(a,b)}^{P,i}$ | <i>continuous</i> | auxiliary variables that allow to violate the preference condition on (a, b) , $\sigma_{(a,b)}^{P,i} \geq 0$, $\forall(a, b) \in \mathcal{L}_P^i, \forall i \in \mathcal{X}$ |
| $\sigma_{(a,b)}^{I,i}$ | <i>continuous</i> | auxiliary variables that allow to violate the preference condition on (ab) $\forall(a, b) \in \mathcal{L}_I^i, \forall i \in \mathcal{X}$ |
| $\tau_{(a,b)}^{I,i}$ | <i>continuous</i> | variable representing the absolute value of $\sigma_{(a,b)}^{I,i}$, $\tau_{(a,b)}^{I,i} \geq 0$, $\forall(a, b) \in \mathcal{L}_P^i, \forall i \in \mathcal{X}$ |
| $\rho_{(a,b)}^{I,i}$ | <i>binary</i> | variable for the modelling of the absolute value of $\sigma_{(a,b)}^{I,i}$, $\forall(a, b) \in \mathcal{L}_P^i, \forall i \in \mathcal{X}$ |

Table 6: Extra variables for the extended mathematical model

| | |
|-----|---|
| N | a sufficiently large positive constant, larger than the largest possible distance |
|-----|---|

Table 7: Extra parameter for the extended mathematical model

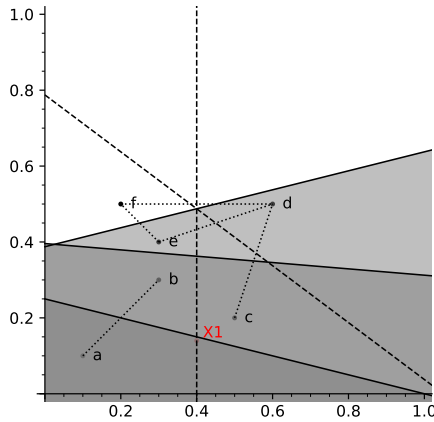


Fig. 4: Example with $a \succ_1 b$, $c \succ_1 d$, $e \succ_1 f$, $f \sim_1 d$ and $e \sim_1 d$. The dashed lines correspond to indifference statements.

than that between X^1 and d , which means that according to this solution $e \succ_1 d$.

3.1.3 Connected decision makers partially incompatible with the model

We now consider the case where there exists a link between the decision makers of the group and where they share parts of the preference parameters of the

$$\begin{array}{l}
\min \quad \sum_{i \in \mathcal{X}} \left(\sum_{(a,b) \in \mathcal{L}_P^i} \sigma_{(a,b)}^{P,i} + \sum_{(a,b) \in \mathcal{L}_I^i} \tau_{(a,b)}^{I,i} \right) \\
\text{s.t.} \\
w_j^i \geq \gamma \quad \forall j \in M, \forall i \in \mathcal{X} \\
\sum_{j=1}^m w_j^i = 1 \\
0 \leq x_j^i \leq 1 \quad \forall j \in M, \forall i \in \mathcal{X} \\
\delta_a^i = \sqrt{\sum_{j=1}^m w_j^i (a_j - x_j^i)^2} \quad \forall a \in \mathcal{A}, \forall i \in \mathcal{X} \\
\delta_a^i + \gamma \leq \delta_b^i + \sigma_{(a,b)}^{P,i} \quad \forall (a,b) \in \mathcal{L}_P^i, \forall i \in \mathcal{X} \\
\delta_b^i = \delta_a^i + \sigma_{(a,b)}^{I,i} \quad \forall (a,b) \in \mathcal{L}_I^i, \forall i \in \mathcal{X} \\
\sigma_{(a,b)}^{I,i} + N\rho_{(a,b)}^{I,i} \geq \tau_{(a,b)}^{I,i} \quad \forall (a,b) \in \mathcal{L}_I^i, \forall i \in \mathcal{X} \\
-\sigma_{(a,b)}^{I,i} + N(1 - \rho_{(a,b)}^{I,i}) \geq \tau_{(a,b)}^{I,i} \quad \forall (a,b) \in \mathcal{L}_I^i, \forall i \in \mathcal{X} \\
\sigma_{(a,b)}^{I,i} \leq \tau_{(a,b)}^{I,i} \quad \forall (a,b) \in \mathcal{L}_I^i, \forall i \in \mathcal{X} \\
-\sigma_{(a,b)}^{I,i} \leq \tau_{(a,b)}^{I,i} \quad \forall (a,b) \in \mathcal{L}_I^i, \forall i \in \mathcal{X}
\end{array} \quad \begin{array}{l} \\ \\ \\ \\ (i) \\ \\ (ii) \\ \\ (iii) \\ \\ (iv) \\ \\ (v) \\ \\ \\ \end{array}$$

Table 8: Extended mathematical model coping with incompatible preference and indifference judgements

model. As already said, this connection could either concern the parameters of the distance or the positions of the decision makers.

If the goal is to create clusters of similarly behaving decision makers, this link between the decision makers has to be established in the distance measure, so that a common definition of the distance allows to model a common behavior regarding the attributes.

To learn such a common model, we propose to impose that all the decision makers share common weights in constraints (i) and (ii) of the extended mathematical program. To do so, it is enough to add a set of constraints which say that:

$$w_j^i = w_j^k \quad \forall j \in M, \forall i, k \in \mathcal{X}.$$

These constraints will generate the same distance measure for each decision maker. This limits the search for the model parameters much more, and it is obvious that some preferential statements of some decision makers may no longer be representable by this common model. In that case, the error variables will allow to find a feasible solution, by violating some of the preference judgements, for certain decision makers. However, with this common distance

measure, it is possible to use distance-based clustering algorithms to determine similar behaviors among the decision makers, in terms of their positions in the attributes space.

We present an illustrative example in which we use the same alternatives as in example from Table 1. This time we have two decision makers $\mathcal{X} = \{1, 2\}$. The first decision maker has expressed the following holistic preferences: $a \succ_1 b$, $c \succ_1 d$, $e \succ_1 f$ and $f \sim_1 d$ while the second one has expressed the following holistic preferences: $b \succ_2 a$, $c \succ_2 d$ and $f \succ_2 e$. The solution found by the elicitation model with common weights for the two decision makers places the first decision maker at $X^1 = (0.39, 0.10)$ and the second at $X^2 = (0.10, 0.40)$, with the weights $w_1^1 = w_1^2 = 0.3$ and $w_2^1 = w_2^2 = 0.7$. Figure 5 illustrates this problem: on the left the constraints for decision maker 1, and on the right the constraints for decision maker 2. We have chosen to represent both decision makers in both Figures for illustrative purposes. Note that for this example, all the holistic preferences of the 2 decision makers are respected.

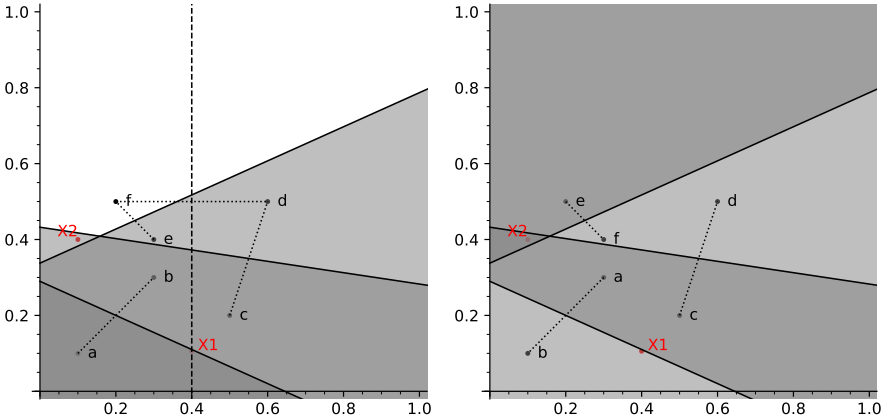


Fig. 5: Group decision problem with shared weights (decision maker 1's constraints on the left, 2's constraints on the right)

Another possibility is to link the decision makers by their position in the attributes space. This comes down to allowing the weights to take independent values among the decision makers, while imposing that the decision makers are located in the same point in the attributes space. Similarly as to the previous model, in constraint (ii) it is necessary to impose that the x_j^i are common to all the decision makers $i \in \mathcal{X}$. To allow for some flexibility in the determination of the preference parameters, we could however impose some tolerance on this common location, by imposing that all the positions of the decision makers are located in a small hypercube (of length $0 \leq \mu \leq 1$, which becomes a parameter of the model).

Thus, to locate the decision makers in this small hypercube, we have to ensure that:

$$|x_j^i - x_j^k| \leq \mu \quad \forall j \in \mathcal{M}, \forall i, k \in \mathcal{X}.$$

To linearize this absolute value of the differences, we add these constraints to the extended mathematical model:

$$x_j^i - x_j^k \leq \mu \quad \forall j \in \mathcal{M}, \forall i, k \in \mathcal{X}$$

$$-x_j^i + x_j^k \leq \mu \quad \forall j \in \mathcal{M}, \forall i, k \in \mathcal{X}$$

To illustrate this, we consider the same example with the same holistic preferences expressed by the two decisions makers as above ($1 : a \succ_1 b, c \succ_1 d, e \succ_1 f$ and $f \sim_1 d$, $2 : b \succ_2 a, c \succ_2 d$ and $f \succ_2 e$). With $\mu = 0.05$, we obtain a solution that places decision maker 1 at $X^1 = (0.29, 0.15)$ with weights $w_1^1 = 0.001, w_2^1 = 0.999$ and the second one at $X^2 = (0.24, 0.20)$ with weights of $w_1^2 = 0.999, w_2^2 = 0.001$. This solution violates the indifference between d and f expressed by decision maker 1. On the left of Figure 6, it looks as if X^1 could be moved further to the right to be positioned on the indifference line separating f and d . However this is not possible, as the positions of X^2 and X^1 are very constrained by μ . Consequently, for decision maker 1 $f \succ_1 d$.

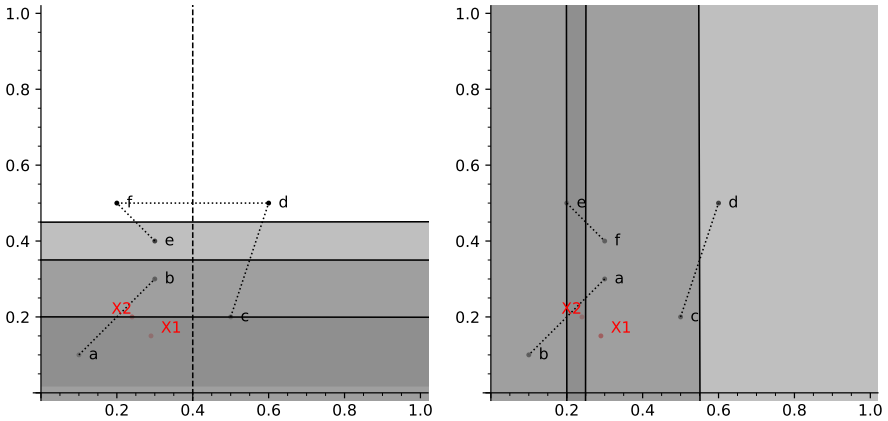


Fig. 6: Group decision problem with close positions (decision maker 1 on the left, 2 on the right)

The third possibility to link the decision makers is through the weights and their positions simultaneously. This constraints the elicitation problem even more than in the two previous variants. However, it allows to determine a common preference model for all the decision makers, which allows to rank the alternatives through that unique model. It is obvious that a lot of incompatibilities may arise in the individual preference judgements through this elicitation model.

In any case, it is obvious that if either the positions of the decision makers, or their weights are known in advance, all these mathematical programs can easily be adapted by transforming the variables related to the known preferences into parameters.

3.2 Resolution and experimental study

Due to the square root in the definition of the weighted Euclidean distance, and the product between the weights and the position of the decision maker, the above mathematical programs are non-linear mixed integer programs.

It can be easily understood that dropping the square root from constraints (ii) does not modify the resolution of these programs, as the square root is a strictly monotonic function, and we only compare distances here. Dropping the square root therefore leads to simpler non-linear programs, which we solve using the non-linear solver in Gurobi [36]. We use the `addGenConstrPow()` function of this solver to model power function constraints as piecewise-linear approximations (we use `FuncPieces=-2` as parameter that bounds the relative error of the approximation).

In this section we propose to study experimentally the behaviour of the proposed elicitation model when facing artificially generated random data, for a single decision maker. The goal is to answer the following questions:

- How do resolution times vary with the number of input preference statements?
- How does the model generalize when confronted with unseen data?
- What is the influence of noise in the input preference judgements on the model?

In the experiments, the decision maker is replaced by a randomly generated model (denoted by M_{DM}), which is used to compare pairs of alternatives (and which generates the holistic preferential statements).

We use two different sets of alternatives:

- a *training* dataset \mathcal{D} of 100 alternatives, from which we will extract pairwise comparisons that will be used for the identification of the model parameters through the second mathematical program (Table 8), without the square root,
- and a *test* dataset \mathcal{D}_{test} composed of 1000 alternatives, which is used to evaluate the performance of the elicited model on unseen data (to evaluate the generalization capacity of the model).

The performance vectors attached to alternatives are randomly drawn as floats (with 2 digits after the decimal point) within the interval $[0, 1]$ using a uniform distribution. We test different sizes of problems in terms of number of attributes (3, 5, 7 and 9 attributes).

In a first step, we generate all possible (unordered) pairs of alternatives from \mathcal{D} . We obtain 4950 pairs with 100 alternatives. They correspond to all the preference queries which could be presented to the decision maker regarding the alternatives of \mathcal{D} . Let \mathcal{Q} denote the set of all (unordered) pairs of alternatives in set \mathcal{D} .

To answer the questions related to resolution time and generalization, different sizes of learning sets (5, 20, 40, 60, 80 and 100 pairs) are generated. For each size we select randomly pairs from \mathcal{Q} and we obtain $\mathcal{L}_P \cup \mathcal{L}_I$ by applying M_{DM} on the selected pairs. Using this information, we generate a model (denoted by M_{cur}) using the second mathematical program from the previous section (without the square root). Two rankings of the alternatives in \mathcal{D}_{test} are then computed using both M_{DM} and M_{cur} models and the *Kendall's* rank correlation (τ) [37] is used to compare them. This measure is used as a similarity indicator for the two rankings, and varies between -1 and 1. If both rankings are identical then $\tau = 1$, while if they are completely reversed then $\tau = -1$. We repeat this process 100 times (using different sets of alternatives and generated models (M_{DM})) and report the averaged results below.

To determine the influence of noise on the model, similar tests are performed. However, we limit ourselves to 100 pairs in the learning set, and insert different levels of noise. More precisely, a certain amount (5%, 10% 15% and 20%) of preference pairs are reversed (compared to those generated with \mathcal{D}_{test}), which could lead to preference statements incompatible with the proposed model. The goal here is to study the influence of this noise on both the quality of the learnt model w.r.t. the learning set, and the test set.

All the experiments are performed on a computing server using 20 cores and 100 GB of RAM.

Figure 7 shows execution times for the various problem sizes (in terms of number of attributes). The abscissa represents the numbers of pairs in the learning set, whereas the ordinate shows execution time in seconds. We can observe that for a given problem size, execution time increases with the size of the learning set. For example, for a learning set containing 100 pairs for problems of 9 attributes, the solver requires about 500 seconds on average to obtain the parameters of the model. Next to that, we can also see that computation times increase with the number of attributes, for a given size of the learning set.

The performance of the Inference of parameters algorithm for various problem sizes is shown on Figure 8. It shows the average Kendall's tau of 100 executions on the test datasets between the ranking obtained through M_{DM} and the one obtained through M_{cur} , for different sizes of the learning set (and different problem sizes). As we can see, Kendall's Tau increases with the size of the learning set. This was expected, since adding preference and indifference constraints to the mathematical program reduces the uncertainty on the variables, and therefore allows to find a model which ranks similarly as M_{DM} . For example, for a problem with 5 attributes, this rank correlation index equals on average about 0.9 when using 100 pairs. We can also see that Kendall's Tau decreases when the number of attributes increases, for a given size of the learning set. This was also expected, as a higher complexity of the problem (in terms of number of attributes) requires more learning examples.

Regarding noise in the preference statements, Figure 9 shows, for learning sets of size 100, Kendall's tau between the ranking obtained through the sought

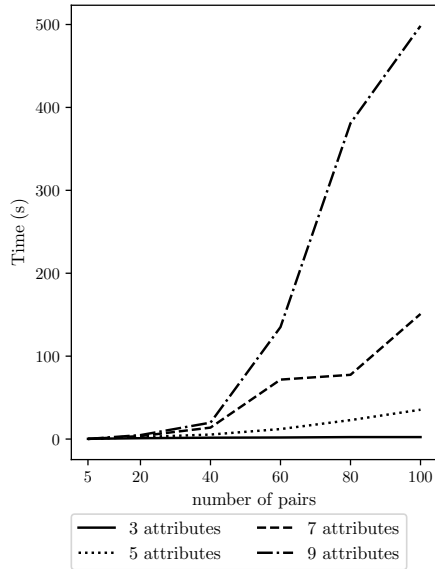


Fig. 7: Mean execution time in seconds for all problem sizes

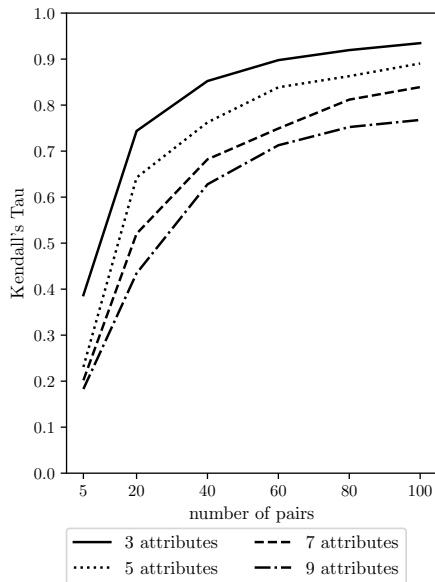


Fig. 8: Mean Kendall tau for all problems

model M_{DM} and M_{cur} , for various levels of noise, and various numbers of attributes, both on the learning set and the test set. It can be observed that

adding noise to the input statements decreases the quality of the model, both on the learning and the test datasets. Obviously, for 0% of noise, Kendall’s tau on the learning set equals 1. Similarly, still for 0% of noise, Kendall’s tau on the test set equals the highest value achieved in Figure 8. Due to higher computational times in this noisy situation, tests were only performed on 3 attributes.

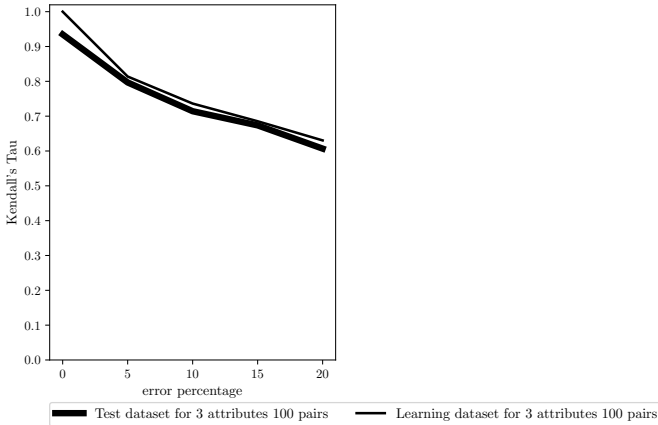


Fig. 9: Mean Kendall tau for the different error percentage and both learning and test dataset

4 Real world application: OOCIT

OOCIT¹ is a set of tools which aims at facilitating the dialogue between various stakeholders concerning urban and industrial territories. It has been developed in the context of the TIGA project [38], whose objective is to reconnect industries with the territories and their inhabitants.

Among the different tasks of the laboratory, here we focus on the task that aims at extracting the priorities of the various stakeholders with regards to the future of their territory. These stakeholders are the industrial companies, the public authorities and the inhabitants of the territory.

These priorities, which might differ from one stakeholder to another, are expressed along different themes (“getting around”, “finding a place to live or set up a business”, “resourcing (nature, culture, sports)”, “eating and drinking”, “protecting yourself”, “work and training”). Inside each of these themes, various features or attributes are contributing to the evaluation of the territories according to each theme.

In the TIGA project, we have observed that, when the various stakeholders were interviewed, they often had difficulties for expressing preferences related

¹<https://www.oocit.fr/public/home>

to some of these attributes. Either they were unable to indicate clear preference directions, or their preferences seemed to be non monotonically increasing or decreasing with the attribute scales. Next to that, these stakeholders also expressed the need to understand the differences of their priorities with those of the other stakeholders.

This led us to develop the approach proposed in this paper, which allows us to position the stakeholders (the decision makers of this proposal) with the territories (the alternatives of this proposal) in a common space. Moreover, it simplifies the comparison of the preferences of the various stakeholders and allows us to find compromise territories for all the stakeholders, according to each theme.

To show the interest of the approach in such a real world problem, we now detail the output of the method on one specific theme, namely “finding a place to live or set up a business”. It is composed of the following five attributes : “population density”, “access to housing”, “security”, “industrial activities” and “available space”. These attributes are the result of the aggregation of several indicators, and their evaluation scales are therefore normalized between 0 and 1. Note that several indicators obtained through the use of open access public statistics. Their evaluation scales have been normalized between 0 and 1 and aggregated with a weighted mean, under the supervision of experts of the territory.

For reasons of confidentiality, we will not detail the characteristics of each of the stakeholders interviewed. We will simply call them stakeholder₁ and stakeholder₂.

Each of these stakeholders responded to a questionnaire, in which s/he was presented with 4 among 64 possible territories, described according to the attributes of the theme “finding a place to live or set up a business”. Each of those attributes is an aggregation of multiple indicators through a weighted sum, resulting in normalized scores in $[0, 1]$.

The selection of these 4 alternatives is the result of an unsupervised hierarchical clustering approach, where the 4 selected territories correspond to the real territories closest to the centroids of the 4 obtained clusters.

To avoid any kind of bias related to the name of the territory we decided to make them anonymous. The 4 territories are summarized in Table 9. The remaining territories are in Table A.

Each stakeholder was then asked to rank those 4 territories according to her or his priorities or preferences (the holistic preference statements). These holistic preferences expressed by the stakeholders are summarized in Table 10.

Choosing the mathematical model to implement (among those presented in Section 2) is obviously not an easy choice. In the context of this application, we have noticed that this choice can depend on several factors. First, if in the discussions with the stakeholders it turns out that they have similar views on their ideal alternatives, it would probably be appropriate to choose a model that links their position in **E**. If, on the other hand, it is observed that they have very different preferences, the choice of a less restrictive model may be a

| territory | popula- tion density (1) | access to housing (2) | security (3) | indus- trial activi- ties (4) | available space (5) |
|-----------|-----------------------------------|--------------------------------|-----------------|--|---------------------------|
| t_1 | 0.29 | 0.40 | 0.46 | 0.19 | 0.76 |
| t_{55} | 0.31 | 0.42 | 0.52 | 0.09 | 0.00 |
| t_3 | 0.64 | 0.65 | 0.53 | 0.46 | 0.23 |
| t_4 | 0.33 | 0.34 | 0.68 | 0.11 | 0.01 |

Table 9: Performance table of the territories used to elicit the preferences

| stakeholder | ranking |
|--------------------------|--|
| stakeholder ₁ | $t_{55} \succ t_4 \succ t_3 \succ t_1$ |
| stakeholder ₂ | $t_3 \succ t_1 \succ t_{55} \succ t_4$ |

Table 10: Holistic preferences of the 2 stakeholders

better solution. It may also be interesting to observe the differences between the holistic preferences expressed and those reproduced by the models (the “errors” discussed in Section 3.1.2). In the case where one model allows to represent the holistic preferences of the decision-makers with fewer errors than another, it would probably be preferable to choose this model to describe the preferences of the respondents. Finally, if the characteristics of the stakeholders are similar or if their areas of expertise are similar, it is probably preferable to choose a model that links them together.

In the following, we detail the results obtained through 3 of the models presented in Section 2 in order to be able to discuss their various implications.

We first use the preference elicitation model with unconnected decision makers (see Section 3.1.2). This leads to the parameters of Table 11. These parameters of the two decision makers are then used to rank the 64 territories presented in Table A1. We show in Table 12 the ranking of the top 20 territories only.

| stakeholder | position | attributes weights |
|--------------------------|--------------------------------|-------------------------------------|
| stakeholder ₁ | (0.60, 0.32, 0.41, 0.23, 0.01) | (0.015, 0.001, 0.001, 0.005, 0.978) |
| stakeholder ₂ | (0.80, 0.80, 0.22, 0.75, 0.57) | (0.159, 0.164, 0.190, 0.256, 0.231) |

Table 11: The model parameters for the 2 stakeholders, considered as unconnected

One can observe that the rankings are quite different from one stakeholder to another. Using the rule from Section 2.3 it is possible to determine a good compromise territory, according to the proposed model. In this case, it is t_{62} , which on average is the closest to each of the stakeholders. If we look at

| stakeholder | Total pre-order of the 64 territories |
|--------------------------|--|
| stakeholder ₁ | $t_{57} \succ t_{60} \succ t_{58} \succ t_{62} \succ t_2 \succ t_{61} \succ t_{33} \succ t_4 \succ t_{55} \succ t_{10} \succ t_{18} \succ t_{17} \succ t_{52} \succ t_{32} \succ t_{59} \succ t_7 \succ t_{35} \succ t_{43} \succ t_{31} \succ t_{14} \succ \dots$ |
| stakeholder ₂ | $t_{44} \succ t_{42} \succ t_{54} \succ t_3 \succ t_{64} \succ t_{59} \succ t_{47} \succ t_{34} \succ t_{62} \succ t_{41} \succ t_{63} \succ t_{45} \succ t_{53} \succ t_{58} \succ t_{50} \succ t_{57} \succ t_1 \succ t_6 \succ t_{29} \succ t_{60} \succ \dots$ |

Table 12: Total pre-order of 64 territories for each of the 2 stakeholders, considered as totally unconnected

the 2 rankings obtained, we also see that t_{62} is positioned in the top 9 for both decision-makers. Regarding their preferential parameters of Table 11, we observe that they are very different between the 2 respondents, which is normal, given that we chose a model that does not link them at all.

As a second preference elicitation model, we consider now that the two stakeholders (or decision makers) use the same weights when comparing the territories. This allows us to use the same definition of the distance for both actors. We execute the program presented in Section 3.1.3 and we obtain the solution presented in Table 13 with the corresponding rankings shown in Table 14.

| stakeholder | position in $[0, 1]^5$ | attributes weights |
|--------------------------|--------------------------------|-------------------------------------|
| stakeholder ₁ | (0.78, 0.79, 0.46, 0.71, 0.06) | (0.028, 0.026, 0.030, 0.022, 0.893) |
| stakeholder ₂ | (0.80, 0.80, 0.23, 0.73, 0.46) | |

Table 13: The model parameters for the 2 stakeholders, with linked weights

| stakeholder | Total pre-order of the 64 territories |
|--------------------------|---|
| stakeholder ₁ | $t_{58} \succ t_{61} \succ t_{57} \succ t_{56} \succ t_{62} \succ t_7 \succ t_{22} \succ t_{27} \succ t_5 \succ t_{60} \succ t_{42} \succ t_{31} \succ t_{39} \succ t_{38} \succ t_1 \succ t_9 \succ t_{40} \succ t_{49} \succ t_{51} \succ t_{23} \succ \dots$ |
| stakeholder ₂ | $t_{41} \succ t_{46} \succ t_{53} \succ t_{28} \succ t_{43} \succ t_{52} \succ t_2 \succ t_{44} \succ t_{26} \succ t_{47} \succ t_{55} \succ t_{63} \succ t_{48} \succ t_{24} \succ t_{20} \succ t_{50} \succ t_{35} \succ t_8 \succ t_{40} \succ t_{45} \succ \dots$ |

Table 14: Total pre-order of 64 territories for each of the 2 stakeholders, with linked weights

We can see that the positions of the two stakeholders are relatively close, while having the same values for the attributes weights. This time, the compromise alternative according to the rule of Section 2.3 is t_3 . As it can be seen in Table 14, this alternative does not appear in the first positions of the pre-orders of the two stakeholders.

As a third possibility we use the inference algorithm which tries to position the decision makers close in the attributes space. We therefore solve the

program presented in Section 3.1.3 with $\mu = 0.05$ and we obtain the solution presented in Tables 15 and 16.

| stakeholder | position | attributes weights |
|--------------------------|-------------------------------|-------------------------------------|
| stakeholder ₁ | (0.59, 0.59, 0.52, 0.0, 0.33) | (0.756, 0.001, 0.001, 0.241, 0.001) |
| stakeholder ₂ | (0.65, 0.65, 0.47, 0.0, 0.33) | (0.001, 0.798, 0.001, 0.199, 0.001) |

Table 15: The model parameters for 2 stakeholders, with linked positions

| stakeholder | Total pre-order of the 64 territories |
|--------------------------|--|
| stakeholder ₁ | $t_{56} \succ t_{59} \succ t_{57} \succ t_{60} \succ t_{62} \succ t_1 \succ t_{22} \succ t_{28} \succ t_3 \succ t_{12} \succ t_{39} \succ t_{18} \succ t_2 \succ t_{27} \succ t_{54} \succ t_{44} \succ t_5 \succ t_9 \succ t_7 \succ t_{63} \succ t_{50} \succ \dots$ |
| stakeholder ₂ | $t_{60} \succ t_{15} \succ t_{59} \succ t_{62} \succ t_{22} \succ t_{56} \succ t_{18} \succ t_7 \succ t_{44} \succ t_{63} \succ t_{12} \succ t_9 \succ t_{17} \succ t_{14} \succ t_{50} \succ t_{34} \succ t_4 \succ t_2 \succ t_{26} \succ t_{38} \succ t_{54} \succ \dots$ |
| | ... |

Table 16: Total pre-order of 64 territories for each of the 2 stakeholders, with linked positions

It can be seen that both stakeholders are obviously positioned very closely in \mathbf{E} , while having relatively different weights in the used distances. According to the rule of Section 2.3 alternative t_{60} is considered as the best compromise alternative for both stakeholders, which is ranked in the top 4 positions of the two pre-orders of Table 16.

The 3 inference techniques lead to different preference models. However, in all 3 cases, the preferences of the 2 decision makers are respected and they are compatible with the learned model. In this particular case, the inference model chosen is the 3rd one, i.e. the one where the positions of the respondents are very similar. The 2 main reasons for this choice are: first, the characteristics of the 2 stakeholders are quite similar and therefore it was important that their positions in \mathbf{E} be close; second, the compromise alternative is found in the top positions of both rankings.

In the OOCIT project, this approach is used as a discussion tool to better understand the preferences of different stakeholders. In the real problem, the number of stakeholders is of course much higher, and the preference modeling and inference methods are applied to all themes. We decided in this simplified version to focus only on two decision makers to underline the differences of the proposed approaches.

5 Conclusion and perspectives

In this article we have presented how to use techniques inspired from the spatial theory of voting in multi-criteria group decision aiding. We have also illustrated how such a spatial preference model allows to identify ideal alternatives representing the decision makers alongside the alternatives, while taking into account non-monotonic preferences. We have also presented various preference elicitation techniques, creating various links between the decision makers that are related to the model parameters.

Such a preference model can be used as a basis for discussions with the stakeholders, to understand their preferences, and how their various behaviors can (or cannot) lead to some consensus. The experiments show also that the proposed model, and the various elicitation techniques, have a good generalization power on unseen data, which allows to use them in real world applications.

However, a certain number of limitations remain to be studied. In particular the important question regarding an incremental preference elicitation of the decision makers' preferences, which should lead to models with a higher generalization power, while requiring fewer preference queries. Next to that, it could be interesting to work on graphical representations of the attributes space, which allow to interpret the preferences contained in the position of the decision maker and the corresponding weights. We can consider for example the PCA (Principal Component Analysis) or MDS (Multi Dimensional Scalling), which represents multidimensional distance matrices in a two-dimensional space. And last but not least, integrating some negotiation techniques, in order to solve certain incompatibilities between the decision makers, could also lead to more robust preference models of the group.

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Conflict of Interest:. The authors declare that they have no conflict of interest.

Appendix A Performance Table of 64 territories

| Territory | Population density | Access to housing | Security | Industrial activities | Available space |
|-----------|--------------------|-------------------|-------------|-----------------------|-----------------|
| t1 | 0.29 | 0.40 | 0.46 | 0.19 | 0.76 |
| t2 | 0.44 | 0.45 | 0.83 | 0.34 | 0.02 |
| t3 | 0.64 | 0.65 | 0.53 | 0.46 | 0.23 |
| t4 | 0.33 | 0.34 | 0.68 | 0.11 | 0.01 |
| t5 | 0.24 | 0.44 | 0.9616 | 0.20 | 0.01 |

Continued on next page

Table A1 – continued from previous page

| Territory | population density | access to housing | security | industrial activities | available space |
|-----------|--------------------|-------------------|----------|-----------------------|-----------------|
| t6 | 0.37 | 0.47 | 0.53 | 0.30 | 0.11 |
| t7 | 0.27 | 0.42 | 0.82 | 0.20 | 0 |
| t8 | 0.35 | 0.51 | 0.64 | 0.29 | 0.06 |
| t9 | 0.32 | 0.47 | 0.91 | 0.34 | 0.18 |
| t10 | 0.33 | 0.44 | 0.49 | 0.19 | 0.03 |
| t11 | 0.26 | 0.42 | 0.64 | 0.19 | 0.06 |
| t12 | 0.17 | 0.38 | 0.58 | 0.15 | 0.03 |
| t13 | 0.36 | 0.46 | 0.771 | 0.21 | 0.10 |
| t14 | 0.25 | 0.26 | 0.63 | 0.15 | 0 |
| t15 | 0.25 | 0.47 | 0.61 | 0.29 | 0.11 |
| t16 | 0.29 | 0.53 | 0.65 | 0.20 | 0.10 |
| t17 | 0.31 | 0.41 | 0.70 | 0.38 | 0 |
| t18 | 0.32 | 0.45 | 0.87 | 0.20 | 0 |
| t19 | 0.33 | 0.47 | 0.84 | 0.16 | 0.04 |
| t20 | 0.29 | 0.39 | 0.45 | 0.20 | 0.11 |
| t21 | 0.29 | 0.36 | 0.64 | 0.28 | 0.21 |
| t22 | 0.29 | 0.45 | 0.89 | 0.26 | 0.09 |
| t23 | 0.38 | 0.51 | 0.66 | 0.27 | 0.10 |
| t24 | 0.26 | 0.37 | 0.31 | 0.32 | 0.06 |
| t25 | 0.28 | 0.41 | 0.65 | 0.13 | 0.21 |
| t26 | 0.34 | 0.43 | 0.66 | 0.33 | 0.14 |
| t27 | 0.30 | 0.42 | 0.75 | 0.16 | 0.63 |
| t28 | 0.36 | 0.43 | 0.56 | 0.26 | 0.05 |
| t29 | 0.36 | 0.43 | 0.56 | 0.20 | 0.35 |
| t30 | 0.29 | 0.29 | 0.49 | 0.19 | 0.05 |
| t31 | 0.26 | 0.43 | 0.10 | 0.20 | 0 |
| t32 | 0.27 | 0.48 | 0.54 | 0.34 | 0.01 |
| t33 | 0.33 | 0.42 | 0.69 | 0.25 | 0 |
| t34 | 0.38 | 0.46 | 0.46 | 0.41 | 0.86 |
| t35 | 0.28 | 0.44 | 0.65 | 0.23 | 0.03 |
| t36 | 0.25 | 0.43 | 0.68 | 0.17 | 0.18 |
| t37 | 0.26 | 0.41 | 0.86 | 0.33 | 0.05 |
| t38 | 0.27 | 0.42 | 0.87 | 0.18 | 0.03 |
| t39 | 0.33 | 0.47 | 0.76 | 0.30 | 0.06 |
| t40 | 0.34 | 0.43 | 0.44 | 0.20 | 0.06 |
| t41 | 0.40 | 0.47 | 0.44 | 0.43 | 0.14 |
| t42 | 0.39 | 0.49 | 0.37 | 0.50 | 0.54 |
| t43 | 0.34 | 0.40 | 0.54 | 0.29 | 0.04 |
| t44 | 0.52 | 0.61 | 0.48 | 0.66 | 0.26 |
| t45 | 0.36 | 0.48 | 0.61 | 0.28 | 0.65 |
| t46 | 0.34 | 0.44 | 0.58 | 0.29 | 0.16 |
| t47 | 0.32 | 0.41 | 0.29 | 0.34 | 0.52 |

Continued on next page

Table A1 – continued from previous page

| Territory | population density | access to housing | security | industrial activities | available space |
|------------|--------------------|-------------------|-------------|-----------------------|-----------------|
| t48 | 0.27 | 0.42 | 0.70 | 0.21 | 0.27 |
| t49 | 0.26 | 0.26 | 0.62 | 0.26 | 0.23 |
| t50 | 0.32 | 0.44 | 0.55 | 0.38 | 0.13 |
| t51 | 0.34 | 0.46 | 0.63 | 0.27 | 0.20 |
| t52 | 0.32 | 0.40 | 0.50 | 0.23 | 0.02 |
| t53 | 0.28 | 0.48 | 0.55 | 0.36 | 0.26 |
| t54 | 0.38 | 0.44 | 0.51 | 0.58 | 0.56 |
| t55 | 0.31 | 0.42 | 0.52 | 0.09 | 0 |
| t56 | 0.27 | 0.38 | 0.27 | 0.19 | 0.25 |
| t57 | 0.68 | 0.78 | 0.53 | 0.25 | 0 |
| t58 | 0.65 | 0.83 | 0.53 | 0.32 | 0 |
| t59 | 0.55 | 0.66 | 0.53 | 0.60 | 0.04 |
| t60 | 0.46 | 0.74 | 0.53 | 0.26 | 0 |
| t61 | 0.40 | 0.55 | 0.53 | 0.24 | 0 |
| t62 | 0.66 | 0.77 | 0.53 | 0.50 | 0 |
| t63 | 0.47 | 0.64 | 0.53 | 0.36 | 0.13 |
| t64 | 0.42 | 0.70 | 0.53 | 0.42 | 0.21 |

Table A1: Performance table of the 64 territories

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