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A COMPACTNESS RESULT FOR A SYSTEM WITH HOLDERIAN CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give a quantization analysis to an elliptic system with Dirichlet condition. An application, we have a compactness result for an elliptic system with Hölderian condition.

Mathematics Subject Classification: 35J60 35B45 35B50

Keywords: blow-up, boundary, system, Dirichlet condition, a priori estimate, analytic domain, Hölder condition.

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open analytic domain Ω of \mathbb{R}^2 .

We consider the following equation:

$$(P) \begin{cases} -\Delta u = V(1 + |x|^{2\beta})e^v & \text{in } \Omega \subset \mathbb{R}^2, \\ -\Delta v = We^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b_1 < +\infty, e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega),$$

$$0 \leq W \leq b_2 < +\infty, e^v \in L^1(\Omega) \text{ and } v \in W_0^{1,1}(\Omega),$$

and,

$$0 \in \partial\Omega, \beta \in [0, 1/2).$$

When $u = v$, the above system is reduced to an equation which was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-17], one can find some existence and compactness results, also for a system.

Among other results, we can see in [6] the following important Theorem,

Theorem A. (Brezis-Merle [6]). Consider the case of one equation; if $(u_i)_i = (v_i)_i$ and $(V_i)_i = (W_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

Theorem B (Brezis-Merle [6]). Consider the case of one equation and assume that $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 \leq V_i \leq b < +\infty$, and,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact set K of Ω ,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

Next, we call energy the following quantity:

$$E = \int_{\Omega} e^{u_i} dy.$$

The boundedness of the energy is a necessary condition to work on the problem (P) as showed in [6], by the following counterexample.

Theorem C (Brezis-Merle [6]). Consider the case of one equation, then there are two sequences $(u_i)_i$ and $(V_i)_i$ of the problem (P) with, $0 \leq V_i \leq b < +\infty$, and,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and

$$\sup_{\Omega} u_i \rightarrow +\infty.$$

When $\beta = 0$, the above system have many properties in the constant and the Lipschitzian cases. Indeed we have (when $\beta = 0$):

In [12], Dupaigne-Farina-Sirakov proved (by an existence result of Montenegro, see [16]) that the solutions of the above system when V and W are constants can be extremal and this condition imply the boundedness of the energy and directly the compactness. Note that in [11], if we assume (in particular) that

$\nabla \log V$ and $\nabla \log W$ and $V > a > 0$ or $W > a' > 0$ and V, W are nonnegative and uniformly bounded then the energy is bounded and we have a compactness result.

Note that in the case of one equation (and $\beta = 0$), we can prove by using the Pohozaev identity that if $+\infty > b \geq V \geq a > 0$, ∇V is uniformly Lipschitzian that the energy is bounded when Ω is starshaped. In [15] Ma-Wei, using the moving-plane method showed that this fact is true for all domain Ω with the same assumptions on V . In [11] De Figueiredo-do O-Ruf extend this fact to a system by using the moving-plane method for a system.

Theorem C, shows that we have not a global compactness to the previous problem with one equation, perhaps we need more information on V to conclude to the boundedness of the solutions. When $\nabla \log V$ is Lipschitz function and $\beta = 0$, Chen-Li and Ma-Wei see [7] and [15], showed that we have a compactness on all the open set. The proof is via the moving plane-Method of Serrin and Gidas-Ni-Nirenberg. Note that in [11], we have the same result for this system when $\nabla \log V$ and $\nabla \log W$ are uniformly bounded. We will see below that for a system we also have a compactness result when V and W are Lipschitzian and $\beta \geq 0$.

Now consider the case of one equation. In this case our equation have nice properties.

If we assume V with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafirir see [17], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see [5].

Here we are interested by the case of a system of this type of equation. First, we give the behavior of the blow-up points on the boundary and in the second time we have a proof of compactness of the solutions to Gelfand-Liouville type system with a Hölderian condition.

Here, we write an extension of Brezis-Merle Problem (see [6]) to a system:

Problem. Suppose that $V_i \rightarrow V$ and $W_i \rightarrow W$ in $C^0(\bar{\Omega})$, with, $0 \leq V_i$ and $0 \leq W_i$. Also, we consider two sequences of solutions $(u_i), (v_i)$ of (P) relatively to $(V_i), (W_i)$ such that,

$$\int_{\Omega} e^{u_i} dx \leq C_1, \quad \int_{\Omega} e^{v_i} dx \leq C_2,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C_3 = C_3(\beta, C_1, C_2, \Omega)?$$

and,

$$\|v_i\|_{L^\infty} \leq C_4 = C_4(\beta, C_1, C_2, \Omega)?$$

In this paper we give a characterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when V_i and W_i are uniformly Lipschitzian. For the behavior of the blow-up points on the boundary, the following condition are enough,

$$0 \leq V_i \leq b_1, \quad 0 \leq W_i \leq b_2,$$

The conditions $V_i \rightarrow V$ and $W_i \rightarrow W$ in $C^0(\bar{\Omega})$ are not necessary.

But for the proof of the compactness for the Gelfand-Liouville type system (Brezis-Merle type problem) we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A_1, \quad \|\nabla W_i\|_{L^\infty} \leq A_2.$$

Our main result are:

Theorem 1.1. *Assume that $\max_{\Omega} u_i \rightarrow +\infty$ and $\max_{\Omega} v_i \rightarrow +\infty$ Where (u_i) and (v_i) are solutions of the probleme (P) with:*

$$0 \leq V_i \leq b_1, \quad \text{and} \quad \int_{\Omega} e^{u_i} dx \leq C_1, \quad \forall i,$$

and,

$$0 \leq W_i \leq b_2, \quad \text{and} \quad \int_{\Omega} e^{v_i} dx \leq C_2, \quad \forall i,$$

then; after passing to a subsequence, there is a function u , there is a number $N \in \mathbb{N}$ and N points $x_1, x_2, \dots, x_N \in \partial\Omega$, such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi + \sum_{j=1}^N \alpha_j \varphi(x_j), \quad \alpha_j \geq 4\pi,$$

for any $\varphi \in C^0(\partial\Omega)$, and,

$$u_i \rightarrow u \quad \text{in} \quad C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

$$\int_{\partial\Omega} \partial_\nu u_i \varphi \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi + \sum_{j=1}^N \beta_j \varphi(x_j), \quad \beta_j \geq 4\pi,$$

for any $\varphi \in C^0(\partial\Omega)$, and,

$$v_i \rightarrow v \text{ in } C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

In the following theorem, we have a proof for the global a priori estimate which concern the problem (P).

Theorem 1.2. *Assume that $(u_i), (v_i)$ are solutions of (P) relatively to $(V_i), (W_i)$ with the following conditions:*

$$x_1 = 0 \in \partial\Omega, \beta \in [0, 1/2),$$

and,

$$0 \leq V_i \leq b_1, \|\nabla V_i\|_{L^\infty} \leq A_1, \text{ and } \int_{\Omega} e^{u_i} \leq C_1,$$

$$0 \leq W_i \leq b_2, \|\nabla W_i\|_{L^\infty} \leq A_2, \text{ and } \int_{\Omega} e^{v_i} \leq C_2,$$

We have,

$$\|u_i\|_{L^\infty} \leq C_3(b_1, b_2, \beta, A_1, A_2, C_1, C_2, \Omega),$$

and,

$$\|v_i\|_{L^\infty} \leq C_4(b_1, b_2, \beta, A_1, A_2, C_1, C_2, \Omega),$$

2. PROOF OF THE THEOREMS

Proof of theorem 1.1:

We have:

$$u_i \in W_0^{1,1}(\Omega).$$

Since $e^{u_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [6]) we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

And,

We have:

$$v_i \in W_0^{1,1}(\Omega).$$

Since $e^{v_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [6]) we have $e^{v_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$v_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Since $V_i e^{v_i}$ and $W_i e^{u_i}$ are bounded in $L^1(\Omega)$, we can extract from those two sequences two subsequences which converge to two nonnegative measures μ_1 and μ_2 . (This procedure is similar to the procedure of Brezis-Merle, we apply corollary 4 of Brezis-Merle paper, see [6]).

If $\mu_1(x_0) < 4\pi$, by a Brezis-Merle estimate for the first equation, we have $e^{u_i} \in L^{1+\epsilon}$ around x_0 , by the elliptic estimates, for the second equation, we have $v_i \in W^{2,1+\epsilon} \subset L^\infty$ around x_0 , and, returning to the first equation, we have $u_i \in L^\infty$ around x_0 .

If $\mu_2(x_0) < 4\pi$, then u_i and v_i are also locally bounded around x_0 .

Thus, we take a look to the case when, $\mu_1(x_0) \geq 4\pi$ and $\mu_2(x_0) \geq 4\pi$. By our hypothesis, those points x_0 are finite.

We will see that inside Ω no such points exist. By contradiction, assume that, we have $\mu_1(x_0) \geq 4\pi$. Let us consider a ball $B_R(x_0)$ which contain only x_0 as nonregular point. Thus, on $\partial B_R(x_0)$, the two sequence u_i and v_i are uniformly bounded. Let us consider:

$$\begin{cases} -\Delta z_i = V_i e^{v_i} & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z_i = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

By the maximum principle we have:

$$z_i \leq u_i$$

and $z_i \rightarrow z$ almost everywhere on this ball, and thus,

$$\int e^{z_i} \leq \int e^{u_i} \leq C,$$

and,

$$\int e^z \leq C.$$

but, z is a solution in $W_0^{1,q}(B_R(x_0))$, $1 \leq q < 2$, of the following equation:

$$\begin{cases} -\Delta z = \mu_1 & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

with, $\mu_1 \geq 4\pi$ and thus, $\mu_1 \geq 4\pi\delta_{x_0}$ and then, by the maximum principle in $W_0^{1,q}(B_R(x_0))$:

$$z \geq -2 \log |x - x_0| + C$$

thus,

$$\int e^z = +\infty,$$

which is a contradiction. Thus, there is no nonregular points inside Ω

Thus, we consider the case where we have nonregular points on the boundary, we use two estimates:

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C_1, \quad \int_{\partial\Omega} \partial_\nu v_i d\sigma \leq C_2,$$

and,

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \|\nabla v_i\|_{L^q} \leq C'_q, \quad \forall i \text{ and } 1 < q < 2.$$

We have the same computations, as in the case of one equation.

We consider a points $x_0 \in \partial\Omega$ such that:

$$\mu_1(x_0) < 4\pi.$$

We consider a test function on the boundary η we extend η by a harmonic function on Ω , we write the equation:

$$-\Delta((u_i - u)\eta) = (1 + |x|^{2\beta})(V_i e^{v_i} - V e^v)\eta + \langle \nabla(u_i - u) | \nabla \eta \rangle = f_i$$

with,

$$\int |f_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

$$-\Delta((v_i - v)\eta) = (W_i e^{u_i} - W e^u)\eta + \langle \nabla(v_i - v) | \nabla \eta \rangle = g_i,$$

with,

$$\int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

By the Brezis-Merle estimate, we have uniformly, $e^{u_i} \in L^{1+\epsilon}$ around x_0 , by the elliptic estimates, for the second equation, we have $v_i \in W^{2,1+\epsilon} \subset L^\infty$ around x_0 , and, returning to the first equation, we have $u_i \in L^\infty$ around x_0 .

We have the same thing if we assume:

$$\mu_2(x_0) < 4\pi.$$

Thus, if $\mu_1(x_0) < 4\pi$ or $\mu_2(x_0) < 4\pi$, we have for $R > 0$ small enough:

$$(u_i, v_i) \in L^\infty(B_R(x_0) \cap \bar{\Omega}).$$

By our hypothesis the set of the points such that:

$$\mu_1(x_0) \geq 4\pi, \quad \mu_2(x_0) \geq 4\pi,$$

is finite, and, outside this set u_i and v_i are locally uniformly bounded. By the elliptic estimates, we have the C^1 convergence to u and v on each compact set of $\bar{\Omega} - \{x_1, \dots, x_N\}$.

Indeed,

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \mu_1(\varphi), \quad \forall \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu_1(x_0) < 4\pi$. For $\epsilon > 0$ small enough set $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$ on the unit disk or one can assume it as an interval. We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} -\Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

Remark: We use the following steps in the construction of $\tilde{\eta}_\epsilon$:

We take a cutoff function η_0 in $B(0, 2)$ or $B(x_0, 2)$:

1- We set $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart $(f, \tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x) - x_0| \leq k_2|x| \leq 1$ for $|x| \leq 1/k_2$ and $|f(x) - x_0| \geq k_1|x| \geq 2$ for $|x| \geq 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on a the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And, $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, since f is Lipschitz. Here H_1 is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} -\Delta \tilde{\eta}_\epsilon = -\Delta \eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finally we set $\tilde{\eta}_\epsilon = -\tilde{\eta}_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have :

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with C_1 depends on Ω .

We use the following estimate, see [8],

$$\|\nabla v_i\|_{L^q} \leq C_q, \|\nabla u_i\|_q \leq C_q, \forall i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (v_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $v \geq 0$ and $\int_{\Omega} e^v < +\infty$ (by Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞ .

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_{\Omega} e^u < +\infty$ (by Fatou lemma). Also, W_i weakly converge to a nonnegative function W in L^∞ .

The function u, v are in $W_0^{1,q}(\Omega)$ solutions of :

$$\begin{cases} -\Delta u = V(1 + |x|^{2\beta})e^v \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

And,

$$\begin{cases} -\Delta v = We^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle's result, see [6], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

According to the corollary 1 of Brezis-Merle's result, see [6], we have $e^{kv} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $v \in C^1(\bar{\Omega})$.

For two vectors f and g we denote by $f \cdot g$ the inner product of f and g .

We can write:

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (1 + |x|^{2\beta})(V_i e^{v_i} - V e^v)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon. \quad (1)$$

$$-\Delta((v_i - v)\tilde{\eta}_\epsilon) = (W_i e^{u_i} - W e^u)\tilde{\eta}_\epsilon - 2\nabla(v_i - v) \cdot \nabla\tilde{\eta}_\epsilon.$$

We use the interior estimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u , we obtain,

$$\int_{\Omega} (1 + |x|^{2\beta})V e^v \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \tilde{\eta}_\epsilon \leq C'\epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (2)$$

We have,

$$\begin{cases} -\Delta u_i = (1 + |x|^{2\beta})V_i e^{v_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_{\Omega} (1 + |x|^{2\beta})V_i e^{v_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \rightarrow \mu_1(\eta_\epsilon) \leq \mu_1(J'_\epsilon) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} |(1 + \gamma|x|^{2\beta})(V_i e^{v_i} - V e^v) \tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$.

Remark: for the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

(Proof of this fact; let's consider $d(x, \partial\Omega) = d(x, z_0)$, $z_0 \in \partial\Omega$, this imply that $(d(x, z_0))^2 \leq (d(x, z))^2$ for all $z \in \partial\Omega$ which it is equivalent to $(z - z_0) \cdot (2x - z - z_0) \leq 0$ for all $z \in \partial\Omega$, let's consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have;

$(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$ if we divide by $(t - t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$, this imply that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0)

With this fact, we can say that $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}$. It is sufficient to work on $\partial\Omega$. Let's consider a charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover $(B(z_k, \epsilon_k))$, $k = 1, \dots, m$, by the area formula the measure of $S \cap B(z_k, \epsilon_k)$ is less than a $k\epsilon$ (a ϵ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary.

We write,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx. \quad (5)$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $\|\nabla\tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in L^q , $1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle's theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q .

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1, \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3(k_1 \|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant C_1 does not depend on ϵ but on Ω .

Step 2.2: Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx$.

We know that, $\Omega_\epsilon \subset \subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla\tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (7)$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |-\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla v\|_{L^\infty} + 2 + C) \quad (8)$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \frac{\epsilon_0}{2}$.

We can use Theorem 1 of [6] to conclude that there are $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|} dx \leq \int_{\Omega} e^{q|u_i - u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0$, $q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i. \quad (9)$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$-\Delta(v_i \eta) = W_i e^{u_i} \eta - 2\nabla v_i \cdot \nabla \eta - v_i \Delta \eta.$$

By the elliptic estimates, $(v_i)_i$ is uniformly bounded in $L^\infty(V_\epsilon(x_0))$. Finally, we have, for some $\epsilon > 0$ small enough,

$$\|v_i\|_{C^{0,\theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$-\Delta(u_i \eta) = (1 + |x|^{2\beta})V_i e^{v_i} \eta - 2\nabla u_i \cdot \nabla \eta - u_i \Delta \eta.$$

By the elliptic estimates, $(u_i)_i$ is uniformly bounded in $L^\infty(V_\epsilon(x_0))$ and also in $C^{0,\theta}$ norm.

If we repeat this procedure another time, we have a boundedness of $(u_i)_i$ and $(v_i)_i$ in the $C^{1,\theta}$ norm, because they are bounded in $W^{2,q} \subset L^{q^*}$ norms with $2q/(2-q) = q^* > 2$.

We have the same computations and conclusion if we consider a regular point x_0 for the measure μ_2 .

We have proved that, there is a finite number of points $\bar{x}_1, \dots, \bar{x}_m$ such that the squence $(u_i)_i$ and $(v_i)_i$ are locally uniformly bounded (in $C^{1,\theta}$, $\theta > 0$) in $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$.

Proof of theorem 1.2:

Without loss of generality, we can assume that 0 is a blow-up point. Since the boundary is an analytic curve $\gamma_1(t)$, there is a neighborhood of 0 such that the curve γ_1 can be extend to a holomorphic map such that $\gamma_1'(0) \neq 0$ (series) and by the inverse mapping one can assume that this map is univalent around 0. In the case when the boundary is a simple Jordan curve the domain is simply connected. In the case that the domains has a finite number of holes it is conformally equivalent to a disk with a finite number of disks removed. Here we consider a general domain. Without loss of generality one can assume that $\gamma_1(B_1^+) \subset \Omega$ and also $\gamma_1(B_1^-) \subset (\Omega)^c$ and $\gamma_1(-1, 1) \subset \partial\Omega$ and γ_1 is univalent. This means that (B_1, γ_1) is a local chart around 0 for Ω and γ_1 univalent. (This fact holds if we assume that we have an analytic domain, (below a graph of an analytic function), we have necessary the condition $\partial\bar{\Omega} = \partial\Omega$ and the graph is analytic, in this case $\gamma_1(t) = (t, \varphi(t))$ with φ real analytic and an example of this fact is the unit disk around the point $(0, 1)$ for example).

By this conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which not contain 0 and on which u_i converge in the C^1 norm to u . Let us consider B_ϵ^+ , the half ball with radius $\epsilon > 0$. Also, one can consider a C^1 domain (a rectangle between two half disks) and by charts its image is a C^1 domain) We know that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Thus we can use integrations by parts (Stokes formula). The Pohozaev identity applied around the blow-up 0:

$$\int_{B_\epsilon^+} \Delta u_i \langle x | \nabla v_i \rangle dx = - \int_{B_\epsilon^+} \Delta v_i \langle x | \nabla u_i \rangle dx + \int_{\partial^+ B_\epsilon^+} g(\nabla u_i, \nabla v_i) d\sigma, \quad (10)$$

Thus,

$$\int_{B_\epsilon^+} V_i(1 + |x|^{2\beta})e^{v_i} \langle x|\nabla v_i \rangle dx = - \int_{B_\epsilon^+} W_i e^{u_i} \langle x|\nabla u_i \rangle dx - \int_{\partial^+ B_\epsilon^+} g(\nabla u_i, \nabla v_i) d\sigma, \quad (11)$$

After integration by parts, we obtain:

$$\begin{aligned} & \int_{B_\epsilon^+} 2V_i(1 + (1 + \beta)|x|^{2\beta})e^{v_i} dx + \int_{B_\epsilon^+} \langle x|\nabla V_i \rangle e^{v_i} dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle V_i d\sigma + \\ & + \int_{B_\epsilon^+} W_i e^{u_i} dx + \int_{B_\epsilon^+} \langle x|\nabla W_i \rangle e^{u_i} dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle W_i d\sigma = \\ & = - \int_{\partial^+ B_\epsilon^+} g(\nabla u_i, \nabla v_i) d\sigma, \end{aligned}$$

Also, for u and v , we have:

$$\begin{aligned} & \int_{B_\epsilon^+} 2V(1 + (1 + \beta)|x|^{2\beta})e^v dx + \int_{B_\epsilon^+} \langle x|\nabla V \rangle e^v dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle V d\sigma + \\ & + \int_{B_\epsilon^+} W e^u dx + \int_{B_\epsilon^+} \langle x|\nabla W \rangle e^u dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle W d\sigma = \\ & = - \int_{\partial^+ B_\epsilon^+} g(\nabla u, \nabla v) d\sigma, \end{aligned}$$

If, we take the difference, we obtain:

$$\begin{aligned} & (1 + o(\epsilon)) \left(\int_{B_\epsilon^+} V_i e^{v_i} dx - \int_{B_\epsilon^+} V e^v dx \right) + \\ & + (1 + o(\epsilon)) \left(\int_{B_\epsilon^+} W_i e^{u_i} dx - \int_{B_\epsilon^+} W e^u dx \right) = \\ & = \alpha_1 + \beta_1 + o(\epsilon) + o(1) = o(1), \end{aligned} \quad (12)$$

a contradiction.

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