



HAL
open science

A compactness result for a system with Holderian condition

Samy Skander Bahoura

► **To cite this version:**

Samy Skander Bahoura. A compactness result for a system with Holderian condition. 2022. hal-03782160

HAL Id: hal-03782160

<https://hal.science/hal-03782160>

Preprint submitted on 21 Sep 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A COMPACTNESS RESULT FOR A SYSTEM WITH HOLDERIAN CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give a quantization analysis to an elliptic system with Dirichlet condition. An application, we have a compactness result for an elliptic system with Hölderian condition.

Mathematics Subject Classification: 35J60 35B45 35B50

Keywords: blow-up, boundary, system, Dirichlet condition, a priori estimate, analytic domain, Hölder condition.

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open analytic domain Ω of \mathbb{R}^2 .

We consider the following equation:

$$(P) \begin{cases} -\Delta u = V(1 + |x|^{2\beta})e^v & \text{in } \Omega \subset \mathbb{R}^2, \\ -\Delta v = We^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b_1 < +\infty, e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega),$$

$$0 \leq W \leq b_2 < +\infty, e^v \in L^1(\Omega) \text{ and } v \in W_0^{1,1}(\Omega),$$

and,

$$0 \in \partial\Omega, \beta \in [0, 1/2).$$

When $u = v$, the above system is reduced to an equation which was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-17], one can find some existence and compactness results, also for a system.

Among other results, we can see in [6] the following important Theorem,

Theorem A.(Brezis-Merle [6]).*Consider the case of one equation; if $(u_i)_i = (v_i)_i$ and $(V_i)_i = (W_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

Theorem B (Brezis-Merle [6]).*Consider the case of one equation and assume that $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 \leq V_i \leq b < +\infty$, and,*

$$\int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact set K of Ω ,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

Next, we call energy the following quantity:

$$E = \int_{\Omega} e^{u_i} dy.$$

The boundedness of the energy is a necessary condition to work on the problem (P) as showed in [6], by the following counterexample.

Theorem C (Brezis-Merle [6]).*Consider the case of one equation, then there are two sequences $(u_i)_i$ and $(V_i)_i$ of the problem (P) with, $0 \leq V_i \leq b < +\infty$, and,*

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and

$$\sup_{\Omega} u_i \rightarrow +\infty.$$

When $\beta = 0$, the above system have many properties in the constant and the Lipschitzian cases. Indeed we have (when $\beta = 0$):

In [12], Dupaigne-Farina-Sirakov proved (by an existence result of Montenegro, see [16]) that the solutions of the above system when V and W are constants can be extremal and this condition imply the boundedness of the energy and directly the compactness. Note that in [11], if we assume (in particular) that

$\nabla \log V$ and $\nabla \log W$ and $V > a > 0$ or $W > a' > 0$ and V, W are nonnegative and uniformly bounded then the energy is bounded and we have a compactness result.

Note that in the case of one equation (and $\beta = 0$), we can prove by using the Pohozaev identity that if $+\infty > b \geq V \geq a > 0$, ∇V is uniformly Lipschitzian that the energy is bounded when Ω is starshaped. In [15] Ma-Wei, using the moving-plane method showed that this fact is true for all domain Ω with the same assumptions on V . In [11] De Figueiredo-do O-Ruf extend this fact to a system by using the moving-plane method for a system.

Theorem C, shows that we have not a global compactness to the previous problem with one equation, perhaps we need more information on V to conclude to the boundedness of the solutions. When $\nabla \log V$ is Lipschitz function and $\beta = 0$, Chen-Li and Ma-Wei see [7] and [15], showed that we have a compactness on all the open set. The proof is via the moving plane-Method of Serrin and Gidas-Ni-Nirenberg. Note that in [11], we have the same result for this system when $\nabla \log V$ and $\nabla \log W$ are uniformly bounded. We will see below that for a system we also have a compactness result when V and W are Lipschitzian and $\beta \geq 0$.

Now consider the case of one equation. In this case our equation have nice properties.

If we assume V with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafirir see [17], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see [5].

Here we are interested by the case of a system of this type of equation. First, we give the behavior of the blow-up points on the boundary and in the second time we have a proof of compactness of the solutions to Gelfand-Liouville type system with a Hölderian condition.

Here, we write an extension of Brezis-Merle Problem (see [6]) to a system:

Problem. Suppose that $V_i \rightarrow V$ and $W_i \rightarrow W$ in $C^0(\bar{\Omega})$, with, $0 \leq V_i$ and $0 \leq W_i$. Also, we consider two sequences of solutions $(u_i), (v_i)$ of (P) relatively to $(V_i), (W_i)$ such that,

$$\int_{\Omega} e^{u_i} dx \leq C_1, \quad \int_{\Omega} e^{v_i} dx \leq C_2,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C_3 = C_3(\beta, C_1, C_2, \Omega)?$$

and,

$$\|v_i\|_{L^\infty} \leq C_4 = C_4(\beta, C_1, C_2, \Omega)?$$

In this paper we give a characterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when V_i and W_i are uniformly Lipschitzian. For the behavior of the blow-up points on the boundary, the following condition are enough,

$$0 \leq V_i \leq b_1, \quad 0 \leq W_i \leq b_2,$$

The conditions $V_i \rightarrow V$ and $W_i \rightarrow W$ in $C^0(\bar{\Omega})$ are not necessary.

But for the proof of the compactness for the Gelfand-Liouville type system (Brezis-Merle type problem) we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A_1, \quad \|\nabla W_i\|_{L^\infty} \leq A_2.$$

Our main result are:

Theorem 1.1. *Assume that $\max_{\Omega} u_i \rightarrow +\infty$ and $\max_{\Omega} v_i \rightarrow +\infty$ Where (u_i) and (v_i) are solutions of the probleme (P) with:*

$$0 \leq V_i \leq b_1, \quad \text{and} \quad \int_{\Omega} e^{u_i} dx \leq C_1, \quad \forall i,$$

and,

$$0 \leq W_i \leq b_2, \quad \text{and} \quad \int_{\Omega} e^{v_i} dx \leq C_2, \quad \forall i,$$

then; after passing to a subsequence, there is a function u , there is a number $N \in \mathbb{N}$ and N points $x_1, x_2, \dots, x_N \in \partial\Omega$, such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi + \sum_{j=1}^N \alpha_j \varphi(x_j), \quad \alpha_j \geq 4\pi,$$

for any $\varphi \in C^0(\partial\Omega)$, and,

$$u_i \rightarrow u \quad \text{in} \quad C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

$$\int_{\partial\Omega} \partial_\nu u_i \varphi \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi + \sum_{j=1}^N \beta_j \varphi(x_j), \quad \beta_j \geq 4\pi,$$

for any $\varphi \in C^0(\partial\Omega)$, and,

$$v_i \rightarrow v \text{ in } C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

In the following theorem, we have a proof for the global a priori estimate which concern the problem (P).

Theorem 1.2. *Assume that $(u_i), (v_i)$ are solutions of (P) relatively to $(V_i), (W_i)$ with the following conditions:*

$$x_1 = 0 \in \partial\Omega, \beta \in [0, 1/2),$$

and,

$$0 \leq V_i \leq b_1, \|\nabla V_i\|_{L^\infty} \leq A_1, \text{ and } \int_{\Omega} e^{u_i} \leq C_1,$$

$$0 \leq W_i \leq b_2, \|\nabla W_i\|_{L^\infty} \leq A_2, \text{ and } \int_{\Omega} e^{v_i} \leq C_2,$$

We have,

$$\|u_i\|_{L^\infty} \leq C_3(b_1, b_2, \beta, A_1, A_2, C_1, C_2, \Omega),$$

and,

$$\|v_i\|_{L^\infty} \leq C_4(b_1, b_2, \beta, A_1, A_2, C_1, C_2, \Omega),$$

2. PROOF OF THE THEOREMS

Proof of theorem 1.1:

We have:

$$u_i \in W_0^{1,1}(\Omega).$$

Since $e^{u_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [6]) we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

And,

We have:

$$v_i \in W_0^{1,1}(\Omega).$$

Since $e^{v_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [6]) we have $e^{v_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$v_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Since $V_i e^{v_i}$ and $W_i e^{u_i}$ are bounded in $L^1(\Omega)$, we can extract from those two sequences two subsequences which converge to two nonnegative measures μ_1 and μ_2 . (This procedure is similar to the procedure of Brezis-Merle, we apply corollary 4 of Brezis-Merle paper, see [6]).

If $\mu_1(x_0) < 4\pi$, by a Brezis-Merle estimate for the first equation, we have $e^{u_i} \in L^{1+\epsilon}$ around x_0 , by the elliptic estimates, for the second equation, we have $v_i \in W^{2,1+\epsilon} \subset L^\infty$ around x_0 , and, returning to the first equation, we have $u_i \in L^\infty$ around x_0 .

If $\mu_2(x_0) < 4\pi$, then u_i and v_i are also locally bounded around x_0 .

Thus, we take a look to the case when, $\mu_1(x_0) \geq 4\pi$ and $\mu_2(x_0) \geq 4\pi$. By our hypothesis, those points x_0 are finite.

We will see that inside Ω no such points exist. By contradiction, assume that, we have $\mu_1(x_0) \geq 4\pi$. Let us consider a ball $B_R(x_0)$ which contain only x_0 as nonregular point. Thus, on $\partial B_R(x_0)$, the two sequence u_i and v_i are uniformly bounded. Let us consider:

$$\begin{cases} -\Delta z_i = V_i e^{v_i} & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z_i = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

By the maximum principle we have:

$$z_i \leq u_i$$

and $z_i \rightarrow z$ almost everywhere on this ball, and thus,

$$\int e^{z_i} \leq \int e^{u_i} \leq C,$$

and,

$$\int e^z \leq C.$$

but, z is a solution in $W_0^{1,q}(B_R(x_0))$, $1 \leq q < 2$, of the following equation:

$$\begin{cases} -\Delta z = \mu_1 & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

with, $\mu_1 \geq 4\pi$ and thus, $\mu_1 \geq 4\pi\delta_{x_0}$ and then, by the maximum principle in $W_0^{1,q}(B_R(x_0))$:

$$z \geq -2 \log |x - x_0| + C$$

thus,

$$\int e^z = +\infty,$$

which is a contradiction. Thus, there is no nonregular points inside Ω

Thus, we consider the case where we have nonregular points on the boundary, we use two estimates:

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C_1, \quad \int_{\partial\Omega} \partial_\nu v_i d\sigma \leq C_2,$$

and,

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \|\nabla v_i\|_{L^q} \leq C'_q, \quad \forall i \text{ and } 1 < q < 2.$$

We have the same computations, as in the case of one equation.

We consider a points $x_0 \in \partial\Omega$ such that:

$$\mu_1(x_0) < 4\pi.$$

We consider a test function on the boundary η we extend η by a harmonic function on Ω , we write the equation:

$$-\Delta((u_i - u)\eta) = (1 + |x|^{2\beta})(V_i e^{v_i} - V e^v)\eta + \langle \nabla(u_i - u) | \nabla \eta \rangle = f_i$$

with,

$$\int |f_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

$$-\Delta((v_i - v)\eta) = (W_i e^{u_i} - W e^u)\eta + \langle \nabla(v_i - v) | \nabla \eta \rangle = g_i,$$

with,

$$\int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

By the Brezis-Merle estimate, we have uniformly, $e^{u_i} \in L^{1+\epsilon}$ around x_0 , by the elliptic estimates, for the second equation, we have $v_i \in W^{2,1+\epsilon} \subset L^\infty$ around x_0 , and, returning to the first equation, we have $u_i \in L^\infty$ around x_0 .

We have the same thing if we assume:

$$\mu_2(x_0) < 4\pi.$$

Thus, if $\mu_1(x_0) < 4\pi$ or $\mu_2(x_0) < 4\pi$, we have for $R > 0$ small enough:

$$(u_i, v_i) \in L^\infty(B_R(x_0) \cap \bar{\Omega}).$$

By our hypothesis the set of the points such that:

$$\mu_1(x_0) \geq 4\pi, \quad \mu_2(x_0) \geq 4\pi,$$

is finite, and, outside this set u_i and v_i are locally uniformly bounded. By the elliptic estimates, we have the C^1 convergence to u and v on each compact set of $\bar{\Omega} - \{x_1, \dots, x_N\}$.

Indeed,

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \mu_1(\varphi), \quad \forall \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu_1(x_0) < 4\pi$. For $\epsilon > 0$ small enough set $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$ on the unit disk or one can assume it as an interval. We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} -\Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

Remark: We use the following steps in the construction of $\tilde{\eta}_\epsilon$:

We take a cutoff function η_0 in $B(0, 2)$ or $B(x_0, 2)$:

1- We set $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart $(f, \tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x) - x_0| \leq k_2|x| \leq 1$ for $|x| \leq 1/k_2$ and $|f(x) - x_0| \geq k_1|x| \geq 2$ for $|x| \geq 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on a the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And, $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, since f is Lipschitz. Here H_1 is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} -\Delta \tilde{\eta}_\epsilon = -\Delta \eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finally we set $\tilde{\eta}_\epsilon = -\tilde{\eta}_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have :

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with C_1 depends on Ω .

We use the following estimate, see [8],

$$\|\nabla v_i\|_{L^q} \leq C_q, \|\nabla u_i\|_q \leq C_q, \forall i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (v_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $v \geq 0$ and $\int_{\Omega} e^v < +\infty$ (by Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞ .

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_{\Omega} e^u < +\infty$ (by Fatou lemma). Also, W_i weakly converge to a nonnegative function W in L^∞ .

The function u, v are in $W_0^{1,q}(\Omega)$ solutions of :

$$\begin{cases} -\Delta u = V(1 + |x|^{2\beta})e^v \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

And,

$$\begin{cases} -\Delta v = We^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle's result, see [6], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

According to the corollary 1 of Brezis-Merle's result, see [6], we have $e^{kv} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $v \in C^1(\bar{\Omega})$.

For two vectors f and g we denote by $f \cdot g$ the inner product of f and g .

We can write:

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (1 + |x|^{2\beta})(V_i e^{v_i} - V e^v)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon. \quad (1)$$

$$-\Delta((v_i - v)\tilde{\eta}_\epsilon) = (W_i e^{u_i} - W e^u)\tilde{\eta}_\epsilon - 2\nabla(v_i - v) \cdot \nabla\tilde{\eta}_\epsilon.$$

We use the interior estimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u , we obtain,

$$\int_{\Omega} (1 + |x|^{2\beta})V e^v \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \tilde{\eta}_\epsilon \leq C' \epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (2)$$

We have,

$$\begin{cases} -\Delta u_i = (1 + |x|^{2\beta})V_i e^{v_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_{\Omega} (1 + |x|^{2\beta})V_i e^{v_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \rightarrow \mu_1(\eta_\epsilon) \leq \mu_1(J'_\epsilon) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} |(1 + \gamma|x|^{2\beta})(V_i e^{v_i} - V e^v) \tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$.

Remark: for the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

(Proof of this fact; let's consider $d(x, \partial\Omega) = d(x, z_0)$, $z_0 \in \partial\Omega$, this imply that $(d(x, z_0))^2 \leq (d(x, z))^2$ for all $z \in \partial\Omega$ which it is equivalent to $(z - z_0) \cdot (2x - z - z_0) \leq 0$ for all $z \in \partial\Omega$, let's consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have;

$(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$ if we divide by $(t - t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$, this imply that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0)

With this fact, we can say that $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}$. It is sufficient to work on $\partial\Omega$. Let's consider a charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover $(B(z_k, \epsilon_k))$, $k = 1, \dots, m$, by the area formula the measure of $S \cap B(z_k, \epsilon_k)$ is less than a $k\epsilon$ (a ϵ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary.

We write,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx. \quad (5)$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $\|\nabla\tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in L^q , $1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle's theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q .

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1, \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3(k_1 \|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant C_1 does not depend on ϵ but on Ω .

Step 2.2: Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx$.

We know that, $\Omega_\epsilon \subset \subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla\tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (7)$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |-\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla v\|_{L^\infty} + 2 + C) \quad (8)$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \frac{\epsilon_0}{2}$.

We can use Theorem 1 of [6] to conclude that there are $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|} dx \leq \int_{\Omega} e^{q|u_i - u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0$, $q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i. \quad (9)$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$-\Delta(v_i \eta) = W_i e^{u_i} \eta - 2\nabla v_i \cdot \nabla \eta - v_i \Delta \eta.$$

By the elliptic estimates, $(v_i)_i$ is uniformly bounded in $L^\infty(V_\epsilon(x_0))$. Finally, we have, for some $\epsilon > 0$ small enough,

$$\|v_i\|_{C^{0,\theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$-\Delta(u_i \eta) = (1 + |x|^{2\beta})V_i e^{v_i} \eta - 2\nabla u_i \cdot \nabla \eta - u_i \Delta \eta.$$

By the elliptic estimates, $(u_i)_i$ is uniformly bounded in $L^\infty(V_\epsilon(x_0))$ and also in $C^{0,\theta}$ norm.

If we repeat this procedure another time, we have a boundedness of $(u_i)_i$ and $(v_i)_i$ in the $C^{1,\theta}$ norm, because they are bounded in $W^{2,q} \subset L^{q^*}$ norms with $2q/(2-q) = q^* > 2$.

We have the same computations and conclusion if we consider a regular point x_0 for the measure μ_2 .

We have proved that, there is a finite number of points $\bar{x}_1, \dots, \bar{x}_m$ such that the sequence $(u_i)_i$ and $(v_i)_i$ are locally uniformly bounded (in $C^{1,\theta}$, $\theta > 0$) in $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$.

Proof of theorem 1.2:

Without loss of generality, we can assume that 0 is a blow-up point. Since the boundary is an analytic curve $\gamma_1(t)$, there is a neighborhood of 0 such that the curve γ_1 can be extend to a holomorphic map such that $\gamma_1'(0) \neq 0$ (series) and by the inverse mapping one can assume that this map is univalent around 0. In the case when the boundary is a simple Jordan curve the domain is simply connected. In the case that the domains has a finite number of holes it is conformally equivalent to a disk with a finite number of disks removed. Here we consider a general domain. Without loss of generality one can assume that $\gamma_1(B_1^+) \subset \Omega$ and also $\gamma_1(B_1^-) \subset (\Omega)^c$ and $\gamma_1(-1, 1) \subset \partial\Omega$ and γ_1 is univalent. This means that (B_1, γ_1) is a local chart around 0 for Ω and γ_1 univalent. (This fact holds if we assume that we have an analytic domain, (below a graph of an analytic function), we have necessary the condition $\partial\bar{\Omega} = \partial\Omega$ and the graph is analytic, in this case $\gamma_1(t) = (t, \varphi(t))$ with φ real analytic and an example of this fact is the unit disk around the point $(0, 1)$ for example).

By this conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which not contain 0 and on which u_i converge in the C^1 norm to u . Let us consider B_ϵ^+ , the half ball with radius $\epsilon > 0$. Also, one can consider a C^1 domain (a rectangle between two half disks) and by charts its image is a C^1 domain) We know that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Thus we can use integrations by parts (Stokes formula). The Pohozaev identity applied around the blow-up 0:

$$\int_{B_\epsilon^+} \Delta u_i \langle x | \nabla v_i \rangle dx = - \int_{B_\epsilon^+} \Delta v_i \langle x | \nabla u_i \rangle dx + \int_{\partial^+ B_\epsilon^+} g(\nabla u_i, \nabla v_i) d\sigma, \quad (10)$$

Thus,

$$\int_{B_\epsilon^+} V_i(1 + |x|^{2\beta})e^{v_i} \langle x|\nabla v_i \rangle dx = - \int_{B_\epsilon^+} W_i e^{u_i} \langle x|\nabla u_i \rangle dx - \int_{\partial^+ B_\epsilon^+} g(\nabla u_i, \nabla v_i) d\sigma, \quad (11)$$

After integration by parts, we obtain:

$$\begin{aligned} & \int_{B_\epsilon^+} 2V_i(1 + (1 + \beta)|x|^{2\beta})e^{v_i} dx + \int_{B_\epsilon^+} \langle x|\nabla V_i \rangle e^{v_i} dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle V_i d\sigma + \\ & + \int_{B_\epsilon^+} W_i e^{u_i} dx + \int_{B_\epsilon^+} \langle x|\nabla W_i \rangle e^{u_i} dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle W_i d\sigma = \\ & = - \int_{\partial^+ B_\epsilon^+} g(\nabla u_i, \nabla v_i) d\sigma, \end{aligned}$$

Also, for u and v , we have:

$$\begin{aligned} & \int_{B_\epsilon^+} 2V(1 + (1 + \beta)|x|^{2\beta})e^v dx + \int_{B_\epsilon^+} \langle x|\nabla V \rangle e^v dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle V d\sigma + \\ & + \int_{B_\epsilon^+} W e^u dx + \int_{B_\epsilon^+} \langle x|\nabla W \rangle e^u dx + \int_{\partial B_\epsilon^+} \langle \nu|x \rangle W d\sigma = \\ & = - \int_{\partial^+ B_\epsilon^+} g(\nabla u, \nabla v) d\sigma, \end{aligned}$$

If, we take the difference, we obtain:

$$\begin{aligned} & (1 + o(\epsilon)) \left(\int_{B_\epsilon^+} V_i e^{v_i} dx - \int_{B_\epsilon^+} V e^v dx \right) + \\ & + (1 + o(\epsilon)) \left(\int_{B_\epsilon^+} W_i e^{u_i} dx - \int_{B_\epsilon^+} W e^u dx \right) = \\ & = \alpha_1 + \beta_1 + o(\epsilon) + o(1) = o(1), \end{aligned} \quad (12)$$

a contradiction.

REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.
- [2] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.
- [3] Bartolucci, D. A "sup+Cinf" inequality for Liouville-type equations with singular potentials. Math. Nachr. 284 (2011), no. 13, 1639-1651.
- [4] Bartolucci, D. A 'sup+Cinf' inequality for the equation $-\Delta u = Ve^u/|x|^{2\alpha}$. Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), no. 6, 1119-1139
- [5] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [6] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).
- [7] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.
- [8] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L^1 . J. Math. Soc. Japan 25 (1973), 565-590.
- [9] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [10] D.G. De Figueiredo, P.L. Lions, R.D. Nussbaum, A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations, J. Math. Pures et Appl., vol 61, 1982, pp.41-63.
- [11] D.G. De Figueiredo. J. M. do O. B. Ruf. Semilinear elliptic systems with exponential nonlinearities in two dimensions. Adv. Nonlinear Stud. 6 (2006), no. 2, 199-213.
- [12] Dupaigne, L. Farina, A. Sirakov, B. Regularity of the extremal solutions for the Liouville system. Geometric partial differential equations, 139-144, CRM Series, 15, Ed. Norm., Pisa, 2013.
- [13] YY. Li, I. Shafrir. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.
- [14] YY. Li. Harnack Type Inequality: the method of moving planes. Commun. Math. Phys. 200,421-444 (1999).
- [15] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.
- [16] Montenegro. M. Minimal solutions for a class of elliptic systems. Bull. London. Math. Soc. 37 (2005), no. 3, 405-416.
- [17] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

DEPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PIERRE ET MARIE CURIE, 2 PLACE JUSSIEU, 75005, PARIS, FRANCE.

Email address: samybahoura@yahoo.fr, samybahoura@gmail.com