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# QUANTITATIVE STABILITY OF BARYCENTERS IN THE WASSERSTEIN SPACE

GUILLAUME CARLIER, ALEX DELALANDE, AND QUENTIN MÉRIGOT

ABSTRACT. Wasserstein barycenters define averages of probability measures in a geometrically meaningful way. Their use is increasingly popular in applied fields, such as image, geometry or language processing. In these fields however, the probability measures of interest are often not accessible in their entirety and the practitioner may have to deal with statistical or computational approximations instead. In this article, we quantify the effect of such approximations on the corresponding barycenters. We show that Wasserstein barycenters depend in a Hölder-continuous way on their marginals under relatively mild assumptions. Our proof relies on recent estimates that quantify the strong convexity of the dual quadratic optimal transport problem and a new result that allows to control the modulus of continuity of the push-forward operation under a (not necessarily smooth) optimal transport map.

## 1. INTRODUCTION

Wasserstein barycenters are Fréchet means in Wasserstein spaces: they define averages of families of probability measures that are consistent with the optimal transport geometry. As such, they average out probability measures in a geometrically meaningful way and appear as a relevant tool to interpolate or summarize measure data. Such notion of barycenter have indeed found many successful applications, for instance in image processing [29], geometry processing [31], language processing [17, 14, 25], statistics [32] or machine learning [15, 21]. We refer the readers to existing surveys [28, 26] for further applications. In such applications however, the probability measures of interest are often not accessible in their entirety. They may be accessible for instance only through noisy samples in a statistical context, or they may be approximated in order to use existing computational methods that estimate Wasserstein barycenters (see e.g. [12, 4, 15, 3]) while paying an affordable computational cost. This means that in addition to the computational error induced by the algorithm used to calculate the barycenter, the practitioner may be subject to an extra statistical or approximation error that corresponds to the approximation of the marginal measures of interest. While works focusing on the computation of Wasserstein barycenters may now come with guarantees on the first type of error (see e.g. [3]), very little is known on the second type of error, which corresponds broadly speaking to a stability error since it quantifies the effect of a perturbation of the marginals on the corresponding barycenters. In this work, we focus on this kind of error and show that the Wasserstein barycenter depends in an Hölder-continuous way on its marginal measures under regularity assumptions on (some of) the latter. In the remaining of this section, we define Wasserstein barycenters and the setting we focus on. We then show that mild regularity assumptions are necessary in order to hope for any stability result. Next, we give the dual formulation of the Wasserstein barycenter problem in our context, that is necessary to present our main assumption. This assumption and our main result are then stated and we give some immediate but useful

consequences of this result. We conclude this section with the principal elements of proof to our main result.

**1.1. Wasserstein barycenters.** Introduced in [1] for finite families of probability measures supported over a Euclidean space, the definition of Wasserstein barycenters have been extended to infinite families of probability measures in [7, 27], possibly supported over a Riemannian manifold in [22, 24]. In this work, we focus on families of probability measures supported over a compact Euclidean domain. Let  $\Omega = B(0, R) \subset \mathbb{R}^d$  be the ball of  $\mathbb{R}^d$  centered at zero and of radius  $R > 0$  and denote  $\mathcal{P}(\Omega)$  the set of Borel probability measures over  $\Omega$ . We endow  $\mathcal{P}(\Omega)$  with the 2-Wasserstein distance  $W_2$  defined for any  $\rho, \mu \in \mathcal{P}(\Omega)$  by

$$W_2(\rho, \mu) = \left( \min_{\gamma \in \Pi(\rho, \mu)} \int_{\Omega \times \Omega} \|x - y\|^2 d\gamma(x, y) \right)^{1/2},$$

where the minimum is taken over the set  $\Pi(\rho, \mu)$  of transport plans between  $\rho$  and  $\mu$ . We equip  $\mathcal{P}(\Omega)$  with the topology induced by  $W_2$  (i.e. the weak topology) and denote  $\mathcal{P}(\mathcal{P}(\Omega))$  the set of corresponding Borel probability measures over  $\mathcal{P}(\Omega)$ . A Wasserstein barycenter of  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$  is then defined as a minimizer  $\mu_{\mathbb{P}}$  of

$$\min \left\{ \frac{1}{2} \int_{\mathcal{P}(\Omega)} W_2^2(\rho, \mu) d\mathbb{P}(\rho), \quad \mu \in \mathcal{P}(\Omega) \right\} = (\mathbb{P})_{\mathbb{P}}.$$

Such a minimizer always exists, and it is uniquely defined whenever  $\mathbb{P}(\mathcal{P}_{a.c.}(\Omega) > 0)$ , where  $\mathcal{P}_{a.c.}(\Omega)$  denotes the set of probability measures over  $\Omega$  that are absolutely continuous with respect to the Lebesgue measure [22, 24].

**1.2. Stability of Wasserstein barycenters.** As mentioned above, the population of interest  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$  may not always be accessible in practice, and one may have to deal with another measure  $\mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$  instead. The stability question that then comes up is the following: can we bound a distance between minimizers  $\mu_{\mathbb{P}}$  of  $(\mathbb{P})_{\mathbb{P}}$  and  $\mu_{\mathbb{Q}}$  of  $(\mathbb{Q})_{\mathbb{Q}}$  in terms of a distance between  $\mathbb{P}$  and  $\mathbb{Q}$ ? While the above-defined 2-Wasserstein distance gives a natural metric to compare  $\mu_{\mathbb{P}}$  and  $\mu_{\mathbb{Q}}$ , there remains to choose a metric in order to compare  $\mathbb{P}$  and  $\mathbb{Q}$ . For this, we will use the following 1-Wasserstein distance over  $\mathcal{P}(\mathcal{P}(\Omega))$ , defined for any  $\mathbb{P}, \mathbb{Q}$  in  $\mathcal{P}(\mathcal{P}(\Omega))$  by

$$\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) = \min_{\gamma \in \Pi(\mathbb{P}, \mathbb{Q})} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} W_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho}).$$

This choice of distance is justified by the fact that Wasserstein distances are naturally defined for probability measures on the compact metric space  $(\mathcal{P}(\Omega), W_2)$  and that they allow to compare measures that have incomparable support. The 1-Wasserstein distance being the weakest of the Wasserstein distances, our bounds are ensured to be the sharpest in terms of this optimal transport geometry. We are thus interested in bounding  $W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}})$  in terms of  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q})$  for  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$ .

**1.2.1. Consistency of Wasserstein barycenters.** Before looking for any quantitative stability result, one may first wonder if barycenters depend at least in a continuous way on their marginals. This question, framed under the notion of *consistency* of Wasserstein barycenters, has been answered positively in [7, 8] in some specific settings and in [24] in the most general setting. Theorem 3 of [24] ensures in particular the following:

**Theorem** (Le Gouic, Loubes). *Let  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$  and a sequence  $(\mathbb{P}_n)_{n \geq 1} \in \mathcal{P}(\mathcal{P}(\Omega))$  be such that*

$$\mathcal{W}_1(\mathbb{P}_n, \mathbb{P}) \xrightarrow{n \rightarrow +\infty} 0.$$

*For all  $n \geq 1$ , denote  $\mu_{\mathbb{P}_n}$  a barycenter of  $\mathbb{P}_n$ . Then the sequence  $(\mu_{\mathbb{P}_n})_{n \geq 1}$  is precompact in  $(\mathcal{P}(\Omega), W_2)$  and any limit is a barycenter of  $\mathbb{P}$ .*

This result ensures the continuity of Wasserstein barycenters with respect to the marginal measures, at least in our setting, so that we can now legitimately look for bounds that quantify this continuity.

1.2.2. *Quantitative stability in dimension  $d = 1$ .* In dimension  $d = 1$ , the derivation of quantitative stability bounds for Wasserstein barycenters is straightforward. Indeed, in such setting  $W_2$  is Hilbertian, which ensures a Lipschitz behavior of the barycenters with respect to their marginals. More precisely, denoting  $F_\rho^{-1}$  the *quantile function* of a measure  $\rho \in \mathcal{P}(\Omega)$  (i.e. the generalized inverse of its cumulative distribution function  $F_\rho$ ), one has for any measures  $\rho, \mu \in \mathcal{P}(\Omega)$ ,  $W_2(\rho, \mu) = \|F_\rho^{-1} - F_\mu^{-1}\|_{L^2([0,1])}$ . This leads for any  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$  to a simple formula for the unique barycenter:

$$\mu_{\mathbb{P}} = \left( \int_{\mathcal{P}(\Omega)} F_\rho^{-1} d\mathbb{P}(\rho) \right)_{\#} \lambda_{[0,1]},$$

where  $\lambda_{[0,1]}$  denotes the Lebesgue measure over  $[0, 1]$ . Using this fact and the triangle inequality, one immediately obtains the following Lipschitz stability result, that actually holds for any families of measures in the set  $\mathcal{P}_2(\mathbb{R})$  of probability measures supported over  $\mathbb{R}$  that admit a finite second moment:

**Proposition.** *Let  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}_2(\mathbb{R}))$  and denote  $\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}$  their respective barycenters. Then*

$$W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) \leq \mathcal{W}_1(\mathbb{P}, \mathbb{Q}).$$

This fact was exploited in [6] to characterize the statistical rate of convergence of empirical Wasserstein barycenters towards their population counterpart in an asymptotic setting for probability measures supported over the real line.

1.2.3. *Quantitative stability in dimension  $d \geq 2$ .* In dimension  $d \geq 2$ , the derivation of any quantitative stability bound turns out to be much more difficult. This may first come from the fact that without any assumption on  $\mathbb{P}$  and  $\mathbb{Q}$ , the barycenters  $\mu_{\mathbb{P}}$  and  $\mu_{\mathbb{Q}}$  may not be uniquely defined, which makes hopeless the derivation of any stability result. Even when uniqueness of the barycenters is ensured, one can easily build examples where no quantitative stability bound holds, see for instance the setting illustrated in Figure 1. This example relies on barycenters with only discrete marginals, and recovers in the limit  $\varepsilon = 0$  the pathological case where the barycenter is not uniquely defined. One may circumvent this issue by ensuring, even in the limit  $\varepsilon = 0$ , uniqueness of the barycenter. As mentioned above, this can be done by imposing that some of the marginal measures are absolutely continuous. Nevertheless, even under such assumption on the marginals, one can easily build an example where the barycenter achieves an Hölder behavior with respect to its marginal, but with an arbitrary low Hölder exponent, see Figure 2. These negative results show that, even in dimension  $d = 2$ , some regularity assumptions on the marginals  $\mathbb{P}, \mathbb{Q}$  are necessary in order to hope to derive stability estimates for their barycenters.

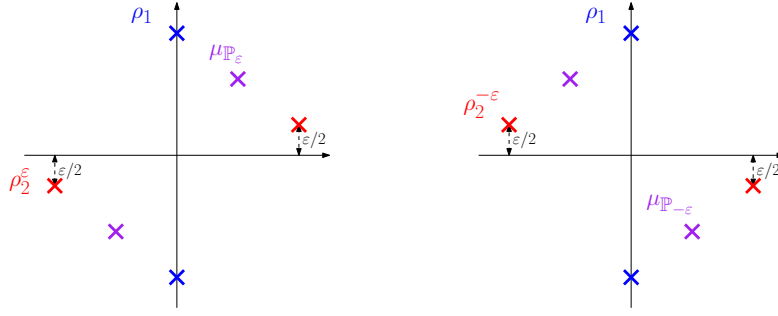


FIGURE 1. Let  $\rho_1 = \frac{1}{2}(\delta_{(0;1)} + \delta_{(0;-1)})$ . For  $\varepsilon > 0$  and  $x_\varepsilon = (1; \varepsilon/2) \in \mathbb{R}^2$ , let  $\rho_2^\varepsilon = \frac{1}{2}(\delta_{x^\varepsilon} + \delta_{-x^\varepsilon})$ . Introduce  $\mathbb{P}_\varepsilon = \frac{1}{2}(\delta_{\rho_1} + \delta_{\rho_2^\varepsilon})$ . Then for  $\varepsilon \leq \frac{1}{2}$ ,  $W_2(\mu_{\mathbb{P}_\varepsilon}, \mu_{\mathbb{P}_{-\varepsilon}}) = 1$  while  $W_1(\mathbb{P}_\varepsilon, \mathbb{P}_{-\varepsilon}) \leq \varepsilon$ .

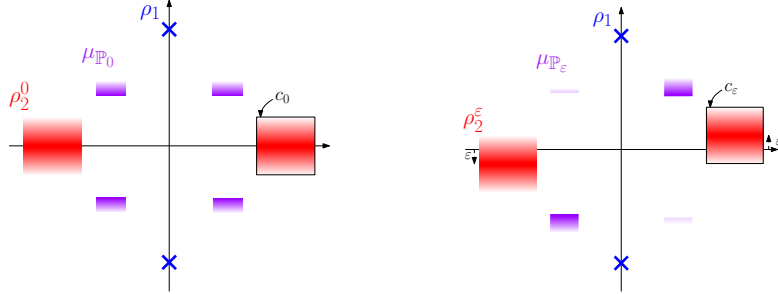


FIGURE 2. Let  $\rho_1 = \frac{1}{2}(\delta_{(0;1)} + \delta_{(0;-1)})$ . For  $a \in (0, 1)$  and  $\varepsilon > 0$ , let  $c_\varepsilon = [1 - \frac{a}{2}; 1 + \frac{a}{2}] \times [-\frac{a}{2} + \varepsilon; \frac{a}{2} + \varepsilon]$  and  $\rho_2^\varepsilon$  the probability measure with density  $\rho_2^\varepsilon(x, y) = \frac{\alpha}{2^{1-2\alpha}a^{1+2\alpha}} \left( |y - \varepsilon|^{2\alpha-1} \mathbf{1}_{c_\varepsilon}(x, y) + |y + \varepsilon|^{2\alpha-1} \mathbf{1}_{-c_\varepsilon}(x, y) \right)$  for some  $\alpha > 0$ . Introduce  $\mathbb{P}_\varepsilon = \frac{1}{2}(\delta_{\rho_1} + \delta_{\rho_2^\varepsilon})$ . Then for  $\varepsilon \leq \frac{a}{2}$ ,  $W_2(\mu_{\mathbb{P}_0}, \mu_{\mathbb{P}_\varepsilon}) \sim \varepsilon^\alpha$  while  $W_1(\mathbb{P}_0, \mathbb{P}_\varepsilon) \leq \varepsilon$ .

1.2.4. *Previous works.* Consistently with the above remarks, previous works having dealt with the stability of Wasserstein barycenters have either worked under stringent assumptions on the marginal measures or regularized the barycenter problem in order to ensure more regular solutions. In [2, 20] for instance, the question of the rate of convergence of the empirical barycenter in a Wasserstein space towards its population counterpart has been answered at the cost of assumptions that require in particular to have guarantees on the regularity of the (unknown) population barycenter (see sub-section 1.5.2 for more details). In [5, 11], a regularization of the barycenter functional has been considered and stability bounds and central limit theorems were deduced for the solutions to this regularized problem. In this work, we do not regularize the barycenter functional and work under less restrictive assumptions on the marginal measures than previous works having dealt with the stability of Wasserstein barycenters. In order to state these assumptions, we first need to introduce the dual problem to  $(P)_{\mathbb{P}}$ .

1.3. **Dual formulation.** Building from [1], we show that  $(P)_{\mathbb{P}}$  admits the following dual formulation with strong duality (the proof is deferred to the appendix, Section A):

**Proposition 1.1** (Dual formulation). *For any  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ , problem  $(P)_{\mathbb{P}}$  satisfies*

$$(P)_{\mathbb{P}} = \frac{1}{2} \int_{\mathcal{P}(\Omega)} M_2(\rho) d\mathbb{P}(\rho) - (D)_{\mathbb{P}},$$

where  $M_2(\rho) = \langle \|\cdot\|^2 | \rho \rangle$  is the second moment of  $\rho$  and where  $(D)_{\mathbb{P}}$  corresponds to the dual value

$$(D)_{\mathbb{P}} = \min \left\{ \int_{\mathcal{P}(\Omega)} \langle \psi_{\rho}^* | \rho \rangle d\mathbb{P}(\rho), \quad (\psi_{\rho})_{\rho} \in L^{\infty}(\mathbb{P}; W^{1,\infty}(\Omega)), \quad \int_{\mathcal{P}(\Omega)} \psi_{\rho}(\cdot) d\mathbb{P}(\rho) = \frac{\|\cdot\|^2}{2} \right\}.$$

In the expression above,  $\psi_{\rho}^*(\cdot) = \sup_{y \in \Omega} \{\langle \cdot | y \rangle - \psi_{\rho}(y)\}$  corresponds to the convex conjugate of  $\psi_{\rho}$  and  $L^{\infty}(\mathbb{P}; W^{1,\infty}(\Omega))$  denotes the set of essentially bounded  $\mathbb{P}$ -measurable mappings from  $\mathcal{P}(\Omega)$  to the Sobolev space  $W^{1,\infty}(\Omega)$  of bounded Lipschitz continuous functions from  $\Omega$  to  $\mathbb{R}$ .

**Remark 1.1.** Note that in the above minimization problem,  $(\psi_{\rho})_{\rho}$  is to be understood as the following mapping, defined  $\mathbb{P}$ -almost everywhere:

$$(\psi_{\rho})_{\rho} : \begin{cases} \mathcal{P}(\Omega) & \rightarrow W^{1,\infty}(\Omega), \\ \rho & \mapsto \psi_{\rho}. \end{cases}$$

**Remark 1.2.** By Kantorovich duality [34], for  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ , the collection of functions  $(\psi_{\rho})_{\rho}$  solving  $(D)_{\mathbb{P}}$  give solutions to the optimal transport problems between  $\mathbb{P}$ -a.e.  $\rho \in \mathcal{P}(\Omega)$  and any barycenter  $\mu_{\mathbb{P}} \in \arg \min (P)_{\mathbb{P}}$ :

$$\begin{aligned} \frac{1}{2} W_2^2(\rho, \mu_{\mathbb{P}}) &= \frac{1}{2} M_2(\rho) + \frac{1}{2} M_2(\mu_{\mathbb{P}}) - (\langle \psi_{\rho}^* | \rho \rangle + \langle \psi_{\rho} | \mu_{\mathbb{P}} \rangle) \\ &= \frac{1}{2} M_2(\rho) + \frac{1}{2} M_2(\mu_{\mathbb{P}}) - \left( \min_{\psi \in \mathcal{C}(\Omega)} \langle \psi^* | \rho \rangle + \langle \psi | \mu_{\mathbb{P}} \rangle \right). \end{aligned} \quad (1)$$

As such,  $\psi_{\rho} = \psi_{\rho}^{**}$  for  $\mathbb{P}$ -a.e.  $\rho$ , so that this function – that we call later on a potential – is convex and Lipschitz continuous with Lipschitz constant smaller than  $R$ . When  $\mathbb{P}(\mathcal{P}_{a.c.}(\Omega) > 0)$  and  $\rho \in \text{spt}(\mathbb{P}) \cap \mathcal{P}_{a.c.}(\Omega)$ , the convex function  $\psi_{\rho}^*$  corresponds to a Brenier potential [10] and its gradients achieves the optimal transport from  $\rho$  to the unique barycenter  $\mu_{\mathbb{P}}$ :

$$(\nabla \psi_{\rho}^*)_{\#} \rho = \mu_{\mathbb{P}}, \quad \text{and} \quad W_2^2(\rho, \mu_{\mathbb{P}}) = \|\nabla \psi_{\rho}^* - \text{id}\|_{L^2(\rho)}^2.$$

**1.4. Assumptions and main result.** The minimization problem that appears in (1) is convex but it is not, in general, globally strongly-convex. Our main result relies on the assumption that one marginal distribution, say  $\mathbb{P}$ , attributes a positive mass to a set of absolutely continuous measures that are such that problem (1) presents a form of local strong-convexity. In particular, denoting  $\mathcal{K}_{\rho} : \psi \mapsto \langle \psi^* | \rho \rangle$  the *Kantorovich functional* associated to  $\rho \in \mathcal{P}(\Omega)$  appearing in the minimization problem (1) (which is convex and whose gradient reads formally  $\nabla \mathcal{K}_{\rho}(\psi) = -(\nabla \psi^*)_{\#} \rho$ , we will make the following assumption:

**Assumption 1.2.** *There exists constants  $\alpha_{\mathbb{P}} > 0$ ,  $c_{\mathbb{P}}, \text{per}_{\mathbb{P}}, m_{\mathbb{P}}, M_{\mathbb{P}} \in (0, +\infty)$  and a measurable set  $S_{\mathbb{P}} \subset \mathcal{P}(\Omega)$  verifying  $\mathbb{P}(S_{\mathbb{P}}) = \alpha_{\mathbb{P}}$  and such that for all  $\rho \in S_{\mathbb{P}}$ ,*

- (1)  $\rho \in \mathcal{P}_{a.c.}(\Omega)$ ,
- (2)  $m_{\mathbb{P}} \leq \rho|_{\text{spt}(\rho)} \leq M_{\mathbb{P}}$ ,
- (3)  $\text{spt}(\rho)$  has a  $\mathcal{H}^{d-1}$ -rectifiable boundary and  $\mathcal{H}^{d-1}(\partial \text{spt}(\rho)) \leq \text{per}_{\mathbb{P}}$ ,
- (4)  $\forall \psi, \tilde{\psi} \in \mathcal{C}(\Omega), \quad c_{\mathbb{P}} \text{Var}_{\rho}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_{\rho}(\tilde{\psi}) - \mathcal{K}_{\rho}(\psi) - \langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\#} \rho \rangle$ ,

where  $\text{spt}(\rho)$  denotes the support of  $\rho$ ,  $\partial \text{spt}(\rho)$  denotes the topological boundary of this support and  $\mathcal{H}^{d-1}$  denotes the  $(d-1)$ -dimensional Hausdorff measure.

Under Assumption 1.2, we prove that Wasserstein barycenters depend in a Hölder-continuous way on their marginals:

**Theorem 1.3.** *Let  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$  and assume that  $\mathbb{P}$  satisfies Assumption 1.2. Let  $\mu_{\mathbb{P}}$  be the barycenter of  $\mathbb{P}$  and  $\mu_{\mathbb{Q}}$  be a barycenter of  $\mathbb{Q}$ . Then*

$$W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/4}} \mathcal{W}_1(\mathbb{P}, \mathbb{Q})^{1/6},$$

where  $\lesssim$  hides the multiplicative constant

$$C_{d,R,m_{\mathbb{P}},M_{\mathbb{P}},\text{per}_{\mathbb{P}},c_{\mathbb{P}}} = C_d(1 + M_{\mathbb{P}})^{1/4}(1 + R)^{\frac{d}{4}+1}\left(1 + \frac{M_{\mathbb{P}}^{1/2} \text{per}_{\mathbb{P}}^{1/3}}{c_{\mathbb{P}}^{1/6} m_{\mathbb{P}}^{1/6}}\right)$$

and  $C_d$  is a constant that depends only on  $d$ .

Before discussing consequences of this result, we make some comments on our main Assumption 1.2. While the conditions (1), (2) and (3) speak for themselves, condition (4) might seem ad hoc. However, conditions under which a measure  $\rho \in \mathcal{P}_{a.c.}(\Omega)$  verifies the local strong-convexity estimate (4) of Assumption 1.2 are given in [16]: for instance, it holds for a measure  $\rho$  that has a density that is bounded away from zero and infinity on a convex support. In the Appendix, we slightly extend this result here to measures supported on a connected union of convex sets:

**Proposition 1.4.** *Let  $\rho \in \mathcal{P}_{a.c.}(\Omega)$  and assume that there exists  $m_{\rho}, M_{\rho} \in (0, +\infty)$  such that  $m_{\rho} \leq \rho \leq M_{\rho}$  on  $\text{spt}(\rho)$ . Assume in addition that  $\rho$  satisfies a Poincaré-Wirtinger inequality and that  $\text{spt}(\rho)$  is a connected finite union of convex sets. Then there exists  $c_{\rho} > 0$  such that for all  $\psi, \tilde{\psi} \in \mathcal{C}(\Omega)$ ,*

$$c_{\rho} \text{Var}_{\rho}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_{\rho}(\tilde{\psi}) - \mathcal{K}_{\rho}(\psi) - \langle \tilde{\psi} - \psi | \nabla \mathcal{K}_{\rho}(\psi) \rangle.$$

We refer to Proposition B.2 of the Appendix for a precise statement and a proof. We conjecture that such strong-convexity estimate actually holds for any a.c. measure satisfying the Poincaré-Wirtinger inequality, maybe with mild additional assumptions on the density and its support. However, this is not the focus of the present article and we leave this for future work.

On a more technical side, we note that the Borel measurability of a set  $S_{\mathbb{P}} \subset \mathcal{P}(\Omega)$  as defined in Assumption 1.2 needs to be checked depending on the application. Obviously, measurability holds when the number of marginals is finite ( $\mathbb{P}$  is discrete) and  $S_{\mathbb{P}}$  is a (finite) subset of these marginals.

## 1.5. Consequences of Theorem 1.3.

1.5.1. *Statistical estimation of barycenter with a finite number of marginals.* For a probability measure  $\rho \in \mathcal{P}(\Omega)$  and an i.i.d. sequence  $(x_j)_{j=1,\dots,n}$  sampled from  $\rho$ , it is well-known that the empirical measure  $\hat{\rho}^n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$  converges weakly to  $\rho$  almost-surely as  $n \rightarrow \infty$  [33]. By Theorem 1 of [19], the rate of this convergence can be controlled in expected Wasserstein distance: there exists a constant  $C_d$  depending only on  $d$  such that

$$\mathbb{E}W_2^2(\hat{\rho}^n, \rho) \leq C_d R^2 \begin{cases} n^{-1/2} & \text{if } d < 4, \\ n^{-1/2} \log(n) & \text{if } d = 4, \\ n^{-2/d} & \text{else,} \end{cases}$$

where the expectation is taken with respect to  $(x_j)_{j=1,\dots,n} \sim \rho^{\otimes n}$ . Theorem 1.3 together with a double use of Jensen's inequality allows to translate these rates to the statistical estimation of a Wasserstein barycenter with a finite number of marginals:

**Corollary 1.5.** *Let  $\mathbb{P}_m = \sum_{i=1}^m \alpha_i \delta_{\rho_i} \in \mathcal{P}(\mathcal{P}(\Omega))$  satisfying Assumption 1.2. For all  $i \in \{1, \dots, m\}$ , denote  $\hat{\rho}_i^n = \frac{1}{n} \sum_{j=1}^n \delta_{x_{i,j}}$  an empirical measure built from an i.i.d. sequence  $(x_{i,j})_{1 \leq j \leq n}$  sampled from  $\rho_i$ . Then the barycenters  $\mu_{\mathbb{P}_m}$  of  $\mathbb{P}_m$  and  $\mu_{\hat{\mathbb{P}}_m^n}$  of  $\hat{\mathbb{P}}_m^n = \sum_{i=1}^m \alpha_i \delta_{\hat{\rho}_i^n}$  verify*

$$\mathbb{E}W_2^2(\mu_{\hat{\mathbb{P}}_m^n}, \mu_{\mathbb{P}_m}) \lesssim \frac{1}{\alpha_{\mathbb{P}_m}^{1/2}} \begin{cases} n^{-1/12} & \text{if } d < 4, \\ n^{-1/12} \log(n)^{1/6} & \text{if } d = 4, \\ n^{-1/(3d)} & \text{else,} \end{cases}$$

where  $\lesssim$  hides a multiplicative constant depending on  $d, R, m_{\mathbb{P}}, M_{\mathbb{P}}, \text{per}_{\mathbb{P}}, c_{\mathbb{P}}$ .

1.5.2. *Convergence rate of empirical barycenters in Wasserstein spaces.* Another statistical question occurs in the setting where the population of marginals  $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$  is only known through samples  $(\rho_i)_{1 \leq i \leq m} \sim \mathbb{P}^{\otimes m}$ . Introducing the plug-in estimator  $\mathbb{P}_m = \frac{1}{m} \sum_{i=1}^m \delta_{\rho_i}$ , it is natural to wonder how well  $\mu_{\mathbb{P}_m}$  approaches  $\mu_{\mathbb{P}}$  in terms of  $m$ . This question, asked in the more general framework of barycenters in Alexandrov spaces, has been the object of recent research [2, 20]. In Wasserstein spaces, the authors of [20] show in particular that  $\mathbb{E}W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m})$  converges at the parametric rate  $m^{-1/2}$  under the assumption that  $\mathbb{P}$  admits a barycenter  $\mu_{\mathbb{P}}$  that it is such that there exists a bi-Lipschitz optimal transport map between any  $\rho \in \text{spt}(\mathbb{P})$  and  $\mu_{\mathbb{P}}$ , and that the Lipschitz constants of these maps and their inverses do not differ by a value more than 1. Under similar assumptions, the authors of [13] derive a strong-convexity estimate of the barycenter functional at its minimum which helps them derive rates of convergence of gradient descent algorithms for the (stochastic) estimation of barycenters. Such assumptions however require to have guarantees on the regularity of a barycenter of  $\mathbb{P}$ , which can be obtained when restricted to specific families of probability measures (e.g. Gaussian measures), but are difficult to get in general (for instance, barycenters of measures with convex support may not have a convex support [30], which hampers a straightforward use of Caffarelli's regularity theory). In contrast, our stability result entails that for barycenters  $\mu_{\mathbb{P}}$  of  $\mathbb{P}$  and  $\mu_{\mathbb{P}_m}$  of  $\mathbb{P}_m$ ,

$$\mathbb{E}W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/4}} \mathbb{E}W_1(\mathbb{P}, \mathbb{P}_m)^{1/6},$$

whenever  $\mathbb{P}$  satisfies Assumption 1.2. This implies that any rate of convergence of  $\mathbb{E}W_1(\mathbb{P}, \mathbb{P}_m)$  w.r.t.  $m$  is readily transferred to  $\mathbb{E}W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m})$ , up to an exponent. However,  $\mathcal{P}(\Omega)$  is an infinite dimensional space and there is no general convergence rate for  $\mathbb{E}W_1(\mathbb{P}, \mathbb{P}_m)$ . Nonetheless, assuming some structure on the population  $\mathbb{P}$  may help to derive bounds. One may use for instance the notion of upper Wasserstein dimension of  $\mathbb{P}$  introduced in [35] (Definition 4), defined from quantities depending on the covering numbers of (subsets of) the support of  $\mathbb{P}$ . Assuming that this dimension is strictly upper bounded by  $s > 0$ , the authors of [35] show that

$$\mathbb{E}W_1(\mathbb{P}, \mathbb{P}_m) \lesssim m^{-1/s},$$

where  $\lesssim$  hides a multiplicative constant that depends on  $R$  and  $s$ .

1.5.3. *Error induced by a discretization of the marginals.* Let  $\rho \in \mathcal{P}(\Omega)$  and let  $h > 0$  be a discretization parameter. Denoting  $(x_i^h)_{1 \leq i \leq N_h}$  an  $h$ -net of  $\Omega$  and  $(V_i^h)_{1 \leq i \leq N_h}$  the



corresponding Voronoi tessellation of  $\Omega$ , it is trivial to verify that the discretization  $\rho^h = \sum_{i=1}^{N_h} \rho(V_i^h) \delta_{x_i^h}$  verifies

$$W_2(\rho, \rho^h) \leq h.$$

Such kind of discretization, with controlled error bound, may be useful in practice for computational purposes. The stability result of Theorem 1.3 allows to translate the error bound made when discretizing the marginals to the corresponding barycenter:

**Corollary 1.6.** *Let  $\mathbb{P}_m = \sum_{i=1}^m \alpha_i \delta_{\rho_i} \in \mathcal{P}(\mathcal{P}(\Omega))$  satisfying Assumption 1.2. Let  $h > 0$  and for all  $i \in \{1, \dots, m\}$ , denote  $\rho_i^h = \sum_{j=1}^{N_h} \rho_i(V_j^h) \delta_{x_j^h}$  a discretization of  $\rho_i$  built from the  $h$ -net  $(x_j^h)_{1 \leq j \leq N_h}$ . Then the barycenters  $\mu_{\mathbb{P}_m}$  of  $\mathbb{P}_m$  and  $\mu_{\mathbb{P}_m^h}$  of  $\mathbb{P}_m^h = \sum_{i=1}^m \alpha_i \delta_{\rho_i^h}$  verify*

$$W_2(\mu_{\mathbb{P}_m^h}, \mu_{\mathbb{P}_m}) \lesssim \frac{1}{\alpha_{\mathbb{P}_m}^{1/4}} h^{1/6},$$

where  $\lesssim$  hides a multiplicative constant depending on  $d, R, m_{\mathbb{P}}, M_{\mathbb{P}}, \text{per}_{\mathbb{P}}, c_{\mathbb{P}}$ .

**1.6. Main elements of proof for Theorem 1.3.** The derivation of Theorem 1.3 is decomposed into two separate sub-problems. Introduce  $(\psi_{\rho})_{\rho}, (\tilde{\psi}_{\tilde{\rho}})_{\tilde{\rho}}$  solutions to the dual problems  $(D)_{\mathbb{P}}, (D)_{\mathbb{Q}}$  with respective populations  $\mathbb{P}, \mathbb{Q}$ . For an optimal  $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$  such that  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} W_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho})$ , recalling that  $\mathbb{P}(S_{\mathbb{P}}) = \alpha_{\mathbb{P}}$ , one can notice the following:

$$\begin{aligned} W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) &= \frac{1}{\alpha_{\mathbb{P}}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} W_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) d\gamma(\rho, \tilde{\rho}) \\ &\leq \frac{1}{\alpha_{\mathbb{P}}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} \left( W_2(\mu_{\mathbb{P}}, (\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho) + W_2((\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho, \mu_{\mathbb{Q}}) \right) d\gamma(\rho, \tilde{\rho}). \end{aligned} \quad (2)$$

Because  $\rho \in S_{\mathbb{P}}$  is absolutely continuous, we may replace  $\mu_{\mathbb{P}}$  by  $(\nabla \psi_{\rho}^*)_{\#} \rho$  in the first term of (2), using Remark 1.2. Bounding the first term of (2) will thus amount to quantify how  $\nabla \psi_{\rho}^*$  deviates from  $\nabla \tilde{\psi}_{\tilde{\rho}}^*$  in terms of  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q})$ . This corresponds to getting a quantitative estimate on the stability of the solutions to  $(D)_{\mathbb{P}}$ , for which the local-strong convexity assumption made for measures in  $S_{\mathbb{P}}$  almost readily gives the following bound, proven in Section 2:

**Proposition 1.7.** *Let  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$  and assume that  $\mathbb{P}$  satisfies Assumption 1.2. Let  $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$  be such that  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} W_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho})$  and let  $(\tilde{\psi}_{\tilde{\rho}})_{\tilde{\rho}}$  be a solution to  $(D)_{\mathbb{Q}}$ . Then the barycenter  $\mu_{\mathbb{P}}$  of  $\mathbb{P}$  satisfies*

$$\frac{1}{\alpha_{\mathbb{P}}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} W_2(\mu_{\mathbb{P}}, (\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho) d\gamma(\rho, \tilde{\rho}) \lesssim \left( \frac{\mathcal{W}_1(\mathbb{P}, \mathbb{Q})}{\alpha_{\mathbb{P}}} \right)^{1/6},$$

where  $\lesssim$  hides the multiplicative constant  $C_{d,R,m_{\mathbb{P}},M_{\mathbb{P}},\text{per}_{\mathbb{P}},c_{\mathbb{P}}} = \left( \frac{C_d R^5 M_{\mathbb{P}}^3 \text{per}_{\mathbb{P}}^2}{m_{\mathbb{P}} c_{\mathbb{P}}} \right)^{1/6}$ , and  $C_d$  is a constant that depends only on  $d$ .

In the second term of (2), we may replace *formally*  $\mu_{\mathbb{Q}}$  by  $(\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \tilde{\rho}$ . Bounding the second term of (2) thus amounts to finding a stability estimate for the push-forward operation under the mapping  $\nabla \tilde{\psi}_{\tilde{\rho}}^*$ . Proposition 3.1 of Section 3 gives such estimates and allows to get the following bound.

**Proposition 1.8.** *Let  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$  and assume that  $\mathbb{P}$  satisfies Assumption 1.2. Let  $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$  be such that  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathbb{W}_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho})$  and let  $(\tilde{\psi}_{\tilde{\rho}}^*)_{\tilde{\rho}}$  be a solution to  $(D)_{\mathbb{Q}}$ . Then any barycenter  $\mu_{\mathbb{Q}}$  of  $\mathbb{Q}$  satisfies*

$$\frac{1}{\alpha_{\mathbb{P}}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} \mathbb{W}_2((\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho, \mu_{\mathbb{Q}}) d\gamma(\rho, \tilde{\rho}) \lesssim \left( \frac{\mathcal{W}_1(\mathbb{P}, \mathbb{Q})}{\alpha_{\mathbb{P}}} \right)^{1/4},$$

where  $\lesssim$  hides the multiplicative constant  $C_{d,R,M_{\mathbb{P}}} = C_d(1 + M_{\mathbb{P}})^{1/4}(1 + R)^{\frac{d+1}{4}}$ , and  $C_d$  is a constant that depends only on  $d$ .

The proof of Theorem 1.3 is then immediate from bound (2) and Propositions 1.7 and 1.8.

## 2. STABILITY OF POTENTIALS

**2.1. Lipschitz behavior of the primal and dual values.** A first immediate fact is the Lipschitz behavior of the primal and dual values with respect to the marginals  $\mathbb{P}, \mathbb{Q}$  (that holds without any assumptions on their regularity):

**Proposition 2.1.** *For any  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$ , the following bounds hold:*

$$\begin{aligned} |(\mathbb{P})_{\mathbb{Q}} - (\mathbb{P})_{\mathbb{P}}| &\leq 3R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}), \\ |(\mathbb{D})_{\mathbb{Q}} - (\mathbb{D})_{\mathbb{P}}| &\leq 4R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}). \end{aligned}$$

*Proof.* Let  $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$  be such that  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathbb{W}_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho})$ . By definition of  $(\mathbb{P})_{\mathbb{P}}$  and using the triangle inequality we have:

$$\begin{aligned} (\mathbb{P})_{\mathbb{P}} &\leq \frac{1}{2} \int_{\mathcal{P}(\Omega)} \mathbb{W}_2^2(\rho, \mu_{\mathbb{Q}}) d\mathbb{P}(\rho) \\ &= \frac{1}{2} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathbb{W}_2^2(\rho, \mu_{\mathbb{Q}}) d\gamma(\rho, \tilde{\rho}) \\ &\leq \frac{1}{2} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathbb{W}_2(\rho, \tilde{\rho}) + \mathbb{W}_2(\tilde{\rho}, \mu_{\mathbb{Q}}))^2 d\gamma(\rho, \tilde{\rho}) \\ &= \frac{1}{2} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathbb{W}_2^2(\rho, \tilde{\rho}) + 2\mathbb{W}_2(\rho, \tilde{\rho})\mathbb{W}_2(\tilde{\rho}, \mu_{\mathbb{Q}}) + \mathbb{W}_2^2(\tilde{\rho}, \mu_{\mathbb{Q}})) d\gamma(\rho, \tilde{\rho}). \end{aligned}$$

Now using that  $\Omega$  is compact and included in a ball centered at the origin and of radius  $R > 0$ , we get the bound

$$\begin{aligned} (\mathbb{P})_{\mathbb{P}} &\leq \frac{1}{2} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (6R\mathbb{W}_2(\rho, \tilde{\rho}) + \mathbb{W}_2^2(\tilde{\rho}, \mu_{\mathbb{Q}})) d\gamma(\rho, \tilde{\rho}) \\ &= 3R \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathbb{W}_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho}) + \frac{1}{2} \int_{\mathcal{P}(\Omega)} \mathbb{W}_2^2(\tilde{\rho}, \mu_{\mathbb{Q}}) d\mathbb{Q}(\tilde{\rho}) \\ &= 3R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) + (\mathbb{P})_{\mathbb{Q}}. \end{aligned}$$

The first inequality is then deduced by symmetry. Next, we can derive the following:

$$\begin{aligned}
 (\text{D})_{\mathbb{P}} &= \frac{1}{2} \int_{\mathcal{P}(\Omega)} M_2(\rho) d\mathbb{P}(\rho) - (\text{P})_{\mathbb{P}}, \\
 &\leq \frac{1}{2} \int_{\mathcal{P}(\Omega)} M_2(\rho) d\mathbb{P}(\rho) - (\text{P})_{\mathbb{Q}} + 3R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) \\
 &\leq \frac{1}{2} \int_{\mathcal{P}(\Omega)} M_2(\tilde{\rho}) d\mathbb{Q}(\tilde{\rho}) - (\text{P})_{\mathbb{Q}} + \frac{1}{2} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} |M_2(\rho) - M_2(\tilde{\rho})| d\gamma(\rho, \tilde{\rho}) + 3R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) \\
 &= (\text{D})_{\mathbb{Q}} + \frac{1}{2} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} |M_2(\rho) - M_2(\tilde{\rho})| d\gamma(\rho, \tilde{\rho}) + 3R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}).
 \end{aligned}$$

We conclude by noticing that for all  $\rho, \tilde{\rho} \in \mathcal{P}(\Omega)$ , one has

$$|M_2(\rho) - M_2(\tilde{\rho})| = |W_2^2(\rho, \delta_0) - W_2^2(\tilde{\rho}, \delta_0)| \leq 2R |W_2(\rho, \delta_0) - W_2(\tilde{\rho}, \delta_0)| \leq 2RW_2(\rho, \tilde{\rho}). \square$$

**2.2. Stability of the dual solutions.** Denote  $(\psi_\rho)_\rho$  and  $(\tilde{\psi}_{\tilde{\rho}})_{\tilde{\rho}}$  solutions to the dual problems  $(\text{D})_{\mathbb{P}}, (\text{D})_{\mathbb{Q}}$  with respective populations  $\mathbb{P}, \mathbb{Q}$ . These families of potentials verify

$$\begin{aligned}
 (\text{D})_{\mathbb{P}} &= \int_{\mathcal{P}(\Omega)} \mathcal{K}_\rho(\psi_\rho) d\mathbb{P}(\rho) \quad \text{and} \quad \int_{\mathcal{P}(\Omega)} \psi_\rho(\cdot) d\mathbb{P}(\rho) = \frac{\|\cdot\|^2}{2}, \\
 (\text{D})_{\mathbb{Q}} &= \int_{\mathcal{P}(\Omega)} \mathcal{K}_{\tilde{\rho}}(\tilde{\psi}_{\tilde{\rho}}) d\mathbb{Q}(\tilde{\rho}) \quad \text{and} \quad \int_{\mathcal{P}(\Omega)} \tilde{\psi}_{\tilde{\rho}}(\cdot) d\mathbb{Q}(\tilde{\rho}) = \frac{\|\cdot\|^2}{2}.
 \end{aligned}$$

We can then show that  $(\tilde{\psi}_{\tilde{\rho}})_{\tilde{\rho}}$  is almost a minimizer for  $(\text{D})_{\mathbb{P}}$  whenever  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q})$  is small:

**Proposition 2.2.** *Let  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$  and  $(\psi_\rho)_\rho, (\tilde{\psi}_{\tilde{\rho}})_{\tilde{\rho}}$  respective solutions of the dual problems  $(\text{D})_{\mathbb{P}}, (\text{D})_{\mathbb{Q}}$ . Let  $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$  be such that  $\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} W_2(\rho, \tilde{\rho}) d\gamma(\rho, \tilde{\rho})$ . Then*

$$\left| \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathcal{K}_\rho(\psi_\rho) - \mathcal{K}_\rho(\tilde{\psi}_{\tilde{\rho}})) d\gamma(\rho, \tilde{\rho}) \right| \leq 5R\mathcal{W}_1(\mathbb{P}, \mathbb{Q}).$$

*Proof.* Using the triangle inequality, we have

$$\begin{aligned}
 \left| \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathcal{K}_\rho(\psi_\rho) - \mathcal{K}_\rho(\tilde{\psi}_{\tilde{\rho}})) d\gamma(\rho, \tilde{\rho}) \right| &\leq \left| \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathcal{K}_\rho(\psi_\rho) - \mathcal{K}_{\tilde{\rho}}(\tilde{\psi}_{\tilde{\rho}})) d\gamma(\rho, \tilde{\rho}) \right| \\
 &\quad + \left| \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathcal{K}_{\tilde{\rho}}(\tilde{\psi}_{\tilde{\rho}}) - \mathcal{K}_\rho(\tilde{\psi}_{\tilde{\rho}})) d\gamma(\rho, \tilde{\rho}) \right|. \tag{3}
 \end{aligned}$$

Using that  $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$ , the first term of sum (3) is equal to

$$\left| \int_{\mathcal{P}(\Omega)} \mathcal{K}_\rho(\psi_\rho) d\mathbb{P}(\rho) - \int_{\mathcal{P}(\Omega)} \mathcal{K}_{\tilde{\rho}}(\tilde{\psi}_{\tilde{\rho}}) d\mathbb{Q}(\tilde{\rho}) \right| = |(\text{D})_{\mathbb{P}} - (\text{D})_{\mathbb{Q}}|,$$

which can be upper bounded by  $4R\mathcal{W}_1(\mathbb{P}, \mathbb{Q})$  using Proposition 2.1. The second term in (3) can then be upper bounded by  $R\mathcal{W}_1(\mathbb{P}, \mathbb{Q})$  using the Kantorovich-Rubinstein duality result, which ensures that for any  $\rho, \tilde{\rho} \in \mathcal{P}(\Omega)$ ,

$$\mathcal{K}_\rho(\tilde{\psi}_{\tilde{\rho}}) - \mathcal{K}_{\tilde{\rho}}(\tilde{\psi}_{\tilde{\rho}}) = \langle \tilde{\psi}_{\tilde{\rho}}^* | \rho - \tilde{\rho} \rangle \leq R\mathcal{W}_1(\rho, \tilde{\rho}) \leq R\mathcal{W}_2(\rho, \tilde{\rho}). \quad \square$$

### 2.3. Proof of Proposition 1.7.

*Proof of Proposition 1.7.* Let  $\rho \in \text{spt}(\mathbb{P})$  and  $\pi \in \Pi(\rho, \mu_{\mathbb{P}})$  be an optimal coupling for the quadratic optimal transport between  $\rho$  and  $\mu_{\mathbb{P}}$ . Then by (1) we have:

$$\int_{\Omega \times \Omega} \langle x|y \rangle d\pi(x, y) = \mathcal{K}_{\rho}(\psi_{\rho}) + \langle \psi_{\rho} | \mu_{\mathbb{P}} \rangle. \quad (4)$$

On the other hand, the Fenchel-Young inequality ensures that for any  $\tilde{\rho} \in \text{spt}(\mathbb{Q})$ , for any pair of points  $x, y \in \Omega$ ,

$$\langle x|y \rangle \leq \tilde{\psi}_{\tilde{\rho}}^*(x) + \tilde{\psi}_{\tilde{\rho}}(y).$$

This ensures

$$\int_{\Omega \times \Omega} \langle x|y \rangle d\pi(x, y) \leq \mathcal{K}_{\rho}(\tilde{\psi}_{\tilde{\rho}}) + \langle \tilde{\psi}_{\tilde{\rho}} | \mu_{\mathbb{P}} \rangle.$$

Injecting (4) into this last inequality then yields

$$\langle \psi_{\rho} - \tilde{\psi}_{\tilde{\rho}} | \mu_{\mathbb{P}} \rangle \leq \mathcal{K}_{\rho}(\tilde{\psi}_{\tilde{\rho}}) - \mathcal{K}_{\rho}(\psi_{\rho}). \quad (5)$$

When  $\rho$  belongs to  $S_{\mathbb{P}}$ , Assumption 1.2 allows us to improve the previous bound:

$$\langle \psi_{\rho} - \tilde{\psi}_{\tilde{\rho}} | \mu_{\mathbb{P}} \rangle + c_{\mathbb{P}} \text{Var}_{\rho}(\tilde{\psi}_{\tilde{\rho}}^* - \psi_{\rho}^*) \leq \mathcal{K}_{\rho}(\tilde{\psi}_{\tilde{\rho}}) - \mathcal{K}_{\rho}(\psi_{\rho}). \quad (6)$$

Hence, weighting (5) and (6) by  $d\gamma(\rho, \tilde{\rho})$  and summing over  $(\rho, \tilde{\rho}) \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$ , we get

$$\begin{aligned} \left\langle \int_{\mathcal{P}(\Omega)} \psi_{\rho} d\mathbb{P}(\rho) - \int_{\mathcal{P}(\Omega)} \tilde{\psi}_{\tilde{\rho}} d\mathbb{Q}(\tilde{\rho}) | \mu_{\mathbb{P}} \right\rangle + c_{\mathbb{P}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} \text{Var}_{\rho}(\tilde{\psi}_{\tilde{\rho}}^* - \psi_{\rho}^*) d\gamma(\rho, \tilde{\rho}) \\ \leq \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} (\mathcal{K}_{\rho}(\tilde{\psi}_{\tilde{\rho}}) - \mathcal{K}_{\rho}(\psi_{\rho})) d\gamma(\rho, \tilde{\rho}). \end{aligned}$$

Using that  $\int_{\mathcal{P}(\Omega)} \psi_{\rho}(\cdot) d\mathbb{P}(\rho) = \int_{\mathcal{P}(\Omega)} \tilde{\psi}_{\tilde{\rho}}(\cdot) d\mathbb{Q}(\tilde{\rho}) = \frac{\|\cdot\|^2}{2}$  and Proposition 2.2, this leads to

$$c_{\mathbb{P}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} \text{Var}_{\rho}(\tilde{\psi}_{\tilde{\rho}}^* - \psi_{\rho}^*) d\gamma(\rho, \tilde{\rho}) \leq 5RW_1(\mathbb{P}, \mathbb{Q}). \quad (7)$$

We now quote a Gagliardo-Nirenberg type inequality, extracted from Proposition 4.1 in [16], that ensures that for any compact domain  $K$  of  $\mathbb{R}^d$  with  $\mathcal{H}^{d-1}$ -rectifiable boundary and  $u, v : K \rightarrow \mathbb{R}$  two Lipschitz functions on  $K$  that are convex on any segment included in  $K$ , there exists a constant  $C_d$  depending only on  $d$  such that

$$\|\nabla u - \nabla v\|_{L^2(K)}^6 \leq C_d \mathcal{H}^{d-1}(\partial K)^2 (\|\nabla u\|_{L^{\infty}(K)} + \|\nabla v\|_{L^{\infty}(K)})^4 \|u - v\|_{L^2(K)}^2.$$

This inequality can be used to bound from below the left-hand side term of equation (7). Indeed, measures belonging to  $S_{\mathbb{P}}$  have a support which has a  $\mathcal{H}^{d-1}$ -rectifiable topological boundary. This ensures the following for any  $\rho \in S_{\mathbb{P}}$  and  $\tilde{\rho} \in \mathcal{P}(\Omega)$ :

$$\left\| \nabla \tilde{\psi}_{\tilde{\rho}}^* - \nabla \psi_{\rho}^* \right\|_{L^2(\rho)}^6 \leq C_d \text{per}_{\mathbb{P}}^2 R^4 \frac{M_{\mathbb{P}}^3}{m_{\mathbb{P}}} \text{Var}_{\rho}(\tilde{\psi}_{\tilde{\rho}}^* - \psi_{\rho}^*), \quad (8)$$

where  $C_d$  is a constant that only depends on the dimension  $d$ . Therefore, using that for any  $\rho \in S_{\mathbb{P}}$  and  $\tilde{\rho} \in \mathcal{P}(\Omega)$ ,

$$W_2(\mu_{\mathbb{P}}, (\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho) = W_2((\nabla \psi_{\rho}^*)_{\#} \rho, (\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho) \leq \left\| \nabla \tilde{\psi}_{\tilde{\rho}}^* - \nabla \psi_{\rho}^* \right\|_{L^2(\rho)},$$

the combination of bounds (7) and (8) with Jensen's inequality yields:

$$\frac{1}{\alpha_{\mathbb{P}}} \int_{S_{\mathbb{P}} \times \mathcal{P}(\Omega)} W_2(\mu_{\mathbb{P}}, (\nabla \tilde{\psi}_{\tilde{\rho}}^*)_{\#} \rho) d\gamma(\rho, \tilde{\rho}) \leq \left( \frac{C_d R^5 M_{\mathbb{P}}^3 \text{per}_{\mathbb{P}}^2}{m_{\mathbb{P}} c_{\mathbb{P}}} \right)^{1/6} \left( \frac{\mathcal{W}_1(\mathbb{P}, \mathbb{Q})}{\alpha_{\mathbb{P}}} \right)^{1/6}. \quad \square$$

### 3. STABILITY OF PUSH-FORWARDS

We prove the following proposition, which is more general than Proposition 1.8, and which entails the latter by a direct use of Jensen's inequality. In this statement,  $p_1 : (x, y) \mapsto x$  and  $p_2 : (x, y) \mapsto y$  are the projections onto the first and second coordinates respectively.

**Proposition 3.1.** *Let  $\rho, \tilde{\rho} \in \mathcal{P}(\Omega)$  and assume that  $\rho$  is absolutely continuous with density bounded above by  $M_{\rho} \in (0, +\infty)$ . Let  $\phi \in \mathcal{C}(\Omega)$  be a convex  $R$ -Lipschitz function. Let  $\tilde{\gamma} \in \mathcal{P}(\Omega \times \Omega)$  be such that  $(p_1)_{\#} \tilde{\gamma} = \tilde{\rho}$  and assume that  $\tilde{\gamma}$  is concentrated on*

$$\partial\phi = \{(x, y) | \phi(x) + \phi^*(y) = \langle x | y \rangle\}.$$

Then

$$W_2((\nabla\phi)_{\#} \rho, (p_2)_{\#} \tilde{\gamma}) \leq C_{d,R,M_{\rho}} W_2(\rho, \tilde{\rho})^{1/4},$$

where  $C_{d,R,M_{\rho}} = C_d(1 + M_{\rho})(1 + R)^{d+1}$ , with  $C_d$  a constant that depends only on  $d$ .

**Remark 3.1.** Whenever  $\phi$  is differentiable  $\tilde{\rho}$ -almost-everywhere, Proposition 3.1 ensures the following stability result for the push-forward operation by  $\nabla\phi$ :

$$W_2((\nabla\phi)_{\#} \rho, (\nabla\phi)_{\#} \tilde{\rho}) \leq C_{d,R,L,M_{\rho}} W_2(\rho, \tilde{\rho})^{1/4}.$$

We will rely on the following lemma, whose proof is deferred to the end of this section. Note that this proof heavily relies on the proof of Lemma 3.2 of [11].

**Lemma 3.2.** *Let  $\phi$  be a convex function over  $\mathbb{R}^d$ . Then for any  $x \in \mathbb{R}^d$  and  $r > 0$ ,*

$$\text{diam}(\partial\phi(B(x, r))) \leq \frac{12}{\omega_d r^d} \|\nabla\phi\|_{L^1(B(x, 4r))},$$

where  $\omega_d$  denotes the volume of the unit ball of  $\mathbb{R}^d$ .

We are now ready to prove Proposition 3.1.

*Proof of Proposition 3.1.* Denote  $\tilde{\gamma} = \tilde{\gamma}_x \otimes \tilde{\rho}$  the disintegration of  $\tilde{\gamma}$  with respect to  $\tilde{\rho}$ . Notice that if  $y \in \text{spt}(\tilde{\gamma}_x)$  then  $y \in \partial\phi(x)$  by assumption. Introduce  $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$  the optimal transport map from  $\rho$  to  $\tilde{\rho}$  and the measure  $\gamma = \tilde{\gamma}_{S(x)} \otimes \rho$ . This measure  $\gamma$  is a coupling between  $\rho$  and  $(p_2)_{\#} \tilde{\gamma}$ , which implies that  $(\nabla\phi, \text{id})_{\#} \gamma$  is a coupling between  $(\nabla\phi)_{\#} \rho$  and  $(p_2)_{\#} \tilde{\gamma}$ . We therefore have the bound:

$$\begin{aligned} W_2^2((\nabla\phi)_{\#} \rho, (p_2)_{\#} \tilde{\gamma}) &\leq \int_{\Omega \times \Omega} \|\nabla\phi(x) - y\|^2 d\gamma(x, y) \\ &= \int_{\Omega \times \Omega} \|\nabla\phi(x) - y\|^2 d\tilde{\gamma}_{S(x)}(y) d\rho(x) \\ &= \int_{x \in \Omega} \int_{y \in \partial\phi(S(x))} \|\nabla\phi(x) - y\|^2 d\tilde{\gamma}_{S(x)}(y) d\rho(x), \end{aligned}$$

where we used that if  $y \in \text{spt}(\tilde{\gamma}_{S(x)})$  then  $y \in \partial\phi(S(x))$  to get to the last line. For  $\eta \in (0, (2R)^{3/4}]$ , we will find an upper bound on the right-hand side by splitting the integral on  $\Omega_{\eta}$  and  $\Omega_{\eta}^c$ , where

$$\Omega_{\eta} = \{x \in \text{spt}(\rho) | \|S(x) - x\|^2 \geq \eta^2\}, \quad \Omega_{\eta}^c = \text{spt}(\rho) \setminus \Omega_{\eta}.$$

Upper bound on  $\Omega_\eta$ . The optimal transport map  $S$  from  $\rho$  to  $\tilde{\rho}$  satisfies

$$\|S - \text{id}\|_{L^2(\rho)} = W_2(\rho, \tilde{\rho}).$$

Then by Markov's inequality,  $\rho(\Omega_\eta) \leq \frac{W_2^2(\rho, \tilde{\rho})}{\eta^2}$ , so that

$$\int_{x \in \Omega_\eta} \int_{y \in \partial\phi(S(x))} \|\nabla\phi(x) - y\|^2 d\tilde{\gamma}_{S(x)}(y) d\rho(x) \leq \int_{x \in \Omega_\eta} 4R^2 d\rho(x) \leq \frac{4R^2}{\eta^2} W_2^2(\rho, \tilde{\rho}). \quad (9)$$

Upper bound on  $\Omega_\eta^c$ . By definition of  $\Omega_\eta$ , for any  $x \in \Omega_\eta^c$ ,  $\|S(x) - x\| \leq \eta$ , i.e.  $S(x) \in B(x, \eta)$ . Then for any such  $x$ ,  $\partial\phi(S(x)) \subset \partial\phi(B(x, \eta))$ . Therefore for any  $g \in \partial\phi(x)$  and  $y \in \partial\phi(S(x))$ , one has

$$\|g - y\| \leq \text{diam}(\partial\phi(B(x, \eta))),$$

which leads to

$$\int_{x \in \Omega_\eta^c} \int_{y \in \partial\phi(S(x))} \|\nabla\phi(x) - y\|^2 d\tilde{\gamma}_{S(x)}(y) d\rho(x) \leq \int_{x \in \Omega_\eta^c} \text{diam}(\partial\phi(B(x, \eta)))^2 d\rho(x). \quad (10)$$

For  $\alpha > 0$ , introduce  $\mathcal{X}_\alpha = \{x \in \Omega_\eta^c \mid \text{diam}(\partial\phi(B(x, \eta))) \geq \eta^\alpha\}$ . We will quantify  $\rho(\mathcal{X}_\alpha)$  by finding an upper bound on its covering number. Let  $(\mathcal{X}_\alpha)_{4\eta}^{\text{pack}} \subset \mathcal{X}_\alpha$  be a maximal  $(4\eta)$ -packing of  $\mathcal{X}_\alpha$ , i.e. a finite subset of  $\mathcal{X}_\alpha$  such that for any  $x, y \in (\mathcal{X}_\alpha)_{4\eta}^{\text{pack}}$ ,  $x \neq y$ ,  $B(x, 4\eta) \cap B(y, 4\eta) = \emptyset$  and such that for any  $z \in \mathcal{X}_\alpha$ , there exists  $x \in (\mathcal{X}_\alpha)_{4\eta}^{\text{pack}}$  such that  $B(z, 4\eta) \cap B(x, 4\eta) \neq \emptyset$ . We denote  $N_{4\eta}^{\text{pack}}(\mathcal{X}_\alpha)$  the cardinal of  $(\mathcal{X}_\alpha)_{4\eta}^{\text{pack}}$ . By definition of  $\mathcal{X}_\alpha$ , for any  $x \in (\mathcal{X}_\alpha)_{4\eta}^{\text{pack}} \subset \mathcal{X}_\alpha$ , one has

$$\eta^\alpha \leq \text{diam}(\partial\phi(B(x, \eta))).$$

Lemma 3.2 then ensures that for any  $c \in \mathbb{R}^d$ ,

$$\eta^\alpha \leq \frac{12}{\omega_d \eta^d} \|\nabla\phi - c\|_{L^1(B(x, 4\eta))}. \quad (11)$$

Choosing  $c = \frac{1}{|B(x, 4\eta)|} \int_{B(x, 4\eta)} \nabla\phi(u) du$ , the Poincaré-Wirtinger inequality ensures

$$\|\nabla\phi - c\|_{L^1(B(x, 4\eta))} \leq 4\eta \int_{B(x, 4\eta)} \|\mathbb{D}^2\phi(u)\|_{1,1} du.$$

Using that for any positive semi-definite  $d \times d$  matrix  $M$ ,  $\|M\|_{1,1} \leq d \text{tr}(M)$ , we then have

$$\|\nabla\phi - c\|_{L^1(B(x, 4\eta))} \leq 4\eta d \int_{B(x, 4\eta)} \Delta\phi(u) du,$$

where  $\Delta$  stands for the Laplace operator. Injecting this last bound into (11) yields

$$\eta^\alpha \lesssim \frac{1}{\eta^{d-1}} \int_{B(x, 4\eta)} \Delta\phi(u) du,$$

where  $\lesssim$  hides multiplicative constants depending on  $d, M_\rho$  and  $R$ . Summing the last bound over  $x \in (\mathcal{X}_\alpha)_{4\eta}^{pack}$  then yields

$$\begin{aligned} N_{4\eta}^{pack}(\mathcal{X}_\alpha) &\lesssim \eta^{1-d-\alpha} \sum_{x \in (\mathcal{X}_\alpha)_{4\eta}^{pack}} \int_{B(x, 4\eta)} \Delta\phi(u) du \\ &\lesssim \eta^{1-d-\alpha} \int_{\Omega+B(0, 4\eta)} \Delta\phi(u) du \\ &\lesssim \eta^{1-d-\alpha} \int_{\partial(\Omega+B(0, 4\eta))} \langle \nabla\phi(u) | n_u \rangle du, \end{aligned}$$

where  $n_u$  is the outward pointing unit normal at  $u \in \partial(\Omega + B(0, 4\eta))$ . Using that  $\|\partial\phi\| \leq R$ , one then has

$$N_{4\eta}^{pack}(\mathcal{X}_\alpha) \lesssim \eta^{1-d-\alpha}.$$

We can now upper bound the  $(8\eta)$ -covering number of  $\mathcal{X}_\alpha$ , denoted by  $N_{8\eta}^{cov}(\mathcal{X}_\alpha)$ , with its  $(4\eta)$ -packing number:

$$N_{8\eta}^{cov}(\mathcal{X}_\alpha) \leq N_{4\eta}^{pack}(\mathcal{X}_\alpha).$$

Hence for any minimal  $(8\eta)$ -covering of  $\mathcal{X}_\alpha$  denoted by  $(\mathcal{X}_\alpha)_{8\eta}^{cov}$ , one has

$$\begin{aligned} \rho(\mathcal{X}_\alpha) &\leq \rho\left(\bigcup_{x \in (\mathcal{X}_\alpha)_{8\eta}^{cov}} B(x, 8\eta)\right) \\ &\leq N_{8\eta}^{cov}(\mathcal{X}_\alpha) M_\rho \omega_d (8\eta)^d \\ &\lesssim \eta^{1-\alpha}. \end{aligned}$$

We can finally write

$$\begin{aligned} \int_{x \in \Omega_\eta^c} \text{diam}(\partial\phi(B(x, \eta)))^2 d\rho(x) &= \int_{x \in \Omega_\eta^c \cap \mathcal{X}_\alpha} \text{diam}(\partial\phi(B(x, \eta)))^2 d\rho(x) \\ &\quad + \int_{x \in \Omega_\eta^c \cap \mathcal{X}_\alpha^c} \text{diam}(\partial\phi(B(x, \eta)))^2 d\rho(x) \\ &\leq 4R^2 \rho(\mathcal{X}_\alpha) + \int_{\Omega_\eta^c \cap \mathcal{X}_\alpha^c} \eta^{2\alpha} d\rho(x) \\ &\lesssim \eta^{1-\alpha} + \eta^{2\alpha}. \end{aligned}$$

Choosing  $\alpha = \frac{1}{3}$  then yields in (10):

$$\int_{x \in \Omega_\eta^c} \int_{y \in \partial\phi(S(x))} \|\nabla\phi(x) - y\|^2 d\tilde{\gamma}_{S(x)}(y) d\rho(x) \lesssim \eta^{2/3}.$$

Combining this last bound with (9) leads to

$$W_2^2((\nabla\phi)_\#\rho, (p_2)_\#\tilde{\gamma}) \lesssim \frac{1}{\eta^2} W_2^2(\rho, \tilde{\rho}) + \eta^{2/3}.$$

Setting  $\eta = W_2(\rho, \tilde{\rho})^{3/4}$  then gives us

$$W_2^2((\nabla\phi)_\#\rho, (p_2)_\#\tilde{\gamma}) \lesssim W_2(\rho, \tilde{\rho})^{1/2}. \quad \square$$

We now prove Lemma 3.2, which is key to the proof of Proposition 3.1.

*Proof of Lemma 3.2.* Let  $x \in \mathbb{R}^d$  and  $r > 0$ . One has by definition:

$$\begin{aligned} \text{diam}(\partial\phi(B(x, r))) &= \sup_{y, y' \in B(x, r)} \sup_{g \in \partial\phi(y), g' \in \partial\phi(y')} \|g - g'\| \\ &\leq \sup_{y, y' \in B(x, r)} \sup_{g \in \partial\phi(y), g' \in \partial\phi(y')} \|g\| + \|g'\| \\ &= 2 \sup_{y \in B(x, r)} \sup_{g \in \partial\phi(y)} \|g\| \\ &= 2 \|\partial\phi\|_{L^\infty(B(x, r))}. \end{aligned}$$

But for any  $y, y' \in \mathbb{R}^d$  and  $g \in \partial\phi(y)$ , the convexity of  $\phi$  entails

$$\langle g | y' - y \rangle \leq |\phi(y') - \phi(y)|.$$

Therefore, choosing  $y \in B(x, r)$  and  $g \in \partial\phi(y)$  such that  $\|\partial\phi\|_{L^\infty(B(x, r))} = \|g\|$ , one has for  $y' = y + r \frac{g}{\|g\|} \in B(y, r) \subset B(x, 2r)$  the following bound:

$$r \|g\| \leq |\phi(y') - \phi(y)| \leq \text{osc}_{B(x, 2r)}(\phi),$$

where  $\text{osc}_K(f) = \sup_{u, v \in K} |f(u) - f(v)|$ . We thus have shown

$$\text{diam}(\partial\phi(B(x, r))) \leq \frac{2}{r} \text{osc}_{B(x, 2r)}(\phi). \quad (12)$$

We conclude exactly as in the proof of Lemma 3.2 of [11], that we report here only for completeness: let  $y_0 \in \arg \min_{B(x, 2r)} \phi$ ,  $y_1 \in \arg \max_{B(x, 2r)} \phi$ ,  $g_1 \in \partial\phi(y_1)$ . Then by convexity of  $\phi$ , for any  $y \in \mathbb{R}^d$  and  $g \in \partial\phi(y)$  one has

$$\phi(y_1) + \langle g_1 | y - y_1 \rangle \leq \phi(y) \leq \phi(y_0) + \langle g | y - y_0 \rangle.$$

It follows that

$$\|g\| \geq \frac{\text{osc}_{B(x, 2r)}(\phi) + \langle g_1 | y - y_1 \rangle}{\|y - y_0\|}.$$

Introducing  $W_r(y_1, g_1) = \{y \in B(y_1, 2r) | \langle g_1 | y - y_1 \rangle \geq 0\} \subset B(x, 4r)$ , one then has

$$\begin{aligned} \|\nabla\phi\|_{L^1(B(x, 4r))} &\geq \int_{W_r(y_1, g_1)} \|\nabla\phi\| \, dy \\ &\geq \int_{W_r(y_1, g_1)} \frac{\text{osc}_{B(x, 2r)}(\phi)}{\|y - y_0\|} \, dy \\ &\geq \text{osc}_{B(x, 2r)}(\phi) \int_{W_r(y_1, g_1)} \frac{1}{\|y - y_1\| + \|y_1 - y_0\|} \, dy \\ &\geq \frac{\text{osc}_{B(x, 2r)}(\phi)}{6r} \int_{B(y_1 + r \frac{g_1}{\|g_1\|}, r)} \, dy \\ &= \omega_d \frac{r^{d-1}}{6} \text{osc}_{B(x, 2r)}(\phi), \end{aligned}$$

where  $\omega_d$  denotes the volume of the unit ball of  $\mathbb{R}^d$  and where we used the fact that  $B(y_1 + r \frac{g_1}{\|g_1\|}, r) \subset W_r(y_1, g_1)$ . Plugging this last bound into (12) finally yields

$$\text{diam}(\partial\phi(B(x, r))) \leq \frac{12}{\omega_d r^d} \|\nabla\phi\|_{L^1(B(x, 4r))}. \quad \square$$



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## APPENDIX A. DUAL FORMULATION

*Proof of Proposition 1.1.* Instead of showing directly the formulation of Proposition 1.1, we will rather show

$$(P)_{\mathbb{P}} = \max \left\{ \int_{\mathcal{P}(\Omega)} \langle \phi_{\rho}^c | \rho \rangle d\mathbb{P}(\rho), \quad (\phi_{\rho})_{\rho} \in L^{\infty}(\mathbb{P}; W^{1,\infty}(\Omega)), \quad \int_{\mathcal{P}(\Omega)} \phi_{\rho}(\cdot) d\mathbb{P}(\rho) = 0 \right\},$$

where for any  $\rho \in \mathcal{P}(\Omega)$ ,  $\phi_{\rho}^c$  denotes the following  $c$ -transform of  $\phi_{\rho}$ :  $\phi_{\rho}^c(x) = \inf_{y \in \Omega} \frac{1}{2} \|x - y\|^2 - \phi_{\rho}(y)$  for all  $x \in \Omega$ . Such a formulation entails the result of Proposition 1.1 by the change of variable  $(\psi_{\rho})_{\rho} = \frac{\|\cdot\|^2}{2} - (\phi_{\rho})_{\rho} \in L^{\infty}(\mathbb{P}; W^{1,\infty}(\Omega))$ .

**Duality.** Let's first show that  $(P)_{\mathbb{P}}$  is equal to the value of the following supremum

$$(\widetilde{P})_{\mathbb{P}} := \sup \left\{ \int_{\mathcal{P}(\Omega)} \langle \phi_{\rho}^c | \rho \rangle d\mathbb{P}(\rho), \quad (\phi_{\rho})_{\rho} \in L^1(\mathbb{P}; \mathcal{C}(\Omega)), \quad \int_{\mathcal{P}(\Omega)} \phi_{\rho}(\cdot) d\mathbb{P}(\rho) = 0 \right\},$$

where  $L^1(\mathbb{P}; \mathcal{C}(\Omega))$  denotes the set of  $\mathbb{P}$ -measurable and Bochner integrable mappings from  $\mathcal{P}(\Omega)$  to the space  $(\mathcal{C}(\Omega), \|\cdot\|_{\infty})$  of continuous function from  $\Omega$  to  $\mathbb{R}$  equipped with the supremum norm. Introduce the functional  $H : \mathcal{C}(\Omega) \rightarrow \mathbb{R}$  defined for all  $\varphi \in \mathcal{C}(\Omega)$  by

$$H(\varphi) = \inf \left\{ - \int_{\mathcal{P}(\Omega)} \langle \phi_{\rho}^c | \rho \rangle d\mathbb{P}(\rho), \quad (\phi_{\rho})_{\rho} \in L^1(\mathbb{P}; \mathcal{C}(\Omega)), \quad \int_{\mathcal{P}(\Omega)} \phi_{\rho}(\cdot) d\mathbb{P}(\rho) = \varphi(\cdot) \right\}.$$

Notice then that  $(\widetilde{P})_{\mathbb{P}} = -H(0)$ . On the other hand, notice that  $H$  has the following convex conjugate: for  $\nu \in \mathcal{P}(\Omega)$ ,

$$\begin{aligned} H^*(\nu) &= \sup \{ \langle \varphi | \nu \rangle - H(\varphi), \quad \varphi \in \mathcal{C}(\Omega) \} \\ &= \sup \left\{ \langle \varphi | \nu \rangle + \int_{\mathcal{P}(\Omega)} \langle \phi_{\rho}^c | \rho \rangle d\mathbb{P}(\rho), \quad \varphi \in \mathcal{C}(\Omega), (\phi_{\rho})_{\rho} \in L^1(\mathbb{P}; \mathcal{C}(\Omega)), \int_{\mathcal{P}(\Omega)} \phi_{\rho}(\cdot) d\mathbb{P}(\rho) = \varphi(\cdot) \right\} \\ &= \sup \left\{ \int_{\mathcal{P}(\Omega)} \langle \phi_{\rho} | \nu \rangle d\mathbb{P}(\rho) + \int_{\mathcal{P}(\Omega)} \langle \phi_{\rho}^c | \rho \rangle d\mathbb{P}(\rho), \quad (\phi_{\rho})_{\rho} \in L^1(\mathbb{P}; \mathcal{C}(\Omega)) \right\} \\ &= \int_{\mathcal{P}(\Omega)} \left( \sup_{\phi_{\rho} \in \mathcal{C}(\Omega)} \langle \phi_{\rho} | \nu \rangle + \langle \phi_{\rho}^c | \rho \rangle \right) d\mathbb{P}(\rho) \\ &= \int_{\mathcal{P}(\Omega)} \frac{1}{2} W_2^2(\nu, \rho) d\mathbb{P}(\rho), \end{aligned}$$

where we used the Kantorovich duality formula (see for instance [34]) to get to the last line. By definition of  $(P)_{\mathbb{P}}$  we then have

$$(P)_{\mathbb{P}} = \inf_{\nu \in \mathcal{P}(\Omega)} H^*(\nu) = -H^{**}(0).$$

Therefore, showing that  $(P)_{\mathbb{P}} = (\widetilde{P})_{\mathbb{P}}$  corresponds to show that  $H(0) = H^{**}(0)$ . Since  $H$  is convex (by concavity of the  $c$ -transform operation), this will follow from the continuity of  $H$  at 0 for the supremum-norm over  $\mathcal{C}(\Omega)$  (Proposition 4.1 of [18]). For this, we can first notice that  $H$  never takes the value  $-\infty$ : for any  $\varphi \in \mathcal{C}(\Omega)$  and  $(\phi_{\rho})_{\rho} \in L^1(\mathbb{P}; \mathcal{C}(\Omega))$  such that  $\int_{\mathcal{P}(\Omega)} \phi_{\rho}(\cdot) d\mathbb{P}(\rho) = \varphi(\cdot)$ , one has

$$\forall \rho \in \mathcal{P}(\Omega), \quad -\phi_{\rho}^c(x) = \sup_{y \in \mathbb{R}^d} \phi_{\rho}(y) - \frac{1}{2} \|x - y\|^2 \geq \phi_{\rho}(0) - \frac{1}{2} \|x\|^2.$$

It follows that

$$H(\varphi) \geq \varphi(0) - \int_{\mathcal{P}(\Omega)} \frac{M_2(\rho)}{2} d\mathbb{P}(\rho) > -\infty.$$

On the other hand, notice that  $H$  is bounded from above in a neighborhood of 0 in  $\mathcal{C}(\Omega)$ : for any  $\varphi \in \mathcal{C}(\Omega)$  such that  $\|\varphi\|_\infty \leq 1$ , one has  $-\varphi^c(x) \leq 1$  for any  $x \in \mathbb{R}^d$  so that

$$H(\varphi) \leq - \int_{\mathcal{P}(\Omega)} \langle (\varphi)^c | \rho \rangle d\mathbb{P}(\rho) \leq 1.$$

A standard convex analysis result (Proposition 2.5 in [18]) then ensures that  $H$  is continuous at 0, so that  $H(0) = H^{**}(0)$  and  $(\mathbb{P})_{\mathbb{P}} = \widetilde{(\mathbb{P})}_{\mathbb{P}}$ .

**Restriction to  $L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$ .** We show here that we can run the supremum  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$  only over  $L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  instead of  $L^1(\mathbb{P}; \mathcal{C}(\Omega))$ , that is

$$\widetilde{(\mathbb{P})}_{\mathbb{P}} = \sup \left\{ \int_{\mathcal{P}(\Omega)} \langle \phi_\rho^c | \rho \rangle d\mathbb{P}(\rho), \quad (\phi_\rho)_\rho \in L^\infty(\mathbb{P}; W^{1,\infty}(\Omega)), \quad \int_{\mathcal{P}(\Omega)} \phi_\rho(\cdot) d\mathbb{P}(\rho) = 0 \right\}.$$

Let  $(\phi_\rho)_\rho \in L^1(\mathbb{P}; \mathcal{C}(\Omega))$  be an admissible solution to  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$ , i.e.  $(\phi_\rho)_\rho$  satisfies

$$\int_{\mathcal{P}(\Omega)} \phi_\rho(\cdot) d\mathbb{P}(\rho) = 0. \tag{13}$$

Then we can build from  $(\phi_\rho)_\rho$  another admissible solution  $(\tilde{\phi}_\rho)_\rho$  that belongs to  $L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  and that performs better at  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$ , i.e. that verifies

$$\int_{\mathcal{P}(\Omega)} \langle \tilde{\phi}_\rho^c | \rho \rangle d\mathbb{P}(\rho) \geq \int_{\mathcal{P}(\Omega)} \langle \phi_\rho^c | \rho \rangle d\mathbb{P}(\rho). \tag{14}$$

Indeed, introduce  $(\hat{\phi}_\rho)_\rho := (\phi_\rho^{cc})_\rho$ . Then for all  $\rho \in \mathcal{P}(\Omega)$ ,  $\hat{\phi}_\rho = \phi_\rho^{cc}$  is obviously  $2R$ -Lipschitz (as a  $c$ -transform) and satisfies  $\hat{\phi}_\rho^c = \phi_\rho^c$  and  $\hat{\phi}_\rho \geq \phi_\rho$  (as a double  $c$ -transform). Using then (13), one has that

$$\alpha(\cdot) := \int_{\mathcal{P}(\Omega)} \hat{\phi}_\rho(\cdot) d\mathbb{P}(\rho) \geq 0,$$

where  $\alpha$  is also  $2R$ -Lipschitz. Now denoting  $\tilde{\phi}_\rho = \hat{\phi}_\rho - \alpha$  for all  $\rho \in \mathcal{P}(\Omega)$ , the mapping  $(\tilde{\phi}_\rho)_\rho \in L^1(\mathbb{P}; \mathcal{C}(\Omega))$  is admissible to  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$  by construction and satisfies  $\tilde{\phi}_\rho \leq \hat{\phi}_\rho$  for all  $\rho \in \mathcal{P}(\Omega)$ , so that  $\tilde{\phi}_\rho^c \geq \hat{\phi}_\rho^c = \phi_\rho^c$  (using that the  $c$ -transform is order-reversing). For each  $\rho \in \mathcal{P}(\Omega)$ , up to subtracting  $\tilde{\phi}_\rho(0)$  to  $\tilde{\phi}_\rho$  (this operation leaves  $(\tilde{\phi}_\rho)_\rho$  admissible to  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$  and does not change its value), one can assume that  $\tilde{\phi}_\rho(0) = 0$ . Noticing that  $\tilde{\phi}_\rho$  is  $4R$ -Lipschitz by construction, we thus have the bound  $\|\tilde{\phi}_\rho\|_{W^{1,\infty}(\Omega)} \leq 4R(1+R)$ . We thus have built an admissible  $(\tilde{\phi}_\rho)_\rho \in L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  from an admissible  $(\phi_\rho)_\rho \in L^1(\mathbb{P}; \mathcal{C}(\Omega))$  that satisfies (14), which shows that we can run the supremum  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$  only over  $L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  instead of  $L^1(\mathbb{P}; \mathcal{C}(\Omega))$ .

**Existence of a maximizer.** There now remains to show that the supremum in  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$  can be replaced by a maximum. Let  $((\phi_\rho^n)_\rho)_{n \geq 0}$  be a maximizing sequence to  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$ , and assume from what precedes that this sequence belongs to  $L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  and satisfies for all  $n \geq 0$

and  $\rho \in \mathcal{P}(\Omega)$ ,  $\|\phi_\rho^n\|_{W^{1,\infty}(\Omega)} \leq 4R(1+R)$ . Further assume that this sequence verifies for all  $n \geq 1$ ,

$$\int_{\mathcal{P}(\Omega)} \langle (\phi_\rho^n)^c | \rho \rangle d\mathbb{P}(\rho) \geq \widetilde{(\mathbb{P})}_{\mathbb{P}} - \frac{1}{n}. \quad (15)$$

For any  $n \geq 0$ , the mapping  $(\rho, x) \mapsto \phi_\rho^n(x)$  is bounded in  $L^2(\mathbb{P} \otimes \lambda)$  where  $\lambda$  denotes the Lebesgue measure over  $\Omega$ . Therefore, by Banach-Alaoglu theorem, the sequence  $((\phi_\rho^n)_\rho)_{n \geq 0}$  (seen as a sequence in  $L^2(\mathbb{P} \otimes \lambda)$ ) admits a weakly converging subsequence in  $L^2(\mathbb{P} \otimes \lambda)$ , that we do not relabel and for which we denote  $(\phi_\rho^\infty)_\rho$  the weak limit in  $L^2(\mathbb{P} \otimes \lambda)$ . Using now Mazur's lemma, we know that there exists a sequence of integers  $(N_n)_{n \geq 0}$  and coefficients  $((\lambda_{n,k})_{n \leq k \leq N_n})_{n \geq 0} \geq 0$  satisfying for all  $n \geq 0$ ,  $\sum_{k=n}^{N_n} \lambda_{n,k} = 1$  such that the sequence  $((\bar{\phi}_\rho^n)_\rho)_{n \geq 0}$  defined for all  $n \geq 0$  and  $\rho \in \mathcal{P}(\Omega)$  by  $\bar{\phi}_\rho^n := \sum_{k=n}^{N_n} \lambda_{n,k} \phi_\rho^k$  converges strongly toward  $(\phi_\rho^\infty)_\rho$  in  $L^2(\mathbb{P} \otimes \lambda)$ . By concavity of the  $c$ -transform operation and equation (15), we then have the bound

$$\begin{aligned} \int_{\mathcal{P}(\Omega)} \langle (\bar{\phi}_\rho^n)^c | \rho \rangle d\mathbb{P}(\rho) &\geq \sum_{k=n}^{N_n} \lambda_{n,k} \int_{\mathcal{P}(\Omega)} \langle (\phi_\rho^k)^c | \rho \rangle d\mathbb{P}(\rho) \\ &\geq \sum_{k=n}^{N_n} \lambda_{n,k} \left( \widetilde{(\mathbb{P})}_{\mathbb{P}} - \frac{1}{k} \right) \\ &\geq \widetilde{(\mathbb{P})}_{\mathbb{P}} - \frac{1}{n}. \end{aligned} \quad (16)$$

The sequence  $((\bar{\phi}_\rho^n)_\rho)_{n \geq 0}$  is therefore also a maximizing sequence of  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$  and it also satisfies for any  $n \geq 0$  and  $\rho \in \mathcal{P}(\Omega)$  the bound

$$\|\bar{\phi}_\rho^n\|_{W^{1,\infty}(\Omega)} \leq 4R(1+R). \quad (17)$$

Since the sequence  $((\bar{\phi}_\rho^n)_\rho)_{n \geq 0}$  strongly converges to  $(\phi_\rho^\infty)_\rho$  in  $L^2(\mathbb{P} \otimes \lambda)$ , one can extract a subsequence (that we do not relabel) such that for  $\mathbb{P}$ -almost-every  $\rho \in \mathcal{P}(\Omega)$ , the sequence  $(\bar{\phi}_\rho^n)_{n \geq 0}$  converges to  $\phi_\rho^\infty$  in  $L^2(\lambda)$ . Using (17) and Arzelà-Ascoli theorem, we deduce that for  $\mathbb{P}$ -almost-every  $\rho \in \mathcal{P}(\Omega)$ , the sequence  $(\bar{\phi}_\rho^n)_{n \geq 0}$  converges uniformly to  $\phi_\rho^\infty$  in  $\mathcal{C}(\Omega)$  and that

$$\|\phi_\rho^\infty\|_{W^{1,\infty}(\Omega)} \leq 4R(1+R).$$

In particular,  $(\phi_\rho^\infty)_\rho$  belongs to  $L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  and we have the limit

$$0 = \int_{\mathcal{P}(\Omega)} \bar{\phi}_\rho^n(\cdot) d\mathbb{P}(\rho) \xrightarrow{n \rightarrow \infty} \int_{\mathcal{P}(\Omega)} \phi_\rho^\infty(\cdot) d\mathbb{P}(\rho),$$

so that  $(\phi_\rho^\infty)_\rho$  is admissible to  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$ . Eventually, for  $\mathbb{P}$ -almost-every  $\rho \in \mathcal{P}(\Omega)$ , we have the limit

$$\langle (\bar{\phi}_\rho^n)^c | \rho \rangle \xrightarrow{n \rightarrow \infty} \langle (\phi_\rho^\infty)^c | \rho \rangle, \quad (18)$$

so that by Lebesgue's dominated convergence theorem and bound (16),

$$\int_{\mathcal{P}(\Omega)} \langle (\phi_\rho^\infty)^c | \rho \rangle d\mathbb{P}(\rho) = \lim_{n \rightarrow +\infty} \int_{\mathcal{P}(\Omega)} \langle (\bar{\phi}_\rho^n)^c | \rho \rangle d\mathbb{P}(\rho) = \widetilde{(\mathbb{P})}_{\mathbb{P}},$$

which proves that  $(\phi_\rho^\infty)_\rho \in L^\infty(\mathbb{P}; W^{1,\infty}(\Omega))$  is a maximizer for  $\widetilde{(\mathbb{P})}_{\mathbb{P}}$ .  $\square$

## APPENDIX B. STRONG-CONVEXITY ESTIMATES

This section gathers occurrences of measures  $\rho$  where the strong-convexity estimate of Assumption 1.2 is verified. These results are mostly extracted from [16].

**B.1. Measures with convex support.**

**Proposition B.1.** *Let  $\rho \in \mathcal{P}_{a.c.}(\Omega)$ . Assume that  $\text{spt}(\rho)$  is convex and that there exists  $m_\rho, M_\rho \in (0, +\infty)$  such that  $m_\rho \leq \rho \leq M_\rho$  on  $\text{spt}(\rho)$ . Let  $\psi, \tilde{\psi} \in \mathcal{C}(\Omega)$ . Then*

$$\langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\#} \rho \rangle + C_{d,R,m_\rho,M_\rho} \text{Var}_\rho(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi),$$

where  $C_{d,R,m_\rho,M_\rho} = \left( e(d+1)2^{d+1} R \text{diam}(\text{spt}(\rho)) \left( \frac{M_\rho}{m_\rho} \right)^2 \right)^{-1}$ .

*Proof.* We only present here a formal sketch of the proof, which heavily relies on computations done in Section 2 of [16]. Assuming that  $\psi$  and  $\tilde{\psi}$  are smooth enough (see Proposition 2.4 of [16]) and introducing for  $t \in [0, 1]$ ,  $\psi^t = (1-t)\psi + t\tilde{\psi}$ , Proposition 2.2 of [16] allows to differentiate  $\mathcal{K}_\rho(\psi^t)$  with respect to  $t$  and to obtain:

$$\begin{aligned} \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi) &= \left. \frac{d}{dt} \mathcal{K}_\rho(\psi^t) \right|_{t=0} + \int_0^1 \int_0^s \frac{d^2}{dt^2} \mathcal{K}_\rho(\psi^t) dt ds \\ &= \langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\#} \rho \rangle + \int_0^1 \int_0^s \int_\Omega \langle \nabla v(\nabla(\psi^t)^*) | \mathbf{D}^2(\psi^t)^* \cdot \nabla v(\nabla(\psi^t)^*) \rangle d\rho dt ds, \end{aligned} \quad (19)$$

where  $v = \tilde{\psi} - \psi$ . Reasonning as in the proof of Proposition 2.4 of [16], the Brascamp-Lieb concentration inequality [9] and the log-concavity of the determinant ensure the following bound:

$$C_{R,m_\rho,M_\rho} \min(t, 1-t)^d \text{Var}_{\mu+\tilde{\mu}}(\tilde{\psi} - \psi) \leq \int_\Omega \langle \nabla v(\nabla(\psi^t)^*) | \mathbf{D}^2(\psi^t)^* \cdot \nabla v(\nabla(\psi^t)^*) \rangle d\rho,$$

where  $C_{R,m_\rho,M_\rho} = \left( e R \text{diam}(\text{spt}(\rho)) \left( \frac{M_\rho}{m_\rho} \right)^2 \right)^{-1}$ ,  $\mu = (\nabla \psi^*)_{\#} \rho$  and  $\tilde{\mu} = (\nabla \tilde{\psi})_{\#} \rho$ . Back to (19), this leads to

$$\langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\#} \rho \rangle + C_{d,R,m_\rho,M_\rho} \text{Var}_{\mu+\tilde{\mu}}(\tilde{\psi} - \psi) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi),$$

where  $C_{d,R,m_\rho,M_\rho} = \left( e(d+1)2^{d+1} R \text{diam}(\text{spt}(\rho)) \left( \frac{M_\rho}{m_\rho} \right)^2 \right)^{-1}$ . We conclude using the convex analysis argument of Proposition 3.1 from [16], which directly ensures

$$\text{Var}_\rho(\tilde{\psi}^* - \psi^*) \leq \text{Var}_{\mu+\tilde{\mu}}(\tilde{\psi} - \psi).$$

We get the general case (without the smoothness assumptions on  $\psi$  and  $\tilde{\psi}$ ) using approximation arguments presented in Proposition 2.5 and 2.7 of [16].  $\square$

**B.2. Measures with connected union of convex sets as support.** We extend Proposition B.1 to the case of a source measure  $\rho$  with a possibly non-convex support. We will assume that  $\text{spt}(\rho)$  can be written as a finite union of convex sets, and that  $\rho$  has a positive weighted Cheeger constant, defined by

$$h(\rho) = \inf_{A \subset \text{spt}(\rho)} \frac{|\partial A|_\rho}{\min(\rho(A), \rho(\text{spt}(\rho) \setminus A))},$$

where  $|\partial A|_\rho = \int_{\partial A \cap \text{int}(\text{spt}(\rho))} \rho(x) d\mathcal{H}^{d-1}(x)$  and where the infimum is taken over Lipschitz domains  $A \subset \text{int}(\text{spt}(\rho))$  with boundary of finite  $\mathcal{H}^{d-1}$ -measure. Quoting [23] (Lemma 5.3), we note that this constant is positive whenever the Poincaré-Wirtinger constant  $C_{PW}(\rho)$  of  $\rho$  is finite, by the inequality

$$h(\rho) \geq \frac{2}{C_{PW}(\rho)}.$$

**Proposition B.2.** *Let  $\rho \in \mathcal{P}_{a.c.}(\Omega)$  such that there exists  $m_\rho, M_\rho \in (0, +\infty)$  verifying  $m_\rho \leq \rho \leq M_\rho$  on  $\text{spt}(\rho)$ . Assume that  $h(\rho) > 0$  and that there exists  $N \geq 1$  distinct convex sets  $(C_i)_{1 \leq i \leq N}$  such that  $\text{spt}(\rho) = \bigcup_{i=1}^N C_i$ . Also assume that*

$$\varepsilon := \min \left( \min_{i,j | C_i \cap C_j \neq \emptyset} \rho(C_i \cap C_j), \min_i \rho(C_i \setminus \bigcup_{j \neq i} C_j) \right) > 0.$$

Then there exists a constant  $c_\rho$  depending on  $\rho$  such that for any  $\psi, \tilde{\psi} \in \mathcal{C}(\Omega)$ ,

$$\langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\#} \rho \rangle + c_\rho \text{Var}_\rho(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi).$$

**Remark B.1** (Constant). The constant  $c_\rho$  of Proposition B.2 is not precised and a look at the proof of this proposition only allows to bound  $c_\rho$  in terms of the second smallest eigenvalue of a weighted graph Laplacian, built from the graph whose vertices are the convex sets from the support of  $\rho$  and whose edge weights are the mass  $\rho$  grants to the intersection of the convex sets. It is actually possible to bound this eigenvalue in terms of the Cheeger constant of  $\rho$ . We do not detail this comparison here but only report that  $c_\rho$  may be written

$$c_\rho = \left( e(d+1)2^{d+1}R^2 \left( \frac{M_\rho}{m_\rho} \right)^2 N \left( N + 4 \left( \frac{M_\rho s_{d-1} R^{d-1} N^2}{\varepsilon^2 h(\rho)} \right)^3 \right) \right)^{-1}.$$

*Proof.* Let's denote for now  $f = \tilde{\psi}^* - \psi^*$ . We will first exploit a discrete Laplacian over  $\mathcal{X} = \text{spt}(\rho)$  in order to upper bound  $\text{Var}_\rho(f)$  by a sum of variances of  $f$  w.r.t. probability measures supported over the convex sets  $(C_i)_i$ . We will then use Proposition B.1 to conclude.

For any  $i \in \{1, \dots, N\}$ , we denote  $\rho_i = \frac{1}{\rho(C_i)} \rho|_{C_i}$  and  $m_i = \int_{C_i} f d\rho_i$ . Then one has the following bound:

$$\begin{aligned} \text{Var}_\rho(f) &= \frac{1}{2} \int_{\mathcal{X} \times \mathcal{X}} (f(x) - f(y))^2 d\rho(x) d\rho(y) \\ &\leq \frac{1}{2} \sum_{i,j} \int_{C_i \times C_j} (f(x) - f(y))^2 d\rho(x) d\rho(y) \\ &= \frac{1}{2} \sum_{i,j} \int_{C_i \times C_j} (f(x) - m_i + m_i - m_j + m_j - f(y))^2 d\rho(x) d\rho(y) \\ &= \left( \sum_i \rho(C_i) \right) \sum_i \int_{C_i} (f(x) - m_i)^2 d\rho(x) + \frac{1}{2} \sum_{i,j} (m_i - m_j)^2 \rho(C_i) \rho(C_j) \\ &= \left( \sum_i \rho(C_i) \right) \sum_i \rho(C_i) \text{Var}_{\rho_i}(f) + \frac{1}{2} \sum_{i,j} (m_i - m_j)^2 \rho(C_i) \rho(C_j). \end{aligned} \quad (20)$$

We now consider the graph  $G = (\{C_i\}_{1 \leq i \leq N}, \{w_{ij}\}_{1 \leq i, j \leq N})$  with vertices  $\{C_i\}_{1 \leq i \leq N}$  and weighted edges  $\{w_{ij}\}_{1 \leq i, j \leq N}$  defined by

$$\forall i, j \in \{1, \dots, N\}, \quad w_{ij} = \rho(C_i \cap C_j).$$

By construction, this graph has a single connected component. We introduce the weighted Laplacian matrix  $L \in \mathbb{R}^{N \times N}$  of  $G$  as follows:

$$\forall i, j \in \{1, \dots, N\}, \quad L_{ij} = \begin{cases} \sum_k w_{ik} & \text{if } i = j, \\ -w_{ij} & \text{else.} \end{cases}$$

Then  $L$  is a symmetric and positive semi-definite matrix. Its null space is made of constant vectors and we denote  $\lambda_2(L)$  its second smallest eigenvalue, which is non-zero. Denoting  $m = (m_i)_{1 \leq i \leq N} \in \mathbb{R}^N$ , we introduce  $\bar{m} = (\frac{1}{N} \sum_i m_i) \mathbf{1}_N \in \mathbb{R}^N$  the constant vector whose coordinates equal the mean of  $m$  (we use  $\mathbf{1}_N = (1)_{1 \leq i \leq N} \in \mathbb{R}^N$ ). Notice that  $m - \bar{m}$  is in the orthogonal to the null space of  $L$ , ensuring the following bound:

$$\begin{aligned} \frac{1}{2} \sum_{i,j} (m_i - m_j)^2 \rho(C_i) \rho(C_j) &\leq N^2 \frac{1}{2} \sum_{i,j} (m_i - m_j)^2 \frac{1}{N^2} \\ &= N \|m - \bar{m}\|^2 \\ &\leq \frac{N}{\lambda_2(L)} \langle m - \bar{m} | L (m - \bar{m}) \rangle \\ &= \frac{N}{\lambda_2(L)} \sum_{i,j} w_{ij} (m_i^2 - m_i m_j) \\ &= \frac{N}{\lambda_2(L)} \sum_{i,j} \frac{w_{ij}}{2} (m_i - m_j)^2. \end{aligned} \quad (21)$$

But for any  $i, j$  such that  $w_{ij} > 0$ , denoting  $m_{i \cap j} = \frac{1}{\rho(C_i \cap C_j)} \int_{C_i \cap C_j} f d\rho$ , one has

$$\frac{1}{2} (m_i - m_j)^2 \leq (m_{i \cap j} - m_i)^2 + (m_{i \cap j} - m_j)^2.$$

And for such  $i, j$ ,

$$\begin{aligned} (m_{i \cap j} - m_i)^2 &= \left( \frac{1}{\rho(C_i \cap C_j)} \int_{C_i \cap C_j} (f - m_i) d\rho \right)^2 \\ &\leq \frac{1}{\rho(C_i \cap C_j)} \int_{C_i} (f - m_i)^2 d\rho \\ &= \frac{\rho(C_i)}{w_{ij}} \mathbb{V}\text{ar}_{\rho_i}(f), \end{aligned}$$

where we used Jensen's inequality and the fact that  $C_i \cap C_j \subset C_i$ . A similar bound can be shown for  $(m_{i \cap j} - m_j)^2$ , and plugging these into (21) yields

$$\begin{aligned} \frac{1}{2} \sum_{i,j} (m_i - m_j)^2 \rho(C_i) \rho(C_j) &\leq \frac{N}{\lambda_2(L)} \sum_i \sum_{j | C_i \cap C_j \neq \emptyset} (\rho(C_i) \mathbb{V}\text{ar}_{\rho_i}(f) + \rho(C_j) \mathbb{V}\text{ar}_{\rho_j}(f)) \\ &\leq \frac{2N^2}{\lambda_2(L)} \sum_i \rho(C_i) \mathbb{V}\text{ar}_{\rho_i}(f). \end{aligned}$$

Injecting this into (20) yields

$$\mathbb{V}\text{ar}_{\rho}(f) \leq \left( N + \frac{2N^2}{\lambda_2(L)} \right) \sum_i \rho(C_i) \mathbb{V}\text{ar}_{\rho_i}(f). \quad (22)$$



Now recalling that  $f = \psi - \tilde{\psi}$ , we have by Proposition B.1 for any  $i \in \{1, \dots, N\}$  that

$$\langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\# \rho_i} \rangle + C_{d,R,m_\rho,M_\rho} \text{Var}_{\rho_i}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_{\rho_i}(\tilde{\psi}) - \mathcal{K}_{\rho_i}(\psi),$$

where  $C_{d,R,m_\rho,M_\rho} = \left( e(d+1)2^{d+1}R^2 \left( \frac{M_\rho}{m_\rho} \right)^2 \right)^{-1}$ . Weighting this last inequality with  $\rho(C_i)$  and summing over  $i \in \{1, \dots, N\}$ , this raises

$$\langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\# \rho} \rangle + \frac{C_{d,R,m_\rho,M_\rho}}{N} \sum_{i=1}^N \rho(C_i) \text{Var}_{\rho_i}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi).$$

Using (22) eventually gives

$$\langle \psi - \tilde{\psi} | (\nabla \psi^*)_{\# \rho} \rangle + c_\rho \text{Var}_\rho(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi),$$

where  $c_\rho = \left( e(d+1)2^{d+1}R^2 \left( \frac{M_\rho}{m_\rho} \right)^2 \left( N^2 + \frac{2N^3}{\lambda_2(L)} \right) \right)^{-1}$ . □

CEREMADE, UNIV. PARIS-DAUPHINE PSL, 75775 PARIS AND MOKAPLAN, INRIA PARIS  
 Email address: [carlier@ceremade.dauphine.fr](mailto:carlier@ceremade.dauphine.fr)

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, 91405, ORSAY, FRANCE  
 AND INRIA SACLAY, ILE-DE-FRANCE  
 Email address: [alex.delalande@inria.fr](mailto:alex.delalande@inria.fr)

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, 91405, ORSAY, FRANCE  
 AND INSTITUT UNIVERSITAIRE DE FRANCE  
 Email address: [quentin.merigot@universite-paris-saclay.fr](mailto:quentin.merigot@universite-paris-saclay.fr)