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# Hydrodynamic limit for the Kob-Andersen model

Assaf Shapira

ABSTRACT. This paper concerns with the hydrodynamic limit of the Kob-Andersen model, an interacting particle system that has been introduced by physicists in order to explain glassy behavior, and widely studied since. We will see that the density profile evolves in the hydrodynamic limit according to a non-degenerate hydrodynamic equation, and understand how the diffusion coefficient decays as density grows.

## 1. Introduction

The Kob-Andersen (KA) model is an interacting particle system on  $\mathbb{Z}^d$ , where each site of the lattice is allowed to contain at most one particle, and particles could jump to an empty neighboring site only under a certain constraint, conserving the total number of particles. More precisely, depending on an integer parameter  $k$ , every particle jumps with rate 1 to each of its neighboring sites, provided that the particle has at least  $k$  empty neighbors both before and after the jump (so for  $k = 1$  we obtain the symmetric simple exclusion process). This model has been introduced in the physics literature ([16]) as one member of a large family of interacting particle systems called *kinetically constrained lattice gases* (KCLGs), which model certain aspects of glassy behavior (see [12, 20]).

In this paper we will study the hydrodynamic limit of the KA model. Consider a finite box with periodic boundary conditions  $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$ , and run the KA dynamics inside  $\mathbb{T}_N^d$ . The configuration at time  $s$  could be described as an empirical measure  $\nu_s^{(N)}$  on the continuous torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ : for a rectangle  $R \subset [0, 1]^d$ , seen as a subset of  $\mathbb{T}^d$ ,  $\nu_s^{(N)}(R)$  will count the number of particles in  $(NR) \cap \mathbb{T}_N^d$ , normalized by  $N^{-d}$  (so that the total mass remains independent of  $N$ ). The initial configuration that we choose will be approximated by some macroscopic profile  $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$ , i.e., the measure  $\nu_0^{(N)}$  will be close to a measure  $\nu_0$  that has density  $\rho_0$  with respect to the Lebesgue measure. A simple example of such initial configuration is given by placing a particle at each site  $x \in \mathbb{Z}^d$  independently at random with probability  $\rho_0(x/N)$ .

In many systems, the relevant time scale over which  $\nu_N^s$  changes macroscopically is the diffusive time scale  $N^2$  (see, e.g., [15, 23]). That is, fixing a time  $t$ , we expect the random measure  $\nu_{N^2 t}^{(N)}$  to satisfy a law of large numbers, converging to some limiting measure  $\nu_t$ . We also expect this limiting measure to have a density with respect to the Lebesgue measure, namely  $\nu_t = \rho(\theta, t)d\theta$ , which solves the diffusion equation:

$$\frac{\partial}{\partial t} \rho = \nabla D(\rho) \nabla \rho, \quad \rho(\theta, 0) = \rho_0(\theta). \quad (1.1)$$

28 The parameter  $D(\rho)$  is the *diffusion coefficient*, and when it is non-zero we obtain indeed a  
 29 macroscopic density profile that changes over diffusive time scales.

30 Hydrodynamic limits of other KCLGs have been analyzed in [13, 4]. They present two  
 31 example of *non-cooperative* KCLGs, in which one is able to identify structures (called *mobile*  
 32 *clusters*) that could move freely in  $\mathbb{Z}^d$  (see Definition A.3). This way, even though particles  
 33 could be blocked, mobile clusters behave effectively in an unconstrained manner. In *cooper-*  
 34 *ative* KCLGs there are no such mobile clusters, so in order to move a particle from one site  
 35 to the other one needs the cooperation of a diverging number of particles. This property has  
 36 a major contribution to the glassy behavior of many KCLGs, and is responsible for the fast  
 37 divergence of time scales. See [25].

38 Unlike the models previously studied in [13, 4], the KA model is cooperative. Due to this  
 39 cooperative nature, the combinatorics behind the KA model becomes much more compli-  
 40 cated. Consider the following question – starting from a stationary measure and assuming  
 41 that there is a particle at the origin, will this particle eventually move, or could it stay at the  
 42 origin forever? When the model is non-cooperative the probability to stay forever at the ori-  
 43 gin is clearly 0 – we know that there is some non-zero density of mobile clusters in  $\mathbb{Z}^d$  which  
 44 diffuse freely, so at some point one of them will reach the origin and move the particle. When  
 45 the model is cooperative, as in the case of the KA model, already this basic question becomes  
 46 much more complicated. In some cooperative models the particle might remain blocked for-  
 47 ever with positive probability, possibly depending on the density of the initial configuration.  
 48 In the case of the KA model, it is shown in [25] that all particles will eventually move with  
 49 probability 1, unless the initial density equals 1.

50 In the context of the hydrodynamic limit, the techniques used in [13, 4] cannot be simply  
 51 adapted to cooperative models. It is shown in Appendix A that cooperative KCLGs are non-  
 52 gradient, a fact which makes the analysis of the hydrodynamic limit much more involved.  
 53 See Appendix A for a more complete discussion. Another property of non-cooperative models  
 54 used in [4] is that the probability for a site to stay blocked forever for the dynamics in  $\mathbb{T}_N^d$   
 55 decreases exponentially fast with the volume  $N^d$ , since it is bounded by the probability that  
 56 no mobile cluster is found in  $\mathbb{T}_N^d$ . In the KA model, on the other hand, even though this  
 57 probability decays to 0, the decay is not fast enough.

58 Recently, a few methods have been developed to overcome some of these difficulties, prov-  
 59 ing diffusive scaling of the relaxation time [17] and of the motion of a tagged particle [5, 9]  
 60 in the stationary setting. In both cases, the behavior is the same as that of the simple exclu-  
 61 sion process, with time scales that are all slowed down by a factor which diverges quickly as  
 62 the density approaches 1. For example, in the case  $k = d = 2$ , the relaxation time at density  $\rho$   
 63 in a box of side  $N$  behaves (roughly) like  $e^{C/(1-\rho)} N^2$ ; and the path of a tagged particle in  $\mathbb{Z}^2$   
 64 converges to a standard Brownian motion as the length scale  $N$  diverges, when time is scaled  
 65 (roughly) as  $e^{C/(1-\rho)} N^2$ .

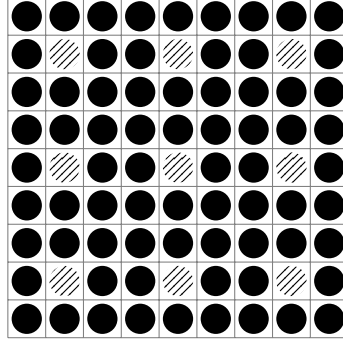


FIGURE 1.1. This is an example of a blocked configuration for the case  $k = d = 2$ . The filled circles represent occupied sites, while sites marked with a line pattern could be either empty or occupied. In this example particles have at most one empty neighbor, so none of them could move.

66 The hydrodynamic limit of the KA model has been studied in the physics literature, both  
 67 heuristically and numerically. In [21] the model has been analyzed, under the (wrong) as-  
 68 sumption that the diffusion coefficient  $D(\rho)$  vanishes for  $\rho > \rho_c \approx 0.88$ . [24] studies the  
 69 diffusion coefficient in two dimensions both numerically and under a mean-field approxima-  
 70 tion. This approximation yields a diffusion coefficient that behaves polynomially in  $\rho$ , and is  
 71 in rather good agreement with numerical results for low densities. [1] provides a perturba-  
 72 tive analysis of the diffusion coefficient in two dimensions, considering finite range effects,  
 73 and obtaining a polynomial in  $\rho$  which approximates  $D(\rho)$  very accurately as long as  $\rho$  is not  
 74 too big. In view of other quantities related to the KA model studied in [17, 9], a natural con-  
 75 jecture for the high density regime is that the diffusion coefficient remains positive whenever  
 76  $\rho < 1$ , and as  $\rho$  tends to 1 it decays (roughly) as  $e^{-C/(1-\rho)}$  (in the case  $k = d = 2$ ). This  
 77 conjecture has been raised in [1] and was supported by numerical simulations.

78 The hydrodynamic limit in its full generality, though, cannot exist for this model – consider,  
 79 for example, the case  $k = d = 2$ , and an initial density  $\rho_0$  bounded above  $\frac{8}{9}$ . Fix  $N \in 3\mathbb{N}$ ,  
 80 and construct the following initial configuration – for every  $x \in \mathbb{T}_N^2$ , if  $x \notin 3\mathbb{Z}^2$  place a  
 81 particle at  $x$  (deterministically). Otherwise, place a particle at  $x$  independently at random  
 82 with probability  $9\rho_0(x/N) - 8$ . See Figure 1.1. These configurations have limiting density  
 83  $\frac{1}{9}(9\rho_0(x/N) - 8) + \frac{8}{9} = \rho_0$ , so one may naively expect that, starting the KA dynamics from  
 84 such a configuration, the particle density will converge to the solution of the hydrodynamic  
 85 equation (1.1) with initial density  $\rho_0$ . However, observing the initial configuration more  
 86 carefully, one sees that it is blocked – no site has two empty neighbors, so the constraint is  
 87 not satisfied. In this case particles do not move, and the dynamics will certainly not follow  
 88 the hydrodynamic limit. Still, since blocked configurations are very rare ([25]), we may hope  
 89 that a hydrodynamic limit does exist in a weaker sense, that would allow us to avoid these  
 90 untypical configurations.

91 The same problem also appears in [13], and they suggest two solutions – the first is to re-  
 92 strict the initial configuration, e.g., to an independent product of Bernoulli random variables

with parameter  $\rho_0(x/N)$ . This prevents the issue discussed above, where the configuration is entirely blocked from the beginning, but one must work harder in order to show that blocked configurations are not created later on during the dynamics. Another approach, also considered in [13], is to permit transitions in which the constraint is not satisfied, but with a vanishing rate. Namely, for some  $\varepsilon > 0$ , we introduce soft constraints, which allow a particle to move with rate 1 when it has  $k$  empty neighbors before and after the jump, and with rate  $\varepsilon$  otherwise. This softening of the constraint enables the system to unblock the blocked configurations, and still the main contribution to the overall dynamics comes from the allowed transitions (where the constraint is satisfied).

This is the approach we will take – consider the KA model with  $\varepsilon$ -soft constraints, which has a hydrodynamic limit with diffusion coefficient  $D^{(\varepsilon)}$ . We analyze this coefficient, showing that, as  $\varepsilon \rightarrow 0$ , it converges to a strictly positive limiting coefficient  $D$ . This result tells us that when  $\varepsilon$  is very small, it has a very mild effect on the hydrodynamic limit; and the role it plays (of unblocking configurations), though crucial for the convergence to the hydrodynamic limit, takes a negligible amount of time compared to the hydrodynamic scale. We also analyze the value of  $D$  at large densities, finding upper and lower bounds for its decay, which match up to sub-leading corrections. The decay that we obtain is of the same type as the corresponding factor in [17, 9]; so in particular for the case  $k = d = 2$ , as conjectured is [1],  $D$  decays (roughly) as  $e^{-C/(1-\rho)}$ .

## 2. Model and main result

The Kob-Andersen model in dimension  $d$  is a Markov process on  $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ , depending on a parameter  $2 \leq k \leq d$ . For a configuration  $\eta \in \Omega$ , we say that  $x \in \mathbb{Z}^d$  is *occupied* if  $\eta(x) = 1$  and *empty* if  $\eta(x) = 0$ . The elements of  $\mathbb{Z}^d$  are called *sites*, and we will consider the (undirected) graph structure given by the edge set

$$\mathcal{E}(\mathbb{Z}^d) = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d, y \in x + \{\pm e_1, \dots, \pm e_d\}\},$$

where  $e_1, \dots, e_d$  are the standard basis vectors. We will sometimes write  $x \sim y$  to denote  $(x, y) \in \mathcal{E}(\mathbb{Z}^d)$ .

For each configuration  $\eta \in \Omega$  and edge  $(x, y) \in \mathcal{E}(\mathbb{Z}^d)$ , we define the constraint

$$c_{x,y}(\eta) = \begin{cases} 1 & \text{if } \sum_{z: y \sim z \neq x} (1 - \eta(z)) \geq k - 1 \text{ and } \sum_{z: x \sim z \neq y} (1 - \eta(z)) \geq k - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

The KA dynamics is then defined as the Markov process whose generator, operating on a local function  $f : \Omega \rightarrow \mathbb{R}$ , is given by

$$\mathcal{L}f(\eta) = \sum_{(x,y) \in \mathcal{E}(\mathbb{Z}^d)} c_{x,y}(\eta) \nabla_{x,y} f(\eta), \quad (2.2)$$

122 where

$$\nabla_{x,y}f(\eta) = f(\eta^{x,y}) - f(\eta),$$

123 and  $\eta^{x,y}$  is the configuration obtained from  $\eta$  by exchanging the occupation at  $x$  and at  $y$ . This  
 124 process, for any  $\rho \in (0, 1)$ , is reversible with respect to the measure  $\mu_\rho$ , which is a product  
 125 measure of Bernoulli random variables with parameter  $\rho$ . This is a consequence of the fact  
 126 that  $c_{x,y}(\eta)$  does not depend on the occupation at  $x$  and at  $y$ . When clear from the context we  
 127 will sometimes omit the subscript  $\rho$ . For more details on the construction of the model see  
 128 [6].

129 As discussed in the introduction, in order to study the hydrodynamic limit we introduce  
 130 the *soft constraint* for some  $\varepsilon \geq 0$ :

$$c_{x,y}^{(\varepsilon)} = \begin{cases} 1 & \text{if } c_{x,y} = 1, \\ \varepsilon & \text{otherwise,} \end{cases} \quad (2.3)$$

131 and the *soft dynamics* defined by the generator

$$\mathcal{L}^{(\varepsilon)}f(\eta) = \sum_{(x,y) \in \mathcal{E}(\mathbb{Z}^d)} c_{x,y}^{(\varepsilon)}(\eta) \nabla_{x,y}f(\eta). \quad (2.4)$$

132 The introduction of the soft constraints allows us to use the general result of [11, 26]. Fix  
 133  $\varepsilon \geq 0$ , and let

$$D^{(\varepsilon)}(\rho) = \frac{1}{2\rho(1-\rho)} \inf_f \mu_\rho \left[ \sum_{\alpha} c_{0,e_\alpha}^{(\varepsilon)} \left( \delta_{\alpha,1} (\eta(e_1) - \eta(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right], \quad (2.5)$$

134 where the infimum is taken over all local functions  $f : \Omega \rightarrow \mathbb{R}$ ; a function is *local* if it depends  
 135 on the occupation of finitely many sites. Note that the sum over  $x$  is well defined thanks to  
 136 the locality of  $f$ . The operator  $\tau_x$  is the translation by  $x$ , that is,

$$\begin{aligned} (\tau_x f)(\eta) &= f(\tau_x \eta), \\ (\tau_x \eta)(y) &= \eta(x + y). \end{aligned}$$

137 In this setting, by [11, 3, 26], the density profile of the soft dynamics converges in the hydro-  
 138 dynamic limit to the solution of the hydrodynamic equation (1.1), with diffusion coefficient  
 139  $D^{(\varepsilon)}(\rho)$ .

140 By equation (2.5) the diffusion coefficient is decreasing with  $\varepsilon$ , and hence converging to a  
 141 limit:

$$D(\rho) = \lim_{\varepsilon \rightarrow 0} D^{(\varepsilon)}(\rho). \quad (2.6)$$

142 When taking  $\varepsilon$  to 0 slowly enough as  $N$  grows to infinity, the density profile converges to the  
 143 solution of the diffusion equation (1.1) with this diffusion coefficient:

144 **Proposition 2.1.** *Fix a smooth initial density profile  $\rho_0 : \mathbb{T}^d \rightarrow (0, 1)$ , and let  $\nu_0$  be the measure*  
 145 *whose density with respect to the Lebesgue measure is  $\rho_0$ . Consider a sequence of initial conditions*

146  $(\eta_0^{(N)})_{N \in \mathbb{N}}$ , with  $\eta_0^{(N)} \in \{0, 1\}^{\mathbb{T}_N^d}$ , such that the associated empirical measures  $\nu_0^{(N)}$  converge to  
 147  $\nu_0$ . Let  $\rho_t(\theta)$  be the solution of the diffusion equation (1.1), with diffusion coefficient  $D$  given by  
 148 equation (2.6), and  $\nu_t$  the measure with density  $\rho_t$ . For  $s \geq 0$ , denote by  $\nu_s^{(\varepsilon, N)}$  the (random)  
 149 empirical measure associated with the Kob-Andersen model on  $\mathbb{T}_N^d$  with  $\varepsilon$ -soft constraints at  
 150 (microscopic) time  $s$ , with the initial configuration  $\eta_0^{(N)}$ .

151 Then there exists a sequence  $(\varepsilon_N)_{N \in \mathbb{N}}$  for which  $\nu_{N^2 s}^{(\varepsilon_N, N)}$  converges in probability to  $\nu_t$  as  $N$   
 152 tends to infinity.

153 **Remark 2.2.** In general, the diffusion coefficient is a matrix given by (see [23, Propositions  
 154 2.1 and 2.2])

$$D_{\alpha\beta} = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{1}{2\rho(1-\rho)} \sum_{x \in \mathbb{Z}^d} x_\alpha x_\beta (\mu_\rho(\eta(0) e^{t\mathcal{L}} \eta(x)) - \rho^2).$$

155 The reason that  $D^{(\varepsilon)}(\rho)$  in equation (2.5) is a real number, is that in our case  $D$  is a scalar  
 156 matrix: the dynamics is invariant under inversion of a single coordinate (i.e.,  $x \mapsto x -$   
 157  $(2x \cdot e_\alpha) e_\alpha$ ), and therefore, if  $\alpha \neq \beta$ , the sum  $\sum_{x \in \mathbb{Z}^d} x_\alpha x_\beta (\mu(\eta(0) e^{t\mathcal{L}} \eta(x)) - \rho^2)$  must vanish.  
 158 That is,  $D$  is a diagonal matrix. Since the dynamics is also invariant under permutation  
 159 of coordinates, all diagonal elements are equal, i.e.,  $D$  is scalar. This fact is useful for the  
 160 analysis of the limiting PDE in the proof of Proposition 2.1.

161 The main result of this paper is that  $D$  is strictly positive, so that the hydrodynamic limit is  
 162 not degenerate, i.e., the density profile evolves over diffusive time scales.

163 **Theorem 2.3.** For all  $\rho \in (0, 1)$ ,

$$D(\rho) \geq \begin{cases} C / \exp(\lambda \log(1/(1-\rho))^2 (1-\rho)^{-1/(d-1)}) & k = 2, \\ C / \exp^{k-1}(\lambda(1-\rho)^{-1/(d-k+1)}) & k \geq 3, \end{cases}$$

$$D(\rho) \leq C' / \exp^{k-1}(\lambda'(1-\rho)^{-1/(d-k+1)}),$$

164 where  $\exp^k(\cdot)$  is the  $k$ -th iterate of the exponential. The constants  $C, C', \lambda, \lambda'$  are all strictly  
 165 positive, and may depend only on  $d$  and  $k$ .

### 166 3. Proof of Proposition 2.1

167 The proof is based on the results of [11, 3, 26], together with the continuity of the solution  
 168  $\rho_t(\theta)$  with respect to the diffusion coefficient.

169 Denote by  $\rho_t^{(\varepsilon)}$  the solution of the diffusion equation (1.1) with diffusion coefficient  $D^{(\varepsilon)}$   
 170 given by equation (2.5). The existence and uniqueness of  $\rho_t$  and  $\rho_t^\varepsilon$ , as well the maximum  
 171 principle, comes from the theory of parabolic equations (see, e.g., [27])<sup>1</sup>. Indeed, since  $D$   
 172 is bounded and isotropic (see Remark 2.2), we may define a continuous increasing positive

<sup>1</sup>Most works treat the equation on  $\mathbb{R}^d$  rather than the torus. Nonetheless, the results we need hold also for the equation on  $\mathbb{T}^d$ , see the discussion in Section 11.5 of [27].

173 function  $\Phi(u) = \int_0^u D(\rho) \, d\rho$ , allowing us to write equation (1.1) as

$$\partial_t \rho = \Delta \Phi(\rho),$$

174 known as the *generalized porous medium equation*, or the *filtration equation*. It is the subject  
175 of [2], and discussed thoroughly in [27].

176 The main tool we use is:

177 **Theorem 3.1** ([11, 3, 26]). *For any smooth test function  $f$  on  $\mathbb{T}^d$ ,*

$$\int f(\theta) \, d\nu_{N^2t}^{(\varepsilon, N)}(\theta) \xrightarrow{N \rightarrow \infty} \int f(\theta) \rho_t^\varepsilon(\theta) \, d\theta$$

178 *in probability.*

179 In addition, we need to know that, for small  $\varepsilon$ , the profile  $\rho_t^{(\varepsilon)}$  is close to  $\rho_t$ . This problem  
180 is analyzed in [2] in a much more complicated setting, where  $D(\rho)$  may approach 0 in some  
181 points of space. Since we only consider the case where  $\rho$  is bounded away from 1, the  
182 assumptions of [2] are easily verified, yielding:

183 **Claim 3.2.** For all  $t > 0$

$$\rho_t^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} \rho_t$$

184 *in  $L^1(\mathbb{T}^d)$ .*

185 In order to prove Proposition 2.1, we will fix a dense countable family  $\{f_m\}_{m \in \mathbb{N}}$  of bounded  
186 functions on  $\mathbb{T}^d$ . Then, as discussed in [15, Chapter 4.1], it suffices to show that

$$P \left[ \left| \int f_m(\theta) \, d\nu_{N^2t}^{(\varepsilon_N, N)}(\theta) - \int f_m(\theta) \rho_t(\theta) \, d\theta \right| > \delta \right] \xrightarrow{N \rightarrow \infty} 0 \quad (3.1)$$

187 for any fixed  $\delta > 0$  and all  $m \in \mathbb{N}$ , for an appropriately chosen sequence  $\{\varepsilon_N\}$ .

188 An immediate corollary of Theorem 3.1 and Claim 3.2 is:

189 **Corollary 3.3.** *Fix  $M > 0$ . Then there exists  $\varepsilon(M), N(M)$  such that, for all  $m \leq M$  and*  
190  *$N \geq N(M)$ ,*

$$P \left[ \left| \int f_m(\theta) \, d\nu_{N^2t}^{(\varepsilon(M), N)}(\theta) - \int f_m(\theta) \rho_t(\theta) \, d\theta \right| > \delta \right] \leq \frac{1}{M}.$$

191 *Moreover, we may assume  $\varepsilon(M) \rightarrow 0$  and  $N(M) \rightarrow \infty$  as  $M \rightarrow \infty$ .*

192 *Proof.* By Claim 3.2, for all  $m$  there exists  $\varepsilon^*(m)$  such that

$$\left| \int f_m(\theta) \rho_t^{(\varepsilon)}(\theta) \, d\theta - \int f_m(\theta) \rho_t(\theta) \, d\theta \right| < \delta/2$$

193 for all  $\varepsilon < \varepsilon^*(m)$ . We will choose  $\varepsilon(M) = \min_{m < M} \varepsilon^*(m) \wedge \frac{1}{M}$ .

194 By Theorem 3.1, for any  $m$  and any  $\varepsilon$  there exists  $N(m, \varepsilon)$ , such that if  $N \geq N(m, \varepsilon)$  then

$$P \left[ \left| \int f_m(\theta) \, d\nu_{N^2t}^{(\varepsilon, N)}(\theta) - \int f_m(\theta) \rho_t^{(\varepsilon)}(\theta) \, d\theta \right| > \delta/2 \right] \leq \frac{1}{M}.$$

195 Define  $N(M) = \max_{m \leq M} N(m, \varepsilon(M)) \vee M$ . This concludes the proof of the corollary.  $\square$



196 We are now ready to choose our sequence  $\varepsilon_N$ :

$$M_N = \max\{M : N \geq N(M)\},$$

$$\varepsilon_N = \varepsilon(M_N).$$

197 Then, indeed,

$$P \left[ \left| \int f_m(\theta) d\nu_{N^{2t}}^{\varepsilon_N, N}(\theta) - \int f_m(\theta) \rho(\theta) d\theta \right| > \delta \right] \leq \frac{1}{M_N}$$

198 for all  $m \leq M_N$ . This concludes the proof of the proposition.  $\square$

#### 199 4. Proof of the lower bound

200 The purpose of this section is to prove

$$D^{(0)} \geq L^{-\lambda}, \tag{4.1}$$

$$L = \begin{cases} C \exp(\lambda \log(1/q)^2 q^{-1/(d-1)}) & k = 2, \\ C \exp^{k-1}(\lambda q^{-1/(d-k+1)}) & k \geq 3, \end{cases} \tag{4.2}$$

201 where for convenience we denote  $1 - \rho = q$ . Throughout the section  $\lambda$  and  $C$  denote generic  
202 positive constants, depending only on  $k$  and  $d$ , that may be updated from one line to the  
203 other. This will prove the first inequality of Theorem 2.3 since  $D \geq D^{(0)}$ .

204 The proof is based on a comparison to the diffusion coefficient of a random walk on the  
205 infinite component of a percolation cluster. The idea behind the proof, is that even though  
206 at small scale particles are blocked, at a large scale there is high probability that somewhere  
207 a droplet containing many empty sites could approach the particle allowing it to move; and  
208 this is the scale which determines the diffusion coefficient. This mechanism is constructed in  
209 [17, 9] using the notion of a *multistep move* – a sequence of exchanges, all allowed for the  
210 KA dynamics, moving a particle with the aid of a nearby vacancies.

211 We start by providing the exact definition of a multistep move (see also [17]):

212 **Definition 4.1** (multistep move). Fix  $\mathcal{M} \subseteq \Omega$  and  $T \in \mathbb{N}$ . A  $T$ -step move  $M$  with domain  $\mathcal{M}$  is  
213 a function from  $\mathcal{M}$  to  $(\Omega \times \mathbb{Z}^d \times \{\pm e_1, \dots, \pm e_d, 0\})^{T+1}$ , described by a sequence of functions  
214  $M = \{\eta_t(\eta), x^t(\eta), e^t(\eta)\}_{t=0}^T$ , such that, for all  $\eta \in \mathcal{M}$ ,

- 215 (1)  $\eta_0(\eta) = \eta$ ,
- 216 (2) for all  $t \in \{1, \dots, T\}$ ,  $\eta_t(\eta) = \eta_{t-1}(\eta)^{x^t, x^t + e^t}$ ,
- 217 (3) for all  $t \in \{1, \dots, T\}$ ,  $c_{x^t, x^t + e^t}(\eta_t(\eta)) = 1$ , where by convention we set  $c_{x, x}(\eta) = 1$  for  
218 all  $x, \eta$ .

219 **Warning:**  $t$  in the above definition does *not* stand for the (continuous) time in which  
220 the process evolves. It is a discrete variable indexing steps in a multistep move.

**Definition 4.2.** Fix a  $T$ -step move  $M$  with domain  $\mathcal{M}$ . Then, for  $t \in \{1, \dots, T\}$ , the loss of information at time  $t$ , denoted  $\text{Loss}_t(M)$ , is defined as

$$2^{\text{Loss}_t(M)} = \sup_{\eta' \in \mathcal{M}} \#\{\eta \in \mathcal{M} : \eta_t(\eta) = \eta_t(\eta'), x^t(\eta) = x^t(\eta'), e^t(\eta) = e^t(\eta')\}.$$

We also set  $\text{Loss}(M) = \sup_t \text{Loss}_t(M)$ .

The multistep move that we will define will allow us to move a particle at  $x$  to the site  $x + Le_\alpha$  ( $\alpha \in \{1, \dots, d\}$ ). The choice of  $L$  in equation (4.2) guarantees that such a multistep move could indeed be applied. Note that  $C, \lambda$  can be chosen such that  $L \in \mathbb{N}$ .

We will therefore consider the coarse grained lattice  $\mathbb{Z}_L^d = L\mathbb{Z}^d$ , and split a generic configuration  $\eta$  in two – the occupation of the sites of  $\mathbb{Z}_L^d$  denoted  $\bar{\eta} \in \bar{\Omega}_L = \{0, 1\}^{\mathbb{Z}_L^d}$ , and that of the sites outside  $\mathbb{Z}_L^d$  denoted  $\hat{\eta} \in \{0, 1\}^{\mathbb{Z}^d \setminus \mathbb{Z}_L^d}$ . We will also split the measure in two, such that  $\bar{\eta}$  distributes according to  $\bar{\mu}$  and  $\hat{\eta}$  according to  $\hat{\mu}$ . Note that both measures  $\bar{\mu}, \hat{\mu}$  are Bernoulli product measure with parameter  $\rho$  (which is implicit in the notation). The coarse grained lattice has a graph structure (isomorphic to  $\mathbb{Z}^d$ ), i.e., two vertices  $i, j$  are connected by an edge if  $i - j \in \{\pm \bar{e}_1, \dots, \pm \bar{e}_d\}$ , where  $\bar{e}_\alpha = Le_\alpha$ . We denote the edge set by  $\mathcal{E}(\mathbb{Z}_L^d)$ .

Next, we will define a multistep move allowing particles to move on the coarse grained lattice  $\mathbb{Z}_L^d$ , that is, it will exchange the occupation at  $Li$  with the occupation at  $Lj$  for some edge  $(i, j) \in \mathcal{E}(\mathbb{Z}_L^d)$ . In order to achieve that, there must be sufficiently many empty sites organized in a proper fashion in the region between  $i$  and  $j$ . We can think of an edge in  $\mathcal{E}(\mathbb{Z}_L^d)$  satisfying this condition as *open*, and otherwise *closed*, defining a percolation process on  $\mathbb{Z}_L^d$ . An important property of this construction is that the conditions we require for an edge to be open or closed will only depend on the occupation *outside*  $\mathbb{Z}_L^d$ , namely  $\hat{\eta}$ .

**Lemma 4.3.** *There exist a percolation process  $\bar{c}(\hat{\eta}) \in \Pi = \{0, 1\}^{\mathcal{E}(\mathbb{Z}_L^d)}$  and  $T$ -step moves  $M^{\pm \bar{e}_1}, \dots, M^{\pm \bar{e}_d}$  such that:*

- (1) *The process  $\bar{c}_{ij}$  is stationary and ergodic (with respect to  $\hat{\mu}$ ), and dominates a supercritical Bernoulli percolation<sup>2</sup> uniformly in  $q$ .*
- (2)  *$T \leq CL^\lambda$ .*
- (3) *For any  $\bar{e} \in \{\pm \bar{e}_1, \dots, \pm \bar{e}_d\}$  the move  $M^{\bar{e}}$  satisfies:*
  - (a) *The domain of  $M^{\bar{e}}$ ,  $\text{Dom } M^{\bar{e}}$ , consists of the configurations in which  $\bar{c}_{0, \bar{e}} = 1$ .*
  - (b)  *$2^{\text{Loss}(M^{\bar{e}})} \leq C L^\lambda$ .*
  - (c) *For any  $\eta \in \text{Dom } M^{\bar{e}}$ , denoting  $M^{\bar{e}} = \{\eta_t(\eta), x^t(\eta), e^t(\eta)\}_{t=0}^T$ , at the final configuration*

$$\eta_T(\eta) = \eta^{0, \bar{e}}.$$

*Proof.* The lemma is proven in [9], Lemmas 3.9 and 3.14. See also [17, Section 3.4.1]. The reader may note that in the proof of [9]  $q$  is assumed small, but since the relevant probabilities

<sup>2</sup>Domination here means that there exists a supercritical Bernoulli percolation process on  $\mathbb{Z}_L^d$  whose set of open edges is contained in the set of open edges given by  $\bar{c}$ .

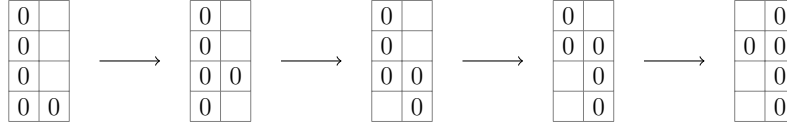


FIGURE 4.1. Droplet propagation. Sites marked with 0 are empty, the other sites could be either empty or occupied. We see that in a sequence of *unconstrained* transitions the empty column moves to the right.

253 estimated are monotone in  $q$  one may discard this assumption by adjusting the constants  $C, \lambda$   
 254 in equation (4.2).  $\square$

255 *Remark 4.4.* The reason for the iterated exponential scaling of  $D(\rho)$  hides in the proof of  
 256 Lemma 4.3, and explained in details in [9, 17, 25]. It is based on induction over both  $k$  and  
 257  $d$ , of two different scales.

258 The first scale,  $l(k, d)$ , is the scale at which cluster of empty sites could typically advance.  
 259 For  $k = 1$ , for example, the constraint is always satisfied and  $l(1, d) = 1$ . Perhaps more  
 260 interesting is the case  $k = d = 2$ , where a column of empty sites of length  $l$  could move  
 261 if there is an empty site in a neighboring column (see Figure 4.1). The probability to have  
 262 a vacancy in the neighboring column is  $1 - (1 - q)^l$ , hence this event becomes likely when  
 263 choosing  $l(2, 2) \approx 1/q$ . This is the scale of the *droplets*, which are those empty clusters of size  
 264  $l$  that are able to move in  $\mathbb{Z}^d$ .

265 The second scale,  $L(k, d)$ , is the typical distance of an arbitrary site to a droplet, so  $L(k, d) \approx$   
 266  $q^{-l(k, d)}$ . If we look at a particle and consider its neighborhood at scale  $L(k, d)$ , we are likely  
 267 to find a droplet, that would be able to move to the vicinity of that particle and help it jump.

268 In order to understand the scaling of  $D(\rho)$ , we should understand the two scales  $l(k, d)$   
 269 and  $L(k, d)$ . Consider the set  $[1, L(k - 1, d - 1)]^d$ . If we empty the entire boundary of this  
 270 set, it could serve as a droplet – take, for example, the surface  $\{0\} \times [1, L(k - 1, d - 1)]^{d-1}$ .  
 271 This is a  $d - 1$  dimensional surface, and each of its sites has an empty neighbor to the right  
 272 coming from  $[1, L(k - 1, d - 1)]^d$ . Therefore, any move for the KA dynamics with parameters  
 273  $k - 1, d - 1$  could be applied to that surface. Since its size is  $L(k - 1, d - 1)$ , it is likely  
 274 to contain a droplet. Hence, using this droplet, we are able to move freely the sites on  
 275 the surface. With slightly more careful analysis, it could be shown that by rearranging the  
 276 sites on  $\{0\} \times [1, L(k - 1, d - 1)]^{d-1}$  the set  $[1, L(k - 1, d - 1)]^d$  could “swallow” this surface,  
 277 thus moving one step to the left. That is,  $[1, L(k - 1, d - 1)]^d$  is, indeed, a droplet; and so  
 278  $l(k, d) \approx L(k - 1, d - 1)$ .

279 The two relations,  $L(k, d) \approx q^{-l(k, d)}$  and  $l(k, d) \approx L(k - 1, d - 1)$ , show that the two scales  
 280 indeed behave as an iterated exponential. The scaling of the diffusion coefficient could then  
 281 be explained heuristically, if we imagine that the particles are mostly blocked, except those in  
 282 the vicinity of a droplet. Since the sites that are able to move have density  $L^{-d}$ , the diffusion  
 283 coefficient scales polynomially in  $L$ .

284 An immediate consequence of point one of Lemma 4.3 is that the graph induced by the  
 285 open edges (i.e., for which  $\bar{c}$  equals 1) has a unique infinite connected component. Let  
 286  $\mathcal{C}$  denote this infinite component. In [10] (see also [22]), it is shown that the diffusion  
 287 coefficient of a random walk on  $\mathcal{C}$  is given by the following variational formula:

$$\bar{D} = \inf_{\psi} \sum_{\alpha} \hat{\mu} [\bar{c}_{0,e_{\alpha}} (\delta_{\alpha,1} + \psi(\tau_{\bar{e}_{\alpha}} \bar{c}) - \psi(\bar{c}))^2 | 0 \in \mathcal{C}, \bar{e}_{\alpha} \in \mathcal{C}],$$

288 where the infimum is taken over local functions  $\psi : \Pi \rightarrow \mathbb{R}$  (namely, functions that depend  
 289 on finitely many edges).

290 The input we need from [10, 8] is the positivity of the diffusion coefficient:

291 **Lemma 4.5.** *There exists  $\bar{D}_0 > 0$  such that for all local  $\psi : \Pi \rightarrow \mathbb{R}$  and all  $\rho \in (0, 1)$ ,*

$$\sum_{\alpha} \hat{\mu} [\bar{c}_{0,e_{\alpha}} (\delta_{\alpha,1} + \psi(\tau_{\bar{e}_{\alpha}} \bar{c}) - \psi(\bar{c}))^2] \geq \bar{D}_0.$$

292 *Proof.* This is a direct consequence of [10, Lemma 2.1] and the first point of Lemma 4.3.  $\square$

293 In order to relate the diffusion coefficient given in equation (2.5) to  $\bar{D}$ , we use the following  
 294 proposition:

295 **Proposition 4.6.** *Fix a local function  $g : \bar{\Omega}_L \times \Pi \rightarrow \mathbb{R}$ . Then there exists a local function*  
 296  *$\psi : \Pi \rightarrow \mathbb{R}$ , such that*

$$\sum_{\alpha=1}^d \hat{\mu} [\bar{c}_{0,\bar{e}_{\alpha}} (\delta_{\alpha,1} + \psi(\tau_{\bar{e}_{\alpha}} \bar{c}) - \psi(\bar{c}))^2] \leq \frac{1}{2\rho(1-\rho)} \times$$

$$\sum_{\alpha=1}^d \bar{\mu} \otimes \hat{\mu} \left[ \bar{c}_{0,\bar{e}_{\alpha}} \left( \delta_{\alpha,1} (\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) - \sum_{i \in \mathbb{Z}_L^d} \bar{\nabla}_{0,\bar{e}_{\alpha}} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right)^2 \right],$$

297 where  $\bar{\nabla}$  is the gradient operating only on  $\bar{\eta}$  (that is,  $\bar{\nabla}_{0,\bar{e}_{\alpha}} g(\tau_i \bar{\eta}, \tau_i \bar{c}) = g(\tau_i \bar{\eta}^{0,\bar{e}_{\alpha}}, \tau_i \bar{c}) - g(\tau_i \bar{\eta}, \tau_i \bar{c})$ ).

298 *Proof.* Note first that the sum  $\sum_{i \in \mathbb{Z}_L^d} \bar{\nabla}_{0,\bar{e}_{\alpha}} g(\tau_i \bar{\eta}, \tau_i \bar{c})$  is finite (and hence well defined) since  $g$   
 299 is local. We are therefore allowed, throughout the proof, to replace it by a sum over a large  
 300 enough torus  $\mathbb{T}_{N,L}^d = \mathbb{Z}_L^d / N\mathbb{Z}_L^d$  for large  $N$  (depending on  $g$ ). We start by writing the left hand  
 301 side of the inequality as

$$\sum_{\alpha=1}^d \hat{\mu} [\bar{c}_{0,\bar{e}_{\alpha}} (\text{I} + \text{II} + \text{III})],$$

$$\text{I} = \delta_{\alpha,1},$$

$$\text{II} = 2\delta_{\alpha,1} (\psi(\tau_{\bar{e}_1} \bar{c}) - \psi(\bar{c})),$$

$$\text{III} = (\psi(\tau_{\bar{e}_{\alpha}} \bar{c}) - \psi(\bar{c}))^2;$$

303 and the right hand side (noting that  $\bar{c}$  depends only on  $\hat{\eta}$  and not on  $\bar{\eta}$ ) as

$$\sum_{\alpha=1}^d \hat{\mu} [\bar{c}_{0,\bar{e}_\alpha} (\mathbf{I}' + \mathbf{II}' + \mathbf{III}')],$$

304

$$\begin{aligned} \mathbf{I}' &= \bar{\mu} [\delta_{\alpha,1} (\bar{\eta}(\bar{e}_1) - \bar{\eta}(0))^2], \\ \mathbf{II}' &= -2\delta_{\alpha,1} \bar{\mu} \left[ (\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) \sum_{i \in \mathbb{T}_{N,L}^d} \bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right], \\ \mathbf{III}' &= \bar{\mu} \left[ \left( \sum_{i \in \mathbb{T}_{N,L}^d} \bar{\nabla}_{0,\bar{e}_\alpha} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right)^2 \right]. \end{aligned}$$

305 We now compare term by term. The term  $\mathbf{I}, \mathbf{I}'$  do not depend on  $\psi$ :  $\mathbf{I}' = \delta_{\alpha,1} 2\rho(1 - \rho)$ , so  
 306 indeed  $\mathbf{I} \leq \frac{1}{2\rho(1-\rho)} \mathbf{I}'$ .

307 For the other terms we need to specify our choice of  $\psi$ :

$$\psi(\bar{c}) = 2\bar{\mu} \left[ \bar{\eta}(0) \sum_{i \in \mathbb{T}_{N,L}^d} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right].$$

308 Fix  $\bar{e} \in \{\bar{e}_1, \dots, \bar{e}_d\}$ , and note that  $\tau_i \bar{c}$  depends only on  $\hat{\eta}$  for any  $i \in \mathbb{Z}_L^d$ . Then

$$\begin{aligned} \psi(\tau_{\bar{e}} \bar{c}) &= 2\bar{\mu} \left[ \bar{\eta}(0) \sum_{i \in \mathbb{T}_{N,L}^d} g(\tau_i \bar{\eta}, \tau_{i+\bar{e}} \bar{c}) \right] = 2\bar{\mu} \left[ \bar{\eta}(\bar{e}) \sum_{i \in \mathbb{T}_{N,L}^d} g(\tau_{i+\bar{e}} \bar{\eta}, \tau_{i+\bar{e}} \bar{c}) \right] \\ &= 2\bar{\mu} \left[ \bar{\eta}(0) \sum_{i \in \mathbb{T}_{N,L}^d} g(\tau_i \bar{\eta}^{0,\bar{e}}, \tau_i \bar{c}) \right], \end{aligned}$$

309 and thus

$$\psi(\tau_{\bar{e}_\alpha} \bar{c}) - \psi(\bar{c}) = \bar{\mu} \left[ 2\bar{\eta}(0) \sum_{i \in \mathbb{T}_{N,L}^d} \bar{\nabla}_{0,\bar{e}_\alpha} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right]. \quad (4.3)$$

310 Observe now that  $\bar{\eta}(0) = \bar{\eta}(\bar{e}_1)$  implies  $\bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c}) = 0$ , and otherwise  $\bar{\eta}(\bar{e}_1) = 1 - \bar{\eta}(0)$ ,  
 311 yielding

$$(\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) \bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c}) = (1 - 2\bar{\eta}(0)) \bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c}).$$

312 Therefore, by equation (4.3),

$$\bar{\mu} \left[ (\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) \sum_{i \in \mathbb{T}_{N,L}^d} \bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right] = \sum_{i \in \mathbb{T}_{N,L}^d} \bar{\mu} [\bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c})] - (\psi(\tau_{\bar{e}_1} \bar{c}) - \psi(\bar{c})),$$

313 and noting that  $\bar{\mu} [\bar{\nabla}_{0,\bar{e}_1} g(\tau_i \bar{\eta}, \tau_i \bar{c})] = 0$  (the gradient of any function has 0 expected value),  
 314 we obtain

$$\Pi = \Pi'.$$

315 Finally, for the last term we use again equation (4.3), together with Jensen's inequality and  
 316 the fact that  $\bar{\eta}(0)^2 \leq 1$ :

$$\text{III} \leq \bar{\mu} \left[ \left( 2\bar{\eta}(0) \sum_{i \in \mathbb{T}_{N,L}^d} \bar{\nabla}_{0,\bar{e}_\alpha} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right)^2 \right] \leq 4 \text{III}'. \quad \square$$

317 **Corollary 4.7.** *For all local  $g : \bar{\Omega}_L \times \Pi \rightarrow \mathbb{R}$ ,*

$$\frac{1}{2\rho(1-\rho)} \sum_{\alpha=1}^d \bar{\mu} \otimes \hat{\mu} \left[ \bar{c}_{0,\bar{e}_\alpha} \left( \delta_{\alpha,1}(\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) - \sum_{i \in \mathbb{Z}_L^d} \bar{\nabla}_{0,\bar{e}_\alpha} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right)^2 \right] \geq \bar{D}_0,$$

318 where  $\bar{D}_0$  is the positive constant given in Lemma 4.5.

319 The next step of the proof is to use the multistep move given in Lemma 4.3 in order to  
 320 compare  $\bar{D}_0$  with  $D^{(0)}$  (recalling equation (2.5)).

321 **Proposition 4.8.** *Fix a local function  $f : \Omega \rightarrow \mathbb{R}$ . Then there exists a local function  $g : \bar{\Omega}_L \times \Pi \rightarrow$   
 322  $\mathbb{R}$  such that*

$$\mu \left( \sum_{\alpha=1}^d c_{0,e_\alpha}(\eta) \left( \delta_{\alpha,1}(\eta(e_\alpha) - \eta(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha}(\tau_x f) \right)^2 \right) \geq \\ L^{-\lambda} \sum_{\alpha=1}^d \bar{\mu} \otimes \hat{\mu} \left[ \bar{c}_{0,\bar{e}_\alpha} \left( \delta_{\alpha,1}(\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) - \sum_{i \in \mathbb{Z}_L^d} \bar{\nabla}_{0,\bar{e}_\alpha} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right)^2 \right].$$

323 *Proof.* Let  $g(\bar{\eta}, \bar{c}) = \mu \left[ \frac{1}{L} \sum_{y \in [L]^d} \tau_y f(\eta) \middle| \bar{\eta}, \bar{c} \right]$ . We use Lemma 4.3 in order to write, for all  
 324  $x \in \mathbb{Z}^d$  and  $\alpha \in \{1, \dots, d\}$ , denoting  $M^{\bar{e}_\alpha} = \{\eta_t(\eta), x^t(\eta), e^t(\eta)\}_{t=0}^T$ ,

$$\bar{\nabla}_{0,\bar{e}_\alpha} \tau_x f = \sum_{t=1}^T \nabla_{x^t, x^t + e^t} \tau_x f(\eta_t) = \sum_{t=1}^T \tau_{x^t} \nabla_{0, e^t} \tau_{x - x^t} f(\eta_t). \quad (4.4)$$

325 We also note that the total particle flow (defined as the change in  $\sum_x x \eta(x)$ ) can be decom-  
 326 posed along the  $T$ -step move. In more details, using the fact that  $\eta_0$  and  $\eta_T$  agree outside  
 327  $\{0, \bar{e}_\alpha\}$ ,

$$\sum_x x(\eta_0(x) - \eta_T(x)) = L e_\alpha (\bar{\eta}(\bar{e}_\alpha) - \bar{\eta}(0)).$$

328 On the other hand, at step  $t$  the configuration changes only at  $x^t$  and  $x^t + e^t$ , therefore

$$\sum_x x(\eta_{t-1}(x) - \eta_t(x)) = e^t(\eta_t(x^t) - \eta_t(x^t + e^t)) = e^t \tau_{x^t}(\eta_t(0) - \eta_t(e^t)),$$

implying

$$\sum_x x(\eta_0(x) - \eta_T(x)) = \sum_{t=1}^T \sum_x x(\eta_{t-1}(x) - \eta_t(x)) = \sum_{t=1}^T e^t \tau_{x^t}(\eta_t(0) - \eta_t(e^t)).$$

That is,

$$Le_\alpha(\bar{\eta}(\bar{e}_\alpha) - \bar{\eta}(0)) = \sum_{t=1}^T e^t \tau_{x^t}(\eta^t(e^t) - \eta_t(0)). \quad (4.5)$$

Using these two identities, the Cauchy-Schwarz inequality, and the properties of the move, we obtain

$$\begin{aligned} & \sum_{\alpha=1}^d \bar{\mu} \otimes \hat{\mu} \left[ \bar{c}_{0, \bar{e}_\alpha} \left( e_1 \cdot e_\alpha(\bar{\eta}(\bar{e}_\alpha) - \bar{\eta}(0)) - \sum_{i \in \mathbb{Z}_L^d} \bar{\nabla}_{0, \bar{e}_\alpha} g(\tau_i \bar{\eta}, \tau_i \bar{c}) \right) \right]^2 \\ & \leq \frac{1}{L^2} \sum_{\alpha=1}^d \mu \left[ \bar{c}_{0, \bar{e}_\alpha} \left( e_1 \cdot Le_\alpha(\bar{\eta}(\bar{e}_1) - \bar{\eta}(0)) - \sum_{i \in \mathbb{Z}_L^d} \bar{\nabla}_{0, \bar{e}_\alpha} \sum_{y \in [L]^d} \tau_{i+y} f(\eta) \right) \right]^2 \\ & = \frac{1}{L^2} \sum_{\alpha=1}^d \mu \left[ \bar{c}_{0, \bar{e}_\alpha} \left( e_1 \cdot \sum_{t=1}^T e^t \tau_{x^t}(\eta_t(e^t) - \eta_t(0)) - \sum_{x \in \mathbb{Z}^d} \sum_{t=1}^T \tau_{x^t} \nabla_{0, e^t} \tau_{x-x^t} f(\eta_t) \right) \right]^2 \\ & \leq \frac{T}{L^2} \sum_{t=1}^T \sum_{\alpha=1}^d \mu \left[ \bar{c}_{0, \bar{e}_\alpha} \tau_{x^t} c_{0, e^t}(\eta_t) \left( e_1 \cdot e^t(\eta_t(e^t) - \eta_t(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0, e^t} \tau_x f(\eta_t) \right) \right]^2 \\ & \leq \frac{T}{L^2} \sum_{t=1}^T \sum_{\alpha=1}^d \sum_{\eta \in \Omega} \mu(\eta) \sum_{\eta' \in \Omega} \mathbb{1}_{\eta' = \eta_t} \sum_{\beta=1}^d \mathbb{1}_{e_\beta = e^t} c_{0, e_\beta}(\eta') \left( e_1 \cdot e_\beta(\eta'(e_\beta) - \eta'(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0, e_\beta} \tau_x f(\eta') \right)^2 \\ & \leq \frac{T^2}{L^2} \sum_{\alpha=1}^d 2^{\text{Loss}(M^{\bar{e}_\alpha})} \sum_{\eta' \in \Omega} \mu(\eta') \sum_{\beta=1}^d c_{0, e_\beta}(\eta') \left( e_1 \cdot e_\beta(\eta'(e_\beta) - \eta'(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0, e_\beta} \tau_x f(\eta') \right)^2. \end{aligned}$$

The result follows by inserting the bounds for  $T$  and  $\text{Loss}(M)$  given in Lemma 4.3.  $\square$

The proof of the lower bound (4.1) follows from Proposition 4.8, Corollary 4.7, and the variational characterization of  $D^{(0)}$  in equation (2.5).  $\square$

## 5. Proof of the upper bound

In order to find the upper bound we will use a process tightly related to the Kob-Andersen model, called the *k-neighbor bootstrap percolation* (see, e.g., [19]). We start by defining this process, and describing some of its basic properties.

### 5.1. Bootstrap percolation.

**Definition 5.1** (bootstrap percolation). Fix  $V \subseteq \mathbb{Z}^d$  and  $A \subseteq \mathbb{Z}^d$ . The *bootstrap percolation in  $V$  starting from  $A$*  is a deterministic process defined for  $t = 0, 1, 2, \dots$  as

$$A_0 = A \cap V,$$

$$A_{t+1} = A_t \cup \{x \in V : \#\{y \in A_t \text{ such that } y \sim x\} \geq k\}.$$

The limit  $\cup_{t \geq 0} A_t$  is called the *span of  $A$  in  $V$* , and denoted by  $[A]^V$ . We say that two sites  $x$  and  $y$  are *connected for the bootstrap percolation in  $V$  starting from  $A$*  if they are connected in  $[A]^V$  (thought of as the subgraph of  $\mathbb{Z}^d$  induced by the set  $[A]^V$ ), that is, if there is a nearest neighbor path  $x = x_1, \dots, x_n = y$  such that  $x_1, \dots, x_n \in [A]^V$ .

For  $\eta \in \Omega$ , we define

$$A_\eta = \{x \in \mathbb{Z}^d : \eta_x = 0\}.$$

We may refer to the bootstrap percolation in  $V$  starting from  $A_\eta$  as the bootstrap percolation starting from  $\eta$ . When context allows we omit the explicit mention of  $V$ ,  $A$ , or both.

We continue with several properties of bootstrap percolation.

**Observation 5.2.** (*monotonicity*). Let  $U \subseteq V \subseteq \mathbb{Z}^d$ , and fix  $A \subseteq B \subseteq \mathbb{Z}^d$ . Then  $[A]^U \subseteq [A]^V$  and  $[A]^U \subseteq [B]^U$ .

The following observation reveals the the connection between bootstrap percolation and the Kob-Andersen model:

**Observation 5.3.** Fix  $\eta \in \Omega$ , and consider a set  $V \subset \mathbb{Z}^d$ . Assume that, for two neighboring sites  $x, y \in V$ , the constraint  $c_{x,y}$  is satisfied in  $V$ , that is,  $c_{x,y}(\eta') = 1$  for any  $\eta'$  that agrees with  $\eta$  on  $V$ . Then  $[A_\eta]^V = [A_{\eta^{x,y}}]^V$ .

*Proof.* Assume without loss of generality that  $\eta(x) = 1$  and  $\eta(y) = 0$ , and note that  $[A_\eta]^V \subseteq [A_\eta \cup \{x\}]^V$ . On the other hand, since  $c_{x,y} = 1$  in  $V$ , the site  $x$  will be added to  $A_\eta$  after a single step of the bootstrap percolation. Denoting the set after that single step by  $A'$ ,  $[A_\eta \cup \{x\}]^V \subseteq [A']^V = [A_\eta]^V$ . Therefore  $[A_\eta]^V = [A_\eta \cup \{x\}]^V$ . The same argument shows that  $[A_{\eta^{x,y}}]^V = [A_\eta \cup \{x\}]^V$ .  $\square$

**Observation 5.4.** Fix  $A \subset \mathbb{Z}^d$ ,  $V \subset \mathbb{Z}^d$ , and  $x \in V$ . Let  $U$  be the set of sites connected to  $x$  in  $[A]^V$ . Then  $[A]^U = U$ .

*Proof.* Let  $(A_t)_{t \geq 0}$  denote the bootstrap percolation in  $V$  starting with  $A$ , and assume by contradiction  $[A]^U \subsetneq U$ . Since  $U \subseteq [A]^V$ , there exists a first time  $t$  for which some  $y \in U \setminus [A]^U$  is contained in  $A_t$ . By minimality,  $A_{t-1} \cap U \subseteq [A]^U$ , and since  $y \notin [A]^U$  it has at most  $k-1$  neighbors in  $A_{t-1} \cap U$ . On the other hand, it has at least  $k$  neighbors in  $A_t$ . Therefore, it must have at least one neighbor in  $V \setminus U$ . This is a contradiction, since  $U$  is a connected component containing  $y$ .  $\square$



371 **Claim 5.5.** Fix  $A \subset \mathbb{Z}^d$ . Consider two sets  $B \subset B' \subset \mathbb{Z}^d$ , a site  $z \in B$ , and any  $S \subset \mathbb{Z}^d$ .  
 372 Assume that  $z$  is connected to  $S$  for the bootstrap percolation in  $B'$ , but not for the bootstrap  
 373 percolation in  $B$ . Then  $z$  is connected to  $\partial B$  for the bootstrap percolation in  $B'$ .

374 *Proof.* Assume that  $z$  is not connected to  $\partial B$  for the bootstrap percolation in  $B'$ , so in partic-  
 375 ular its connected component in  $[A]^{B'}$ , denoted  $U$ , is entirely contained in  $B$ . By Observation  
 376 5.4 and monotonicity of the bootstrap percolation,  $U = [A]^U \subseteq [A]^B$ . This is a contradiction,  
 377 since by assumption  $U \cap S \neq \emptyset$  but  $[A]^B \cap S = \emptyset$ .  $\square$

378 **5.2. Analysis of the test function.** We will prove the upper bound by estimating the expres-  
 379 sion inside the infimum in equation (2.5) for a carefully chosen function  $f$ .

380 Recall  $q = 1 - \rho$ . The test function we will construct will depend on a scale

$$l = \left\lfloor \exp^{k-2}(\lambda q^{-\frac{1}{d-k+1}}) \right\rfloor. \quad (5.1)$$

381 Throughout the section  $\lambda$  and  $C$  denote generic positive constants.

382 **Definition 5.6** (relevant sites). Fix  $\eta \in \Omega$ . A site  $x \in [-2l, 2l]^d$  is called *relevant* if it is not  
 383 connected to  $\{0, 1\} \times [-2l, 2l]^{d-1}$  for the bootstrap percolation in  $[-2l, 2l]^d$ ; and otherwise it  
 384 is called *irrelevant*. Denote the set of relevant sites by  $\mathcal{R}(\eta)$ .

385 We divide the box  $[-l, l]^d$  in two parts – the left part  $\Lambda_- = [-l, 0] \times [-l, l]^{d-1}$ , and the right  
 386 part  $\Lambda_+ = [0, l] \times [-l, l]^{d-1}$  (see Figure 5.1). The test function we consider is

$$f(\eta) = \frac{1}{2(2l+1)^{d-1}} \left( \sum_{x \in \Lambda_+ \cap \mathcal{R}} \eta(x) - \sum_{x \in \Lambda_- \cap \mathcal{R}} \eta(x) \right). \quad (5.2)$$

387 Hence, the purpose of this section is to prove that for  $\varepsilon$  small enough

$$\mu \left[ \sum_{\alpha=1}^d c_{0,e_\alpha}^{(\varepsilon)} \left( \delta_{\alpha,1} (\eta(e_1) - \eta(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right] \leq e^{-\lambda l}.$$

388 **Remark 5.7.** The choice of  $f$  in equation (5.2) seems mysterious at first sight – Observation  
 389 5.3 explains the use of bootstrap percolation, but the introduction of relevant sites and the  
 390 exact form of  $f$  are not that clear.

391 One way to gain more intuition on this choice of  $f$  is to look more carefully at the varia-  
 392 tional principle (2.5). Ignoring the contribution of  $f$ , we are left with the term

$$c_{0,e_\alpha} (\delta_{\alpha,1} (\eta(e_1) - \eta(0)))^2.$$

393 This could be though of as a contribution of the instantaneous current between the origin and  
 394  $e_1$ . The appearance of this term is not surprising – if typically the system has large currents,  
 395 it is natural to expect the diffusion coefficient to be large.

396 However, the typical instantaneous current is not sufficient to understand the behavior of  
 397 the diffusion coefficient – correlations in space and time could also have an important effect.

For example, take the Kob-Andersen model with  $k = d = 2$ , and consider a configuration in which the sites  $e_1, e_2$  and  $e_1 + e_2$  are empty, and all other sites (in a large neighborhood of the origin) are occupied. The particle at the origin could jump one step to the right, but any attempt to jump further is not allowed by the constraint. Therefore, if we wait for some time it is likely to jump back to the left. Thus, we see that an instantaneous right current can cause at a later time a current to the left. The role of the function  $f$  in equation (2.5) is to compensate for this effect, by adding to  $\delta_{\alpha,1}(\eta(e_1) - \eta(0))$  an effective current in the opposite direction.

The example of the last paragraph demonstrates the following heuristic picture – typically, most particles are confined to a very small region; they can move back and forth but never too far. Assume for simplicity that the origin is occupied, and consider the particle there. In view of the heuristic described above, this particle will remain for a very long time in a certain region that we may refer to as the *attainable region*. Recalling Observation 5.3, it is reasonable to approximate this attainable region by the set of sites connected to the origin for the bootstrap percolation in some (large) box. Hence, being *relevant* roughly represents a small attainable region.

As long as this attainable region remains small, we expect that any instantaneous current to the right caused by the particle at the origin will be canceled shortly after by a jump to the left. For a good choice of  $f$ , this fact (assuming  $c_{0,e_1}(\eta) = 1$ ) should be expressed as

$$\eta(e_1) - \eta(0) \approx \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_1} \tau_x f(\eta).$$

In the following we will see that the function  $f$  defined in equation (5.2) satisfies this approximated relation. The error term corresponds to the possibility that the attainable region is, in fact, large. When the notion of "small" or "large" attainable region is determined according to the scale  $l$  given in equation (5.1), we obtain the upper bound of  $D$ .

First, observe that since  $f$  depends on  $(4l+1)^d$  sites and its maximum is smaller than  $l+1$ ,

$$\mu \left[ \sum_{\alpha=1}^d \varepsilon \left( \delta_{\alpha,1}(\eta(e_1) - \eta(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right] \leq d\varepsilon(1 + (4l+1)^d(l+1))^2 = O(\varepsilon).$$

Therefore, since  $c_{0,e_\alpha}^{(\varepsilon)} = (1 - \varepsilon)c_{0,e_\alpha} + \varepsilon$ , it suffices to prove

$$\mu \left[ \sum_{\alpha=1}^d c_{0,e_\alpha} \left( \delta_{\alpha,1}(\eta(e_1) - \eta(0)) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right] \leq e^{-\lambda l}. \quad (5.3)$$

Since the analysis of  $f$  will require us to understand when particles enter or exit different boxes (and in particular  $\Lambda_\pm$ ), we will need to introduce some notation. First, for a set  $\Lambda \subset \mathbb{Z}^d$ , we say that an (undirected) edge  $(x, y)$  is on the *boundary* of  $\Lambda$ , and write  $(x, y) \in \bar{\partial}\Lambda$ , if one

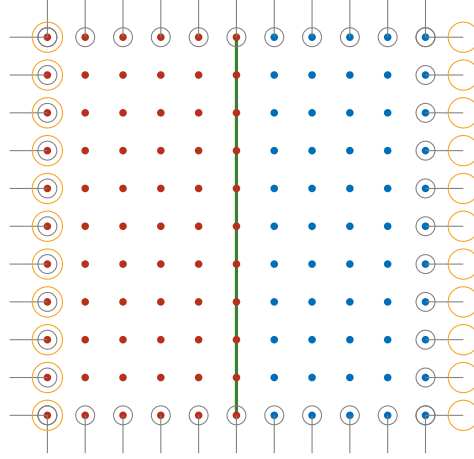


FIGURE 5.1. A sketch of the different sets in  $\Lambda_l$  in two dimensions. The left part  $\Lambda_-$  are the red vertices; the right part  $\Lambda_+$  consists of the blue vertices; the boundary  $\bar{\partial}\Lambda$  is given by the gray edges; the inner boundary  $\partial\Lambda$  contains the sites circled in gray; the boundary in the  $e_1$  direction  $\partial^1\Lambda$  is circled in orange; and the set  $\Lambda_l^1$  is represented by the green line crossing the box. In this picture, for any  $x_0 \in \Lambda_l^1$  (i.e., on the green line), the site  $x_0^{+1}$  is the orange circle to its right (at distance  $l + 1$ ); and the site  $x_0^{-1}$  is the orange circle to its left (at distance  $l$ ).

vertex is in  $\Lambda$  and the other outside  $\Lambda$ . The (inner) boundary  $\partial\Lambda$  are the sites in  $\Lambda$  that have a neighbor outside  $\Lambda$ .

For  $\alpha = 1, \dots, d$  we define the boundary in the  $e_\alpha$  direction

$$\partial^\alpha\Lambda = \{x : (x, x - e_\alpha) \in \bar{\partial}\Lambda\}.$$

We will write  $\Lambda_l = [-l, l]^d$  (and  $\Lambda_{2l} = [-2l, 2l]^d$ ), as well as

$$\Lambda_l^\alpha = [-l, l]^{\alpha-1} \times \{0\} \times [-l, l]^{d-\alpha}.$$

Finally, for  $x_0 \in \Lambda_l^\alpha$ , we denote the two boundary sites above and below  $x_0$  as

$$x_0^{+\alpha} = x_0 + (l + 1)e_\alpha,$$

$$x_0^{-\alpha} = x_0 - le_\alpha.$$

Note that  $x_0^{\pm\alpha} \in \partial^\alpha\Lambda_l$ . See Figure 5.1.

We are now ready to start the analysis of  $f$ . In the next proposition we will see, for fixed  $x$ , what is the contribution of  $\nabla_{0,e_1}\tau_x f$ :

**Proposition 5.8.** Fix an edge  $(x, x - e)$  and configuration  $\eta$  such that  $c_{0,e} = 1$  and  $\nabla_{0,e}\tau_x f \neq 0$ . Then one of the following holds:

- (1)  $0 \in x + (\Lambda_{2l+1} \setminus \Lambda_{2l-2})$  (equivalently  $x \in \Lambda_{2l+1} \setminus \Lambda_{2l-2}$ ), and there exists  $y \in x + \partial\Lambda_l$  such that the bootstrap percolation in  $x + \Lambda_{2l}$  connects  $y$  to  $x + \partial\Lambda_{2l-2}$ , either for  $\eta$  or  $\eta^{0,e}$ . In this case  $|\nabla_{0,e}\tau_x f| \leq Cl$ . See Figure 5.2.

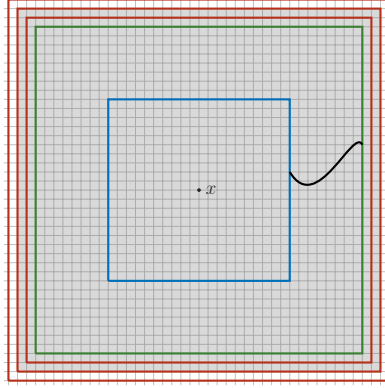


FIGURE 5.2. Illustration of the first case in Proposition 5.8.  $x + \Lambda_{2l}$  is filled in gray. The origin is on one of the three red lines, which represent  $x + \Lambda_{2l+1} \setminus \Lambda_{2l-2}$ . The blue square is  $x + \partial\Lambda_l$  and the green square is  $x + \partial\Lambda_{2l-2}$ . These two squares must be connected for the bootstrap percolation.

439 (2)  $(0, e) \in x + \bar{\partial}\Lambda_l$  (equivalently  $(x, x - e) \in \bar{\partial}\Lambda_l$ ) and  $-x$  is relevant for  $\tau_x\eta$ . In this case

$$\begin{aligned} \nabla_{0,e}\tau_x f &= \frac{\eta(e) - \eta(0)}{2(2l+1)^{d-1}} \times \begin{cases} 1 & 0 \in x + \Lambda_+ \cap \partial[-l, l]^d \\ -1 & e \in x + \Lambda_+ \cap \partial[-l, l]^d \\ -1 & 0 \in x + \Lambda_- \cap \partial[-l, l]^d \\ 1 & e \in x + \Lambda_- \cap \partial[-l, l]^d \end{cases} \\ &= \frac{\eta(e) - \eta(0)}{2(2l+1)^{d-1}} \times \begin{cases} 1 & x \in \Lambda_- \cap \partial[-l, l]^d \\ -1 & x - e \in \Lambda_- \cap \partial[-l, l]^d \\ -1 & x \in \Lambda_+ \cap \partial[-l, l]^d \\ 1 & x - e \in \Lambda_+ \cap \partial[-l, l]^d \end{cases}. \end{aligned} \quad (5.4)$$

440 *Proof.*  $f$  could only change when the set of relevant sites changes, or when a relevant site  
441 changes its occupation.

442 The first case corresponds to point 1 – for the set of relevant sites for  $\tau_x\eta$  to change,  $[A_\eta]^{x+\Lambda_{2l}}$   
443 must change, and by Observation 5.3 this is only possible if  $c_{0,e}$  is only satisfied with the help  
444 of sites outside  $x + \Lambda_{2l}$ . This means that at least one of the vertices 0 or  $e$  is in  $x + \partial\Lambda_{2l}$ , and  
445 in particular  $0 \in x + (\Lambda_{2l+1} \setminus \Lambda_{2l-2})$ .

446 To understand the second implication, we may assume without loss of generality that there  
447 is some site  $z \in \Lambda_l$  which is connected to  $\{0, 1\} \times [-2l, 2l]^{d-1}$  in  $[A_\eta]^{x+\Lambda_{2l}}$  but not in  $[A_{\eta^{0,e}}]^{x+\Lambda_{2l}}$ .  
448 By monotonicity of bootstrap percolation and using again Observation 5.3,  $z$  cannot be con-  
449 nected to  $\{0, 1\} \times [-2l, 2l]^{d-1}$  in  $[A_\eta]^{\Lambda_{2l-2}}$ . Then, by Claim 5.5,  $z$  is connected to  $\partial\Lambda_{2l-2}$  in  
450  $[A_\eta]^{x+\Lambda_{2l}}$ . To finish the first point, we only need the rough bound  $|f(\eta)| \leq \frac{|\Lambda_+| + |\Lambda_-|}{2(2l+1)^{d-1}}$ .

451 In the second case, we note first if a particle jumps inside  $\Lambda_l$ , and it is originally in  $\Lambda_+ \cap \mathcal{R}$ ,  
452 then it will remain in  $\Lambda_+ \cap \mathcal{R}$  (and analogously for  $\Lambda_-$ ). Therefore,  $f$  could only change if  
453 a particle jumps into or out of  $\Lambda_l$ , so for  $\tau_x f$  to change we must require  $(0, e) \in x + \bar{\partial}\Lambda_l$ .

Moreover, its (shifted) position  $-x$  must be relevant for the shifted configuration  $\tau_x \eta$ . Then,  $\nabla_{(0,e)} \tau_x f$  is given by following carefully the four options: moving into  $\Lambda_+$ , out of  $\Lambda_+$ , into  $\Lambda_-$ , or out of  $\Lambda_-$ .  $\square$

In order to control the contribution coming from the first case of Proposition 5.8, we give a name to the event that appears there:

**Definition 5.9.** Fix an edge  $(x, x - e)$ . Then  $E_{x, x-e}$  is the event, that there exist  $y \in x + \partial\Lambda_l$  and  $z \in x + \partial\Lambda_{2l-2}$  such that the bootstrap percolation in  $x + \Lambda_{2l}$  connects  $y$  to  $z$ , either for  $\eta$  or  $\eta^{0,e}$ .

An important tool we will use in order to bound the probability of this event is the following lemma:

**Lemma 5.10** ([7, Lemma 5.1]). Let  $l' < 10l$ , and fix  $y, z \in \Lambda_{l'}$ . Then, assuming that the constant  $\lambda$  in equation (5.1) is small enough,

$$\begin{aligned} \mu(z \text{ connected to } y \text{ in } [A_\eta]^{\Lambda_{l'}}) &\leq \left( C \|z - y\|_\infty^{d-1} q \right)^{\lambda \|z - y\|_\infty} & k = 2, \\ \mu(z \text{ connected to } y \text{ in } [A_\eta]^{\Lambda_{l'}}) &\leq q^{\lambda \|z - y\|_\infty} & k \geq 3. \end{aligned}$$

**Claim 5.11.** Fix an edge  $(x, x - e)$ . Then

$$\mu(E_{x, x-e}) \leq C e^{-\lambda l}.$$

*Proof.* First, note that there are  $C l^{d-1}$  possible choices of  $y$  and  $C l^{d-1}$  choices of  $z$ . Note that  $\|z - y\|_\infty \geq l - 2$ . By Lemma 5.10, the probability for  $y$  to be connected to  $z$  is bounded by  $(C l^{d-1} q)^\lambda$  for  $k = 2$  and  $q^\lambda$  for  $k \geq 3$ ; both of which are, indeed, smaller than  $C e^{-\lambda l}$ .  $\square$

The last claim covers the first case of Proposition 5.8, and we now move to the second.

When considering this case, we will use certain cancellations in directions perpendicular to  $e_1$ . More precisely, a particle jumping from 0 to  $e_\alpha$  (for  $\alpha \neq 1$ ) will enter  $x + \Lambda_l$  (for an appropriate choice of  $x$ ) but exit  $x' + \Lambda_l$  (for an appropriate choice of  $x'$ ). We then expect that  $\nabla_{0, e_\alpha} \tau_x f$  and  $\nabla_{0, e_\alpha} \tau_{x'} f$  will cancel out, which is indeed the case unless one of the sites is relevant and the other irrelevant. We therefore introduce the following event:

**Definition 5.12.** Fix  $\alpha \in \{2, \dots, d\}$  and  $x_0 \in \Lambda_l^\alpha$ . Let  $E_\alpha(x_0)$  be the event, that either  $-x_0^{+\alpha}$  is relevant for  $\tau_{x_0^{+\alpha}} \eta$ , or  $-x_0^{-\alpha}$  is relevant for  $\tau_{x_0^{-\alpha}} \eta$ , but not both.

**Claim 5.13.** Fix  $\alpha \in \{2, \dots, d\}$  and  $x_0 \in \Lambda_l^\alpha$ . Then for all  $\eta \in E_\alpha(x_0)$ , the origin is connected to  $\partial\Lambda_l$  in  $[A_\eta]^{\Lambda_{3l}}$ . Moreover,

$$\mu(E_\alpha(x_0)) \leq C e^{-\lambda l}.$$

*Proof.* We will prove that the origin is connected to  $\partial\Lambda_l$  in  $[A_\eta]^{\Lambda_{3l}}$  for the case where  $-x_0^{+\alpha}$  is relevant for  $\tau_{x_0^{+\alpha}} \eta$ , but  $-x_0^{-\alpha}$  is irrelevant for  $\tau_{x_0^{-\alpha}} \eta$ . The complementing case is analogous.

Let  $S = x_0 + \{0, 1\} \times \mathbb{Z}^{d-1}$ ,  $B_- = x_0^{-\alpha} + \Lambda_{2l}$ ,  $B_+ = x_0^{+\alpha} + \Lambda_{2l}$ . Saying that  $-x_0^{+\alpha}$  is relevant for  $\tau_{x_0^{+\alpha}}\eta$  is the same as saying that 0 is connected to  $B_+ \cap S$  in  $[A_\eta]^{B_+}$ ; and saying that  $-x_0^{-\alpha}$  is irrelevant for  $\tau_{x_0^{-\alpha}}\eta$  is the same as saying that 0 is not connected to  $B_- \cap S$  in  $[A_\eta]^{B_-}$ .

In particular, setting  $z = 0$ ,  $B = B_-$  and  $B' = \Lambda_{3l}$ ,  $A_\eta$  satisfies the conditions of Claim 5.5. Therefore 0 is connected to  $\partial B_-$  in  $[A_\eta]^{B'}$ , which implies the result since  $0 \in \Lambda_l \subset B_-$ .

The probability estimate follows from Lemma 5.10.  $\square$

**Claim 5.14.** Fix  $\alpha \in \{2, \dots, d\}$ , and a configuration  $\eta$  such that  $\eta \notin \bigcup_{x_0 \in \Lambda_l^\alpha} E_\alpha(x_0)$  and  $c_{0,e_\alpha}(\eta) = 1$ . Then

$$\sum_{x \in \partial^\alpha \Lambda_l} \nabla_{0,e_\alpha} \tau_x f = 0.$$

*Proof.* We split the sum according to the projection of  $x$  on  $\Lambda_l^\alpha$  –

$$\sum_{x \in \partial^\alpha \Lambda_l} \nabla_{0,e_\alpha} \tau_x f = \sum_{x_0 \in \Lambda_l^\alpha} \left( \nabla_{0,e_\alpha} \tau_{x_0^{+\alpha}} f + \nabla_{0,e_\alpha} \tau_{x_0^{-\alpha}} f \right).$$

Fix one of these summands. If  $-x_0^{+\alpha}$  is irrelevant for  $\tau_{x_0^{+\alpha}}\eta$  and  $-x_0^{-\alpha}$  is irrelevant for  $\tau_{x_0^{-\alpha}}\eta$ , then by the Proposition 5.8

$$\nabla_{0,e_\alpha} \tau_{x_0^{+\alpha}} f = \nabla_{0,e_\alpha} \tau_{x_0^{-\alpha}} f = 0.$$

Otherwise, since  $\eta \notin E_\alpha(x_0)$ , both must be relevant, hence

$$\begin{aligned} \nabla_{0,e_\alpha} \tau_{x_0^{+\alpha}} f &= \frac{\eta(e_\alpha) - \eta(0)}{2(2l+1)^{d-1}} \times \begin{cases} -1 & x_0 \in \Lambda_-, \\ 1 & x_0 \in \Lambda_+; \end{cases} \\ \nabla_{0,e_\alpha} \tau_{x_0^{-\alpha}} f &= \frac{\eta(e_\alpha) - \eta(0)}{2(2l+1)^{d-1}} \times \begin{cases} 1 & x_0 \in \Lambda_-, \\ -1 & x_0 \in \Lambda_+; \end{cases} \end{aligned}$$

and their sum is 0.  $\square$

**Claim 5.15.** Fix  $\alpha \in \{2, \dots, d\}$ . Then

$$\mu \left[ c_{0,e_\alpha} \left( \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right] \leq C e^{-\lambda l}.$$

*Proof.* We split in the different cases described in Proposition 5.8:

$$\mu \left[ c_{0,e_\alpha} \left( \sum_{x \in \mathbb{Z}^d} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right] \leq 2\mu \left[ c_{0,e_\alpha} \left( \sum_{x \in \Lambda_{2l+1} \setminus \Lambda_{2l-2}} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right] + 2\mu \left[ c_{0,e_\alpha} \left( \sum_{x \in \partial^\alpha \Lambda_l} \nabla_{0,e_\alpha} \tau_x f \right)^2 \right].$$

We can bound the first term using Claim 5.11:

$$\mu \left[ c_{0,e_\alpha} \left( \sum_{x \in \Lambda_{2l+1} \setminus \Lambda_{2l-2}} \mathbb{1}_{E(x, x-e_\alpha)} C l \right)^2 \right] \leq C l^d \mu \left[ \sum_x \mathbb{1}_{E(x, x-e_\alpha)} \right] \leq C e^{-\lambda l}.$$

498 The second term, according to Claim 5.14, vanishes unless  $\eta \in E_\alpha(x_0)$  for some  $x_0 \in \Lambda_l^\alpha$ , so  
 499 we are left with an error term which by Claim 5.13 is bounded by

$$\mu \left[ \left( \frac{|\partial^\alpha \Lambda_l|}{2(2l+1)^{d-1}} \right)^2 \sum_{x_0 \in \Lambda_l^\alpha} \mathbb{1}_{E_\alpha(x_0)} \right] \leq Ce^{-\lambda l}. \quad \square$$

500 The next step is to consider the direction  $e_1$ :

501 *Claim 5.16.* Fix  $x \in \partial^1[-l, l]^d$ . Then  $-x$  is irrelevant for  $\tau_x \eta$  with probability smaller than  
 502  $Ce^{-\lambda l}$ .

503 *Proof.* For  $-x$  to be irrelevant it must be connected to one of  $2(4l+1)^{d-1}$  sites on  $\{0, 1\} \times$   
 504  $[-2l, 2l]^{d-1}$ . All of these sites are at distance at least  $l-2$  from  $x$ , and the statement follows  
 505 by direct application of Lemma 5.10.  $\square$

506 *Claim 5.17.* For  $e = e_1$ ,

$$\mu \left[ c_{0,e} \left( \eta(e) - \eta(0) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e} \tau_x f \right)^2 \right] \leq Ce^{-\lambda l}.$$

507 *Proof.* The proof of the claim consists in showing that each site on  $\partial^1 \Lambda_l$  contributes  $\frac{\eta(e) - \eta(0)}{|\partial^1 \Lambda_l|}$   
 508 to the sum, up to a small error term.

509 First, using Proposition 5.8, we write

$$\begin{aligned} \mu \left[ c_{0,e} \left( \eta(e) - \eta(0) - \sum_{x \in \mathbb{Z}^d} \nabla_{0,e} \tau_x f \right)^2 \right] &\leq 2\mu \left[ c_{0,e} \left( \sum_{x \in \Lambda_{2l+1} \setminus \Lambda_{2l-2}} \nabla_{0,e} \tau_x f \right)^2 \right] \\ &\quad + 2\mu \left[ c_{0,e} \left( \eta(e) - \eta(0) - \sum_{x \in \partial^1 \Lambda_l} \nabla_{0,e} \tau_x f \right)^2 \right]. \end{aligned}$$

510 The first term, just as in the proof of Claim 5.15, is bounded by  $Ce^{-\lambda l}$  according to Claim  
 511 5.11.

512 In order to bound the second term, we start by assuming that all sites of  $\partial^1 \Lambda_l$  are relevant.  
 513 In this case,

$$\sum_{x \in \partial^1 \Lambda_l} \nabla_{0,e} \tau_x f = \sum_{x \in \partial^1 \Lambda_l} \frac{\eta(e) - \eta(0)}{2(2l+1)^{d-1}} = \eta(e) - \eta(0),$$

514 SO

$$\mu \left[ c_{0,e} \left( \eta(e) - \eta(0) - \sum_{x \in \partial^1 \Lambda_l} \nabla_{0,e} \tau_x f \right)^2 \mathbb{1}_{-\partial^1 \Lambda_l \subseteq \mathcal{R}} \right] = 0.$$

515 Finally, by Claim 5.16 the probability that  $\partial^1 \Lambda_l$  contains irrelevant sites is smaller than  $Ce^{-\lambda l}$ :

$$\mu \left[ c_{0,e} \left( \eta(e) - \eta(0) - \sum_{x \in \partial^1 \Lambda_l} \nabla_{0,e} \tau_x f \right)^2 \mathbb{1}_{-\partial^1 \Lambda_l \not\subseteq \mathcal{R}} \right] \leq Ce^{-\lambda l}.$$

516 The claim thus follows by summing the contribution of the three terms.  $\square$

517 All that is left is to combine Claims 5.15 and 5.17, proving inequality (5.3) and hence the  
 518 second part of Theorem 2.3.  $\square$

## 519 6. Further problems

- 520 • Prove convergence to a hydrodynamic limit without the soft constraint from a more  
 521 restricted family of initial states (as in [13]).
- 522 • Improve the bounds on the diffusion coefficient, and in particular find matching up-  
 523 per and lower bound without a logarithmic correction. In the case of the closely  
 524 related Fredrickson-Andersen model, where similar bounds have been obtained for  
 525 the spectral gap ([18]), the logarithmic correction could be removed, and, moreover,  
 526 the exact constant multiplying  $1/(1 - \rho)^{d-k+1}$  could be identified [14].
- 527 • Study qualitative properties of the diffusion coefficient – is it decreasing in  $\rho$ ? Is it  
 528 continuous? Smooth? Is  $D^{(0)} = D$ ?
- 529 • Understand the hydrodynamic limit of more KCLGs. The comparison argument of  
 530 Section 4 could be used in order to estimate the diffusion coefficient whenever an ap-  
 531 propriate multistep move could be constructed, and may be useful in larger generality  
 532 than presented here.
- 533 • The bounds on the diffusion coefficient may have consequences other than the hy-  
 534 drodynamic limit – in general, we expect the correlation  $\mu(\eta(0)e^{t\mathcal{L}}\eta(x)) - \rho^2$  to be-  
 535 have like  $\rho(1 - \rho)(4\pi t D)^{-d/2} e^{-\frac{x^2}{4tD}}$  (see, e.g., [23]). It has been shown in [6] that  
 536 in the Kob-Andersen model, for  $x = 0$ , this correlation decays at least as fast as  
 537  $C(\log t)^5/t$  for some unidentified constant  $C$ . Any progress towards the predicted  
 538  $\rho(1 - \rho)(4\pi t D)^{-d/2} e^{-\frac{x^2}{4tD}}$  (for  $D$  is as in equation (2.6)) would be an interesting re-  
 539 sult.

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## 544 Appendix A. The gradient condition in cooperative models

545 In this appendix we will see that cooperative kinetically constrained lattice gas models  
 546 (KCLGs) are non-gradient.

547 A general KCLG is a Markov process with configuration space  $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ , determined  
 548 by a set of constraints giving each edge  $(x, y) \in \mathcal{E}(\mathbb{Z}^d)$  a rate  $c_{x,y}(\eta) \in \{0\} \cup [1, \infty)$ , for any  
 549 configuration  $\eta \in \Omega$ . We will make the following assumptions:

- 550 (1) The model is homogeneous, i.e., the constraint is translation invariant.



- (2) The constraint  $c_{x,y}$  depends only on the configuration outside  $x$  and  $y$ .
- (3) The constraints have finite range, i.e.,  $c_{x,y}$  depends only on the occupation of sites in the box  $x + \Lambda_R$ , where  $R$  is called the *range*.
- (4) The constraint is non-degenerate, i.e., for every edge  $(x, y)$  of  $\mathbb{Z}^d$  there exist a configuration  $\eta$  such that  $c_{x,y}(\eta) > 0$  and  $\eta'$  such that  $c_{x,y}(\eta') = 0$ .
- (5) For fixed  $x, y$  the constraint  $c_{x,y}(\eta)$  is a decreasing function of  $\eta$ , i.e., adding more empty sites could only help the constraint to be satisfied.

With such constraints, the process is given by a generator as in equation (2.2).

**Definition A.1** (connected configurations). Fix a KCLG and two configurations  $\eta, \eta'$ . We say that  $\eta'$  is *connected* to  $\eta$  if there exists a sequence of configuration  $\eta_0, \dots, \eta_T$  such that  $\eta_0 = \eta$ ,  $\eta_T = \eta'$ , and for all  $t \in \{0, \dots, T-1\}$  there exist  $x_{t+1} \sim y_{t+1}$  such that  $\eta_{t+1} = \eta_t^{x_{t+1}, y_{t+1}}$ , with  $c_{x_{t+1}, y_{t+1}}(\eta_t) \geq 1$ . For any fixed  $e \in \{\pm e_1, \dots, \pm e_d\}$ , we say that  $\eta'$  is *e-connected* to  $\eta$  if, in addition,  $y_{t+1} = x_{t+1} + e$  and  $\eta_t(x_t) = 0$ , namely, all transitions move a vacancy in the direction  $e$  (or, equivalently, a particle in the direction  $-e$ ). Note that  $\eta'$  is connected to  $\eta$  if and only if  $\eta$  is connected to  $\eta'$ ; and  $\eta'$  is *e-connected* to  $\eta$  if and only if  $\eta$  is  $(-e)$ -connected to  $\eta'$ .

**Definition A.2.** Let  $A \subseteq \mathbb{Z}^d$ . The configuration  $\eta_A$  is defined as

$$\eta_A(x) = \begin{cases} 0 & x \in A, \\ 1 & \text{otherwise.} \end{cases}$$

KCLGs could be either *cooperative* or *non-cooperative* (see [6, Definition 1.1]). We remind here that a non-cooperative model is a model in which there exists a *mobile cluster*, defined as follows:

**Definition A.3** (mobile cluster). Let  $A$  be a finite non-empty subset of  $\mathbb{Z}^d$ . We say that  $A$  is a *mobile cluster* if:

- (1) For all  $z \in \mathbb{Z}^d$ , the configuration  $\eta_A$  is connected to the configuration  $\eta_{z+A}$ .
- (2) For every edge  $(x, y)$ , there exists a translation  $z \in \mathbb{Z}^d$  such that  $c_{x,y}(\eta_{z+A}) \geq 1$ .

Gradient models, in our context, are interacting particle systems with conserved number of particles, in which the current is a gradient of some local function. This property significantly simplifies the analysis of their hydrodynamic limits (see, e.g., [15, Definition 2.5]). The purpose of this appendix is to prove the following result:

**Theorem A.4.** *Cooperative KCLGs are non-gradient.*

In order to prove that a model is non-gradient, we will consider the model on a torus, and show that the integral of the current does not always vanish:

582 **Lemma A.5.** Consider a KCLG, and assume that for  $N$  large enough, there exists a configuration  
 583 on the torus  $\eta \in \{0, 1\}^{\mathbb{T}_N^d}$ , such that

$$\sum_{x, y \in \mathbb{T}_N^d} (x - y)(\eta(x) - \eta(y))c_{x, y}(\eta) \neq 0.$$

584 Then the model is non-gradient.

585 *Proof.* Assume that the model is gradient. That is, by [15, Definition 2.5], for some  $n_0 \in \mathbb{N}$ ,  
 586 for any  $1 \leq i \leq d, 1 \leq n \leq n_0$  there exist a cylinder function  $h_{i, n}$  and a finite range function  
 587  $p_{i, n}$  satisfying  $\sum_{x \in \mathbb{T}_N^d} p_{i, n}(x) = 0$ , such that the current is given by

$$W_{0, e_i}(\eta) = \sum_{n=1}^{n_0} \sum_{x \in \mathbb{T}_N^d} p_{i, n}(x) \tau_x h_{i, n}(\eta)$$

588 for every  $i$ .

589 Then for any  $e \in \{e_1, \dots, e_d\}$

$$\begin{aligned} \sum_{y \in \mathbb{T}_N^d} W_{y, y+e}(\eta) &= \sum_{y \in \mathbb{T}_N^d} \tau_{-y} \sum_{n=1}^{n_0} \sum_{x \in \mathbb{T}_N^d} p_{i, n}(x) \tau_x h_{i, n}(\eta) \\ &= \sum_{x \in \mathbb{T}_N^d} \sum_{z \in \mathbb{T}_N^d} \sum_{n=1}^{n_0} p_{i, n}(x) \tau_z h_{i, n}(\eta) \\ &= 0. \end{aligned}$$

590 This concludes the proof, recalling

$$W_{x, x+e}(\eta) = (\eta(x) - \eta(x+e))c_{x, y}(\eta). \quad \square$$

591 The construction of such  $\eta$  for a cooperative KCLG is based on the notion of reachable sites:

592 **Definition A.6** (reachable sites and  $e$ -stretch). We say that a site is *reachable* from a configu-  
 593 ration  $\eta$  if it is empty for some  $\eta'$  which is connected to  $\eta$ . For  $e \in \{\pm e_1, \dots, \pm e_d\}$  we say that  
 594 a site is  $e$ -reachable for a configuration  $\eta$  if it is empty for some  $\eta'$  which is  $e$ -connected to  $\eta$ .  
 595 The  $e$ -stretch of  $\eta$  is defined as

$$\sup \{e \cdot x : x \text{ is } e\text{-reachable}\}.$$

596 By the definition of non-cooperative models, it is immediate that if  $\eta$  contains a mobile  
 597 cluster then for every site  $x$  there exists  $\eta'$  connected to  $\eta$  for which  $\eta'(x) = 0$ . In the next  
 598 proposition we will see that if we require  $e$ -connectivity the converse is also true –

599 **Proposition A.7.** Assume that for all  $e \in \{\pm e_1, \dots, \pm e_d\}$  there exists a finite subset  $A_e$  of  $\mathbb{Z}^d$ ,  
 600 such that the  $e$ -stretch of  $\eta_{A_e}$  is infinite. Then the model is non-cooperative.

Before proving this proposition, we will see how it implies Theorem A.4. Consider a cooperative KCLG, so by Proposition A.7 for some  $e \in \{\pm e_1, \dots, \pm e_d\}$  and any  $L \in \mathbb{N}$ , configurations that are entirely filled outside  $\Lambda_L$  have finite  $e$ -stretch. We will assume without loss of generality that  $e = e_1$ .

Since the model is non-degenerate, there exists a configuration  $\eta_0$  for which  $c_{0,e_1}(\eta_0) = 1$ . Since the model has finite range  $R$ , we may assume that this configuration is entirely filled outside  $\Lambda_R$ ; and since the constraint does not depend on the occupation at 0 and  $e_1$  we assume  $\eta_0(0) = 0$  and  $\eta_0(e_1) = 1$ . We will now construct a sequence of configuration starting at  $\eta_0$ , so that  $\eta_{i+1}$  is obtained from  $\eta_i$  by moving a 0 to the right, i.e.,  $\eta_{i+1} = \eta_i^{x_i, x_i+e_1}$  for some  $x_i$  such that  $c_{x_i, x_i+e_1}(\eta_i) > 0$ ,  $\eta_i(x_i) = 0$ , and  $\eta_i(x_i + e_1) = 1$ . When, for some  $i$ , more than one such choice of  $x$  is possible, we choose one arbitrarily. We stop when none of the sites satisfy the required conditions.

Since the  $e_1$ -stretch of  $\eta_0$  is finite the construction must stop at some step  $n < \infty$ . On the other hand, we chose  $\eta_0$  such that  $c_{0,e_1}(\eta_0) \geq 1$ ,  $\eta_0(0) = 0$ , and  $\eta_0(e_1) = 1$ , so  $n \geq 1$ . Hence, for the configuration  $\eta = \eta_n$ , for all  $x \in \mathbb{Z}^d$

$$c_{x, x+e_1}(\eta)(1 - \eta(x))\eta(x + e_1) = 0,$$

but for  $x^* = x_{n-1}$  (using  $c_{x^*, x^*+e_1}(\eta_n) = c_{x^*, x^*+e_1}(\eta_{n-1}) = 1$ ), we know that

$$c_{x^*, x^*+e_1}(\eta)\eta(x^*)(1 - \eta(x^* + e_1)) \geq 1.$$

That is,

$$\sum_{x \in \mathbb{Z}^d} (\eta(x) - \eta(x + e_1))c_{x, x+e_1}(\eta) \geq 1.$$

Since  $\eta$  is filled outside  $\Lambda_{R+n}$ , we may as well sum over  $x$  in a large enough torus  $\mathbb{T}_{100R+n}^d$ . Therefore, by Lemma A.5, the model is indeed non-gradient.  $\square$

620

We return to the proof of Proposition A.7.

**Claim A.8.** Fix a finite non-empty  $A \subset \mathbb{Z}^d$ , and  $e \in \{\pm e_1, \dots, \pm e_d\}$ . Assume that the  $e$ -stretch of  $\eta_A$  is infinite. Then there exists a finite non-empty  $A' \subset \mathbb{Z}^d$  and a strictly positive integer  $n$ , such that  $\eta_{A'}$  is  $e$ -connected to  $\eta_{ne+A'}$ .

*Proof.* First, we may assume without loss of generality that  $A$  has the minimal possible size, among sets for which the  $e$ -stretch of  $\eta_A$  is infinite; and for notational convenience we also assume  $e = e_1$ . Set  $k = |A|$ , and fix  $L$  such that  $A \subset \Lambda_L$ .

We will start by showing the following property:

**Claim A.9.** For all  $j < k$ , there exists  $s^{(j)}$  such that for all  $B \subset (-\infty, 0] \times \mathbb{Z}^{d-1}$  with  $|B| = j$ , the  $e_1$ -stretch of  $\eta_B$  is at most  $s^{(j)}$ . In particular, there exists  $L^{(j)}$  such that the maximal possible  $e_1$ -stretch for such a set is obtained for some  $B \subset [-L^{(j)}, 0] \times \mathbb{Z}^{d-1}$ .

632 *Proof.* For  $j = 1$  choosing  $s^{(1)} = L^{(1)} = 0$  suffices since no particle could move. For  $j > 1$ ,  
 633 let  $L^{(j)} = j(h^{(j-1)} + R)$  and  $s^{(j)}$  the maximal  $e_1$ -stretch of  $\eta_B$  for any  $B \subset [-L^{(j)}, 0] \times \mathbb{Z}^{d-1}$ .  
 634 Note that  $s^{(j)}$  is well defined since particles cannot move in directions orthogonal to  $e_1$ , so we  
 635 may assume without loss of generality that  $B \subset [-L^{(j)}, 0] \times [-jR, jR]^{d-1}$ ; and it is finite since  
 636  $j < k$ .

637 Assume now that for some  $B \subset (-\infty, 0] \times \mathbb{Z}^{d-1}$  of size  $j$  the  $e_1$ -stretch of  $\eta_B$  is more than  
 638  $s^{(j)}$ . We can assume without loss of generality that  $0 \in B$ , and by construction there must be  
 639 a site  $x \in B$  outside  $[-L^{(j)}, 0] \times \mathbb{Z}^{d-1}$ . Due to our choice of  $L^{(j)}$ , the set  $B$  could be separated  
 640 by a strip of width  $h^{(j-1)} + R$ , namely, there exists  $n \in \mathbb{Z}$  such that

$$\begin{aligned} B &= B_- \cup B_+, \\ B_- &\subset (-\infty, n] \times \mathbb{Z}^d, \\ B_+ &\subset (n + h^{(j-1)} + R, 0] \times \mathbb{Z}^d. \end{aligned}$$

641 However, since the  $e_1$ -stretch of  $\eta_{B_-}$  is at most  $h^{(j-1)}$ , it would never be able to influence  
 642 transitions to the right of  $n + h^{(j-1)} + R$ , thus the  $e_1$ -stretch of  $B$  cannot be larger than that  
 643 of  $B_-$ , which is a contradiction.  $\square$

644 As a result of this claim, there exists  $s < \infty$ , such that for any set  $B$  of size strictly less than  
 645  $k$ , the  $e_1$ -stretch of  $B$  is at most  $s$  plus its maximal  $e_1$  coordinate.

646 Since the  $e_1$ -stretch of  $\eta_A$  is infinite, there exists an  $e_1$ -reachable site  $x$  with  $e \cdot x > \binom{(2L+1)^{d-1}k(s+R)}{k} +$   
 647  $s + 1$ . Consider a sequence of  $T$  flips which empties that site. We denote the set of empty  
 648 sites at step  $t$  by  $A_t$ , so that  $A_0 = A$  and  $A_T \ni x$ ; and  $a_t$  denotes the rightmost coordinate  
 649 of  $A_t$  (i.e.,  $a_t = \max_{y \in A_t} \{e_1 \cdot y\}$ ). Assume now that at some time  $t$  we are able to identify a  
 650 non-empty set  $\tilde{A}_t$  whose rightmost coordinate is  $\tilde{a}_t$ , such that all sites of  $A_t \setminus \tilde{A}_t$  are at least  
 651  $s + R$  to the right of  $\tilde{a}_t$ , i.e.,  $a_t < e_1 \cdot y - s - R$  for all  $y \in A_t \setminus \tilde{A}_t$ . We then know that the 0's  
 652 coming from  $\tilde{A}_t$  will never be able to reach distance  $R$  from the sites of  $A_t \setminus \tilde{A}_t$ , thus the set  
 653  $A_t \setminus \tilde{A}_t$  moves as if these sites were filled. In particular, it could not go further than distance  
 654  $s$ , hence  $a_t > \binom{(2L+1)^{d-1}k(h+s)}{k} + 1$ . That means that for at least  $\binom{(2L+1)^{d-1}k(s+R)}{k} + 1$  times  $t$   
 655 with different values of  $a_t$ ,

$$A_t \subset [a_t - k(s + R), a_t] \times [-L, L]^{d-1}.$$

656 This box has volume  $(2L + 1)^{d-1}k(s + R)$ , so by the pigeonhole principle there exist  $t$  and  $t'$   
 657 with  $a_t < a_{t'}$  such that  $A_t - a_t e_1 = A_{t'} - a_{t'} e_1$ . This finishes the proof by taking  $A' = A_t - a_t e_1$   
 658 and  $n = a_{t'} - a_t$ , and using the translation invariance of the model.  $\square$

659 *Claim A.10.* Fix any finite  $B \subset \mathbb{Z}^d$  and  $e \in \mathbb{Z}^d$ , and assume that there exists a finite non-empty  
 660  $A \subset \mathbb{Z}^d$  such that the  $e$ -stretch of  $\eta_A$  is infinite. Then there exist a finite non-empty set  $A' \subset \mathbb{Z}^d$   
 661 such that for all  $m \in \mathbb{N}$ , the configuration  $\eta_{A'}$  is  $e$ -connected to a configuration  $\eta_m$  in which  
 662 all the sites of  $me + B$  are empty. Moreover, we can assume that no site after  $me + B$  is empty,  
 663 i.e.,  $\eta_m(x) = 1$  whenever  $x \cdot e > m + \sup_{y \in B} y \cdot e$ .

664 *Proof.* By the Claim A.8 there exists  $L \in \mathbb{N}$ ,  $A'' \subset \Lambda_L$ , and  $n \in \mathbb{N}$ , such that  $\eta_{A''}$  is  $e$ -connected  
 665 to  $\eta_{ne+A''}$ . Note that we may, equivalently, choose any  $A''$  which is a translation of  $A_\eta$  for any  
 666  $\eta$  in the path connecting  $\eta_{A''}$  with  $\eta_{ne+A''}$ . We will therefore assume without loss of generality  
 667 that  $0 \in A''$ , but  $e \cdot x < 0$  for all  $x \in A \setminus \{0\}$ .

668 Denote  $B = \{b_1, \dots, b_k\}$ , with  $e \cdot b_1 \geq \dots \geq e \cdot b_k$ , and consider the union

$$A_0 = \bigcup_{i=1}^k (b_i + A'' - inLe).$$

669 This union is disjoint, since  $A'' \subset \Lambda_L$ , and by repeating  $L$  times the sequence of flips required  
 670 to move  $A''$  to  $ne + A''$ , we can move  $b_1 + A'' - nLe$  to  $b_1 + A''$ , reaching a configuration in  
 671 which  $b_1$  is empty. Then, repeating this sequence again  $2L$  times we can move  $b_2 + A'' - 2nLe$   
 672 to  $b_2 + A''$ . This is allowed since during the first sequence we do not changes the configuration  
 673 at the sites of  $b_2 + A'' - 2nLe$ ; and the in the resulting configuration both  $b_1$  and  $b_2$  are empty.  
 674 We continue in the same manner, until we reach a configuration  $\eta'_0$  in which the sites of  $B$   
 675 are all empty.

676 Consider now for  $j = 0, \dots, n-1$  the set

$$A_j = A_0 - knLje + je.$$

677 As before, applying repeatedly the sequence that allowed us to move  $A''$  we can reach a  
 678 configuration  $\eta_j$  (connected to  $\eta'_{A_j}$ ) in which the sites of  $je + B$  are empty. Furthermore,  $A_j$   
 679 and  $A_{j'}$  are disjoint for  $j \neq j'$ , so, indeed, taking

$$A' = \bigcup_{j=0}^{n-1} A_j,$$

680 for  $j = 0, \dots, n-1$ , the configuration  $\eta_{A'}$  is  $e$ -connected to a configuration  $\eta_j$  for which the  
 681 sites of  $je + B$  are empty. Finally, since  $A'$  is a disjoint union of copies of  $A''$ , we can translate  
 682 each of them by  $ne$ , and if we do that in the right order (starting with  $b_1 + A'' - nLe$  and  
 683 ending with  $b_k + A'' - knL(n-1)e + (n-1)e$ ) they will never intersect. Hence  $\eta_{ne+A'}$  is  
 684  $e$ -connected to  $\eta_{A'}$ , and the result follows.  $\square$

685 *Claim A.11.* Fix  $e \in \{\pm e_1, \dots, \pm e_d\}$  and  $L \in \mathbb{N}$ . Assume that there exists a finite non-empty  
 686  $A \subset \mathbb{Z}^d$  such that the  $e$ -stretch of  $\eta_A$  is infinite. Then there exists  $L'$  and  $A' \subset \Lambda_{L'}$  such that  
 687 for all  $x \in [L', \infty] \times [-L, L]^{d-1}$  and every configuration  $\eta$  for which the sites of  $A'$  are empty,  
 688  $\eta$  is connected to  $\eta^{x, x+e}$ .

689 *Proof.* We assume without loss of generality that  $e = e_1$ . The first observation needed in order  
 690 to prove this claim, is that there is a configuration for which the constraint  $c_{x, x+e_1}$  is satisfied,  
 691 but none of the sites to the right of  $x$  are empty, i.e.,  $x + [1, \infty] \times \mathbb{Z}^{d-1}$  is entirely occupied.  
 692 This is true since, if the  $e_1$ -stretch of  $\eta_A$  is infinite for finite  $A$ , at some point the rightmost 0  
 693 has to move to the right.

694 We then find a finite non-empty  $B_0 \subset [-\infty, 0] \times \mathbb{Z}^{d-1} \setminus \{0\}$  such that  $c_{0,e_1}(\eta_{B_0}) = 1$ . Let

$$B = \bigcup_{z \in \{0\} \times [-L, L]^{d-1}} (z + B_0).$$

695 Then, in particular,  $c_{x,x+e_1}(\eta_B) = 0$  for  $x \in \{0\} \times [-L, L]^{d-1}$ .

696 We now apply Claim A.10 to find a finite non-empty set  $A' \subset \mathbb{Z}^d$  such that for all  $m \in \mathbb{N}$ ,  
 697 the configuration  $\eta_{A'}$  is  $e$ -connected to a configuration  $\eta_m$  in which all the sites of  $me + B$  are  
 698 empty. We define  $L'$  such that  $A' \subset \Lambda_{L'}$ , and then, for every  $x \in [L', \infty] \times [-L, L]^{d-1}$ , taking  
 699  $m = e_1 \cdot x$  yields  $c_{x,x+e_1}(\eta_m) = 1$ . Therefore, if we take any configuration  $\eta$  for which  $A'$  is  
 700 empty, by performing the same transitions that connected  $\eta_{A'}$  to  $\eta_m$ , we reach a configuration  
 701 for which  $c_{x,x+e_1} = 1$ . Note that this is done without changing the configuration neither at  $x$   
 702 nor at  $x + e_1$ . We then exchange  $x$  and  $x + e_1$ , and fold back all the transitions we have done  
 703 before, reaching the configuration  $\eta^{x,x+e_1}$ .  $\square$

704 *Claim A.12.* Assume that for all  $e \in \{e_1, \dots, e_d\}$  there exists a finite set  $A_e \subset \mathbb{Z}^d$  such that the  
 705  $e$ -stretch of  $\eta_{A_e}$  is infinite, and fix  $e' \in \{e_1, \dots, e_d\}$ . Then there exists  $L \in \mathbb{N}$  and  $A \subset \Lambda_L$  such  
 706 that for any  $\eta$  in which the sites of  $A$  are empty, and any  $x \in [L+1, \infty]^d$ , the configuration  
 707  $\eta^{x,x+e'}$  is connected to  $\eta$ .

708 *Proof.* Without loss of generality we fix  $e = e_1$ . By Claim A.11 we can define  $L_1 \in \mathbb{N}$  and  
 709  $A_1 \subset \Lambda_{L_1}$  be such that for all  $x_1 \in [L_1, \infty] \times \{0\}^{d-1}$  and every configuration  $\eta$  for which  
 710 the sites of  $A_1$  are empty,  $\eta$  is connected to  $\eta^{x_1, x_1+e_1}$ . Using Claim A.10 we can find  $L_2 \in$   
 711  $\mathbb{N}$  and  $A_2 \subset \Lambda_{L_2}$  such that, for every  $x_2 \in \{0\} \times [L_2, \infty] \times \{0\}^{d-1}$ , the configuration  $\eta_{A_2}$   
 712 is connected to a configuration  $\eta$  in which the sites of  $x_2 + A_1$  are empty, and during the  
 713 sequence of configurations connecting the two only edges of  $[-\infty, -L_2]^d$  were flipped. We  
 714 continue in the same manner, for  $i = 1, \dots, d$ , to construct  $L_i$  and  $A_i \subset \Lambda_{L_i}$  such that for all  
 715  $x_i \in \{0\}^{i-1} \times [L_i, \infty] \times \{0\}^{d-i}$ , the configuration  $\eta_{A_i}$  is connected to a configuration in which  
 716 the sites of  $x_i + A_{i-1}$  are empty, and during the sequence of configurations connecting the  
 717 two only edges of  $[-\infty, -L_i]^d$  were flipped.

718 Let  $L = L_d$ ,  $A = A_d$ , and fix  $\eta$  in which the sites of  $A$  are empty and  $x \in [L+1, \infty]^d$ . We  
 719 write  $x = x_1 + \dots + x_d$  for  $x_i \in \{0\}^{i-1} \times [L_i, \infty] \times \{0\}^{d-i}$ . By our construction of  $A$ ,  $\eta$  is  
 720 connected to a configuration  $\eta'$  in which the set  $A_1 + x_2 + \dots + x_d$  is empty, and during the  
 721 sequence of configurations connecting the two the sites  $x$  and  $x + e_1$  remained untouched.  
 722 Then, by the construction of  $A_1$ , we can connect  $\eta'$  to  $\eta'^{x,x+e_1}$ . All that is left is to rewind the  
 723 steps leading to  $\eta'$ , and the proof is complete.  $\square$

724 *Claim A.13.* Assume that for all  $e \in \{e_1, \dots, e_d\}$  there exists a finite set  $A_e \subset \mathbb{Z}^d$  such that the  
 725  $e$ -stretch of  $\eta_{A_e}$  is infinite. Then there exists  $L \in \mathbb{N}$  and  $A \subset \Lambda_L$  such that for any  $\eta$  in which  
 726 the sites of  $A$  are empty, any  $x \in [L+1, \infty]^d$ , and any  $e' \in \{e_1, \dots, e_d\}$ , the configuration  
 727  $\eta^{x,x+e'}$  is connected to  $\eta$ .

728 *Proof.* The only difference between this claim and Claim A.12 is that now  $e'$  is chosen after  $A$  is  
 729 fixed. In order to achieve that, we apply Claim A.12  $d$  times, with  $e' = e_i$  for all  $i \in \{1, \dots, d\}$ ,  
 730 obtaining  $d$  numbers  $L_1, \dots, L_d \in \mathbb{N}$  and  $d$  sets  $A_1 \in \Lambda_{L_1}, \dots, A_d \in \Lambda_{L_d}$ . Taking  $L = \max_i L_i$   
 731 and  $A = \cup_{i=1}^d A_i$  will suffice – fix  $\eta$  in which the sites of  $A$  are empty, every  $x \in [L+1, \infty]^d$   
 732 and  $i \in \{1, \dots, d\}$ . In particular  $x \in [L_i+1, \infty]^d$ , and that the sites of  $A_i$  are empty in  $\eta$ , so  
 733 by construction of  $A_i$  we know that  $\eta^{x, x+e_i}$  is connected to  $\eta$ .  $\square$

734 We are now ready to prove Proposition A.7.

735 *Proof of Proposition A.7.* We assume that for all  $e \in \{\pm e_1, \dots, \pm e_d\}$  there exists a finite set  
 736  $A_e \subset \mathbb{Z}^d$  such that the  $e$ -stretch of  $\eta_{A_e}$  is infinite, and construct a mobile cluster  $A$ .

737 First, use Claim A.13 in order to find  $L_+ \in \mathbb{N}$  and  $A_+ \subset \Lambda_{L_+}$  such that for any  $\eta$  in which  
 738 the sites of  $A_+$  are empty, any  $x \in [L_++1, \infty]^d$ , and any  $e \in \{e_1, \dots, e_d\}$ , the configuration  
 739  $\eta^{x, x+e}$  is connected to  $\eta$ . Similarly (by flipping  $\mathbb{Z}^d$ ), we can find  $L_- \in \mathbb{N}$  and  $A_- \subset \Lambda_{L_-}$   
 740 such that for any  $\eta$  in which the sites of  $A_-$  are empty, any  $x \in [-\infty, -L_- - 1]^d$ , and any  
 741  $e \in \{-e_1, \dots, -e_d\}$ , the configuration  $\eta^{x, x+e}$  is connected to  $\eta$ . It will be more convenient to  
 742 consider translations of these sets,

$$\begin{aligned} A'_+ &= A_+ - (L_+ + 2)e_1 - \dots - (L_+ + 2)e_d, \\ A'_- &= A_- + (L_- + 2)e_1 + \dots + (L_- + 2)e_d. \end{aligned}$$

743 This way, for any  $\eta$  in which the sites of  $A'_+$  are empty, any  $x \in [2, \infty]^d$ , and any  $e \in$   
 744  $\{\pm e_1, \dots, \pm e_d\}$ , the configuration  $\eta^{x, x+e}$  is connected to  $\eta$ ; and for any  $\eta$  in which the sites  
 745 of  $A'_-$  are empty, any  $x \in [-\infty, -2]^d$ , and any  $e \in \{\pm e_1, \dots, \pm e_d\}$ , the configuration  $\eta^{x, x+e}$  is  
 746 connected to  $\eta$ . Let

$$A = A'_+ \cup A'_-$$

747 We will show that it is a mobile cluster. Since already  $A'_+$  allows us to flip edges in its vicinity,  
 748 we only need to show that  $\eta_A$  is connected to  $\eta_{e+A}$  for all  $e \in \{\pm e_1, \dots, \pm e_d\}$ . To do that,  
 749 we note that, since the sites of  $A'_-$  are all in  $[2, \infty]$ , the configuration  $\eta_A$  is connected to  
 750  $\eta_{A'_+ \cup (e+A'_-)}$ . In this new configuration the sites of  $e+A'_-$  are empty, and since the sites of  $A'_+$   
 751 are all in  $[-\infty, -2]^d + e$  it is connected to  $\eta_{(e+A'_+) \cup (e+A'_-)} = \eta_{e+A}$ .  $\square$

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