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The calculation of the electromagnetic self-force of a non-Lorentz-contractible uniformly charged spherical shell in arbitrary rectilinear motion

G. Vaman*

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We write the electromagnetic self-force of a non-Lorentz-contractible uniformly charged shell of radius a as a series in powers of a , and we calculate the first three terms of this expansion. The method of calculation presented here allows the exact consideration of all linear and non-linear terms in velocity and its derivatives corresponding to a given power of a . Our calculation is entirely done in the laboratory frame of reference.

Keywords: self-force, electromagnetic field, spherical shell.

I. INTRODUCTION

The calculation of the electromagnetic self-force of an extended charged body became important recently with the development of high-power lasers [1] - [3]. The experimental results of the interaction of an ultra-intense laser beam with an electron beam or a plasma slab were interpreted by comparing the post-collision spectra with the spectra corresponding to some theoretical models. The theoretical models considered were both classical and quantum but, as far as we know, they have not considered the finite dimensions of the charged particles. Taking into account the spatial extension of a charged radiating body not only eliminates the divergencies that appear in point-like models but also provides a new parameter of equal importance to velocity and its derivatives for fitting the experimental data. Considering the spatial extension is also important in the study of charged or finite-mass bodies moving in external electromagnetic or gravitational fields because higher electromagnetic and gravitational multipoles can give contributions to dynamics of the same order of magnitude as the monopoles. [4, 5].

Technically, the theoretical calculation of the electromagnetic self-force is not simple, and the existing models were developed using some reasonable approximations. A short survey of the existing models and results can be found, for example, in [6]. We present in this paper a

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method of calculation of the electromagnetic self-force of a non-Lorentz-contractible spherical shell. We suppose that, in the laboratory frame of reference, the shell has a spherical shape of radius a , it is uniformly charged with a charge density $e/4\pi a^2$, and its center undergoes an arbitrary rectilinear motion along the x -axis. We denote the trajectory of its center by $w(t)\mathbf{i}$, where \mathbf{i} is the unit vector along the x -axis and, for simplicity, we suppose that, at the current time t , the center of the shell is located at the origin $\mathbf{r} = 0$, so $w(t) = 0$. Our calculation is done entirely in the laboratory frame. We write the self-force as a series in powers of the shell's radius a and calculate the first three terms of this series. In contrast to other similar calculations from the literature, the method presented here is exact for arbitrary velocities because, for a given power of a , we add all the terms, linear and non-linear in velocity and its derivatives. Although we only give here the results for the first three terms, the next terms can be easily calculated using the same algorithm. In the second section of this paper we present our method of calculation, and in the third section we discuss our results and compare them with the other results existing in the literature.

II. CALCULATION OF THE ELECTROMAGNETIC SELF-FORCE

The electromagnetic self-force acting on a charged extended particle can be written in terms of the Lorentz force density

$$\mathbf{F} = \int d\mathbf{r} \left(\rho(\mathbf{r}, t) \mathbf{E}_s(\mathbf{r}, t) + \frac{1}{c} \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}_s(\mathbf{r}, t) \right), \quad (1)$$

where the self-fields are given in terms of the electromagnetic potentials as $\mathbf{E}_s(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$, $\mathbf{B}_s(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$. As done in [7], we write the electromagnetic potentials in terms of retarded charge and current densities $\phi(\mathbf{r}, t) = \int d\mathbf{r}' \frac{\rho(\mathbf{r}', t')}{R}$, $\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}', t')}{R}$, $R = |\mathbf{r} - \mathbf{r}'|$, $t' = t - R/c$, and expand the retarded charge and current densities in Taylor series around $t' = t$. We obtain

$$\mathbf{F}(t) = \mathbf{F}^1(t) + \mathbf{F}^2(t), \quad (2)$$

$$\mathbf{F}^1(t) = - \int d\mathbf{r} \rho(\mathbf{r}, t) \nabla \phi(\mathbf{r}, t) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n! c^{n+2}} \frac{n+1}{n+2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{n-1} \cdot \frac{\partial^{n+1}}{\partial t^{n+1}} (\rho(\mathbf{r}, t) \mathbf{v}(t)), \quad (3)$$

$$\mathbf{F}^2(t) = -\frac{1}{c} \int d\mathbf{r} \rho(\mathbf{r}, t) \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! c^{n+2}} \frac{n-1}{n+2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}', t) |\mathbf{r} - \mathbf{r}'|^{n-3} \cdot \frac{\partial^{n+1}}{\partial t^{n+1}} (\rho(\mathbf{r}, t) \mathbf{v}(t) \cdot (\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')). \quad (4)$$

For rectilinear motion, the second term from the r.h.s. of Eq. (1) gives no contribution. It is convenient to write the charge density of the uniformly charged shell of radius a as a Taylor expansion in ascending powers of $w(t)$ [8]

$$\rho(\mathbf{r}, t) = \frac{e}{4\pi a^2} \delta(|\mathbf{r} - \mathbf{i}w(t)| - a) = \frac{e}{4\pi a^2} \sum_{p=0}^{\infty} \frac{(-1)^p w^p(t)}{p!} \partial_x^p \delta(r - a), \quad (5)$$

where δ is the Dirac delta function and ∂_x is the partial derivative with respect to x . Although this expansion suggests that we consider only small oscillations $w(t)$ about a fixed point, during our calculation we shall perform the summation over p , and our final results are valid for an arbitrary rectilinear motion. As we consider the charge distribution as being rigid, the current density is given by $\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \dot{w}(t) \mathbf{i}$.

We introduce Eq. (5) in Eqs. (3), (4) and perform the radial integration as usual

$$\int d\mathbf{r} \delta(r - a) f(r, \theta, \phi) = a^2 \int_{S_a} dS_a f(a, \theta, \phi), \quad (6)$$

$$\int d\mathbf{r} \partial_x^p \delta(r - a) f(r, \theta, \phi) = (-1)^p \int d\mathbf{r} \delta(r - a) \partial_x^p f(r, \theta, \phi) = (-1)^p a^2 \int_{S_a} dS_a \partial_x^p f(r, \theta, \phi)|_{r=a}, \quad (7)$$

where

$$dS_a = \sin \theta d\theta d\phi, \quad (8)$$

$\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$, is the spherical surface element and $f(r, \theta, \phi)$ is an arbitrary function written in terms of spherical coordinates. For calculating $F_x^2(t)$, we also need the formula for the higher derivative of a product

$$\partial_x^p (f(\mathbf{r})g(\mathbf{r})) = \sum_{k=0}^p \frac{p!}{k!(p-k)!} \partial_x^k f(\mathbf{r}) \partial_x^{p-k} g(\mathbf{r}), \quad (9)$$

from which, after noticing that only the first two derivatives with respect to x of $(x - x')^2$ are different from zero, it follows

$$\begin{aligned} \partial_x^p [|\mathbf{r} - \mathbf{r}'|^{n-3} (x - x')^2] &= (x - x')^2 \partial_x^p |\mathbf{r} - \mathbf{r}'|^{n-3} \\ &+ 2p(x - x') \partial_x^{p-1} |\mathbf{r} - \mathbf{r}'|^{n-3} + p(p-1) \partial_x^{p-2} |\mathbf{r} - \mathbf{r}'|^{n-3}. \end{aligned} \quad (10)$$

For the derivatives of the type $\partial_x^p |\mathbf{r} - \mathbf{r}'|^{n-1}$ we use [8]

$$\frac{\partial^p}{\partial x^p} |\mathbf{r} - \mathbf{r}'|^{n-1} = \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{b+\frac{p}{2}} 2^{2b} p!}{(2b)! (\frac{p}{2} - b)!} \left(\frac{1-n}{2} \right)_{b+\frac{p}{2}} |\mathbf{r} - \mathbf{r}'|^{n-p-2b-1} (x-x')^{2b}, \quad (11)$$

when p even, and a similar formula for p odd. These derivatives can be inferred with the method used in [10] for the calculation of $\partial_x^p r^n$. Noting that the integrals of the type $I = \int dS_a \int dS'_a \frac{(x-x')^k}{|\mathbf{r}-\mathbf{r}'|^n}$ are different from zero only for k even (because, when we change $\mathbf{r} \rightarrow \mathbf{r}'$, I becomes $(-1)^k I$), one obtains

$$F_x^1(t) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{n=0}^{\infty} \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{n+b+\frac{p}{2}+1} 2^{2b}}{n!(2b)! (\frac{p}{2} - b)! c^{n+2}} \frac{n+1}{(n+2)(p+1)} \cdot \left(\frac{1-n}{2} \right)_{b+\frac{p}{2}} \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^{p-n+2b+1}}, \quad (12)$$

$$F_x^2(t) = f_{x1}(t) + f_{x2}(t) + f_{x3}(t), \quad (13)$$

where

$$f_{x1}(t) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{n=0}^{\infty} \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{n+b+\frac{p}{2}} 2^{2b}}{n!(2b)! (\frac{p}{2} - b)! c^{n+2}} \frac{n-1}{(n+2)(p+1)} \left(\frac{3-n}{2} \right)_{b+\frac{p}{2}} \cdot \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int dS_a \int dS'_a \frac{(x-x')^{2b+2}}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^{p-n+2b+3}}, \quad (14)$$

$$f_{x2}(t) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{n=0}^{\infty} \sum_{b=0}^{\frac{p}{2}-1} \frac{(-1)^{n+b+\frac{p}{2}} 2^{2b+2}}{n!(2b+1)! (\frac{p}{2} - b - 1)! c^{n+2}} \frac{n-1}{(n+2)(p+1)} \left(\frac{3-n}{2} \right)_{b+\frac{p}{2}} \cdot \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int dS_a \int dS'_a \frac{(x-x')^{2b+2}}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^{p-n+2b+3}}, \quad (15)$$

$$f_{x3}(t) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{n=0}^{\infty} \sum_{b=0}^{\frac{p}{2}-1} \frac{(-1)^{n+b+\frac{p}{2}-1} 2^{2b}}{n!(2b)! (\frac{p}{2} - b - 1)! c^{n+2}} \frac{n-1}{(n+2)(p+1)} \left(\frac{3-n}{2} \right)_{b+\frac{p}{2}-1} \cdot \frac{d^{n+2} w^{p+1}(t)}{dt^{n+2}} \int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^{p-n+2b+1}}, \quad (16)$$

and dS'_a is the same spherical surface element written with primed variables.

As we assumed that $w(t) = 0$, it follows that $\frac{d^p w^k(t)}{dt^p}$ is different from zero only for $p \geq k$. With this remark, it follows that the terms in the r.h.s. of equations (12) - (16) are different

from zero only for $n \geq p - 1$. As a Pochhammer symbol of the type $(-n)_k$, where n is a positive integer, is different from zero only for $k \leq n$, we see that the terms obtained for $n = p - 1$ in Eqs. (12) - (16) are zero due to the Pochhammer symbols $(\frac{1-n}{2})_{b+p/2}$, $(\frac{3-n}{2})_{b+p/2}$ and $(\frac{3-n}{2})_{b+p/2-1}$. We note that, when we give n increasing values from p to infinity, our equations (12) - (16) are giving us the series expansion of the electromagnetic self-force in powers of a . We shall calculate here the first three non-zero terms in the expansion of the self-force in powers of a , of orders a^{-1} , a^0 and a^1 , that can be obtained for $n = p$, $p + 1$ and $p + 2$ respectively. We write the first three terms of this expansion as

$$F_x(t) = F_x\left(\frac{1}{a}\right) + F_x(a^0) + F_x(a) + \mathcal{O}(a^2). \quad (17)$$

In the r.h.s. of the above equation, for shortness, we dropped the argument t of the self-force, and this is the notation that we shall use in the rest of our paper for the first terms in the expansion of the self-force. The double surface integrals that appear during our calculation can be calculated by using the expansion [8]

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|^n} \Big|_{r=r'=a} &= \frac{1}{a^n 2^{\frac{n}{2}}} \sum_{k=0}^{\infty} \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=-k+2q}^{k-2q} \frac{(k + \frac{n}{2} - 1)!}{k! (\frac{n}{2} - 1)!} \frac{(2k - 4q + 1)k!}{2^q q! (2k - 2q + 1)!!} \\ &\quad \cdot \frac{(k - 2q - m)!}{(k - 2q + m)!} P_{k-2q}^m(\cos \theta) P_{k-2q}^m(\cos \theta') e^{im(\phi - \phi')}, \end{aligned} \quad (18)$$

that was obtained from

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|^n} \Big|_{r=r'=a} = a^{-n} 2^{-n/2} (1 - \cos \theta_{12})^{-n/2} = \frac{1}{a^n 2^{n/2}} \sum_{k=0}^{\infty} \frac{(\cos \theta_{12})^k (k + \frac{n}{2} - 1)!}{k! (\frac{n}{2} - 1)!}, \quad (19)$$

where

$$\cos \theta_{12} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2), \quad (20)$$

and [9]

$$(\cos \theta_{12})^k = \sum_{l=0 \text{ or } 1}^k \frac{(2l+1) k! P_l(\cos \theta_{12})}{2^{\frac{k-l}{2}} (\frac{k-l}{2})! (k+l+1)!!}, \quad (21)$$

where k and l in the above formula have the same parity. For the derivatives of the type $\frac{d^p}{dt^p} w^k(t)$ when $w(t) = 0$, that appear in our calculation, we use Eqs. (4) - (6) from [10] and

$$\begin{aligned} \frac{d^{k+3} w^k(t)}{dt^{k+3}} &= \frac{k(k+3)!}{24} \dot{w}^{k-1}(t) \ddot{w}(t) + \frac{k(k-1)(k+3)!}{12} \dot{w}^{k-2}(t) \ddot{w}(t) \dot{w}(t) \\ &\quad + \frac{k(k-1)(k-2)(k+3)!}{48} \dot{w}^{k-3}(t) \ddot{w}^3(t). \end{aligned} \quad (22)$$

A. The term of order $1/a$

We put $n = p$ in Eqs. (12) - (16) and use [10]

$$\frac{d^{p+2}w^{p+1}(t)}{dt^{p+2}} = \frac{(p+1)(p+2)!}{2}\dot{w}^p(t)\ddot{w}(t), \quad (23)$$

and [8]

$$\int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r}-\mathbf{r}'|^{2b+1}}_{|r=r'=a} = \frac{16\pi^2}{a(2b+1)}. \quad (24)$$

After changing the summation index $p \rightarrow 2p$ and writing the Pochhammer symbols in terms of Gamma functions according to their definition $(a)_k = \Gamma(a+k)/\Gamma(a)$, one obtains

$$F_x^1\left(\frac{1}{a}\right) = -\frac{e^2\ddot{w}}{2ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^{2p} (2p+1)^2}{\Gamma\left(\frac{1}{2}-p\right)} \sum_{b=0}^p \frac{(-1)^b 2^{2b} (b-\frac{1}{2})!}{(2b+1)!(p-b)!}, \quad (25)$$

$$F_x^2\left(\frac{1}{a}\right) = f_{x1}\left(\frac{1}{a}\right) + f_{x2}\left(\frac{1}{a}\right) + f_{x3}\left(\frac{1}{a}\right), \quad (26)$$

$$f_{x1}\left(\frac{1}{a}\right) = \frac{e^2\ddot{w}}{2ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p (4p^2-1)\beta^{2p}}{\Gamma\left(\frac{3}{2}-p\right)} \sum_{b=0}^p \frac{(-1)^b 2^{2b} (\frac{1}{2}+b)!}{(2b)!(p-b)!(2b+3)}, \quad (27)$$

$$f_{x2}\left(\frac{1}{a}\right) = \frac{2e^2\ddot{w}}{ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^{2p} (4p^2-1)}{\Gamma\left(\frac{3}{2}-p\right)} \sum_{b=0}^{p-1} \frac{(-1)^b 2^{2b} (\frac{1}{2}+b)!}{(2b+1)!(p-b-1)!(2b+3)} \quad (28)$$

$$f_{x3}\left(\frac{1}{a}\right) = -\frac{e^2\ddot{w}}{2ac^2} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^{2p} (4p^2-1)}{\Gamma\left(\frac{3}{2}-p\right)} \sum_{b=0}^{p-1} \frac{(-1)^b 2^{2b} (-\frac{1}{2}+b)!}{(2b)!(p-b-1)!(2b+1)!}. \quad (29)$$

The sums over b in Eqs. (25) - (29) can be done immediately in terms of Gamma functions. To perform the remaining sums over p , we use for the Gamma functions from the denominator the formula [12]

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}, \quad (30)$$

with the help of which the remaining series over p can be written in term of Gaussian hypergeometric functions. One obtains the final result

$$F_x\left(\frac{1}{a}\right) = \frac{e^2\ddot{w}(t)}{ac^2\beta^3} \left(\frac{1}{2} \ln \frac{1+\beta}{1-\beta} - \frac{\beta}{1-\beta^2} \right), \quad (31)$$

where $\beta = \dot{w}(t)/c$. This result was also obtained in the preprint [8] by using a slightly different method. We note that this first term in the expansion of the self-force is proportional to the acceleration $\ddot{w}(t)$, and the proportionality coefficient is the negative of the Abraham's longitudinal electromagnetic mass [13, 14].

B. The term of order a^0

By putting $n = p + 1$ in Eqs. (12) - (16), one obtains

$$F_x^1(a^0) = \frac{e^2}{16\pi^2 c^3} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{b+\frac{p}{2}} 2^{2b}}{c^p (p+1)! (2b)! \left(\frac{p}{2} - b\right)! (p+3)(p+1)} \left(-\frac{p}{2}\right)_{b+\frac{p}{2}} \cdot \frac{d^{p+3} w^{p+1}(t)}{dt^{p+3}} \int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r}-\mathbf{r}'|_{r=r'=a}^{2b}}, \quad (32)$$

$$F_x^2(a^0) = f_{x1}(a^0) + f_{x2}(a^0) + f_{x3}(a^0), \quad (33)$$

where

$$f_{x1}(a^0) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}} \frac{(-1)^{b+\frac{p}{2}+1} 2^{2b}}{(p+1)! (2b)! \left(\frac{p}{2} - b\right)! c^{p+3}} \frac{p}{(p+3)(p+1)} \frac{d^{p+3} w^{p+1}(t)}{dt^{p+3}} \cdot \left(1 - \frac{p}{2}\right)_{b+\frac{p}{2}} \int dS_a \int dS'_a \frac{(x-x')^{2b+2}}{|\mathbf{r}-\mathbf{r}'|_{r=r'=a}^{2b+2}}, \quad (34)$$

$$f_{x2}(a^0) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}-1} \frac{(-1)^{b+\frac{p}{2}+1} 2^{2b+2}}{(p+1)! (2b+1)! \left(\frac{p}{2} - b - 1\right)! c^{p+3}} \frac{p}{(p+3)(p+1)} \frac{d^{p+3} w^{p+1}(t)}{dt^{p+3}} \cdot \left(1 - \frac{p}{2}\right)_{b+\frac{p}{2}} \int dS_a \int dS'_a \frac{(x-x')^{2b+2}}{|\mathbf{r}-\mathbf{r}'|_{r=r'=a}^{2b+2}}, \quad (35)$$

$$f_{x3}(a^0) = \frac{e^2}{16\pi^2} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \sum_{b=0}^{\frac{p}{2}-1} \frac{(-1)^{b+\frac{p}{2}} 2^{2b}}{(p+1)! (2b)! \left(\frac{p}{2} - b - 1\right)! c^{p+3}} \frac{p}{(p+3)(p+1)} \frac{d^{p+3} w^{p+1}(t)}{dt^{p+3}} \cdot \left(1 - \frac{p}{2}\right)_{b+\frac{p}{2}-1} \int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r}-\mathbf{r}'|_{r=r'=a}^{2b}}. \quad (36)$$

As the Pochhammer symbol $\left(-\frac{p}{2}\right)_{b+\frac{p}{2}}$ is different from zero only for $b \leq 0$, it follows that the only non-zero contribution in Eq. (32) is obtained for $b = 0$. With this observation, the remaining sum over p can be performed easily [11] and one obtains

$$F_x^1(a^0) = \frac{2e^2}{3c^3} \gamma^6 \left[\dot{w}(t)(\beta^2 + 1) + \ddot{w}^2(t) \dot{w}(t) \frac{3\gamma^2}{c^2} (\beta^2 + 2) \right]. \quad (37)$$

The Pochhammer symbol $\left(1 - \frac{p}{2}\right)_{b+\frac{p}{2}}$ is different from zero only for $b \leq -1$, so $f_{x1}(a^0) = f_{x2}(a^0) = 0$. The Pochhammer symbol $\left(1 - \frac{p}{2}\right)_{b+\frac{p}{2}-1}$ is different from zero only for $b \leq 0$, so

the only non-zero contribution in Eq. (36) is obtained for $b = 0$. Performing the summation over p [11], one obtains

$$F_x^2(a^0) = f_{x3}(a^0) = -\frac{2e^2}{c^3}\gamma^6\beta \left[\ddot{w}^2(t)\frac{\gamma^2}{c}(2\beta^2 + 1) + \dot{w}(t)\frac{2\beta}{3} \right]. \quad (38)$$

From Eqs. (37), (38) one obtains

$$F_x(a^0) = \frac{2e^2}{3c^3}\gamma^4\ddot{v}(t) + \frac{2e^2}{c^5}\gamma^6v(t)\dot{v}^2(t), \quad (39)$$

where $\dot{w}(t) = v(t)$ is the velocity of the shell's centre along the x -axis. This is the space part of the relativistic Lorentz- Abraham - Dirac (LAD) radiation reaction force [6], written for the case of rectilinear motion.

C. The term of order a^1

We put $n = p + 2$ in Eqs. (12) - (16), change the summation index $p \rightarrow 2p$, perform the double angular integrals using Eq. (A24) from Appendix and use Eq. (22) for the derivatives $\frac{d^{p+4}w^{p+1}(t)}{dt^{p+4}}$. One obtains

$$\begin{aligned} F_x^1(a) = & -\frac{e^2a}{18c^4}\ddot{w}(t)\sum_{p=0}^{\infty}\beta^{2p}(2p+3)^3\left(p+\frac{1}{2}\right)!\cdot S(p) \\ & -\frac{2e^2a}{9c^4}\frac{\ddot{w}(t)\dot{w}(t)}{\dot{w}(t)}\sum_{p=0}^{\infty}\beta^{2p}p(2p+3)^2\left(p+\frac{1}{2}\right)!\cdot S(p) \\ & -\frac{e^2a}{18c^4}\frac{\ddot{w}^3(t)}{\dot{w}^2(t)}\sum_{p=0}^{\infty}\beta^{2p}p(2p-1)(2p+3)^2\left(p+\frac{1}{2}\right)!\cdot S(p), \end{aligned} \quad (40)$$

where [11]

$$S(p) \equiv \sum_{b=0}^p \frac{2^{2b}}{(2b+1)!(p-b)!\left(\frac{1}{2}-b\right)!} = \frac{p+1}{\left(p+\frac{1}{2}\right)!}, \quad (41)$$

$$F_x^2(a) = \frac{4e^2a}{3c^4}\sum_{p=0}^{\infty}\frac{\left(p-\frac{1}{2}\right)!}{(2p+2)!c^{2p}(2p+4)}\frac{d^{2p+4}w^{2p+1}(t)}{dt^{2p+4}}(S^1(p) + 4S^2(p) + S^3(p)), \quad (42)$$

where [11]

$$S^1(p) \equiv \sum_{b=0}^p \frac{2^{2b}}{(2b)!(p-b)!\left(-b-\frac{1}{2}\right)!(2b+3)} = \frac{1}{4\left(p+\frac{3}{2}\right)!}, \quad (43)$$

$$S^2(p) \equiv \sum_{b=0}^{p-1} \frac{2^{2b}}{(2b+1)!(p-b-1)!\left(-b-\frac{1}{2}\right)!(2b+3)} = \frac{p}{4\left(p+\frac{1}{2}\right)!}, \quad (44)$$

$$S^3(p) \equiv \sum_{b=0}^{p-1} \frac{2^{2b}}{(2b)!(p-b-1)!(\frac{1}{2}-b)!(1+2b)} = \frac{p}{(p-\frac{1}{2})!}. \quad (45)$$

After performing the sums over p , we obtain

$$\begin{aligned} F_x^1(a) = & -\frac{e^2 a}{18c^4} \ddot{w}(t) \gamma^8 (\beta^4 + 14\beta^2 + 9) - \frac{4e^2 a}{9c^5} \ddot{w}(t) \dot{w}(t) \gamma^{10} \beta (\beta^4 + 22\beta^2 + 25) \\ & - \frac{e^2 a}{9c^6} \ddot{w}^3(t) \gamma^{12} (5\beta^6 + 159\beta^4 + 291\beta^2 + 25), \end{aligned} \quad (46)$$

$$\begin{aligned} F_x^2(a) = & \frac{2e^2 a}{3c^4} \left[\ddot{w}(t) \frac{\gamma^8}{12} (\beta^4 + 22\beta^2 + 1) + \ddot{w}(t) \dot{w}(t) \frac{2\gamma^{10} \beta}{3c} (\beta^4 + 34\beta^2 + 13) \right. \\ & \left. + \ddot{w}^3(t) \frac{\gamma^{12}}{6c^2} (5\beta^6 + 243\beta^4 + 219\beta^2 + 13) \right]. \end{aligned} \quad (47)$$

So, for the term of order a^1 in the expansion of the total self-force, we obtain the result

$$F_x(a) = -\frac{4e^2 a \gamma^6}{9c^4} \ddot{w}(t) - \frac{16e^2 a \gamma^8 \beta}{3c^5} \ddot{w}(t) \dot{w}(t) - \frac{4e^2 a \gamma^{10}}{9c^6} (7\beta^2 + 1) \ddot{w}^3(t). \quad (48)$$

In the limit of low velocities, we have

$$F_x(a) \xrightarrow{\beta \rightarrow 0} -\frac{4e^2 a}{9c^4} \ddot{w}(t) - \frac{4e^2 a}{9c^6} \ddot{w}^3(t). \quad (49)$$

III. DISCUSSION AND CONCLUSIONS

We have studied here the electromagnetic self-force of a non-Lorentz-contractible uniformly charged spherical shell of radius a in arbitrary rectilinear motion along the x -axis. The whole calculation was done in the laboratory frame of reference. The self-force was expanded in powers of a , and we calculated the first three terms of this expansion. Our final result is

$$\begin{aligned} F_x(t) = & F_x \left(\frac{1}{a} \right) + F_x(a^0) + F_x(a^1) + \mathcal{O}(a^2) \\ = & \frac{e^2 \ddot{w}(t)}{ac^2 \beta^3} \left(\frac{1}{2} \ln \frac{1+\beta}{1-\beta} - \frac{\beta}{1-\beta^2} \right) \\ & + \frac{2e^2}{3c^3} \gamma^4 \dot{w}(t) + \frac{2e^2}{c^4} \gamma^6 \beta \dot{w}^2(t) \\ & - \frac{4e^2 a \gamma^6}{9c^4} \ddot{w}(t) - \frac{16e^2 a \gamma^8 \beta}{3c^5} \ddot{w}(t) \dot{w}(t) - \frac{4e^2 a \gamma^{10}}{9c^6} (7\beta^2 + 1) \ddot{w}^3(t) + \mathcal{O}(a^2). \end{aligned} \quad (50)$$

The term of order $1/a$ is proportional to the acceleration and the proportionality coefficient is the negative of the Abraham's longitudinal electromagnetic mass [13, 14], the term of order a^0 is the well-known LAD force, and the term of order a^1 , as far as we know, was not yet

reported elsewhere. Still, the term $-\frac{4e^2a}{9c^4}\ddot{w}(t)$, that is obtained in the limit $\beta \rightarrow 0$, can be found in other papers, too. For example, in [7] it was written the series expansion of the electromagnetic self-force of a charged object with spherical symmetry, in the proper frame of reference, neglecting the non-linear terms in velocity's derivatives. For a spherical shell in rectilinear motion, Jackson's result can be written using our notations as [15]

$$F_x(t) = -\frac{4e^2}{3c^2} \sum_{n=0}^{\infty} \frac{(-1)^n (2a)^{n-1}}{(n+1)!c^n} \frac{d^{n+2}w(t)}{dt^{n+2}}. \quad (51)$$

The first three terms of the above series are

$$F_x(t) = -\frac{2e^2}{3c^2a}\ddot{w}(t) + \frac{2e^2}{3c^3}\dot{w}(t) - \frac{4e^2a}{9c^4}\ddot{w}(t) + \mathcal{O}(a^2). \quad (52)$$

In the limit $\beta \rightarrow 0$, if we neglect the non-linear terms, our result Eq. (50) coincides with Jackson's result.

Nodvick studied in [16] the general motion of a Lorentz-contractible extended charged object, using an entirely covariant formalism. His result for translational motion can be particularized for a Lorentz-contractible shell in rectilinear motion as follows [17]

$$\begin{aligned} F_x(t) = & -\frac{e^2}{2ac^2}\ddot{w}(t) + \frac{2e^2}{3c^3}\dot{w}(t) - \frac{2e^2}{3c^3}\beta\ddot{w}^2(t) \\ & + \frac{4e^2a}{3c^5}\beta\ddot{w}(t)\dot{w}(t) + \frac{4e^2a}{3c^6}\ddot{w}(t)^3 - \frac{4e^2a}{9c^4}\ddot{w}(t) + \mathcal{O}(a^2). \end{aligned} \quad (53)$$

We note that, although Nodvick's result is valid for a Lorentz-deformable shell, the term $-\frac{4e^2a}{9c^4}\ddot{w}(t)$ appears in it, too. The other two terms of order a are of the same type as ours, but with differences in their coefficients. These differences can appear both because of the fact that his shell is Lorentz-contractible and because of the fact that the expansions that he used for the fields are in powers of $1/c$, so we do not expect all the non-linear terms to appear in Nodvick's result.

Page [18] also studied the self-force of a Lorentz-contractible spherical shell and he obtained in his Eq. (32) the contribution of order a

$$F_x^{\text{Page}}(a) = -\frac{4e^2a\gamma^5}{9c^4}\ddot{w}(t). \quad (54)$$

We notice a difference in the power of γ compared to our result Eq. (41). The other non-linear terms of order a are missing in Page's result, but this was to be expected because he used the expansions of the fields up to the fourth order in $1/c$.

Although we have calculated only the first three terms in the expansion of the self-force in powers of a , higher order terms can be easily calculated using the same algorithm. Unfortunately, we are not able yet to calculate the general term of the expansion so that we have the closed form of the self-force in the laboratory frame. In connection with the convergence of the expansion of the electromagnetic self-force in powers of a , we can only say that it must be studied for each individual trajectory, because a term of order n of this expansion depends on $n + 4$ parameters: the radius a , the velocity and its first $n + 2$ derivatives. To generalize this method for the study of a Lorentz-contractible shell, we need an expansion of the type (A2) for the inverse distance between two points on an oblate spheroid, raised to an arbitrary power. The study of such an expansion will be presented in a future paper.

Appendix A: The calculation of the integral $I = \int dS_a \int dS'_a \frac{(x-x')^{2b}}{|\mathbf{r}-\mathbf{r}'|^{2b-1}}$

Using the usual spherical parametrization on a sphere of radius a , we have

$$(x - x')^{2b} = a^{2b} (\cos \theta - \cos \theta')^{2b}. \quad (\text{A1})$$

After using the expansion [8]

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|_{r=r'=a}^n} &= \frac{1}{a^n 2^{n/2}} \sum_{k=0}^{\infty} \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=-k+2q}^{k-2q} \frac{(k + \frac{n}{2} - 1)!}{(\frac{n}{2} - 1)!} \frac{(2k - 4q + 1)}{2^q q! (2k - 2q + 1)!!} \\ &\cdot \frac{(k - 2q - m)!}{(k - 2q + m)!} P_{k-2q}^m(\cos \theta) P_{k-2q}^m(\cos \theta') e^{im(\phi - \phi')}, \end{aligned} \quad (\text{A2})$$

we perform the angular integrals as follows [8, 19]

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' e^{im(\phi - \phi')} = 4\pi^2, \quad (\text{A3})$$

$$\int_0^\pi d\theta \sin \theta \int_0^\pi d\theta' \sin \theta' (\cos \theta - \cos \theta')^{2b} P_l^m(\cos \theta) P_l^m(\cos \theta') = \begin{cases} \frac{(-1)^k 2^{2b+2} (b!)^2}{(2b+1)(b-l)!(b+l+1)!}, & b \geq l \\ 0, & b < l \end{cases}. \quad (\text{A4})$$

One obtains

$$I = \frac{4\pi^2 a^{2b+5/2} (b!)^2}{(b - \frac{3}{2})! (2b+1)} \sum_{q=0}^{\infty} \frac{1}{2^q q!} (S_k^1 + S_k^2), \quad (\text{A5})$$

where

$$S_k^1 = \sum_{i=0}^b \frac{(-1)^i (i + 2q + b - \frac{3}{2})!}{(2i + 2q + 1)!! (b-i)! (b+i+1)!}, \quad (\text{A6})$$

$$S_k^2 = 2 \sum_{i=0}^b \frac{i(-1)^i (i + 2q + b - \frac{3}{2})!}{(2i + 2q + 1)!!(b - i)!(b + i + 1)!}. \quad (\text{A7})$$

We note that, to calculate $S_k^{1,2}$, we have to calculate a sum of the type

$$\tilde{S} = \sum_{i=0}^b \frac{x^i (i + 2q + b - \frac{3}{2})!}{(2i + 2q + 1)!!(b - i)!(b + i + 1)!} \quad (\text{A8})$$

and its derivative with respect to x . For this, we write the factorials in the above equation in terms of Pochhammer symbols, as follows [12]

$$\left(i + 2q + b - \frac{3}{2}\right)! = \Gamma\left(2q + b - \frac{1}{2} + i\right) = \Gamma\left(2q + b - \frac{1}{2}\right) \left(2q + b - \frac{1}{2}\right)_i, \quad (\text{A9})$$

$$(2i + 2q + 1)!! = \frac{2^{i+q+1}}{\sqrt{\pi}} \Gamma\left(q + \frac{3}{2}\right) \left(q + \frac{3}{2}\right)_i, \quad (\text{A10})$$

$$(b - i)! = \frac{(-1)^i \Gamma(1 + b)}{(-b)_i}, \quad (\text{A11})$$

$$(b + 1 + i)! = \Gamma(b + 2)(b + 2)_i. \quad (\text{A12})$$

Introducing Eqs. (A9) - (A12) in Eq. (A8), one obtains

$$\tilde{S} = \frac{\sqrt{\pi} \Gamma\left(2q + b - \frac{1}{2}\right)}{2^{q+1} \Gamma\left(q + \frac{3}{2}\right) \Gamma(b + 1) \Gamma(b + 2)} {}_3F_2\left(-b, 1, 2q + b - \frac{1}{2}; q + \frac{3}{2}, b + 2; -\frac{x}{2}\right). \quad (\text{A13})$$

Taking the derivative with respect to x of the above equation, one obtains

$$\begin{aligned} \sum_{i=0}^b \frac{ix^i (i + 2q + b - \frac{3}{2})!}{(2i + 2q + 1)!!(b - i)!(b + i + 1)!} &= \frac{x\sqrt{\pi}}{2^{q+2}} \frac{b \Gamma\left(2q + b + \frac{1}{2}\right)}{\Gamma\left(q + \frac{5}{2}\right) \Gamma(b + 1) \Gamma(b + 3)} \\ &\cdot {}_3F_2\left(-b + 1, 2, 2q + b + \frac{1}{2}; q + \frac{5}{2}, b + 3; -\frac{x}{2}\right). \end{aligned} \quad (\text{A14})$$

Putting $x = -1$ in Eqs. (A13), (A14), one obtains

$$S_k^1 = \frac{\sqrt{\pi} \Gamma\left(2q + b - \frac{1}{2}\right)}{\Gamma\left(q + \frac{3}{2}\right) \Gamma(b + 1) \Gamma(b + 2)} {}_3F_2\left(-b, 1, 2q + b - \frac{1}{2}; q + \frac{3}{2}, b + 2; \frac{1}{2}\right), \quad (\text{A15})$$

$$S_k^2 = - \frac{\sqrt{\pi} b \Gamma\left(2q + b + \frac{1}{2}\right)}{2^{q+1} \Gamma\left(q + \frac{5}{2}\right) \Gamma(b + 1) \Gamma(b + 3)} {}_3F_2\left(-b + 1, 2q + b + \frac{1}{2}; q + \frac{5}{2}, b + 3; \frac{1}{2}\right). \quad (\text{A16})$$

Now, we come back to Eq. (A5) and perform the summation over q by writing the hypergeometric functions in $S_k^{1,2}$ according to their definitions and changing the order of summation, as follows

$$\begin{aligned} \sum_{q=0}^{\infty} \frac{1}{2^q q!} S_k^1 &= \sum_{q=0}^{\infty} \frac{\sqrt{\pi} \Gamma(2q + b - \frac{1}{2})}{2^{2q+1} q! \Gamma(q + \frac{3}{2}) \Gamma(b+1) \Gamma(b+2)} \sum_{i=0}^b \frac{(-b)_i (2q + b - \frac{1}{2})_i}{(q + \frac{3}{2})_i (b+2)_i 2^i} \\ &= \frac{\sqrt{\pi}}{2\Gamma(b+1)\Gamma(b+2)} \sum_{i=0}^{\infty} \frac{(-b)_i \Gamma(b+i-\frac{1}{2})}{(b+2)_i 2^i \Gamma(i+\frac{3}{2})} {}_2F_1\left(\frac{b+i}{2} - \frac{1}{4}, \frac{b+i}{2} + \frac{1}{4}; i + \frac{3}{2}; 1\right) \\ &= \frac{\pi(-1)^{b+1}}{2^{b-\frac{1}{2}} \Gamma(b+1) \Gamma(b+2) \Gamma(\frac{5}{2}-b)} {}_3F_2\left(1, -b, b - \frac{1}{2}; b+2, \frac{5}{2}-b; 1\right), \end{aligned} \quad (\text{A17})$$

where, before performing the last summation over i , we used [12]

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad (\text{A18})$$

Using in Eq. (A17) [12]

$${}_3F_2(1, a, b; 2-a, 2-b; 1) = \frac{\sqrt{\pi} \Gamma(2-a) \Gamma(2-b) \Gamma(\frac{3}{2}-a-b)}{2\Gamma(\frac{3}{2}-a) \Gamma(\frac{3}{2}-b) \Gamma(2-a-b)}, \quad (\text{A19})$$

one obtains

$$\sum_{q=0}^{\infty} \frac{1}{2^q q!} S_k^1 = \frac{(-1)^{b+1} \pi 2^{\frac{3}{2}-b}}{3\Gamma(b+1) \Gamma(\frac{3}{2}+b) \Gamma(2-b)}. \quad (\text{A20})$$

We note that, because of the function $\Gamma(2-b)$ from the denominator, the above sum is different from zero only for $b = 0, 1$. Using exactly the same technique and the formulas [12]

$${}_3F_2\left(\begin{matrix} -n, a, b, \\ c, d, \end{matrix}; 1\right) = \frac{(c-a)_n (d-a)_n}{(c)_n (d)_n} {}_3F_2\left(\begin{matrix} -n, a, a+b-c-d-n+1 \\ a-c-n+1, a-d-n+1, \end{matrix}; 1\right), \quad (\text{A21})$$

$${}_3F_2\left(-b+1, 2, -2; -2b+1, \frac{1}{2}; 1\right) = \frac{1+2b}{1-2b}, \quad (\text{A22})$$

one obtains

$$\sum_{q=0}^{\infty} \frac{1}{2^q q!} S_k^2 = -\frac{2^{\frac{5}{2}-b}}{3(1-2b)} \frac{(b-\frac{1}{2})!}{(b!)^2}. \quad (\text{A23})$$

Introducing (A20), (A23) in (A5), after simplifications, one obtains

$$I = \frac{64\pi^2 a}{3(1+2b)}. \quad (\text{A24})$$

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