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ON A DIFFERENTIAL INTERMEDIATE VALUE PROPERTY

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

ABSTRACT. Liouville closed H -fields are ordered differential fields whose ordering and derivation interact in a natural way and where every linear differential equation of order 1 has a nontrivial solution. (The introduction gives a precise definition.) For a Liouville closed H -field K with small derivation we show: K has the Intermediate Value Property for differential polynomials iff K is elementarily equivalent to the ordered differential field of transseries. We also indicate how this applies to Hardy fields.

INTRODUCTION

Throughout this introduction K is an ordered differential field, that is, an ordered field equipped with a derivation $\partial: K \rightarrow K$. (We usually write f' instead of ∂f , for $f \in K$.) Its constant field

$$C := \{f \in K : f' = 0\}$$

yields the (convex) valuation ring

$$\mathcal{O} := \{f \in K : |f| \leq c \text{ for some } c \in C\}$$

of K , with maximal ideal

$$\mathfrak{o} := \{f \in K : |f| < c \text{ for all } c > 0 \text{ in } C\}.$$

(It may help to think of the elements of K as germs of real valued functions and of $f \in \mathcal{O}g$ and $f \in \mathfrak{o}g$ as $f = O(g)$ and $f = o(g)$, respectively.) The above definitions exhibit C , \mathcal{O} , and \mathfrak{o} as definable in K in the sense of model theory.

Key example: the ordered differential field \mathbb{T} of **transseries**, which contains \mathbb{R} as an ordered subfield, and where $C = \mathbb{R}$. We refer to [3] for the rather elaborate construction of \mathbb{T} and for any fact about \mathbb{T} that gets mentioned without proof.

Other important examples are Hardy fields. (Hardy [6] proved a striking theorem on logarithmic-exponential functions. Bourbaki [5] put this into the general setting of what they called Hardy fields.) Here we can give a definition from scratch that doesn't take much space. Notation: \mathcal{C} is the ring of germs at $+\infty$ of continuous real-valued functions on halflines $(a, +\infty)$, $a \in \mathbb{R}$. For $r = 1, 2, \dots$, let \mathcal{C}^r be the subring of \mathcal{C} consisting of the germs at $+\infty$ of r -times continuously differentiable real-valued functions on such halflines. This yields the subring

$$\mathcal{C}^{<\infty} := \bigcap_{r \in \mathbb{N}_{\geq 1}} \mathcal{C}^r$$

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of \mathcal{C} , and $\mathcal{C}^{<\infty}$ is naturally a *differential* ring. For a germ $f \in \mathcal{C}$ we let f also denote any real valued function representing this germ, if this causes no ambiguity. A real number is identified with the germ of the corresponding constant function: $\mathbb{R} \subseteq \mathcal{C}$.

A **Hardy field** is by definition a differential subfield of $\mathcal{C}^{<\infty}$. *Examples:*

$$\mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{R}(x), \quad \mathbb{R}(x, e^x), \quad \mathbb{R}(x, e^x, \log x), \quad \mathbb{R}(\Gamma, \Gamma', \Gamma'', \dots),$$

where x denotes the germ at $+\infty$ of the identity function on \mathbb{R} . All these are actually *analytic* Hardy fields, that is, its elements are germs of real analytic functions.

Let H be a Hardy field. Then H is an *ordered* differential field: for $f \in H$, either $f(x) > 0$ eventually (in which case we set $f > 0$), or $f(x) = 0$, eventually, or $f(x) < 0$, eventually; this is because $f \neq 0$ in H implies f has a multiplicative inverse in H , so f cannot have arbitrarily large zeros. Also, if $f' < 0$, then f is eventually strictly decreasing; if $f' = 0$, then f is eventually constant; if $f' > 0$, then f is eventually strictly increasing.

In order to state the main result of this paper we need a bit more terminology: an **H -field** is a K (that is, an ordered differential field) such that:

- for all $f \in K$, if $f > 0$, then $f' > 0$;
- $\mathcal{O} = C + \mathcal{o}$ (so C maps isomorphically onto the residue field \mathcal{O}/\mathcal{o}).

We also say that K **has small derivation** if for all $f \in \mathcal{o}$ we have $f' \in \mathcal{o}$. Hardy fields have small derivation, and any Hardy field containing \mathbb{R} is an H -field.

An H -field K is said to be **Liouville closed** if it is real closed and for every $f \in K$ there are $g, h \in K^\times$ such that $f = g' = h'/h$. The ordered differential field \mathbb{T} is a Liouville closed H -field with small derivation. Any Hardy field $H \supseteq \mathbb{R}$ has a smallest (with respect to inclusion) Liouville closed Hardy field extension $\text{Li}(H)$. (The notions of “ H -field” and “Liouville closed H -field” are introduced in [1]. The capital H is in honor of Hardy, Hausdorff, and Hahn, who pioneered various aspects of our topic about a century ago, as did Du Bois-Reymond and Borel even earlier.)

Now a very strong property: we say K **has DIVP** (the Differential Intermediate Value Property) if for every polynomial $P \in K[Y_0, \dots, Y_r]$ and all $f < g$ in K with

$$P(f, f', \dots, f^{(r)}) < 0 < P(g, g', \dots, g^{(r)})$$

there exists $y \in K$ such that $f < y < g$ and $P(y, y', \dots, y^{(r)}) = 0$. (Existentially closed ordered differential fields have DIVP by [9] and [10, Proposition 1.5]; this has limited interest for us since the ordering and derivation in those structures do not interact.) Actually, DIVP is a bit of an afterthought: in [3] we considered instead two robust but rather technical properties, ω -freeness and newtonianity, and proved that \mathbb{T} is ω -free and newtonian. (One can think of newtonianity as a variant of differential-henselianity.) Afterwards we saw that “ ω -free + newtonian” is equivalent to DIVP, for Liouville closed H -fields. Our aim is to establish this equivalence: Theorem 2.7, the main result of this short paper.

We did not consider DIVP in [3], but it is surely an appealing property and easier to grasp than the more fundamental notions of ω -freeness and newtonianity. (The latter make sense in a wider setting of valued differential fields where the valuation does not necessarily arise from an ordering, as is the case for H -fields.)

Besides [3] we shall rely on [7], which focuses on a particular ordered differential subfield of \mathbb{T} , namely \mathbb{T}_g , consisting of the so-called *grid-based* transseries; see also [3, Appendix A]. We summarize what we need from [7] as follows:

\mathbb{T}_g is a newtonian ω -free Liouville closed H -field with small derivation, and \mathbb{T}_g has DIVP. We alert the reader that the terms *newtonian* and *ω -free* do not occur in [7], and that \mathbb{T}_g there is denoted by \mathbb{T} .

We call attention to the fact that K is a Liouville closed H -field iff $K \models \text{LiH}$ for a set LiH (independent of K) of sentences in the language of ordered differential fields. Also, for H -fields, “ ω -free” is expressible by a single sentence in the language of ordered differential fields, and “newtonian” as well as “DIVP” by a set of sentences in this language. The reason that “ ω -free + newtonian” is central in [3] are various theorems proved there, which are also relevant here. To state these theorems, we consider an H -field K below as an \mathcal{L} -structure, where

$$\mathcal{L} := \{0, 1, +, -, \times, \partial, <, \preceq\}$$

is the language of ordered valued differential fields. The symbols $0, 1, +, -, \times, \partial, <$ name the usual primitives of K , and \preceq encodes its valuation: for $a, b \in K$,

$$a \preceq b \quad :\Longleftrightarrow \quad a \in \mathcal{O}b.$$

We can now summarize what we need from [3, Chapters 15, 16] as follows:

The theory of newtonian ω -free Liouville closed H -fields is model complete, and is the model companion of the theory of H -fields. The theory of newtonian ω -free Liouville closed H -fields whose derivation is small is complete and has \mathbb{T} as a model.

For an H -field K its valuation ring \mathcal{O} and so the binary relation \preceq on K can be defined in terms of the other primitives by an *existential* formula independent of K . However, by [3, Corollary 16.2.6] this cannot be done by a universal such formula and so for the model completeness above we cannot drop \preceq from the language \mathcal{L} .

Corollary 0.1. *Every newtonian ω -free Liouville closed H -field has DIVP.*

Proof. Let K be a newtonian ω -free Liouville closed H -field. If the derivation of K is small, then DIVP follows from the results from [7] quoted earlier and the above completeness result from [3]. Suppose the derivation of K is not small. Replacing the derivation ∂ of K by a multiple $\phi^{-1}\partial$ with $\phi > 0$ in K transforms K into its so-called compositional conjugate K^ϕ , which is still a newtonian ω -free Liouville closed H -field, and K has DIVP iff K^ϕ does. By 4.4.7 and 9.1.5 in [3] we can choose $\phi > 0$ in K such that the derivation $\phi^{-1}\partial$ of K^ϕ is small. \square

This gives one direction of Theorem 2.7. In the rest of this paper we prove a strong version, Corollary 2.6, of the other direction, without using [7] but relying heavily on various parts of [3] with detailed references. Theorem 2.7 and the results quoted above from [3] yield the result stated in the abstract: a Liouville closed H -field with small derivation is elementarily equivalent to \mathbb{T} iff it has DIVP.

Connection to Hardy fields. Every Hardy field H extends to a Hardy field $H(\mathbb{R}) \supseteq \mathbb{R}$, and $H(\mathbb{R})$ is in particular an H -field. We refer to [4] for a discussion of the conjecture that *any Hardy field containing \mathbb{R} extends to a newtonian ω -free Hardy field*. At the end of 2019 we finished the proof of this conjecture by considerably refining material in [3] and [8]; this amounts to a rather complete extension theory of Hardy fields. Note that every Hardy field extends to a maximal Hardy field, by Zorn, and so having established this conjecture we now know that all maximal Hardy fields are elementarily equivalent to \mathbb{T} , as ordered differential fields. Since \mathcal{C} has the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, there are at most $2^{\mathfrak{c}}$

many maximal Hardy fields, and we also have a proof that there are exactly that many. (We thank Ilijas Farah for a useful hint on this point.) These remarks on Hardy fields serve as an announcement. A rather voluminous work containing the proof of the conjecture is currently being prepared for publication. We also hope to include there a proof of DIVP for newtonian ω -free H -fields that does not depend as in the present paper on it being true for \mathbb{T}_g , whose proof in [7] uses the particular nature of \mathbb{T}_g .

We have a second conjecture about Hardy fields in [4], whose proof is not yet finished at this time (May 2021): *for any maximal Hardy field H and countable subsets $A < B$ in H there exists $y \in H$ such that $A < y < B$* . This means that the underlying ordered set of a maximal Hardy field is an η_1 -set in the sense of Hausdorff. Together with the (now established) first conjecture and results from [3] it implies: *all maximal Hardy fields are back-and-forth equivalent as ordered differential fields, and thus isomorphic assuming CH, the Continuum Hypothesis*.

1. PRELIMINARIES

In order to make free use of the valuation-theoretic tools from [3] and to make this paper self-contained modulo references to specific results from the literature we provide more background in this section before returning to DIVP.

Notation and terminology. Throughout, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Given an additively written abelian group A we let $A^\neq := A \setminus \{0\}$. Rings are commutative with identity 1, and for a ring R we let R^\times be the multiplicative group of units (consisting of the $a \in R$ such that $ab = 1$ for some $b \in R$). A *differential ring* will be a ring R containing (an isomorphic copy of) \mathbb{Q} as a subring and equipped with a derivation $\partial: R \rightarrow R$; note that then $C_R := \{a \in R : \partial(a) = 0\}$ is a subring of R , called the ring of constants of R , and that $\mathbb{Q} \subseteq C_R$. If R is a field, then so is C_R . An ordered differential field is in particular a differential ring.

Let R be a differential ring and $a \in R$. When its derivation ∂ is clear from the context we denote $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$ by $a', a'', \dots, a^{(n)}, \dots$, and if $a \in R^\times$, then a^\dagger denotes a'/a , so $(ab)^\dagger = a^\dagger + b^\dagger$ for $a, b \in R^\times$. In Section 2 we need to consider the function $\omega = \omega_R: R \rightarrow R$ given by $\omega(z) = -2z' - z^2$, and the function $\sigma = \sigma_R: R^\times \rightarrow R$ given by $\sigma(y) = \omega(z) + y^2$ for $z := -y^\dagger$.

We have the differential ring $R\{Y\} = R[Y, Y', Y'', \dots]$ of differential polynomials in an indeterminate Y over R . We say that $P = P(Y) \in R\{Y\}$ has order at most $r \in \mathbb{N}$ if $P \in R[Y, Y', \dots, Y^{(r)}]$.

For $\phi \in R^\times$ we let R^ϕ be the *compositional conjugate of R by ϕ* : the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ instead of ∂ . We then have an R -algebra isomorphism

$$P \mapsto P^\phi : R\{Y\} \rightarrow R^\phi\{Y\}$$

with $P^\phi(y) = P(y)$ for all $y \in R$; see [3, Section 5.7].

For a field K we have $K^\times = K^\neq$, and a (Krull) valuation on K is a surjective map $v: K^\times \rightarrow \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v: K \rightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$ by $v(0) := \infty$, where the ordering on Γ is extended to a total ordering on Γ_∞ by $\gamma < \infty$ for all $\gamma \in \Gamma$.

Let K be a *valued field*: a field (also denoted by K) together with a valuation ring \mathcal{O} of that field. This yields a valuation $v: K^\times \rightarrow \Gamma$ on the underlying field

such that $\mathcal{O} = \{a \in K : va \geq 0\}$ as explained in [3, Section 3.1]. We introduce various binary relations on the set K by defining for $a, b \in K$:

$$\begin{aligned} a \asymp b &:\Leftrightarrow va = vb, & a \preccurlyeq b &:\Leftrightarrow va \geq vb, & a \prec b &:\Leftrightarrow va > vb, \\ a \succcurlyeq b &:\Leftrightarrow b \preccurlyeq a, & a \succ b &:\Leftrightarrow b \prec a, & a \sim b &:\Leftrightarrow a - b \prec a. \end{aligned}$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$, and that \sim is an equivalence relation on K^\times . We also let $\mathfrak{o} = \{a \in K : va > 0\}$ be the maximal ideal of \mathcal{O} , so \mathcal{O}/\mathfrak{o} is the residue field of the valued field K . A convex subgroup Δ of the value group Γ of v gives rise to the Δ -coarsening of the valued field K ; see [ADH, 3.4].

H -fields and pre- H -fields. As in [3], a *valued differential field* is a valued field K with residue field of characteristic zero and equipped with a derivation $\partial: K \rightarrow K$. An *ordered valued differential field* is a valued differential field K equipped with an ordering on K making K an ordered field. We consider any H -field K as an ordered valued differential field whose valuation ring is the convex hull in K of its constant field C , in accordance with construing it as an \mathcal{L} -structure as specified in the introduction.

A *pre- H -field* is by definition an ordered valued differential subfield of an H -field. By [3, Sections 10.1, 10.3, 10.5], an ordered valued differential field K is a pre- H -field iff the valuation ring \mathcal{O} of K is convex in K , $f' > 0$ for all $f > \mathcal{O}$ in K , and $f' \prec g^\dagger$ for all $f, g \in K^\times$ with $f \preccurlyeq 1$ and $g \prec 1$. Any Hardy field H is construed as a pre- H -field by taking the convex hull of \mathbb{Q} in H as its valuation ring, giving rise to the so-called “natural valuation” on H as an ordered field. At the end of Section 9.1 in [3] we give $\mathbb{Q}(\sqrt{2+x^{-1}})$ as an example of a Hardy field that is not an H -field. Any ordered differential field K with the trivial valuation ring $\mathcal{O} = K$ is a pre- H -field (so the valuation ring of a pre- H -field K is not always the convex hull in K of its constant field, in contrast to Hardy fields and H -fields). If K is a pre- H -field whose valuation ring is nontrivial, then the valuation topology on K equals its order topology, by [3, Lemma 2.4.1].

Let K be a pre- H -field. Then the derivation of K and its valuation $v: K^\times \rightarrow \Gamma$ induce an operation $\psi: \Gamma^\neq \rightarrow \Gamma$, given by $\psi(vf) = v(f^\dagger)$ for $f \neq 1$ in K^\times ; the pair (Γ, ψ) is called the H -asymptotic couple of K ; see [3, Section 9.1]. Below we assume some familiarity with (Γ, ψ) , and properties of K based on it, such as K having *asymptotic integration* and K having a *gap* [3, Sections 9.1, 9.2]. The *flattening* of K is the Γ^b -coarsening of K where $\Gamma^b = \{vf : f \in K^\times, f' \prec f\}$, with associated binary relations \asymp^b, \preccurlyeq^b etc.; see [ADH, 9.4].

2. DIVP

In this section K is a pre- H -field. We let \mathcal{O} be its valuation ring, with maximal ideal \mathfrak{o} , and corresponding valuation $v: K^\times \rightarrow \Gamma = v(K^\times)$. Let (Γ, ψ) be its H -asymptotic couple, and $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^\neq\}$. Recall that “ K has DIVP” means: for all $P(Y) \in K\{Y\}$ and $f < g$ in K with $P(f) < 0 < P(g)$ there is a $y \in K$ such that $f < y < g$ and $P(y) = 0$. Restricting this to P of order $\leq r$, where $r \in \mathbb{N}$, gives the notion of r -DIVP. Thus K having 0-DIVP is equivalent to K being real closed as an ordered field. In particular, if K has 0-DIVP, then $\Gamma = v(K^\times)$ is divisible. From [3, Section 2.4] recall our convention that $K^> = \{a \in K : a > 0\}$, and similarly with $<$ replacing $>$.

Lemma 2.1. *Suppose $\Gamma \neq \{0\}$ and K has 1-DIVP. Then $\partial K = K$, $(K^>)^{\dagger} = (K^<)^{\dagger}$ is a convex subgroup of K , Ψ has no largest element, and Ψ is convex in Γ .*

Proof. We have $y' = 0$ for $y = 0$, and y' takes arbitrarily large positive values in K as y ranges over $K^{>\mathcal{O}} = \{a \in K : a > \mathcal{O}\}$, since by [3, Lemma 9.2.6] the set $(\Gamma^<)'$ is coinital in Γ . Hence y' takes all positive values on $K^>$, and therefore also all negative values on $K^<$. Thus $\partial K = K$. Next, let $a, b \in K^>$, and suppose $s \in K$ lies strictly between a^{\dagger} and b^{\dagger} . Then $s = y^{\dagger}$ for some $y \in K^>$ strictly between a and b ; this follows by noting that for $y = a$ and $y = b$ the signs of $sy - y'$ are opposite.

Let $\beta \in \Psi$ and take $a \in K$ with $v(a') = \beta$. Then $a \succ 1$, since $a \preccurlyeq 1$ would give $v(a') > \Psi$. Hence for $\alpha = va < 0$ we have $\alpha + \alpha^{\dagger} = \beta$, so $\alpha^{\dagger} > \beta$. Thus Ψ has no largest element. Therefore the set Ψ is convex in Γ . \square

Thus the ordered differential field \mathbb{T}_{\log} of logarithmic transseries [3, Appendix A] does not have 1-DIVP, although it is a newtonian ω -free H -field.

Does DIVP imply that K is an H -field? No: take an \aleph_0 -saturated elementary extension of \mathbb{T} and let Δ be as in [3, Example 10.1.7]. Then the Δ -coarsening of K is a pre- H -field with DIVP and nontrivial value group, and has a gap, but it is not an H -field. On the other hand:

Lemma 2.2. *Suppose K has 1-DIVP and has no gap. Then K is an H -field.*

Proof. In [3, Section 11.8] we defined

$$I(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\},$$

a convex \mathcal{O} -submodule of K . Since K has no gap, we have

$$\partial \mathcal{O} \subseteq I(K) = \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}.$$

Also $\Gamma \neq \{0\}$, and so (Γ, ψ) has asymptotic integration by Lemma 2.1. We show that K is an H -field by proving $I(K) = \partial \mathcal{O}$, so let $g \in I(K)$, $g < 0$. Since $(\Gamma^>)'$ has no least element we can take positive $f \in \mathcal{O}$ such that $f' \succ g$. Since $f' < 0$, this gives $f' < g$. Since $(\Gamma^>)'$ is cofinal in Γ we can also take positive $h \in \mathcal{O}$ such that $h' \prec g$, which in view of $h' < 0$ gives $g < h'$. Thus $f' < g < h'$, and so 1-DIVP yields $a \in \mathcal{O}$ with $g = a'$. \square

We refer to Sections 11.6 and 14.2 of [3] for the definitions of λ -freeness and r -newtonianity ($r \in \mathbb{N}$). From the introduction we recall that $\omega(z) := -2z' - z^2$. Below, compositionally conjugating an H -field K means replacing it by some K^{ϕ} with $\phi \in K^>$; this preserves most relevant properties like being an H -field, being λ -free, r -DIVP, and r -newtonianity, and it replaces Ψ by $\Psi - v\phi$.

Lemma 2.3. *Suppose K is an H -field, $\Gamma \neq \{0\}$, and K has 1-DIVP. Then K is λ -free and 1-newtonian, and the subset $\omega(K)$ of K is downward closed.*

Proof. Note that K has (asymptotic) integration, by Lemma 2.1. Assume towards a contradiction that K is not λ -free. We arrange by compositional conjugation that K has small derivation, so K has an element $x \succ 1$ with $x' = 1$, hence $x > C$. A construction in the beginning of [3, Section 11.5] yields by [3, Lemma 11.5.2] a pseudocauchy sequence (λ_{ρ}) in K with certain properties including $\lambda_{\rho} \sim x^{-1}$ for all ρ . As K is not λ -free, (λ_{ρ}) has a pseudolimit $\lambda \in K$ by [3, Corollary 11.6.1]. Then $s := -\lambda \sim -x^{-1}$, and s creates a gap over K by [3, Lemma 11.5.14]. Now note that for $P := Y' + sY$ we have $P(0) = 0$ and $P(x^2) = 2x + sx^2 \sim x$, so by 1-DIVP we have $P(y) = 1$ for some $y \in K$, contradicting [3, Lemma 11.5.12].

Let $P \in K\{Y\}$ of order at most 1 have Newton degree 1; we have to show that P has a zero in \mathcal{O} . We know that K is λ -free, so by [3, Proposition 13.3.6] we can pass to an elementary extension, compositionally conjugate, and divide by an element of K^\times to arrange that K has small derivation and $P = D + R$ where $D = cY + d$ or $D = cY'$ with $c, d \in C$, $c \neq 0$, and where $R \prec^b 1$. Then $R(a) \prec^b 1$ for all $a \in \mathcal{O}$. If $D = cY + d$, then we can take $a, b \in C$ with $D(a) < 0$ and $D(b) > 0$, which in view of $R(a) \prec D(a)$ and $R(b) \prec D(b)$ gives $P(a) < 0$ and $P(b) > 0$, and so P has a zero strictly between a and b , and thus a zero in \mathcal{O} . Next, suppose $D = cY'$. Then we take $t \in \mathcal{O}^\neq$ with $v(t^\dagger) = v(t)$, that is, $t' \asymp t^2$, so

$$P(t) = ct' + R(t), \quad P(-t) = -ct' + R(-t), \quad R(t), R(-t) \prec t'.$$

Hence $P(t)$ and $P(-t)$ have opposite signs, so P has a zero strictly between t and $-t$, and thus P has a zero in \mathcal{O} .

From $\omega(z) = -z^2 - 2z'$ we see that $\omega(z) \rightarrow -\infty$ as $z \rightarrow +\infty$ and as $z \rightarrow -\infty$ in K , so $\omega(K)$ is downward closed by 1-IVP. \square

For results involving r -DIVP for $r \geq 2$ we need a variant of [3, Lemma 11.8.31]. To state this variant we introduce as in [3, Section 11.8] the sets

$$\Gamma(K) := \{a^\dagger : a \in K \setminus \mathcal{O}\} \subseteq K^>, \quad \Lambda(K) := -\Gamma(K)^\dagger \subseteq K.$$

The superscripts \uparrow, \downarrow used in the statement of Lemma 2.4 below indicate upward, respectively downward, closure in the ordered set K , as in [3, Section 2.1].

Lemma 2.4. *Let K be an H -field with asymptotic integration. Then*

$$K^> = \text{I}(K)^> \cup \Gamma(K)^\uparrow, \quad \sigma(K^> \setminus \Gamma(K)^\uparrow) \subseteq \omega(\Lambda(K))^\downarrow.$$

Proof. If $a \in K$, $a > \text{I}(K)$, then $a \geq b^\dagger$ for some $b \in K^{\succ 1}$, and thus $a \in \Gamma(K)^\uparrow$. Next, let $s \in K^> \setminus \Gamma(K)^\uparrow$; we have to show $\sigma(s) \in \omega(\Lambda(K))^\downarrow$. Note that $s \in \text{I}(K)^>$ by what we just proved. From [3, 10.2.7 and 10.5.8] we obtain an immediate H -field extension L of K and $a \in L^{\succ 1}$ with $s = (1/a)'$. As in the proof of [3, 11.8.31] with L instead of K this gives $\sigma(s) \in \omega(\Lambda(L))^\downarrow$, where \downarrow indicates here the downward closure in L . It remains to note that ω is increasing on $\Lambda(L)$ by the remark preceding [3, 11.8.21], and that $\Lambda(K)$ is cofinal in $\Lambda(L)$ by [3, 11.8.14]. \square

The concept of ω -freeness is introduced in [3, Section 11.7]. If K has asymptotic integration, then by [3, 11.8.30]: K is ω -free $\Leftrightarrow K = \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$.

The next lemma also mentions the differential field extension $K[i]$ of K where $i^2 = -1$, as well as linear differential operators over differential fields like K and $K[i]$; for this we refer to [3, Sections 5.1, 5.2].

Lemma 2.5. *Suppose K is an H -field, $\Gamma \neq \{0\}$, $r \geq 2$, and K has r -DIVP. Then the following hold, with (i), (ii), (iii) using only the case $r = 2$:*

- (i) $K = \omega(K) \cup \sigma(K^>) = \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$;
- (ii) K is ω -free and $\omega(K) = \omega(\Lambda(K))^\downarrow$;
- (iii) for all $a \in K$ the operator $\partial^2 - a$ splits over $K[i]$;
- (iv) K is r -newtonian.

Proof. For (i) we use the end of [3, Section 11.7] to replace K with a compositional conjugate so that $0 \in \Psi$. Then K has small derivation, and we have $a \in K^>$ such that $a \not\asymp 1$ and $a^\dagger \asymp 1$. Replacing a by a^{-1} if necessary this gives $a^\dagger = -\phi$ with $\phi \asymp 1$, $\phi > 0$, so $a \prec 1$. Then $\phi^{-1}a^\dagger = -1$; replacing K by K^ϕ and renaming

the latter as K this means $a^\dagger = -1$. Let $f \in K$; to get $f \in \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$, note first that $1 = (1/a)^\dagger \in \Gamma(K)$, so $0 \in \Lambda(K)$. Also $\omega(\Lambda(K))^\downarrow \subseteq \omega(K)$ by Lemma 2.3.

If $f \leq 0$, then $\omega(0) = 0$ gives $f \in \omega(\Lambda(K))^\downarrow$. So assume $f > 0$; we first show that then $f \in \sigma(K^>)$. Now for $y \in K^>$, $f = \sigma(y)$ is equivalent (by multiplying with y^2) to $P(y) = 0$, where

$$P(Y) := 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in K\{Y\}.$$

See also [3, Section 13.7]. We have $P(0) = 0$ and $P(y) \rightarrow +\infty$ as $y \rightarrow +\infty$ (because of the term y^4). In view of 2-DIVP it will suffice to show that for some $y > 0$ in K we have $P(y) < 0$. Now with $y \in K^>$ and $z := -y^\dagger$ we have

$$\begin{aligned} P(y) &= y^2(\sigma(y) - f) = y^2(\omega(z) + y^2 - f), \text{ hence} \\ P(a) &= a^2(\omega(1) + a^2 - f) = a^2(-1 + a^2 - f) < 0, \end{aligned}$$

so $f \in \sigma(K^>)$. By the second inclusion of Lemma 2.4 this yields $f \in \omega(\Lambda(K))^\downarrow$ or $f \in \sigma(\Gamma(K)^\dagger)$. But we have $\sigma(\Gamma(K)^\dagger) \subseteq \sigma(\Gamma(K))^\uparrow$, because σ is increasing on $\Gamma(K)^\dagger$ by the remark preceding [3, 11.8.30]. This concludes the proof of (i), and then (ii) follows, using for its second part also the fact stated just before [3, 11.8.29] that we have $\omega(K) < \sigma(\Gamma(K))$.

Now (iii) follows from $K = \omega(K) \cup \sigma(K^>)$ by [3, Section 5.2, (5.2.1)]. As to (iv), let $P \in K\{Y\}$ of order at most r have Newton degree 1; we have to show that P has a zero in \mathcal{O} . For this we repeat the argument in the proof of Lemma 2.3 so that it applies to our P , using ω -freeness instead of λ -freeness, [3, Proposition 13.3.13] instead of [3, Proposition 13.3.6], and r -DIVP instead of 1-DIVP. \square

Corollary 2.6. *If K is an H -field, $\Gamma \neq \{0\}$, and K has DIVP, then K is ω -free and newtonian.*

There are non-Liouville closed H -fields with nontrivial derivation that have DIVP; see [2, Section 14]. By Lemma 2.3 and Lemma 2.5(iii), Liouville closed H -fields having 2-DIVP are *Schwarz closed* as defined in [3, Section 11.8].

Theorem 2.7. *Let K be a Liouville closed H -field. Then*

$$K \text{ has DIVP} \iff K \text{ is } \omega\text{-free and newtonian}.$$

Proof. The forward direction is part of Corollary 2.6. The backward direction is Corollary 0.1. \square

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