



HAL
open science

On a differential intermediate value property

Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven

► **To cite this version:**

Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven. On a differential intermediate value property. *Revista de la Unión Matemática Argentina*, 2022, 64 (1), pp.1-10. 10.33044/revuma.2892 . hal-03780812

HAL Id: hal-03780812

<https://hal.science/hal-03780812v1>

Submitted on 19 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON A DIFFERENTIAL INTERMEDIATE VALUE PROPERTY

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

ABSTRACT. Liouville closed H -fields are ordered differential fields whose ordering and derivation interact in a natural way and where every linear differential equation of order 1 has a nontrivial solution. (The introduction gives a precise definition.) For a Liouville closed H -field K with small derivation we show: K has the Intermediate Value Property for differential polynomials iff K is elementarily equivalent to the ordered differential field of transseries. We also indicate how this applies to Hardy fields.

INTRODUCTION

Throughout this introduction K is an ordered differential field, that is, an ordered field equipped with a derivation $\partial: K \rightarrow K$. (We usually write f' instead of ∂f , for $f \in K$.) Its constant field

$$C := \{f \in K : f' = 0\}$$

yields the (convex) valuation ring

$$\mathcal{O} := \{f \in K : |f| \leq c \text{ for some } c \in C\}$$

of K , with maximal ideal

$$\mathfrak{o} := \{f \in K : |f| < c \text{ for all } c > 0 \text{ in } C\}.$$

(It may help to think of the elements of K as germs of real valued functions and of $f \in \mathcal{O}g$ and $f \in \mathfrak{o}g$ as $f = O(g)$ and $f = o(g)$, respectively.) The above definitions exhibit C , \mathcal{O} , and \mathfrak{o} as definable in K in the sense of model theory.

Key example: the ordered differential field \mathbb{T} of **transseries**, which contains \mathbb{R} as an ordered subfield, and where $C = \mathbb{R}$. We refer to [3] for the rather elaborate construction of \mathbb{T} and for any fact about \mathbb{T} that gets mentioned without proof.

Other important examples are Hardy fields. (Hardy [6] proved a striking theorem on logarithmic-exponential functions. Bourbaki [5] put this into the general setting of what they called Hardy fields.) Here we can give a definition from scratch that doesn't take much space. Notation: \mathcal{C} is the ring of germs at $+\infty$ of continuous real-valued functions on halflines $(a, +\infty)$, $a \in \mathbb{R}$. For $r = 1, 2, \dots$, let \mathcal{C}^r be the subring of \mathcal{C} consisting of the germs at $+\infty$ of r -times continuously differentiable real-valued functions on such halflines. This yields the subring

$$\mathcal{C}^{<\infty} := \bigcap_{r \in \mathbb{N}^{\geq 1}} \mathcal{C}^r$$

Date: May 2021.

The first-named author was partially supported by NSF Grant DMS-1700439. We thank Allen Gehret for commenting on an earlier version of this paper.

of \mathcal{C} , and $\mathcal{C}^{<\infty}$ is naturally a *differential* ring. For a germ $f \in \mathcal{C}$ we let f also denote any real valued function representing this germ, if this causes no ambiguity. A real number is identified with the germ of the corresponding constant function: $\mathbb{R} \subseteq \mathcal{C}$.

A **Hardy field** is by definition a differential subfield of $\mathcal{C}^{<\infty}$. *Examples:*

$$\mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{R}(x), \quad \mathbb{R}(x, e^x), \quad \mathbb{R}(x, e^x, \log x), \quad \mathbb{R}(\Gamma, \Gamma', \Gamma'', \dots),$$

where x denotes the germ at $+\infty$ of the identity function on \mathbb{R} . All these are actually *analytic* Hardy fields, that is, its elements are germs of real analytic functions.

Let H be a Hardy field. Then H is an *ordered* differential field: for $f \in H$, either $f(x) > 0$ eventually (in which case we set $f > 0$), or $f(x) = 0$, eventually, or $f(x) < 0$, eventually; this is because $f \neq 0$ in H implies f has a multiplicative inverse in H , so f cannot have arbitrarily large zeros. Also, if $f' < 0$, then f is eventually strictly decreasing; if $f' = 0$, then f is eventually constant; if $f' > 0$, then f is eventually strictly increasing.

In order to state the main result of this paper we need a bit more terminology: an **H -field** is a K (that is, an ordered differential field) such that:

- for all $f \in K$, if $f > C$, then $f' > 0$;
- $\mathcal{O} = C + \mathfrak{o}$ (so C maps isomorphically onto the residue field \mathcal{O}/\mathfrak{o}).

We also say that K **has small derivation** if for all $f \in \mathfrak{o}$ we have $f' \in \mathfrak{o}$. Hardy fields have small derivation, and any Hardy field containing \mathbb{R} is an H -field.

An H -field K is said to be **Liouville closed** if it is real closed and for every $f \in K$ there are $g, h \in K^\times$ such that $f = g' = h'/h$. The ordered differential field \mathbb{T} is a Liouville closed H -field with small derivation. Any Hardy field $H \supseteq \mathbb{R}$ has a smallest (with respect to inclusion) Liouville closed Hardy field extension $\text{Li}(H)$. (The notions of “ H -field” and “Liouville closed H -field” are introduced in [1]. The capital H is in honor of Hardy, Hausdorff, and Hahn, who pioneered various aspects of our topic about a century ago, as did Du Bois-Reymond and Borel even earlier.)

Now a very strong property: we say K **has DIVP** (the Differential Intermediate Value Property) if for every polynomial $P \in K[Y_0, \dots, Y_r]$ and all $f < g$ in K with

$$P(f, f', \dots, f^{(r)}) < 0 < P(g, g', \dots, g^{(r)})$$

there exists $y \in K$ such that $f < y < g$ and $P(y, y', \dots, y^{(r)}) = 0$. (Existentially closed ordered differential fields have DIVP by [9] and [10, Proposition 1.5]; this has limited interest for us since the ordering and derivation in those structures do not interact.) Actually, DIVP is a bit of an afterthought: in [3] we considered instead two robust but rather technical properties, \mathfrak{o} -freeness and newtonianity, and proved that \mathbb{T} is \mathfrak{o} -free and newtonian. (One can think of newtonianity as a variant of differential-henselianity.) Afterwards we saw that “ \mathfrak{o} -free + newtonian” is equivalent to DIVP, for Liouville closed H -fields. Our aim is to establish this equivalence: Theorem 2.7, the main result of this short paper.

We did not consider DIVP in [3], but it is surely an appealing property and easier to grasp than the more fundamental notions of \mathfrak{o} -freeness and newtonianity. (The latter make sense in a wider setting of valued differential fields where the valuation does not necessarily arise from an ordering, as is the case for H -fields.)

Besides [3] we shall rely on [7], which focuses on a particular ordered differential subfield of \mathbb{T} , namely \mathbb{T}_g , consisting of the so-called *grid-based* transseries; see also [3, Appendix A]. We summarize what we need from [7] as follows:

\mathbb{T}_g is a newtonian ω -free Liouville closed H -field with small derivation, and \mathbb{T}_g has DIVP. We alert the reader that the terms *newtonian* and *ω -free* do not occur in [7], and that \mathbb{T}_g there is denoted by \mathbb{T} .

We call attention to the fact that K is a Liouville closed H -field iff $K \models \text{LiH}$ for a set LiH (independent of K) of sentences in the language of ordered differential fields. Also, for H -fields, “ ω -free” is expressible by a single sentence in the language of ordered differential fields, and “newtonian” as well as “DIVP” by a set of sentences in this language. The reason that “ ω -free + newtonian” is central in [3] are various theorems proved there, which are also relevant here. To state these theorems, we consider an H -field K below as an \mathcal{L} -structure, where

$$\mathcal{L} := \{0, 1, +, -, \times, \partial, <, \preceq\}$$

is the language of ordered valued differential fields. The symbols $0, 1, +, -, \times, \partial, <$ name the usual primitives of K , and \preceq encodes its valuation: for $a, b \in K$,

$$a \preceq b \quad :\iff \quad a \in \mathcal{O}b.$$

We can now summarize what we need from [3, Chapters 15, 16] as follows:

The theory of newtonian ω -free Liouville closed H -fields is model complete, and is the model companion of the theory of H -fields. The theory of newtonian ω -free Liouville closed H -fields whose derivation is small is complete and has \mathbb{T} as a model.

For an H -field K its valuation ring \mathcal{O} and so the binary relation \preceq on K can be defined in terms of the other primitives by an *existential* formula independent of K . However, by [3, Corollary 16.2.6] this cannot be done by a universal such formula and so for the model completeness above we cannot drop \preceq from the language \mathcal{L} .

Corollary 0.1. *Every newtonian ω -free Liouville closed H -field has DIVP.*

Proof. Let K be a newtonian ω -free Liouville closed H -field. If the derivation of K is small, then DIVP follows from the results from [7] quoted earlier and the above completeness result from [3]. Suppose the derivation of K is not small. Replacing the derivation ∂ of K by a multiple $\phi^{-1}\partial$ with $\phi > 0$ in K transforms K into its so-called compositional conjugate K^ϕ , which is still a newtonian ω -free Liouville closed H -field, and K has DIVP iff K^ϕ does. By 4.4.7 and 9.1.5 in [3] we can choose $\phi > 0$ in K such that the derivation $\phi^{-1}\partial$ of K^ϕ is small. \square

This gives one direction of Theorem 2.7. In the rest of this paper we prove a strong version, Corollary 2.6, of the other direction, without using [7] but relying heavily on various parts of [3] with detailed references. Theorem 2.7 and the results quoted above from [3] yield the result stated in the abstract: a Liouville closed H -field with small derivation is elementarily equivalent to \mathbb{T} iff it has DIVP.

Connection to Hardy fields. Every Hardy field H extends to a Hardy field $H(\mathbb{R}) \supseteq \mathbb{R}$, and $H(\mathbb{R})$ is in particular an H -field. We refer to [4] for a discussion of the conjecture that *any Hardy field containing \mathbb{R} extends to a newtonian ω -free Hardy field*. At the end of 2019 we finished the proof of this conjecture by considerably refining material in [3] and [8]; this amounts to a rather complete extension theory of Hardy fields. Note that every Hardy field extends to a maximal Hardy field, by Zorn, and so having established this conjecture we now know that all maximal Hardy fields are elementarily equivalent to \mathbb{T} , as ordered differential fields. Since \mathcal{C} has the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, there are at most $2^{\mathfrak{c}}$

many maximal Hardy fields, and we also have a proof that there are exactly that many. (We thank Ilijas Farah for a useful hint on this point.) These remarks on Hardy fields serve as an announcement. A rather voluminous work containing the proof of the conjecture is currently being prepared for publication. We also hope to include there a proof of DIVP for newtonian ω -free H -fields that does not depend as in the present paper on it being true for \mathbb{T}_g , whose proof in [7] uses the particular nature of \mathbb{T}_g .

We have a second conjecture about Hardy fields in [4], whose proof is not yet finished at this time (May 2021): *for any maximal Hardy field H and countable subsets $A < B$ in H there exists $y \in H$ such that $A < y < B$* . This means that the underlying ordered set of a maximal Hardy field is an η_1 -set in the sense of Hausdorff. Together with the (now established) first conjecture and results from [3] it implies: *all maximal Hardy fields are back-and-forth equivalent as ordered differential fields, and thus isomorphic assuming CH, the Continuum Hypothesis*.

1. PRELIMINARIES

In order to make free use of the valuation-theoretic tools from [3] and to make this paper self-contained modulo references to specific results from the literature we provide more background in this section before returning to DIVP.

Notation and terminology. Throughout, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Given an additively written abelian group A we let $A^\neq := A \setminus \{0\}$. Rings are commutative with identity 1, and for a ring R we let R^\times be the multiplicative group of units (consisting of the $a \in R$ such that $ab = 1$ for some $b \in R$). A *differential ring* will be a ring R containing (an isomorphic copy of) \mathbb{Q} as a subring and equipped with a derivation $\partial: R \rightarrow R$; note that then $C_R := \{a \in R : \partial(a) = 0\}$ is a subring of R , called the ring of constants of R , and that $\mathbb{Q} \subseteq C_R$. If R is a field, then so is C_R . An ordered differential field is in particular a differential ring.

Let R be a differential ring and $a \in R$. When its derivation ∂ is clear from the context we denote $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$ by $a', a'', \dots, a^{(n)}, \dots$, and if $a \in R^\times$, then a^\dagger denotes a'/a , so $(ab)^\dagger = a^\dagger + b^\dagger$ for $a, b \in R^\times$. In Section 2 we need to consider the function $\omega = \omega_R: R \rightarrow R$ given by $\omega(z) = -2z' - z^2$, and the function $\sigma = \sigma_R: R^\times \rightarrow R$ given by $\sigma(y) = \omega(z) + y^2$ for $z := -y^\dagger$.

We have the differential ring $R\{Y\} = R[Y, Y', Y'', \dots]$ of differential polynomials in an indeterminate Y over R . We say that $P = P(Y) \in R\{Y\}$ has order at most $r \in \mathbb{N}$ if $P \in R[Y, Y', \dots, Y^{(r)}]$.

For $\phi \in R^\times$ we let R^ϕ be the *compositional conjugate of R by ϕ* : the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ instead of ∂ . We then have an R -algebra isomorphism

$$P \mapsto P^\phi : R\{Y\} \rightarrow R^\phi\{Y\}$$

with $P^\phi(y) = P(y)$ for all $y \in R$; see [3, Section 5.7].

For a field K we have $K^\times = K^\neq$, and a (Krull) valuation on K is a surjective map $v: K^\times \rightarrow \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v: K \rightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$ by $v(0) := \infty$, where the ordering on Γ is extended to a total ordering on Γ_∞ by $\gamma < \infty$ for all $\gamma \in \Gamma$.

Let K be a *valued field*: a field (also denoted by K) together with a valuation ring \mathcal{O} of that field. This yields a valuation $v: K^\times \rightarrow \Gamma$ on the underlying field

such that $\mathcal{O} = \{a \in K : va \geq 0\}$ as explained in [3, Section 3.1]. We introduce various binary relations on the set K by defining for $a, b \in K$:

$$\begin{aligned} a \succ b &:\Leftrightarrow va = vb, & a \preccurlyeq b &:\Leftrightarrow va \geq vb, & a \prec b &:\Leftrightarrow va > vb, \\ a \succcurlyeq b &:\Leftrightarrow b \preccurlyeq a, & a \succ b &:\Leftrightarrow b \prec a, & a \sim b &:\Leftrightarrow a - b \prec a. \end{aligned}$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$, and that \sim is an equivalence relation on K^\times . We also let $\mathfrak{o} = \{a \in K : va > 0\}$ be the maximal ideal of \mathcal{O} , so \mathcal{O}/\mathfrak{o} is the residue field of the valued field K . A convex subgroup Δ of the value group Γ of v gives rise to the Δ -coarsening of the valued field K ; see [ADH, 3.4].

H -fields and pre- H -fields. As in [3], a *valued differential field* is a valued field K with residue field of characteristic zero and equipped with a derivation $\partial: K \rightarrow K$. An *ordered valued differential field* is a valued differential field K equipped with an ordering on K making K an ordered field. We consider any H -field K as an ordered valued differential field whose valuation ring is the convex hull in K of its constant field C , in accordance with construing it as an \mathcal{L} -structure as specified in the introduction.

A *pre- H -field* is by definition an ordered valued differential subfield of an H -field. By [3, Sections 10.1, 10.3, 10.5], an ordered valued differential field K is a pre- H -field iff the valuation ring \mathcal{O} of K is convex in K , $f' > 0$ for all $f > \mathcal{O}$ in K , and $f' \prec g^\dagger$ for all $f, g \in K^\times$ with $f \preccurlyeq 1$ and $g \prec 1$. Any Hardy field H is construed as a pre- H -field by taking the convex hull of \mathbb{Q} in H as its valuation ring, giving rise to the so-called “natural valuation” on H as an ordered field. At the end of Section 9.1 in [3] we give $\mathbb{Q}(\sqrt{2+x^{-1}})$ as an example of a Hardy field that is not an H -field. Any ordered differential field K with the trivial valuation ring $\mathcal{O} = K$ is a pre- H -field (so the valuation ring of a pre- H -field K is not always the convex hull in K of its constant field, in contrast to Hardy fields and H -fields). If K is a pre- H -field whose valuation ring is nontrivial, then the valuation topology on K equals its order topology, by [3, Lemma 2.4.1].

Let K be a pre- H -field. Then the derivation of K and its valuation $v: K^\times \rightarrow \Gamma$ induce an operation $\psi: \Gamma^\neq \rightarrow \Gamma$, given by $\psi(vf) = v(f^\dagger)$ for $f \neq 1$ in K^\times ; the pair (Γ, ψ) is called the H -asymptotic couple of K ; see [3, Section 9.1]. Below we assume some familiarity with (Γ, ψ) , and properties of K based on it, such as K having *asymptotic integration* and K having a *gap* [3, Sections 9.1, 9.2]. The *flattening* of K is the Γ^b -coarsening of K where $\Gamma^b = \{vf : f \in K^\times, f' \prec f\}$, with associated binary relations \succ^b, \preccurlyeq^b etc.; see [ADH, 9.4].

2. DIVP

In this section K is a pre- H -field. We let \mathcal{O} be its valuation ring, with maximal ideal \mathfrak{o} , and corresponding valuation $v: K^\times \rightarrow \Gamma = v(K^\times)$. Let (Γ, ψ) be its H -asymptotic couple, and $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^\neq\}$. Recall that “ K has DIVP” means: for all $P(Y) \in K\{Y\}$ and $f < g$ in K with $P(f) < 0 < P(g)$ there is a $y \in K$ such that $f < y < g$ and $P(y) = 0$. Restricting this to P of order $\leq r$, where $r \in \mathbb{N}$, gives the notion of r -DIVP. Thus K having 0-DIVP is equivalent to K being real closed as an ordered field. In particular, if K has 0-DIVP, then $\Gamma = v(K^\times)$ is divisible. From [3, Section 2.4] recall our convention that $K^> = \{a \in K : a > 0\}$, and similarly with $<$ replacing $>$.

Lemma 2.1. *Suppose $\Gamma \neq \{0\}$ and K has 1-DIVP. Then $\partial K = K$, $(K^>)^\dagger = (K^<)^\dagger$ is a convex subgroup of K , Ψ has no largest element, and Ψ is convex in Γ .*

Proof. We have $y' = 0$ for $y = 0$, and y' takes arbitrarily large positive values in K as y ranges over $K^{>\mathcal{O}} = \{a \in K : a > \mathcal{O}\}$, since by [3, Lemma 9.2.6] the set $(\Gamma^<)'$ is coinital in Γ . Hence y' takes all positive values on $K^>$, and therefore also all negative values on $K^<$. Thus $\partial K = K$. Next, let $a, b \in K^>$, and suppose $s \in K$ lies strictly between a^\dagger and b^\dagger . Then $s = y^\dagger$ for some $y \in K^>$ strictly between a and b ; this follows by noting that for $y = a$ and $y = b$ the signs of $sy - y'$ are opposite.

Let $\beta \in \Psi$ and take $a \in K$ with $v(a') = \beta$. Then $a \succ 1$, since $a \preccurlyeq 1$ would give $v(a') > \Psi$. Hence for $\alpha = va < 0$ we have $\alpha + \alpha^\dagger = \beta$, so $\alpha^\dagger > \beta$. Thus Ψ has no largest element. Therefore the set Ψ is convex in Γ . \square

Thus the ordered differential field \mathbb{T}_{\log} of logarithmic transseries [3, Appendix A] does not have 1-DIVP, although it is a newtonian ω -free H -field.

Does DIVP imply that K is an H -field? No: take an \aleph_0 -saturated elementary extension of \mathbb{T} and let Δ be as in [3, Example 10.1.7]. Then the Δ -coarsening of K is a pre- H -field with DIVP and nontrivial value group, and has a gap, but it is not an H -field. On the other hand:

Lemma 2.2. *Suppose K has 1-DIVP and has no gap. Then K is an H -field.*

Proof. In [3, Section 11.8] we defined

$$I(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\},$$

a convex \mathcal{O} -submodule of K . Since K has no gap, we have

$$\partial\mathcal{O} \subseteq I(K) = \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}.$$

Also $\Gamma \neq \{0\}$, and so (Γ, ψ) has asymptotic integration by Lemma 2.1. We show that K is an H -field by proving $I(K) = \partial\mathcal{O}$, so let $g \in I(K)$, $g < 0$. Since $(\Gamma^>)'$ has no least element we can take positive $f \in \mathcal{O}$ such that $f' \succ g$. Since $f' < 0$, this gives $f' < g$. Since $(\Gamma^>)'$ is cofinal in Γ we can also take positive $h \in \mathcal{O}$ such that $h' \prec g$, which in view of $h' < 0$ gives $g < h'$. Thus $f' < g < h'$, and so 1-DIVP yields $a \in \mathcal{O}$ with $g = a'$. \square

We refer to Sections 11.6 and 14.2 of [3] for the definitions of λ -freeness and r -newtonianity ($r \in \mathbb{N}$). From the introduction we recall that $\omega(z) := -2z' - z^2$. Below, compositionally conjugating an H -field K means replacing it by some K^ϕ with $\phi \in K^>$; this preserves most relevant properties like being an H -field, being λ -free, r -DIVP, and r -newtonianity, and it replaces Ψ by $\Psi - v\phi$.

Lemma 2.3. *Suppose K is an H -field, $\Gamma \neq \{0\}$, and K has 1-DIVP. Then K is λ -free and 1-newtonian, and the subset $\omega(K)$ of K is downward closed.*

Proof. Note that K has (asymptotic) integration, by Lemma 2.1. Assume towards a contradiction that K is not λ -free. We arrange by compositional conjugation that K has small derivation, so K has an element $x \succ 1$ with $x' = 1$, hence $x > C$. A construction in the beginning of [3, Section 11.5] yields by [3, Lemma 11.5.2] a pseudocauchy sequence (λ_ρ) in K with certain properties including $\lambda_\rho \sim x^{-1}$ for all ρ . As K is not λ -free, (λ_ρ) has a pseudolimit $\lambda \in K$ by [3, Corollary 11.6.1]. Then $s := -\lambda \sim -x^{-1}$, and s creates a gap over K by [3, Lemma 11.5.14]. Now note that for $P := Y' + sY$ we have $P(0) = 0$ and $P(x^2) = 2x + sx^2 \sim x$, so by 1-DIVP we have $P(y) = 1$ for some $y \in K$, contradicting [3, Lemma 11.5.12].

Let $P \in K\{Y\}$ of order at most 1 have Newton degree 1; we have to show that P has a zero in \mathcal{O} . We know that K is λ -free, so by [3, Proposition 13.3.6] we can pass to an elementary extension, compositionally conjugate, and divide by an element of K^\times to arrange that K has small derivation and $P = D + R$ where $D = cY + d$ or $D = cY'$ with $c, d \in C$, $c \neq 0$, and where $R \prec^b 1$. Then $R(a) \prec^b 1$ for all $a \in \mathcal{O}$. If $D = cY + d$, then we can take $a, b \in C$ with $D(a) < 0$ and $D(b) > 0$, which in view of $R(a) \prec D(a)$ and $R(b) \prec D(b)$ gives $P(a) < 0$ and $P(b) > 0$, and so P has a zero strictly between a and b , and thus a zero in \mathcal{O} . Next, suppose $D = cY'$. Then we take $t \in \mathcal{O}^\neq$ with $v(t^\dagger) = v(t)$, that is, $t' \asymp t^2$, so

$$P(t) = ct' + R(t), \quad P(-t) = -ct' + R(-t), \quad R(t), R(-t) \prec t'.$$

Hence $P(t)$ and $P(-t)$ have opposite signs, so P has a zero strictly between t and $-t$, and thus P has a zero in \mathcal{O} .

From $\omega(z) = -z^2 - 2z'$ we see that $\omega(z) \rightarrow -\infty$ as $z \rightarrow +\infty$ and as $z \rightarrow -\infty$ in K , so $\omega(K)$ is downward closed by 1-IVP. \square

For results involving r -DIVP for $r \geq 2$ we need a variant of [3, Lemma 11.8.31]. To state this variant we introduce as in [3, Section 11.8] the sets

$$\Gamma(K) := \{a^\dagger : a \in K \setminus \mathcal{O}\} \subseteq K^\succ, \quad \Lambda(K) := -\Gamma(K)^\dagger \subseteq K.$$

The superscripts \uparrow, \downarrow used in the statement of Lemma 2.4 below indicate upward, respectively downward, closure in the ordered set K , as in [3, Section 2.1].

Lemma 2.4. *Let K be an H -field with asymptotic integration. Then*

$$K^\succ = \text{I}(K)^\succ \cup \Gamma(K)^\uparrow, \quad \sigma(K^\succ \setminus \Gamma(K)^\uparrow) \subseteq \omega(\Lambda(K))^\downarrow.$$

Proof. If $a \in K$, $a > \text{I}(K)$, then $a \geq b^\dagger$ for some $b \in K^{\succ 1}$, and thus $a \in \Gamma(K)^\uparrow$. Next, let $s \in K^\succ \setminus \Gamma(K)^\uparrow$; we have to show $\sigma(s) \in \omega(\Lambda(K))^\downarrow$. Note that $s \in \text{I}(K)^\succ$ by what we just proved. From [3, 10.2.7 and 10.5.8] we obtain an immediate H -field extension L of K and $a \in L^{\succ 1}$ with $s = (1/a)'$. As in the proof of [3, 11.8.31] with L instead of K this gives $\sigma(s) \in \omega(\Lambda(L))^\downarrow$, where \downarrow indicates here the downward closure in L . It remains to note that ω is increasing on $\Lambda(L)$ by the remark preceding [3, 11.8.21], and that $\Lambda(K)$ is cofinal in $\Lambda(L)$ by [3, 11.8.14]. \square

The concept of ω -freeness is introduced in [3, Section 11.7]. If K has asymptotic integration, then by [3, 11.8.30]: K is ω -free $\Leftrightarrow K = \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$.

The next lemma also mentions the differential field extension $K[i]$ of K where $i^2 = -1$, as well as linear differential operators over differential fields like K and $K[i]$; for this we refer to [3, Sections 5.1, 5.2].

Lemma 2.5. *Suppose K is an H -field, $\Gamma \neq \{0\}$, $r \geq 2$, and K has r -DIVP. Then the following hold, with (i), (ii), (iii) using only the case $r = 2$:*

- (i) $K = \omega(K) \cup \sigma(K^\succ) = \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$;
- (ii) K is ω -free and $\omega(K) = \omega(\Lambda(K))^\downarrow$;
- (iii) for all $a \in K$ the operator $\partial^2 - a$ splits over $K[i]$;
- (iv) K is r -newtonian.

Proof. For (i) we use the end of [3, Section 11.7] to replace K with a compositional conjugate so that $0 \in \Psi$. Then K has small derivation, and we have $a \in K^\succ$ such that $a \neq 1$ and $a^\dagger \asymp 1$. Replacing a by a^{-1} if necessary this gives $a^\dagger = -\phi$ with $\phi \asymp 1$, $\phi > 0$, so $a \prec 1$. Then $\phi^{-1}a^\dagger = -1$; replacing K by K^ϕ and renaming

the latter as K this means $a^\dagger = -1$. Let $f \in K$; to get $f \in \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$, note first that $1 = (1/a)^\dagger \in \Gamma(K)$, so $0 \in \Lambda(K)$. Also $\omega(\Lambda(K))^\downarrow \subseteq \omega(K)$ by Lemma 2.3.

If $f \leq 0$, then $\omega(0) = 0$ gives $f \in \omega(\Lambda(K))^\downarrow$. So assume $f > 0$; we first show that then $f \in \sigma(K^\succ)$. Now for $y \in K^\succ$, $f = \sigma(y)$ is equivalent (by multiplying with y^2) to $P(y) = 0$, where

$$P(Y) := 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in K\{Y\}.$$

See also [3, Section 13.7]. We have $P(0) = 0$ and $P(y) \rightarrow +\infty$ as $y \rightarrow +\infty$ (because of the term y^4). In view of 2-DIVP it will suffice to show that for some $y > 0$ in K we have $P(y) < 0$. Now with $y \in K^\succ$ and $z := -y^\dagger$ we have

$$\begin{aligned} P(y) &= y^2(\sigma(y) - f) = y^2(\omega(z) + y^2 - f), \text{ hence} \\ P(a) &= a^2(\omega(1) + a^2 - f) = a^2(-1 + a^2 - f) < 0, \end{aligned}$$

so $f \in \sigma(K^\succ)$. By the second inclusion of Lemma 2.4 this yields $f \in \omega(\Lambda(K))^\downarrow$ or $f \in \sigma(\Gamma(K)^\dagger)$. But we have $\sigma(\Gamma(K)^\dagger) \subseteq \sigma(\Gamma(K))^\uparrow$, because σ is increasing on $\Gamma(K)^\uparrow$ by the remark preceding [3, 11.8.30]. This concludes the proof of (i), and then (ii) follows, using for its second part also the fact stated just before [3, 11.8.29] that we have $\omega(K) < \sigma(\Gamma(K))$.

Now (iii) follows from $K = \omega(K) \cup \sigma(K^\succ)$ by [3, Section 5.2, (5.2.1)]. As to (iv), let $P \in K\{Y\}$ of order at most r have Newton degree 1; we have to show that P has a zero in \mathcal{O} . For this we repeat the argument in the proof of Lemma 2.3 so that it applies to our P , using ω -freeness instead of λ -freeness, [3, Proposition 13.3.13] instead of [3, Proposition 13.3.6], and r -DIVP instead of 1-DIVP. \square

Corollary 2.6. *If K is an H -field, $\Gamma \neq \{0\}$, and K has DIVP, then K is ω -free and newtonian.*

There are non-Liouville closed H -fields with nontrivial derivation that have DIVP; see [2, Section 14]. By Lemma 2.3 and Lemma 2.5(iii), Liouville closed H -fields having 2-DIVP are *Schwarz closed* as defined in [3, Section 11.8].

Theorem 2.7. *Let K be a Liouville closed H -field. Then*

$$K \text{ has DIVP} \iff K \text{ is } \omega\text{-free and newtonian.}$$

Proof. The forward direction is part of Corollary 2.6. The backward direction is Corollary 0.1. \square

REFERENCES

- [1] M. Aschenbrenner, L. van den Dries, *H-fields and their Liouville extensions*, Math. Z. **242** (2002), 543–588.
- [2] M. Aschenbrenner, L. van den Dries, *Liouville closed H-fields*, J. Pure Appl. Algebra **197** (2005), 83–139.
- [3] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries*, Annals of Mathematics Studies, vol. 195, Princeton University Press, Princeton, NJ, 2017.
- [4] ———, *On numbers, germs, and transseries*, in: B. Sirakov et al. (eds.): *Proceedings of the International Congress of Mathematicians, Rio de Janeiro 2018*, vol. 2, pp. 19–42, World Scientific Publishing Co., Singapore, 2018.
- [5] N. Bourbaki, *Fonctions d'une Variable Réelle*, Chapitre V, *Étude Locale des Fonctions*, Hermann, Paris, 1976.
- [6] G. H. Hardy, *Orders of Infinity*, 2nd ed., Cambridge Univ. Press, Cambridge, 1924.

- [7] J. van der Hoeven, *Transseries and Real Differential Algebra*, Lecture Notes in Math., vol. 1888, Springer-Verlag, New York, 2006.
- [8] J. van der Hoeven, *Transserial Hardy fields*, *Astérisque* **323** (2009), 453–487.
- [9] M. Singer, *The model theory of ordered differential fields*, *J. Symbolic Logic* **43** (1978), no. 1, 82–91.
- [10] S. Spodzieja, *A geometric model of an arbitrary differentially closed field of characteristic zero*, *Trans. Amer. Math. Soc.* **374** (2021), no. 1, 663–686.

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC, UNIVERSITÄT WIEN, 1090 WIEN, AUSTRIA

Email address: `matthias.aschenbrenner@univie.ac.at`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, U.S.A.

Email address: `vddries@math.uiuc.edu`

CNRS, LIX, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE

Email address: `vdhoeven@lix.polytechnique.fr`