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# ON A DIFFERENTIAL INTERMEDIATE VALUE PROPERTY

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

ABSTRACT. Liouville closed  $H$ -fields are ordered differential fields whose ordering and derivation interact in a natural way and where every linear differential equation of order 1 has a nontrivial solution. (The introduction gives a precise definition.) For a Liouville closed  $H$ -field  $K$  with small derivation we show:  $K$  has the Intermediate Value Property for differential polynomials iff  $K$  is elementarily equivalent to the ordered differential field of transseries. We also indicate how this applies to Hardy fields.

## INTRODUCTION

Throughout this introduction  $K$  is an ordered differential field, that is, an ordered field equipped with a derivation  $\partial: K \rightarrow K$ . (We usually write  $f'$  instead of  $\partial f$ , for  $f \in K$ .) Its constant field

$$C := \{f \in K : f' = 0\}$$

yields the (convex) valuation ring

$$\mathcal{O} := \{f \in K : |f| \leq c \text{ for some } c \in C\}$$

of  $K$ , with maximal ideal

$$\mathfrak{o} := \{f \in K : |f| < c \text{ for all } c > 0 \text{ in } C\}.$$

(It may help to think of the elements of  $K$  as germs of real valued functions and of  $f \in \mathcal{O}g$  and  $f \in \mathfrak{o}g$  as  $f = O(g)$  and  $f = o(g)$ , respectively.) The above definitions exhibit  $C$ ,  $\mathcal{O}$ , and  $\mathfrak{o}$  as definable in  $K$  in the sense of model theory.

Key example: the ordered differential field  $\mathbb{T}$  of **transseries**, which contains  $\mathbb{R}$  as an ordered subfield, and where  $C = \mathbb{R}$ . We refer to [3] for the rather elaborate construction of  $\mathbb{T}$  and for any fact about  $\mathbb{T}$  that gets mentioned without proof.

Other important examples are Hardy fields. (Hardy [6] proved a striking theorem on logarithmic-exponential functions. Bourbaki [5] put this into the general setting of what they called Hardy fields.) Here we can give a definition from scratch that doesn't take much space. Notation:  $\mathcal{C}$  is the ring of germs at  $+\infty$  of continuous real-valued functions on halflines  $(a, +\infty)$ ,  $a \in \mathbb{R}$ . For  $r = 1, 2, \dots$ , let  $\mathcal{C}^r$  be the subring of  $\mathcal{C}$  consisting of the germs at  $+\infty$  of  $r$ -times continuously differentiable real-valued functions on such halflines. This yields the subring

$$\mathcal{C}^{<\infty} := \bigcap_{r \in \mathbb{N}^{\geq 1}} \mathcal{C}^r$$

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of  $\mathcal{C}$ , and  $\mathcal{C}^{<\infty}$  is naturally a *differential* ring. For a germ  $f \in \mathcal{C}$  we let  $f$  also denote any real valued function representing this germ, if this causes no ambiguity. A real number is identified with the germ of the corresponding constant function:  $\mathbb{R} \subseteq \mathcal{C}$ .

A **Hardy field** is by definition a differential subfield of  $\mathcal{C}^{<\infty}$ . *Examples:*

$$\mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{R}(x), \quad \mathbb{R}(x, e^x), \quad \mathbb{R}(x, e^x, \log x), \quad \mathbb{R}(\Gamma, \Gamma', \Gamma'', \dots),$$

where  $x$  denotes the germ at  $+\infty$  of the identity function on  $\mathbb{R}$ . All these are actually *analytic* Hardy fields, that is, its elements are germs of real analytic functions.

Let  $H$  be a Hardy field. Then  $H$  is an *ordered* differential field: for  $f \in H$ , either  $f(x) > 0$  eventually (in which case we set  $f > 0$ ), or  $f(x) = 0$ , eventually, or  $f(x) < 0$ , eventually; this is because  $f \neq 0$  in  $H$  implies  $f$  has a multiplicative inverse in  $H$ , so  $f$  cannot have arbitrarily large zeros. Also, if  $f' < 0$ , then  $f$  is eventually strictly decreasing; if  $f' = 0$ , then  $f$  is eventually constant; if  $f' > 0$ , then  $f$  is eventually strictly increasing.

In order to state the main result of this paper we need a bit more terminology: an  **$H$ -field** is a  $K$  (that is, an ordered differential field) such that:

- for all  $f \in K$ , if  $f > C$ , then  $f' > 0$ ;
- $\mathcal{O} = C + \mathfrak{o}$  (so  $C$  maps isomorphically onto the residue field  $\mathcal{O}/\mathfrak{o}$ ).

We also say that  $K$  **has small derivation** if for all  $f \in \mathfrak{o}$  we have  $f' \in \mathfrak{o}$ . Hardy fields have small derivation, and any Hardy field containing  $\mathbb{R}$  is an  $H$ -field.

An  $H$ -field  $K$  is said to be **Liouville closed** if it is real closed and for every  $f \in K$  there are  $g, h \in K^\times$  such that  $f = g' = h'/h$ . The ordered differential field  $\mathbb{T}$  is a Liouville closed  $H$ -field with small derivation. Any Hardy field  $H \supseteq \mathbb{R}$  has a smallest (with respect to inclusion) Liouville closed Hardy field extension  $\text{Li}(H)$ . (The notions of “ $H$ -field” and “Liouville closed  $H$ -field” are introduced in [1]. The capital  $H$  is in honor of Hardy, Hausdorff, and Hahn, who pioneered various aspects of our topic about a century ago, as did Du Bois-Reymond and Borel even earlier.)

Now a very strong property: we say  $K$  **has DIVP** (the Differential Intermediate Value Property) if for every polynomial  $P \in K[Y_0, \dots, Y_r]$  and all  $f < g$  in  $K$  with

$$P(f, f', \dots, f^{(r)}) < 0 < P(g, g', \dots, g^{(r)})$$

there exists  $y \in K$  such that  $f < y < g$  and  $P(y, y', \dots, y^{(r)}) = 0$ . (Existentially closed ordered differential fields have DIVP by [9] and [10, Proposition 1.5]; this has limited interest for us since the ordering and derivation in those structures do not interact.) Actually, DIVP is a bit of an afterthought: in [3] we considered instead two robust but rather technical properties,  $\mathfrak{o}$ -freeness and newtonianity, and proved that  $\mathbb{T}$  is  $\mathfrak{o}$ -free and newtonian. (One can think of newtonianity as a variant of differential-henselianity.) Afterwards we saw that “ $\mathfrak{o}$ -free + newtonian” is equivalent to DIVP, for Liouville closed  $H$ -fields. Our aim is to establish this equivalence: Theorem 2.7, the main result of this short paper.

We did not consider DIVP in [3], but it is surely an appealing property and easier to grasp than the more fundamental notions of  $\mathfrak{o}$ -freeness and newtonianity. (The latter make sense in a wider setting of valued differential fields where the valuation does not necessarily arise from an ordering, as is the case for  $H$ -fields.)

Besides [3] we shall rely on [7], which focuses on a particular ordered differential subfield of  $\mathbb{T}$ , namely  $\mathbb{T}_g$ , consisting of the so-called *grid-based* transseries; see also [3, Appendix A]. We summarize what we need from [7] as follows:

$\mathbb{T}_g$  is a newtonian  $\omega$ -free Liouville closed  $H$ -field with small derivation, and  $\mathbb{T}_g$  has DIVP. We alert the reader that the terms *newtonian* and  *$\omega$ -free* do not occur in [7], and that  $\mathbb{T}_g$  there is denoted by  $\mathbb{T}$ .

We call attention to the fact that  $K$  is a Liouville closed  $H$ -field iff  $K \models \text{LiH}$  for a set LiH (independent of  $K$ ) of sentences in the language of ordered differential fields. Also, for  $H$ -fields, “ $\omega$ -free” is expressible by a single sentence in the language of ordered differential fields, and “newtonian” as well as “DIVP” by a set of sentences in this language. The reason that “ $\omega$ -free + newtonian” is central in [3] are various theorems proved there, which are also relevant here. To state these theorems, we consider an  $H$ -field  $K$  below as an  $\mathcal{L}$ -structure, where

$$\mathcal{L} := \{0, 1, +, -, \times, \partial, <, \preceq\}$$

is the language of ordered valued differential fields. The symbols  $0, 1, +, -, \times, \partial, <$  name the usual primitives of  $K$ , and  $\preceq$  encodes its valuation: for  $a, b \in K$ ,

$$a \preceq b \quad :\iff \quad a \in \mathcal{O}b.$$

We can now summarize what we need from [3, Chapters 15, 16] as follows:

*The theory of newtonian  $\omega$ -free Liouville closed  $H$ -fields is model complete, and is the model companion of the theory of  $H$ -fields. The theory of newtonian  $\omega$ -free Liouville closed  $H$ -fields whose derivation is small is complete and has  $\mathbb{T}$  as a model.*

For an  $H$ -field  $K$  its valuation ring  $\mathcal{O}$  and so the binary relation  $\preceq$  on  $K$  can be defined in terms of the other primitives by an *existential* formula independent of  $K$ . However, by [3, Corollary 16.2.6] this cannot be done by a universal such formula and so for the model completeness above we cannot drop  $\preceq$  from the language  $\mathcal{L}$ .

**Corollary 0.1.** *Every newtonian  $\omega$ -free Liouville closed  $H$ -field has DIVP.*

*Proof.* Let  $K$  be a newtonian  $\omega$ -free Liouville closed  $H$ -field. If the derivation of  $K$  is small, then DIVP follows from the results from [7] quoted earlier and the above completeness result from [3]. Suppose the derivation of  $K$  is not small. Replacing the derivation  $\partial$  of  $K$  by a multiple  $\phi^{-1}\partial$  with  $\phi > 0$  in  $K$  transforms  $K$  into its so-called compositional conjugate  $K^\phi$ , which is still a newtonian  $\omega$ -free Liouville closed  $H$ -field, and  $K$  has DIVP iff  $K^\phi$  does. By 4.4.7 and 9.1.5 in [3] we can choose  $\phi > 0$  in  $K$  such that the derivation  $\phi^{-1}\partial$  of  $K^\phi$  is small.  $\square$

This gives one direction of Theorem 2.7. In the rest of this paper we prove a strong version, Corollary 2.6, of the other direction, without using [7] but relying heavily on various parts of [3] with detailed references. Theorem 2.7 and the results quoted above from [3] yield the result stated in the abstract: a Liouville closed  $H$ -field with small derivation is elementarily equivalent to  $\mathbb{T}$  iff it has DIVP.

**Connection to Hardy fields.** Every Hardy field  $H$  extends to a Hardy field  $H(\mathbb{R}) \supseteq \mathbb{R}$ , and  $H(\mathbb{R})$  is in particular an  $H$ -field. We refer to [4] for a discussion of the conjecture that *any Hardy field containing  $\mathbb{R}$  extends to a newtonian  $\omega$ -free Hardy field*. At the end of 2019 we finished the proof of this conjecture by considerably refining material in [3] and [8]; this amounts to a rather complete extension theory of Hardy fields. Note that every Hardy field extends to a maximal Hardy field, by Zorn, and so having established this conjecture we now know that all maximal Hardy fields are elementarily equivalent to  $\mathbb{T}$ , as ordered differential fields. Since  $\mathcal{C}$  has the cardinality  $\mathfrak{c} = 2^{\aleph_0}$  of the continuum, there are at most  $2^{\mathfrak{c}}$

many maximal Hardy fields, and we also have a proof that there are exactly that many. (We thank Ilijas Farah for a useful hint on this point.) These remarks on Hardy fields serve as an announcement. A rather voluminous work containing the proof of the conjecture is currently being prepared for publication. We also hope to include there a proof of DIVP for newtonian  $\omega$ -free  $H$ -fields that does not depend as in the present paper on it being true for  $\mathbb{T}_g$ , whose proof in [7] uses the particular nature of  $\mathbb{T}_g$ .

We have a second conjecture about Hardy fields in [4], whose proof is not yet finished at this time (May 2021): *for any maximal Hardy field  $H$  and countable subsets  $A < B$  in  $H$  there exists  $y \in H$  such that  $A < y < B$* . This means that the underlying ordered set of a maximal Hardy field is an  $\eta_1$ -set in the sense of Hausdorff. Together with the (now established) first conjecture and results from [3] it implies: *all maximal Hardy fields are back-and-forth equivalent as ordered differential fields, and thus isomorphic assuming CH, the Continuum Hypothesis*.

## 1. PRELIMINARIES

In order to make free use of the valuation-theoretic tools from [3] and to make this paper self-contained modulo references to specific results from the literature we provide more background in this section before returning to DIVP.

**Notation and terminology.** Throughout,  $m, n$  range over  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Given an additively written abelian group  $A$  we let  $A^\neq := A \setminus \{0\}$ . Rings are commutative with identity 1, and for a ring  $R$  we let  $R^\times$  be the multiplicative group of units (consisting of the  $a \in R$  such that  $ab = 1$  for some  $b \in R$ ). A *differential ring* will be a ring  $R$  containing (an isomorphic copy of)  $\mathbb{Q}$  as a subring and equipped with a derivation  $\partial: R \rightarrow R$ ; note that then  $C_R := \{a \in R : \partial(a) = 0\}$  is a subring of  $R$ , called the ring of constants of  $R$ , and that  $\mathbb{Q} \subseteq C_R$ . If  $R$  is a field, then so is  $C_R$ . An ordered differential field is in particular a differential ring.

Let  $R$  be a differential ring and  $a \in R$ . When its derivation  $\partial$  is clear from the context we denote  $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$  by  $a', a'', \dots, a^{(n)}, \dots$ , and if  $a \in R^\times$ , then  $a^\dagger$  denotes  $a'/a$ , so  $(ab)^\dagger = a^\dagger + b^\dagger$  for  $a, b \in R^\times$ . In Section 2 we need to consider the function  $\omega = \omega_R: R \rightarrow R$  given by  $\omega(z) = -2z' - z^2$ , and the function  $\sigma = \sigma_R: R^\times \rightarrow R$  given by  $\sigma(y) = \omega(z) + y^2$  for  $z := -y^\dagger$ .

We have the differential ring  $R\{Y\} = R[Y, Y', Y'', \dots]$  of differential polynomials in an indeterminate  $Y$  over  $R$ . We say that  $P = P(Y) \in R\{Y\}$  has order at most  $r \in \mathbb{N}$  if  $P \in R[Y, Y', \dots, Y^{(r)}]$ .

For  $\phi \in R^\times$  we let  $R^\phi$  be the *compositional conjugate of  $R$  by  $\phi$* : the differential ring with the same underlying ring as  $R$  but with derivation  $\phi^{-1}\partial$  instead of  $\partial$ . We then have an  $R$ -algebra isomorphism

$$P \mapsto P^\phi : R\{Y\} \rightarrow R^\phi\{Y\}$$

with  $P^\phi(y) = P(y)$  for all  $y \in R$ ; see [3, Section 5.7].

For a field  $K$  we have  $K^\times = K^\neq$ , and a (Krull) valuation on  $K$  is a surjective map  $v: K^\times \rightarrow \Gamma$  onto an ordered abelian group  $\Gamma$  (additively written) satisfying the usual laws, and extended to  $v: K \rightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$  by  $v(0) := \infty$ , where the ordering on  $\Gamma$  is extended to a total ordering on  $\Gamma_\infty$  by  $\gamma < \infty$  for all  $\gamma \in \Gamma$ .

Let  $K$  be a *valued field*: a field (also denoted by  $K$ ) together with a valuation ring  $\mathcal{O}$  of that field. This yields a valuation  $v: K^\times \rightarrow \Gamma$  on the underlying field

such that  $\mathcal{O} = \{a \in K : va \geq 0\}$  as explained in [3, Section 3.1]. We introduce various binary relations on the set  $K$  by defining for  $a, b \in K$ :

$$\begin{aligned} a \succ b &:\Leftrightarrow va = vb, & a \preccurlyeq b &:\Leftrightarrow va \geq vb, & a \prec b &:\Leftrightarrow va > vb, \\ a \succcurlyeq b &:\Leftrightarrow b \preccurlyeq a, & a \succ b &:\Leftrightarrow b \prec a, & a \sim b &:\Leftrightarrow a - b \prec a. \end{aligned}$$

It is easy to check that if  $a \sim b$ , then  $a, b \neq 0$ , and that  $\sim$  is an equivalence relation on  $K^\times$ . We also let  $\mathfrak{o} = \{a \in K : va > 0\}$  be the maximal ideal of  $\mathcal{O}$ , so  $\mathcal{O}/\mathfrak{o}$  is the residue field of the valued field  $K$ . A convex subgroup  $\Delta$  of the value group  $\Gamma$  of  $v$  gives rise to the  $\Delta$ -coarsening of the valued field  $K$ ; see [ADH, 3.4].

**$H$ -fields and pre- $H$ -fields.** As in [3], a *valued differential field* is a valued field  $K$  with residue field of characteristic zero and equipped with a derivation  $\partial: K \rightarrow K$ . An *ordered valued differential field* is a valued differential field  $K$  equipped with an ordering on  $K$  making  $K$  an ordered field. We consider any  $H$ -field  $K$  as an ordered valued differential field whose valuation ring is the convex hull in  $K$  of its constant field  $C$ , in accordance with construing it as an  $\mathcal{L}$ -structure as specified in the introduction.

A *pre- $H$ -field* is by definition an ordered valued differential subfield of an  $H$ -field. By [3, Sections 10.1, 10.3, 10.5], an ordered valued differential field  $K$  is a pre- $H$ -field iff the valuation ring  $\mathcal{O}$  of  $K$  is convex in  $K$ ,  $f' > 0$  for all  $f > \mathcal{O}$  in  $K$ , and  $f' \prec g^\dagger$  for all  $f, g \in K^\times$  with  $f \preccurlyeq 1$  and  $g \prec 1$ . Any Hardy field  $H$  is construed as a pre- $H$ -field by taking the convex hull of  $\mathbb{Q}$  in  $H$  as its valuation ring, giving rise to the so-called “natural valuation” on  $H$  as an ordered field. At the end of Section 9.1 in [3] we give  $\mathbb{Q}(\sqrt{2+x^{-1}})$  as an example of a Hardy field that is not an  $H$ -field. Any ordered differential field  $K$  with the trivial valuation ring  $\mathcal{O} = K$  is a pre- $H$ -field (so the valuation ring of a pre- $H$ -field  $K$  is not always the convex hull in  $K$  of its constant field, in contrast to Hardy fields and  $H$ -fields). If  $K$  is a pre- $H$ -field whose valuation ring is nontrivial, then the valuation topology on  $K$  equals its order topology, by [3, Lemma 2.4.1].

Let  $K$  be a pre- $H$ -field. Then the derivation of  $K$  and its valuation  $v: K^\times \rightarrow \Gamma$  induce an operation  $\psi: \Gamma^\neq \rightarrow \Gamma$ , given by  $\psi(vf) = v(f^\dagger)$  for  $f \neq 1$  in  $K^\times$ ; the pair  $(\Gamma, \psi)$  is called the  $H$ -asymptotic couple of  $K$ ; see [3, Section 9.1]. Below we assume some familiarity with  $(\Gamma, \psi)$ , and properties of  $K$  based on it, such as  $K$  having *asymptotic integration* and  $K$  having a *gap* [3, Sections 9.1, 9.2]. The *flattening* of  $K$  is the  $\Gamma^b$ -coarsening of  $K$  where  $\Gamma^b = \{vf : f \in K^\times, f' \prec f\}$ , with associated binary relations  $\succ^b, \preccurlyeq^b$  etc.; see [ADH, 9.4].

## 2. DIVP

In this section  $K$  is a pre- $H$ -field. We let  $\mathcal{O}$  be its valuation ring, with maximal ideal  $\mathfrak{o}$ , and corresponding valuation  $v: K^\times \rightarrow \Gamma = v(K^\times)$ . Let  $(\Gamma, \psi)$  be its  $H$ -asymptotic couple, and  $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^\neq\}$ . Recall that “ $K$  has DIVP” means: for all  $P(Y) \in K\{Y\}$  and  $f < g$  in  $K$  with  $P(f) < 0 < P(g)$  there is a  $y \in K$  such that  $f < y < g$  and  $P(y) = 0$ . Restricting this to  $P$  of order  $\leq r$ , where  $r \in \mathbb{N}$ , gives the notion of  $r$ -DIVP. Thus  $K$  having 0-DIVP is equivalent to  $K$  being real closed as an ordered field. In particular, if  $K$  has 0-DIVP, then  $\Gamma = v(K^\times)$  is divisible. From [3, Section 2.4] recall our convention that  $K^> = \{a \in K : a > 0\}$ , and similarly with  $<$  replacing  $>$ .

**Lemma 2.1.** *Suppose  $\Gamma \neq \{0\}$  and  $K$  has 1-DIVP. Then  $\partial K = K$ ,  $(K^>)^\dagger = (K^<)^\dagger$  is a convex subgroup of  $K$ ,  $\Psi$  has no largest element, and  $\Psi$  is convex in  $\Gamma$ .*

*Proof.* We have  $y' = 0$  for  $y = 0$ , and  $y'$  takes arbitrarily large positive values in  $K$  as  $y$  ranges over  $K^{>\mathcal{O}} = \{a \in K : a > \mathcal{O}\}$ , since by [3, Lemma 9.2.6] the set  $(\Gamma^<)'$  is coinital in  $\Gamma$ . Hence  $y'$  takes all positive values on  $K^>$ , and therefore also all negative values on  $K^<$ . Thus  $\partial K = K$ . Next, let  $a, b \in K^>$ , and suppose  $s \in K$  lies strictly between  $a^\dagger$  and  $b^\dagger$ . Then  $s = y^\dagger$  for some  $y \in K^>$  strictly between  $a$  and  $b$ ; this follows by noting that for  $y = a$  and  $y = b$  the signs of  $sy - y'$  are opposite.

Let  $\beta \in \Psi$  and take  $a \in K$  with  $v(a') = \beta$ . Then  $a \succ 1$ , since  $a \preccurlyeq 1$  would give  $v(a') > \Psi$ . Hence for  $\alpha = va < 0$  we have  $\alpha + \alpha^\dagger = \beta$ , so  $\alpha^\dagger > \beta$ . Thus  $\Psi$  has no largest element. Therefore the set  $\Psi$  is convex in  $\Gamma$ .  $\square$

Thus the ordered differential field  $\mathbb{T}_{\log}$  of logarithmic transseries [3, Appendix A] does not have 1-DIVP, although it is a newtonian  $\omega$ -free  $H$ -field.

Does DIVP imply that  $K$  is an  $H$ -field? No: take an  $\aleph_0$ -saturated elementary extension of  $\mathbb{T}$  and let  $\Delta$  be as in [3, Example 10.1.7]. Then the  $\Delta$ -coarsening of  $K$  is a pre- $H$ -field with DIVP and nontrivial value group, and has a gap, but it is not an  $H$ -field. On the other hand:

**Lemma 2.2.** *Suppose  $K$  has 1-DIVP and has no gap. Then  $K$  is an  $H$ -field.*

*Proof.* In [3, Section 11.8] we defined

$$I(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\},$$

a convex  $\mathcal{O}$ -submodule of  $K$ . Since  $K$  has no gap, we have

$$\partial\mathcal{O} \subseteq I(K) = \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}.$$

Also  $\Gamma \neq \{0\}$ , and so  $(\Gamma, \psi)$  has asymptotic integration by Lemma 2.1. We show that  $K$  is an  $H$ -field by proving  $I(K) = \partial\mathcal{O}$ , so let  $g \in I(K)$ ,  $g < 0$ . Since  $(\Gamma^>)'$  has no least element we can take positive  $f \in \mathcal{O}$  such that  $f' \succ g$ . Since  $f' < 0$ , this gives  $f' < g$ . Since  $(\Gamma^>)'$  is cofinal in  $\Gamma$  we can also take positive  $h \in \mathcal{O}$  such that  $h' \prec g$ , which in view of  $h' < 0$  gives  $g < h'$ . Thus  $f' < g < h'$ , and so 1-DIVP yields  $a \in \mathcal{O}$  with  $g = a'$ .  $\square$

We refer to Sections 11.6 and 14.2 of [3] for the definitions of  $\lambda$ -freeness and  $r$ -newtonianity ( $r \in \mathbb{N}$ ). From the introduction we recall that  $\omega(z) := -2z' - z^2$ . Below, compositionally conjugating an  $H$ -field  $K$  means replacing it by some  $K^\phi$  with  $\phi \in K^>$ ; this preserves most relevant properties like being an  $H$ -field, being  $\lambda$ -free,  $r$ -DIVP, and  $r$ -newtonianity, and it replaces  $\Psi$  by  $\Psi - v\phi$ .

**Lemma 2.3.** *Suppose  $K$  is an  $H$ -field,  $\Gamma \neq \{0\}$ , and  $K$  has 1-DIVP. Then  $K$  is  $\lambda$ -free and 1-newtonian, and the subset  $\omega(K)$  of  $K$  is downward closed.*

*Proof.* Note that  $K$  has (asymptotic) integration, by Lemma 2.1. Assume towards a contradiction that  $K$  is not  $\lambda$ -free. We arrange by compositional conjugation that  $K$  has small derivation, so  $K$  has an element  $x \succ 1$  with  $x' = 1$ , hence  $x > C$ . A construction in the beginning of [3, Section 11.5] yields by [3, Lemma 11.5.2] a pseudocauchy sequence  $(\lambda_\rho)$  in  $K$  with certain properties including  $\lambda_\rho \sim x^{-1}$  for all  $\rho$ . As  $K$  is not  $\lambda$ -free,  $(\lambda_\rho)$  has a pseudolimit  $\lambda \in K$  by [3, Corollary 11.6.1]. Then  $s := -\lambda \sim -x^{-1}$ , and  $s$  creates a gap over  $K$  by [3, Lemma 11.5.14]. Now note that for  $P := Y' + sY$  we have  $P(0) = 0$  and  $P(x^2) = 2x + sx^2 \sim x$ , so by 1-DIVP we have  $P(y) = 1$  for some  $y \in K$ , contradicting [3, Lemma 11.5.12].

Let  $P \in K\{Y\}$  of order at most 1 have Newton degree 1; we have to show that  $P$  has a zero in  $\mathcal{O}$ . We know that  $K$  is  $\lambda$ -free, so by [3, Proposition 13.3.6] we can pass to an elementary extension, compositionally conjugate, and divide by an element of  $K^\times$  to arrange that  $K$  has small derivation and  $P = D + R$  where  $D = cY + d$  or  $D = cY'$  with  $c, d \in C$ ,  $c \neq 0$ , and where  $R \prec^b 1$ . Then  $R(a) \prec^b 1$  for all  $a \in \mathcal{O}$ . If  $D = cY + d$ , then we can take  $a, b \in C$  with  $D(a) < 0$  and  $D(b) > 0$ , which in view of  $R(a) \prec D(a)$  and  $R(b) \prec D(b)$  gives  $P(a) < 0$  and  $P(b) > 0$ , and so  $P$  has a zero strictly between  $a$  and  $b$ , and thus a zero in  $\mathcal{O}$ . Next, suppose  $D = cY'$ . Then we take  $t \in \mathcal{O}^\neq$  with  $v(t^\dagger) = v(t)$ , that is,  $t' \asymp t^2$ , so

$$P(t) = ct' + R(t), \quad P(-t) = -ct' + R(-t), \quad R(t), R(-t) \prec t'.$$

Hence  $P(t)$  and  $P(-t)$  have opposite signs, so  $P$  has a zero strictly between  $t$  and  $-t$ , and thus  $P$  has a zero in  $\mathcal{O}$ .

From  $\omega(z) = -z^2 - 2z'$  we see that  $\omega(z) \rightarrow -\infty$  as  $z \rightarrow +\infty$  and as  $z \rightarrow -\infty$  in  $K$ , so  $\omega(K)$  is downward closed by 1-IVP.  $\square$

For results involving  $r$ -DIVP for  $r \geq 2$  we need a variant of [3, Lemma 11.8.31]. To state this variant we introduce as in [3, Section 11.8] the sets

$$\Gamma(K) := \{a^\dagger : a \in K \setminus \mathcal{O}\} \subseteq K^\succ, \quad \Lambda(K) := -\Gamma(K)^\dagger \subseteq K.$$

The superscripts  $\uparrow, \downarrow$  used in the statement of Lemma 2.4 below indicate upward, respectively downward, closure in the ordered set  $K$ , as in [3, Section 2.1].

**Lemma 2.4.** *Let  $K$  be an  $H$ -field with asymptotic integration. Then*

$$K^\succ = \text{I}(K)^\succ \cup \Gamma(K)^\uparrow, \quad \sigma(K^\succ \setminus \Gamma(K)^\uparrow) \subseteq \omega(\Lambda(K))^\downarrow.$$

*Proof.* If  $a \in K$ ,  $a > \text{I}(K)$ , then  $a \geq b^\dagger$  for some  $b \in K^{\succ 1}$ , and thus  $a \in \Gamma(K)^\uparrow$ . Next, let  $s \in K^\succ \setminus \Gamma(K)^\uparrow$ ; we have to show  $\sigma(s) \in \omega(\Lambda(K))^\downarrow$ . Note that  $s \in \text{I}(K)^\succ$  by what we just proved. From [3, 10.2.7 and 10.5.8] we obtain an immediate  $H$ -field extension  $L$  of  $K$  and  $a \in L^{\succ 1}$  with  $s = (1/a)'$ . As in the proof of [3, 11.8.31] with  $L$  instead of  $K$  this gives  $\sigma(s) \in \omega(\Lambda(L))^\downarrow$ , where  $\downarrow$  indicates here the downward closure in  $L$ . It remains to note that  $\omega$  is increasing on  $\Lambda(L)$  by the remark preceding [3, 11.8.21], and that  $\Lambda(K)$  is cofinal in  $\Lambda(L)$  by [3, 11.8.14].  $\square$

The concept of  $\omega$ -freeness is introduced in [3, Section 11.7]. If  $K$  has asymptotic integration, then by [3, 11.8.30]:  $K$  is  $\omega$ -free  $\Leftrightarrow K = \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$ .

The next lemma also mentions the differential field extension  $K[i]$  of  $K$  where  $i^2 = -1$ , as well as linear differential operators over differential fields like  $K$  and  $K[i]$ ; for this we refer to [3, Sections 5.1, 5.2].

**Lemma 2.5.** *Suppose  $K$  is an  $H$ -field,  $\Gamma \neq \{0\}$ ,  $r \geq 2$ , and  $K$  has  $r$ -DIVP. Then the following hold, with (i), (ii), (iii) using only the case  $r = 2$ :*

- (i)  $K = \omega(K) \cup \sigma(K^\succ) = \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$ ;
- (ii)  $K$  is  $\omega$ -free and  $\omega(K) = \omega(\Lambda(K))^\downarrow$ ;
- (iii) for all  $a \in K$  the operator  $\partial^2 - a$  splits over  $K[i]$ ;
- (iv)  $K$  is  $r$ -newtonian.

*Proof.* For (i) we use the end of [3, Section 11.7] to replace  $K$  with a compositional conjugate so that  $0 \in \Psi$ . Then  $K$  has small derivation, and we have  $a \in K^\succ$  such that  $a \neq 1$  and  $a^\dagger \asymp 1$ . Replacing  $a$  by  $a^{-1}$  if necessary this gives  $a^\dagger = -\phi$  with  $\phi \asymp 1$ ,  $\phi > 0$ , so  $a \prec 1$ . Then  $\phi^{-1}a^\dagger = -1$ ; replacing  $K$  by  $K^\phi$  and renaming



the latter as  $K$  this means  $a^\dagger = -1$ . Let  $f \in K$ ; to get  $f \in \omega(\Lambda(K))^\downarrow \cup \sigma(\Gamma(K))^\uparrow$ , note first that  $1 = (1/a)^\dagger \in \Gamma(K)$ , so  $0 \in \Lambda(K)$ . Also  $\omega(\Lambda(K))^\downarrow \subseteq \omega(K)$  by Lemma 2.3.

If  $f \leq 0$ , then  $\omega(0) = 0$  gives  $f \in \omega(\Lambda(K))^\downarrow$ . So assume  $f > 0$ ; we first show that then  $f \in \sigma(K^\succ)$ . Now for  $y \in K^\succ$ ,  $f = \sigma(y)$  is equivalent (by multiplying with  $y^2$ ) to  $P(y) = 0$ , where

$$P(Y) := 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in K\{Y\}.$$

See also [3, Section 13.7]. We have  $P(0) = 0$  and  $P(y) \rightarrow +\infty$  as  $y \rightarrow +\infty$  (because of the term  $y^4$ ). In view of 2-DIVP it will suffice to show that for some  $y > 0$  in  $K$  we have  $P(y) < 0$ . Now with  $y \in K^\succ$  and  $z := -y^\dagger$  we have

$$\begin{aligned} P(y) &= y^2(\sigma(y) - f) = y^2(\omega(z) + y^2 - f), \text{ hence} \\ P(a) &= a^2(\omega(1) + a^2 - f) = a^2(-1 + a^2 - f) < 0, \end{aligned}$$

so  $f \in \sigma(K^\succ)$ . By the second inclusion of Lemma 2.4 this yields  $f \in \omega(\Lambda(K))^\downarrow$  or  $f \in \sigma(\Gamma(K)^\dagger)$ . But we have  $\sigma(\Gamma(K)^\dagger) \subseteq \sigma(\Gamma(K))^\uparrow$ , because  $\sigma$  is increasing on  $\Gamma(K)^\dagger$  by the remark preceding [3, 11.8.30]. This concludes the proof of (i), and then (ii) follows, using for its second part also the fact stated just before [3, 11.8.29] that we have  $\omega(K) < \sigma(\Gamma(K))$ .

Now (iii) follows from  $K = \omega(K) \cup \sigma(K^\succ)$  by [3, Section 5.2, (5.2.1)]. As to (iv), let  $P \in K\{Y\}$  of order at most  $r$  have Newton degree 1; we have to show that  $P$  has a zero in  $\mathcal{O}$ . For this we repeat the argument in the proof of Lemma 2.3 so that it applies to our  $P$ , using  $\omega$ -freeness instead of  $\lambda$ -freeness, [3, Proposition 13.3.13] instead of [3, Proposition 13.3.6], and  $r$ -DIVP instead of 1-DIVP.  $\square$

**Corollary 2.6.** *If  $K$  is an  $H$ -field,  $\Gamma \neq \{0\}$ , and  $K$  has DIVP, then  $K$  is  $\omega$ -free and newtonian.*

There are non-Liouville closed  $H$ -fields with nontrivial derivation that have DIVP; see [2, Section 14]. By Lemma 2.3 and Lemma 2.5(iii), Liouville closed  $H$ -fields having 2-DIVP are *Schwarz closed* as defined in [3, Section 11.8].

**Theorem 2.7.** *Let  $K$  be a Liouville closed  $H$ -field. Then*

$$K \text{ has DIVP} \iff K \text{ is } \omega\text{-free and newtonian.}$$

*Proof.* The forward direction is part of Corollary 2.6. The backward direction is Corollary 0.1.  $\square$

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