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Adaptive Precision Sparse Matrix-Vector Product and its Application to Krylov Solvers

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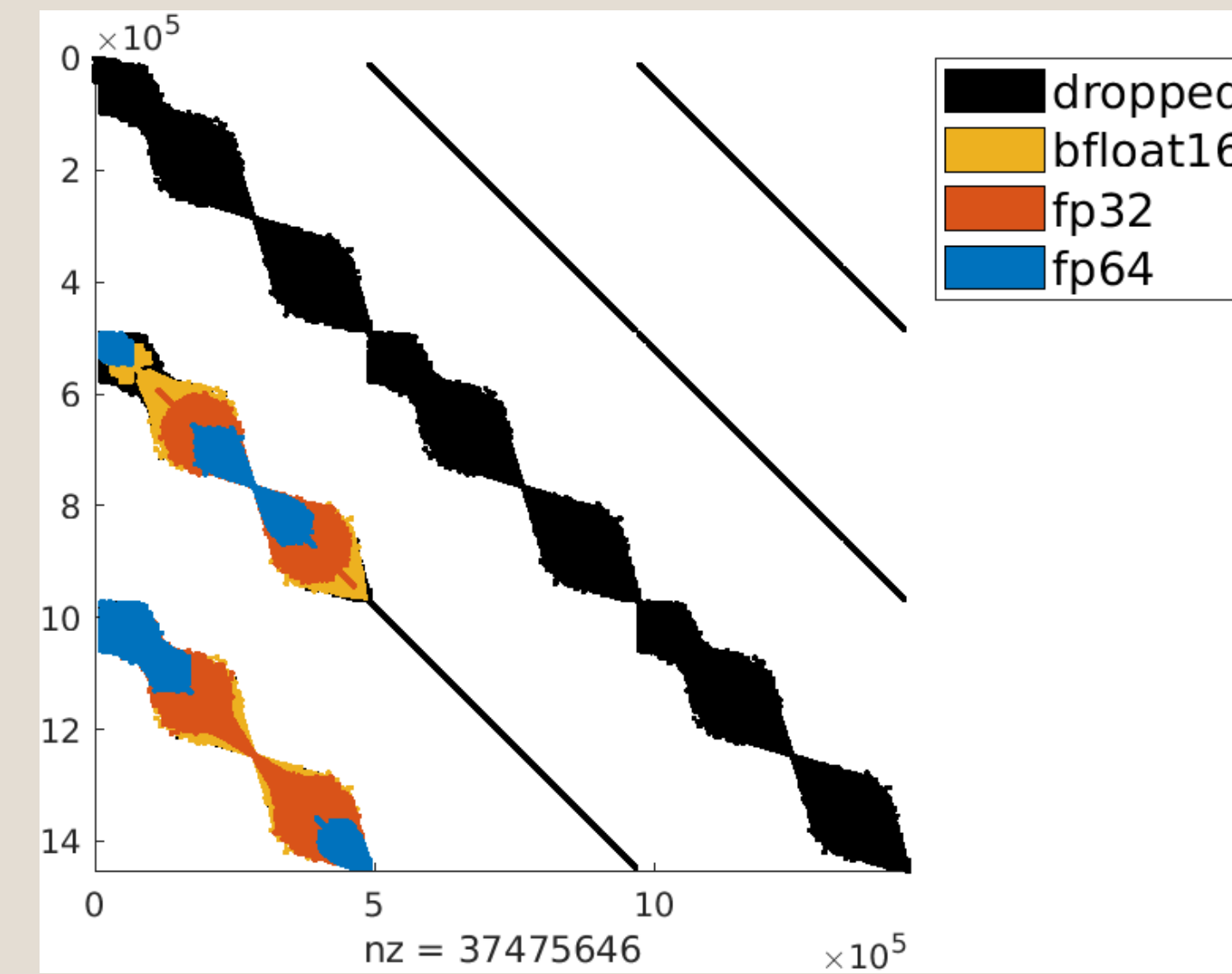


Today's floating-point landscape

	Bits		$u = 2^{-t}$	
	Signif. (t)	Exp. Range		
fp8-e4m3	4	3	$10^{\pm 2}$	1×10^{-1}
fp8-e5m2	5	2	$10^{\pm 5}$	3×10^{-1}
bfloat16	8	8	$10^{\pm 38}$	4×10^{-3}
fp16	11	5	$10^{\pm 5}$	5×10^{-4}
fp32	24	8	$10^{\pm 38}$	6×10^{-8}
fp64	53	11	$10^{\pm 308}$	1×10^{-16}
fp128	113	15	$10^{\pm 4932}$	1×10^{-34}

- Low precision increasingly supported
- **Great benefits:**
 - Reduced **storage**
 - Reduced **energy**
 - Increased **speed**
- **Some limitations:**
 - Low accuracy (large u)
 - Narrow range

An example



- Targeting an fp64 accuracy using precisions bfloat16, fp32 and fp64
- 66% elements dropped
 - 11% converted to bf16
 - 12% converted to fp32
 - only 11% stay in fp64

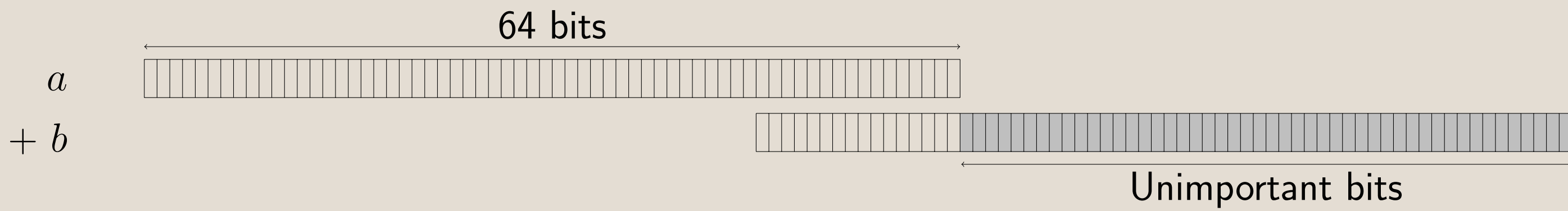
Mixed precision algorithms

Objectives:

- **Performance benefits** of low precisions
- **Accuracy and stability** of high precisions

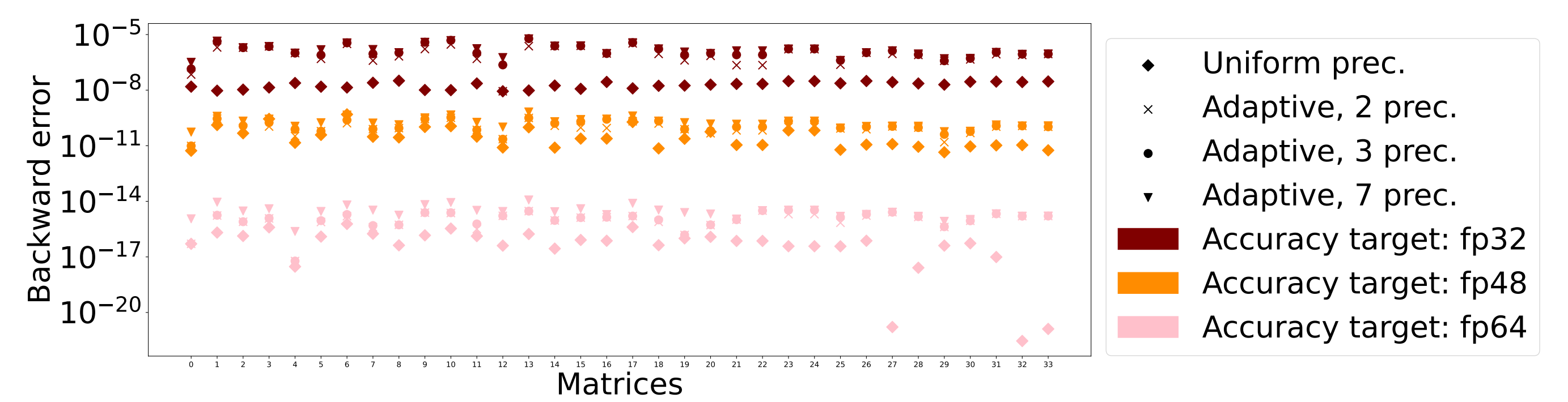
Opportunity for mixed precision:

⇒ **all computations are not equally "important"!**



⇒ We can **adapt the precisions to the data at hand**

Measured error



Adaptive methods achieve an **accuracy similar** to that of the uniform ones

Uniform vs adaptive precision SpMV

$$A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n$$

Uniform precision SpMV

```
for i = 1: m do
  y_i = 0
  for j in nnz_i(A) do
    y_i = y_i + a_ij x_j in precision ε
  end for
end for
```

$$|\hat{y}_i - y_i| \leq \#nnz_i(A) \varepsilon \sum_{j \in nnz_i(A)} |a_{ij} x_j|$$

Idea: For each row i of A , split the set of j -indices into q buckets B_{ik} and sum the corresponding elements a_{ij} in precision u_k

Adaptive precision SpMV

```
for i = 1: m do
  for k = 1: q do
    y_i^{(k)} = 0
    for j in B_{ik} do
      y_i^{(k)} ← y_i^{(k)} + a_ij x_j in precision u_k
    end for
  end for
  y_i = sum_{k=1}^q y_i^{(k)} in precision u_q
end for
```

$$|\hat{y}_i^{(k)} - y_i^{(k)}| \leq \#B_{ik} u_k \sum_{j \in B_{ik}} |a_{ij} x_j|$$

Theorem 1: Adaptive SpMV error bound

Given an accuracy target ϵ the buckets B_{ik} must be built as

$$B_{ik} = \{j \in nnz_i(A) : |a_{ij} x_j| \in (\epsilon \beta_i / u_{k+1}, \epsilon \beta_i / u_k]\}$$

to get

$$|\hat{y}_i - y_i| \leq n_i \epsilon \beta_i$$

with:

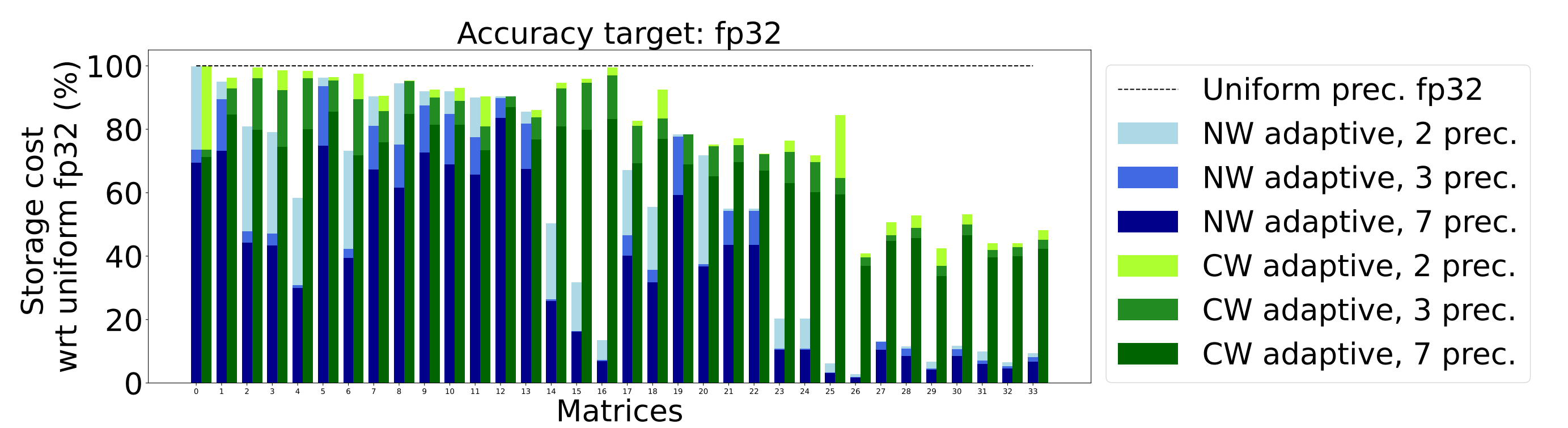
- $\beta_i = \sum_j |a_{ij} x_j|$ for **componentwise (CW) error**: $\forall i |\hat{y}_i - y_i| \leq O(\epsilon) \sum_j |a_{ij} x_j|$
- $\beta_i = \|A\| \|x\|$ for **normwise (NW) error**: $\|\hat{y} - y\| \leq O(\epsilon) \|A\| \|x\|$

Experimental settings

- 34 matrices from SuiteSparse collection and industrial partners with at most 166M non-zeros.
- Machine: Intel Xeon E5-2690v3, 24 cores @2.60GHz, 193Go Memory
- Different **accuracy targets**
 - fp32
 - fp48
 - fp64
- Different **sets of precision formats**
 - 2 precisions: fp32, fp64
 - 3 precisions: bfloat16, fp32, fp64
 - 7 precisions: bfloat16, fp24, fp32, fp40, fp48, fp56, fp64

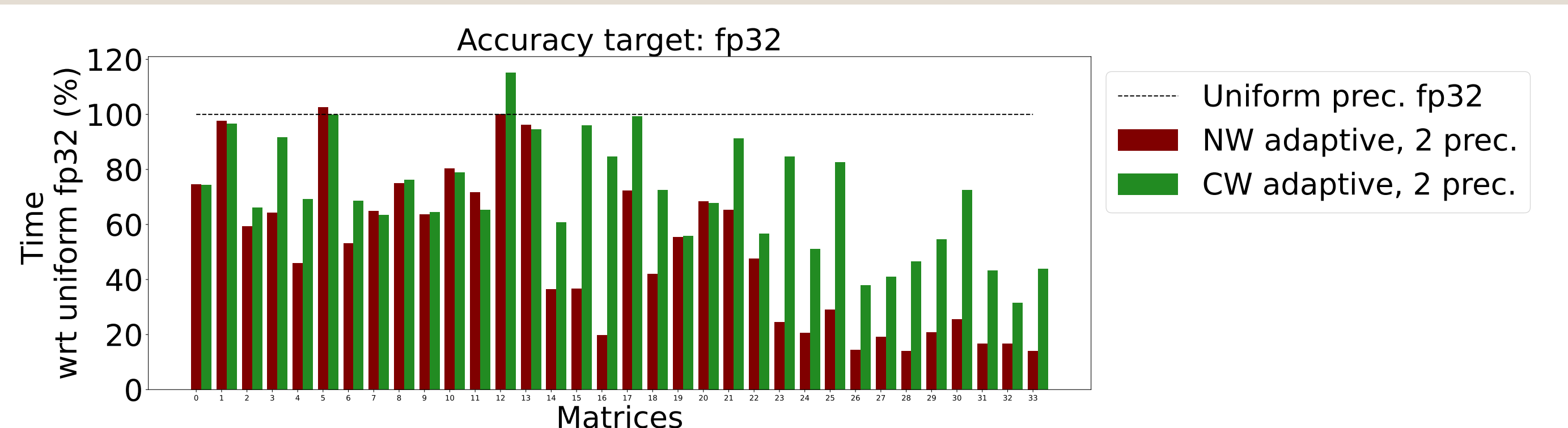
Emulated formats				
	Bits		$u = 2^{-t}$	
Format	Signif. (t)	Exponent Range		
fp24	16	8	$10^{\pm 38}$	2×10^{-5}
fp40	29	11	$10^{\pm 308}$	2×10^{-9}
fp48	37	11	$10^{\pm 308}$	8×10^{-12}
fp56	45	11	$10^{\pm 308}$	3×10^{-14}

Storage gains



Adaptive methods achieve **significant storage gains**, up to a factor **36x**

Time gains



Storage gains translate into **time gains**, up to a factor **7x**

Adaptive SpMV within GMRES

GMRES

```
r = b - Ax_0
beta = ||r||_2
q_1 = r / beta
for k = 1, 2, ... do
  y = Aq_k
  for j = 1: k do
    h_{jk} = q_j^T y
    y = y - h_{jk} q_j
  end for
  h_{k+1,k} = ||y||_2
  q_{k+1} = y / h_{k+1,k}
  Solve min_{c_k} ||HC_k - beta e_1||_2
  x_k = x_0 + Q_k c_k
end for
```

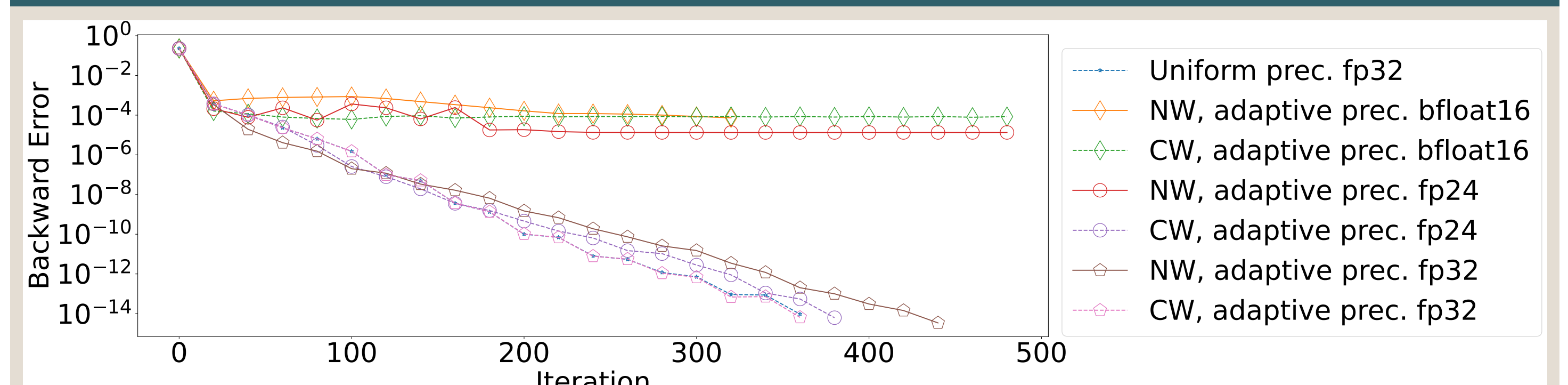
GMRES-based iterative refinement

```
for i = 1, 2, ... do
  r_i = b - Ax_{i-1} in high precision
  Solve Ad_i = r_i by GMRES in lower precision
  x_i = x_{i-1} + d_i
end for
```

The bottleneck of GMRES is SpMV

⇒ But **how does the adaptive method affect the convergence?**

Impact on GMRES convergence



For reasonable accuracy targets, adaptive SpMV **does not affect the convergence scheme**