# Transport equations in Hölder space by vanishing viscosity and applications

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**Abstract**

We obtain a sharp *limit Hölder continuity* of the solution for the transport equations thanks to a vanishing viscosity analysis. We also derive the same control for parabolic equations and for inviscid Burgers' equation.

Eventually, we provide existence and uniqueness of a Hölder continuous solution, under some structural hypothesis on the coefficients (possibly lying in a negative Besov space).

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# 1 Introduction

## 1.1 Statement of the problem

For a given $d \in \mathbb{N}$, we consider the following $d$-dimensional Cauchy problem:

$$\begin{cases} \partial_t u(t,x) + \langle b(t,x), \nabla u(t,x)\rangle = f(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = g(x),\ x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

We suppose that the transport coefficient, $b(t,\cdot)$, $t \in \mathbb{R}_+$, lies in the non-homogeneous Besov Hölder space $B^{-\beta}_{\infty,\infty}$, for a finite $\beta \in \mathbb{R}$.

One of our main result in this article, we derive a *limit Hölder continuity modulus* of the mollified and viscous version of the solution $u$ of (1.1), this is a generalisation of the control of [CJ23]. Under some structural assumptions on $b$, we show that the limit solution is unique if $-\beta \in (0,1)$ is large enough. The solution of transport equation (1.1) is approached with the second order parabolic equations whose second order term $\nu$, called viscosity, goes to 0,

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + \langle b_m(t,x), \nabla u^{m,\nu}(t,x)\rangle - \nu\Delta u^{m,\nu}(t,x) = f_m(t,x),\ (t,x) \in (0,T] \times \mathbb{R}^d, \\ u^{m,\nu}(0,x) = g_m(x),\ x \in \mathbb{R}^d; \end{cases} \tag{1.2}$$

the function $b_m$ is a mollified version of the distributional valued $b$, also $g_m$ and $f_m$ stand respectively for a mollified version of $g$ and $f$.

We write the solution as a perturbation of a PDE with constant components. These constants correspond to the first order term $b_m$ taking at a *freezing point* throw the corresponding flow, as done in [CDRHM18]. However, to estimate the *limit Hölder norm*, we have to distinguish two regimes, like for Schauder estimates in a parabolic context, the *diagonal* and the *off-diagonal* ones. In each regime, the choice of *freezing points* changes in order to get a negligible first order contribution when $\nu \to 0$, or with a time decomposition trick.

One of the crucial consequences of our analysis is a general meaning of a classic product of distributions, $\langle b(t,x), \nabla u(t,x)\rangle$, where $b$ and $\nabla u$ have negative regularity. This is written as a weak limit of a sub-sequence of a smooth parabolic approximation. The price to pay in this representation is

that we do not have usual uniqueness of the limit in this rough case.

Thanks to the techniques developed for the transport equation (1.1), we also succeed to extend our analysis to parabolic equation (1.2) without vanishing viscosity. Our result includes negative regularity for $b$ without use of rough path theory*.

The *a priori* estimates being independent on $b$, we are able to use a fixed-point argument to handle with the inviscid Burgers' equation

$$\begin{cases} \partial_t u(t,x) + u(t,x)\partial_x u(t,x) = f(t,x), \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = g(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

see *e.g.* [BK21] for more details on this equation. We deduce, some kind of uniqueness of the solution $u$ of (1.3) in case of the existence of Hölder continuous solution (true before a blow-up time). Precisely, on the one hand there is uniqueness of selection whatever the choice of vanishing viscosity, and on the other hand there is uniqueness whatever the way to mollify; nevertheless we do not succeed to get both uniquenesses for the same solution (which should imply usual uniqueness).

## 1.2 Existing results

### 1.2.1 On the transport equations

The Lipschitz framework is classic *via* the characteristic method. Indeed if $b \in L^\infty([0,T]; C^1(\mathbb{R}^d, \mathbb{R}^d))$, considering the ODE $\dot{X}_t = b(t, X_t)$, thanks to the Cauchy–Lipschitz theorem, there is a unique solution $u \in L^\infty([0,T]; C^1(\mathbb{R}^d, \mathbb{R}))$ of the transport equation (1.1). Out of this regular context, the analysis has to be more involved.

For instance, a meaning of the equation (1.1), when the coefficients are in a suitable Sobolev space, can be given by a renormalisation procedure developed by DiPerna and Lions [DPL89]. When $b$ is divergence free and Log-Lipschitz continuous, [BC94] establish that there is a unique solution with a loss of regularity in the Sobolev spaces. If $b \in L^1([0,T]; W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}))$ and $div(b) \in L^1([0,T]; L^\infty(\mathbb{R}^d, \mathbb{R}))$, they establish that the Cauchy problem (1.1) is well-posed in $L^\infty$.

When, $b$ is only supposed to have bounded variations in space, Ambrosio [Amb04] extends this result for $b \in L^1([0,T]; BV_{loc}(\mathbb{R}^d, \mathbb{R}))$, $div(b)_- \in L^1([0,T]; L^\infty(\mathbb{R}^d, \mathbb{R}))$.

For other references on transport equation in the non Lipschitz case, see for instance to [MS18], [Xia19].

If $b$ is only Hölder continuous then the Cauchy problem (1.1) is not well-posed any-longer, this is illustrated by the well-known counter-example

$$b(x) = \frac{1}{1-\gamma} sign(x)(|x| \wedge R)^\gamma, \ \gamma \in (0,1). \tag{1.4}$$

With a multiplicative noise, Flandoli Gubinelli and Priola [FGP10], see also [FGP12] and [MO17], establish that the following Stochastic Partial Differential Equation

$$\begin{cases} d_t u + \langle b, \nabla u \rangle dt + \nabla u \circ dW_t = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{1.5}$$

with $b \in L^\infty([0,T], C^\alpha_b)$ and $div(b) \in L^p$ is well-posed. Here, the symbol $\circ$ corresponds to the stochastic Stratonovich integral. This is a typical consequence of the regularisation by the noise, let us mention [FF13], [AF11], [Cat16].

This stochastic approach seems to be hopeless to get uniqueness by zero limit noise selection, indeed from [AF09] : if $b$ is defined as in (1.4), then the following equation

$$\partial_t u^\varepsilon + b \cdot \partial_x u^\varepsilon = \varepsilon \nabla u^\varepsilon \circ dW_t,$$

*see [Lyo98].

has a weak convergence of the corresponding probability $P^\varepsilon$ towards $\frac{\delta_{u_1}+\delta_{u_2}}{2}$, when $\varepsilon \to 0$, where $u_1$ and $u_2$ are two different solutions of the associated transport equation. Other counter-examples are stated in [Dep03].

In [CCS20], the authors show that there is $b \in L^p_{loc}$, $p \in [1, \frac{4}{3}]$, such that

$$\partial_t u^m + b_m \cdot \nabla u^m = 0,$$

where $b_m$ is a regularisation of $b$, for $d = 3$, and such that there is no uniqueness of bounded distributional solutions when $m \to +\infty$. The conclusion is the same, in [DLG22], for a compactly supported divergence-free vector field $b \in L^\infty$. To put it another way, there is no smooth selection principle by regularisation.

Finally, let us mention [CCS22], where the authors build a first order coefficients $b$ Hölder continuous such that the vanishing viscosity procedure from a parabolic approximation yields to several different solutions, uniqueness fails to be true in $L^2$. In spite of this counter-example, there is no contradiction with our uniqueness result, stated in Theorem 1, as we only consider uniqueness in a Hölder space, defined further. The non-uniqueness seems to occur only in non-smooth functional spaces. Then we are able to affirm that a selection principle by vanishing viscosity may happen, which is a partial positive answer to the question (Q3) in [CCS20].

We propose in this article a new approach to handle with the determinist transport equation (1.1), by *vanishing viscous solution*, see e.g. [Eva98].

For the best author's knowledge, the notion of vanishing viscosity has been already used for several classes of evolution PDEs, e.g. hyperbolic ones in [BB05], but it was not developed to establish the regularity control of the solution of a general transport equation. Finally, with this method, we also deduce existence and uniqueness of a Hölder continuous solution of the inviscid Burgers' equation.

### 1.2.2 On the parabolic equations

Historically, a *full* control of the parabolic equation (1.2) with bounded Hölder continuous coefficients, called Schauder estimates was first proved by Friedman [Fri64] thanks to a *parametrix* approach. Let us mention also the major references of the parabolic equation: Ladyzenskaja, Solonnikov and Ural'ceva [LSU68], Krylov [Kry96] and Lieberman [Lie96].

The first article handle with unbounded coefficients in (1.2) is due to Krylov and Priola [KP10]. In parabolic and elliptic framework, they get the parabolic bootstrap through Schauder estimates. Let us also mention [KKL75], for a first partial result in a unbounded context; when coefficients are "merely" measurable in time is handled in [Lor11].

In a degenerate framework, with some Hörmander conditions, Lunardi establishes, in her work [Lun97], Schauder estimates for a linear $b$. For a fully non-linear Hölder continuous drift, $b$, Chaudru de Raynal et al. [CDRHM18] get the corresponding controls, see also [Pri09]. To extend our result to a degenerate chain, the principal points would be to control the regularity gain of the flow $\boldsymbol{\theta}$ associated with $b$ through the chain which is not direct when the non-degenerate components of the drift are distributional valued. This regularity control is crucial as the *proxy* density does depend on this flow.

When $b$ is a tempered distribution, in particular when $b \in B^{-\beta}_{\infty,\infty}$ with $\beta < \frac{2}{3}$ some controls of the solution of (1.2) are established in [DD16] and [CC18], in order to build a "polymer measure", an important object to study some stochastic partial differential equations such as the Kardar-Parisi-Zhang equation.

In [Dan07] and [BCD11], when $b$ lies in a more general Besov space, $B^{-\beta}_{p,q}$, $p, q \neq +\infty$, some *a priori* estimates in Besov space is established for the solution of some parabolic equations. In particular, these controls are crucial in some approach to handle with the Navier-Stokes equation.

The paper is organised as following. The notations and the definitions used are gathered in Section 2. A first result about the transport equation under existence assumption is stated in Section 3. We provide a discussion on the consequence in term of distributions product in Section 3.2. The complete analysis is detailed in Section 4. In particular, we write an extension on the structural assumption (denoted by **(A)** in the following) in Section 4.10.2. Comments and results on the parabolic equation and as well as the full proof are featured in Section 5.

Our last result is stated and as well the proof of the regularity of a solution to the inviscid Burgers' equation in Section 6.

We gather in Section 7, some results about regularity controls on the solution of linear parabolic of second order and on the non-linear Burgers' case.

Also, in Appendix, some property of the Besov spaces are developed in Sections A-C. Precisely, in Section A, we establish that the space $C^\infty_b$ functions are dense in the space of multi-differentiated Hölder continuous functions. Some inequalities over the norm of Besov-Hölder distributions with their derivative are established in Section B. Also in Section C, we detail why the limit of regularised distribution in Besov-Hölder space does not depend on the choice of mollification procedure.

Eventually, some precise and long comments of the analysis are postponed in Section D.

# 2 Notations and Definitions

From now on, we denote by $C > 0$ and $c > 1$ generic constants that may change from line to line but only depends on known parameters such as $\gamma$, $d$. Importantly, these constants do not depend on $\beta$.

For $\varepsilon, \tilde{\varepsilon} > 0$, we use the usual notation of asymptotic domination:

$$\varepsilon \ll \tilde{\varepsilon}, \ \textit{if} \ \frac{\varepsilon}{\tilde{\varepsilon}} \longrightarrow 0. \tag{2.1}$$

We also write $\xrightarrow[(\varepsilon,\tilde{\varepsilon})\to(0,0)]{(2.1)}$ or $\lim\limits^{(2.1)}_{(\varepsilon,\tilde{\varepsilon})\to(0,0)}$, the limit, up to some subsequence selection, under the condition (2.1).

## 2.1 Tensor and Differential notations

For any $z \in \mathbb{R}^d$, we use the decomposition $z = z_1 e_1 + \dots z_d e_d$, where $(e_1, \dots, e_d)$ is the canonical base of $\mathbb{R}^d$.

We usually use the notation $\partial_t$ for the derivative in time $t \in [0, T]$ also $\partial_{z_k}$, $k \in \mathbb{N}$, is the derivative in the variable $z_k$.

The gradient in space is denoted by $\nabla$, in other words $\nabla = \partial_{z_1} e_1 + \dots + \partial_{z_d} e_d$, the divergence by $\nabla\cdot = div$ and is defined for any $\mathbb{R}$-function $f : \mathbb{R}^d \mapsto \mathbb{R}$ by $\nabla \cdot f = \sum_{k=1}^d \partial_{z_k} f$.

The symbol "$\cdot$" between two tensors is the usual tensor contraction. For example, if $M \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ and $N \in \mathbb{R}^d$ then $M \cdot N$ is a $d \times d$ matrix. If the two considered tensors are vectors then "$\cdot$" matches with the scalar product which is also denoted by $\langle \cdot, \cdot \rangle$.

For any $\mathbb{R}^d \mapsto \mathbb{R}$, we define the Hessian matrix $D^2_z f = \left(\partial_{z_i}\partial_{z_j} f\right)_{1\le i,j\le d}$, and the usual Laplacian operator $\Delta f = \sum_{1\le i\le d} \partial^2_{z_i} f$.

More generally, for any $k \in \mathbb{N}$, $D^k_z f$ denotes the order $k$ tensor $(\partial_{z_{i_1}} \dots \partial_{z_{i_k}} f)_{(i_1,\dots,i_k)\in[\![1,d]\!]^k}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d_0$, we write $D^\alpha_z f = \partial^{\alpha_1}_{z_1} \dots \partial^{\alpha_k}_{z_k} f$, in particular if, for $i \in [\![1, d]\!]$, $\alpha_i = 0$, there is no derivative in $z_i$ in the expression of $D^\alpha_z f$.

We also denote for any $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$, the order of this multi-index by $|\alpha| = \sum_{i=1}^m \alpha_i$.

## 2.2 Associated Hölder, Besov spaces

### 2.2.1 Hölder spaces

For any, $\tilde{\gamma} \in (0,1)$, $\|\cdot\|_{C^\delta(\mathbb{R}^m,\mathbb{R}^\ell)}$, $m \in \{1,d\}$, $\ell \in \{1,d,d\otimes d\}$[†] is the usual homogeneous Hölder norm, see e.g. Lunardi [Lun95] or Krylov [Kry96]. Precisely, for all $\psi \in C^\delta(\mathbb{R}^m,\mathbb{R}^\ell)$, we set the semi-norm:

$$\|\psi\|_{C^\delta} = [\psi]_\delta := \sup_{(x,y)\in(\mathbb{R}^m)^2, x\neq y} \frac{|\psi(x)-\psi(y)|}{|x-y|^\delta}, \tag{2.2}$$

the notation $|\cdot|$ is the Euclidean norm on the considered space. We denote by:

$$C_b^\delta(\mathbb{R}^m,\mathbb{R}^\ell) := \{\psi \in C^\delta(\mathbb{R}^m,\mathbb{R}^\ell) : \|\psi\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^\ell)} < +\infty\},$$

the associated subspace with bounded elements (non-homogeneous Hölder space). The corresponding Hölder norm is defined by:

$$\|\psi\|_{C_b^\delta(\mathbb{R}^m,\mathbb{R}^\ell)} := \|\psi\|_{C^\delta(\mathbb{R}^m,\mathbb{R}^\ell)} + \|\psi\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^\ell)}. \tag{2.3}$$

For the sake of notational simplicity, from now on we write:

$$\|\psi\|_{L^\infty} := \|\psi\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^\ell)},\ \|\psi\|_{C^\delta} := \|\psi\|_{C^\delta(\mathbb{R}^d,\mathbb{R}^\ell)},\ \|\psi\|_{C_b^\delta} := \|\psi\|_{C_b^\delta(\mathbb{R}^d,\mathbb{R}^\ell)}.$$

For time dependent functions, $\varphi_1 \in L^\infty\big([0,T],C_b^\delta(\mathbb{R}^m,\mathbb{R}^\ell)\big)$ and $\varphi_2 \in L^\infty\big([0,T],C^\delta(\mathbb{R}^m,\mathbb{R}^\ell)\big)$, we define the norms:

$$\begin{aligned}
\|\varphi_1\|_{L^\infty(C_b^\delta)} &:= \sup_{t\in[0,T]} \|\varphi_1(t,\cdot)\|_{C_b^\delta(\mathbb{R}^m,\mathbb{R}^\ell)},\\
\|\varphi_2\|_{L^\infty(C^\delta)} &:= \sup_{t\in[0,T]} \|\varphi_2(t,\cdot)\|_{C^\delta(\mathbb{R}^m,\mathbb{R}^\ell)},\\
\|\varphi_2\|_{L^\infty} &:= \sup_{t\in[0,T]} \|\varphi_2(t,\cdot)\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^\ell)}.
\end{aligned}$$

The test functions for some weak formulations of different solutions will be in $C_0^\infty(\mathbb{R}^d,\mathbb{R})$, which corresponds to the space of smooth functions infinitely differentiable with bounded derivatives and with a compact support.

For any sequence of functions $(\psi^{m,\varepsilon})_{m,\varepsilon>0}$ lying in $C^\delta(\mathbb{R}^d,\mathbb{R})$, we denote, for any $\eta>0$ by

$$[\psi^{m,\varepsilon}]_{\delta,\eta} := \sup_{(x,y)\in(\mathbb{R}^m)^2, |x-y|\leq\eta} \frac{|\psi^{m,\varepsilon}(x)-\psi^{m,\varepsilon}(y)|}{|x-y|^\delta}, \tag{2.4}$$

the local Hölder modulus without the limit.

For any function $F:\mathbb{R}_+^2\to\mathbb{R}_+$ s.t.

$$F(\varepsilon,m) \ll 1, \tag{2.5}$$

and similar to [GT83] and [CJ23], we introduce the *limit Hölder continuity* as the modulus of continuity defined by

$$[\psi^{m,\varepsilon}]_{\delta,\eta}^{(2.5)} := \overset{(2.5)}{\lim_{(m,\varepsilon)\to(+\infty,0)}} \sup_{(x,y)\in(\mathbb{R}^m)^2, |x-y|\leq\eta} \frac{|\psi^{m,\varepsilon}(x)-\psi^{m,\varepsilon}(y)|}{|x-y|^\delta}. \tag{2.6}$$

**Remark 1.** *This semi-norm reveals a partial regularity of a limit function, when $\varepsilon\to 0$. The condition $|x-y|\leq\varepsilon$ allows to avoid some potential singularities arising; without this criterion, we fail to use the cut-locus trick and so to prove the a priori control of the Hölder modulus, where obviously*

$$\max([\psi^{m,\varepsilon}]_{\delta,0}^{(2.5)}, [\psi^{m,\varepsilon}]_{\delta,\eta}) \leq [\psi^{m,\varepsilon}]_\delta.$$

[†] we write $\mathbb{R}^{d\otimes d}$ for $\mathbb{R}^d\otimes\mathbb{R}^d$ the space of square matrices of size $d$.

The above modulus, can be replaced through all the article by a fractional derivative defined, for any $\gamma \in (0,1)$, for all $\varphi \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$ and $x \in \mathbb{R}^d$:

$$D^\gamma \varphi(x) := \lim_{h \to 0} \sum_{k=0}^{+\infty} \binom{\gamma}{k} \frac{\varphi(x) - \varphi(x - kh)}{h^\gamma}, \tag{2.7}$$

with

$$\binom{\gamma}{k} := \frac{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-k+1)}{k!}, \tag{2.8}$$

the generalised binomial coefficient.

This operator is called the Grünwald-Letnikov derivative, see for instance [Die10].

### 2.2.2 Thermic characterization of the Besov space

We define the Besov spaces thanks to a thermic characteristic, see Triebel [Tri83] Section 2.6.4. For all $\alpha \in \mathbb{R}$, $q \in (0, +\infty]$, $p \in (0, \infty]$,

$$\|f\|_{B^\alpha_{p,q}} := \|\varphi(D)f\|_{L^p(\mathbb{R}^d)} + \|f\|_{\ddot{B}^\alpha_{p,q}}, \text{ with } \|f\|_{\ddot{B}^\alpha_{p,q}} := \Big( \int_0^1 \frac{dv}{v} v^{(m-\frac{\alpha}{2})q} \|\partial_v^m h_v \star f\|^q_{L^p(\mathbb{R}^d)} \Big)^{\frac{1}{q}}, \tag{2.9}$$

where we define the heat kernel

$$h_v(z) := \frac{1}{(2\pi v)^{d/2}} \exp\left( -\frac{|z|^2}{2v} \right), \tag{2.10}$$

and $\varphi(D)f := (\varphi \hat{f})^\vee$ with $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(0) \neq 0$, $\hat{f}$ and $(\varphi \hat{f})^\vee$ respectively denote the Fourier transform of $f$ and the inverse Fourier transform of $\varphi \hat{f}$. Note that, when $\alpha > d(\frac{1}{p} - 1)_+ = d \max(0, \frac{1}{p} - 1)$ then in (2.9), it possible to replace $\|\varphi(D)f\|_{L^p(\mathbb{R}^d)}$ by $\|f\|_{L^p(\mathbb{R}^d)}$.

When $p = q = +\infty$, we naturally write:

$$\|f\|_{\ddot{B}^\alpha_{p,q}} = \sup_{v \in [0,1]} v^{m - \frac{\alpha}{2}} \|\partial_v^m h_v \star f\|_{L^\infty(\mathbb{R}^d)},$$

and if $\alpha > d(\frac{1}{p} - 1)_+$,

$$\|f\|_{B^\alpha_{p,q}} = \|f\|_{L^\infty} + \sup_{v \in [0,1]} v^{m - \frac{\alpha}{2}} \|\partial_v^m h_v \star f\|_{L^\infty(\mathbb{R}^d)}.$$

We carefully point out that the homogeneous term $\|f\|_{\ddot{B}^\alpha_{p,q}}$ does not define a norm associated to a Banach space. To consider the whole homogeneous Besov space we have to consider $v \in \mathbb{R}_+$ in the definition (2.9), for $\alpha < 0$, see e.g. Theorem 2.34 in [BCD11]. Somehow, for the inhomogeneous norm defined in (2.9), the contribution of the heat kernel convolution for $v > 1$ is "hidden" in the inhomogeneous term $\|\varphi(D)f\|_{L^p(\mathbb{R}^d)}$.

For $\alpha > 0$, the homogeneous and respectively inhomogeneous Hölder spaces match with Besov space, namely $C^\alpha = \dot{B}^\alpha_{\infty,\infty}$ and $C^\alpha_b = B^\alpha_{\infty,\infty}$, see [Tri83] for details.

Our analysis tackles with inhomogeneous Besov spaces, in order to extend our analysis to the homogeneous ones some sophisticated changes should be performed as the homogeneous Besov spaces are *a priori* not Banach spaces; and we should consider the realisation of the space of homogeneous Besov spaces as a space of distributions defined quotiented by polynomials, see e.g. Proposition 3.8 in [LR02] to make it a Banach space.

If $\alpha < 0$ then it is known that $\dot{B}^\alpha_{p,q} \subset B^\alpha_{p,q}$, i.e. there is a constant $C > 0$ such that, for any $\alpha < 0$,

$$\| \cdot \|_{B^\alpha_{p,q}} \leq -\frac{C}{\alpha} \| \cdot \|_{\dot{B}^\alpha_{p,q}}. \tag{2.11}$$

We also introduce the distributions that can be approached by a mollification procedure. We put a *tilde* in order to mean that we consider the closure of $C_b^\infty$ in the considered space[‡]. Namely, for any $(\alpha, p, q) \in \mathbb{R} \times (1, +\infty] \times (1, +\infty]$ we define:

$$\tilde{B}^\alpha_{p,q} := \mathbf{cl}_{B^\alpha_{p,q}}(C_b^\infty),\ \tilde{\dot{B}}^\alpha_{p,q} := \mathbf{cl}_{\dot{B}^\alpha_{p,q}}(C_b^\infty). \tag{2.12}$$

**Remark 2.** *If there is $\psi \in C^\gamma$, $\gamma \in (0,1)$, such that $b = D^\alpha\psi$, $\alpha \in \mathbb{N}_0^d$, so $b \in \dot{B}^{-\beta}_{\infty,\infty}$ with $-\beta = -|\alpha| + \gamma$, and in Appendix Section A, we show that $b \in \tilde{\dot{B}}^{-\beta}_{\infty,\infty}$.*

*The last constraint on b, being the derivative of a Hölder function, is quiet natural when we consider the structure theorem of the tempered distributions $\mathcal{S}'$, see Theorem 8.3.1 in [Fri98]. We recall indeed that any $b \in \mathcal{S}'$ writes $b = D^\alpha\psi$ where $\alpha \in \mathbb{N}_0^d$ and $\psi$ is a continuous function with polynomial growth.*

*Out of the Besov-Hölder space, namely if $1 \le p, q < +\infty$, then $\tilde{B}^\alpha_{p,q} = B^\alpha_{p,q}$ and $\tilde{\dot{B}}^\alpha_{p,q} = \dot{B}^\alpha_{p,q}$, see Theorem 4.1.3 in [AH96], Proposition 2.27 and Proposition 2.74 in [BCD11]. For more Besov properties, we also mention [Pee76] and [Jaw77].*

*We might consider the non-homogeneous low-frequency cut-off in the Littlewood-Paley characterisation instead of usual mollification by convolution, as performed in the current paper, and adapt Lemma 2.73 in [BCD11] for the space $B^{-\beta}_{\infty,\infty}$.*

### 2.2.3 Besov duality

In our analysis, we thoroughly use the Besov duality. The full proof of the duality of Besov spaces is established for example in Proposition 3.6 in [LR02] thanks to a Littlewood-Paley decomposition.

**Proposition 1.** *For all $1 \le p, q \le +\infty$ and $\alpha \in \mathbb{R}$, we have for all $\varphi, \psi \in \mathcal{S}'$:*

$$\Big| \int_{\mathbb{R}^d} \varphi(y)\psi(y)dy \Big| \le C_{d,p,q,\alpha} \|\varphi\|_{B^\alpha_{p,q}} \|\psi\|_{B^{-\alpha}_{p',q'}},$$

*with $1 \le p', q' \le +\infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

*Sketch of the proof.* Let us suppose that w.l.o.g. that $1 < p, q < +\infty$ (the analysis is identical if we suppose that $1 < p', q' < +\infty$). It is known that $B^\alpha_{p,q}(\mathbb{R}^d, \mathbb{R})$ and $B^{-\alpha}_{p',q'}(\mathbb{R}^d, \mathbb{R})$ are in duality (Proposition 3.6 in [LR02]). Precisely, $B^\alpha_{p,q}$ is the dual of the closure of the Schwartz class $\mathcal{S}$ in $B^{-\alpha}_{p',q'}$. But $\mathcal{S}$ is dense in $B^\alpha_{p,q}$ (see for instance 4.1.3. in [AH96]).

□

It is possible to adapt this result to homogeneous spaces, see for instance Proposition 2.29 in [BCD11].

### 2.2.4 Usual tools for the Gaussian function

One of the reason to use the thermic representation of the Besov space comes from a well-known and important result about the Gaussian function: for any $\delta > 0$, there is $C_\delta = C_\delta(\delta) > 1$ such that:

$$\forall x \in \mathbb{R}^d,\ |x|^\delta e^{-|x|^2} \le C_\delta e^{-C_\delta^{-1}|x|^2}. \tag{2.13}$$

Furthermore, we often use the cancellation principle: for all $f \in C^\gamma$, $\gamma \in (0,1)$, $x \in \mathbb{R}^d$ and $\sigma > 0$

$$\nabla_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} f(y)dy = \int_{\mathbb{R}^d} \nabla_x e^{-\frac{|x-y|^2}{2\sigma}} [f(y) - f(x)]dy, \tag{2.14}$$

[‡]As in Proposition 3.6 in [LR02] for the closure Schwartz space, but we do not need the mollified versions of the considered distributions to be rapidly decreasing functions.

as the Gaussian function, up to a renormalisation by a multiplicative constant, is a probabilistic distribution, hence $\nabla_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} dy = 0$. Hence, we obtain,

$$\begin{aligned}(2\pi\sigma)^{\frac{d}{2}}\Big|\nabla_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} f(y)dy\Big| &\leq (2\pi\sigma)^{\frac{d}{2}}[f]_\gamma \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} \frac{|y-x|}{\sigma}|y-x|^\gamma dy \\ &\leq C_\gamma(2\pi\sigma)^{\frac{d}{2}}[f]_\gamma \sigma^{\frac{\gamma-1}{2}} \int_{\mathbb{R}^d} e^{-C_\gamma^{-1}\frac{|x-y|^2}{2\sigma}} dy \\ &= C_\gamma[f]_\gamma \sigma^{\frac{\gamma-1}{2}}.\end{aligned}$$

The penultimate identity comes from the absorbing property (2.13).

### 2.3 Different definitions of a solution to the transport equation problem

As said in the introduction, we need to carefully defined the suitable notion of solution, no strong solution can be in a negative Besov space or even in a Hölder space, implying a product of distributions which is obviously not point-wisely defined.

**Definition 1** (mild vanishing viscous)**.** *A function $u$ is said to be a mild vanishing viscous solution in $L^\infty\big([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R})\big)$ of equation (1.1) if for a sequence $(b_m)_{m\in\mathbb{N}}$ in $L^\infty([0,T];C_b^\infty(\mathbb{R}^d,\mathbb{R}^d))$ such that there is $\beta\in\mathbb{R}$,*

$$\forall \varepsilon>0,\ \lim_{m\to+\infty}\|b_m-b\|_{L^\infty([0,T];B^{-\beta-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))}=0, \tag{2.15}$$

*for any $t\in[0,T]$, there exists a sub-sequence of $(u^{m,\nu}(t,\cdot))_{(m,\nu)\in\mathbb{R}_+^2}$ lying in $C_b^\gamma(\mathbb{R}^d,\mathbb{R})$ converging in the space $C_b^{\gamma-\tilde\varepsilon}(K,\mathbb{R})$, $0<\tilde\varepsilon<\gamma$, for any compact subset $K\subset\mathbb{R}^d$, when $\nu\to 0$ and $m\to+\infty$ towards $u(t,\cdot)\in C_b^\gamma(K,\mathbb{R})$ and satisfying, for all $m\in\mathbb{N}$ and $\nu\in\mathbb{R}_+$,*

$$\begin{cases}\partial_t u^{m,\nu}(t,x)+\langle b_m(t,x),\nabla u^{m,\nu}(t,x)\rangle-\nu\Delta u^{m,\nu}(t,x)=f_m(t,x),\ (t,x)\in(0,T]\times\mathbb{R}^d,\\ u^{m,\nu}(0,x)=g_m(x),\ x\in\mathbb{R}^d,\end{cases} \tag{2.16}$$

where $(f_m,g_m)\underset{m\to+\infty}{\longrightarrow}(f,g)$ in $L^\infty([0,T];C^\gamma(\mathbb{R}^d,\mathbb{R}))\times C^\gamma(\mathbb{R}^d,\mathbb{R})$.

We point out that such a sequence $(b_m)_{m\geq0}$ exists if $b\in L^\infty([0,T];\tilde B^{-\beta}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$; or in particular if $b$ is the derivative of a bounded Hölder continuous function but in this former case the limit result (2.15) has to be in the homogeneous space $L^\infty([0,T];\dot B^{-\beta}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$, see Appendix Section A, and identity (2.15) is implied by (2.11).

Moreover, it is important to notice that the choice of sub-sequence may depend on the current time $t$. We do not succeed in getting uniform continuity in time $t$ (only boundedness in $L^\infty$), thus the impossibility to apply a suitable compact argument in time, we need to consider the problem at a fixed $t$. Nevertheless, for the sake of simplicity we write for the sub-sequence $u^{m,\nu}(t,\cdot)$ instead of a notation of the kind $u^{m_t,\nu_t}(t,\cdot)$.

**Remark 3.** *We could consider another formulation of mild vanishing viscous solution, where the considered function is*

$$w^{m,\nu}(t,x)=\int_0^t u^{m,\nu}(s,x)ds. \tag{2.17}$$

*Under structural assumption and the result stated in Theorem 2 below, we have that $(t,x)\mapsto w^{m,\nu}(t,x)$ lies, uniformly in $(m,\nu)$, in $C_b^1([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R}))$, by the Arzelà-Ascoli theorem, we obtain a convergence in all compacts $[0,T]\times K$ of $[0,T]\times\mathbb{R}^d$ towards a function $w\in C_b^1([0,T];C_b^\gamma(K,\mathbb{R}))$.*

Let us define an alternative form of solution which is a mixed version between *mild* and *weak* notions of solutions.

**Definition 2** (mild-weak solution)**.** *A function $u$ is a mild-weak solution in $L^\infty([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R}))$ of equation* (1.1) *if $u$ is a mild vanishing viscous solution, and such that for any function $\varphi \in C_0^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$, we have, up to a sub-sequence selection, for any $t\in[0,T]$,*

$$\lim_{m\to\infty,\nu\to 0}\int_{\mathbb{R}^d}\Big\{\varphi(t,y)u^{m,\nu}(t,y)+\int_0^t\big\{-\partial_t\varphi(s,y)u^{m,\nu}(s,y)+\langle b_m(s,y),\nabla u^{m,\nu}(s,y)\rangle\varphi(s,y)\big\}ds\Big\}dy$$
$$=\int_{\mathbb{R}^d}\varphi(0,y)g(y)dy+\int_{\mathbb{R}^d}\int_0^t\varphi(s,y)f(s,y)ds\,dy. \tag{2.18}$$

The distributional formulation allows to give a sense to the potential irregularities of $b$ and of $\nabla u^{\nu,m}$ when $\nu\to 0$, $m\to+\infty$, and to consider the whole space $[0,T]\times\mathbb{R}^d$ (instead of any compact), the cut-off of $\mathbb{R}^d$ required for compact argument is included in the test function $\varphi$.

**Definition 3** (weak solution)**.** *A function $u$ is a weak solution in $L^\infty\big([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R})\big)$ of equation* (1.1) *if $u$ is a mild vanishing viscous solution and satisfies, for any function $\varphi\in C_0^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$,*

$$\begin{aligned}&\int_{\mathbb{R}^d}\Big\{\varphi(t,y)u(t,y)+\int_0^t\big\{-\partial_t\varphi(s,y)u(s,y)+\langle b(s,y),\nabla u(s,y)\rangle\varphi(s,y)\big\}ds\Big\}dy\\ =\;&\int_{\mathbb{R}^d}\varphi(0,y)g(y)dy+\int_{\mathbb{R}^d}\int_0^t\varphi(s,y)f(s,y)ds\,dy.\end{aligned} \tag{2.19}$$

**Remark 4.** *We cannot hope to define classic solution in our irregular context. Indeed, even if roughly speaking $\partial_t u+\langle b,\nabla u\rangle$ is supposed to lie in $L^\infty(C_b^\gamma)$, we cannot a priori define point-wisely the classic scalar product between $b$ and $\nabla u$. If $b$ has a blow up at a point $x_0\in\mathbb{R}^d$ then, as $u$ is solution of* (2.16)*, $\lim_{x\to x_0}\int_0^t\langle b(s,x),\nabla u(s,x)\rangle ds$ is necessary finite for any $t\in[0,T]$ but we cannot give a meaning of $\langle b(t,x_0),\nabla u(t,x_0)\rangle$ in a point-wise sense. Roughly speaking, to handle distributional drift we have to stay in a distributional formulation of the solution.*

**Definition 4** (Uniqueness)**.** *There is a unique solution of* (1.1) *if for two solutions $u^{m,\nu}$ and $\bar u^{m,\bar\nu}$ of*

$$\begin{cases}\partial_t u^{m,\nu}(t,x)+\langle b_m(t,x),\nabla u^{m,\nu}(t,x)\rangle-\nu\Delta u^{m,\nu}(t,x)=f_m(t,x),\;\;(t,x)\in(0,T]\times\mathbb{R}^d,\\ u^{m,\nu}(0,x)=g_m(x),\;\;x\in\mathbb{R}^d,\end{cases}$$

*and respectively*

$$\begin{cases}\partial_t \bar u^{m,\bar\nu}(t,x)+\langle \bar b_m(t,x),\nabla \bar u^{m,\bar\nu}(t,x)\rangle-\bar\nu\Delta \bar u^{m,\bar\nu}(t,x)=f_m(t,x),\;\;(t,x)\in(0,T]\times\mathbb{R}^d,\\ \bar u^{m,\bar\nu}(0,x)=g_m(x),\;\;x\in\mathbb{R}^d,\end{cases}$$

*for $\nu,\bar\nu>0$, with*

$$\forall\varepsilon>0,\;\lim_{m\to+\infty}\|b_m-b\|_{L^\infty([0,T];B^{-\beta-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))}+\lim_{n\to+\infty}\|\bar b_n-b\|_{L^\infty([0,T];B^{-\beta-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))}=0, \tag{2.20}$$

*converging, up to sub-sequence selection, towards two Hölder continuous solutions $u$, $\bar u$, when $(m,\nu,\bar\nu)\to(+\infty,0,0)$, then $u=\bar u$.*

The mollification procedure associated with $\bar b_n$ can be different from the one performed for $b_m$.

# 3 Statement under existence assumption

When $b$ lies in a Hölder-Besov space, we succeed in obtaining a *maximum principle* like for the *limit Hölder continuity*, matching the regularity of $f$ and $g$. The type of solution strongly depends on the regularity of $b$, however to be sure that our approximation converges towards a true Hölder continuous solution, we need to introduce an assumption framework.

**(A)** We suppose that there is a function $\psi : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d,\ b(t,x) = (\nabla \psi(x))^{-1},$$

satisfying

$$\forall (\tau, x) \in [0,T] \times \mathbb{R},\ \frac{\nabla \psi(x)}{\nabla \psi\big(\psi^{-1}(c\tau + \psi(x))\big)} < +\infty,$$

and

$$\psi_m^{-1}\big(C(t-s) + \psi_m(x)\big) \underset{m \to +\infty}{\longrightarrow} \psi^{-1}\big(C(t-s) + \psi(x)\big) \text{ in } C^1(\mathbb{R}^d, \mathbb{R}^d),$$

with $\psi_m$ a mollified version of $\psi$, e.g. $\psi_m := \rho_m \star \psi$.

For more details, see Section 3.1 further, in particular, we develop the case $\psi(x) = |x|^{1+\beta}$ in Section 3.1.2.

We could suppose that $b$ does depend on the time $t$, if $b$ is smooth enough in time, thanks to the time decomposition performed in Section 4.9. The only use of this assumption is to get the convergence of the solution $u^{m,\nu}$ toward $u$ in a suitable Hölder space.

**Theorem 1** (Rough transport equation in Hölder spaces)**.** *For $\beta \in \mathbb{R}^*$ and $0 < \gamma < 1$ be given, let $f \in L^\infty([0,T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}^d, \mathbb{R})$. For a distribution $b \in L^\infty([0,T], \tilde{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d))$, then for $u^{m,\nu}$ defined in* (2.16) *we have then,*

$$\begin{aligned} \lim_{n \to +\infty} \sup_{t \in [0,T]} [u^{m,\nu}(t,\cdot)]_{\gamma,(\nu T/n)^{1/2}} &\leq \lim_{n \to +\infty} \int_0^T [f(t,\cdot)]_{\gamma,(\nu T/n)^{1/2}} + \lim_{n \to +\infty} [g]_{\gamma,(\nu T/n)^{1/2}}, \\ \|u^{m,\nu}\|_{L^\infty} &\leq T\|f\|_{L^\infty} + \|g\|_{L^\infty}. \end{aligned} \tag{3.1}$$

*Moreover under if* **(A)** *is in force there is a mild vanishing viscosity solution $u \in L^\infty([0,T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of* (1.1) *and*

*i)* ***Incompressibility.*** *If $\beta < \gamma$ and $\nabla \cdot b = 0$ then the solution $u$ is also a weak and a mild-weak solution.*

*ii)* ***Positive regularity.*** *If $\beta < -1 + \gamma$, namely if $b \in L^\infty([0,T]; C_b^\beta(\mathbb{R}^d, \mathbb{R}^d))$, $\beta = -\tilde{\gamma} > 1 - \gamma$, the solution $u$ is also a weak solution and if we also suppose that*

$$\nu \|\nabla^2 g_m\|_{L^\infty} + \nu^{\frac{\gamma}{2}} m^{1-\tilde{\gamma}} \ll 1, \tag{3.2}$$

*then $\partial_t u(t,\cdot) \in B^{-1+\gamma}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R})$, for any $t \in (0,T]$.*

*iii)* ***Greater regularity.*** *If $\beta = -\tilde{\gamma} > \frac{1}{1+\gamma}$, and if*

$$\nu \|\nabla^2 g_m\|_{L^\infty} + m^{1-2\tilde{\gamma}} \nu^{\frac{\gamma-1}{2}} + m^{1-\tilde{\gamma}} \nu^{\frac{\gamma}{2}} \ll 1, \tag{3.3}$$

*the solution is unique*[§]*.*

Except for the control of the time derivative and for uniqueness, there is no condition on the vanishing viscosity unlike in Theorem 2. This is due to the possibility to use the time decomposition trick which allows to get estimates independent on $b$, see Section 4.10.2.

**Remark 5.** *We derive a limit Hölder modulus in a more general setting than [CJ23] and [DEL23] where $f$ is supposed to be null, and the consider modulus is a* log*-Hölder and an Osgood modulus, see also [DE23]. Let us insist that estimate* (3.1) *does not depend on the condition* **(A)** *which only allows to consider the limit solution $u^{m,\nu}$ in a Hölder space.*

[§]Namely the limit solution does not depend on the choice of sequence $(m,\nu)$ and on the way to mollify $b$.

**Remark 6.** *Importantly, the above controls* (3.1) *do not depend on the drift b; which seems to be paradoxical with the usual gradient control of the transport equation by characteristic method. This independence on b is crucial to consider very rough coefficients as well as non-linear equation such as the inviscid Burgers' equation studied in Section 6.*

*While for the usual weak solution, there is no more such infinite regularisation effect. The* ***Incompressibility*** *framework, i.e.* $\nabla \cdot b = 0$*, allows to still consider a negative regularity of b. Such divergence free condition for non-smooth distributions already exists for instance for Leray's solution of Navier-Stokes equation [Ler34].*

*In the last case, i.e.* ***Positive regularity****, the considered drift b is supposed to be Hölder continuous in space, in particular lying in* $L^\infty([0,T]; C_b^\alpha(\mathbb{R}^d, \mathbb{R}^d))$*,* $\alpha > 1-\gamma$ *which is the Bony's para-product assumption, see Section 3.2 below for more details.*

*We even obtain uniqueness for the* ***Greater regularity*** *case, we use a kind of regularisation by turbulence which makes negligible the second order term. The condition on g means that the initial function has to be smooth enough such that* $\nu\|\nabla^2 g_m\|_{L^\infty}$ *is negligible, obvious if* $g \in C^2(\mathbb{R}^d, \mathbb{R})$*. Again, this a partial positive answer to the question (Q3) in [CCS20], but we fail to consider uniqueness in a rough (negative regularity) case, see Remark 7 further.*

*In this Hölder case, the product* $\langle b, \nabla u\rangle$ *falls into the usual case of the para-product by the Bony's microlocal analysis [Bon81]. In the negative regularity case, we again obtain a result on the limit Hölder modulus of the product of distributions.*

For the sake of completeness, we introduce some examples of coefficient $b$ where the existence of solution $u \in L^\infty([0,T], C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d))$ is granted. In particular, in the following section, we detail a framework taking into account the Coulombian interactions.

## 3.1 The flow $\theta_{s,t}$ under hypothesis (A)

### 3.1.1 A structural example in $\mathbb{R}^d$

If the coefficient of (1.1) is, for any $x \in \mathbb{R}^d$,

$$b(x) = c(\nabla\psi(x))^{-1},$$

where $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is a function such that [¶]

$$\forall(\tau, x) \in [0,T] \times \mathbb{R},\ \frac{\nabla\psi(x)}{\nabla\psi(\psi^{-1}(c\tau + \psi(x)))} < +\infty.$$

Then the associated flow is

$$\theta_{s,t}(x) = \psi^{-1}(c(t-s) + \psi(x)).$$

Indeed, differentiating w.r.t. $s$ gives

$$\dot\theta_{s,t}(x) = c\nabla(\psi^{-1})(c(t-s)+\psi(x)) = \frac{c}{\nabla\psi \circ \psi^{-1}(c(t-s)+\psi(x))} = b(\psi^{-1}(c(t-s)+\psi(x))) = b(\theta_{s,t}(x)).$$

Furthermore,

$$\nabla\theta_{s,t}(x) = \frac{\nabla\psi(x)}{\nabla\psi(\psi^{-1}(c(t-s)+\psi(x)))},$$

which is in $L^\infty$.

Recalling that we obtain the example in Section 3.1.2 below, taking $\psi(x) = |x|^{1+\beta}$.

[¶] Which is granted if there is $c > 0$ such that $\forall x, y \in \mathbb{R}^d$, $c|y|^2 < \langle y, \nabla\psi(x)y\rangle < c^{-1}|y|^2$ yielding that $b \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

### 3.1.2 A polynomial example

For a given constant $c > 0$, and $\beta \geq 0$, if the transport coefficient write[‖]

$$b(x) = cx|x|^{-(1+\beta)} \in B^{-\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d), \tag{3.4}$$

then we can see that a flow can be written as

$$\theta_{s,t}(x) = \frac{x}{|x|}\Big(c(1+\beta)(t-s) + |x|^{1+\beta}\Big)^{\frac{1}{1+\beta}}. \tag{3.5}$$

Indeed, differentiating in $s$,

$$\dot{\theta}_{s,t}(x) = -\frac{x}{|x|}c\Big(c(1+\beta)(t-s) + |x|^{1+\beta}\Big)^{-\frac{\beta}{1+\beta}}.$$

and

$$\begin{aligned} b(\theta_{s,t}(x)) &= c\frac{x}{|x|}\Big(c(1+\beta)(t-s) + |x|^{1+\beta}\Big)^{-\frac{\beta}{1+\beta}} \\ &= -\dot{\theta}_{s,t}(x). \end{aligned}$$

Furthermore, taking the gradient of the flow,

$$\nabla\theta_{s,t}(x) = \frac{(\beta(1-\beta)(t-s) + |x|^{1+\beta})^{-\frac{\beta}{1+\beta}}}{|x|^{-\beta}}. \tag{3.6}$$

We have, for $\beta \geq 0$, and

$$\lim_{x\to 0} \nabla\theta_{s,t}(x) = \begin{cases} 0 \text{ if } s < t, \\ 1 \text{ if } s = t. \end{cases} \quad \text{and} \quad \lim_{|x|\to+\infty} \nabla\theta_{s,t}(x) = 1. \tag{3.7}$$

In other words, for any $0 \leq s \leq t$, $\nabla\theta_{s,t} \in L^\infty$.

In order to illustrate an example of a singular $b$ satisfying all conditions **(A)**, let us consider, for any $n > 0$,

$$b^n(x) = c\frac{x}{|x|}(|x| + n^{-1})^{-\beta},$$

see Appendix Section C for the non importance of the mollification procedure.

We deduce from (3.5), the "almost" associated flow,

$$\tilde{\theta}^n_{s,t}(x) = \frac{x}{|x|}\Big(c(1+\beta)(t-s) + |x|^{1+\beta} + n^{-1}\Big)^{\frac{1}{1+\beta}}.$$

We indeed get,

$$\begin{aligned} \dot{\tilde{\theta}}^n_{s,t}(x) &= -\frac{x}{|x|}c\Big(c(1+\beta)(t-s) + |x|^{1+\beta} + n^{-1}\Big)^{-\frac{\beta}{1+\beta}} \\ &= -b^n(\theta^n_{s,t}(x)). \end{aligned}$$

Next, we have the following gradient identity

$$\nabla\tilde{\theta}^n_{s,t}(x) = \frac{(\beta(1-\beta)(t-s) + |x|^{1+\beta} + n^{-1})^{-\frac{\beta}{1+\beta}}}{|x|^{-\beta}} \in L^\infty(\mathbb{R}^d, \mathbb{R}^d). \tag{3.8}$$

[‖] The definition of the norm $|x|$ does not change the analysis. The term $x|x|^{-1}$ stands for a multidimensional version of the sign function.

However, we get from (3.5), before transportation $\tilde{\theta}^n_{t,t}(x) = \frac{x}{|x|}(|x|^{1+\beta} + n^{-1})^{\frac{1}{1+\beta}}$ which goes to $x$ in $C^1(\mathbb{R}^d, \mathbb{R}^d)$ as $n \to +\infty$.

In order to get the right terminal condition of the flow, we define the corrected flow $\theta^n_{s,t}$ by

$$\tilde{\theta}^n_{s,t}(x) =: \theta^n_{s,t}\big(\frac{x}{|x|}(|x|^{1+\beta} + n^{-1})^{\frac{1}{1+\beta}}\big).$$

We equivalently[**] define, for any $y \in \mathbb{R}^d$,

$$\theta^n_{s,t}(y) := \tilde{\theta}^n_{s,t}\big(\frac{y}{|y|}(|y|^{1+\beta} - n^{-1})^{\frac{1}{1+\beta}}\big).$$

The analysis for this example follows exactly the computations below, see Section 4.7 further.

### 3.2 On the product of distributions

The sense of some particular products of distributions is very challenging, and is related with many long-standing problems. For instance, Hairer in [Hai14] introduce a regularity structure theory which after some renormalisation allows to handle with products of distribution, and to give a meaning of stochastic partial differential equation such as KPZ [Hai13]. However, such renormalisation leads to blowing-up constants which is not the case in Theorem 2; the price that we have to pay is the potential non-uniqueness of the limit.

From the different formulations above, we define different meanings of the product $\langle b, \nabla u\rangle$. First of all, let us remark that by rough *a priori* controls, see Lemma 3 below,

$$\begin{aligned}[\Delta u^{m,\nu}(t,\cdot)]_\gamma &\leq 2^{1-\gamma}\|\nabla^2 u^{m,\nu}(t,\cdot)\|^{1-\gamma}_{L^\infty}\|\nabla^3 u^{m,\nu}(t,\cdot)\|^{\gamma}_{L^\infty} \\ &\leq Cm^{2+\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big)\exp(Cm^{1-\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}).\end{aligned}$$

Hence, if

$$\nu \ll Cm^{-(2+\gamma)}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big)^{-1}\exp(-Ctm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}), \tag{3.9}$$

then, up to subsequence choice, $\nu\Delta u^{m,\nu}(t,\cdot) \underset{(m,\nu)\to(+\infty,0)}{\longrightarrow} 0$ in $C^\gamma_b(\mathbb{R}^d, \mathbb{R})$.

Also, for a given $t \in [0,T]$, we see from the definition of *mild vanishing viscous* solution, up to subsequence choice according to the condition (3.9), that

$$\lim_{(m,\nu)\to(+\infty,0)}\int_0^t \langle b_m(s,\cdot), \nabla u^{m,\nu}(s,\cdot)\rangle ds = g - u(t,\cdot) - \int_0^t f(s,\cdot)ds \in C^\gamma_b(K, \mathbb{R}), \tag{3.10}$$

for any compact $K \subset \mathbb{R}^d$. We highly point out that $b$ lies in any arbitrary negative regularity in space $L^\infty([0,T]; B^{-\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d))$, $\beta \in \mathbb{R}_+$, and $\nabla u(s,\cdot) \in B^{-1+\gamma}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

In other words, thanks to the time averaging, we get a new para-product condition. Indeed, in general from Bony's microlocal analysis [Bon81], for all $\varphi \in B^{\alpha_1}_{\infty,\infty}$ and $\psi \in B^{\alpha_2}_{\infty,\infty}$, we have

$$\phi\psi \in B^{\alpha_1\wedge\alpha_2}_{\infty,\infty}, \text{ if } \alpha_1 + \alpha_2 > 0. \tag{3.11}$$

However, the uniqueness of the limit in (3.10) seems to be false in general.

Also in the weak formulation, from Theorem 2, we obtain, if $\nabla\cdot b = 0$ and $-\beta = \tilde{\gamma} < \gamma$ (regularity condition weaker than (3.11)), a distributional meaning of $\langle b(s,\cdot), \nabla u(s,\cdot)\rangle$, but we still do not know in this case if the limit is unique.

[**] $y = \frac{x}{|x|}(|x|^{1+\beta} + n^{-1})^{\frac{1}{1+\beta}}$ implies $\frac{y}{|y|} = \frac{x}{|x|}$ and $|y|^{1+\beta} = |x|^{1=\beta} + n^{-1}$ which implies that $x = \frac{y}{|y|}(|y|^{1+\beta} - n^{-1})^{\frac{1}{1+\beta}}$.

Finally, we point out that in the **Positive regularity** framework, the hypothesis matches with the Bony's para-product condition (3.11). Indeed, for any $t \in [0,T]$, $b(t,\cdot) \in B^{\alpha}_{\infty,\infty}$, $\alpha = -\beta$, and $\nabla u(t,\cdot) \in B^{-1+\gamma}_{\infty,\infty}$ with $\alpha - 1 + \gamma > 0$. We are able quantify the regularity, $\langle b(t,\cdot), \nabla u(t,\cdot)\rangle \in B^{-1+\gamma}_{\infty,\infty}$ by para-product detailed further in Section 4.5. Moreover, the time averaging version $\int_0^t \langle b(s,\cdot), \nabla u(s,\cdot)\rangle ds$ in the sense of (3.10) is $\gamma$-Hölder. We remark, as $\alpha \wedge (-1+\gamma) = -1+\gamma$ then there is a $+1$ gain of regularity comparing with the usual para-product result.

# 4 Proof of Theorem 2

## 4.1 Parabolic approximation procedure

Let us first smoothen the drift and the source functions of the parabolic approximation,

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + \langle b_m, \nabla u^{m,\nu}\rangle(t,x) - \nu \Delta u^{m,\nu}(t,x) = f_m(t,x), \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ u^{m,\nu}(0,x) = g_m(x), \ x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

where the mollified functions are defined by

$$\begin{aligned} b_m(t,x) &:= \int_{\mathbb{R}^d} \rho_m(x-y) b(t,y) dy, \\ f_m(t,x) &:= \int_{\mathbb{R}^d} \rho_m(x-y) f(t,y) dy, \\ g_m(t,x) &:= \int_{\mathbb{R}^d} \rho_m(x-y) g(y) dy, \end{aligned} \tag{4.2}$$

for $\rho_m(\cdot) := m^d \rho(m\cdot)$ where $\rho$ is a non-negative smooth function $\rho_m$, such that $\int_{\mathbb{R}^d} \rho_m(x-y)dy = 1$. In particular, we choose $\rho = h_1$, the heat kernel defined in (2.9). In Appendix Sections A and C, we see that the limit of $b_m$ does not depend on the choice of the mollification procedure, whereas the limit of $u^{m,\nu}$ potentially does.

In our analysis, we use some point-wise controls of the mollified functions or distributions whose blowing-up in the regularisation parameter $m$ is stated below.

**Lemma 1.** *For all $m > 1$, and $\beta < 0$, if $b \in L^\infty([0,T], B^{-\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d))$, we have for any $(t,x) \in [0,T] \times \mathbb{R}^d$:*

$$\begin{aligned} |b_m(t,x)| &\leq C m^{\beta} \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}, \\ |\nabla b_m(t,x)| &\leq C m^{1+\beta} \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}, \end{aligned} \tag{4.3}$$

*where $\nabla b_m$ stands for the Jacobian matrix of $b_m$; also if $\beta \leq 0$,*

$$\begin{aligned} |b_m(t,x)| &\leq \|b\|_{L^\infty}, \\ |\nabla b_m(t,x)| &\leq C m^{1+\beta} \|b\|_{L^\infty(C^{-\beta})}. \end{aligned}$$

*Proof of Lemma 1.* From the mollification definition (4.2), we see, from (2.10), that $\rho_m = h_{m^{-2}}$, and from our scaling choice, we get for any $m > 1$

$$\begin{aligned} \big|b_m(t,x)\big| &= \big| \int_{\mathbb{R}^d} h_{m^{-2}}(x-y) b(t,y) dy \big| \\ &\leq m^{\beta} \sup_{\tilde{m}^{-2} \in [0,1],\ x \in \mathbb{R}^d} \tilde{m}^{-\beta} \big| \int_{\mathbb{R}^d} h_{\tilde{m}^{-2}}(x-y) b(t,y) dy \big|. \end{aligned}$$

We readily get by the thermic definition of the Besov norm (2.9):

$$|b_m(t,x)| \leq m^{\beta} \|b\|_{L^\infty(\ddot{B}^{-\beta}_{\infty,\infty})} \leq m^{\beta} \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}.$$

For the second inequality, it is known that for any $t \in [0,T]$, $\nabla b(t,\cdot) \in B^{-1-\beta}_{\infty,\infty}$, see Theorem 9 of Chapter 3 in [Pee76], in particular if $\beta \in (0,-1)$ see Corollary 6 in Appendix Section B. Hence,

$$\begin{aligned}\big|\nabla_x b_m(t,x)\big| &\leq m^{1+\beta} \sup_{\tilde{m}^{-2}\in[0,1],\ \tilde{x}\in\mathbb{R}^d} \tilde{m}^{-\beta}\big|\int_{\mathbb{R}^d} h_{\tilde{m}^{-2}}(\tilde{x}-y)\nabla_y b(t,y)dy\big| \\ &= m^{1+\beta}\|\nabla b\|_{L^\infty(\ddot{B}^{-1-\beta}_{\infty,\infty})}.\end{aligned}$$

We deduce that there is a constant $C = C(d) > 0$ such that:

$$\big|\nabla_x b_m(t,x)\big| \leq C m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})},$$

see Corollary 6 in Appendix Section B.

The two last inequalities, i.e. for the case $\beta \leq 0$, are standard. □

### 4.1.1 *Proxy* choice

We approximate the Cauchy problem around the flow associated to the smooth function $b_m$, which is unique by Cauchy-Lipschitz theorem. Namely, let us consider the unique function defined for any *freezing* point $(\tau,\xi) \in [0,T]\times\mathbb{R}^d$ by,

$$\theta^m_{s,\tau}(\xi) := \xi + \int_s^\tau b_m(\tilde{s}, \theta^m_{\tilde{s},\tau}(\xi))d\tilde{s},\ \ s \in [0,\tau]. \tag{4.4}$$

In other words, for any $t \in [0,\tau]$,

$$\dot{\theta}^m_{t,\tau}(\xi) = -b_m(t,\theta^m_{t,\tau}(\xi)),\ \ \theta^m_{\tau,\tau}(\xi) = \xi.$$

We again rewrite the system of linear parabolic PDEs (4.2),

$$\partial_t u^{m,\nu}(t,x) + b_m(t,\theta^m_{t,\tau}(\xi))\cdot\nabla u^{m,\nu}(t,x) - \nu\Delta u^{m,\nu}(t,x) = b^m_\Delta[\tau,\xi](t,x)\cdot\nabla u^{m,\nu}(t,x) + f_m(t,x), \tag{4.5}$$

where we have

$$b^m_\Delta[\tau,\xi](t,x) := b_m(t,\theta^m_{t,\tau}(\xi)) - b_m(t,x). \tag{4.6}$$

For such a fixed freezing point $(\tau,\xi) \in [0,T]\times\mathbb{R}^d$, we use the corresponding Duhamel formula:

$$u^{m,\nu}(t,x) = \hat{P}^{\tau,\xi}g_m(t,x) + \hat{G}^{\tau,\xi}f_m(t,x) + \hat{G}^{\tau,\xi}\big(b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big)(t,x), \tag{4.7}$$

where we define, for any $f \in C^{1,2}_0((0,T]\times\mathbb{R}^d,\mathbb{R})$, the Green operator associated with the perturbed parabolic equation with constant coefficients (4.5),

$$\forall (t,x) \in (0,T]\times\mathbb{R}^d,\ \ \hat{G}^{\tau,\xi}f(t,x) := \int_0^t\int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s,t,x,y)f(s,y)\ dy\ ds, \tag{4.8}$$

and for any $g \in C^2_0(\mathbb{R}^d,\mathbb{R})$, the associated semi-group

$$\hat{P}^{\tau,\xi}g(t,x) := \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(0,t,x,y)g(y)\ dy, \tag{4.9}$$

where the perturbed heat kernel is

$$\hat{p}^{\tau,\xi}(s,t,x,y) := \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}}\exp\Big(-\frac{\Big|x+\int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(\xi))d\tilde{s} - y\Big|^2}{4\nu(t-s)}\Big). \tag{4.10}$$

We carefully point out that, from definition (4.4), if $(\tau,\xi)=(t,x)$,

$$\hat{p}^{t,x}(s,t,x,y)=\frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}}\exp\Big(-\frac{\big|\theta^m_{s,t}(x)-y\big|^2}{4\nu(t-s)}\Big).$$

We have for each $\alpha\in\mathbb{N}^d$ that there is a constant $C_\alpha>1$ s.t.

$$\begin{aligned}|D^\alpha\hat{p}^{\tau,\xi}(s,t,x,y)| &\leq \frac{C_\alpha[\nu(t-s)]^{-\frac{|\alpha|}{2}}}{(4\pi\nu(t-s))^{\frac{d}{2}}}\exp\Big(-C_\alpha^{-1}\frac{\big|x+\int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(\xi))d\tilde{s}-y\big|^2}{4\nu(t-s)}\Big)\\ &=: C[\nu(t-s)]^{-\frac{|\alpha|}{2}}\bar{p}^{\tau,\xi}(s,t,x,y),\end{aligned}\tag{4.11}$$

and also, after the derivatives we can choose $(\tau,\xi)=(t,x)$, and $\gamma\in[0,1]$,

$$\begin{aligned}|D^\alpha\hat{p}^{t,x}(s,t,x,y)|\times\big|y-x-\int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(\xi))d\tilde{s}\big|^\gamma &= |D^\alpha\hat{p}^{t,x}(s,t,x,y)|\times\big|y-\theta^m_{s,t}(x)\big|^\gamma\\ &\leq C[\nu(t-s)]^{-\frac{|\alpha|}{2}+\frac{\gamma}{2}}\bar{p}^{t,x}(s,t,x,y),\end{aligned}\tag{4.12}$$

from absorbing property (2.13). It also clear, for any $0\leq s<t$, that

$$\partial_t\hat{p}^{\tau,\xi}(s,t,x,y)=\nu\Delta\hat{p}^{\tau,\xi}(s,t,x,y)-\langle b_m(t,\theta^m_{t,\tau}(\xi)),\nabla\hat{p}^{\tau,\xi}(s,t,x,y)\rangle,\tag{4.13}$$

which naturally implies that the function $u^{m,\nu}$ defined in (4.7) is indeed solution to (4.1) and to (4.5).

Finally, we will marginally use the "pure" heat kernel already defined in (5.3),

$$\tilde{p}(s,t,x,y)=\hat{p}^{\tau,\xi}\Big(s,t,x-\int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(\xi))d\tilde{s},y\Big)=\frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}}\exp\Big(-\frac{|x-y|^2}{4\nu(t-s)}\Big),\tag{4.14}$$

recalling the corresponding Green operator

$$\forall(t,x)\in(0,T]\times\mathbb{R}^d,\ \tilde{G}f(t,x):=\int_0^t\int_{\mathbb{R}^d}\tilde{p}(s,t,x,y)f(s,y)\,dy\,ds,\tag{4.15}$$

and the associated semi-group

$$\tilde{P}g(t,x):=\int_{\mathbb{R}^d}\tilde{p}(0,t,x,y)g(y)\,dy.\tag{4.16}$$

## 4.2 Convergence of the solution

To prove that $u^{m,\nu}$ indeed converges toward $u$, the analysis is similar to the one perform in Section 4.4 replacing $\bar{u}^{m,\bar{\nu}}$ by the solution

$$\begin{cases}\partial_t u(t,x)+\langle b(t,x),\nabla u(t,x)\rangle=f(t,x),\ t\in[0,T),\\ \bar{u}_n(0,x)=g(x),\end{cases}\tag{4.17}$$

supposed to exist in $L^\infty([0,T],C^\gamma(\mathbb{R}^d,\mathbb{R}^d))$ because the flow $\theta_{s,\tau}(\xi):=\xi+\int_s^\tau b(\tilde{s},\theta_{\tilde{s},\tau}(\xi))d\tilde{s},\ s\in[0,\tau]$ is assumed to be Lipschitz in space.

Under assumption (3.9), we have $\nu\Delta u^{m,\nu}(t,\cdot)\underset{(m,\nu)\to(+\infty,0)}{\longrightarrow}0$ in $C^\gamma_b(\mathbb{R}^d,\mathbb{R})$. Also under the assumption on the limit flow in **(A)**, $\forall\varepsilon>0$,

$$\|u-u^{m,\nu}\|_{L^\infty(C^{\gamma-\varepsilon})}\underset{(m,\nu)\to(+\infty,0)}{\longrightarrow}0.\tag{4.18}$$

## 4.3 Compactness arguments

In this section, we suppose that hypothesis **(A)** is in force, which yields existence of uniform control of $u^{m,\nu}$, because the associated flow $\theta_{s,t}$ is Lipschitz continuous, see Section 3.1.

### 4.3.1 Mild vanishing viscous

In order to pass to the limit $m \to +\infty$, $\nu \to 0$, according to the vanishing condition (4.45), we consider a subsequence given by (4.18) or by the usual Arzelà-Ascoli theorem. However, this former result is available for uniform continuous function in a compact space. From the lack of smoothness, uniformly on $\nu$, in space of $u^{m,\nu}(t,\cdot)$ (only $\gamma$-Hölder continuous, $\gamma < 1$), we are stuck at a convergence in a compact subset of $\mathbb{R}^d$. For instance, the analysis performed in [Hon22] to get rid of the compactness convergence criterion for quasi-linear equations does not work here as there is no hope to obtain any strong formulation of the PDE (1.1).

We do not succeed to obtain any positive regularity on $t$, which would imply dependency on $b$ and so we cannot exploit uniform continuity in time to get a convergence of a sub-sequence of $u^{m,\nu}$ in $[0,T]\times\mathbb{R}^d$. Thus the convergence at any given time in all compacts set of the *mild vanishing viscous* solution in Theorem 2.

Nevertheless, we are still able to include a truncation procedure into a weak formulation in order to obtain a convergence in a distributional meaning and not a point-wise one as for the *mild vanishing viscous* solution.

### 4.3.2 Truncation procedure

The method is highly inspired by the one in [Hon22], we also consider a smooth cut-off $\vartheta_{y,R} \in \mathcal{D}$ supported in a ball $B_d(y,R) = \{x \in \mathbb{R}^d; |x-y| \le R\}$, $y \in \mathbb{R}^d$ and defined by

$$\vartheta_{y,R}(x) = \vartheta_y(\frac{x}{R}), \tag{4.19}$$

where $\vartheta_y : \mathbb{R}^d \to [0,1]^d$ is function lying in $C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ s.t.

$$\vartheta_y(x) = \begin{cases} x, & \text{if } |x-y|<1, \\ 0, & \text{if } |x-y|>2. \end{cases}$$

The corresponding truncated function is, for any $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u^{m,\nu}_{y,R}(t,x) := u^{m,\nu}(t,\vartheta_{y,R}(x)). \tag{4.20}$$

We highlight the particular case

$$u^{m,\nu}_{x,R}(t,x) = u^{m,\nu}(t,x). \tag{4.21}$$

The above truncation solution (4.20) naturally appears when we write a weak formulation of the parabolic equation (2.16).

### 4.3.3 Weak solution of the parabolic approximating equation

For any smooth function $\varphi_R$ supported on $B_d(0,R)$, a d-ball of radius $R>0$ and center $(0,\dots,0)\in\mathbb{R}^d$. We consider a weak formulation of the parabolic solution $u^{m,\nu}$, for any $(t,x)\in[0,T]\times\mathbb{R}^d$:

$$\int_0^t\int_{\mathbb{R}^d}\Big\{-\partial_t\varphi_R(s,y)u^{m,\nu}(s,y)+\varphi_R(s,y)\langle b_m(s,y),\nabla u^{m,\nu}(s,y)\rangle+\nu\Delta\varphi_R(s,y)u^{m,\nu}(s,y)\Big\}dy\,ds$$
$$=\int_{\mathbb{R}^d}\varphi_R(0,y)g_m(y)dy-\int_{\mathbb{R}^d}\varphi_R(t,y)u^{m,\nu}(t,y)dy+\int_0^t\int_{\mathbb{R}^d}\varphi_R(s,y)f_m(s,y)dy\,ds,$$

where in l.h.s. the limit of the first order term, $\langle b_m(s,y), \nabla u^{m,\nu}(s,y)\rangle$, has *a priori* no point-wise limit neither in term of the usual distributional meaning of Schwartz. Indeed, as already enunciated in Section 3.2, the usual distribution theory does not provide any interpretation of a product of distributions, to get any limit result we have to thoroughly use the PDE.

By the cut-off definition, we equivalently have

$$\int_0^t \int_{\mathbb{R}^d} \Big\{ -\partial_t \varphi_R(s,y) u^{m,\nu}_{0,R}(s,y) + \varphi_R(s,y)\langle b_m(s,y), \nabla u^{m,\nu}_{0,R}(s,y)\rangle + \nu \Delta \varphi_R(s,y) u^{m,\nu}_{0,R}(s,y)\Big\} dy\, ds$$
$$= \int_{\mathbb{R}^d} \varphi_R(0,y) g_m(y) dy - \int_{\mathbb{R}^d} \varphi_R(s,y) u^{m,\nu}_{0,R}(s,y) dy + \int_0^t \int_{\mathbb{R}^d} \varphi_R(s,y) f_m(s,y) dy\, ds. \tag{4.22}$$

Now, from compact argument developed in Section 4.3.1, we have that $u^{m,\nu}_{0,R}(s,\cdot)$ converges in $C^\gamma_b(K,\mathbb{R}^d)$, $K = B_d(0,R)$, towards a function $u_{0,R}(s,\cdot)$ when $(m,\nu) \to (+\infty, 0)$ and the condition (4.45) is satisfied. In other words, $u_{0,R}(s,\cdot)$ is a *mild vanishing viscous* solution of (1.1).

### 4.3.4 Mild-weak solution of the transport equation

To get a *mild-weak* solution we have to pass to the limit in the weak formulation (4.22) of the mollified parabolic equation (2.16). In equation (4.22), up to a sub-sequence selection, except for the first order term $\varphi_R(s,y)\langle b(s,y), \nabla u^{m,\nu}_{0,R}(s,y)\rangle$, each contribution obviously has the good converge property by the Arzelà-Ascoli theorem. In particular, from (4.101), we have

$$\nu \int_0^t \int_{\mathbb{R}^d} \Delta \varphi_R(s,y) u^{m,\nu}_{0,R}(s,y) dy\, ds \xrightarrow[(m,\nu)\to(+\infty,0)]{(4.45)} 0. \tag{4.23}$$

To deal with the *drift* part, we write by integration by parts

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle b_m(t,y), \nabla u^{m,\nu}_{0,R}(t,y)\rangle \varphi_R(t,y) dy \\ = & \int_{\mathbb{R}^d} \nabla \varphi_R(t,y)\cdot b_m(t,y) u^{m,\nu}_{0,R}(t,y) dy + \int_{\mathbb{R}^d} \varphi_R(t,y) \nabla\cdot b_m(t,y) u^{m,\nu}_{0,R}(t,y) dy \\ =: & \; B_1 + B_2. \end{aligned} \tag{4.24}$$

By the Besov duality property, see Proposition 1, we write

$$|B_1| \le \|b_m\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^\infty(B^{-\beta}_{1,1})}, \tag{4.25}$$

and

$$|B_2| \le \|\nabla\cdot b_m\|_{L^\infty(B^{-1-\beta}_{\infty,\infty})} \|\varphi_R u^{m,\nu}_{0,R}\|_{L^\infty(B^{1+\beta}_{1,1})}, \tag{4.26}$$

with $B_2 = 0$ for $b$ incompressible.

**Control of $\|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^\infty(B^\beta_{1,1})}$**

Let us prove that $\|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^\infty(B^\beta_{1,1})}$ is controlled uniformly in $(m,\nu)$. By the thermic representation of the Besov norms (2.9), we have

$$\begin{aligned} \|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^\infty(B^\beta_{1,1})} &= \|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^1} + \|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^\infty(\ddot{B}^\beta_{1,1})} \\ &= \|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^1} + \int_0^1 \frac{1}{v} v^{1+\frac{\beta}{2}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \partial_v h_v(z-y) u^{m,\nu}_{0,R}(t,y) \nabla \varphi_R(t,y) dy \Big| dz\, dv. \end{aligned}$$

The first contribution in the r.h.s. above is obviously bounded uniformly in $(m,\nu)$ by

$$\|\nabla \varphi_R u^{m,\nu}_{0,R}\|_{L^1} \le \|\nabla \varphi_R\|_{L^1} \|u^{m,\nu}_{0,R}\|_{L^\infty} \le C \|\nabla \varphi_R\|_{L^1} \big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\big),$$

by uniform estimate (4.101).

For the second one, we need to deeply use the already known regularity of $u_{0,R}^{m,\nu}$. By triangular inequality, we obtain

$$\begin{aligned}\|\nabla\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\ddot{B}^\beta_{1,1})} &\leq \int_0^1 \frac{1}{v}v^{1-\frac{\beta}{2}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v h_v(z-y)\cdot\big\{[u_{0,R}^{m,\nu}(t,y)-u_{0,R}^{m,\nu}(t,z)]\nabla\varphi_R(t,y)\\ &\quad +u_{0,R}^{m,\nu}(t,z)[\nabla\varphi(t,y)-\nabla\varphi(t,z)]\big\}dy\Big|dz\,dv\\ &\leq \int_0^1 \frac{1}{v}v^{1-\frac{\beta}{2}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v h_v(z-y)\cdot\big\{[u_{0,R}^{m,\nu}(t,y)-u_{0,R}^{m,\nu}(t,z)]\nabla\varphi_R(t,y)\\ &\quad +u_{0,R}^{m,\nu}(t,z)(y-z)\cdot\int_0^1 D^2\varphi_R(t,z+\mu(y-z))d\mu\big\}dy\Big|dz\,dv,\end{aligned}$$

by Taylor expansion. Next, with the exponential absorbing property (2.13),

$$\begin{aligned}&\|\nabla\varphi u_{0,R}^{m,\nu}\|_{L^\infty(\ddot{B}^\beta_{1,1})}\\ \leq\ & C\|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)}\int_0^1\frac{1}{v}v^{-\frac{\beta}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}h_{C^{-1}v}(z-y)|y-z|^\gamma|\nabla\varphi_R(t,y)|dy\,dz\,dv\\ &+C\|u_{0,R}^{m,\nu}\|_{L^\infty}\int_0^1\frac{1}{v}v^{-\frac{\beta}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}h_{C^{-1}v}(z-y)|y-z|\int_0^1\big|D^2\varphi_R(t,z+\mu(y-z))\big|d\mu dy\,dz\,dv\\ \leq\ & C\|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)}\|\nabla\varphi_R\|_{L^1}\int_0^1\frac{1}{v}v^{\frac{\gamma-\beta}{2}}dv+C\|u_{0,R}^{m,\nu}\|_{L^\infty}\|D^2\varphi_R\|_{L^1}\int_0^1\frac{1}{v}v^{\frac{1-\beta}{2}}dv,\end{aligned}$$

which is finite if $\beta<\gamma$.

**Control of $\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B^{1+\beta}_{1,1})}$**

The analysis is similar as before, replacing $\beta$ by $1+\beta$ and $\nabla\varphi_R$ by $\varphi_R$:

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B^{1+\beta}_{1,1})}=\|\varphi_R u_{0,R}^{m,\nu}\|_{L^1}+\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\ddot{B}^{1+\beta}_{1,1})}.$$

We readily get

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^1}\leq\|\varphi_R\|_{L^1}\|u_{0,R}^{m,\nu}\|_{L^\infty}\leq\|\varphi_R\|_{L^1}\big(T\|f\|_{L^\infty}+\|g\|_{L^\infty}\big),$$

and

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\ddot{B}^{1+\beta}_{1,1})}\leq C\|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)}\|\varphi_R\|_{L^1}\int_0^1\frac{1}{v}v^{\frac{\gamma-\beta-1}{2}}dv+C\|u_{0,R}^{m,\nu}\|_{L^\infty}\|\nabla\varphi_R\|_{L^1}\int_0^1\frac{1}{v}v^{\frac{-\beta}{2}}dv,$$

this is finite if $1+\beta<\gamma\Longleftrightarrow\beta<-1+\gamma<0$. Let us carefully notice that if there is the incompressible assumption $\nabla\cdot b=0$, then $B_2=0$ and this former constraint disappears. Thus the different cases considered in Theorem 2.

The Bolzano-Weierstrass theorem then yields the result.

### 4.3.5 Weak solution

The difficulty for the usual weak solution, here, is to prove that, up to a subsequence extraction,

$$\begin{aligned}\lim_{m\to\infty}\int_{\mathbb{R}^d}\int_0^T\langle b_m(t,y),\nabla u_{0,R}^{m,\nu}(t,y)\rangle\varphi_R(t,y)dy\,dt &= \int_0^T\int_{\mathbb{R}^d}\langle b(t,y),\nabla u_{0,R}(t,y)\rangle\varphi_R(t,y)dy\,dt\\ &= \int_0^T\int_{\mathbb{R}^d}\langle b(t,y),\nabla u(t,y)\rangle\varphi_R(t,y)dy\,dt,\end{aligned}\tag{4.27}$$

where $b \in L^\infty([0,T];\tilde{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ is the drift of the initial Cauchy problem (1.1) and coincides with the limit of $b_m$ in $L^\infty([0,T];B^{-\beta-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$, for any $0<\varepsilon$ when $m\to\infty$; also $u(s,\cdot)\in C^\gamma_b(K,\mathbb{R})$, $K=B_d(0,R)$, is the limit of $u^{m,\nu}(s,\cdot)$, up to a subsequence selection possibly depending on the current time $s$, in $C^{\gamma-\tilde{\varepsilon}}_b(\mathbb{R}^d,\mathbb{R})$ for any $0<\tilde{\varepsilon}<\gamma$, see Section 4.3.1.

Let us recall that $\tilde{B}^{-\beta}_{\infty,\infty}$ is the closure space of $C^\infty_b$ in $B^{-\beta}_{\infty,\infty}$, from Appendix Section C, we still can take the regular sequence $(b_m)_{m\geq 1}$ defined in (4.2) to approximate $b$ in $L^\infty([0,T];B^{-\beta-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$, for any $0<\varepsilon$; whereas the considered solution $u(s,\cdot)$ may depend on the choice of mollification. In other words, we have:

$$\lim_{m\to\infty}\|b_m-b\|_{L^\infty(B^{-\beta-\varepsilon}_{\infty,\infty})}=0. \tag{4.28}$$

For all $m>0$, $t\in[0,T]$, we write by integration by parts that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d}\langle b_m(t,y),\nabla u^{m,\nu}_{0,R}(t,y)\rangle\varphi_R(t,y)dy-\int_{\mathbb{R}^d}\langle b(t,y),\nabla u_{0,R}(t,y)\rangle\varphi_R(t,y)dy\Big|\\
\leq\ &\Big|\int_{\mathbb{R}^d}[b_m-b](t,y)u^{m,\nu}(t,y)\cdot\nabla\varphi_R(t,y)dy\Big|+\Big|\int_{\mathbb{R}^d}b(t,y)[u-u^{m,\nu}](t,y)\cdot\nabla\varphi_R(t,y)dy\Big|\\
&+\Big|\int_{\mathbb{R}^d}\nabla\cdot[b_m-b](t,y)u^{m,\nu}(t,y)\varphi_R(t,y)dy\Big|+\Big|\int_{\mathbb{R}^d}\nabla\cdot b(t,y)[u-u^{m,\nu}](t,y)\ \varphi_R(t,y)dy\Big|\\
=:\ &\tilde{B}_1+\tilde{B}_2+\tilde{B}_3+\tilde{B}_4.
\end{aligned}\tag{4.29}$$

To deal with the first contribution, we aim to use (4.28). We have by the Besov duality result of Proposition 1:

$$\tilde{B}_1\leq\|b_m-b\|_{L^\infty(B^{-\beta-\varepsilon}_{\infty,\infty})}\|u^{m,\nu}\nabla\varphi_R\|_{L^\infty(B^{\beta+\varepsilon}_{1,1})}, \tag{4.30}$$

it then remains to control $\|u^{m,\nu}\nabla\varphi_R\|_{L^\infty(B^{\beta+\varepsilon}_{1,1})}$. Similar computations as in Section 4.3.4 yields that $\|u^{m,\nu}\nabla\varphi_R\|_{L^\infty(B^{\beta+\varepsilon}_{1,1})}$ is finite if $\beta<\gamma-\varepsilon$.

Also for the third $\tilde{B}_3$ and the fourth term $\tilde{B}_4$, which are null if $\nabla\cdot b=0$.

Hence, from (4.28), we obtain

$$\tilde{B}_1\xrightarrow[(m,\nu)\to(+\infty,0)]{(4.45)}0. \tag{4.31}$$

Now, let us handle with the second term in (4.29). We aim here to use the convergence of $u^{m,\nu}(s,\cdot)$ towards $u(s,\cdot)$ in the ball $B_d(0,R)$.

Again by the Besov duality result of Proposition 1, we have:

$$\begin{aligned}
\tilde{B}_2\ &=\ \Big|\int_{\mathbb{R}^d}b(t,y)[u-u^{m,\nu}](t,y)\cdot\nabla\varphi_R(t,y)dy\Big|\\
&\leq\ \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\|(u^{m,\nu}-u)\nabla\varphi_R\|_{L^\infty(B^{\beta}_{1,1})}.
\end{aligned}\tag{4.32}$$

We have, from Section 4.2,

Arzelà-Ascoli theorem, we have

$$\|\nabla\varphi_R(u^{m,\nu}-u)\|_{L^\infty}\leq\|\nabla\varphi_R\|_{L^\infty}\|(u^{m,\nu}-u)\|_{L^\infty}\xrightarrow[(m,\nu)\to(+\infty,0)]{(4.45)}0.$$

For the homogenous part of the Besov norm, we also mimic the analysis in the previous section replacing $u^{m,\nu}$ by $(u^{m,\nu}-u)$, for any $\varepsilon\in(0,\gamma)$:

$$\begin{aligned}
&\|(u^{m,\nu}-u)\nabla\varphi_R\|_{L^\infty(\ddot{B}^{\beta}_{1,1})}\\
\leq\ &C\|(u^{m,\nu}-u)\|_{L^\infty(C^{\gamma-\varepsilon})}\int_0^1\frac{1}{v}v^{-\frac{\beta}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}h_{C^{-1}v}(z-y)|y-z|^{\gamma-\varepsilon}|\nabla\varphi_R(t,y)|dy\,dz\,dv\\
&+C\|(u^{m,\nu}-u)\|_{L^\infty}\int_0^1\frac{1}{v}v^{-\frac{\beta}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}h_{C^{-1}v}(z-y)|y-z|\int_0^1\big|D^2\varphi_R(t,z+\mu(y-z))\big|d\mu dy\,dz\,dv\\
\leq\ &C\|(u^{m,\nu}-u)\|_{L^\infty(C^{\gamma-\varepsilon})}\|\nabla\varphi_R\|_{L^1}\int_0^1\frac{1}{v}v^{\frac{\gamma-\varepsilon-\beta}{2}}dv+C\|(u^{m,\nu}-u)\|_{L^\infty}\|D^2\varphi_R\|_{L^1}\int_0^1\frac{1}{v}v^{\frac{1-\beta}{2}}dv,
\end{aligned}$$

which is finite as soon as $\beta < \gamma - \varepsilon < 0$, also by Arzelà-Ascoli theorem, we have the converging result $\|(u^{m,\nu} - u)\|_{L^\infty(C^{\gamma-\varepsilon})} \xrightarrow{(m,\nu)\to(+\infty,0)} 0$.

Therefore, we even get

$$\tilde{B}_2 \xrightarrow{(m,\nu)\to(+\infty,0)} 0. \tag{4.33}$$

The last contribution $\tilde{B}_4$, null if $\nabla \cdot b = 0$, is similar replacing $\nabla\varphi_R$ by $\varphi_R$ and $\beta$ by $\beta + 1$. Namely, we have

$$\begin{aligned} \tilde{B}_4 &\leq \|\nabla b\|_{L^\infty(B^{-1-\beta}_{\infty,\infty})} \|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(B^{1+\beta}_{1,1})} \\ &= \|\nabla b\|_{L^\infty(B^{-1-\beta}_{\infty,\infty})} (\|(u^{m,\nu} - u)\varphi_R\|_{L^1} + \|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(\ddot{B}^{1+\beta}_{1,1})}), \end{aligned} \tag{4.34}$$

with

$$\|(u^{m,\nu} - u)\varphi_R\|_{L^1} \leq \|u^{m,\nu} - u\|_{L^\infty} \|\varphi_R\|_{L^1},$$

and

$$\begin{aligned} &\|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(\ddot{B}^{1+\beta}_{1,1})} \\ = &\int_0^1 \frac{1}{v} v^{1-\frac{1+\beta}{2}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \partial_v h_v(z-y)(u^{m,\nu} - u)(t,y)\varphi_R(t,y)dy \Big| dz\, dv \\ \leq &\; C\|u^{m,\nu} - u\|_{L^\infty(C^{\gamma-\varepsilon})} \|\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\varepsilon-1-\beta}{2}} dv + C\|u^{m,\nu} - u\|_{L^\infty} \|D^2\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{-\beta}{2}} dv, \end{aligned}$$

which is finite if $\beta < -1 + \gamma - \varepsilon$ and by (4.18), we deduce

$$\tilde{B}_4 \xrightarrow{(m,\nu)\to(+\infty,0)} 0. \tag{4.35}$$

Hence, from (4.29), (4.31), (4.33) and (4.35) we deduce (4.27).

### 4.4 Uniqueness

Let us insist that uniqueness of *vanishing viscous* solution does not mean uniqueness of usual solution. Indeed, this question arises for the uniqueness of the limits of any sub-sequence of $(u^{m,\nu})_{m,\nu\geq 0}$ and for the non-dependency of the limit on the regularisation procedure; also the smooth selection principle established here does not depend on the choice of the vanishing sequence $(\nu)$.

Let us suppose that there are two *vanishing viscous* solutions $u$ and $\bar{u}$ of (1.1) satisfying estimates (3.1). We then consider the associated mollified version $(u^{m,\nu})_{m\geq 0}$ and $(\bar{u}^{m,\bar{\nu}})_{m\geq 0}$ solutions, for any $x \in \mathbb{R}^d$, to

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + \langle b_m(t,x), \nabla u^{m,\nu}(t,x)\rangle - \bar{\nu}\Delta u^{m,\nu}(t,x) = f_m(t,x),\ t \in [0,T), \\ u^{m,\nu}(0,x) = g_m(x), \end{cases} \tag{4.36}$$

where $b_m$ is a mollified version of $b$ as in (4.2) by a convolution with the Gaussian mollifier $\rho_m$, and

$$\begin{cases} \partial_t \bar{u}^{m,\bar{\nu}}(t,x) + \langle \bar{b}_m(t,x), \nabla \bar{u}^{m,\bar{\nu}}(t,x)\rangle - \bar{\nu}\Delta \bar{u}^{m,\bar{\nu}}(t,x) = f_m(t,x),\ t \in [0,T), \\ \bar{u}_n(0,x) = g_m(x), \end{cases} \tag{4.37}$$

where $\bar{b}_m$ is a mollified version of $b$ which is potentially defined differently as in (4.2), and such that

$$\forall 0 < \varepsilon < 1,\ \lim_{m\to+\infty} \|\bar{b}_m - b\|_{L^\infty([0,T];C^{\bar{\gamma}-\varepsilon}(\mathbb{R}^d,\mathbb{R}^d))} = 0. \tag{4.38}$$

From the linearity of the equations, we then derive that $U_m := u^{m,\nu} - \bar{u}^{m,\bar{\nu}}$ solves the following Cauchy problem for any $(t,x) \in [0,T) \times \mathbb{R}^d$:

$$\begin{cases} \partial_t U_m(t,x) + \langle \bar{b}_m, \nabla U_m \rangle(t,x) - \bar{\nu} \Delta U_m(t,x) = (\nu - \bar{\nu}) \Delta u^{m,\nu}(t,x) - \langle [b_m - \bar{b}_m], \nabla u^{m,\nu} \rangle(t,x), \\ U_m(0,x) = 0. \end{cases} \tag{4.39}$$

By uniform control (4.101), we directly derive that

$$\begin{aligned} \|U_m(t,\cdot)\|_{L^\infty} &\leq \int_0^t \|\langle [b_m - \bar{b}_m], \nabla u^{m,\nu} \rangle(s,\cdot)\|_{L^\infty} ds + \int_0^t (\nu - \bar{\nu}) \|\Delta u^{m,\nu}(s,\cdot)\|_{L^\infty} ds \\ &\leq T \|b_m - \bar{b}_m\|_{L^\infty} \|\nabla u^{m,\nu}\|_{L^\infty} + T|\nu - \bar{\nu}| \|\Delta u^{m,\nu}\|_{L^\infty}. \end{aligned} \tag{4.40}$$

It is clear that if $b$ is $\tilde{\gamma}$-Hölder continuous then $\|b_m - \bar{b}_m\|_{L^\infty} \leq C m^{-\tilde{\gamma}}$. To take advantage of the convergence of $\|b_m - \bar{b}_m\|_{L^\infty}$ towards 0, we need to use other a priori controls.

**Lemma 2.** *If $u^{m,\nu}$ is solution of (4.36), then, for any $(t,x) \in [0,T] \times \mathbb{R}^d$, we get the gradient estimate*

$$|\nabla u^{m,\nu}(t,x)| \leq \int_0^t \|\nabla f_m(s,\cdot)\|_{L^\infty} ds + \|\nabla g_m\|_{L^\infty} + C m^{1-\tilde{\gamma}} \|b_m\|_{L^\infty(C^{\tilde{\gamma}})} \Big( t \|f_m\|_{L^\infty} + \|g_m\|_{L^\infty} \Big) t^{\frac{1}{2}},$$

*and he Hessian estimate*

$$\begin{aligned} |\nabla^2 u^{m,\nu}(t,x)| &\leq \Big( C \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f_m\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \Big) \\ &\quad + C m^{1-\tilde{\gamma}} \|b_m\|_{L^\infty(C^{\tilde{\gamma}})} \nu^{-\frac{1}{2}} t^{\frac{1}{2}} \big( t \|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty} \big) \\ &\quad + C m^{2(1-\tilde{\gamma})} \|b_m\|^2_{L^\infty(C^{\tilde{\gamma}})} \big( t \|f_m\|_{L^\infty} + \|g_m\|_{L^\infty} \big) \nu^{-\frac{1}{2}} t^{\frac{1}{2}}. \end{aligned}$$

The proof is deferred in Section 7.3.

Hence,

$$\begin{aligned} & \|U_m(t,\cdot)\|_{L^\infty} \\ \leq\ & T m^{-\tilde{\gamma}} \|b\|_{L^\infty(C^{\tilde{\gamma}})} \Big( t \|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty} + C \|b_m\|_{L^\infty(C^{\tilde{\gamma}})} \Big( t \|f_m\|_{L^\infty} + \|g_m\|_{L^\infty} \Big) \nu^{\frac{\tilde{\gamma}}{2}-1} t^{\frac{\tilde{\gamma}}{2}} \Big) \\ & + T|\nu - \bar{\nu}| \Bigg( \Big( C \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f_m\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \Big) \\ & + C \|b_m\|_{L^\infty(C^{\tilde{\gamma}})} \nu^{\frac{\tilde{\gamma}}{2}-1} t^{\frac{\tilde{\gamma}}{2}} \big( t \|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty} \big) + C \|b_m\|^2_{L^\infty(C^{\tilde{\gamma}})} \big( t \|f_m\|_{L^\infty} + \|g_m\|_{L^\infty} \big) \nu^{\tilde{\gamma}-2} t^{\tilde{\gamma}} \Bigg). \end{aligned}$$

From this estimate, to prove uniqueness, we need to consider

$$(m^{-\tilde{\gamma}} + \nu^{\frac{\tilde{\gamma}}{2}})(\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty}) + m^{-\tilde{\gamma}} \nu^{\frac{\tilde{\gamma}}{2}-1} + \nu \|\nabla^2 g_m\|_{L^\infty} + \nu^{\tilde{\gamma}-1} \ll 1, \tag{4.41}$$

the first and the third term are indeed negligible if $\tilde{\gamma} > \max(1-\gamma, \frac{1}{2})$.

We also write for the two last terms,

$$\nu^{\frac{\gamma-1}{2}} \ll m^{-1+2\tilde{\gamma}}, \text{ and } m^{1-\tilde{\gamma}} \ll \nu^{-\frac{\gamma}{2}}.$$

Combining the two terms, we get

$$m^{1-\tilde{\gamma}} \ll \nu^{-\frac{\gamma}{2}} \ll \big( m^{\frac{2(-1+2\tilde{\gamma})}{1-\gamma}} \big)^{\frac{\gamma}{2}} = m^{\frac{\gamma(-1+2\tilde{\gamma})}{1-\gamma}}.$$

Hence, we have to suppose that

$$1-\tilde{\gamma}<\frac{\gamma(-1+2\tilde{\gamma})}{1-\gamma}.$$

This is equivalent to

$$1-\tilde{\gamma}-\gamma+\gamma\tilde{\gamma}<\gamma(-1+2\tilde{\gamma}),$$

and

$$1<\tilde{\gamma}\gamma+\tilde{\gamma}=\tilde{\gamma}(1+\gamma),$$

thus the condition $\tilde{\gamma}>\frac{1}{1+\gamma}>\max(1-\gamma,\frac{1}{2})$ of Theorem 1.

We can have the equality case (useful for the Burgers' equation): $\tilde{\gamma}=\gamma$, if $P(\gamma)=\gamma^2+\gamma-1>0$ which means that $\gamma>\frac{-1+\sqrt{5}}{2}\asymp 0.6180$.

**Remark 7.** *We fail to get uniqueness for negative Besov regularity of $b$. Indeed, from the analysis performed in Section 4.9, and because we do not differentiate $U_m$ in order to take unsuccessfully take advantage of Grönwall's lemma, we need to consider the term by Besov duality*

$$\begin{aligned}\|\nabla\tilde{p}(s,t,x,\cdot)U_m(s,\cdot)\|_{\ddot{B}^{-\tilde{\gamma}}_{1,1}} &= \int_0^1 v^{-1}v^{1+\frac{\tilde{\gamma}}{2}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v h_v(z-y)\nabla\tilde{p}(s,t,x,y)U_m(s,y)dy\Big|dz\,dv\\ &\leq C\|U_m(s,\cdot)\|_{L^\infty}\int_0^1 v^{-1}v^{-\frac{\tilde{\gamma}}{2}}[\nu(t-s)]^{-\frac{1}{2}}dv,\end{aligned}$$

*which is finite if only $\tilde{\gamma}<-1$, that means that $b$ has to be Lipschitz continuous.*

### 4.5 Control of $\|\partial_t u(t,\cdot)\|_{B^{-1+\gamma}_{\infty,\infty}}$

If $b\in L^\infty([0,T];C^{\tilde{\gamma}}(\mathbb{R}^d,\mathbb{R}^d))$, $0<1-\gamma<\tilde{\gamma}$, we derive an upper-bound of $\|\partial_t u(t,\cdot)\|_{B^{-1+\gamma}_{\infty,\infty}}$ by the equation (1.1) and by para-product result. But first of all, let us precise why we have, point-wisely, with the viscous condition (4.45),

$$\lim_{(m,\nu)\to(+\infty,0)}\nu\Delta u^{m,\nu}(t,\cdot)=0. \tag{4.42}$$

Recalling that, from Lemma 3,

$$\|\nabla^2 u^{m,\nu}(t,\cdot)\|_{L^\infty}\leq\Big(m^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)+Ctm^{2-\tilde{\gamma}}\|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})}O_m(t)\Big)\exp(tm^{1-\tilde{\gamma}}\|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})}). \tag{4.43}$$

Hence, for

$$\nu\ll\Big(m^{2-\gamma}\big(T\|f\|_{L^\infty}+[g]_\gamma\big)+CTm^{2-\tilde{\gamma}}\|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})}O_m(t)\Big)^{-1}\exp(-Tm^{1-\tilde{\gamma}}\|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})}),$$

we deduce (4.42).

We are able to take the limit of equation (2.16), up to sub-sequence selection as explained in Section 4.2, for any $t\in(0,T]$,

$$\lim_{(m,\nu)\to(+\infty,0)}\partial_t u^{m,\nu}(t,\cdot)=\lim_{(m,\nu)\to(+\infty,0)}\langle b_m(t,\cdot),\nabla u^{m,\nu}(t,\cdot)\rangle+f(t,\cdot). \tag{4.44}$$

But from para-product (3.11), we know that $\langle b_m(t,\cdot),\nabla u^{m,\nu}(t,\cdot)\rangle\in B^{-1+\gamma}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$, $\nabla u^{m,\nu}(t,\cdot)$ being in $B^{-1+\gamma}_{\infty,\infty}$, the result then follows.

## 4.6 Control of the *limit Hölder modulus*

In this section, let us suppose that $b \in L^\infty([0,T], B^{-\beta}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$.

**Proposition 2** (Partial result for the transport equation)**.** *For $\gamma \in (0,1)$, $\beta \in \mathbb{R}^*$ be given. For all $b \in L^\infty([0,T], \tilde{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$, $f \in L^\infty([0,T]; C^\gamma_b(\mathbb{R}^d,\mathbb{R}))$ and $g \in C^\gamma_b(\mathbb{R}^d,\mathbb{R})$, if the conditions on the vanishing viscosity $0 < \nu < T^{-1}$, for a given constant $C > 0$ depending only on $(\gamma, d)$,*

$$1 \ll \nu^{-1}T^{-1}m^{-\frac{2-\gamma}{1-\gamma}}(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)^{-\frac{2}{1-\gamma}} \exp\Big(-c\frac{m^{1+\beta}T\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}}{1-\gamma}\Big), \tag{4.45}$$

*are satisfied, then $u^{m\nu}$ satisfies*

$$\begin{aligned} [u^{m,\nu}]^{(4.45)}_{\gamma,\nu^{1/2}T^{1/2}} &\le T \sup_{t\in[0,T]} [f(t,\cdot)]^{(4.45)}_{\gamma,\nu^{1/2}T^{1/2}} + [g]^{(4.45)}_{\gamma,\nu^{1/2}T^{1/2}}, \\ \|u\|_{L^\infty} &\le T\|f\|_{L^\infty} + \|g\|_{L^\infty}, \end{aligned} \tag{4.46}$$

**Remark 8.** *The exponential criterion in* (4.45) *relies on an a priori control by $\|u^{m,\nu}\|_{L^\infty}$, see Section D.1 for more details. This condition prevents us to hope any balance between $m$ and $\nu$ required to get usual uniqueness. Indeed, when we expand the computations, we see only polynomial dependency on $(m,\nu)$ in the upper-bounds. But the contribution on $\nu$ goes in the wrong way, and cannot be overwhelmed by polynomial converging terms in $m$, because at the best $m \sim |\ln(\nu)|$ from* (4.45)*.*

*Even for $b$ lying in a Hölder space, namely with a positive regularity, we cannot avoid an exponential criterion like in* (4.45) *by the current analysis, see again Section D.1.*

### 4.6.1 Some a priori controls

It is well-known that the unique solution $u^{m,\nu}$ of (4.1) is smooth, see [Fri64]. Some *a priori* controls, potentially blowing up in $m$ and $\nu$, are required in our analysis.

**Lemma 3.** *For $u^{m,\nu}$ strong solution of* (4.1)*, we have*

$$\|\nabla u^{m,\nu}(t,\cdot)\|_{L^\infty} \le \Big(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big)\exp\Big(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big) =: O_m(t), \tag{4.47}$$

*and*

$$\|\nabla^2 u^{m,\nu}(t,\cdot)\|_{L^\infty} \le Cm^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big)\exp(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}) =: O^{(2)}_m(t), \tag{4.48}$$

*also*

$$\|\nabla^3 u^{m,\nu}(t,\cdot)\|_{L^\infty} \le Cm^{3-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big)\exp(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}). \tag{4.49}$$

The proof is postponed in Section 7.1.

## 4.7 First part of the control of the limit modulus of continuity: the *cut locus* trick

For any $(t,x,x') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, we choose the associated *freezing* points $\big((\tau,\xi),(\tau',\xi')\big) \in \big([0,T]\times\mathbb{R}^d\big)^2$. Like in [CDRHM18], we take $\tau = \tau'$. The previous Duhamel formula (4.7) yields

$$\begin{aligned} & |u^{m,\nu}(t,x) - u^{m,\nu}(t,x')| \\ \le\ & |\hat{G}^{\tau,\xi}f_m(t,x) - \hat{G}^{\tau,\xi'}f_m(t,x')| + |\hat{P}^{\tau,\xi}g_m(t,x) - \hat{P}^{\tau,\xi'}g_m(t,x')| \\ & + \Big|\int_0^t\int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s,t,x,y)[b_m(s,\theta^m_{s,\tau}(\xi)) - b_m(s,y)]\cdot\nabla u^{m,\nu}(s,y)dy\,ds \\ & - \int_0^t\int_{\mathbb{R}^d} \hat{p}^{\tau',\xi'}(s,t,x,y)[b_m(s,\theta^m_{s,\tau'}(\xi')) - b_m(s,y)]\cdot\nabla u^{m,\nu}(s,y)dy\,ds\Big| \\ =:\ & |\hat{G}^{\tau,\xi}f_m(t,x) - \hat{G}^{\tau,\xi'}f_m(t,x')| + |\hat{P}^{\tau,\xi}g_m(t,x) - \hat{P}^{\tau,\xi'}g_m(t,x')| + |R^{\tau,\xi,\xi'}(t,x,x')|. \end{aligned} \tag{4.50}$$

However, our analysis need different choices of *freezing* point which yields extra contributions in the above inequality, the final Duhamel like identity is stated in (4.84) further.

**Remark 9.** *It seems to be useless to proceed with any integration by parts to transfer the gradient from* $u^{m,\nu}$ *to* $\hat{p}^{\tau,\xi}(s,t,x,y)b^m_\Delta[\tau,\xi]$ *and to upper-bound by* $\|u^{m,\nu}\|_{L^\infty(C^\gamma_b)}$*, a well-controlled norm, instead of* $\|\nabla u^{m,\nu}\|_{L^\infty}$*, a blowing up term.*

*Indeed, in diagonal regime,* $\hat{p}^{\tau,\xi}(s,t,x,y)$ *yields a term of the type* $[\nu(t-s)]^{-\frac{1}{2}}$ *which has to smoothen by Lipschitz norms* $\|u^{m,\nu}(t,\cdot)\|_{C^1}$ *in order to get a contribution of* $|x-x'|^\gamma$ *and a positive contribution of* $\nu$*. For more details, we provide in Section D.1 a full computation associated with the remainder* $R^{\tau,\xi,\xi'}(t,x,x')$ *which necessarily implies to upper-bound by a Lipschitz norm of* $u^{m,\nu}$*.*

*This last value is finite but increases exponentially with* $m$*, see Lemma 3. This exponential blowing-up, without the time decomposition trick introduced in Section 5, yields the limit criterion* (4.45) *of* $(m,\nu)$*, and prevents us to get any balance between* $m$ *and* $\nu$ *to obtain uniqueness of solution.*

### 4.7.1 Main terms

For the main contributions associated with $f$ and $g$, we have to suppose that the *diagonal* regime specified further is in force. Precisely, we assume that there are $\alpha_1,\alpha_2>0$ such that

$$|x-x'|\le \nu^{\alpha_1}T^{\alpha_2}. \tag{4.51}$$

In this case, we choose the *freezing* parameters to be $\tau=t$ and $\xi=\xi'=x$.

Let us carefully point out that condition (4.51) leads to the limit modulus of continuity of the type of (2.6) when $\nu\to 0$, instead of the usual Hölder modulus. We precise the reverse condition in Section 4.10.1

This constraint is crucial, in order to avoid the sensitivity of the flow on different points $x,x'$. However, the assumption (4.51) yields to consider the limit Hölder continuity defined in (2.6).

**Semi-group**

We readily derive by change of variables:

$$\begin{aligned}
&|\hat{P}^{\tau,\xi}g_m(t,x)-\hat{P}^{\tau,\xi'}g_m(t,x')|\Big|_{\tau=t,\xi=\xi'=x}\\
=\;&\Big|\int_{\mathbb{R}^d}[\hat{p}^{\tau,\xi}(0,t,x,y)-\hat{p}^{\tau,\xi'}(0,t,x',y)]g_m(y)dy\Big|\Big|_{\tau=t,\xi=\xi'=x}\\
=\;&\Big|\int_{\mathbb{R}^d}[\hat{p}^{t,x}(0,t,0,y)[g_m(x+y)-g_m(x'+y)]dy\Big|\\
\le\;&[g]_{\gamma,\nu^{\alpha_1}(t-s)^{\alpha_2}}|x-x'|^\gamma.
\end{aligned}\tag{4.52}$$

**Green operator**

We also get by change of variables:

$$\begin{aligned}
&|\hat{G}^{\tau,\xi}f_m(t,x)-\hat{G}^{\tau,\xi'}f_m(t,x')|\Big|_{\tau=t,\xi=\xi'=x}\\
=\;&\Big|\int_0^t\int_{\mathbb{R}^d}[\hat{p}^{t,x}(s,t,x,y)-\hat{p}^{t,x}(s,t,x',y)]f_m(s,y)dy\,ds\Big|\\
=\;&\Big|\int_0^t\int_{\mathbb{R}^d}[\hat{p}^{t,x}(s,t,0,y)[f_m(s,x-y)-f_m(s,x'-y)]ds\,dy\Big|\\
\le\;&\int_0^t[f(s,\cdot)]_{\gamma,\nu^{\alpha_1}(t-s)^{\alpha_2}}ds|x-x'|^\gamma.
\end{aligned}\tag{4.53}$$

### 4.7.2 Remainder term

To analyse the Hölder modulus of the remainder term, the core of the *a priori* controls, we separate the *diagonal* regime from the *off-diagonal* one, as performed in [CDRHM18]. This strategy is natural in view with the vanishing viscous solution selected by a parabolic approximation.

However, in the vanishing viscosity context, we have to carefully track the dependency on $\nu$ which yields, for our first approach, to consider a unusual criterion of *diagonal / off-diagonal* regime, unlike for the standard parabolic scaling.

Specifically, for any $x, x' \in \mathbb{R}^d$ and for given parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, to be tailored further, we call *off-diagonal* regime the case $|x - x'| > \nu^{\alpha_1}(t-s)^{\alpha_2} \Leftrightarrow s > t_0$ with

$$t_0 := t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}, \tag{4.54}$$

called the *cut locus* time.

On the contrary the *diagonal* regime holds when $|x-x'| \leq \nu^{\alpha_1}(t-s)^{\alpha_2} \Leftrightarrow s \leq t_0$. This regime can be in force only if $|x - x'|$ is small enough, specifically, only if $|x - x'| \leq \nu^{\alpha_1} T^{\alpha_2}$, thus the definition of the limit modulus of continuity (2.6).

The point $t_0$ can be regarded as a *cut-locus* point where we "catch" the shortest path from $u^{m,\nu}(t,x)$ to $u^{m,\nu}(t,x')$ if $t \in [0, t_0]$ and we choose another way if $t_0 < t$.

This procedure yields an extra contribution in (4.50), this is detailed in Sections 4.7.6, 4.7.7.

We specify in Section 4.7.8 below why it is possible to choose different *freezing* parameters for the remainder term according to the current regime; meanwhile the semi-group and the Green operator dealt in Section 4.7.1 has somehow to stay in the *diagonal* regime.

### 4.7.3 Diagonal regime

If $|x-x'| \leq \nu^{\alpha_1}(t-s)^{\alpha_2} \Leftrightarrow s \leq t_0 = t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x-x'|^{\frac{1}{\alpha_2}}$, the points $x$ and $x'$ are supposed to be closed from each other, then we pick $\xi = \xi' = x$, also $\tau = t$, then we define the associated space

$$A(x, x', \nu, t)(s) := \{|x - x'| \leq \nu^{\alpha_1}(t-s)^{\alpha_2}\}, \tag{4.55}$$

with the indicator function

$$\mathbb{1}_{A(x,x',\nu,t)}(s) := \begin{cases} 1 \text{ if } |x - x'| \leq \nu^{\alpha_1}(t-s)^{\alpha_2}, \\ 0 \text{ if } |x - x'| > \nu^{\alpha_1}(t-s)^{\alpha_2}, \end{cases}$$

also the associated remainder term

$$\begin{aligned} R_A^{\tau,\xi,\xi'}(t,x,x') \quad := \quad & \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \Big[ \hat{p}^{\tau,\xi}(s,t,x,y)[b_m(s, \theta^m_{s,\tau}(\xi)) - b_m(s,y)] \cdot \nabla u^{m,\nu}(s,y) \\ & - \hat{p}^{\tau',\xi'}(s,t,x,y)[b_m(s, \theta^m_{s,\tau'}(\xi')) - b_m(s,y)] \cdot \nabla u^{m,\nu}(s,y) \Big] dy\, ds, \end{aligned} \tag{4.56}$$

which equivalently write,

$$\begin{aligned} & R_A^{\tau,\xi,\xi'}(t,x,x') \\ := \quad & \int_0^{t_0} \int_{\mathbb{R}^d} \Big[ \hat{p}^{\tau,\xi}(s,t,x,y)[b_m(s, \theta^m_{s,\tau}(\xi)) - b_m(s,y)] \cdot \nabla u^{m,\nu}(s,y) \\ & - \hat{p}^{\tau',\xi'}(s,t,x',y)[b_m(s, \theta^m_{s,\tau'}(\xi')) - b_m(s,y)] \cdot \nabla u^{m,\nu}(s,y) \Big] dy\, ds \\ =: \quad & \hat{G}^{\tau,\xi}_{0,t_0} \Big\{ \big(b_m(\cdot, \theta^m_{\cdot,\tau}(\xi)) - b_m\big) \cdot \nabla u^{m,\nu} \Big\}(t,x) - \hat{G}^{\tau,\xi'}_{0,t_0} \Big\{ \big(b_m(\cdot, \theta^m_{\cdot,\tau}(\xi')) - b_m\big) \cdot \nabla u^{m,\nu} \Big\}(t,x'). \end{aligned} \tag{4.57}$$

A simple change of variable gives

$$\begin{aligned}
&|R_A^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\\
=\;& \Big|\int_0^{t_0}\int_{\mathbb{R}^d}\hat{p}^{\tau,\xi}(s,t,0,y)\Big\{\big(b_m(s,\theta^m_{s,\tau}(\xi))-b_m(s,x+y)\big)\cdot\nabla u^{m,\nu}(s,x+y)\\
&-\big(b_m(s,\theta^m_{s,\tau}(\xi'))-b_m(s,x'+y)\big)\cdot\nabla u^{m,\nu}(s,x'+y)\Big\}dy\,ds\Big|\Big|_{\tau=t,\xi=\xi'=x},
\end{aligned}$$

$$\begin{aligned}
&|R_A^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\\
=\;& \Big|\int_0^{t_0}\int_{\mathbb{R}^d}\hat{p}^{\tau,\xi}(s,t,0,y)\Big\{b_m(s,\theta^m_{s,\tau}(x))\cdot\big(\nabla u^{m,\nu}(s,x+y)-\nabla u^{m,\nu}(s,x'+y)\big)\\
&+\big(b_m(s,x'+y)-b_m(s,x+y)\big)\cdot\nabla u^{m,\nu}(s,x'+y)\\
&-b_m(s,x+y)\cdot\big(\nabla u^{m,\nu}(s,x+y)-\nabla u^{m,\nu}(s,x'+y)\big)\Big\}dy\,ds\Big|\Big|_{\tau=t,\xi=\xi'=x}.
\end{aligned}$$

Hence, we readily derive that

$$|R_A^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\le C|x-x'|\|b_m\|_{L^\infty(C^1_b)}\|\nabla u^{m,\nu}\|_{L^\infty(C^1_b)}\int_0^{t_0}ds. \tag{4.58}$$

Recalling that we have to use the *a priori* controls of the gradient and the Hessian in Lemma 3,

$$\begin{aligned}
\|u^{m,\nu}(t,\cdot)\|_{C^1} &\le \Big(T\|f_m\|_{L^\infty(C^1)}+\|g_m\|_{C^1}\Big)\exp\Big(m^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big)=O_m(t),\\
\|\nabla^2u^{m,\nu}(t,\cdot)\|_{L^\infty} &\le Cm^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)\exp(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})})=O^{(2)}_m(t).
\end{aligned} \tag{4.59}$$

From the definition of the *diagonal* regime, $|x-x'|\le|x-x'|^\gamma\nu^{\alpha_1(1-\gamma)}(t-s)^{(1-\gamma)\alpha_2}$, we deduce

$$|R_A^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\le C|x-x'|^\gamma\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}m^{1+\beta}(O_m(t)+O^{(2)}_m(t))\int_0^{t_0}\nu^{\alpha_1(1-\gamma)}(t-s)^{(1-\gamma)\alpha_2}ds. \tag{4.60}$$

If $(1-\gamma)\alpha_2>-1\Leftrightarrow\alpha_2>-\frac{1}{1-\gamma}>-1$, the above time integral is finite, and after integration,

$$|R_A^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\le C|x-x'|^\gamma\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}m^{1+\beta}O_m(t)\nu^{\alpha_1(1-\gamma)}t^{1+(1-\gamma)\alpha_2}. \tag{4.61}$$

To consider vanishing viscous, it is necessary to have $\alpha_1(1-\gamma)>0\Leftrightarrow\alpha_1>0$ for $\gamma<1$.

To sum up, we consider the constraints

$$\begin{aligned}
\alpha_1 &> 0,\\
\alpha_2 &> -\frac{1}{1-\gamma}.
\end{aligned} \tag{4.62}$$

### 4.7.4 Off-diagonal regime

In this case, $x$ and $x'$ are supposed to be "far away" from each other, then we pick $\xi=x$ and $\xi'=x'$. If $|x-x'|>\nu^{\alpha_1}(t-s)^{\alpha_2}\Leftrightarrow s>t-\nu^{-\frac{\alpha_1}{\alpha_2}}|x-x'|^{\frac{1}{\alpha_2}}$, we recall the corresponding *cut locus* time

$$t_0=t-\nu^{-\frac{\alpha_1}{\alpha_2}}|x-x'|^{\frac{1}{\alpha_2}}.$$

In this case, we choose as *freezing* parameters $\xi = x$ and $\xi' = x'$. In the *off-diagonal* regime, the associated space is

$$A^c(x,x',\nu,t)(s) := \{|x-x'| > \nu^{\alpha_1}(t-s)^{\alpha_2}\}, \tag{4.63}$$

the indicator function

$$\mathbb{1}_{A^c(x,x',\nu,t)}(s) = 1 - \mathbb{1}_{A(x,x',\nu,t)}(s) = \begin{cases} 1 \text{ if } |x-x'| > \nu^{\alpha_1}(t-s)^{\alpha_2}, \\ 0 \text{ if } |x-x'| \leq \nu^{\alpha_1}(t-s)^{\alpha_2}, \end{cases}$$

and the associated remainder terms are

$$\begin{aligned} & R_{A^c}^{\tau,\xi,\xi'}(t,x,x') \\ := \;& R^{\tau,\xi,\xi'}(t,x,x') - R_{A}^{\tau,\xi,\xi'}(t,x,x') \\ = \;& \int_{t_0}^t \int_{\mathbb{R}^d} \Big[ \hat{p}^{\tau,\xi}(s,t,x,y)\big(b_m(s,\theta_{s,\tau}^m(\xi)) - b_m(s,y)\big)\cdot \nabla u^{m,\nu}(s,y) \\ & - \hat{p}^{\tau',\xi'}(s,t,x',y)\big(b_m(s,\theta_{s,\tau}^m(\xi')) - b_m(s,y)\big)\cdot \nabla u^{m,\nu}(s,y)\Big] dy\, ds \\ =: \;& \hat{G}_{t_0,t}^{\tau,\xi}\Big\{\big(b_m(\cdot,\theta_{\cdot,\tau}^m(\xi)) - b_m\big)\cdot\nabla u^{m,\nu}\Big\}(t,x) - \hat{G}_{t_0,t}^{\tau,\xi'}\Big\{\big(b_m(\cdot,\theta_{\cdot,\tau}^m(\xi')) - b_m\big)\cdot\nabla u^{m,\nu}\Big\}(t,x'). \end{aligned}$$

The analysis is direct by triangular inequality

$$|R_{A^c}^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=x,\xi'=x'} \leq 2C\|b_m\|_{L^\infty(C^1)}\|u^{m,\nu}\|_{L^\infty(C^1)} \sup_{x\in\mathbb{R}^d}\int_{t_0}^t\int_{\mathbb{R}^d} \bar{p}^{t,x}(s,t,x,y)|y-\theta_{s,t}^m(x)|dy\, ds,$$

the absorbing property of the exponential (2.13) gives

$$\begin{aligned} |R_{A^c}^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=x,\xi'=x'} &\leq Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}O_m(t)\int_{t_0}^t \big(\nu(t-s)\big)^{\frac{1}{2}} ds \\ &\leq Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}O_m(t)\int_{t_0}^t \nu^{\frac{1}{2}}(t-s)^{\frac{1-\varepsilon}{2}}\nu^{-\frac{\alpha_1\varepsilon}{2\alpha_2}}|x-x'|^{\frac{\varepsilon}{2\alpha_2}} ds, \end{aligned} \tag{4.64}$$

for a given $0<\varepsilon<1$, and because the *off-diagonal* regime is in force, $|x-x'| \geq \nu^{\alpha_1}(t-s)^{\alpha_2}$.

The solution $u^{m,\nu}$ is supposed to have a $\gamma$ regularity in the limit modulus of continuity (2.4) then

$$\gamma = \frac{\varepsilon}{2\alpha_2} \;\Rightarrow\; \varepsilon = 2\alpha_2\gamma > 0.$$

Also, we have to suppose that $\varepsilon < 1$ (for positive time contribution purpose) which implies

$$\alpha_2 < \frac{1}{2\gamma}. \tag{4.65}$$

Let us consider equality between the two regimes from (4.60) and (4.64), i.e.

$$(1-\gamma)\alpha_2 = \frac{1-2\alpha_2\gamma}{2} \;\Leftrightarrow\; \alpha_2 = \frac{1}{2} < \frac{1}{2\gamma}, \tag{4.66}$$

which is exactly the parabolic scale, and also

$$\varepsilon = \gamma.$$

This is another way to see the difficulty to get a Lipschitz control, in our framework, because $\varepsilon$ has to be strictly lower than 1.

$$|R_{A^c}^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=x,\xi'=x'} \leq Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}O_m(t)\int_{t_0}^t \nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}}|x-x'|^\gamma ds, \tag{4.67}$$

### 4.7.5 Final control of $R^{\tau,\xi,\xi'}(t,x,x')$

We then conclude

$$\begin{aligned}&|R^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\\ \leq\ & Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}|x-x'|^\gamma(O_m(t)+O_m^{(2)}(t))\int_0^t(\nu^{\alpha_1(1-\gamma)}+\nu^{\frac12-\alpha_1\gamma})(t-s)^{\frac{1-\gamma}{2}}ds.\end{aligned}$$

Let us also consider the equality of the exponent of $\nu$,

$$\alpha_1(1-\gamma)=\frac12-\alpha_1\gamma\ \Leftrightarrow\ \alpha_1=\frac12,$$

the usual parabolic scale $\alpha_1=\alpha_2=\frac12$ is finally in force. We deduce,

$$|R^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x}\leq Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}|x-x'|^\gamma(O_m(t)+O_m^{(2)}(t))\int_0^t\nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}}ds. \tag{4.68}$$

### 4.7.6 On the discontinuous choice of freezing parameters : consequence for $u^{m,\nu}(t,x')$

Let us carefully point out that even if the solution $u^{m,\nu}$ **does not** depend on the corresponding freezing parameter $\xi$, the choice of $\xi$ in this section **does** depend on the current time variable of integration $s$.

Therefore, like for the approach developed in [CDRHM18], the *cut locus* time yields an additional contribution.

Previously, in the Hölder norm controls, we considered two points $(x,x')\in\mathbb{R}^d\times\mathbb{R}^d$. Let us specify how to write the solution $u^{m,\nu}(t,x')$ with the different choices of freezing parameter $\xi'$ depending on the time variable of integration $s$. To do so, we first rewrite the theoretical representation of the solution where the initial time is $r\in[0,T]$, and the initial function is replaced by $u^{m,\nu}(r,x')$,

$$u^{m,\nu}(t,x')=P_r^{m,\nu}u^{m,\nu}(r,x')+G_r^{m,\nu}f_m(t,x'), \tag{4.69}$$

where $P_{r,t}^{m,\nu}$ and $G_{r,t}^{m,\nu}$ stand respectively for the semi-group and the Green operator associated with the Cauchy problem

$$\begin{cases}\partial_t u^{m,\nu}(t,x)+\langle b_m(t,x),\nabla u^{m,\nu}(t,x)\rangle-\nu\Delta u^{m,\nu}(t,x)=f_m(t,x),\ (t,x)\in(r,T]\times\mathbb{R}^d,\\ u^{m,\nu}(r,x)=u^{m,\nu}(r,x),\ x\in\mathbb{R}^d.\end{cases} \tag{4.70}$$

We also write

$$u^{m,\nu}(t,x)=\hat P_r^{\tau,\xi}u^{m,\nu}(r,\cdot)(t,x)+\hat G_r^{\tau,\xi}f_m(t,x)+\hat G_r^{\tau,\xi}\big(b_\Delta^m[\tau,\xi]\cdot\nabla u^{m,\nu}\big)(t,x), \tag{4.71}$$

where the operators are defined by

$$\forall(t,x)\in(0,T]\times\mathbb{R}^d,\ \hat G_r^{\tau,\xi}f_m(t,x):=\int_r^t\int_{\mathbb{R}^d}\hat p^{\tau,\xi}(s,t,x,y)f(s,y)dy\,ds, \tag{4.72}$$

and

$$\hat P_r^{\tau,\xi}g_m(t,x):=\int_{\mathbb{R}^d}\hat p^{\tau,\xi}(r,t,x,y)g_m(y)dy. \tag{4.73}$$

Let us recall the definition of the transition time

$$t_0:=t-\nu^{-\frac{\alpha_1}{\alpha_2}}|x-x'|^{\frac{1}{\alpha_2}}=t-\nu^{-1}|x-x'|^2. \tag{4.74}$$

If $t_0 \le 0 \Leftrightarrow t \le \nu^{-1}|x-x'|^2$, the *off-diagonal* regime is in force, then we pick $\xi' = x'$ and there is no intricate choice of the *freezing* parameter. However, we cannot have the same control as (4.52), we indeed need to handle with the flow regularity by the choice of different *proxy*, see Section 4.10.1 further.

However, if $t_0 > 0 \Leftrightarrow t > \nu^{-1}|x-x'|^2$, we need to be more subtle to handle with the dependency on $s$ for the value choice of $\xi' \in \mathbb{R}^d$. From now on, we suppose that $t_0 > 0$. In this case, we can get identity (4.52), but the condition $|x-x'|^2 < \nu t$ implies that we do consider the *limit Hölder continuity* defined in (2.6) instead of the usual Hölder modulus.

We next differentiate (4.71)w.r.t. $r$

$$0 = \partial_r\big(\hat{P}_r^{\tau,\xi'} u^{m,\nu}(r,\cdot)\big)(t,x') + \partial_r \hat{G}_r^{\tau,\xi'} f_m(t,x') + \partial_r \hat{G}_r^{\tau,\xi'}\big(b_\Delta^m[\tau,\xi']\cdot\nabla u^{m,\nu}\big)(t,x'). \tag{4.75}$$

We integrate the variable $r$ between $[t_0,t]$ with the *proxy* parameter $\xi' \in \mathbb{R}^d$,

$$0 = u^{m,\nu}(t,x') - [P_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') - G_{t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{t_0}^{\tau,\xi'}\big(b_\Delta^m[\tau,\xi']\cdot\nabla u^{m,\nu}\big)(t,x'),$$

which yields for $t \in [t_0,T]$

$$u^{m,\nu}(t,x') = [\hat{P}_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') + G_{t_0}^{\tau,\xi'} f_m(t,x') + \hat{G}_{t_0}^{\tau,\xi'}\big(b_\Delta^m[\tau,\xi']\cdot\nabla u^{m,\nu}\big)(t,x'). \tag{4.76}$$

Next, we integrate in time between $[0,t_0]$ with a different *freezing* parameter $\tilde{\xi}' \in \mathbb{R}^d$,

$$\begin{aligned} 0 &= [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x') - \hat{P}_0^{\tau,\tilde{\xi}'} g_m(t,x') + \hat{G}_{t_0}^{\tau,\tilde{\xi}'} f_m(t,x') - \hat{G}_0^{\tau,\tilde{\xi}'} f_m(t,x') \\ &\quad + \hat{G}_{t_0}^{\tau,\tilde{\xi}'}\big(b_\Delta^m[\tau,\tilde{\xi}']\cdot\nabla u^{m,\nu}\big)(t,x') - \hat{G}_0^{\tau,\tilde{\xi}'}\big(b_\Delta^m[\tau,\tilde{\xi}']\cdot\nabla u^{m,\nu}\big)(t,x'). \end{aligned}$$

Hence,

$$\begin{aligned} u^{m,\nu}(t,x') &= [\hat{P}_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') + G_{t_0}^{\tau,\xi'} f_m(t,x') + \hat{G}_{t_0}^{\tau,\xi'}\big(b_\Delta^m[\tau,\xi']\cdot\nabla u^{m,\nu}\big)(t,x') \\ &\quad - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x') + \hat{P}_0^{\tau,\tilde{\xi}'} g_m(t,x') - \hat{G}_{t_0}^{\tau,\tilde{\xi}'} f_m(t,x') + \hat{G}_0^{\tau,\tilde{\xi}'} f_m(t,x') \\ &\quad - \hat{G}_{t_0}^{\tau,\tilde{\xi}'}\big(b_\Delta^m[\tau,\tilde{\xi}']\cdot\nabla u^{m,\nu}\big)(t,x') + \hat{G}_0^{\tau,\tilde{\xi}'}\big(b_\Delta^m[\tau,\tilde{\xi}']\cdot\nabla u^{m,\nu}\big)(t,x'). \end{aligned}$$

Defining

$$\forall (t',x) \in [0,t]\times\mathbb{R}^d,\ \hat{G}_{r,t'}^{\tau,\xi} f_m(t,x) := \int_r^{t'}\int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s,t,x,y) f(s,y)\,dy\,ds, \tag{4.77}$$

we write

$$\begin{aligned} u^{m,\nu}(t,x') &= [\hat{P}_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x') + G_{t_0}^{\tau,\xi'} f_m(t,x') \\ &\quad + \hat{G}_{t_0}^{\tau,\xi'}\big(b_\Delta^m[\tau,\xi']\cdot\nabla u^{m,\nu}\big)(t,x') + \hat{P}_0^{\tau,\tilde{\xi}'} g_m(x') + \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') \\ &\quad + \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'}\big(b_\Delta^m[\tau,\tilde{\xi}']\cdot\nabla u^{m,\nu}\big)(t,x'). \end{aligned} \tag{4.78}$$

There is an extra contribution $[\hat{P}_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x')$ due to the discontinuous *freezing* choice, the other terms match with the ones appearing in the above computations.

### 4.7.7 Extra contribution $[\hat{P}_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x')$

Thanks to a change of variables, we readily obtain

$$\begin{aligned} &[\hat{P}_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x') \\ &= \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi'}(t_0,t,x',y) u^{m,\nu}(t_0,y)dy - \int_{\mathbb{R}^d} \hat{p}^{\tau,\tilde{\xi}'}(t_0,t,x',y) u^{m,\nu}(t_0,y)dy \\ &= \int_{\mathbb{R}^d} \tilde{p}(t_0,t,x',y)\Big[u^{m,\nu}\big(t_0,y+\int_{t_0}^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau'}(\xi'))d\tilde{s}\big)dy - u^{m,\nu}\big(t_0,y+\int_{t_0}^t b_m(\tilde{s},\theta^m_{\tilde{s},\tilde{\tau}'}(\tilde{\xi}')d\tilde{s})\big)\Big]dy, \end{aligned}$$

recalling that $\tilde{p}(t_0,t,x',y)$ stands for the usual heat kernel defined in (5.3). Therefore,

$$\begin{aligned}
&\Big|[\hat{P}_{t_0}^{\tau,\xi'}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'}u^{m,\nu}(t_0,\cdot)](t,x')\Big| \\
\leq\ & \|u^{m,\nu}\|_{L^\infty(C^1)}\int_{t_0}^t \big|b_m(\tilde{s},\theta_{\tilde{s},\tau}^m(\xi')) - b_m(\tilde{s},\theta_{\tilde{s},\tau}^m(\tilde{\xi}')\big|d\tilde{s} \\
\leq\ & \|u^{m,\nu}\|_{L^\infty(C^1)}\|b_m\|_{L^\infty(C^1)}\int_{t_0}^t |\theta_{\tilde{s},\tau'}^m(\xi') - \theta_{\tilde{s},\tau}^m(\tilde{\xi}')|d\tilde{s}.
\end{aligned}$$

Additionally, we get the *a priori* control for the flow.

**Lemma 4.** *For all $(x,x')\in\mathbb{R}^d\times\mathbb{R}^d$ and $0\leq s\leq\tau\leq T$:*

$$\sup_{\tilde{s}\in[0,\tau]}|\theta_{s,\tau}^m(x) - \theta_{s,\tau}^m(x')| \ \leq\ |x-x'|\exp\big(\|b_m\|_{L^\infty(C^1)}\tau\big).$$

The proof is postponed in Section 7.2.

We then deduce for $(\tau,\xi',\tilde{\xi}') = (t,x',x)$ that

$$\begin{aligned}
&\Big|[\hat{P}_{t_0}^{\tau,\xi'}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'}u^{m,\nu}(t_0,\cdot)](t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & \|u^{m,\nu}\|_{L^\infty(C^1)}\|b_m\|_{L^\infty(C^1)}\int_{t_0}^t |x-x'|\exp\big(\|b_m\|_{L^\infty(C^1)}t\big)d\tilde{s} \\
\leq\ & \|u^{m,\nu}\|_{L^\infty(C^1)}\|b_m\|_{L^\infty(C^1)}|x-x'|(t-t_0)\exp\big(\|b_m\|_{L^\infty(C^1)}t\big)d\tilde{s}.
\end{aligned}$$

Recalling that, see Lemma 3,

$$\|\nabla u^{m,\nu}(t,\cdot)\|_{L^\infty} \leq \Big(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big)\exp\Big(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big) =: O_m(t).$$

Therefore,

$$\begin{aligned}
&\Big|[\hat{P}_{t_0}^{\tau,\xi'}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'}u^{m,\nu}(t_0,\cdot)](t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & m^{1+\beta}O_m(t)\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}|x-x'|(t-t_0)\exp\big(\|b_m\|_{L^\infty(C^1)}t\big). \qquad (4.79)
\end{aligned}$$

Because $|x-x'| = \nu^{\alpha_1}(t-t_0)^{\alpha_1}$, we get

$$\begin{aligned}
&\Big|[\hat{P}_{t_0}^{\tau,\xi'}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'}u^{m,\nu}(t_0,\cdot)](t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & |x-x'|^\gamma m^{1+\beta}O_m(t)\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{1-\gamma}{2}+1}\exp\big(\|b_m\|_{L^\infty(C^1)}t\big). \qquad (4.80)
\end{aligned}$$

As we have supposed that $t_0\geq 0$, it follows that

$$\begin{aligned}
&\Big|[\hat{P}_{t_0}^{\tau,\xi'}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'}u^{m,\nu}(t_0,\cdot)](t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & t^{\frac{1-\gamma}{2}+1}|x-x'|^\gamma m^{1+\beta}O_m(t)\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\nu^{\frac{1-\gamma}{2}}\exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}t\big). \qquad (4.81)
\end{aligned}$$

We finally derive, from definition of $O_m(t)$ and by exponential absorption (2.13),

$$\begin{aligned}
&\Big|[\hat{P}_{t_0}^{\tau,\xi'}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'}u^{m,\nu}(t_0,\cdot)](t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & Ct^{\frac{1-\gamma}{2}}|x-x'|^\gamma\nu^{\frac{1-\gamma}{2}}\Big(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big)\exp\Big(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big). \qquad (4.82)
\end{aligned}$$

The above term goes to 0 with the vanishing condition

$$\nu \ll T^{-1}\Big(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big)^{-\frac{2}{1-\gamma}} \exp\big(-C\frac{m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}T}{1-\gamma}\big). \tag{4.83}$$

### 4.7.8 Justification of the *freezing* point change

Eventually, let us explicitly write the difference of the two final Duhamel formulas associated with $u^{m,\nu}(t,x)$ and $u^{m,\nu}(t,x')$.

For any $(t,x,x') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ and $(\tau,\xi,\xi',\tilde{\xi}') \in [0,T]\times\mathbb{R}^d$, we write from (4.78),

$$\begin{aligned} u^{m,\nu}(t,x) - u^{m,\nu}(t,x') &= \Big[\hat{P}^{\tau,\xi}g_m(t,x) - \hat{P}^{\tau,\tilde{\xi}'}g_m(t,x')\Big] + \Big[\hat{G}^{\tau,\xi}_{t_0,t}f_m(t,x) - \hat{G}^{\tau,\xi'}_{t_0,t}f_m(t,x')\Big] \\ &\quad + \Big[\hat{G}^{\tau,\xi}_{0,t_0}f_m(t,x) - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0}f_m(t,x')\Big] \\ &\quad + \Big[\tilde{G}^{\tau,\xi}_{t_0,t}\big[b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big](t,x) - \tilde{G}^{\tau,\xi'}_{t_0,t}\big[b^{m'}_\Delta[\tau,\xi']\cdot\nabla u^{m,\nu}\big](t,x')\Big] \\ &\quad + \Big[\tilde{G}^{\tau,\xi}_{0,t_0}\big[b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big](t,x) - \tilde{G}^{\tau,\xi'}_{0,t_0}\big[b^{m'}_\Delta[\tau,\xi']\cdot\nabla u^{m,\nu}\big](t,x')\Big] \\ &\quad + [\hat{P}^{\tau,\xi'}_{t_0}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}^{\tau,\tilde{\xi}'}_{t_0}u^{m,\nu}(t_0,\cdot)](t,x'). \end{aligned}$$

Because the l.h.s. of the first equality does not depend on $(\xi,\xi')$, we can get the infimum over these *freezing* points, namely

$$\begin{aligned} & u^{m,\nu}(t,x) - u^{m,\nu}(t,x') \\ = \inf_{\xi,\xi',\tilde{\xi}'\in\mathbb{R}} \Bigg\{ & \Big[\hat{P}^{\tau,\xi}g_m(t,x) - \hat{P}^{\tau,\tilde{\xi}'}g_m(t,x')\Big] + \Big[\hat{G}^{\tau,\xi}_{t_0,t}f_m(t,x) - \hat{G}^{\tau,\xi'}_{t_0,t}f_m(t,x')\Big] \\ & + \Big[\hat{G}^{\tau,\xi}_{0,t_0}f_m(t,x) - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0}f_m(t,x')\Big] \\ & + \Big[\tilde{G}^{\tau,\xi}_{t_0,t}\big[b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big](t,x) - \tilde{G}^{\tau,\xi'}_{t_0,t}\big[b^{m'}_\Delta[\tau,\xi']\cdot\nabla u^{m,\nu}\big](t,x')\Big] \\ & + \Big[\tilde{G}^{\tau,\xi}_{0,t_0}\big[b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big](t,x) - \tilde{G}^{\tau,\xi'}_{0,t_0}\big[b^{m'}_\Delta[\tau,\xi']\cdot\nabla u^{m,\nu}\big](t,x')\Big] \\ & + [\hat{P}^{\tau,\xi'}_{t_0}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}^{\tau,\tilde{\xi}'}_{t_0}u^{m,\nu}(t_0,\cdot)](t,x') \Bigg\}. \end{aligned}$$

However, we aim to control the source functions term only in the *diagonal* regime, see Section D.2 below for details. We rewrite,

$$\begin{aligned} & \Big[\hat{G}^{\tau,\xi}_{t_0,t}f_m(t,x) - \hat{G}^{\tau,\xi'}_{t_0,t}f_m(t,x')\Big] + \Big[\hat{G}^{\tau,\xi}_{0,t_0}f_m(t,x) - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0}f_m(t,x')\Big] \\ = & \Big[\hat{G}^{\tau,\xi}f_m(t,x) - \hat{G}^{\tau,\tilde{\xi}'}f_m(t,x')\Big] + \Big[\hat{G}^{\tau,\xi'}_{0,t_0}f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0}f_m(t,x')\Big], \end{aligned}$$

and,

$$\begin{aligned} & u^{m,\nu}(t,x) - u^{m,\nu}(t,x') \\ = \inf_{\xi,\xi',\tilde{\xi}'\in\mathbb{R}} \Bigg\{ & \Big[\hat{P}^{\tau,\xi}g_m(t,x) - \hat{P}^{\tau,\tilde{\xi}'}g_m(t,x')\Big] + \Big[\hat{G}^{\tau,\xi}f_m(t,x) - \hat{G}^{\tau,\tilde{\xi}'}f_m(t,x')\Big] \\ & + \Big[\hat{G}^{\tau,\xi'}_{0,t_0}f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0}f_m(t,x')\Big] \\ & + \Big[\tilde{G}^{\tau,\xi}_{t_0,t}\big[b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big](t,x) - \tilde{G}^{\tau,\xi'}_{t_0,t}\big[b^{m'}_\Delta[\tau,\xi']\cdot\nabla u^{m,\nu}\big](t,x')\Big] \\ & + \Big[\tilde{G}^{\tau,\xi}_{0,t_0}\big[b^m_\Delta[\tau,\xi]\cdot\nabla u^{m,\nu}\big](t,x) - \tilde{G}^{\tau,\xi'}_{0,t_0}\big[b^{m'}_\Delta[\tau,\xi']\cdot\nabla u^{m,\nu}\big](t,x')\Big] \\ & + [\hat{P}^{\tau,\xi'}_{t_0}u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}^{\tau,\tilde{\xi}'}_{t_0}u^{m,\nu}(t_0,\cdot)](t,x') \Bigg\}, \end{aligned} \tag{4.84}$$

because we have $\hat{G}^{\tau,\tilde{\xi}'} f_m(t,x') = \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x') + \hat{G}^{\tau,\tilde{\xi}'}_{t_0,t} f_m(t,x')$.

Hence, taking $(\tau,\xi,\xi',\tilde{\xi}') = (t,x,x',x)$ yields the previous terms already controlled plus a new extra contribution $\Big[\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big]$.

### 4.7.9 Control of the new extra contribution $\Big[\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big]$

This last extra term is tackled similarly as the first one, see Section 4.7.7.

$$\begin{aligned}
&\Big[\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big] \\
&= \int_0^{t_0}\int_{\mathbb{R}^d} \tilde{p}(s,t,x',y)\Big[f_m\big(s, y + \int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau'}(\xi'))d\tilde{s}\big) - f_m\big(s, y + \int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tilde{\tau}'}(\xi')d\tilde{s})\big)\Big]dy\,ds.
\end{aligned}$$

We readily obtain,

$$\begin{aligned}
&\Big|\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big| \\
\leq\ & \|f_m\|_{L^\infty(C^1)} \int_0^{t_0} \Big| \int_s^t b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(\xi')) - b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(\tilde{\xi}')d\tilde{s}\Big| ds \\
\leq\ & \|f_m\|_{L^\infty(C^1)}\|b_m\|_{L^\infty(C^1)} \int_0^{t_0}\int_s^t |\theta^m_{\tilde{s},\tau'}(\xi') - \theta^m_{\tilde{s},\tau}(\tilde{\xi}')|d\tilde{s}\,ds.
\end{aligned}$$

Again, we use, for any $(x,x') \in \mathbb{R}^d \times \mathbb{R}^d$, the control of the flow,

$$\sup_{\tilde{s}\in[0,\tau]} |\theta^m_{s,\tau}(x) - \theta^m_{s,\tau}(x')| \ \leq\ |x-x'| \exp\big(\|b_m\|_{L^\infty(C^1)}\tau\big), \tag{4.85}$$

see Lemma 4.

We then deduce for $(\tau,\xi',\tilde{\xi}') = (t,x',x)$ that

$$\begin{aligned}
&\Big|\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & \|f_m\|_{L^\infty(C^1)}\|b_m\|_{L^\infty(C^1)} \int_0^{t_0}\int_s^t |x-x'| \exp\big(\|b_m\|_{L^\infty(C^1)}t\big)d\tilde{s}\,ds \\
\leq\ & \frac{1}{2} m^{1-\gamma}\|f\|_{L^\infty(C^\gamma)} m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} |x-x'|t^2 \exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}t\big).
\end{aligned} \tag{4.86}$$

From the definition of $t_0$ in (4.54), $|x-x'| = \nu^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}}$, we get

$$\begin{aligned}
&\Big|\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\
\leq\ & |x-x'|^\gamma \frac{\nu^{\frac{1-\gamma}{2}} t^{\frac{1-\gamma}{2}+2}}{2} m^{2+\beta-\gamma}\|f\|_{L^\infty(C^\gamma)}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}t\big) \\
\leq\ & C|x-x'|^\gamma \nu^{\frac{1-\gamma}{2}} t^{\frac{1-\gamma}{2}+1} m^{1-\gamma}\|f\|_{L^\infty(C^\gamma)} \exp\big(cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}t\big),
\end{aligned} \tag{4.87}$$

which goes to 0 if

$$\nu \ll (m^{1-\gamma}T^{\frac{1-\gamma}{2}+1}\|f\|_{L^\infty(C^\gamma)})^{-\frac{2}{1-\gamma}} \exp\big(-c\frac{m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}T}{1-\gamma}\big). \tag{4.88}$$

We postpone, in Section D.2, a full comment on the strategy associated with the choice of *freezing* point for the source functions terms.

### 4.7.10 Control of $[u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}}$

We have from the final Duhamel formula (4.78) combined with the estimates of each contribution stated in (4.68) (4.82), (4.87):

$$\begin{aligned}
&[u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}}\\
\leq\ & \int_0^t [f(s,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}}ds + [g]_{\gamma,\nu^{1/2}T^{1/2}}\\
&+Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}(O_m(t)+O_m^{(2)}(t))\int_0^t \nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}}ds\\
&+Ct^{\frac{1-\gamma}{2}}\nu^{\frac{1-\gamma}{2}}\Big(T\|f_m\|_{L^\infty(C^1)}+\|g_m\|_{C^1}\Big)\exp\Big(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big)\\
&+C\nu^{\frac{1-\gamma}{2}}t^{\frac{1-\gamma}{2}+1}m^{1-\gamma}\|f\|_{L^\infty(C^\gamma)}\exp\big(cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}t\big),
\end{aligned}\tag{4.89}$$

where we recall that

$$\begin{aligned}
O_m(t) &= \Big(t\|f_m\|_{L^\infty(C^1)}+\|g_m\|_{C^1}\Big)\exp\Big(m^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big)\\
&\leq Cm^{1-\gamma}(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma)\exp\Big(m^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big),
\end{aligned}$$

and

$$O_m^{(2)}(t)=Cm^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)\exp(Cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}).$$

Then, for $\nu \ll T$ and because $\frac{1-\gamma^2}{4} > \frac{\gamma(1-\gamma)}{4}$, we obtain

$$\begin{aligned}
[u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}} &\leq \int_0^t [f(s,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}}ds + [g]_{\gamma,\nu^{1/2}T^{1/2}}\\
&\quad +Ct^{\frac{1-\gamma}{2}}\nu^{\frac{1-\gamma}{2}}m^{2-\gamma}(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma)\exp\Big(cm^{1+\beta}t\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big).
\end{aligned}$$

Finally, we can also impose the condition on the viscosity

$$\nu \ \ll\ T^{-1}m^{-\frac{2-\gamma}{1-\gamma}}(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma)^{-\frac{2}{1-\gamma}}\exp\Big(-c\frac{m^{1+\beta}T\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}}{1-\gamma}\Big),\tag{4.90}$$

the required limit modulus of continuity control (3.1) is then established when $(m,\nu)\to(+\infty,0)$ according to the above condition.

We fail to obtain a uniqueness of a viscous selection principle with this first analysis. Indeed in the *a priori* controls of (2.16) we have to suppose that $\nu$ goes to 0 much faster than $m$ towards $+\infty$. This constraint prevents us to take advantage of the convergence of $b_m$ towards $b$ to balance the blow up in the viscosity $\nu$ occurring in the computations to get uniqueness.

## 4.8 The last part of the control of Hölder modulus: the *time decomposition locus* trick

The beginning of the analysis for the parabolic case is similar as for the transport equation performed in Section 4, except that the goal of the regularisation is different. We aim here to raise the time contribution instead of the viscosity one. This allows us to conclude, thanks to a time decomposition trick, without any vanishing viscosity.

We recall,

$$|R^{\tau,\xi,\xi'}(t,x,x')|\big|_{\tau=t,\xi=\xi'=x} \leq Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}|x-x'|^\gamma(O_m(t)+O_m^{(2)}(t))\int_0^t \nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}}ds.\tag{4.91}$$

### 4.8.1 Adapting the controls of the extra contributions

We have to change the parameters $(\alpha_1, \alpha_2)$ into (4.81) and (4.87), which gives

$$\begin{aligned} & \Big|[\hat{P}^{\tau,\xi'}_{t_0} u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}^{\tau,\tilde{\xi}'}_{t_0} u^{m,\nu}(t_0,\cdot)](t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\ \leq \;& |x-x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \nu^{\frac{1-\gamma}{2}} (t-t_0)^{\frac{(1-\gamma)}{2}+1} \exp\big(\|b_m\|_{L^\infty(C^1)} t\big), \end{aligned} \tag{4.92}$$

and

$$\begin{aligned} & \Big|\hat{G}^{\tau,\xi'}_{0,t_0} f_m(t,x') - \hat{G}^{\tau,\tilde{\xi}'}_{0,t_0} f_m(t,x')\Big|\Big|_{(\tau,\xi',\tilde{\xi}')=(t,x',x)} \\ \leq \;& |x-x'|^\gamma m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} t\big) \int_0^{t_0} \int_s^t \frac{\nu^{\frac{1-\gamma}{2}} (t-t_0)^{\frac{(1-\gamma)}{2}}}{2} d\tilde{s}\, ds. \end{aligned} \tag{4.93}$$

### 4.8.2 Gathering the controls

To put in a nutshell, gathering all the previous estimates (4.52), (4.53), (4.68), (4.92), (4.93) into (4.84)

$$\begin{aligned} & [u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}} \\ \leq \;& \int_0^t [f(s,\cdot)]_{\gamma,\nu^{1/2}T^{1/2}} ds + [g]_{\gamma,\nu^{1/2}T^{1/2}} \\ & + C m^{1+\beta} \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} (O_m(t) + O^{(2)}_m(t)) \int_0^t \nu^{\frac{1-\gamma}{2}} (t-s)^{\frac{1-\gamma}{2}} ds \\ & + C m^{1+\beta} O_m(t) \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \nu^{\frac{1-\gamma}{2}} (t-t_0)^{\frac{(1-\gamma)}{2}+1} \exp\big(\|b_m\|_{L^\infty(C^1)} t\big) \\ & + m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} t\big) \int_0^t \int_s^t \frac{\nu^{\frac{1-\gamma}{2}} (t-t_0)^{\frac{(1-\gamma)}{2}}}{2} d\tilde{s}\, ds, \end{aligned} \tag{4.94}$$

with

$$\begin{aligned} O_m(t) &= \Big(t\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big) \exp\Big(m^{1+\beta} t \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big) \\ &\leq C\Big(m^{1-\gamma}(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)\Big) \exp\Big(m^{1+\beta} t \|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\Big). \end{aligned}$$

Now, the idea is to make negligible the terms involving positive time contribution in the time integral.

## 4.9 The time decomposition of the Cauchy problem

We cut the Cauchy problem in small intervals of $[0,T]$, and we see, from computations below, that the first order term $b^m_\Delta \cdot \nabla u^{m,\nu}$ and the extra terms are negligible when the size of the time intervals goes to 0.

Let $n \in \mathbb{N}$, and for each $k \in [\![0,n]\!]$,

$$\tau_k := \frac{k}{n} T,$$

we also denote, for all $x \in \mathbb{R}^3$, $k \in [\![0, n-1]\!]$ and $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{cases} u^{m,\nu}_{k+1}(t,x) & := u^{m,\nu}(t,x), \\ u^{m,\nu}_1(0,x) & := g_m(x). \end{cases}$$

The associated Cauchy problems write

If $k \in [\![1, n-1]\!]$

$$\begin{cases} \partial_t u_{k+1}^{m,\nu}(t,x) + \langle b_m, \nabla u_{k+1}^{m,\nu}\rangle(t,x) = \nu\Delta u_{k+1}^{m,\nu}(t,x) + f_m(t,x),\ t \in (\tau_k, \tau_{k+1}] \\ u_{k+1}^{m,\nu}(\tau_k, x) = u_k^{m,\nu}(\tau_k, x), \end{cases}$$

if $k = 0$

$$\begin{cases} \partial_t u_1^{m,\nu}(t,x) + \langle b_m, \nabla u_1^{m,\nu}\rangle(t,x) = \nu\Delta u_1^{m,\nu}(t,x) + f_m(t,x),\ t \in (0, \tau_1), \\ u_1^{m,\nu}(0, x) = g_m(x). \end{cases}$$

Using the Duhamel formulation (4.7) around the consider heat like equation, we get for any $t \in (\tau_k, \tau_{k+1}]$, if $k \in [\![1, n-1]\!]$,

$$u_{k+1}^{m,\nu}(t,x) = \hat{P}_{\tau_k}^{\tau,\xi} u_k^{m,\nu}(t,x) + \hat{G}_{\tau_k}^{\tau,\xi} f_m(t,x) + \hat{G}_{\tau_k}^{\tau,\xi}\big(\langle b_\Delta^m[\tau,\xi], \nabla u_k^{m,\nu}\rangle\big)(t,x), \tag{4.95}$$

and if $k = 0$,

$$u_1^{m,\nu}(t,x) = \hat{P}_0^{\tau,\xi} g_m(t,x) + \hat{G}_0^{\tau,\xi} f_m(t,x) + \hat{G}_0^{\tau,\xi}\big(\langle b_\Delta^m[\tau,\xi], \nabla u_1^{m,\nu}\rangle\big)(t,x).$$

Next, we use the corresponding Hölder control (4.94), including the extra contributions coming from the *cut locus* argument,

$$\begin{aligned} & [u_{k+1}^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \\ \leq\ & \int_{\tau_k}^t [f(s,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} ds + [u_k^{m,\nu}(\tau_k,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \\ & + Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}(O_m(t) + O_m^{(2)}(t)) \int_{\tau_k}^t \nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}} ds \\ & + Cm^{1+\beta} O_m(t)\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{(1-\gamma)}{2}+1} \exp\big(\|b_m\|_{L^\infty(C^1)} t\big) \\ & + m^{2+\beta-\gamma}\|f\|_{L^\infty(C^\gamma)}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} t\big) \int_{\tau_k}^t \int_{\tau_k}^t \frac{\nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{(1-\gamma)}{2}}}{2} d\tilde{s}\, ds. \end{aligned} \tag{4.96}$$

Let us carefully point out that we suppose that $(t - t_0) \leq (t - \tau_k)$, otherwise there is no off-diagonal regime and we cannot use the *cut locus* technique. This condition allows use to replace the Hölder modulus in the r.h.s. in (4.52) by the *limit Hölder continuity* defined in (2.6).

Hence, recalling that $\tau_{k+1} - \tau_k = \frac{T}{n}$,

$$\begin{aligned} & [u_{k+1}^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \\ \leq\ & \int_{\tau_k}^t [f(s,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} ds + [u_k^{m,\nu}(\tau_k,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \\ & + Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}(O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \Big(\frac{T}{n}\Big)^{\frac{1-\gamma}{2}} \int_{\tau_k}^t ds \\ & + Cm^{1+\beta} O_m(t)\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \nu^{\frac{1-\gamma}{2}} \Big(\frac{T}{n}\Big)^{\frac{1-\gamma}{2}} (t-\tau_k) \exp\big(\|b_m\|_{L^\infty(C^1)} t\big) \\ & + C\nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma}\|f\|_{L^\infty(C^\gamma)}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} \exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})} t\big)(t-\tau_k)^{\frac{(1-\gamma)}{2}+1} \int_{\tau_k}^t ds. \end{aligned} \tag{4.97}$$

For any $t \in [0,T]$, let $\kappa_n(t) = \lfloor \frac{nt}{T} \rfloor \in [\![0, n]\!]$, such that $\tau_{\kappa_n(t)} \leq t < \tau_{\kappa_n(t)+1}$, and iterating the above inequality,

$$[u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \leq [g]_{\gamma,\nu^{1/2}(T/n)^{1/2}} + \Lambda_{\tau_{\kappa_n(t)},t} + \sum_{k=0}^{\kappa(t)-1} \Lambda_{\tau_k,\tau_{k+1}},$$

with, for any $0 \le r \le t \le T$,

$$\begin{aligned}\Lambda_{r,t} \;:=\; & \int_r^t [f(s,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} ds \\ & + Cm^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}(O_m(t)+O_m^{(2)}(t))\nu^{\frac{1-\gamma}{2}}\Big(\frac{T}{n}\Big)^{\frac{1-\gamma}{2}}\int_r^t ds \\ & + Cm^{1+\beta}O_m(t)\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\nu^{\frac{1-\gamma}{2}}\Big(\frac{T}{n}\Big)^{\frac{1-\gamma}{2}}(t-r)\exp\big(\|b_m\|_{L^\infty(C^1)}t\big) \\ & + C\nu^{\frac{1-\gamma}{2}}m^{2+\beta-\gamma}\|f\|_{L^\infty(C^\gamma)}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\exp\big(m^{1+\beta}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}t\big)\Big(\frac{T}{n}\Big)^{\frac{(1-\gamma)}{2}+1}\int_r^t ds,\end{aligned} \tag{4.98}$$

which goes to $\int_r^t [f(s,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} ds$ when $n \to +\infty$ as soon as

$$1 \gg \nu^{\frac{1-\gamma}{2}}(1+\frac{T}{n}m^{1-\gamma}\|f\|_{L^\infty(C^\gamma)})\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}\exp\big(Cm^{2+\beta-\gamma}\|b\|_{L^\infty(B^{-\beta}_{\infty,\infty})}T\big)\Big(\frac{T}{n}\Big)^{\frac{(1-\gamma)}{2}}T, \tag{4.99}$$

always true for given $m < +\infty$, $\nu > 0$ and letting $n \to +\infty$.

By Chasles equality and by telescopic sum, the identity below readily comes

$$[u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \le [g]_{\gamma,\nu^{1/2}(T/n)^{1/2}} + \Lambda_{0,t}.$$

The l.h.s. does not depend on $n$, then we are able to pass to the limit $n \to +\infty$,

$$\lim_{n\to+\infty}[u^{m,\nu}(t,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} \le \lim_{n\to+\infty}[g]_{\gamma,\nu^{1/2}(T/n)^{1/2}} + \lim_{n\to+\infty}\int_0^t [f(s,\cdot)]_{\gamma,\nu^{1/2}(T/n)^{1/2}} ds. \tag{4.100}$$

## 4.10 Another control of uniform norm

We have directly by the Feynman-Kac formulation the uniform control, see for example [KS91] (also used in the analysis performed in [Hon22]), or from maximum principle for linear parabolic equation see e.g. [Lie96], the control in $L^\infty$ of the parabolic solution $u^{m,\nu}$.

This maximum principle can also be derived from our strategy.

Indeed, by a similar way as for the Hölder control performed in the previous section, by Duhamel formula (4.95), after choosing $(\tau,\xi) = (t,x)$, we get

$$\|u^{m,\nu}_{k+1}(t,\cdot)\|_{L^\infty} \le \|u^{m,\nu}_k(\tau_k,\cdot)\|_{L^\infty} + \int_{\tau_k}^t \|f(s,\cdot)\|_{L^\infty} ds + C\|b_m\|_{L^\infty(C^1)}\|\nabla u^{m,\nu}_k\|_{L^\infty}\int_{\tau_k}^t (t-s)^{\frac{1}{2}} ds.$$

Iterating this inequality,

$$\|u^{m,\nu}\|_{L^\infty} \le \|g\|_{L^\infty} + \int_0^T \|f(s,\cdot)\|_{L^\infty} ds + C\frac{T^{\frac{3}{2}}}{n^{\frac{1}{2}}}\|b_m\|_{L^\infty(C^1)}\|\nabla u^{m,\nu}\|_{L^\infty}.$$

We obviously deduce when $n \to +\infty$, the following estimate,

$$\|u^{m,\nu}\|_{L^\infty} \le \|g\|_{L^\infty} + \int_0^T \|f(s,\cdot)\|_{L^\infty} ds. \tag{4.101}$$

*In fine*, the time decomposition trick allows to retrieve the powerful uniform estimate given by Feynman-Kac formula or usual maximum principle method.

### 4.10.1 Comment on the full Hölder modulus

In order to write the full Hölder modulus instead the *limit* one, we have to restart the computations of Section 4.7.1 without supposing (4.51), i.e. in the case

$$|x - x'| > \nu^{\frac{1}{2}} T^{\frac{1}{2}}.$$

This situation prevent us to perform the same distinction *diagonal/off-diagonal* regimes performed in Section 4.7.2. In other words, the situation (4.55) cannot happen, and the only possible regime is (4.63) which implies the choice $(\tau, \xi, \xi') = (t, x, x')$.

**Semi-group**

We readily derive by change of variables:

$$\begin{aligned} & |\hat{P}^{\tau,\xi} g_m(t,x) - \hat{P}^{\tau,\xi'} g_m(t,x')| \Big|_{\tau=t,\xi=\xi'=x'} \\ = & \Big| \int_{\mathbb{R}^d} \tilde{p}(0,t,0,y)[g_m(y - \theta_{0,t}(x)) - g_m(y - \theta_{0,t}(x'))]dy \Big| \\ \leq & \|\nabla\theta_{0,t}(\cdot)\|_{L^\infty} [g]_\gamma |x - x'|^\gamma. \end{aligned} \tag{4.102}$$

**Green operator**

We also get by change of variables:

$$\begin{aligned} & |\hat{G}^{\tau,\xi} f_m(t,x) - \hat{G}^{\tau,\xi'} f_m(t,x')| \Big|_{\tau=t,\xi=\xi'=x'} \\ = & \Big| \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s,t,0,y)[f_m(s, y - \theta_{s,t}(x)) - f_m(s, y - \theta_{s,t}(x'))]ds\, dy \Big| \\ \leq & \int_0^t \|\nabla\theta_{s,t}(\cdot)\|_{L^\infty} [f(s,\cdot)]_\gamma ds |x - x'|^\gamma. \end{aligned} \tag{4.103}$$

In other word, we derive a suitable Hölder modulus control if $\theta_{s,t}$ is Lipschitz continuous uniformly in $(m, \nu)$, which is guaranteed by assumption **(A)**, see Section 3.1.

### 4.10.2 On a $b$ depending on $t$

In this section, we develop why we can extend the hypothesis **(A)** to involved a time dependency:

We say that **(A')** is in force if there is a function $\psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $C^1$ in time, such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d,\ b(t,x) = (\nabla\psi(t,x))^{-1},$$

satisfying

$$\forall (\tau, x) \in [0,T] \times \mathbb{R},\ \frac{\nabla\psi(t,x)}{\nabla\psi(t, \psi^{-1}(t, c\tau + \psi(t,x)))} < +\infty,$$

and

$$\psi_m^{-1}(t, \tau + \psi_m(t,x)) \underset{m\to+\infty}{\longrightarrow} \psi^{-1}(t, \tau + \psi(t,x)) \text{ in } C^1(\mathbb{R}^d, \mathbb{R}^d),$$

with $\psi_m$ a mollified version of $\psi$.

This new assumption is a by-product of **(A)**, because for any $t \in [\tau_k, \tau_{k+1}]$, $k \in [\![0, n-1]\!]$, we derive

$$\forall x \in \mathbb{R}^d,\ |b_m(t,x) - b_m(\tau_k, x)| \leq C(t - \tau_k)\|\partial_t b_m\|_{L^\infty} \leq C\frac{T}{n} m^{-\beta} \|\partial_t b\|_{L^\infty}.$$

This above inequality means that we can assume that $b$ does depend on $t$, and at each time interval $t \in [\tau_k, \tau_{k+1}]$ we approximate $b(t,\cdot)$ by the distribution constant in time $b(\tau_k, \cdot)$.

**Corollary 2.** *The conclusions of Theorem 1 is still true replacing condition* **(A)** *by* **(A')** *if* $b \in C^1([0,T], \tilde{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d))$, $\beta \in \mathbb{R}$.

# 5 Analysis of the parabolic equation

## 5.1 Some *a priori* attempts

### 5.1.1 Peano's heuristic

The Peano counter-example yields a heuristic of the expected minimum regularity of $b$. The threshold comes from a regularisation by noise argument, for other explanation see [Fla11], and [CdRM17] for a degenerate case. Similarly to (1.4), let us consider $b(x) = \text{sign}(x)(|x| \wedge R)^\gamma$, $\gamma \in \mathbb{R}$, and the associated flow,

$$\frac{dX_t}{dt} = \text{sign}(X_t)|X_t|^\alpha, \ X_0 = 0. \tag{5.1}$$

There are an infinite number of solutions written, for any $t^* \in [0, T]$, by :

$$X_t = \pm c_\alpha (t - t^*)^{\frac{1}{1-\alpha}} \mathbb{1}_{[t^*,T]}(t). \tag{5.2}$$

The associated stochastic problem is

$$d\tilde{X}_t = \text{sign}(\tilde{X}_t)|\tilde{X}_t|^\alpha dt + \nu dW_t,$$

where $(W_t)_{t\geq 0}$ stands for a Brownian motion. Parabolic equation (1.2) is the determinist counterpart of this SDE, where the solution is given by a stochastic representation, the Feynman-Kac formula, see for instance [KS91].

There is a critical time when the noise overwhelms the singular drift, see [DF14], after this time the SDE solution fluctuates around a solution of the ODE (5.1).

As a consequence, we aim to compare the time scaling between the Brownian motion, i.e. $t^{\frac{1}{2}}$, with a solution of the ODE given in (5.2), i.e. $t^{\frac{1}{1-\alpha}}$. That is to say, to take advantage of the regularisation by noise before the critical point, the condition is

$$t^{\frac{1}{1-\alpha}} < t^{\frac{1}{2}},$$

which yields for a small time,

$$\frac{1}{1-\alpha} > \frac{1}{2} \ \Leftrightarrow \alpha > -1.$$

In other words, for $\tilde{\gamma} = \alpha = -1 + \tilde{\gamma}$, $\tilde{\gamma} > 0$, the expected minimum regularity of the drift is $b(t, \cdot) \in B^{-1+\tilde{\gamma}}_{\infty,\infty} = C^{-1+\tilde{\gamma}}$.

There are numerous articles dealing with SDE and the associated parabolic equation with irregular drift, but in all of them there is a "macro" distance with the above Peano heuristic.

The case $\tilde{\gamma} > 1/3$ is dealt in [DD16] in dimension 1 and [CC18] for the multidimensional version; the authors thoroughly use rough path and para-control.

Another notion of solution of stochastic equation, called virtual, is introduced in [FIR17], where the constraint is $\tilde{\gamma} > 1/2$. Let us notice that under this constraint, $\tilde{\gamma} < 1/2$, there is no hope to obtain strong solution of the SDE, see for instance the counter-examples presented in [BC01], [Bar82].

### 5.1.2 On the limitation of the para-product

The last constraint, $\tilde{\gamma} > \frac{1}{2}$, appears naturally with *a priori* computations with usual tools. Indeed, from Duhamel formula the solution of the parabolic equation (1.2) writes

$$u^{m,\nu}(t,x) = \tilde{P} g_m(x) + \tilde{G} f_m(t,x) + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s,t,x,y) \langle b_m(y), \nabla u^{m,\nu}(s,y) \rangle dy \, ds,$$

where

$$\tilde{p}(s,t,x,y) := \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\Big( - \frac{|x-y|^2}{4\nu(t-s)}\Big), \tag{5.3}$$

stands for the the standard heat kernel, also

$$\forall (t,x) \in (0,T]\times\mathbb{R}^d,\ \tilde{G}f_m(t,x) := \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s,t,x,y) f_m(s,y)\ dy\ ds, \tag{5.4}$$

is the corresponding Green operator, and

$$\tilde{P}g_m(t,x) := \int_{\mathbb{R}^d} \tilde{p}(0,t,x,y) g_m(y) dy, \tag{5.5}$$

the associated semi-group.

Let us suppose that $\nabla u^{m,\nu}(s,\cdot) \in C^\delta$, $\delta \geq 0$, therefore from the above Duhamel's formula, we should have:

$$x \mapsto \nabla \int_t^T \int_{\mathbb{R}^d} \tilde{p}(t,s,x,y) \langle b_m(s,y), \nabla u^{m,\nu}(s,y)\rangle dy\, ds \in C^\delta.$$

However, from the para-product result, derived by Bony's microlocal analysis [Bon81], see also [GIP15], if $\nabla u^{m,\nu}(s,\cdot) \in C^\delta$ and $b_m(s,\cdot) \in C^{-\tilde{\gamma}}$, such that $\delta - \tilde{\gamma} > 0 \Leftrightarrow \tilde{\gamma} < \delta$ then $\langle b_m(\cdot), \nabla u^{m,\nu}(s,\cdot)\rangle \in C^{-\tilde{\gamma}}$. Hence, with some common computations of the heat kernel, we obtain that

$$\nabla \int_t^T \int_{\mathbb{R}^d} \tilde{p}(t,s,x,y) \langle b_m(y), \nabla u^{m,\nu}(s,y)\rangle dy\, ds \approx (T-t)^{\frac{1-\tilde{\gamma}}{2}}.$$

Also, thanks to the typical equivalence of the space-time with a parabolic scaling in the analysis of the parabolic bootstrap, see for instance [CDRHM18], where $(T-t)^2 \approx |x-x'|$, we deduce

$$x \mapsto \nabla \int_t^T \int_{\mathbb{R}^d} \tilde{p}(t,s,x,y) \langle b_m(y), \nabla u^{m,\nu}(s,y)\rangle dy\, ds \in C^{1-\tilde{\gamma}}.$$

Then we readily derive that $\tilde{\gamma} \leq 1-\delta$, namely $\delta \leq 1-\tilde{\gamma}$. Combining with the para-product constraint $\tilde{\gamma} < \delta$ yields $\tilde{\gamma} \leq 1-\tilde{\gamma}$ and so $\tilde{\gamma} < 1/2$ which is exactly the same regularity constraint of [FIR17].

This heuristic shows up the difficulty to use para-product results in such a rough framework. To handle with regularity $\tilde{\gamma} > 1/2$, we then have to capitalise on other techniques. Specifically, we thoroughly exploit the fact that $u^{m,\nu}$ is solution of the parabolic equation (1.2), which allows to consider cases out of the Bony's para-product scope.

The previous analysis strongly relies on the vanishing result but seems to be incompatible with the parabolic equation. We developed, here, a new technique based on a time decomposition trick, detailed in Section 4.9 below.

## 5.2 The types of solution to the Cauchy problem

### 5.2.1 Parabolic equation

For $0 < \gamma < 1$, and for a given viscosity $\nu > 0$, we define some solutions of the parabolic equation.

**Definition 5** (mild solution)**.** *We say that $u$ is a mild solution in $L^\infty\big([0,T]; C_b^\gamma(\mathbb{R}^d,\mathbb{R})\big)$ of equation (1.1) if there is a sequence[††] $(b_m)_{m\in\mathbb{R}_+}$ in $L^\infty([0,T]; C_b^\infty(\mathbb{R}^d,\mathbb{R}^d))$ such that there is $\tilde{\gamma} \in \mathbb{R}$,*

$$\forall \varepsilon > 0,\ \lim_{m\to+\infty} \|b_m - b\|_{L^\infty([0,T];B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))} = 0, \tag{5.6}$$

[††]We have such a sequence $(b_m)_{m\geq 0}$, if $b \in L^\infty([0,T]; \tilde{B}^{-\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}))$.

*for any $t \in [0,T]$, there exists a sub-sequence of $u^{m,\nu}(t,\cdot)_{m\in\mathbb{R}_+}$ lying in $C_b^\gamma(\mathbb{R}^d,\mathbb{R})$ converging, for any compact subset $K \subset \mathbb{R}^d$, when $m \to +\infty$, towards $u(t,\cdot) \in C_b^\gamma(K,\mathbb{R})$ and satisfying, for any $m \in \mathbb{R}_+$,*

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + \langle b_m(t,x), \nabla u^{m,\nu}(t,x)\rangle - \nu\Delta u^{m,\nu}(t,x) = f_m(t,x),\ (t,x) \in (0,T]\times\mathbb{R}^d, \\ u^{m,\nu}(0,x) = g_m(x),\ x \in \mathbb{R}^d, \end{cases} \tag{5.7}$$

*where $(f_m, g_m) \underset{m\to+\infty}{\longrightarrow} (f,g)$ in $L^\infty([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R}))\times C_b^\gamma(\mathbb{R}^d,\mathbb{R})$.*

Let us now, recall the notion of usual *weak* solution.

**Definition 6** (weak solution)**.** *A function $u$ is a weak solution in $L^\infty\big([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R})\big)$ of equation (1.1) if $u$ is a mild solution such that for any function $\varphi \in C_0^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$:*

$$\int_{\mathbb{R}^d}\Big\{\varphi(t,y)u(t,y) + \int_0^t \big\{ -\partial_t\varphi(s,y)u(s,y) + \langle b(s,y),\nabla u(s,y)\rangle\varphi(s,y) + \nu u(s,y)\Delta\varphi(s,y)\big\}ds\Big\}dy$$
$$= \int_{\mathbb{R}^d}\varphi(0,y)g(y)dy + \int_{\mathbb{R}^d}\int_0^t \varphi(s,y)f(s,y)ds\,dy. \tag{5.8}$$

Again, this formulation allows us to give a usual distributional meaning of the product $\langle b, \nabla u\rangle$.

### 5.3 Main result on the parabolic equation

When $b$ lies in Hölder-Besov space, we succeed in obtaining the same regularity of the solution as for $f$ and $g$. The type of solution strongly depends on the regularity of $b$.

**Theorem 3** (Rough parabolic equation in Hölder spaces)**.** *For $\tilde\gamma \in \mathbb{R}^*$ and $0 < \gamma < 1$, be given. Let $f \in L^\infty([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}^d,\mathbb{R})$. For a distribution $b \in C_b^1([0,T],\tilde B^{\tilde\gamma}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$, then, for $u^{m,\nu}$ defined in (5.7), we have*

$$\begin{aligned} \lim_{n\to+\infty}\sup_{t\in[0,T]}[u^{(m,\nu)}(t,\cdot)]_{\gamma,(\nu T/n)^{1/2}} &\le \lim_{n\to+\infty}\int_0^T [f(s,\cdot)]_{\gamma,(\nu T/n)^{1/2}} + [g]_{\gamma,(\nu T/n)^{1/2}} \\ \|u\|_{L^\infty} &\le \int_0^T \|f(s,\cdot)\|_{L^\infty} + \|g\|_{L^\infty}. \end{aligned} \tag{5.9}$$

*Moreover, if* **(A')** *is in force, then there is a mild solution $u \in L^\infty([0,T];C_b^\gamma(\mathbb{R}^d,\mathbb{R}))$ of (1.1); also if $\tilde\gamma < \gamma$ and $\nabla\cdot b = 0$ then the solution $u$ is also a weak solution.*

We do not consider the positive regular case, i.e. $\tilde\gamma < 0$, whose control is well-known, the Schauder estimates are even in force, see [Fri64].

**Remark 10.** *The control (5.9) exactly matches with the limit Hölder estimates of the solution of the heat equation, independently of the dimension, and above all of $b$. As consequence, the mild solution has no condition on $\tilde\gamma$, there is to say we define a solution beyond the Peano condition, $\tilde\gamma > -1$, developed in Section 5.1.1.*

*In the incompressible case, $\nabla\cdot b$, the condition $\tilde\gamma < \gamma < 1$ corresponds "almost" to the Peano's heuristic[‡‡] in Section 5.1.1. However, we fail to obtain uniqueness of the solution out of the usual Hölder continuous case (i.e. $\tilde\gamma < 0$) handled in [Fri64], for more information see Remark 7 further.*

*The case $\tilde\gamma = 0$ can be considered, replacing the condition $b \in L^\infty([0,T],\tilde B^0_{\infty,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ by $b \in L^\infty([0,T],L^\infty(\mathbb{R}^d,\mathbb{R}^d))$. Considering the Besov space $B^0_{\infty,\infty}$ would yield some refinements involving some logarithm corrections.*

*Again, the case $\tilde\gamma < 0$ is the usual framework, see for instance [Fri64], [Kry96].*

The proof is a direct consequence of the analysis performed in Section 4, letting $m$ going to $+\infty$. Nevertheless, there is no vanishing viscosity which would restore uniqueness.

[‡‡]The true condition is $\tilde\gamma < 1$, but this is the case, here, as $\gamma$ which can be arbitrarily close to 1.

# 6 Inviscid Burgers' equation

The controls (4.46) of the vanishing viscous solution of the PDE (1.1) being independent on the first order term $b$, we can expect to obtain some fixed-point argument to consider that $b$ being the solution $u$ itself in dimension 1[§§].

Such a Cauchy problem thus defined is called the inviscid Burgers' equation,

$$\begin{cases} \partial_t u(t,x) + u(t,x)\partial_x u(t,x) = f(t,x), \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = g(x), \ x \in \mathbb{R}. \end{cases} \tag{6.1}$$

For more information about the Burgers' equation and the corresponding turbulence phenomenon, we refer to the recent book [BK21].

## 6.1 Statement about the Inviscid Burgers' equation

We obtain a different notion of uniqueness for this equation because the convergence of the mollified first order term, being the solution itself, is more intricate comparing with the transport equation case.

**Definition 7** (Turbulent uniqueness)**.** *There is a turbulent unique solution if there are two solutions $u^{m,\nu}$ and $u^{m,\bar{\nu}}$ of*

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + u_m^{m,\nu}(t,x)\partial_x u^{m,\nu}(t,x) - \nu\partial_{xx}^2 u^{m,\nu}(t,x) = f_m(t,x), \ (t,x) \in (0,T] \times \mathbb{R}, \\ u^{m,\nu}(0,x) = g_m(x), \ x \in \mathbb{R}, \end{cases} \tag{6.2}$$

*and respectively*

$$\begin{cases} \partial_t u^{m,\bar{\nu}}(t,x) + u_m^{m,\bar{\nu}}(t,x)\partial_x u^{m,\bar{\nu}}(t,x) - \bar{\nu}\partial_{xx}^2 u^{m,\bar{\nu}}(t,x) = f_m(t,x), \ (t,x) \in (0,T] \times \mathbb{R}, \\ u^{m,\bar{\nu}}(0,x) = g_m(x), \ x \in \mathbb{R}, \end{cases} \tag{6.3}$$

*for $\nu, \bar{\nu} > 0$, and for any $(t,x) \in [0,T] \times \mathbb{R}$, where $u_m^{m,\nu}$ and $u_m^{m,\bar{\nu}}$ stand respectively for a mollified version of $u^{m,\nu}$ and $u^{m,\bar{\nu}}$, such that, for any $(t,x) \in [0,T] \times \mathbb{R}$,*

$$\forall \varepsilon > 0, \ \lim_{n \to +\infty} \|u_n^{m,\nu} - u^{m,\nu}\|_{L^\infty([0,T];C_b^{\gamma-\varepsilon})} = \lim_{n \to +\infty} \|u_n^{m,\bar{\nu}} - u^{m,\bar{\nu}}\|_{L^\infty([0,T];C_b^{\gamma-\varepsilon})} = 0,$$

*and,*

$$|u_m^{m,\nu}(t,x) - u_m^{m,\bar{\nu}}(t,x)| \leq |u^{m,\bar{\nu}}(t,x) - u^{m,\bar{\nu}}(t,x)|, \tag{6.4}$$

*converging, up to sub-sequence selection, towards two Hölder continuous solutions $u$, $\bar{u}$, when $(m,\nu,\bar{\nu}) \to (+\infty,0,0)$ then $u = \bar{u}$.*

The last condition (6.4) simply means that the mollification procedure behaves like a convolution with a smooth kernel (like performed in (4.2)).

Let us insist that the difference with the uniqueness introduced in Definition 4 is that the considered regularisation procedure for the first order terms $u^{m,\nu}$ and $u^{m,\bar{\nu}}$ is the same in equations (6.2) and (6.3); this explains the definition (6.4).

In particular, uniqueness introduced in Definition 4 (for the transport equation) yields turbulent uniqueness. We detail in Remark 13 why we have to handle with such a turbulent uniqueness or the viscous uniqueness, defined below, instead of the classic uniqueness[¶¶] for the inviscid Burgers' equation.

[§§] It is possible to adapt the analysis for a more general dimension $d \geq 1$ by a reformulation of the product $\langle u(t,x), \nabla u(t,x) \rangle$.

[¶¶] Definition 4.

**Definition 8** (Viscous uniqueness)**.** *There is a viscous unique solution if there are two solutions $u^{m,\nu}$ and $u^{m,\bar{\nu}}$ of*

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + u_m^{m,\nu}(t,x)\partial_x u^{m,\nu}(t,x) - \nu\partial_{xx}^2 u^{m,\nu}(t,x) = f_m(t,x), \ (t,x) \in (0,T]\times\mathbb{R}, \\ u^{m,\nu}(0,x) = g_m(x), \ x \in \mathbb{R}, \end{cases} \tag{6.5}$$

*and respectively*

$$\begin{cases} \partial_t \bar{u}^{m,\nu}(t,x) + \bar{u}_m^{m,\nu}(t,x)\partial_x \bar{u}^{m,\nu}(t,x) - \nu\partial_{xx}^2 \bar{u}^{m,\nu}(t,x) = f_m(t,x), \ (t,x) \in (0,T]\times\mathbb{R}, \\ \bar{u}^{m,\nu}(0,x) = g_m(x), \ x \in \mathbb{R}, \end{cases} \tag{6.6}$$

*for $\nu > 0$, with*

$$\forall \varepsilon > 0, \ \lim_{n\to+\infty} \|u_n^{m,\nu} - u^{m,\nu}\|_{L^\infty([0,T];C_b^{\gamma-\varepsilon})} = \lim_{n\to+\infty} \|\bar{u}_n^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty([0,T];C_b^{\gamma-\varepsilon})} = 0,$$

*converging, up to sub-sequence selection, towards two Hölder continuous solutions $u$, $\bar{u}$, when $(m,\nu,\bar{\nu}) \to (+\infty,0,0)$ then $u = \bar{u}$.*

Importantly, the above equations have the same viscosity $\nu > 0$, but the mollification procedure of the first order coefficient may be different. This uniqueness definition is, somehow, the complementary of the *turbulent* uniqueness in the usual uniqueness introduced in Definition 4.

Replacing $b$ by $u$ in the different definitions of solution in Section 5.2, we establish the last result of this paper.

**Theorem 4** (Uniqueness of Hölder solution of the inviscid Burgers' equation)**.** *For $\gamma \in (0,1)$ be given. For all $f \in L^\infty([0,T];C_b^\gamma(\mathbb{R},\mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R},\mathbb{R})$, if there is a mild vanishing viscosity solution $u \in L^\infty([0,T];C_b^\gamma(\mathbb{R},\mathbb{R}))$ then $u$ satisfies*

$$\begin{aligned} \lim_{n\to+\infty} \sup_{t\in[0,T]} [u^{m,\nu}(t,\cdot)]_{\gamma,(\nu T/n)^{1/2}} &\leq \lim_{n\to+\infty} \int_0^T [f(s,\cdot)]_{\gamma,(\nu T/n)^{1/2}} + [g]_{\gamma,(\nu T/n)^{1/2}}, \\ \|u^{m,\nu}\|_{L^\infty} &\leq \int_0^T \|f(s,\cdot)\|_{L^\infty} + \|g\|_{L^\infty}. \end{aligned} \tag{6.7}$$

*i)* ***Good regularity.*** *If $\gamma > \frac{1}{2}$ then the considered mild vanishing viscosity solution is also a mild-weak and a weak solution, and if*

$$\nu\|\nabla^2 g_m\|_{L^\infty} + \nu^{\frac{\gamma}{2}} m^{1-\gamma} \ll 1, \tag{6.8}$$

*then $\partial_t u(t,\cdot) \in B^{-1+\gamma}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$, $\forall t \in (0,T]$.*

*ii)* ***Fast vanishing viscosity.*** *If, for a constant $C > 0$ big enough,*

$$\exp\Big(CT(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1})\exp\Big(Cmt\big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\big)\Big)\Big)\Big((1+m)\nu^{\frac{\gamma}{2}} + \nu\|\nabla^2 g_m\|_{L^\infty}\Big) \ll 1, \tag{6.9}$$

*then the solution is turbulent unique.*

*iii)* ***Slow vanishing viscosity.*** *If $\|\partial_x f\|_{L^\infty} + \|\partial_x g\|_{L^\infty} < +\infty$ and, for a constant $C > 0$ big enough,*

$$\exp\Big(CT\Big(T\|\partial_x f\|_{L^\infty} + \|\partial_x g\|_{L^\infty}\Big)\exp\Big(CT\nu^{-1}(T\|f\|_{L^\infty} + \|g\|_{L^\infty})^2\Big)\Big)m^{-\gamma} \ll 1, \tag{6.10}$$

*then the solution is viscous unique.*

It is well-known that there is a exploding time $T^*$, such that $u$ is "smooth" before $T^*$. We could imagine some structure condition on the associated flow, or like in Section 3.1.

Let us remark that the condition ii) in Theorem 1 is satisfied if the considered *a priori* regularity of the solution is strong enough (*a priori* not the condition i) in Theorem 1, as $u$ is not incompressible, except if $u$ is also solution of the Euler equation), as we have $-\gamma < -1+\gamma \iff \gamma > \frac{1}{2}$.

**Remark 11.** *Without considering the regularity condition $\gamma > \frac{1}{2}$ to get a weak solution, we may have pathologic situation. Specifically, let us consider the steady-state non-linear problem*

$$u(x)u'(x) = \frac{1}{2}\mathrm{sgn}(x), \tag{6.11}$$

*whose $x \mapsto \sqrt{|x|}$ is solution*[***] *which is as expected $\frac{1}{2}$-Hölder continuous. In other words, if $\gamma = \frac{1}{2}$, we can explicitly find a $\gamma$-Hölder steady-state solution of the inviscid Burgers' equation with source function being in $B^0_{\infty,\infty}(\mathbb{R},\mathbb{R})$ but $C^\infty_b$ almost everywhere and being the limit of a $C^\infty_b$ function, e.g.* tanh.

**Remark 12.** *With our current approach, we fail to provide any Lipschitz control of a solution of the inviscid Burgers' equation* (6.1) *for the same reason as for the transport equation* (1.1). *This is not surprising by the well-known blowing-up of the gradient of a solution of the inviscid Burgers' equation* (6.1).

**Remark 13.** *The conditions* (6.9) *and* (6.10) *are not compatible, that is to say there is no uniqueness in the sense of Definition 4, being the combination of turbulent and viscous uniqueness. Condition* (6.9) *to get turbulent uniqueness means that $\nu$ goes to 0 exponentially faster than $m$ goes to $+\infty$; whereas condition* (6.10) *to get viscous uniqueness implies that $m$ goes to $+\infty$ exponentially faster than $\nu$ goes to 0.*

*We insist that turbulent uniqueness or viscous uniqueness of a such smooth solution of the inviscid Burgers' equation is not a contradiction with the usual counter-example built by characteristics, because we only consider solution selected by a vanishing viscosity approximation for a given mollification procedure. Somehow, this selection principle allows to avoid the blow-up time appearing in the characteristic building for a given vanishing viscous path.*

*Actually, from the mild vanishing viscous solution, we see that for any $t \in [0,T]$, the solution $u(t,\cdot)$ given by the limit of a sub-sequence of $u^{m,\nu}(t,\cdot)$ depends on the mollification choice, moreover the sub-sequence choice also depends on the current time $t$. In other words, $u(t,\cdot)$ seems to avoid the time of blowing-up thanks to a different choice of sub-sequence at each current time.*

## 6.2 Proof of Theorem 4

To establish this result, we consider the mollified version of Burgers' equation, for all $m \in \mathbb{R}_+$ and $\nu > 0$,

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + u^{m,\nu}_m(t,x)\partial_x u^{m,\nu}(t,x) - \nu\partial^2_{xx} u^{m,\nu}(t,x) = f_m(t,x),\ (t,x) \in [0,T)\times\mathbb{R}, \\ u_m(0,x) = g_m(x),\ x \in \mathbb{R}, \end{cases} \tag{6.12}$$

where $u^{m,\nu}_m$ stands for a mollified version of $u^{m,\nu}$, such that

$$\forall \varepsilon > 0,\ \lim_{n\to+\infty} \|u^{m,\nu}_n - u^{m,\nu}\|_{L^\infty([0,T];C^{\gamma-\varepsilon}_b)} = 0,$$

It is direct from Theorem 3 in [Hon22] that there is a smooth solution of (6.12).

We then perform the same computations as for the transport equation, where $-\tilde{\gamma} = \gamma$ and we change in the viscous condition $\|b\|_{L^\infty(B^{-\tilde{\gamma}}_{\infty,\infty})}$ by an upper-bound of $\|u\|_{L^\infty}$, namely by $T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma$ given by Feynman-Kac formula or from the time decomposition trick, see Section 4.10. Finally, we are able to take the limit, thanks to a compact argument, of a suitable sub-sequence yields the result, all the computations readily derives from the analysis of Theorems 2, 3 and 1.

[***] Also the function $x \mapsto -\sqrt{|x|}$, but this non-uniqueness should be related with some ergodic properties.

### 6.2.1 Turbulent uniqueness

To establish uniqueness, let us consider a regularised Burgers' equation with another viscosity $\bar{\nu}$,

$$\begin{cases} \partial_t u^{m,\bar{\nu}}(t,x) + u_m^{m,\bar{\nu}}(t,x)\partial_x u^{m,\bar{\nu}}(t,x) - \bar{\nu}\partial_{xx}^2 u^{m,\bar{\nu}}(t,x) = f_m(t,x),\ (t,x) \in [0,T)\times\mathbb{R}, \\ u^{m,\bar{\nu}}(0,x) = g_m(x),\ x \in \mathbb{R}. \end{cases} \tag{6.13}$$

We highlight that $u_m^{m,\bar{\nu}}$ is also a regularisation version of $u_m^{m,\bar{\nu}}$ such that (6.4) is in force and,

$$\forall \varepsilon > 0,\ \lim_{n\to+\infty} \|\bar{u}_n^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty([0,T];C_b^{\gamma-\varepsilon})} = 0,$$

Like for $u^{m,\nu}$, we suppose that there is a constant $C > 0$ such that for any $(m,\bar{\nu}) \in \mathbb{R}_+^2$,

$$\|u^{m,\bar{\nu}}\|_{L^\infty(C_b^\gamma)} \le C(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma),$$

under some specific asymptotic conditions on $(m,\bar{\nu})$. We still write $U_m := u^{m,\nu} - u^{m,\bar{\nu}}$ which is solution of

$$\partial_t U_m(t,x) + u_m^{m,\nu}(t,x)\partial_x U_m(t,x) - \bar{\nu}\Delta U_m(t,x) = -[u_m^{m,\nu} - u_m^{m,\bar{\nu}}](t,x)\partial_x u^{m,\bar{\nu}}(t,x) + (\nu-\bar{\nu})\Delta u^{m,\nu}(t,x), \tag{6.14}$$

with $U_m(0,x) = 0$.

Adapting inequality (4.40) yields

$$\|U_m(t,\cdot)\|_{L^\infty} \le \int_0^t \|[u_m^{m,\nu} - u_m^{m,\bar{\nu}}](s,\cdot)\partial_x u_m^{m,\bar{\nu}}(s,\cdot)\|_{L^\infty} ds + t|\nu-\bar{\nu}|\|\Delta u^{m,\nu}\|_{L^\infty}.$$

We then get by triangular inequality and by (6.4), we can suppose w.l.o.g. that $\bar{\nu} \le \nu$ (if not we switch the roles of $\nu$ and $\bar{\nu}$),

$$\|U_m(t,\cdot)\|_{L^\infty} \le \bar{C}\int_0^t \|U_m(s,\cdot)\|_{L^\infty}\|\partial_x u_m^{m,\bar{\nu}}\|_{L^\infty} ds + t\nu\|\Delta u^{m,\nu}\|_{L^\infty}.$$

Recalling the Hessian estimate of Lemma 3,

$$\begin{aligned} & |\nabla^2 u^{m,\nu}(t,x)| \\ \le\ & \Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty}\Big) + C\|u_m^{m,\nu}\|_{L^\infty(C^1)}\Big(t\|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma}\Big)\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}} \\ \le\ & \Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty}\Big) + Cm\|u^{m,\nu}\|_{L^\infty}\Big(t\|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma}\Big)\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}. \end{aligned}$$

The *a priori* control of $\|u^{m,\nu}\|_{L^\infty(C^\gamma)}$ obtained in (4.101)yields

$$\begin{aligned} |\nabla^2 u^{m,\nu}(t,x)| \ \le\ & \Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty}\Big) \\ & + Cm\Big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\Big)\Big(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma}\Big)\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|U_m(t,\cdot)\|_{L^\infty} \ \le\ & C\int_0^t \|U_m(s,\cdot)\|_{L^\infty}\|\partial_x u_m^{m,\bar{\nu}}\|_{L^\infty} ds + T\nu\Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \\ & + Cm\Big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\Big)\Big(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma}\Big)\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\Big). \end{aligned}$$

By Grönwall's inequality, we derive

$$\begin{aligned} \|U_m(t,\cdot)\|_{L^\infty} \ \le\ & e^{CT\|\partial_x u_m^{m,\bar{\nu}}\|_{L^\infty}} T\Big(C\nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \nu\|\nabla^2 g_m\|_{L^\infty} \\ & + Cm\Big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\Big)\Big(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma}\Big)\nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}\Big). \end{aligned}$$

Recalling the gradient estimate from Lemma 3

$$\begin{aligned}\|\nabla u^{m,\nu}(t,\cdot)\|_{L^\infty} &\leq \Big(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big)\exp\Big(Cmt\|u^{m,\nu}\|_{L^\infty}\Big)\\ &\leq \Big(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\Big)\exp\Big(Cmt\big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\big)\Big).\end{aligned}$$

Hence, we get

$$\begin{aligned}&\|U_m(t,\cdot)\|_{L^\infty} \qquad (6.15)\\ \leq\ & T\exp\bigg(CT(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1})\exp\Big(Cmt\big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\big)\Big)\bigg)\\ &\bigg(C\nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \nu\|\nabla^2 g_m\|_{L^\infty} + Cm\Big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\Big)\Big(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma}\Big)\nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}\bigg),\end{aligned}$$

which goes to 0 if

$$\exp\bigg(CT(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1})\exp\Big(Cmt\big(T\|f\|_{L^\infty} + \|g\|_{L^\infty}\big)\Big)\bigg)\bigg((1+m)\nu^{\frac{\gamma}{2}} + \nu\|\nabla^2 g_m\|_{L^\infty}\bigg) \ll 1.$$

Turbulent uniqueness of *vanishing viscous* solution is then established.

### 6.2.2 Viscous uniqueness

Let us consider another regularised Burgers' equation with different mollification procedure but with the same viscosity $\nu > 0$,

$$\begin{cases}\partial_t \bar{u}^{m,\nu}(t,x) + \bar{u}^{m,\nu}_m(t,x)\partial_x \bar{u}^{m,\nu}(t,x) - \nu\partial^2_{xx}\bar{u}^{m,\nu}(t,x) = f_m(t,x),\ (t,x)\in[0,T)\times\mathbb{R},\\ \bar{u}^{m,\nu}(0,x) = g_m(x),\ x\in\mathbb{R},\end{cases} \qquad (6.16)$$

with $\nu > 0$, and where $\bar{u}^{m,\nu}_m$ is a regularisation of $\bar{u}^{m,\nu}$ (not necessarily a defined by a convolution) such that

$$\sup_{(m,\nu)\in\mathbb{R}^2_+}\|\bar{u}^{m,\nu}\|_{L^\infty(C^\gamma_b)} \leq C(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) < +\infty,$$

and for any $0 < \varepsilon < 1$,

$$\lim_{n\to+\infty}\|\bar{u}^{m,\nu}_n - \bar{u}^{m,\nu}\|_{L^\infty} = \lim_{n\to+\infty}\|\bar{u}^{m,\nu}_n - \bar{u}^{m,\nu}\|_{L^\infty(C^{\gamma-\varepsilon}_b)} = 0. \qquad (6.17)$$

We still write $U_m := u^{m,\nu} - \bar{u}^{m,\nu}$ which is solution of

$$\begin{cases}\partial_t U_m(t,x) + u^{m,\nu}_m(t,x)\partial_x U_m(t,x) - \nu\Delta U_m(t,x) = -[u^{m,\nu}_m - \bar{u}^{m,\nu}_m](t,x)\partial_x\bar{u}^{m,\nu}(t,x),\\ U_m(0,x) = 0.\end{cases} \qquad (6.18)$$

We again adapt inequality (4.40) yields

$$\|U_m(t,\cdot)\|_{L^\infty} \leq \int_0^t \|(u^{m,\nu}_m - \bar{u}^{m,\nu}_m)\partial_x\bar{u}^{m,\nu}_m(s,\cdot)\|_{L^\infty}ds.$$

Let us denotes

$$\bar{u}^{m,\nu}_{m,m}(t,x) := \int_{\mathbb{R}}\rho_m(x-y)\bar{u}^{m,\nu}(t,y)dy, \qquad (6.19)$$

such that

$$\|\bar{u}^{m,\nu}_{m,m} - \bar{u}^{m,\nu}_m\|_{L^\infty} \leq Cm^{-\gamma}\|\bar{u}^{m,\nu}\|_{L^\infty(C^\gamma)},$$

and

$$\|u^{m,\nu}_m - \bar{u}^{m,\nu}_{m,m}\|_{L^\infty} \leq \|u^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty} = \|U_m(t,\cdot)\|_{L^\infty}.$$

We then get by triangular inequality,

$$\begin{aligned}\|U_m(t,\cdot)\|_{L^\infty} &\le \int_0^t \Big(\|(u_m^{m,\nu}-\bar u_{m,m}^{m,\nu})(s,\cdot)\|_{L^\infty}+\|(\bar u_{m,m}^{m,\nu}-\bar u_m^{m,\nu})(s,\cdot)\|_{L^\infty}\Big)\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}ds\\ &\le \bar C\int_0^t\Big(Cm^{-\gamma}\|\bar u^{m,\nu}\|_{L^\infty(C^\gamma)}+\|U_m(s,\cdot)\|_{L^\infty}\Big)\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}ds,\end{aligned}$$

and from (6.7), we get,

$$\|U_m(t,\cdot)\|_{L^\infty}\le\bar C\int_0^t\Big(Cm^{-\gamma}\big(T\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)+\|U_m(s,\cdot)\|_{L^\infty}\Big)\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}ds.$$

Also by Grönwall's inequality, we derive

$$\|U_m(t,\cdot)\|_{L^\infty}\le e^{CT\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}}TCm^{-\gamma}(T\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma)\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}.$$

From exponential absorbing property,

$$\begin{aligned}\|U_m(t,\cdot)\|_{L^\infty} &\le e^{CT\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}}TCm^{-\gamma}(T\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma)\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}\\ &\le e^{CT\|\partial_x\bar u_m^{m,\nu}\|_{L^\infty}}m^{-\gamma}(T\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma).\end{aligned}\tag{6.20}$$

We need a new *estimate* of the gradient to avoid any blowing up terms in $m$ which cannot be balanced by the $m^{-\gamma}$ in front of the exponential; the singularity has to be in $\nu$.

**Lemma 5.** *For $u^{m,\nu}$ solution to* (6.12)*, we have*

$$\|\partial_x u^{m,\nu}\|_{L^\infty}\le 2\Big(T\|\partial_x f_m\|_{L^\infty}+\|\partial_x g_m\|_{L^\infty}\Big)\exp\Big(C^2\nu^{-1}\pi(T\|f\|_{L^\infty}+\|g\|_{L^\infty})^2T\Big).$$

The proof is postponed in Section 7.4.
We deduce from (6.20),

$$\begin{aligned}\|U_m(t,\cdot)\|_{L^\infty} &\le \exp\Big(CT2\Big(T\|\partial_x f_m\|_{L^\infty}+\|\partial_x g_m\|_{L^\infty}\Big)\exp\Big(C^2\nu^{-1}\pi(T\|f\|_{L^\infty}+\|g\|_{L^\infty})^2T\Big)\Big)\\ &\quad\times m^{-\gamma}(T\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma),\end{aligned}$$

which goes to 0 under condition (6.10).

# 7 Proofs of some *a priori* controls

## 7.1 Proof of Lemma 3

### 7.1.1 Gradient estimates

Let us precise the control previously used:

$$\|\nabla u^{m,\nu}(t,\cdot)\|_{L^\infty}\le\Big(T\|f_m\|_{L^\infty(C^1)}+\|g_m\|_{C^1}\Big)\exp\Big(Cm^{1-\tilde\gamma}t\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\Big)=O_m(t).\tag{7.1}$$

We directly have from Duhamel formula (4.7):

$$\begin{aligned}&|\nabla u^{m,\nu}(t,x)|\\ \le\ &\Big(t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}\Big)\\ &+\Big|\int_0^t\int_{\mathbb{R}^d}\nabla\hat p^{\tau,\xi}(s,t,x,y)[b_m(s,\theta^m_{s,\tau}(\xi))-b_m(s,y)]\cdot\nabla u^{m,\nu}(s,y)dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)}\\ \le\ &\Big(t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}\Big)\\ &+C\|b_m\|_{L^\infty(C^1)}\int_0^t\int_{\mathbb{R}^d}[\nu(t-s)]^{-\frac12}\bar p^{\tau,\xi}(s,t,x,y)|\theta^m_{s,\tau}(\xi)-y|\|\nabla u^{m,\nu}(s,\cdot)\|_{L^\infty}dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)}.\end{aligned}$$

By absorbing property (2.13), we derive

$$\begin{aligned}|\nabla u^{m,\nu}(t,x)| &\leq \Big(t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}\Big)\\ &\quad +Cm^{1-\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\int_0^t\int_{\mathbb{R}^d}\bar p^{\tau,\xi}(s,t,x,y)\|\nabla u^{m,\nu}(s,\cdot)\|_{L^\infty}dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)}\\ &\leq \Big(t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}\Big)+Cm^{1-\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\int_0^t\|\nabla u^{m,\nu}(s,\cdot)\|_{L^\infty}ds.\end{aligned} \tag{7.2}$$

Finally, Grönwall's lemma yields the result.

This useful *a priori* control allows to avoid any blow-up when $\nu\to 0$. However, to be able to prove uniqueness, we also need another estimate stated in Lemma 2 and proved in Section 7.3.

### 7.1.2 Hessian estimates

We perform a similar argument, but for the second derivatives we have to put a second derivatives on $[b_m(s,\theta_t(\xi))-b_m(s,y)]\cdot\nabla u(s,y)$. Indeed, if we twice differentiate $\hat p^{\tau,\xi}(s,t,x,y)$ there is no possibility to smoothen the blowing up the contribution of $\nu$ by Hölder control (or even Lipschitz).

We obtain by Leibniz rules

$$\begin{aligned}&|\nabla^2 u^{m,\nu}(t,x)|\\ &\leq \Big(t\|\nabla^2 f_m\|_{L^\infty}+\|\nabla^2 g_m\|_{L^\infty}\Big)+\Big|\int_0^t\int_{\mathbb{R}^d}\nabla\hat p^{\tau,\xi}(s,t,x,y)\nabla b(s,y)\cdot\nabla u(s,y)dy\,ds\Big|\Big|_{\xi=x}\\ &\quad +\Big|\int_0^t\int_{\mathbb{R}^d}\nabla\hat p^{\tau,\xi}(s,t,x,y)[b_m(s,\theta^m_{s,\tau}(\xi))-b_m(s,y)]\nabla^2 u(s,y)dy\,ds\Big|\Big|_{\xi=x}\\ &\leq m^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)\\ &\quad +C\int_0^t\int_{\mathbb{R}^d}[\nu(t-s)]^{-\frac12}\bar p^{\tau,\xi}(s,t,x,y)|\theta^m_{s,\tau}(\xi)-y|\|\nabla b_m\otimes\nabla u^{m,\nu}(s,\cdot)\|_{L^\infty(C^1)}dy\,ds\\ &\quad +C\|b_m\|_{L^\infty(C^1)}\int_0^t\int_{\mathbb{R}^d}[\nu(t-s)]^{-\frac12}\bar p^{\tau,\xi}(s,t,x,y)|\theta^m_{s,\tau}(\xi)-y|\|\nabla^2 u^{m,\nu}(s,\cdot)\|_{L^\infty}dy\,ds.\end{aligned}$$

Next, with Leibniz rules and absorbing property (2.13),

$$\begin{aligned}&|\nabla^2 u^{m,\nu}(t,x)|\\ &\leq m^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)+Cm^{2+\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\int_0^t\int_{\mathbb{R}^d}\bar p^{\tau,\xi}(s,t,x,y)\|\nabla u^{m,\nu}(s,\cdot)\|_{L^\infty}dy\,ds\\ &\quad +Cm^{1-\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\int_0^t\|\nabla^2 u^{m,\nu}(s,\cdot)\|_{L^\infty}ds\\ &\leq m^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)+Ctm^{2+\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\|\nabla u^{m,\nu}\|_{L^\infty}\\ &\quad +Cm^{1-\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}\int_0^t\|\nabla^2 u^{m,\nu}(s,\cdot)\|_{L^\infty}ds.\end{aligned} \tag{7.3}$$

We finally get by Grönwall's lemma and by identity (7.1)

$$\begin{aligned}&\|\nabla^2 u^{m,\nu}(t,\cdot)\|_{L^\infty}\\ &\leq \Big(m^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)+Ctm^{2+\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}O_m(t)\Big)\exp(tm^{1-\tilde\gamma}\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})}).\end{aligned} \tag{7.4}$$

We also write by exponential absorbing property:

$$\|\nabla^2 u^{m,\nu}(t,\cdot)\|_{L^\infty}\leq Cm^{2-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)}+[g]_\gamma\big)\exp(Cm^{1+\tilde\gamma}t\|b\|_{L^\infty(B^{\tilde\gamma}_{\infty,\infty})})=:O^{(2)}_m(t). \tag{7.5}$$

We insist on the fact that the above inequality does not depend on $\nu$.

### 7.1.3 Third derivatives estimates

In this section, we detail how to control the third derivatives of $u^{m,\nu}$. We use the same method as for the Hessian, the additional derivative is also put on $[b_m(s,\theta_t(\xi)) - b_m(s,y)] \cdot \nabla u(s,y)$.

$$\begin{aligned}
|\nabla^3 u^{m,\nu}(t,x)| &\leq \Big(t\|\nabla^3 f_m\|_{L^\infty} + \|\nabla^3 g_m\|_{L^\infty}\Big) + \Big| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s,t,x,y) \nabla^2 b(s,y) \cdot \nabla u(s,y) dy\, ds \Big| \Big|_{\xi=x} \\
&\quad + \Big| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s,t,x,y) \nabla b(s,y) \cdot \nabla^2 u(s,y) dy\, ds \Big| \Big|_{\xi=x} \\
&\quad + \Big| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s,t,x,y) [b_m(s,\theta^m_{s,\tau}(\xi)) - b_m(s,y)] \nabla^3 u(s,y) dy\, ds \Big| \Big|_{\xi=x}.
\end{aligned}$$

We readily obtain the upper-bound

$$\begin{aligned}
|\nabla^3 u^{m,\nu}(t,x)| &\leq C m^{3-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big) \\
&\quad + C \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s,t,x,y) |\theta^m_{s,\tau}(\xi) - y| \\
&\quad \times (\|\nabla^2 b_m \cdot \nabla u^{m,\nu}(s,\cdot)\|_{L^\infty(C^1)} + \|\nabla b_m \cdot \nabla^2 u^{m,\nu}(s,\cdot)\|_{L^\infty(C^1)}) dy\, ds \\
&\quad + C\|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s,t,x,y) |\theta^m_{s,\tau}(\xi) - y| \|\nabla^3 u^{m,\nu}(s,\cdot)\|_{L^\infty} dy\, ds,
\end{aligned}$$

and

$$\begin{aligned}
|\nabla^3 u^{m,\nu}(t,x)| &\leq C m^{3-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big) + C t m^{3+\tilde{\gamma}} \|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})} \|\nabla u^{m,\nu}\|_{L^\infty} \\
&\quad + C t m^{2+\tilde{\gamma}} \|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})} \|\nabla^2 u^{m,\nu}\|_{L^\infty} + C m^{2+\tilde{\gamma}} \|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})} \int_0^t \|\nabla^3 u^{m,\nu}(s,\cdot)\|_{L^\infty} ds.
\end{aligned}$$

Eventually, by Grönwall's lemma, identities (7.1) and (7.4) imply

$$\begin{aligned}
\|\nabla^3 u^{m,\nu}(t,\cdot)\|_{L^\infty} &\leq C\Big(m^{3-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big) + t m^{2+\tilde{\gamma}} \|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})} \big(m O_m(t) + m O^{(2)}_m(t)\big)\Big) \\
&\quad \times \exp(C t m^{1-\tilde{\gamma}} \|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})}) \\
&\leq C m^{3-\gamma}\big(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma\big) \exp(C m^{1+\tilde{\gamma}} t \|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})}), \qquad (7.6)
\end{aligned}$$

by exponential absorbing property.

## 7.2 Proof of Lemma 4

This section is devoted to the regularity of the flow

$$\theta^m_{s,\tau}(x) := x + \int_s^\tau b_m(\tilde{s}, \theta^m_{\tilde{s},\tau}(x)) d\tilde{s}.$$

By definition, we have

$$\begin{aligned}
|\theta^m_{s,\tau}(x) - \theta^m_{s,\tau}(x')| &\leq |x - x'| + \Big| \int_s^\tau b_m(\tilde{s}, \theta^m_{\tilde{s},\tau}(x)) - b_m(s, \theta^m_{\tilde{s},\tau}(x')) ds \Big| \\
&\leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_s^\tau |\theta^m_{\tilde{s},\tau}(x) - \theta^m_{\tilde{s},\tau}(x)| d\tilde{s},
\end{aligned}$$

which is not the suitable inequality to apply directly Grönwall's lemma. To do so, we use a sup formulation, namely for any $r \leq \tau$, we write similarly to above

$$\begin{aligned}\sup_{s\in[0,r]}|\theta^m_{s,\tau}(x)-\theta^m_{s,\tau}(x')| &\leq |x-x'|+\int_0^r \big|b_m(\tilde{s},\theta^m_{\tilde{s},\tau}(x))-b_m(s,\theta^m_{\tilde{s},\tau}(x'))\big|ds\\ &\leq |x-x'|+\|b_m\|_{L^\infty(C^1)}\int_0^r|\theta^m_{\tilde{s},\tau}(x)-\theta^m_{\tilde{s},\tau}(x)|d\tilde{s}\\ &\leq |x-x'|+\|b_m\|_{L^\infty(C^1)}\int_0^r\sup_{\hat{s}\in[0,\tilde{s}]}|\theta^m_{\hat{s},\tau}(x)-\theta^m_{\hat{s},\tau}(x)|d\tilde{s}.\end{aligned}$$

We are now in position to use Grönwall's lemma, for $r=\tau$

$$\sup_{\tilde{s}\in[0,\tau]}|\theta^m_{s,\tau}(x)-\theta^m_{s,\tau}(x')| \leq |x-x'|\exp\big(\|b_m\|_{L^\infty(C^1)}\tau\big).$$

## 7.3 Proof of Lemma 2

### 7.3.1 Gradient estimates

By integration by parts

$$\begin{aligned}&|\nabla u^{m,\nu}(t,x)|\\ \leq\ & t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}\\ &+\Big|\int_0^t\int_{\mathbb{R}^d}\nabla^2\hat{p}^{\tau,\xi}(s,t,x,y)[b_m(s,\theta^m_{s,\tau}(\xi))-b_m(s,y)]u^{m,\nu}(s,y)dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)}\\ &+\Big|\int_0^t\int_{\mathbb{R}^d}\nabla\hat{p}^{\tau,\xi}(s,t,x,y)\nabla\cdot b_m(s,y)u^{m,\nu}(s,y)dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)},\end{aligned}$$

and by exponential absorbing property and because $\nabla\cdot b=0$,

$$\begin{aligned}|\nabla u^{m,\nu}(t,x)| &\leq t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}\\ &\quad +C\|b_m\|_{L^\infty(C^1)}\|u^{m,\nu}\|_{L^\infty}\int_0^t\int_{\mathbb{R}^d}[\nu(t-s)]^{-\frac{1}{2}}\bar{p}^{\tau,\xi}(s,t,x,y)dy\,ds\Big|_{(\tau,\xi)=(t,x)}\\ &\leq t\|\nabla f_m\|_{L^\infty}+\|\nabla g_m\|_{L^\infty}+Cm^{1-\tilde{\gamma}}\|b_m\|_{L^\infty(C^{\tilde{\gamma}})}\Big(t\|f_m\|_{L^\infty}+\|g_m\|_{L^\infty}\Big)\nu^{-\frac{1}{2}}t^{\frac{1}{2}}.\end{aligned} \tag{7.7}$$

The last identity comes from the uniform norm estimates stated in Theorem 3.

### 7.3.2 Hessian estimates

Like in Section 7.3.1, we integrate by parts

$$\begin{aligned}|\nabla^2 u^{m,\nu}(t,x)| &\leq \Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)}+\|\nabla^2 g_m\|_{L^\infty}\Big)\\ &\quad +\Big|\int_0^t\int_{\mathbb{R}^d}\nabla^3\hat{p}^{\tau,\xi}(s,t,x,y)[b_m(s,\theta^m_{s,\tau}(\xi))-b_m(s,y)]\\ &\quad [u^{m,\nu}(s,y)-u^{m,\nu}(s,\theta^m_{s,\tau}(\xi))]dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)}\\ &\quad +\Big|\int_0^t\int_{\mathbb{R}^d}\nabla^2\hat{p}^{\tau,\xi}(s,t,x,y)\nabla\cdot b_m(s,y)[u^{m,\nu}(s,y)-u^{m,\nu}(s,\theta^m_{s,\tau}(\xi))]dy\,ds\Big|\Big|_{(\tau,\xi)=(t,x)},\end{aligned}$$

and by exponential absorbing property after choosing the freezing parameters,

$$\begin{aligned}
& |\nabla^2 u^{m,\nu}(t,x)| \\
\leq\ & \Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty}\Big) \\
& + C\|b_m\|_{L^\infty(C^1)}\|u^{m,\nu}\|_{L^\infty(C^1)}\int_0^t\int_{\mathbb{R}^d}[\nu(t-s)]^{-\frac{1}{2}}\bar{p}^{\tau,\xi}(s,t,x,y)dy\,ds\Big|_{(\tau,\xi)=(t,x)} \\
\leq\ & \Big(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty}\Big) \\
& + Cm^{1-\tilde{\gamma}}\|b_m\|_{L^\infty(C^{\tilde{\gamma}})}\nu^{-\frac{1}{2}}t^{\frac{1}{2}}\big(t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty}\big) \\
& + Cm^{2(1-\tilde{\gamma})}\|b_m\|^2_{L^\infty(C^{\tilde{\gamma}})}\big(t\|f_m\|_{L^\infty} + \|g_m\|_{L^\infty}\big)\nu^{-\frac{1}{2}}t^{\frac{1}{2}}.
\end{aligned} \tag{7.8}$$

## 7.4 Proof of Lemma 5

From the usual Duhamel formula around the heat equation,

$$\begin{aligned}
|\partial_x u^{m,\nu}(t,x)| &\leq \Big(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty}\Big) + \Big|\int_0^t\int_{\mathbb{R}^d}\partial_x\tilde{p}(s,t,x,y)u_m^{m,\nu}(s,y)\partial_x u^{m,\nu}(s,y)dy\,ds\Big| \\
&\leq \Big(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty}\Big) \\
&\quad + C\|u^{m,\nu}\|_{L^\infty}\int_0^t\int_{\mathbb{R}^d}[\nu(t-s)]^{-\frac{1}{2}}\tilde{p}(s,t,x,y)\|\partial_x u^{m,\nu}(s,\cdot)\|_{L^\infty}dy\,ds.
\end{aligned}$$

By the well known $L^\infty$ control, see Section 4.10, we obtain

$$\begin{aligned}
& |\partial_x u^{m,\nu}(t,x)| \\
\leq\ & \Big(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty}\Big) + C(T\|f\|_{L^\infty} + \|g\|_{L^\infty})\nu^{-\frac{1}{2}}\int_0^t(t-s)^{-\frac{1}{2}}\|\partial_x u^{m,\nu}(s,\cdot)\|_{L^\infty}ds.
\end{aligned} \tag{7.9}$$

Here, it is not possible to directly use Grönwall's lemma due to the "$t$" is in the integral. We have to consider Grönwall-Henry's lemma, cf. [Hen81] chapter 7 Lemma 7.1.1. .

**Lemma 6** (Lemma of Grönwall-Henry)**.** *Let $T > 0$, a positive a constant $K > 0$ and a non-negative function $\alpha$ such that the function $\varphi : [0,T] \longrightarrow \mathbb{R}_+$ satisfying for any $0 < t < Ts$*

$$\varphi(t) \leq \alpha(t) + K\int_0^t(t-s)^{-1+\tilde{\gamma}}\varphi(s)ds, \tag{7.10}$$

*then*

$$\varphi(t) \leq \alpha(t) + \theta\int_0^t E'_{\tilde{\gamma}}(\theta(t-s))\alpha(s)ds, \tag{7.11}$$

*with for any $r \in [0,T]$,*

$$\begin{aligned}
\theta &= (K\Gamma(\tilde{\gamma}))^{\frac{1}{\tilde{\gamma}}}, \\
E_{\tilde{\gamma}}(r) &= \sum_{n=0}^{+\infty}\frac{r^{n\tilde{\gamma}}}{\Gamma(n\tilde{\gamma}+1)},
\end{aligned}$$

*and if $\alpha$ is a non-decreasing function, then, for any $0 < t < T$,*

$$\varphi(t) \leq \alpha(t)E_{\tilde{\gamma}}(\theta t). \tag{7.12}$$

The last inequality (7.12) is readily derived from (7.11).

The only needed case, here, is $\tilde{\gamma} = \frac{1}{2}$, for the sake of completeness, we detail the useful proof of exercise 1 in [Hen81].

**Lemma 7.** *For any $t > 0$,*

$$e^t \leq E_{1/2}(t) \leq 2e^t.$$

*Proof of Lemma 7.* Let us recall that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, then we get, for any $r \geq 0$, by differentiating,

$$\begin{aligned} \partial_r E_{1/2}(r) &= r^{-1} \sum_{n=1}^{+\infty} \frac{n}{2} \frac{r^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = r^{-1} \sum_{n=1}^{+\infty} \frac{r^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \\ &= (\pi r)^{-\frac{1}{2}} + E_{1/2}(r). \end{aligned}$$

By integrating this equation, for any $t \geq 0$, we get

$$E_{1/2}(t) = e^t + e^t \pi^{-\frac{1}{2}} \int_0^t r^{-\frac{1}{2}} e^{-r} dr.$$

The lower bound of the lemma is direct, the upper-bound comes from

$$E_{1/2}(t) \leq e^t + e^t \pi^{-\frac{1}{2}} \int_0^{+\infty} r^{-\frac{1}{2}} e^{-r} dr = e^t + e^t \pi^{-\frac{1}{2}} \Gamma(\frac{1}{2}) = 2e^t. \tag{7.13}$$

□

Coming back to inequality (7.9), we identify the notations in Lemma 6, $\tilde{\gamma} = \frac{1}{2}$, and $\theta = C^2 \nu^{-1} \pi (T\|f\|_{L^\infty} + \|g\|_{L^\infty})^2$, then from Lemmas 6 and 7,

$$\|\partial_x u^{m,\nu}(t,\cdot)\|_{L^\infty} \leq 2\Big(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty}\Big) \exp\Big(C^2 \nu^{-1} \pi (T\|f\|_{L^\infty} + \|g\|_{L^\infty})^2 t\Big).$$

# A Convergence of the mollified distribution

**Proposition 3.** *For any $\psi \in C^\gamma(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0,1]$, we have for all $\vartheta \in \mathbb{N}_0^d$ and $\theta \in \mathbb{N}_0$ s.t. $|\vartheta| = \theta$, that $h_{m^{-2}} \star D^\vartheta \psi \in C_b^\infty$ converges towards $D^\vartheta \psi$ in $\dot{B}^{-\theta}_{\infty,\infty}$ as $m \to +\infty$. More precisely, we have:*

$$\|h_{m^{-2}} \star D^\vartheta \psi - D^\vartheta \psi\|_{\dot{B}^{-\theta}_{\infty,\infty}} \leq C[\psi]_\gamma m^{-\gamma}, \tag{A.1}$$

*and $D^\vartheta \psi \in \dot{B}^{\gamma-\theta}_{\infty,\infty}$ In particular, if $|\vartheta| = 0$, $h_{m^{-2}} \star \psi \in C_b^\infty$ converges towards $\psi$ in $L^\infty$ as $m \to +\infty$, and:*

$$\|h_{m^{-2}} \star \psi - \psi\|_{L^\infty} \leq C[\psi]_\gamma m^{-\gamma}. \tag{A.2}$$

**Remark 14.** *Actually, $\gamma \notin \{0,1\}$ is not a restrictive condition as changing $\vartheta$ into $\tilde{\vartheta} \in \mathbb{N}_0^d$ such that $|\tilde{\vartheta}| = |\vartheta| + 1$ yields the same result.*

*In particular, Proposition 3 is available for the Dirac distribution $\delta \in \dot{B}^{-d}_{\infty,\infty}$ regarded as the distributional derivative of the sign function (also regarded as the derivative of the absolute value), and for any derivative of the Dirac distribution by the same argument.*

*Proof of Proposition 3.* Let us write $\varphi = D^\vartheta \psi$, with $\psi \in C^\gamma$

$$\begin{aligned} \|h_{m^{-2}} \star \varphi - \varphi\|_{\dot{B}^{-\vartheta}_{\infty,\infty}} &= \|D^\vartheta [h_{m^{-2}} \star \psi - \psi]\|_{\dot{B}^{-\vartheta}_{\infty,\infty}} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\vartheta}{2}} \|\partial_v h_v \star D^\vartheta [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\vartheta}{2}} \|\partial_v D^\vartheta h_v \star [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty}, \end{aligned}$$

by integration by parts in convolutions. Next, we can explicitly write,

$$\begin{aligned}
&\|h_{m^{-2}} \star \varphi - \varphi\|_{\dot{B}^{-\vartheta}_{\infty,\infty}} \\
= &\sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} v^{1-\frac{-\vartheta}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m^{-2}}(y-x)[\psi(x)-\psi(y)]dx\,dy\Big| \\
\leq &\ C[\psi]_\gamma \sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x)|x-y|^\gamma dx\,dy \\
\leq &\ C[\psi]_\gamma m^{-\gamma} \sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x)dx\,dy,
\end{aligned}$$

by exponential absorbing property (2.13). Integrating in space finally yields

$$\|h_{m^{-2}} \star \varphi - \varphi\|_{\dot{B}^{-\vartheta}_{\infty,\infty}} \leq C[\psi]_\gamma m^{-\gamma}.$$

Inequality (A.2) is direct with similar arguments. □

**Corollary 5.** *For any $\psi \in \dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$, $\gamma\in(0,1]$ we have, for all $\vartheta\in\mathbb{N}_0^d$ and $\theta\in\mathbb{N}_0$ s.t. $|\vartheta|=\theta$, that $h_{m^{-2}}\star D^\vartheta\psi \in C^\infty_b(\mathbb{R}^d,\mathbb{R})$ converges towards $D^\vartheta\psi$ in $\dot{B}^{-\theta+\gamma-\varepsilon}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$, for any $\varepsilon\in(0,1)$ s.t. $\theta-\gamma+\varepsilon>0$, as $m\to+\infty$. More precisely, we have:*

$$\|h_{m^{-2}}\star D^\vartheta\psi - D^\vartheta\psi\|_{\dot{B}^{-\theta+\gamma-\varepsilon}_{\infty,\infty}} \leq C[\psi]_\gamma m^{-\varepsilon}. \tag{A.3}$$

*Proof of Corollary 5.* We still use the thermic representation, and by convolution property we have:

$$\begin{aligned}
&\|h_{m^{-2}} \star \varphi - \varphi\|_{\dot{B}^{-\theta+\gamma-\varepsilon}_{\infty,\infty}} \\
= &\sup_{v\in\mathbb{R}_+} v^{1-\frac{-\theta+\gamma-\theta}{2}} \|\partial_v h_v \star D^\vartheta[h_{m^{-2}}\star\psi-\psi]\|_{L^\infty(\mathbb{R}^d)} \\
= &\sup_{v\in\mathbb{R}_+} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v D^\vartheta h_v \star [h_{m^{-2}}\star\psi-\psi]\|_{L^\infty(\mathbb{R}^d)} \\
= &\sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m^{-2}}(y-x)[\psi(x)-\psi(y)]dx\,dy\Big|.
\end{aligned}$$

For a given $v\in(0,+\infty)$, we compare the regular contribution $v$ with the mollification contribution. In other words, we consider two possibilities.

- If $m^{-2}<v$, then:

$$\begin{aligned}
&v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^\vartheta[h_{m^{-2}}\star\psi-\psi]\|_{L^\infty(\mathbb{R}^d)} \\
= &\sup_{z\in\mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m^{-2}}(y-x)[\psi(x)-\psi(y)]dx\,dy\Big| \\
\leq &\ C[\psi]_\gamma v^{\frac{-\gamma+\varepsilon}{2}} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x)|x-y|^\gamma dx\,dy \\
\leq &\ C[\psi]_\gamma v^{\frac{-\gamma+\varepsilon}{2}} m^{-\gamma} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x)dx\,dy \\
\leq &\ C[\psi]_\gamma m^{-\varepsilon}.
\end{aligned} \tag{A.4}$$

• If $m^{-2} \geq v$, then:

$$\begin{aligned}
& v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^\vartheta [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty(\mathbb{R}^d)} \\
= & \sup_{z\in\mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_v h_v(z-y) D^\vartheta h_{m^{-2}}(y-x)[\psi(x)-\psi(y)]dx\,dy\Big| \\
\leq & C[\psi]_\gamma m^{\frac{\theta}{2}} v^{\frac{\theta-\gamma+\varepsilon}{2}} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{c^{-1}m^{-1}}(y-x)|x-y|^\gamma dx\,dy \\
\leq & C[\psi]_\gamma m^{\theta-\gamma} v^{\frac{\theta-\gamma+\varepsilon}{2}} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x)dx\,dy \\
\leq & C[\psi]_\gamma m^{-\varepsilon}. \qquad (A.5)
\end{aligned}$$

The result follows from (A.4) and (A.5). □

Proposition 3 and Corollary 5 are more precise forms of the well known convergence in the distributional sense.

**Proposition 4.** *For any $\psi \in \dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$, $\gamma \in (0,1]$ we have for any $\vartheta \in \mathbb{N}_0^d$ that $h_{m^{-2}} \star D^\vartheta\psi \in C_b^\infty(\mathbb{R}^d,\mathbb{R})$ converges towards $D^\vartheta\psi$ in distributional sense as $m \to +\infty$. More precisely, we have for any $\eta \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$:*

$$\sup_{x\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \eta(x-y)\big[h_{m^{-2}} \star D^\vartheta\psi(y) - D^\vartheta\psi(y)\big]dy\Big| \leq C[\psi]_\gamma m^{-\gamma}. \qquad (A.6)$$

**Remark 15.** *We precise that $\eta$ is not supposed to be a Gaussian kernel, as in Proposition 3.*

*Proof.* We directly write by convolution property:

$$\begin{aligned}
& \sup_{x\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \eta(x-y)\big[h_{m^{-2}} \star D^\vartheta\psi(y) - D^\vartheta\psi(y)\big]dy\Big| \\
= & \sup_{x\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D^\vartheta\eta(x-y)\big[h_{m^{-2}} \star \psi(y) - \psi(y)\big]dy\Big| \\
\leq & Cm^{-\gamma}[\psi]_\gamma \sup_{x\in\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \big|D^\vartheta\eta(x-y)\big|dy\Big| \\
\leq & Cm^{-\gamma}[\psi]_\gamma,
\end{aligned}$$

the penultimate inequality is consequence of inequality (A.2). □

# B Properties of derivatives of Besov distributions

**Proposition 5.** *For any $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\nabla\varphi \in \dot{B}^{-1+\gamma}_{\infty,\infty}(\mathbb{R}^d)$, $\gamma \in (0,1)$, there is a constant $C > 1$ such that:*

$$\|\nabla\varphi\|_{\dot{B}^{-1+\gamma}_{\infty,\infty}} \leq C\|\varphi\|_{\dot{B}^\gamma_{\infty,\infty}}.$$

*Proof of Proposition 5.* We first write by the thermic representation of the Besov norm and by integration by parts,

$$\begin{aligned}
\|\nabla\varphi\|_{\dot{B}^{-1+\gamma}_{\infty,\infty}} &= \sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} v^{1-\frac{-1+\gamma}{2}} \Big| \int_{\mathbb{R}^d} \partial_v h_v(z-y)\nabla\varphi(y)dy\Big| \\
&= \sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} v^{\frac{3-\gamma}{2}} \Big| \int_{\mathbb{R}^d} \nabla\cdot\partial_v h_v(z-y)[\varphi(y)-\varphi(z)]dy\Big|,
\end{aligned}$$

by absorbing property (2.13) we derive

$$\begin{aligned}\|\nabla\varphi\|_{\dot{B}^{-1+\gamma}_{\infty,\infty}} &\leq C[\varphi]_\gamma \sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} v^{\frac{3-\gamma}{2}} \int_{\mathbb{R}^d} v^{-\frac{3}{2}} h_{C^{-1}v}(z-y)|y-z|^\gamma dy \\ &\leq C[\varphi]_\gamma \sup_{v\in\mathbb{R}_+,\ z\in\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y)dy \\ &\leq C[\varphi]_\gamma.\end{aligned}$$

□

We also derive the corresponding inequality for the inhomogeneous case.

**Corollary 6.** *For any $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\nabla\varphi \in B^{-1+\gamma}_{\infty,\infty}(\mathbb{R}^d)$, $\gamma \in (0,1)$, there is a constant $c > 1$ such that:*

$$\|\nabla\varphi\|_{B^{-1+\gamma}_{\infty,\infty}} \leq c\|\varphi\|_{B^\gamma_{\infty,\infty}}.$$

*Proof of Corollary 6.* From inequality (2.11), we have $\|\nabla\varphi\|_{B^{-1+\gamma}_{\infty,\infty}} \leq \frac{C}{1-\gamma}\|\nabla\varphi\|_{\dot{B}^{-1+\gamma}_{\infty,\infty}}$. Moreover, it is direct that

$$\begin{aligned}\|\varphi\|_{\dot{B}^\gamma_{\infty,\infty}} = \sup_{v\in\mathbb{R}_+} v^{(1-\frac{\gamma}{2})}\|\partial_v h_v \star \varphi\|_{L^\infty} &\leq \|\varphi\|_{\ddot{B}^\gamma_{\infty,\infty}} + \sup_{v>1} v^{(1-\frac{\alpha}{2})}\|\partial_v h_v \star \varphi\|_{L^\infty} \\ &\leq \|\varphi\|_{\ddot{B}^\gamma_{\infty,\infty}} + \|\varphi\|_{L^\infty} \\ &= \|\varphi\|_{B^\gamma_{\infty,\infty}}.\end{aligned}$$

In other words, we deduce by Proposition 5,

$$\|\nabla\varphi\|_{B^{-1+\gamma}_{\infty,\infty}} \leq \frac{C}{1-\gamma}\|\nabla\varphi\|_{\dot{B}^{-1+\gamma}_{\infty,\infty}} \leq \|\varphi\|_{\dot{B}^\gamma_{\infty,\infty}} \leq C\|\varphi\|_{B^\gamma_{\infty,\infty}}.$$

□

**Remark 16.** *In all generality, the reverse inequality of the above results are not true. For example, any constant function lies in $B^\gamma_{\infty,\infty}$ but its derivative is 0, hence the $B^{-1+\gamma}_{\infty,\infty}$ norm is null, and the corresponding Besov norm equivalence obviously fails to be true.*

*For an equivalence version of this result, we need to consider extra Besov norms, see for instance [KMM07] identity (3.54) for a Triebel-Lizorkin spaces version.*

# C On the freedom of the mollification choice

In this section, we detail why if there is a sequence of smooth function converging toward a $B^{\tilde{\gamma}}_{\infty,\infty}$ distribution then the mollification procedure (4.2) converges also toward the distribution. In other words, if there is a sequence $(\bar{b}_n)_{n\geq 1}$ lying in $L^\infty([0,T];C^\infty_b(\mathbb{R}^d,\mathbb{R}^d))$ such that

$$\lim_{n\to\infty} \|\bar{b}_n - b\|_{L^\infty(B^{-\tilde{\gamma}+\varepsilon}_{\infty,\infty})} = 0, \tag{C.1}$$

for any $0 < \varepsilon$, then

$$\lim_{m\to\infty} \|b_m - b\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} = 0, \tag{C.2}$$

where $b_m$ is defined in (4.2) by:

$$b_m(t,x) = b(t,\cdot) \star \rho_m(x),$$

with for any $z \in \mathbb{R}^d$, $\rho_m(z) := m^d\rho(zm)$ for $\rho(z) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{2}}$. Indeed, we readily write by triangular inequality:

$$\|b_m - b\|_{L^\infty(B^{-\tilde{\gamma}+\varepsilon}_{\infty,\infty})} \leq \|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} + \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} + \|\bar{b}_n - b\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})}. \tag{C.3}$$

The firs term in the r.h.s. above write:

$$\|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} = \|\rho_m \star (b - \bar{b}_n)\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})}.$$

Hence, we obtain

$$\|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} = \sup_{v\in[0,1]} v^{\frac{\tilde{\gamma}-\varepsilon}{2}} \|\rho_m \star h_v \star (b - \bar{b}_n)\|_{L^\infty} \leq \|\bar{b}_n - b\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})}, \tag{C.4}$$

by triangular inequality.

Also, for the second term in (C.3), let us deal with the corresponding homogeneous norm,

$$\|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} = \|\varphi(D)(\rho_m \star \bar{b}_n - \bar{b}_n)\|_{L^\infty} + \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(\ddot{B}^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})}.$$

It is direct that

$$\|\varphi(D)(\rho_m \star \bar{b}_n - \bar{b}_n)\|_{L^\infty} \leq C\|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty} \leq Cm^{-1}\|D\bar{b}_n\|_{L^\infty}. \tag{C.5}$$

Next,

$$\begin{aligned}
&\|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(\ddot{B}^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} \\
= &\sup_{v\in[0,1],\ t\in[0,T],\ z\in\mathbb{R}^d} v^{\frac{\tilde{\gamma}+\varepsilon}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_v(z-y)\rho_m(y-x)[\bar{b}_n(t,x) - \bar{b}_n(t,y)]dx\ dy\Big| \\
\leq &\|D\bar{b}_n\|_{L^\infty} \sup_{v\in[0,1],\ t\in[0,T],\ z\in\mathbb{R}^d} v^{\frac{\tilde{\gamma}+\varepsilon}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{C^{-1}v}(z-y)\rho_m(y-x)|x-y|dx\ dy\Big| \\
\leq &C\|D\bar{b}_n\|_{L^\infty} m^{-1} \sup_{v\in[0,1],\ t\in[0,T],\ z\in\mathbb{R}^d} v^{\frac{\tilde{\gamma}+\varepsilon}{2}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_{C^{-1}v}(z-y)\rho_{C^{-1}m}(y-x)dx\ dy\Big| \\
= &C\|D\bar{b}_n\|_{L^\infty} m^{-1}.
\end{aligned} \tag{C.6}$$

Let us choose $m \gg \|D\bar{b}_n\|_{L^\infty}$ which yields that $\lim_{m,n\to\infty} \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B^{-\tilde{\gamma}-\varepsilon}_{\infty,\infty})} = 0$.

Finally, gathering identities (C.1), (C.3), (C.4), (C.5) and (C.6) yields the limit property (C.2).

# D Comments on the strategy of the *cut locus*

## D.1 Comments on the necessity of using the norm $\|u^{m,\nu}\|_{L^\infty(C^1)}$

Let us rewrite one of the term of $R_A^{\tau,\xi,\xi'}(t,x,x')$ in r.h.s. in (4.56),

$$\begin{aligned}
\mathbf{R} \ := \ &\Big| \int_0^t \int_{\mathbb{R}^d} \nabla\hat{p}^{\tau,\xi}(s,t,0,y) \\
&\cdot\big(b_m(s,x'+y) - b_m(s,x+y)\big)\big(u^{m,\nu}(s,x'+y) - u^{m,\nu}(s,\theta^m_{s,\tau}(\xi))\big)dy\,ds\Big|\Big|_{\tau=t,\xi=\xi'=x} \\
\leq \ &C|x-x'|\|b_m\|_{L^\infty(C^1)} \int_0^t \|u^{m,\nu}(s,\cdot)\|_{C^\gamma} \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s,t,0,y) \times |\theta^m_{s,\tau}(\xi) - x' - y|^\gamma dy\,ds.
\end{aligned}$$

Next, by exponential absorption,

$$\begin{aligned}
\mathbf{R} \ \leq \ &C|x-x'|\|b_m\|_{L^\infty(C^1)} \int_0^t \|u^{m,\nu}(s,\cdot)\|_{C^\gamma} \Big( [\nu(t-s)]^{\frac{\gamma-1}{2}} + [\nu(t-s)]^{-\frac{1}{2}}|x-x'|^\gamma\Big) ds \\
\leq \ &C|x-x'|^\gamma \nu^{\alpha_1(1-\gamma)}\|b\|_{L^\infty(B^{\tilde{\gamma}}_{\infty,\infty})} m^{1-\tilde{\gamma}} \int_0^t \|u^{m,\nu}(s,\cdot)\|_{C^\gamma} \\
&\times (t-s)^{(1-\gamma)\alpha_2} \Big( [\nu(t-s)]^{\frac{\gamma-1}{2}} + [\nu(t-s)]^{-\frac{1}{2}} \nu^{\alpha_1\gamma}(t-s)^{\alpha_2\gamma}\Big) ds.
\end{aligned}$$

Reordering the contributions yields

$$\begin{aligned}\mathbf{R} &\leq C|x-x'|^{\gamma}\|b\|_{L^{\infty}(B^{\tilde{\gamma}}_{\infty,\infty})}m^{1-\tilde{\gamma}}\\ &\quad\times\int_0^t\|u^{m,\nu}(s,\cdot)\|_{C^{\gamma}}\Big((t-s)^{(1-\gamma)\alpha_2+\frac{\gamma-1}{2}}\nu^{\alpha_1(1-\gamma)+\frac{\gamma-1}{2}}+\nu^{\alpha_1-\frac{1}{2}}(t-s)^{\alpha_2-\frac{1}{2}}\Big)ds.\end{aligned}$$

Then the required, assumption on parameters is for this control (from the second additive term above)

$$\alpha_1>\frac{1}{2},\ \alpha_2>-\frac{1}{2},$$

which is incompatible with the the *off-diagonal* regime.

We could consider the case $\alpha_1=\frac{1}{2}$, but this case yields no viscosity contribution and makes the previous upper-bounds blowing up in $m$ (except for the usual framework, i.e. for $b$ Lipschitz continuous); there is no possibility to obtain a *regularisation by turbulence* for such a choice.

If we suppose that $b$ is $\gamma$-Hölder in space, in order to avoid any blowing-up in $m$, we are able to write

$$\begin{aligned}|\mathbf{R}| &\leq C|x-x'|^{\gamma}\|b\|_{L^{\infty}(C^{\gamma})}\int_0^t\|u^{m,\nu}(s,\cdot)\|_{C^{\gamma}}\\ &\quad\int_{\mathbb{R}^d}\mathbb{1}_{A(x,x',\nu,t)}(s)[\nu(t-s)]^{-\frac{1}{2}}\bar{p}^{\tau,\xi}(s,t,0,y)\times|\theta^m_{s,\tau}(x)-x'-y|^{\gamma}dy\,ds\\ &\leq C|x-x'|^{\gamma}\|b\|_{L^{\infty}(C^{\gamma})}\int_0^t\mathbb{1}_{A(x,x',\nu,t)}(s)\|u^{m,\nu}(s,\cdot)\|_{C^{\gamma}}\Big([\nu(t-s)]^{\frac{\gamma-1}{2}}+[\nu(t-s)]^{-\frac{1}{2}}|x-x'|^{\gamma}\Big)ds.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathbf{R}| &\leq C|x-x'|^{\gamma}\|b\|_{L^{\infty}(C^{\gamma})}\int_0^t\mathbb{1}_{A(x,x',\nu,t)}(s)\|u^{m,\nu}(s,\cdot)\|_{C^{\gamma}}\\ &\quad\Big([\nu(t-s)]^{\frac{\gamma-1}{2}}+\nu^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\nu^{\alpha_1\gamma}(t-s)^{\alpha_2\gamma}\Big)ds,\end{aligned}$$

which goes to $+\infty$ when $\nu\to0$, except if $\gamma=1$. In other words, we need to consider the norms $\|b_m\|_{L^{\infty}(C^1)}$ and $\|u^{m,\nu}\|_{L^{\infty}(C^1)}$ on the one hand to smoothen the blowing-up in $\nu$ and to get a suitable control by $|x-x'|$ which allows to overwhelm $\nu$ in the *diagonal* regime.

## D.2 Comments on the choice of *freezing* point for the source functions terms

It is crucial to fix the same *freezing* point for the terms associated with source functions. In our context, it may be unavoidable. To fully explain this choice, let us develop the computations associated with these terms for the same choice of $\xi$ and $\xi'$ as for $A$, the *off-diagonal* regime, i.e. $\xi=x$ and $\xi'=x'$.

To deal with the semi-group, we consider an analysis of the type (or equivalent controls),

$$\begin{aligned}&|\hat{P}^{\tau,\xi}g_m(t,x)-\hat{P}^{\tau,\xi'}g_m(t,x')|\Big|_{\tau=t,\xi=x,\xi'=x'}\\ =\ &\Big|\int_{\mathbb{R}^d}[\hat{p}^{\tau,\xi}(0,t,x,y)-\hat{p}^{\tau,\xi'}(0,t,x',y)]g_m(s,y)dy\Big|_{\tau=t,\xi=x,\xi'=x'}\\ =\ &\Big|\int_{\mathbb{R}^d}\tilde{p}(0,t,0,y)\Big[g_m\big(t,\theta^m_{0,t}(x)+y\big)-g_m\big(t,\theta^m_{0,t}(x')+y\big)\Big]dy\\ \leq\ &[g]_{\gamma}|\theta^m_{0,t}(x)-\theta^m_{0,t}(x')|^{\gamma}.\end{aligned}$$

Similarly, for the Green operator,

$$\begin{aligned}
& |\hat{G}^{\tau,\xi} f_m(t,x) - \hat{G}^{\tau,\xi'} f_m(t,x')| \Big|_{\tau=t,\xi=\xi'=x} \\
= & \Big| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{\tau,\xi}(s,t,x,y) - \hat{p}^{\tau,\xi'}(s,t,x',y)] f_m(s,y) dy\, ds \Big| \Big|_{\tau=t,\xi=x,\xi'=x'} \\
= & \Big| \int_0^t \int_{\mathbb{R}^d} [\tilde{p}(s,t,0,y)[f_m(s,\theta^m_{s,t}(x)+y) - f_m(s,\theta^m_{s,t}(x')+y)] dy\, ds \Big| \\
\leq & \|f\|_{L^\infty(C^\gamma)} \int_0^t |\theta^m_{s,t}(x) - \theta^m_{s,t}(x')|^\gamma ds.
\end{aligned}$$

In other words, we see in the both controls above that we only upper-bound by the flow associated with $b_m$ which is *a priori* not controlled uniformly on $m$ in a suitable spatial Hölder space, see (4.85).

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