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Regularisation by turbulence for transport equations in Hölder space

Igor Honoré*

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Abstract

We consider a transport equation whose the coefficient b can be in a negative Besov space, we provide controls in γ -Hölder spaces, $\gamma \in (0, 1)$, of some kind of *vanishing viscosity* solutions based on a parabolic approximation. The regularity in γ -Hölder space of the solution exactly matches the one of the source functions, but the regularisation occurs on the coefficients irregularity. Indeed, we define some kind of solutions of the transport equation which do not require b to be Lipschitz continuous.

If b lies in a α -Hölder space, $\alpha > 1 - \gamma$, then we establish that there is a weak solution in a γ -Hölder space.

If b is supposed to be divergence free, then we obtain the same result for b having a negative regularity in space, precisely in $L^\infty(B_{\infty, \infty}^{-\beta})$ for $\beta < \gamma$.

Finally, if we consider a “very weak” solution, called in the paper *mild vanishing viscous*, also in a γ -Hölder space, then there is no regularity constraint on b . In this case, there is somehow a *mild infinite regularisation* by turbulence.

The *vanishing viscosity* allows to overwhelm the potential blowing up of the rough coefficients, this is what we call a *regularisation by turbulence*. However, we do not get uniqueness of the considered built solution; we even conjecture that there is no unique selection of the parabolic approximation in such a rough framework.

Importantly, as a by-product of our analysis, we can give a meaning of a product of distributions. For b lying in a γ -Hölder space, we obtain the same condition as for the usual Bony’s paraproduct; but in a weaker solution framework, the product is defined beyond the paraproduct condition and even with no constraint at all in the *mild vanishing viscous* context. We also obtain that the time averaging of the distributions product is γ -Hölder continuous. These strong results happens because one of the distribution is the gradient of a solution, in a certain sense, of the transport equation.

Thanks to our analysis, we also get a Hölder control of a solution of the inviscid Burgers’ equation. The vanishing viscous procedure seems to avoid the well-known time of the regularity blowing-up of the solution built by characteristics.

Keywords: Regularisation by turbulence, Transport equations, Besov spaces, Product of distributions, Paraproduct, Inviscid Burgers’ equation.

1 Introduction

1.1 Statement of the problem

For given $d \in \mathbb{N}$, we consider the following d -dimensional Cauchy problem:

$$\begin{cases} \partial_t u(t, x) + \langle b(t, x), \nabla u(t, x) \rangle = f(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0, x) = g(x), & x \in \mathbb{R}^d. \end{cases} \quad (1.1)$$

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For a finite $\beta \in \mathbb{R}$, we suppose that the transport coefficient, $b(t, \cdot)$, for any $t \in \mathbb{R}_+$, lies in the non-homogeneous Besov Hölder space $B_{\infty, \infty}^{-\beta}$; when $\beta > 0$, this can be regarded as a Hölder space with negative regularity.

We establish in this article that there is a solution u which is Hölder continuous and potentially not unique. To give a meaning of the product $\langle b(t, x), \nabla u(t, x) \rangle$, we introduce a kind of vanishing viscosity solution. We indeed consider usual second order parabolic equation whose the second order term, called viscosity ν , goes to 0,

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + \langle b_m(t, x), \nabla u^{m, \nu}(t, x) \rangle - \nu \Delta u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

the function b_m is a mollified version of the distributional values b . The uniform control in L^∞ is direct by a probabilistic representation of the solution, see [Hon22]. Next, we write the solution as a perturbation of a PDE with constant components. These constants correspond to the first order term b_m taking at a *freezing point* throw the corresponding flow, as done in [CDRHM18]. However, to estimate the Hölder norm, we have to distinguish two regimes, as usual in a parabolic context, the *diagonal* and the *off-diagonal* ones. In each regime, the choice of *freezing points* changes in order to get a negligible first order contribution when $\nu \rightarrow 0$. This phenomenon can be regarded as a *regularisation by turbulence*. As for the fluid mechanics, we can define an associated Reynolds number which goes to $+\infty$ when $\nu \rightarrow 0$, corresponding to a turbulence regime, see for instance [FMRT01].

This approach allows to obtain a non-unique smooth selection principle which does not work in a usual regularisation limit, see [CCS20]. We postulate that, in our framework, uniqueness fails in general as soon as b, f and g are not Lipschitz continuous.

One of the crucial consequences of our analysis is that, we obtain a first general meaning of a classical product of distributions. Indeed, we succeed in giving a meaning of $\langle b(t, x), \nabla u(t, x) \rangle$ where b and ∇u have negative regularity. This is written as a weak limit of a sub-sequence of a smooth parabolic approximation. The price to pay in this representation is that we do not have uniqueness of the limit.

Finally, thanks to this method, we also deduce existence (without uniqueness) of a Hölder continuous solution of the inviscid Burgers' equation.

The paper is organised as following. In Section 1.2, we recall some well-known results on the transport equation. The notations and the assumptions used are gathered in Section 2. Our main result is stated in Section 3 with some comments. The complete analysis is detailed in Section 4. We provide the statement and the proof of the regularity of a solution to the inviscid Burgers' equation in Section 5. Finally, we gather an Appendix some well-known results about regularity controls on the solution of linear parabolic of second order in Section A, and about some property of the Besov spaces in Sections B-D. Precisely, in Section B, we establish that the space C_b^∞ functions are dense in the space of multi-differentiated Hölder continuous functions. Some inequalities over the norm of Besov-Hölder distributions with their derivative are established in Section C. Eventually, in Section D, we detail why the limit of regularised distribution in Besov-Hölder space does not depend on the choice of mollification procedure.

1.2 Existing results on the transport equation

The Lipschitz framework is classical *via* the characteristic method. Indeed if $b \in L^\infty([0, T]; C^1(\mathbb{R}^d, \mathbb{R}^d))$, considering the ODE $\dot{X}_t = b(t, X_t)$, thanks to the Cauchy-Lipschitz theorem we know that there is a unique solution $u \in L^\infty([0, T]; C^1(\mathbb{R}^d, \mathbb{R}))$ of the transport equation (1.1). Out of this regular context, the analysis has to be more involved.

For instance, a meaning of the equation (1.1), when the coefficients are in a suitable Sobolev space, can be given by a renormalisation procedure developed by DiPerna and Lions [DPL89]. If $b \in L^1([0, T]; W_{loc}^{1,1}(\mathbb{R}^d, \mathbb{R}))$ and $div(b) \in L^1([0, T]; L^\infty(\mathbb{R}^d, \mathbb{R}))$, they establish that the Cauchy problem

(1.1) is well-posed in L^∞ . When, b is only supposed to have bounded variations in space, Ambrosio [Amb04] extends this result for $b \in L^1([0, T]; BV_{loc}(\mathbb{R}^d, \mathbb{R}))$, $div(b)_- \in L^1([0, T]; L^\infty(\mathbb{R}^d, \mathbb{R}))$.

However, if b is only Hölder continuous then the Cauchy problem (1.1) is not well-posed any longer, see the well-known counter-example specified in Section 2.4. With a multiplicative noise, Flandoli Gubinelli and Priola [FGP10], see also [FGP12] and [MO17], establish that the following Stochastic Partial Differential Equation

$$\begin{cases} d_t u + \langle b, \nabla u \rangle dt + \nabla u \circ dW_t = 0, \\ u(0, \cdot) = u_0(\cdot), \end{cases} \quad (1.3)$$

with $b \in L^\infty([0, T], C_b^\alpha)$ and $div(b) \in L^p$ is well-posed. Here, the symbol \circ corresponds to the stochastic Stratonovich integral. This is a typical consequence of the regularisation by the noise, see also [FF13], [AF11], [Cat16].

For other references on transport equation in the non Lipschitz case, see for instance to [MS18], [Xia19].

We propose in this article a new approach to handle the determinist transport equation (1.1). We consider *vanishing viscous solution*, see e.g. [Eva98], which is different to the *viscous solution* introduced by Crandall and Lions [CL83] for the Hamilton-Jacobi equation. We do not consider upper or lower solution, but a smooth well-posed parabolic equation and we take the limit, up to sub-sequence selection, of the mollification parameter and of the viscosity.

For the best author's knowledge, the notion of vanishing viscosity has been already used for several classes of evolution PDEs, e.g. hyperbolic ones in [BB05], but it was not developed to establish the regularity control of the solution of a general transport equation.

2 Notations and Definitions

From now on, we denote by $C > 0$ and $c > 1$ generic constants that may change from line to line but only depends on known parameters such as γ, d . Importantly, this constant does not depend on β .

We also write, for $\varepsilon, \tilde{\varepsilon} > 0$, the usual notation of asymptotic domination:

$$\varepsilon \ll \tilde{\varepsilon}, \text{ if } \frac{\varepsilon}{\tilde{\varepsilon}} \rightarrow 0. \quad (2.1)$$

We also write $\xrightarrow[(\varepsilon, \tilde{\varepsilon}) \rightarrow (0, 0)]{(2.1)} 0$, the limit, up to some subsequence selection, under the condition (2.1).

2.1 Tensor and Differential notations

For any $z \in \mathbb{R}^d$, we use the decomposition $z = z_1 e_1 + \dots + z_d e_d$, where (e_1, \dots, e_d) is the canonical base of \mathbb{R}^d .

We usually use the notation ∂_t for the derivative in time $t \in [0, T]$ also ∂_{z_k} , $k \in \mathbb{N}$, is the derivative in the variable z_k .

The gradient in space is denoted by ∇ or by D , in other words $\nabla = \partial_{z_1} e_1 + \dots + \partial_{z_d} e_d$.

The divergence write $\nabla \cdot = div$ and is defined for any \mathbb{R} -function $f : \mathbb{R}^d \mapsto \mathbb{R}$ by $\nabla \cdot f = \sum_{k=1}^d \partial_{z_k} f$.

From now on, the symbol “ \cdot ” between two tensors is the usual tensor contraction. For example, if $M \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ and $N \in \mathbb{R}^d$ then $M \cdot N$ is a $d \times d$ matrix. If the two considered tensors are vectors then “ \cdot ” matches the scalar product which is also denoted by $\langle \cdot, \cdot \rangle$.

For any $\mathbb{R}^d \mapsto \mathbb{R}$, we define the Hessian matrix $D_z^2 f = (\partial_{z_i} \partial_{z_j} f)_{1 \leq i, j \leq d}$, and the usual Laplacian operator $\Delta f = \sum_{1 \leq i, j \leq d} \partial_{z_i} \partial_{z_j} f$.

More generally, for any $k \in \mathbb{N}$, $D_z^k f$ denotes the order k tensor $(\partial_{z_{i_1}} \dots \partial_{z_{i_k}} f)_{(i_1, \dots, i_k) \in \llbracket 1, d \rrbracket^k}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, we write $D_z^\alpha f = \partial_{z_1}^{\alpha_1} \dots \partial_{z_d}^{\alpha_d} f$, in particular if for $i \in \llbracket 1, d \rrbracket$, $\alpha_i = 0$ there is no derivative in z_i in the expression of $D_z^\alpha f$.

We also denote for any $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, the order of this multi-index by $|\alpha| = \sum_{i=1}^m \alpha_i$.

2.2 Associated Hölder, Besov spaces

In this section, we provide some useful notations and functional spaces.

2.2.1 Hölder spaces

For all $k \in \mathbb{N}$ and $\beta \in (0, 1)$, $\|\cdot\|_{C^{k+\beta}(\mathbb{R}^m, \mathbb{R}^\ell)}$, $m \in \{1, d\}$, $\ell \in \{1, d, d \otimes d\}^*$ is the usual homogeneous Hölder norm, see e.g. Lunardi [Lun95] or Krylov [Kry96]. Precisely, for all $\psi \in C^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell)$, $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, we set the semi-norm:

$$\begin{aligned} \|\psi\|_{C^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell)} &:= \sum_{i=1}^k \sup_{|\alpha|=i} \|D^\alpha \psi\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^\ell)} + \sup_{|\alpha|=k} [D^\alpha \psi]_\delta, \\ [D^\alpha \psi]_\delta &:= \sup_{(x,y) \in (\mathbb{R}^m)^2, x \neq y} \frac{|D^\alpha \psi(x) - D^\alpha \psi(y)|}{|x - y|^\delta}, \end{aligned} \quad (2.2)$$

the notation $|\cdot|$ is the Euclidean norm on the considered space. We denote by:

$$C_b^{k+\beta}(\mathbb{R}^m, \mathbb{R}^\ell) := \{\psi \in C^{k+\beta}(\mathbb{R}^m, \mathbb{R}^\ell) : \|\psi\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^\ell)} < +\infty\},$$

the associated subspace with bounded elements (non-homogeneous Hölder space). The corresponding Hölder norm is defined by:

$$\|\psi\|_{C_b^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell)} := \|\psi\|_{C^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell)} + \|\psi\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^\ell)}. \quad (2.3)$$

For the sake of notational simplicity, from now on we write:

$$\|\psi\|_{L^\infty} := \|\psi\|_{L^\infty(\mathbb{R}^d, \mathbb{R}^\ell)}, \quad \|\psi\|_{C^{k+\delta}} := \|\psi\|_{C^{k+\delta}(\mathbb{R}^d, \mathbb{R}^\ell)}, \quad \|\psi\|_{C_b^\delta} := \|\psi\|_{C_b^\delta(\mathbb{R}^d, \mathbb{R}^\ell)}.$$

For time dependent functions, $\varphi_1 \in L^\infty([0, T], C_b^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell))$ and $\varphi_2 \in L^\infty([0, T], C^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell))$, we define the norms:

$$\|\varphi_1\|_{L^\infty(C_b^{k+\delta})} := \sup_{t \in [0, T]} \|\varphi_1(t, \cdot)\|_{C_b^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell)}, \quad \|\varphi_2\|_{L^\infty(C^{k+\delta})} := \sup_{t \in [0, T]} \|\varphi_2(t, \cdot)\|_{C^{k+\delta}(\mathbb{R}^m, \mathbb{R}^\ell)}.$$

The test functions for some weak formulation of different solutions will be in $C_0^\infty(\mathbb{R}^d, \mathbb{R})$ which corresponds to smooth functions infinitely differentiable with bounded derivatives and with a compact support.

2.2.2 Thermic characterization of the Besov space

We define the Besov spaces thanks to a thermic characteristic, see Triebel [Tri83] Section 2.6.4. For all $\alpha \in \mathbb{R}$, $q \in (0, +\infty]$, $p \in (0, \infty]$,

$$\|f\|_{B_{p,q}^\alpha} := \|\varphi(D)f\|_{L^p(\mathbb{R}^d)} + \|f\|_{\dot{B}_{p,q}^\alpha}, \quad \text{with } \|f\|_{\dot{B}_{p,q}^\alpha} := \left(\int_0^1 \frac{dv}{v} v^{(m-\frac{\alpha}{2})q} \|\partial_v^m h_v \star f\|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}, \quad (2.4)$$

where we define the heat kernel

$$h_v(z) := \frac{1}{(2\pi v)^{d/2}} \exp\left(-\frac{|z|^2}{2v}\right), \quad (2.5)$$

and $\varphi(D)f := (\varphi \hat{f})^\vee$ with $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(0) \neq 0$, \hat{f} and $(\varphi \hat{f})^\vee$ respectively denote the Fourier transform of f and the inverse Fourier transform of $\varphi \hat{f}$. Note that, when $\alpha > d(\frac{1}{p} - 1)_+$ then in (2.4), we can replace $\|\varphi(D)f\|_{L^p(\mathbb{R}^d)}$ by $\|f\|_{L^p(\mathbb{R}^d)}$.

*we write $\mathbb{R}^{d \otimes d}$ for $\mathbb{R}^d \otimes \mathbb{R}^d$ the space of square matrices of size d .

When $p = q = +\infty$, we naturally write:

$$\|f\|_{\dot{B}_{p,q}^\alpha} = \sup_{v \in [0,1]} v^{m-\frac{\alpha}{2}} \|\partial_v^m h_v \star f\|_{L^\infty(\mathbb{R}^d)},$$

and if $\alpha > d(\frac{1}{p} - 1)_+$,

$$\|f\|_{B_{p,q}^\alpha} = \|f\|_{L^\infty} + \sup_{v \in [0,1]} v^{m-\frac{\alpha}{2}} \|\partial_v^m h_v \star f\|_{L^\infty(\mathbb{R}^d)}.$$

We carefully point out that the homogeneous term $\|f\|_{\dot{B}_{p,q}^\alpha}$ does not define a norm associated to a Banach space. To consider the whole homogeneous Besov space we need to consider $v \in \mathbb{R}_+$ in the definition (2.4),

$$\|f\|_{\dot{B}_{p,q}^\alpha} := \left(\int_0^{+\infty} \frac{dv}{v} v^{(m-\frac{\alpha}{2})q} \|\partial_v^m h_v \star f\|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}, \quad (2.6)$$

where we do not consider the first term in (2.4), corresponding of the inhomogeneous part, of (2.4), and the parameter v lies in \mathbb{R}_+ for the homogeneous norm. Somehow, for the inhomogeneous norm defined in (2.4), the contribution of the heat kernel convolution for $v > 1$ is “hidden” in the inhomogeneous term $\|\varphi(D)f\|_{L^p(\mathbb{R}^d)}$.

For $\alpha > 0$, the homogeneous and respectively inhomogeneous Hölder spaces match with Besov space, namely $C^\alpha = \dot{B}_{\infty,\infty}^\alpha$ and $C_b^\alpha = B_{\infty,\infty}^\alpha$, see [Tri83] for details.

Our analysis tackles with inhomogeneous Besov spaces, in order to extend our analysis to the homogeneous ones some sophisticated changes should be performed as the homogeneous Besov spaces are *a priori* not Banach spaces; and we should consider the realisation of the space of homogeneous Besov spaces as a space of distributions defined quotiented by polynomials, see e.g. Proposition 3.8 in [LR02] to make it a Banach space.

If $\alpha < 0$ then it is known that $\dot{B}_{p,q}^\alpha \subset B_{p,q}^\alpha$, i.e. there is a constant $C > 0$ such that for any $\alpha < 0$:

$$\|\cdot\|_{B_{p,q}^\alpha} \leq -\frac{C}{\alpha} \|\cdot\|_{\dot{B}_{p,q}^\alpha}. \quad (2.7)$$

We also introduce the distributions that can be approached by a mollification procedure. We put a *tilda* in order to mean that we consider the closure of C_b^∞ in the considered space[†]. Namely, for any $(\alpha, p, q) \in \mathbb{R} \times (1, +\infty] \times (1, +\infty]$ we define:

$$\tilde{B}_{p,q}^\alpha := \mathbf{cl}_{B_{p,q}^\alpha}(C_b^\infty), \quad \tilde{\dot{B}}_{p,q}^\alpha := \mathbf{cl}_{\dot{B}_{p,q}^\alpha}(C_b^\infty). \quad (2.8)$$

Remark 1. Importantly, when $b \in \dot{B}_{\infty,\infty}^{-\beta}$ and such that $b = D^\alpha \psi$, $\alpha \in \mathbb{N}_0^d$ with $\psi \in C^\gamma$, $\gamma \in (0, 1)$, which yields that $\beta = -|\alpha| + \gamma$, in Appendix Section B, we show that $b \in \tilde{\dot{B}}_{\infty,\infty}^{-\beta}$.

The last constraint on b , being the derivative of a Hölder function, is quiet natural when we consider the structure theorem of the tempered distributions \mathcal{S}' , see Theorem 8.3.1 in [Fri98]. We recall indeed that any $b \in \mathcal{S}'$ writes $b = D^\alpha \psi$ where $\alpha \in \mathbb{N}_0^d$ and ψ is a continuous function with polynomial growth.

Out of the Besov-Hölder space, namely if $1 \leq p, q < +\infty$, then $\tilde{B}_{p,q}^\alpha = B_{p,q}^\alpha$ and $\tilde{\dot{B}}_{p,q}^\alpha = \dot{B}_{p,q}^\alpha$, see Theorem 4.1.3 in [AH96], Proposition 2.27 and Proposition 2.74 in [BCD11]. For more Besov properties, we also mention [Pee76] and [Jaw77].

We could consider the non-homogeneous low-frequency cut-off in the Littlewood-Paley characterisation instead of usual mollification by convolution, as performed in the current paper, and adapt Lemma 2.73 in [BCD11] for the space $B_{\infty,\infty}^{-\beta}$.

[†]As in Proposition 3.6 in [LR02] for the closure Schwartz space, but we do not need the mollified versions of the considered distributions to be rapidly decreasing functions.

2.2.3 Besov duality

In our analysis, we thoroughly use the Besov duality. The full proof of the duality of Besov spaces is established for example in Proposition 3.6 in [LR02] thanks to a Littlewood-Paley decomposition.

Proposition 1. *For all $1 \leq p, q \leq +\infty$ and $\alpha \in \mathbb{R}$, we have for all $\varphi, \psi \in \mathcal{S}'$:*

$$\left| \int_{\mathbb{R}^d} \varphi(y)\psi(y)dy \right| \leq C_{d,p,q,\alpha} \|\varphi\|_{B_{p,q}^\alpha} \|\psi\|_{B_{p',q'}^{-\alpha}},$$

with $1 \leq p', q' \leq +\infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Sketch of the proof. Let us suppose that w.l.o.g. that $1 < p, q < +\infty$ (the analysis is identical if we suppose that $1 < p', q' < +\infty$). It is known that $B_{p,q}^\alpha(\mathbb{R}^d, \mathbb{R})$ and $B_{p',q'}^{-\alpha}(\mathbb{R}^d, \mathbb{R})$ are in duality (Proposition 3.6 in [LR02]). Precisely, $B_{p,q}^\alpha$ is the dual of the closure of the Schwartz class \mathcal{S} in $B_{p',q'}^{-\alpha}$. But \mathcal{S} is dense in $B_{p,q}^\alpha$ (see for instance 4.1.3. in [AH96]). \square

The homogeneous counterpart of this result requires additional assumptions on the considered distributions, see for instance Proposition 2.29 in [BCD11].

2.2.4 Usual tools for the Gaussian function

Finally, one of the reason to use the thermic representation of the Besov space comes from a well-known and important result about the Gaussian function: for all $\delta > 0$, there is $C_\delta = C_\delta(\delta) > 1$ such that:

$$\forall x \in \mathbb{R}^d, |x|^\delta e^{-|x|^2} \leq C_\delta e^{-C_\delta^{-1}|x|^2}. \quad (2.9)$$

Furthermore, we will also often use the cancellation principle: for all $f \in C^\gamma$, $\gamma \in (0, 1)$, $x \in \mathbb{R}^d$ and $\sigma > 0$

$$D_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} f(y)dy = \int_{\mathbb{R}^d} D_x e^{-\frac{|x-y|^2}{2\sigma}} [f(y) - f(x)]dy, \quad (2.10)$$

as the Gaussian function, up to a renormalisation by a multiplicative constant, is a probabilistic distribution, hence $D_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} dy = 0$. Hence, we obtain,

$$\begin{aligned} (2\pi\sigma)^{\frac{d}{2}} \left| D_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} f(y)dy \right| &\leq (2\pi\sigma)^{\frac{d}{2}} [f]_\gamma \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} \frac{|y-x|}{\sigma} |y-x|^\gamma dy \\ &\leq C_\gamma (2\pi\sigma)^{\frac{d}{2}} [f]_\gamma \sigma^{\frac{\gamma-1}{2}} \int_{\mathbb{R}^d} e^{-C_\gamma^{-1} \frac{|x-y|^2}{2\sigma}} dy \\ &= C_\gamma [f]_\gamma \sigma^{\frac{\gamma-1}{2}}. \end{aligned}$$

The penultimate identity comes from the absorbing property (2.9).

2.3 Different definitions of solution to the Cauchy problem

As said in the introduction, we need to carefully defined the kind of solution, no strong solution can be define in negative Besov space or even in Hölder implying a product of distributions which is obviously no point-wisely defined.

Definition 1 (mild vanishing viscous). *A function u is said to be a mild vanishing viscous solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if for a sequence $(b_m)_{m \in \mathbb{R}_+}$ in $L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$ such that there is $\beta \in \mathbb{R}$,*

$$\forall \varepsilon > 0, \lim_{m \rightarrow +\infty} \|b_m - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} = 0, \quad (2.11)$$

for any $t \in [0, T]$, there exists a sub-sequence of $(u^{m,\nu}(t, \cdot))_{(m,\nu) \in \mathbb{R}_+^2}$ lying in $C_b^\gamma(\mathbb{R}^d, \mathbb{R})$ converging in the space $C_b^{\gamma-\tilde{\varepsilon}}(K, \mathbb{R})$, $0 < \tilde{\varepsilon} < \gamma$ for any compact subset $K \subset \mathbb{R}^d$, when $\nu \rightarrow 0$ and $m \rightarrow +\infty$ towards $u(t, \cdot) \in C_b^\gamma(K, \mathbb{R})$ and such that for any $m \in \mathbb{R}_+$ and any $\nu \in \mathbb{R}_+$,

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + \langle b_m(t, x), \nabla u^{m,\nu}(t, x) \rangle - \nu \Delta u^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m,\nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (2.12)$$

where $(f_m, g_m) \xrightarrow{m \rightarrow +\infty} (f, g)$ in $L^\infty([0, T]; C^\gamma(\mathbb{R}^d, \mathbb{R}))$. We point out that such a sequence $(b_m)_{m \geq 0}$ exists if $b \in L^\infty([0, T]; \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}))$; or in particular if b is the derivative of a bounded Hölder continuous function but in this former case the limit result (2.11) has to be in the homogeneous space $L^\infty([0, T]; \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}))$, see Appendix Section B, and implies (2.11) by (2.7).

Moreover, it is important to notice that the choice of sub-sequence may depend on the current time t . We do not succeed in getting any regularity in time t (except boundedness in L^∞) which yields uniform continuity in time, hence to apply a suitable compact argument, we need to consider the problem at a fixed time. Nevertheless, for the sake of simplicity we write for the sub-sequence $u^{m,\nu}(t, \cdot)$ instead of a notation of the kind $u^{m_t, \nu}(t, \cdot)$.

Remark 2. We could consider another formulation of mild vanishing viscous solution, where the considered function is

$$w^{m,\nu}(t, x) = \int_0^t u^{m,\nu}(s, x) ds. \quad (2.13)$$

From estimates stated in Theorem 2 below, we have that $(t, x) \mapsto w^{m,\nu}(t, x)$ lies uniformly in (m, ν) in $C_b^1([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$, by the Arzelà-Ascoli theorem, we obtain a convergence in all compacts $[0, T] \times K$ of $[0, T] \times \mathbb{R}^d$ towards a function $w \in C_b^1([0, T]; C_b^\gamma(K, \mathbb{R}))$.

To obtain such a kind of solutions, some constraints on ν and m in terms of convergence are required. Indeed, in our analysis, some blow-up contributions in m has to be overwhelmed by a vanishing viscosity contribution ν .

Let us define an alternative form of solution which is a mixed version between *mild* and *weak* solution.

Definition 2 (mild-weak solution). A function u is a mild-weak solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if u is a mild vanishing viscous for a sequence $(b_m)_{m \in \mathbb{R}_+} \in L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}))$ such that there is $\beta \in \mathbb{R}$,

$$\forall \varepsilon > 0, \quad \lim_{m \rightarrow +\infty} \|b_m - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta-\varepsilon}(\mathbb{R}^d))} = 0, \quad (2.14)$$

there exists a sub-sequence of $(u^{m,\nu})_{(m,\nu) \in \mathbb{R}_+^2}$ in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ strong solution of (2.12) converging, for any compact $K \subset \mathbb{R}^d$, in $L^\infty([0, T]; C_b^{\gamma-\tilde{\varepsilon}}(\mathbb{R}^d, \mathbb{R}))$, $0 < \tilde{\varepsilon} < \gamma$, towards the Hölder continuous $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and such that for any function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$, we have, up to a sub-sequence selection for any $t \in [0, T]$:

$$\begin{aligned} \lim_{m \rightarrow \infty, \nu \rightarrow 0} \int_{\mathbb{R}^d} \left\{ \varphi(t, y) u^{m,\nu}(t, y) + \int_0^t \left\{ -\partial_t \varphi(s, y) u^{m,\nu}(s, y) + \langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle \varphi(s, y) \right\} ds \right\} dy \\ = \int_{\mathbb{R}^d} \varphi(0, y) g(y) dy + \int_{\mathbb{R}^d} \int_0^t \varphi(s, y) f(s, y) ds dy. \end{aligned} \quad (2.15)$$

The distributional formulation allows to give a sense to the potential irregularities of b and of $u^{\nu, m}$ when $\nu \rightarrow 0$, $m \rightarrow +\infty$, and to consider the whole space $[0, T] \times \mathbb{R}^d$, the cut-off of \mathbb{R}^d required to use the Arzelà-Ascoli theorem is included in the test function φ .

We define now the usual *weak* solution.

Definition 3 (weak solution). *A function u is a weak solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if u is a mild vanishing viscous for any function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$:*

$$\begin{aligned} & \int_{\mathbb{R}^d} \left\{ \varphi(t, y)u(t, y) + \int_0^t \left\{ -\partial_t \varphi(s, y)u(s, y) + \langle b(s, y), \nabla u(s, y) \rangle \varphi(s, y) \right\} ds \right\} dy \\ = & \int_{\mathbb{R}^d} \varphi(0, y)g(y)dy + \int_{\mathbb{R}^d} \int_0^t \varphi(s, y)f(s, y)ds dy. \end{aligned} \quad (2.16)$$

Remark 3. *We cannot hope to define classical solution in our irregular context. Indeed, even if roughly speaking $\partial_t u + \langle b, \nabla u \rangle$ is supposed to lie in $L^\infty(C_b^\gamma)$, we cannot a priori define point-wisely the classic scalar product between b and ∇u . If b has a blow up at a point $x_0 \in \mathbb{R}^d$ then, as u is solution of (2.12), $\lim_{x \rightarrow x_0} \int_0^t \langle b(s, x), \nabla u(s, x) \rangle ds$ is necessary finite for any $t \in [0, T]$ but we cannot give a meaning of $\langle b(t, x_0), \nabla u(t, x_0) \rangle$ in a point-wise sense. Roughly speaking, to handle distributional drift we have to stay in a distributional formulation of the solution.*

2.4 On the non-uniqueness

We fail to obtain a uniqueness of a viscous selection principle. Indeed in the *a priori* controls of (2.12) we have to suppose that ν goes to 0 much faster than m towards $+\infty$. This constraint prevents us to take advantage of the convergence of b_m towards b to balance the blow up in the viscosity ν occurring in the computation to get uniqueness.

The non-uniqueness is not surprising even in a Hölder framework where there are usual counter-examples of non-uniqueness of solution in the Hölder space.

For instance in [FGP10] they recall that the following transport coefficient yields non-uniqueness of strong solution,

$$b(x) = \frac{1}{1-\gamma} \text{sign}(x)(|x| \wedge R)^\gamma, \quad \gamma \in (0, 1), \quad (2.17)$$

for a given constant $R > 0$. With this counterexample, [AF09] get non-uniqueness of the zero noise limit of the corresponding SPDE,

$$\partial_t u^\varepsilon + b \cdot \partial_x u^\varepsilon = \varepsilon \nabla u^\varepsilon \circ dW_t.$$

Until now other method of approximations, such as a limit of mollification of the coefficient performed by [CCS20] and [DLG22], also imply non-uniqueness of the solution. We finally can mention other counter-examples stated in [Dep03].

3 Main results

When b lies in Hölder-Besov space, we succeed in obtaining the same regularity of the solution as for f and for g . The type of solution strongly depends on the regularity of b .

Theorem 2 (Existence of solution to rough transport equation). *For $\gamma \in (0, 1)$, $\beta > 0$ be given. For all $b \in L^\infty([0, T], B_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, $f \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}^d, \mathbb{R})$, there is a mild vanishing viscosity solution $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of (1.1) satisfying*

$$\begin{aligned} \|u\|_{L^\infty(C^\gamma)} & \leq T \|f\|_{L^\infty(C^\gamma)} + \|g\|_\gamma, \\ \|u\|_{L^\infty} & \leq T \|f\|_{L^\infty} + \|g\|_{L^\infty}, \end{aligned} \quad (3.1)$$

the conditions on the vanishing viscosity $0 < \nu < T^{-1}$, are for a given constant $C > 0$ depending only

on (γ, d) ,

$$\begin{aligned} \nu &\ll (m^{2+\beta-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma}+2} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})})^{-\frac{4}{1-\gamma^2}} \exp\left(-4 \frac{m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} T}{1-\gamma^2}\right), \\ \nu &\ll \left(C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} T^{\frac{2-\gamma^2+\gamma}{2\gamma}} (m^{1-\gamma} (T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)) (1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\ &\quad \times \exp\left(-\frac{8m^{1+\beta} T \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}}{\gamma(1-\gamma)}\right). \end{aligned} \quad (3.2)$$

Moreover, with additional conditions on b , we have:

i) **Incompressibility.** If $\beta < \gamma$ and $\nabla \cdot b = 0$ then the solution u is also a mild-weak and a weak solution.

ii) **Positive continuity.** If $\beta < -1 + \gamma$, namely if $b \in L^\infty([0, T]; C_b^{\tilde{\gamma}}(\mathbb{R}^d, \mathbb{R}))$, $\tilde{\gamma} > 1 - \gamma$, then the solution u is also a mild-weak and a weak solution, if

$$\nu \ll (T\|f\|_{L^\infty} + [g]_\gamma)^{-1} \left(m^{2-\gamma} + CTm^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \right)^{-1} \exp(-2Tm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}), \quad (3.3)$$

then $\partial_t u(t, \cdot) \in B_{\infty,\infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$.

Remark 4. For the mild vanishing viscous solution there is no restriction on b , roughly speaking there is an “infinite regularisation by turbulence” over the coefficients.

While for the usual weak solution, there is no more such infinite regularisation effect. The **Incompressibility** framework, i.e. $\nabla \cdot b = 0$, allows to still consider a negative regularity of b . Such divergence free condition for non-smooth distributions already exists for instance for Leray’s solution of Navier-Stokes equation [Ler34].

In the last case, i.e. **Positive continuity**, the considered drift b is suppose to be Hölder continuous in space, in particular lying in $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^d, \mathbb{R}^d))$, $\alpha > 1 - \gamma$ which is the Bony’s paraproduct assumption, see Section 3.1 below for more details.

Importantly, the above controls (3.1) do not depend on the drift b . This is crucial to consider very rough coefficients as well as non-linear equation such as the inviscid Burgers’ equation studied in Section 5.

Remark 5. The exponential criterion in (3.2) relies on an a priori control by $\|u^{m,\nu}\|_{L^\infty}$, see Section 4.2.7 for more details. This condition prevents us to hope any balance between m and ν required to get usual uniqueness. Indeed, when we expand the computations, we can see only polynomial dependency on (m, ν) in the upper-bounds. But the contribution on ν goes in the wrong way, and cannot be overwhelmed by polynomial converging terms in m , because at the best $m \sim |\ln(\nu)|$ from (3.2).

Even for b lying in a Hölder space, namely with a positive regularity, we fail to avoid an exponential criterion like in (3.2), see again Section 4.2.7.

3.1 On the product of distributions

The sense of a product of distributions is very challenging, and can allows to deal with long-standing problems. For instance, Hairer in [Hai14] introduce a regularity theory which after some renormalisation allows to handle with products of distribution, and to give a meaning of stochastic partial differential equation such as KPZ [Hai13]. However, such renormalisation leads to blowing-up constants which is not the case in Theorem 2; the price that we have to pay is the potential non-uniqueness of the limit.

From the different formulations above, we see that we can define different meaning of the product $\langle b, \nabla u \rangle$.

First of all, let us remark that by rough *a priori* controls, see Appendix Sections A.2 and A.3,

$$\begin{aligned} [\Delta u^{m,\nu}(t, \cdot)]_\gamma &\leq 2^{1-\gamma} \|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty}^{1-\gamma} \|\nabla^3 u^{m,\nu}(t, \cdot)\|_{L^\infty}^\gamma \\ &\leq C m^{2+\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(Ctm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}). \end{aligned}$$

Hence, if $\nu \ll C m^{2+\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(Ctm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})})$, implied by (3.2), then, up to subsequence choice, $\nu \Delta u^{m,\nu}(t, \cdot) \xrightarrow{(m,\nu) \rightarrow (+\infty, 0)} 0$ in $C_b^\gamma(\mathbb{R}, \mathbb{R})$.

Also, for a given $t \in [0, T]$, we see from the definition of *mild vanishing viscous* solution, up to subsequence choice according to the condition (3.2), that

$$\lim_{(m,\nu) \rightarrow (+\infty, 0)} \int_0^t \langle b_m(s, \cdot), \nabla u^{m,\nu}(s, \cdot) \rangle ds = g - u(t, \cdot) + \int_0^t f(s, \cdot) ds \in C_b^\gamma(K, \mathbb{R}), \quad (3.4)$$

for any compact $K \subset \mathbb{R}^d$. We highly point out that b lies in any arbitrary negative regularity in space $L^\infty([0, T]; B_{\infty,\infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, $\beta \in \mathbb{R}_+$, and $\nabla u(s, \cdot) \in B_{\infty,\infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$.

In other words, thanks to the time averaging we get a new para-product condition. Indeed, in general from Bony's microlocal analysis [Bon81], see also [GIP15], for all $\varphi \in B_{\infty,\infty}^{\alpha_1}$ and $\psi \in B_{\infty,\infty}^{\alpha_2}$, we have

$$\phi\psi \in B_{\infty,\infty}^{\alpha_1 \wedge \alpha_2}, \text{ if } \alpha_1 + \alpha_2 > 0. \quad (3.5)$$

However, the uniqueness of the limit in (3.4) seems to be false in general, which is consistent with the non-uniqueness of classic solution for b Hölder continuous, see example (2.17).

Also in the weak formulation, from Theorem 2, we obtain, if $\nabla \cdot b = 0$ and $\beta < \gamma$, a distributional meaning of $\langle b(s, \cdot), \nabla u(s, \cdot) \rangle$, but we still do not know in this case if the limit is unique. But the regularity condition is still weaker than (3.5).

Finally, we point out that in the **Positive continuity** framework, we meet the Bony's paraproduct condition (3.5). Indeed, for any $t \in [0, T]$, $b(t, \cdot) \in B_{\infty,\infty}^\alpha$, $\alpha = -\beta$, and $\nabla u(t, \cdot) \in B_{\infty,\infty}^{-1+\gamma}$ with $\alpha - 1 + \gamma > 0$. We can quantify the regularity, $\langle b(t, \cdot), \nabla u(t, \cdot) \rangle \in B_{\infty,\infty}^{-1+\gamma}$ by paraproduct detailed further in Section 4.7.6. Moreover, the time averaging version $\int_0^t \langle b(s, \cdot), \nabla u(s, \cdot) \rangle ds$ in the sense of (3.4) is γ -Hölder. We remark, as $\alpha \wedge (-1 + \gamma) = -1 + \gamma$ then there is a +1 gain of regularity comparing with the usual paraproduct result.

4 Proof of Theorem 2

4.1 Parabolic approximation procedure

Let us first smoothen the drift and the source functions of the parabolic approximation,

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + \langle b_m, \nabla u^{m,\nu} \rangle(t, x) - \nu \Delta u^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m,\nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (4.1)$$

where the mollified functions are defined by

$$\begin{aligned} b_m(t, x) &:= \int_{\mathbb{R}^d} \rho_m(x - y) b(t, y) dy, \\ f_m(t, x) &:= \int_{\mathbb{R}^d} \rho_m(x - y) f(t, y) dy, \\ g_m(t, x) &:= \int_{\mathbb{R}^d} \rho_m(x - y) g(y) dy, \end{aligned} \quad (4.2)$$

for $\rho_m(\cdot) := m^d \rho(m \cdot)$ where ρ is a non-negative smooth function ρ_m , such that $\int_{\mathbb{R}^d} \rho_m(x - y) dy = 1$. In particular, we choose $\rho = h_1$ the heat kernel defined in 2.4. In Appendix Sections D and B, we see

that the limit of b_m does not depend on the choice of the mollification procedure, whereas the limit of $u^{m,\nu}$ potentially does.

In our analysis, we use some point-wise controls of the mollified functions or distributions whose the blowing-up in the regularisation parameter m is stated below.

Lemma 1. *For all $m > 1$, and $\beta > 0$, if $b \in L^\infty([0, T], B_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, we have for any $(t, x) \in [0, T] \times \mathbb{R}^d$:*

$$\begin{aligned} |b_m(t, x)| &\leq Cm^\beta \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}, \\ |Db_m(t, x)| &\leq Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}, \end{aligned} \quad (4.3)$$

where Db_m stands for the Jacobian matrix of b_m ; also if $\beta \leq 0$,

$$\begin{aligned} |b_m(t, x)| &\leq \|b\|_{L^\infty}, \\ |Db_m(t, x)| &\leq Cm^{1-\beta} \|b\|_{L^\infty(C^\beta)}. \end{aligned}$$

Proof of Lemma . From the mollification definition (4.2), we see, from (2.5), that $\rho_m = h_{m-2}$, and from our scaling choice, we get for any $m > 1$

$$\begin{aligned} |b_m(t, x)| &= \left| \int_{\mathbb{R}^d} h_{m-2}(x-y)b(t, y)dy \right| \\ &\leq m^\beta \sup_{\tilde{m}^{-2} \in [0, 1], x \in \mathbb{R}^d} \tilde{m}^{-\beta} \left| \int_{\mathbb{R}^d} h_{\tilde{m}-2}(x-y)b(t, y)dy \right|. \end{aligned}$$

We readily get by the thermic definition of the Besov norm (2.4):

$$|b_m(t, x)| \leq m^\beta \|b\|_{L^\infty(\dot{B}_{\infty, \infty}^{-\beta})} \leq m^\beta \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}.$$

For the second inequality, it is known that for any $t \in [0, T]$, $Db(t, \cdot) \in B_{\infty, \infty}^{-1-\beta}$, see Theorem 9 of Chapter 3 in [Pee76], in particular case $\beta \in (0, -1)$ see Corollary 8 in Appendix Section C. Hence,

$$\begin{aligned} |D_x b_m(t, x)| &\leq m^{1+\beta} \sup_{\tilde{m}^{-2} \in [0, 1], \tilde{x} \in \mathbb{R}^d} \tilde{m}^{-\beta} \left| \int_{\mathbb{R}^d} h_{\tilde{m}-2}(\tilde{x}-y)D_y b(t, y)dy \right| \\ &= m^{1+\beta} \|Db\|_{L^\infty(B_{\infty, \infty}^{-1-\beta})}. \end{aligned}$$

We deduce that there is a constant $C = C(d) > 0$ such that:

$$|D_x b_m(t, x)| \leq Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}.$$

The two last equations of the lemma, i.e. for the case $\beta \leq 0$, are standard. \square

4.1.1 L^∞ control

We have directly by the Feynman-Kac formulation the uniform control, see for example from the analysis performed in [Hon22], or from maximum principle for linear parabolic equation see e.g. [Lie96].

$$\|u^{m,\nu}\|_{L^\infty} \leq T\|f\|_{L^\infty} + \|g\|_{L^\infty}. \quad (4.4)$$

For the sake of completeness, we provide in Section 4.6 a way to get exactly the same upper-bound when $\nu \rightarrow 0$.

4.1.2 Proxy choice

To derive the others estimates from Duhamel formulation, we approximate the Cauchy problem around the flow associated to the smooth function b_m , which is unique by Cauchy-Lipschitz theorem. Namely, let us consider the unique function defined for any *freezing* point $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$ by,

$$\theta_{s,\tau}^m(x) := x + \int_s^\tau b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x)) d\tilde{s}, \quad s \in [0, \tau]. \quad (4.5)$$

In other words, for any $t \in [0, \tau]$,

$$\dot{\theta}_{t,\tau}^m(\xi) = -b_m(t, \theta_{t,\tau}^m(\xi)), \quad \theta_{\tau,\tau}^m(\xi) = \xi.$$

We can again rewrite the system of linear parabolic PDEs (4.2),

$$\partial_t u^{m,\nu}(t, x) + b_m(t, \theta_{t,\tau}^m(\xi)) \cdot \nabla u^{m,\nu}(t, x) - \nu \Delta u^{m,\nu}(t, x) = b_\Delta^m[\tau, \xi](t, x) \cdot \nabla u^{m,\nu}(t, x) + f_m(t, x), \quad (4.6)$$

where we define

$$b_\Delta^m[\tau, \xi](t, x) := b_m(t, \theta_{t,\tau}^m(\xi)) - b_m(t, x). \quad (4.7)$$

For such a fixed freezing point $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$, we use the corresponding Duhamel formula:

$$u^{m,\nu}(t, x) = \hat{P}^{\tau,\xi} g_m(t, x) + \hat{G}^{\tau,\xi} f_m(t, x) + \hat{G}^{\tau,\xi} (b_\Delta^m[\tau, \xi] \cdot \nabla u^{m,\nu})(t, x), \quad (4.8)$$

where we define, for any $f \in C_0^{1,2}((0, T] \times \mathbb{R}^d, \mathbb{R})$, the Green operator associated with the perturbed parabolic equation with constant coefficients (4.1),

$$\forall (t, x) \in (0, T] \times \mathbb{R}^d, \quad \hat{G}^{\tau,\xi} f_m(t, x) := \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) f(s, y) dy ds, \quad (4.9)$$

and for any $g \in C_0^2(\mathbb{R}^d, \mathbb{R})$, the associated semi-group

$$\hat{P}^{\tau,\xi} g_m(t, x) := \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(0, t, x, y) g_m(y) dy, \quad (4.10)$$

where the perturbed heat kernel is

$$\hat{p}^{\tau,\xi}(s, t, x, y) := \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{\left|x + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right). \quad (4.11)$$

We carefully point out that, from definition (4.5), if $\xi = x$

$$\hat{p}^{t,x}(s, t, x, y) = \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{|\theta_{s,t}^m(x) - y|^2}{4\nu(t-s)}\right).$$

We have for each $\alpha \in \mathbb{N}^d$ that there is a constant $C_\alpha > 1$ s.t.

$$\begin{aligned} |D^\alpha \hat{p}^{\tau,\xi}(s, t, x, y)| &\leq \frac{C_\alpha [\nu(s-t)]^{-\frac{|\alpha|}{2}}}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-C_\alpha^{-1} \frac{\left|x + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right) \\ &=: C [\nu(s-t)]^{-\frac{|\alpha|}{2}} \hat{p}^{\tau,\xi}(s, t, x, y), \end{aligned} \quad (4.12)$$

and also, after the derivative we can choose $(\tau, \xi) = (t, x)$, and $\gamma \in [0, 1]$,

$$\begin{aligned} |D^\alpha \hat{p}^{t,x}(s, t, x, y)| \times \left|y - x - \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s}\right|^\gamma &= |D^\alpha \hat{p}^{t,x}(s, t, x, y)| \times |y - \theta_{s,t}^m(x)|^\gamma \\ &\leq C [\nu(s-t)]^{-\frac{|\alpha|}{2} + \frac{\gamma}{2}} \hat{p}^{t,x}(s, t, x, y), \end{aligned} \quad (4.13)$$

from absorbing property (2.9). It also clear that for any $0 \leq s < t$ that

$$\partial_t \hat{p}^{\tau, \xi}(s, t, x, y) = \nu \Delta \hat{p}^{\tau, \xi}(s, t, x, y) - \langle b_m(t, \theta_{t, \tau}^m(\xi)), \nabla \hat{p}^{\tau, \xi}(s, t, x, y) \rangle, \quad (4.14)$$

which naturally implies that the function $u^{m, \nu}$ defined in (4.8) is indeed solution to (4.1) and to (4.6).

Finally, we will marginally use the ‘‘pure’’ heat kernel

$$\tilde{p}(s, t, x, y) := \hat{p}^{\tau, \xi}\left(s, t, x - \int_s^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau}^m(\xi)) d\tilde{s}, y\right) = \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4\nu(t-s)}\right), \quad (4.15)$$

the corresponding Green operator

$$\forall (t, x) \in (0, T] \times \mathbb{R}^d, \quad \tilde{G}f_m(t, x) := \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, t, x, y) f(s, y) dy ds, \quad (4.16)$$

and the associated semi-group

$$\tilde{P}g_m(t, x) := \int_{\mathbb{R}^d} \tilde{p}(0, t, x, y) g_m(y) dy. \quad (4.17)$$

4.2 Control of Hölder modulus

For any $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, we choose the associated *freezing* points (τ, ξ) and (τ', ξ') , but we take $\tau = \tau'$ like in [CDRHM18]. The previous Duhamel formula (4.8) yields

$$\begin{aligned} & |u^{m, \nu}(t, x) - u^{m, \nu}(t, x')| \\ & \leq |\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x')| + |\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x')| \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) [b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) dy ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(s, t, x, y) [b_m(s, \theta_{s, \tau'}^m(\xi')) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) dy ds \right| \\ & =: |\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x')| + |\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x')| + |R^{\tau, \xi, \xi'}(t, x, x')|. \end{aligned} \quad (4.18)$$

However, our analysis need different choices of *freezing* point which yields extra contributions in the above inequality, the final Duhamel like identity is stated in (4.56) further.

By integration by parts, we can rewrite the remainder term

$$\begin{aligned} & R^{\tau, \xi, \xi'}(t, x, x') \\ & = \left\{ \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau, \xi}(s, t, x, y) \cdot [b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)] [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))] dy ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau, \xi'}(s, t, x', y) \cdot [b_m(s, \theta_{s, \tau}^m(\xi')) - b_m(s, y)] [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))] dy ds \right\} \\ & \quad + \left\{ \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) \nabla \cdot b_m(s, y) [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))] dy ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(s, t, x', y) \nabla \cdot b_m(s, y) [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))] dy ds \right\} \\ & =: R_1^{\tau, \xi, \xi'}(t, x, x') + R_2^{\tau, \xi, \xi'}(t, x, x'). \end{aligned} \quad (4.19)$$

Actually, this integration by parts is not essential, we could use a point-wise *a priori* control of $\nabla u^{m, \nu}$. However we aim to track as sharp as possible the required *a priori* controls in the upper-bounds. To be more specific, we unsuccessfully tried to only upper-bound by $\|u^{m, \nu}\|_{L^\infty(C_b^\gamma)}$ instead of $\|u^{m, \nu}\|_{L^\infty(C_b^1)}$. This last value is finite but increases exponentially with m , see Appendix Section A. This exponential blowing-up yields the limit criterion (3.2) of (m, ν) , and prevents us to get any balance between m and ν to conclude with uniqueness of solution.

4.2.1 Main terms

For the main contributions associated with f and g , we choose the *freezing* parameters to be $\tau = t$ and $\xi = \xi' = x$, we choose the same parameters as in the *diagonal* regime specified further.

Semi-group

We readily derive by change variables:

$$\begin{aligned}
|\hat{P}^{\tau,\xi} g_m(t, x) - \hat{P}^{\tau,\xi} g_m(t, x')|_{\tau=t, \xi=\xi'=x} &= \left| \int_{\mathbb{R}^d} [\hat{p}^{\tau,\xi}(0, t, x, y) - \hat{p}^{\tau,\xi'}(0, t, x', y)] g_m(s, y) dy \right|_{\tau=t, \xi=\xi'=x} \\
&= \left| \int_{\mathbb{R}^d} [\hat{p}^{t,x}(0, t, 0, y)] [g_m(s, x + y) - g_m(s, x' + y)] dy \right| \\
&\leq [g]_{\gamma} |x - x'|^{\gamma}.
\end{aligned} \tag{4.20}$$

Green operator

We also get by change variables:

$$\begin{aligned}
|\hat{G}^{\tau,\xi} f_m(t, x) - \hat{G}^{\tau,\xi} f_m(t, x')|_{\tau=t, \xi=\xi'=x} &= \left| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{t,x}(s, t, x, y) - \hat{p}^{t,x'}(s, t, x', y)] f_m(s, y) dy ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{t,x}(s, t, 0, y)] [f_m(s, x - y) - f_m(s, x' - y)] ds dy \right| \\
&\leq \|f\|_{L^\infty(C^\gamma)} |x - x'|^{\gamma} t.
\end{aligned} \tag{4.21}$$

4.2.2 Remainder term

To analysis of the Hölder modulus of the remainder term, which is the core of the *a priori* controls, we need to separate the *diagonal* regime from the *off-diagonal* one, as performed in [CDRHM18]. This strategy is natural in view with the vanishing viscous solution selected by the parabolic approximation.

However, in the vanishing viscosity context, we have to carefully track the dependency on ν which yields to consider an other criterion of *diagonal* / *off-diagonal* regime, usual in the parabolic framework.

Specifically, for any $x, x' \in \mathbb{R}^d$ and for given parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, to be tailored further, we call *off-diagonal* regime the case $|x - x'| > \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s > t_0$ with

$$t_0 := t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}; \tag{4.22}$$

on the contrary the *diagonal* regime holds when $|x - x'| \leq \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s \leq t_0$.

The point t_0 can be regarded as a *cut-locus* point where we “catch” the shortest way from $u^{m,\nu}(t, x)$ to $u^{m,\nu}(t, x')$ if $t \in [0, t_0]$ and we choose another way if $t_0 < t$. We carefully point out that this procedure yields an extra contribution in (4.18), this is detailed in Sections 4.3, 4.4.

We specify in Section 4.5 below why we can choose different *freezing* parameters for the remainder term according to the current regime; meanwhile the semi-group and the Green operator dealt in Section 4.2.1 can somehow stay in the *diagonal* regime.

4.2.3 Diagonal regime

If $|x - x'| \leq \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s \leq t_0 = t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}$, $\xi = \xi' = x$ then we define the associated space

$$A(x, x', \nu, t)(s) := \{|x - x'| \leq \nu^{\alpha_1} (t - s)^{\alpha_2}\}, \tag{4.23}$$

with the indicator function

$$\mathbb{1}_{A(x,x',\nu,t)}(s) := \begin{cases} 1 & \text{if } |x - x'| \leq \nu^{\alpha_1}(t - s)^{\alpha_2}, \\ 0 & \text{if } |x - x'| > \nu^{\alpha_1}(t - s)^{\alpha_2}, \end{cases}$$

and the associated remainder terms

$$\begin{aligned} & R_{1,A}^{\tau,\xi,\xi'}(t, x, x') \\ := & \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau,\xi}(s, t, x, y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\ & \left. - \nabla \hat{p}^{\tau,\xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi')) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds, \\ & R_{2,A}^{\tau,\xi,\xi'}(t, x, x') \\ := & \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \left[\hat{p}^{\tau,\xi}(s, t, x, y) \nabla \cdot b_m(s, y) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\ & \left. - \hat{p}^{\tau,\xi'}(s, t, x', y) \nabla \cdot b_m(s, y) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds. \end{aligned} \quad (4.24)$$

Let us remark that, if $t_0 \geq 0$, we can equivalently write the above terms by

$$\begin{aligned} & R_{1,A}^{\tau,\xi,\xi'}(t, x, x') \\ = & \int_0^{t_0} \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau,\xi}(s, t, x, y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\ & \left. - \nabla \hat{p}^{\tau,\xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi')) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds \\ =: & \nabla \cdot \hat{G}_{0,t_0}^{\tau,\xi} \left\{ (b_m(\cdot, \theta_{\cdot,\tau}^m(\xi)) - b_m) (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi))) \right\} (t, x) \\ & - \nabla \cdot \hat{G}_{t_0,t}^{\tau,\xi'} \left\{ (b_m(\cdot, \theta_{\cdot,\tau}^m(\xi')) - b_m) (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi'))) \right\} (t, x'), \end{aligned} \quad (4.25)$$

and by

$$\begin{aligned} & R_{2,A}^{\tau,\xi,\xi'}(t, x, x') \\ = & \int_0^{t_0} \int_{\mathbb{R}^d} \left[\hat{p}^{\tau,\xi}(s, t, x, y) \nabla \cdot b_m(s, y) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\ & \left. - \hat{p}^{\tau,\xi'}(s, t, x', y) \nabla \cdot b_m(s, y) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds \\ =: & \hat{G}_{0,t_0}^{\tau,\xi} \left\{ \nabla \cdot b_m (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi))) \right\} (t, x) - \hat{G}_{t_0,t}^{\tau,\xi'} \left\{ \nabla \cdot b_m (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi'))) \right\} (t, x'). \end{aligned} \quad (4.26)$$

Remainder term $R_{1,A}^{\tau,\xi,\xi'}(t, x, x')$

By change of variable[‡], we directly get

$$\begin{aligned} & \left| R_{1,A}^{\tau,\xi,\xi'}(t, x, x') \right|_{\tau=t, \xi=\xi'=x} \\ = & \left| \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \right. \\ & \cdot \left\{ (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, x + y)) (u^{m,\nu}(s, x + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\ & \left. \left. - (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, x' + y)) (u^{m,\nu}(s, x' + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right\} dy ds \right|_{\tau=t, \xi=\xi'=x}, \end{aligned}$$

[‡]Specifically, we choose the new variable $y' = y - x$.

by parity of the Gaussian density, specifying that

$$\hat{p}^{\tau,\xi}(s, t, 0, y) = \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{\left|\int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi))d\tilde{s} - y\right|^2}{4\nu(t-s)}\right),$$

and expanding the terms above, in the r.h.s., gives us

$$\begin{aligned} & |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\ &= \left| \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot \left\{ b_m(s, \theta_{s,\tau}^m(\xi))(u^{m,\nu}(s, x+y) - u^{m,\nu}(s, x'+y)) \right. \right. \\ & \quad \left. \left. + (b_m(s, x+y) - b_m(s, x'+y))u^{m,\nu}(s, \theta_{s,\tau}^m(\xi)) \right. \right. \\ & \quad \left. \left. + b_m(s, x'+y)u^{m,\nu}(s, x'+y) - b_m(s, x+y)u^{m,\nu}(s, x+y) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x}, \end{aligned}$$

and putting together the corresponding contribution

$$\begin{aligned} & |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\ &= \left| \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot \left\{ b_m(s, \theta_{s,\tau}^m(\xi))(u^{m,\nu}(s, x+y) - u^{m,\nu}(s, x'+y)) \right. \right. \\ & \quad \left. \left. + (b_m(s, x+y) - b_m(s, x'+y))u^{m,\nu}(s, \theta_{s,\tau}^m(\xi)) + (b_m(s, x'+y) - b_m(s, x+y))u^{m,\nu}(s, x'+y) \right. \right. \\ & \quad \left. \left. - b_m(s, x'+y)(u^{m,\nu}(s, x+y) - u^{m,\nu}(s, x'+y)) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x}. \end{aligned}$$

We finally get

$$\begin{aligned} & |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\ &= \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot \left\{ (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, x'+y))(u^{m,\nu}(s, x+y) - u^{m,\nu}(s, x'+y)) \right. \right. \\ & \quad \left. \left. + (b_m(s, x'+y) - b_m(s, x+y))(u^{m,\nu}(s, x'+y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x}, \end{aligned}$$

hence by exponential absorption property (2.9),

$$\begin{aligned} & |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\ &\leq C|x-x'| \|b_m\|_{L^\infty(C^1)} \|u^{m,\nu}\|_{L^\infty(C^1)} \\ & \quad \times \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} (\nu(t-s))^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, 0, y) \times |\theta_{s,\tau}^m(x) - x' - y| dy ds \\ &\leq C|x-x'| \|b_m\|_{L^\infty(C^1)} \|u^{m,\nu}\|_{L^\infty(C^1)} \\ & \quad \times \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} (\nu(t-s))^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, 0, y) (|\theta_{s,\tau}^m(x) - x - y| + |x' - x|) dy ds \Big|_{\tau=t,\xi=\xi'=x}, \end{aligned} \tag{4.27}$$

where we recall from (4.12) that

$$\bar{p}^{\tau,\xi}(s, t, 0, y) = \frac{C}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-C^{-1} \frac{\left|\int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi))d\tilde{s} - y\right|^2}{4\nu(t-s)}\right).$$

Let us remark thanks to Grönwall's lemma we get, see Appendix Section A:

$$\|u^{m,\nu}(t, \cdot)\|_{C^1} \leq \left(T\|f^m\|_{L^\infty(C^1)} + \|g^m\|_{C^1}\right) \exp\left(m^{1+\beta}t\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right) =: O_m(t). \tag{4.28}$$

For $(\tau, \xi) = (t, x)$, recalling that $\int_s^t b_m(\tilde{s}, \theta_{\tilde{s}, t}(x)) d\tilde{s} = \theta_{s, t}^m(x) - x$, and by absorption property (2.9), we deduce

$$\begin{aligned} & |R_{1,A}^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ & \leq C|x - x'| \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} m^{1+\beta} O_m(t) \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) \left(1 + [\nu(t-s)]^{-\frac{1}{2}} |x - x'|\right) ds \\ & \leq C|x - x'|^\gamma \nu^{\alpha_1(1-\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} m^{1+\beta} O_m(t) \int_0^t (t-s)^{(1-\gamma)\alpha_2} \left(1 + \nu^{-\frac{1}{2}}(s-t)^{-\frac{1}{2}} \nu^{\alpha_1}(t-s)^{\alpha_2}\right) ds, \end{aligned}$$

because in the current *diagonal* regime, $|x - x'|^{1-\gamma} \leq \nu^{\alpha_1(1-\gamma)}(t-s)^{\alpha_2(1-\gamma)}$.

If $(1-\gamma)\alpha_2 > -1$ and $(1-\gamma)\alpha_2 - \frac{1}{2} + \alpha_2 > -1 \Leftrightarrow \alpha_2 > -\frac{1}{2(2-\gamma)} > -\frac{1}{1-\gamma} > -1$, the above time integral is finite and

$$\begin{aligned} & |R_1^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ & \leq C|x - x'|^\gamma \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} m^{1+\beta} O_m(t) \left(\nu^{\alpha_1(1-\gamma)} t^{1+(1-\gamma)\alpha_2} + \nu^{-\frac{1}{2}+\alpha_1(2-\gamma)} t^{\frac{1}{2}+\alpha_2(2-\gamma)} \right). \quad (4.29) \end{aligned}$$

To consider vanishing viscous, it is necessary to have $\alpha_1(1-\gamma) > 0$ and $-\frac{1}{2} + \alpha_1(1-\gamma) + \alpha_1 > 0 \Leftrightarrow \alpha_1 > \frac{1}{2(2-\gamma)} > 0$ for $\gamma < 1$.

To sum up, we consider the constraint

$$\begin{aligned} \alpha_1 & > \frac{1}{2(2-\gamma)}, \\ \alpha_2 & > -\frac{1}{2(2-\gamma)}. \end{aligned} \quad (4.30)$$

Remainder term $R_{2,A}^{\tau, \xi, \xi'}(t, x, x')$

Also by change of variables and by similar computations as for the first remainder term $R_1^{\tau, \xi, \xi'}(t, x, x')$:

$$\begin{aligned} & |R_{2,A}^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ & = \left| \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \nabla \cdot b_m(s, x+y) (u^{m, \nu}(s, x+y) - u^{m, \nu}(s, \theta_{s,t}^m(x))) dy ds \right. \\ & \quad \left. - \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \nabla \cdot b_m(s, x'+y) (u^{m, \nu}(s, x'+y) - u^{m, \nu}(s, \theta_{s,t}^m(x))) dy ds \right| \\ & = \left| \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \cdot \left\{ -\nabla \cdot b_m(s, x'+y) (u^{m, \nu}(s, x+y) - u^{m, \nu}(s, x'+y)) \right. \right. \\ & \quad \left. \left. + (\nabla \cdot b_m(s, x'+y) - \nabla \cdot b_m(s, x+y)) (u^{m, \nu}(s, x'+y) - u^{m, \nu}(s, \theta_{s,t}^m(x))) \right\} dy ds \right| \\ & \leq C|x - x'| (\|\nabla b_m\|_{L^\infty} + \|\nabla b_m\|_{L^\infty(C^1)}) \|u^{m, \nu}\|_{L^\infty(C^1)} \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) ds. \end{aligned}$$

Next, we obtain from (2.9) and (4.28),

$$\begin{aligned} & |R_{2,A}^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ & \leq C|x - x'| \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t) \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) (1 + [\nu(t-s)])^{\frac{1}{2}} ds. \end{aligned}$$

By assumption of the *diagonal* regime, we get

$$\begin{aligned}
& |R_{2,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
& \leq C|x-x'|^\gamma \nu^{\alpha_1(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t) \int_0^t (t-s)^{\alpha_2(1-\gamma)} (1 + [\nu(t-s)])^{\frac{1}{2}} ds \\
& \leq C|x-x'|^\gamma \nu^{\alpha_1(1-\gamma)} t^{1+\alpha_2(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t), \tag{4.31}
\end{aligned}$$

for $\nu t \leq \nu T \leq 1$. The above inequality satisfies the vanishing viscosity analysis if $\alpha_1 > 0$ and if $\alpha_2 > \frac{-1}{1-\gamma}$ which is already required for the first remainder (because $\frac{1}{2(2-\gamma)} > 0$ and $\frac{-1}{2(2-\gamma)} > \frac{-1}{1-\gamma}$).

4.2.4 Off-diagonal regime

If $|x-x'| > \nu^{\alpha_1}(t-s)^{\alpha_2} \Leftrightarrow s > t - \nu^{-\frac{\alpha_1}{\alpha_2}}|x-x'|^{\frac{1}{\alpha_2}}$, we recall the corresponding *cut locus* point

$$t_0 := t - \nu^{-\frac{\alpha_1}{\alpha_2}}|x-x'|^{\frac{1}{\alpha_2}}.$$

In this case, we choose as *freezing* parameters $\xi = x$ and $\xi' = x'$. In the *off-diagonal* regime, the associated space is

$$A^c(x, x', \nu, t)(s) := \{|x-x'| > \nu^{\alpha_1}(t-s)^{\alpha_2}\}, \tag{4.32}$$

the indicator function

$$\mathbb{1}_{A^c(x, x', \nu, t)}(s) = 1 - \mathbb{1}_{A(x, x', \nu, t)}(s) = \begin{cases} 1 & \text{if } |x-x'| > \nu^{\alpha_1}(t-s)^{\alpha_2}, \\ 0 & \text{if } |x-x'| \leq \nu^{\alpha_1}(t-s)^{\alpha_2}, \end{cases}$$

and the associated remainder terms are

$$\begin{aligned}
& R_{1,A^c}^{\tau,\xi,\xi'}(t, x, x') = R_1^{\tau,\xi,\xi'}(t, x, x') - R_{1,A}^{\tau,\xi,\xi'}(t, x, x') \\
& = \int_0^t \mathbb{1}_{A^c(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau,\xi}(s, t, x, y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\
& \quad \left. - \nabla \hat{p}^{\tau,\xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi')) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds,
\end{aligned}$$

and

$$\begin{aligned}
R_{2,A^c}^{\tau,\xi,\xi'}(t, x, x') & = R_2^{\tau,\xi,\xi'}(t, x, x') - R_{2,A}^{\tau,\xi,\xi'}(t, x, x') \\
& = \int_0^t \mathbb{1}_{A^c(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \left[\hat{p}^{\tau,\xi}(s, t, x, y) \nabla \cdot b_m(s, y) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\
& \quad \left. - \hat{p}^{\tau,\xi'}(s, t, x', y) \nabla \cdot b_m(s, y) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds.
\end{aligned}$$

Like in the *diagonal* regime, we can also rewrite the above remainder terms

$$\begin{aligned}
& R_{1,A^c}^{\tau,\xi,\xi'}(t, x, x') \\
& = \int_{t_0}^t \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau,\xi}(s, t, x, y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\
& \quad \left. - \nabla \hat{p}^{\tau,\xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi')) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds \\
& =: \nabla \cdot \hat{G}_{t_0,t}^{\tau,\xi} \left\{ (b_m(\cdot, \theta_{\cdot,\tau}^m(\xi)) - b_m) (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi))) \right\} (t, x) \\
& \quad - \nabla \cdot \hat{G}_{t_0,t}^{\tau,\xi'} \left\{ (b_m(\cdot, \theta_{\cdot,\tau}^m(\xi')) - b_m) (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi'))) \right\} (t, x'), \tag{4.33}
\end{aligned}$$

and

$$\begin{aligned}
& R_{2,A^c}^{\tau,\xi,\xi'}(t,x,x') \\
&= \int_{t_0}^t \int_{\mathbb{R}^d} \left[\hat{p}^{\tau,\xi}(s,t,x,y) \nabla \cdot b_m(s,y) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi))) \right. \\
&\quad \left. - \hat{p}^{\tau,\xi'}(s,t,x',y) \nabla \cdot b_m(s,y) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi'))) \right] dy ds \\
&=: \hat{G}_{t_0,t}^{\tau,\xi} \left\{ \nabla \cdot b_m(u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi))) \right\} (t,x) - \hat{G}_{t_0,t}^{\tau,\xi'} \left\{ \nabla \cdot b_m(u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi'))) \right\} (t,x').
\end{aligned} \tag{4.34}$$

Remainder term $R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x')$

By triangular inequality

$$\begin{aligned}
& |R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=x,\xi'=x'} \\
&\leq 2C \|b_m\|_{L^\infty(C^1)} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \sup_{x \in \mathbb{R}^d} \int_{t_0}^t \int_{\mathbb{R}^d} (\nu(t-s))^{-\frac{1}{2}} \bar{p}^{t,x}(s,t,x,y) |y - \theta_{s,\tau}^m(\xi)|^{1+\gamma} dy ds.
\end{aligned}$$

Next, by absorbing property of the exponential (2.9) we get

$$\begin{aligned}
|R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=x,\xi'=x'} &\leq Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \int_{t_0}^t (\nu(t-s))^{\frac{\gamma}{2}} ds \\
&\leq \frac{m^{1+\beta}}{2+\gamma} C\nu^{\frac{\gamma}{2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} (t-t_0)^{1+\frac{\gamma}{2}}.
\end{aligned}$$

Finally by definition of the *cut locus* point (4.22):

$$\begin{aligned}
|R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=x,\xi'=x'} &\leq \frac{m^{1+\beta}}{1+\gamma} C\nu^{\frac{\gamma}{2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} (\nu^{-\frac{\alpha_1}{\alpha_2}} |x-x'|^{\frac{1}{\alpha_2}})^{\frac{2+\gamma}{2}} \\
&\leq \frac{m^{1+\beta}}{1+\gamma} C\nu^{\frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} |x-x'|^{\frac{2+\gamma}{2\alpha_2}}. \tag{4.35}
\end{aligned}$$

Hence, we need to have a second constraint:

$$\begin{aligned}
\gamma &= \frac{2+\gamma}{2\alpha_2}, \\
0 &< \frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}.
\end{aligned} \tag{4.36}$$

In particular, the parameter α_2 is then determined by $\alpha_2 = \frac{2+\gamma}{2\gamma}$.

Remainder term $R_{2,A^c}^{\tau,\xi,\xi'}(t,x,x')$

We also derive by triangular inequality,

$$\begin{aligned}
& |R_{2,A^c}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=x,\xi'=x'} \\
&\leq \left| \int_0^t \mathbf{1}_{A^c(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s,t,0,y) \nabla \cdot b_m(s,x+y) (u^{m,\nu}(s,x+y) - u^{m,\nu}(s,\theta_{s,t}^m(x))) dy ds \right| \\
&\quad + \left| \int_0^t \mathbf{1}_{A^c(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x'}(s,t,0,y) \nabla \cdot b_m(s,x'+y) (u^{m,\nu}(s,x'+y) - u^{m,\nu}(s,\theta_{s,t}^m(x'))) dy ds \right| \\
&\leq 2C \|\nabla \cdot b_m\|_{L^\infty} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \int_{t_0}^t [\nu(t-s)]^{\frac{\gamma}{2}} ds,
\end{aligned}$$

by similar absorbing arguments previously performed; and finally

$$|R_{2,A^c}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=x,\xi'=x'} \leq \frac{m^{1+\beta}}{1+\gamma} C \nu^{\frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} |x-x'|^{\frac{2+\gamma}{2\alpha_2}}, \quad (4.37)$$

which yields the same constraints as (4.36).

4.2.5 Sum up on the constraints on α_1, α_2

Recalling from (4.36)

$$\begin{aligned} \alpha_2 &= \frac{2+\gamma}{2\gamma}, \\ 0 &< \frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}, \end{aligned} \quad (4.38)$$

and from (4.30), that we recall

$$\begin{aligned} \alpha_1 &> \frac{1}{2(2-\gamma)}, \\ \alpha_2 &> -\frac{1}{2(2-\gamma)}, \end{aligned} \quad (4.39)$$

the second is satisfied by the choice $\alpha_2 = \frac{2+\gamma}{2\gamma} > 0 > -\frac{1}{2(2-\gamma)}$.

Next, combining (4.38) with (4.39):

$$\frac{\gamma}{2(2-\gamma)} < \alpha_1 \gamma < \frac{\gamma}{2}. \quad (4.40)$$

This possible if

$$1 < 2 - \gamma \iff \gamma < 1,$$

which means that there is no possibility, with our strategy, to obtain a suitable Lipschitz control of the solution $u^{m,\nu}$.

Furthermore, we also need to suppose that

$$\alpha_1 < \frac{1}{2}. \quad (4.41)$$

4.2.6 Choice of α_1

Finally, we calibrate α_1 such that the “worst” contribution of ν in the *diagonal* regime in (4.29) matches with the *off-diagonal* one in (4.35), namely

$$-\frac{1}{2} + \alpha_1(2-\gamma) = \frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2},$$

as $\frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2} = \frac{\gamma}{2} - \alpha_1\gamma$, we deduce

$$\alpha_1 = \frac{1+\gamma}{4}. \quad (4.42)$$

We point out that the constraint (4.41) is indeed satisfied.

Let us detail that, with the choice (4.42), the conditions in (4.40) are satisfied, namely that $\frac{\gamma}{2(2-\gamma)} < \frac{\gamma(1+\gamma)}{4} < \frac{\gamma}{2}$, the second inequality is direct as $\frac{1+\gamma}{2} < 1$ for any $\gamma < 1$. In order to prove the first inequality, we equivalently need to get $\Gamma(\gamma) := (2-\gamma)(1+\gamma) > 2$. Differentiating this function readily gives $\Gamma'(\gamma) = -1 - \gamma + 2 - \gamma = 1 - 2\gamma$, hence $\inf_{\gamma \in (0,1)} \Gamma(\gamma) = \min_{\gamma \in [0,1]} \Gamma(\gamma) = 2$.

Remark 6. *The condition to have a diagonal regime is then $|x-x'| \leq \nu^{\frac{1+\gamma}{4}}(t-s)^{\frac{2+\gamma}{2\gamma}}$, which differs from the usual parabolic scale where (α_1, α_2) is replaced by $(\frac{1}{2}, \frac{1}{2})$; but recalling that $\alpha_1 = \frac{1}{2}$ is not allowed in (4.41). We do not seek for any parabolic bootstrap of regularity, unlike [CDRHM18], our goal is above all to control as sharp as possible the dependency on ν .*

4.2.7 Comments on the necessity of using the norm $\|u^{m,\nu}\|_{L^\infty(C^1)}$

At this stage, we can justify why we upper-bound by the blowing-up term $\|u^{m,\nu}\|_{L^\infty(C^1)}$ in the *diagonal* regime, which is overwhelmed by the viscosity ν but prevents to get uniqueness, see Remark 5; instead of the well-controlled $\|u^{m,\nu}\|_{L^\infty(C^\gamma)}$.

Let us rewrite one of the term of $R_1^{\tau,\xi,\xi'}(t, x, x')$ in r.h.s. in (4.27),

$$\begin{aligned}
& \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot (b_m(s, x' + y) - b_m(s, x + y)) (u^{m,\nu}(s, x' + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) dy ds \right|_{\tau=t, \xi=\xi'=x} \\
& \leq C|x - x'| \|b_m\|_{L^\infty(C^1)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \hat{p}^{\tau,\xi}(s, t, 0, y) \times |\theta_{s,\tau}^m(\xi) - x' - y|^\gamma dy ds \\
& \leq C|x - x'| \|b_m\|_{L^\infty(C^1)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left([\nu(t-s)]^{\frac{\gamma-1}{2}} + [\nu(t-s)]^{-\frac{1}{2}} |x - x'|^\gamma \right) ds \\
& \leq C|x - x'|^\gamma \nu^{\alpha_1(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \\
& \quad \times (t-s)^{(1-\gamma)\alpha_2} \left([\nu(t-s)]^{\frac{\gamma-1}{2}} + [\nu(t-s)]^{-\frac{1}{2}} \nu^{\alpha_1\gamma} (t-s)^{\alpha_2\gamma} \right) ds \\
& = C|x - x'|^\gamma \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} \\
& \quad \times \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left((t-s)^{(1-\gamma)\alpha_2 + \frac{\gamma-1}{2}} \nu^{\alpha_1(1-\gamma) + \frac{\gamma-1}{2}} + \nu^{\alpha_1 - \frac{1}{2}} (t-s)^{\alpha_2 - \frac{1}{2}} \right) ds.
\end{aligned}$$

Then the required, assumption on parameters is for this control

$$\alpha_1 > \frac{1}{2}, \quad \alpha_2 > -\frac{1}{2},$$

which combined with the constraint on the *off-diagonal* regime (4.36) is absurd.

We could consider the case $\alpha_1 = \frac{1}{2}$, but this case yields no viscosity contribution and make the previous upper-bounds blowing up with m (except for the usual framework, i.e. for b Lipschitz continuous); there is no possibility to obtain a *regularisation by turbulence* for such a choice.

If we suppose that b is γ -Hölder in space, in order to avoid any blowing-up in m , from (4.27), we can write

$$\begin{aligned}
& |R_1^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
& \leq C|x - x'|^\gamma \|b\|_{L^\infty(C^\gamma)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \hat{p}^{\tau,\xi}(s, t, 0, y) \times |\theta_{s,\tau}^m(x) - x' - y|^\gamma dy ds \\
& \leq C|x - x'|^\gamma \|b\|_{L^\infty(C^\gamma)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left([\nu(t-s)]^{\frac{\gamma-1}{2}} + [\nu(t-s)]^{-\frac{1}{2}} |x - x'|^\gamma \right) ds \\
& \leq C|x - x'|^\gamma \|b\|_{L^\infty(C^\gamma)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left([\nu(t-s)]^{\frac{\gamma-1}{2}} + \nu^{-\frac{1}{2}} (s-t)^{-\frac{1}{2}} \nu^{\alpha_1\gamma} (t-s)^{\alpha_2\gamma} \right) ds,
\end{aligned}$$

which goes to $+\infty$ when $\nu \rightarrow 0$, except if $\gamma = 1$. In other words, we need to consider the norms $\|b_m\|_{L^\infty(C^1)}$ and $\|u^{m,\nu}\|_{L^\infty(C^1)}$ on the one hand to smoothen the blowing-up in ν and to get the suitable $|x - x'|$ which allows to overwhelm ν in the *diagonal* regime.

Finally, we could rewrite the analysis performed before this current section without doing an integration by parts in (4.19), and by upper-bounding with $\|\nabla u^{m,\nu}\|_{L^\infty}$. We choose to keep this separation of the remainder term defined in (4.19) in order to track precisely where the regime helps us to overuse the suitable *a priori* regularity of $u^{m,\nu}$ in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$.

4.3 On the discontinuous freezing choice

Let us carefully point out that even if the solution $u^{m,\nu}$ does not depend on the corresponding freezing parameter ξ , the choice of ξ in this section does depend on the current time variable of integration s .

Therefore, like for the approach developed in [CDRHM18], the *cut locus* point yields an additional contribution.

Previously, in the Hölder norm controls, we considered two points $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$. Let us specify how to write the solution $u^{m,\nu}(t, x')$ with the different choices of freezing parameter ξ' depending on the time variable of integration s . To do so, we first rewrite the theoretical representation of the solution where the time horizon is $r \in [0, T]$, and the initial function is replaced by $u^{m,\nu}(r, x')$,

$$u^{m,\nu}(t, x') = P_r^{m,\nu} u^{m,\nu}(r, x') + G_r^{m,\nu} f_m(t, x'), \quad (4.43)$$

where $P_{r,t}^{m,\nu}$ and $G_{r,t}^{m,\nu}$ stand respectively for the semi-group and the Green operator associated with the Cauchy problem

$$\begin{cases} \partial_t v^{m,\nu}(t, x) + \langle b_m(t, x), \nabla v^{m,\nu}(t, x) \rangle - \nu \Delta v^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in (r, T] \times \mathbb{R}^d, \\ v^{m,\nu}(r, x) = u^{m,\nu}(r, x), & x \in \mathbb{R}^d. \end{cases} \quad (4.44)$$

We also write

$$u^{m,\nu}(t, x) = \hat{P}_r^{\tau,\xi} g_m(t, x) + \hat{G}_r^{\tau,\xi} f_m(t, x) + \hat{G}_r^{\tau,\xi} (b_\Delta^m[\tau, \xi] \cdot \nabla u^{m,\nu})(t, x), \quad (4.45)$$

where the operators are defined by

$$\forall (t, x) \in (0, T] \times \mathbb{R}^d, \quad \hat{G}_r^{\tau,\xi} f_m(t, x) := \int_r^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) f(s, y) dy ds, \quad (4.46)$$

and

$$\hat{P}_r^{\tau,\xi} g_m(t, x) := \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(r, t, x, y) g_m(y) dy. \quad (4.47)$$

Let us define the transition time

$$t_0 := t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}} = t - \nu^{-\frac{\gamma(1+\gamma)}{2(2+\gamma)}} |x - x'|^{\frac{2\gamma}{2+\gamma}}. \quad (4.48)$$

If $t_0 \leq 0 \Leftrightarrow t \leq \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}$, the *off-diagonal* regime is in force, then we pick $\xi' = x'$ and there is no intricate choice of the *freezing* parameter.

However, if $t_0 > 0 \Leftrightarrow t > \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}$, we need to be more subtle to handle with the dependency on s for the value choice of $\xi' \in \mathbb{R}^d$. From now on, we suppose that $t_0 > 0$.

We next differentiate (4.43) w.r.t. r

$$0 = \partial_r (\hat{P}_r^{\tau,\xi'} u^{m,\nu}(r, \cdot))(t, x') + \partial_r \hat{G}_r^{\tau,\xi'} f_m(t, x') + \partial_r \hat{G}_r^{\tau,\xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m,\nu})(t, x). \quad (4.49)$$

We integrate the variable r between $[t_0, t]$ with the *proxy* parameter $\xi' \in \mathbb{R}^d$,

$$0 = u^{m,\nu}(t, x') - [P_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0, \cdot)](t, x') - G_{t_0}^{\tau,\xi'} f_m(t, x') - \hat{G}_{t_0}^{\tau,\xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m,\nu})(t, x),$$

which yields for $t \in [t_0, T]$

$$u^{m,\nu}(t, x') = [P_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0, \cdot)](t, x') + G_{t_0}^{\tau,\xi'} f_m(t, x') + \hat{G}_{t_0}^{\tau,\xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m,\nu})(t, x). \quad (4.50)$$

Next, we integrate in time between $[0, t_0]$ with a different *freezing* parameter $\tilde{\xi}' \in \mathbb{R}^d$,

$$\begin{aligned} 0 &= [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') - \hat{P}_0^{\tau,\tilde{\xi}'} g_m(x') + \hat{G}_{t_0}^{\tau,\tilde{\xi}'} f_m(t, x') - \hat{G}_0^{\tau,\tilde{\xi}'} f_m(t, x') \\ &\quad + \hat{G}_{t_0}^{\tau,\tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m,\nu})(t, x) - \hat{G}_0^{\tau,\tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m,\nu})(t, x). \end{aligned}$$

Hence,

$$\begin{aligned}
u^{m,\nu}(t, x') &= [P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') + G_{t_0}^{\tau, \xi'} f_m(t, x') + \hat{G}_{t_0}^{\tau, \xi'} (b_{\Delta}^m[\tau, \xi'] \cdot \nabla u^{m,\nu})(t, x) \\
&\quad - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') + \hat{P}_0^{\tau, \tilde{\xi}'} g_m(x') - \hat{G}_{t_0}^{\tau, \tilde{\xi}'} f_m(t, x') + \hat{G}_0^{\tau, \tilde{\xi}'} f_m(t, x') \\
&\quad - \hat{G}_{t_0}^{\tau, \tilde{\xi}'} (b_{\Delta}^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m,\nu})(t, x) + \hat{G}_0^{\tau, \tilde{\xi}'} (b_{\Delta}^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m,\nu})(t, x).
\end{aligned}$$

Defining

$$\forall (t', x) \in [0, t] \times \mathbb{R}^d, \quad \hat{G}_{r, t'}^{\tau, \xi} f_m(t, x) := \int_r^{t'} \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) f(s, y) dy ds, \quad (4.51)$$

we can write

$$\begin{aligned}
&u^{m,\nu}(t, x') \\
&= [P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') + G_{t_0}^{\tau, \xi'} f_m(t, x') \\
&\quad + \hat{G}_{t_0}^{\tau, \xi'} (b_{\Delta}^m[\tau, \xi'] \cdot \nabla u^{m,\nu})(t, x) + \hat{P}_0^{\tau, \tilde{\xi}'} g_m(x') + \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') + \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} (b_{\Delta}^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m,\nu})(t, x).
\end{aligned} \quad (4.52)$$

There is an extra contribution $[P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x')$ due to the discontinuous *freezing* choice, the others terms match with the ones appearing our above computations.

4.4 Extra contribution $[P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x')$

Thanks to a change of variables, we readily obtain

$$\begin{aligned}
&[P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') \\
&= \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(t_0, t, x', y) u^{m,\nu}(t_0, y) dy - \int_{\mathbb{R}^d} \hat{p}^{\tau, \tilde{\xi}'}(t_0, t, x', y) u^{m,\nu}(t_0, y) dy \\
&= \int_{\mathbb{R}^d} \tilde{p}(t_0, t, x', y) \left[u^{m,\nu}(t_0, y + \int_{t_0}^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\xi')) d\tilde{s}) dy - u^{m,\nu}(t_0, y + \int_{t_0}^t b_m(\tilde{s}, \theta_{\tilde{s}, \tilde{\tau}'}^m(\tilde{\xi}') d\tilde{s}) \right] dy,
\end{aligned}$$

recalling that $\tilde{p}(t_0, t, x', y)$ stands for the usual heat kernel defined in (4.15). Therefore,

$$\begin{aligned}
&\left| [P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') \right| \\
&\leq \|u^{m,\nu}\|_{L^\infty(C^1)} \int_{t_0}^t |b_m(\tilde{s}, \theta_{\tilde{s}, \tau}^m(\xi')) - b_m(\tilde{s}, \theta_{\tilde{s}, \tau}^m(\tilde{\xi}'))| d\tilde{s} \\
&\leq \|u^{m,\nu}\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_{t_0}^t |\theta_{\tilde{s}, \tau'}^m(\xi') - \theta_{\tilde{s}, \tau}^m(\tilde{\xi}')| d\tilde{s}.
\end{aligned}$$

We have, see Lemma 2 in Appendix Section A.4, for any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$:

$$\sup_{\tilde{s} \in [0, \tau]} |\theta_{\tilde{s}, \tau}^m(x) - \theta_{\tilde{s}, \tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau).$$

We then deduce for $(\tau, \xi', \tilde{\xi}') = (t, x, x')$ that

$$\begin{aligned}
&\left| [P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x, x')} \\
&\leq \|u^{m,\nu}\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_{t_0}^t |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} t) d\tilde{s} \\
&\leq \|u^{m,\nu}\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} |x - x'| (t - t_0) \exp(\|b_m\|_{L^\infty(C^1)} t) d\tilde{s}.
\end{aligned}$$

Consequently,

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x, x')} \\ & \leq m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} |x - x'| (t - t_0) \exp(\|b_m\|_{L^\infty(C^1)} t). \end{aligned} \quad (4.53)$$

Because $|x - x'| = \nu^{\alpha_1} (t - t_0)^{\alpha_1}$, we get

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x, x')} \\ & \leq |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{(1-\gamma)\alpha_1} (t - t_0)^{(1-\gamma)\alpha_2 + 1} \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & \leq t^{(1-\gamma)\alpha_2 + 1} |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{(1-\gamma)\alpha_1} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t), \end{aligned}$$

as we have supposed that $t_0 \geq 0$. Taking,

$$(\alpha_1, \alpha_2) = \left(\frac{1 + \gamma}{4}, \frac{2 + \gamma}{2\gamma} \right),$$

we finally derive,

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x, x')} \\ & \leq \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1} |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & = \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1} |x - x'|^\gamma m^{1+\beta} O_m(2t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}. \end{aligned} \quad (4.54)$$

From definition of $O_m(t)$ in (4.28), we obtain the vanishing condition

$$\nu \ll (m^{1+\beta-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1})^{-\frac{4}{1-\gamma^2}} \exp\left(-8 \frac{m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} T}{1-\gamma^2}\right). \quad (4.55)$$

4.5 Justification of the freezing point change

For any $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and $(\tau, \xi, \xi', \tilde{\xi}') \in [0, T] \times \mathbb{R}^3$, we write from (4.52),

$$\begin{aligned} u^{m, \nu}(t, x) - u^{m, \nu}(t, x') &= \left[\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \tilde{\xi}'} g_m(t, x') \right] + \left[\hat{G}_{t_0, t}^{\tau, \xi} f_m(t, x) - \hat{G}_{t_0, t}^{\tau, \xi'} f_m(t, x') \right] \\ &+ \left[\hat{G}_{0, t_0}^{\tau, \xi} f_m(t, x) - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\ &+ \left[\tilde{G}_{t_0, t}^{\tau, \xi} [b_\Delta^m[\tau, \xi] \cdot \nabla u^{m, \nu}](t, x) - \tilde{G}_{t_0, t}^{\tau, \xi'} [b_\Delta^{m'}[\tau, \xi'] \cdot \nabla u^{m, \nu}](t, x') \right] \\ &+ \left[\tilde{G}_{0, t_0}^{\tau, \xi} [b_\Delta^m[\tau, \xi] \cdot \nabla u^{m, \nu}](t, x) - \tilde{G}_{0, t_0}^{\tau, \xi'} [b_\Delta^{m'}[\tau, \xi'] \cdot \nabla u^{m, \nu}](t, x') \right] \\ &+ [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x'). \end{aligned}$$

Because the l.h.s. of the first equality does not depend on (ξ, ξ') , we can get the infimum over these *freezing* points, namely

$$\begin{aligned}
u^{m,\nu}(t, x) - u^{m,\nu}(t, x') &= \inf_{\xi, \xi', \tilde{\xi}' \in \mathbb{R}} \left\{ \left[\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \tilde{\xi}'} g_m(t, x') \right] + \left[\hat{G}_{t_0, t}^{\tau, \xi} f_m(t, x) - \hat{G}_{t_0, t}^{\tau, \xi'} f_m(t, x') \right] \right. \\
&\quad + \left[\hat{G}_{0, t_0}^{\tau, \xi} f_m(t, x) - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\
&\quad + \left[\tilde{G}_{t_0, t}^{\tau, \xi} [b_{\Delta}^m[\tau, \xi] \cdot \nabla u^{m,\nu}](t, x) - \tilde{G}_{t_0, t}^{\tau, \xi'} [b_{\Delta}^{m'}[\tau, \xi'] \cdot \nabla u^{m,\nu}](t, x') \right] \\
&\quad + \left[\tilde{G}_{0, t_0}^{\tau, \xi} [b_{\Delta}^m[\tau, \xi] \cdot \nabla u^{m,\nu}](t, x) - \tilde{G}_{0, t_0}^{\tau, \xi'} [b_{\Delta}^{m'}[\tau, \xi'] \cdot \nabla u^{m,\nu}](t, x') \right] \\
&\quad \left. + [P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') \right\}.
\end{aligned}$$

However, we aim to control the source functions term only in the *diagonal* regime, see Section 4.5.2 below for details.

From definitions (4.25), (4.26), (4.33) and (4.34), we have $\hat{G}^{\tau, \tilde{\xi}'} f_m(t, x') = \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') + \hat{G}_{t_0, t}^{\tau, \tilde{\xi}'} f_m(t, x')$, and we can rewrite,

$$\begin{aligned}
&\left[\hat{G}_{t_0, t}^{\tau, \xi} f_m(t, x) - \hat{G}_{t_0, t}^{\tau, \xi'} f_m(t, x') \right] + \left[\hat{G}_{0, t_0}^{\tau, \xi} f_m(t, x) - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\
&= \left[\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \tilde{\xi}'} f_m(t, x') \right] + \left[\hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right],
\end{aligned}$$

which readily gives

$$\begin{aligned}
u^{m,\nu}(t, x) - u^{m,\nu}(t, x') &= \inf_{\xi, \xi', \tilde{\xi}' \in \mathbb{R}} \left\{ \left[\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \tilde{\xi}'} g_m(t, x') \right] + \left[\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \right. \\
&\quad + \left[\hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\
&\quad + \left[\tilde{G}_{t_0, t}^{\tau, \xi} [b_{\Delta}^m[\tau, \xi] \cdot \nabla u^{m,\nu}](t, x) - \tilde{G}_{t_0, t}^{\tau, \xi'} [b_{\Delta}^{m'}[\tau, \xi'] \cdot \nabla u^{m,\nu}](t, x') \right] \\
&\quad + \left[\tilde{G}_{0, t_0}^{\tau, \xi} [b_{\Delta}^m[\tau, \xi] \cdot \nabla u^{m,\nu}](t, x) - \tilde{G}_{0, t_0}^{\tau, \xi'} [b_{\Delta}^{m'}[\tau, \xi'] \cdot \nabla u^{m,\nu}](t, x') \right] \\
&\quad \left. + [P_{t_0}^{\tau, \xi'} u^{m,\nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m,\nu}(t_0, \cdot)](t, x') \right\}. \tag{4.56}
\end{aligned}$$

Hence, taking $(\tau, \xi, \xi', \tilde{\xi}') = (t, x, x', x)$ yields the previous terms already controlled with a new extra contribution $\left[\hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right]$.

4.5.1 Control of the new extra contribution $\left[\hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right]$

This last extra term is tackled similarly as the first one in Section 4.4.

$$\begin{aligned}
&\left[\hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\
&= \int_0^{t_0} \int_{\mathbb{R}^d} \tilde{p}(s, t, x', y) \left[f_m(s, y + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\xi')) d\tilde{s}) - f_m(s, y + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s}, \tilde{\tau}'}^m(\xi')) d\tilde{s}) \right] dy ds.
\end{aligned}$$

We readily obtain,

$$\begin{aligned}
&\left| \hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right| \\
&\leq \|f_m\|_{L^\infty(C^1)} \int_0^{t_0} \left| \int_s^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\xi')) - b_m(\tilde{s}, \theta_{\tilde{s}, \tilde{\tau}'}^m(\xi')) d\tilde{s} \right| ds \\
&\leq \|f_m\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_0^{t_0} \int_s^t |\theta_{\tilde{s}, \tau'}^m(\xi') - \theta_{\tilde{s}, \tilde{\tau}'}^m(\tilde{\xi}')| d\tilde{s} ds.
\end{aligned}$$

We know that, for any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$:

$$\sup_{\tilde{s} \in [0, \tau]} |\theta_{s, \tau}^m(x) - \theta_{s, \tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau),$$

see Lemma 2 in Appendix Section A.4.

We then deduce for $(\tau, \xi', \tilde{\xi}') = (t, x, x')$ that

$$\begin{aligned} & \left| \hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x, x')} \\ & \leq \|f_m\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_0^{t_0} \int_s^t |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} t) d\tilde{s} ds \\ & \leq \frac{1}{2} m^{1-\gamma} \|f\|_{L^\infty(C^\gamma)} m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} |x - x'| t^2 \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t). \end{aligned} \quad (4.57)$$

From the definition of t_0 in (4.22), $|x - x'| = \nu^{\alpha_1} (t - t_0)^{\alpha_1}$, we get

$$\begin{aligned} & \left| \hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x, x')} \\ & \leq |x - x'|^\gamma \frac{\nu^{(1-\gamma)\alpha_1} t^{(1-\gamma)\alpha_2 + 2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t^2 \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) \\ & = |x - x'|^\gamma \frac{\nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t). \end{aligned}$$

Hence, the vanishing constraint is

$$\nu \ll (m^{2+\beta-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 2} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})})^{-\frac{4}{1-\gamma^2}} \exp\left(-4 \frac{m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} T}{1-\gamma^2}\right). \quad (4.58)$$

4.5.2 Comments on the choice of *freezing* point for the source functions terms

It is crucial to fix the same *freezing* point for the terms associated with source functions. In our context, it may be unavoidable. To fully explain this choice, let us develop the computations associated with these terms for the same choice of ξ and ξ' as for A in the *off-diagonal* regime, i.e. $\xi = x$ and $\xi' = x'$.

To deal with the semi-group, we can consider an analysis of the type (or equivalent controls),

$$\begin{aligned} & \left| \hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x') \right|_{\tau=t, \xi=x, \xi'=x'} \\ & = \left| \int_{\mathbb{R}^d} [\hat{p}^{\tau, \xi}(0, t, x, y) - \hat{p}^{\tau, \xi'}(0, t, x', y)] g_m(s, y) dy \right|_{\tau=t, \xi=x, \xi'=x'} \\ & = \left| \int_{\mathbb{R}^d} \tilde{p}(0, t, 0, y) [g_m(t, \theta_{0, t}^m(x) + y) - g_m(t, \theta_{0, t}^m(x') + y)] dy \right| \\ & \leq [g]_\gamma |\theta_{0, t}^m(x) - \theta_{0, t}^m(x')|^\gamma. \end{aligned}$$

Similarly for the Green operator,

$$\begin{aligned} & \left| \hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x') \right|_{\tau=t, \xi=\xi'=x} \\ & = \left| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{\tau, \xi}(s, t, x, y) - \hat{p}^{\tau, \xi'}(s, t, x', y)] f_m(s, y) dy ds \right|_{\tau=t, \xi=x, \xi'=x'} \\ & = \left| \int_0^t \int_{\mathbb{R}^d} [\tilde{p}(s, t, 0, y)] [f_m(s, \theta_{s, t}^m(x) + y) - f_m(s, \theta_{s, t}^m(x') + y)] dy ds \right| \\ & \leq \|f\|_{L^\infty(C^\gamma)} \int_0^t |\theta_{s, t}^m(x) - \theta_{s, t}^m(x')|^\gamma ds. \end{aligned}$$

In other words, we can see in both above controls that we only upper-bound by the flow associated with b_m which is *a priori* not controlled uniformly on m in a suitable spatial Hölder space.

4.5.3 Final Hölder control

Gathering (4.29) and (4.35) with

$$(\alpha_1, \alpha_2) = \left(\frac{1+\gamma}{4}, \frac{2+\gamma}{2\gamma} \right), \quad (4.59)$$

we have from the final Duhamel formula (4.52) combined with the estimates of each contribution stated (4.29), (4.31), (4.35), (4.37), (4.54), (4.57):

$$\begin{aligned} & \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\ & \leq t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + C\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{2+\beta} O_m(2t) \left(\nu^{\frac{1-\gamma^2}{4}} t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} + \nu^{\frac{\gamma(1-\gamma)}{4}} t^{\frac{1}{2} + \frac{4-\gamma^2}{2\gamma}} \right) \\ & \quad + \frac{m^{1+\beta}}{1+\gamma} C \nu^{\frac{\gamma(1-\gamma)}{4}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \\ & \quad + \frac{\nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t) \\ & \quad + \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1} m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(\|b_m\|_{L^\infty(C^1)} t), \end{aligned} \quad (4.60)$$

where we recall that

$$\begin{aligned} O_m(t) & = \left(t\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right) \\ & \leq C \left(m^{1-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \right) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right). \end{aligned}$$

Then for $\nu \ll T$ and because $\frac{1-\gamma^2}{4} > \frac{\gamma(1-\gamma)}{4}$ we obtain

$$\begin{aligned} \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} & \leq t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \\ & \quad + C \nu^{\frac{\gamma(1-\gamma)}{4}} m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \left(t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} O_m(2t) (1 + \|f\|_{L^\infty(C^\gamma)}) + \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \right). \end{aligned}$$

For a “small” $\kappa \in (0, 1)$, we choose

$$\nu \leq \kappa^{\frac{4}{\gamma(1-\gamma)}} \left(1 + C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \right)^{-\frac{4}{\gamma(1-\gamma)}}, \quad (4.61)$$

which yields by circular argument

$$\begin{aligned} & \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\ & \leq (1-\kappa)^{-1} \left(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + C \nu^{\frac{\gamma(1-\gamma)}{4}} m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} O_m(2t) (1 + \|f\|_{L^\infty(C^\gamma)}) \right). \end{aligned}$$

Finally, we also impose a given “small” $\tilde{\kappa} \in (0, 1)$,

$$\begin{aligned} \nu & \leq \tilde{\kappa}^{\frac{4}{\gamma(1-\gamma)}} \left(C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} T^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} O_m(2T) (1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\ & = \tilde{\kappa}^{\frac{4}{\gamma(1-\gamma)}} \left(C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} T^{\frac{2-\gamma^2+\gamma}{2\gamma}} (m^{1-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)) (1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\ & \quad \times \exp\left(-\frac{8m^{1+\beta} T \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}}{\gamma(1-\gamma)}\right), \end{aligned} \quad (4.62)$$

which obviously implies condition (4.61) for $\kappa = \tilde{\kappa}$.

Under assumptions (4.61) and (4.62), we deduce for any $(\kappa, \tilde{\kappa}) \in (0, 1)^2$

$$\|u^{m,\nu}\|_{L^\infty(C^\gamma)} \leq (1 - \kappa)^{-1}(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + \tilde{\kappa}), \quad (4.63)$$

the required Hölder control (3.1) is then established when $\kappa, \kappa' \rightarrow 0$ and $(m, \nu) \rightarrow (+\infty, 0)$ according to conditions (3.2), up to a convergence argument developed in Section 4.7.

4.6 Another control of uniform norm

In this section, we provide another way to get uniform control without stochastic representation; these computations lead some dependency on m overwhelmed by ν and can be useful in other contexts.

For the L^∞ , control we choose the *freezing* point as for the *diagonal* regime, i.e. $(\tau, \xi) = (t, x)$. The terms associated with the source functions are dealt easily:

$$\left| \hat{P}^{\tau,\xi} g_m(t, x) \right|_{\tau=t, \xi=x} \leq \|g\|_{L^\infty}, \quad \left| \hat{G}^{\tau,\xi} f_m(t, x) \right|_{\tau=t, \xi=x} \leq t\|f\|_{L^\infty}.$$

The remainder terms are controlled in a very similar way as previously in Section 4.2.4

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) \cdot \nabla u^{m,\nu}(s, y) dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \leq 2C \|b_m\|_{L^\infty(C^1)} \|\nabla u^{m,\nu}\|_{L^\infty} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) |y - \theta_{s,\tau}^m(\xi)|^\gamma dy ds \Big|_{\tau=t, \xi=x} \\ & \leq C m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla u^{m,\nu}\|_{L^\infty} \int_0^t (\nu(t-s))^{\frac{\gamma}{2}} ds, \end{aligned}$$

by (2.9), and we conclude by (4.28) that

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) \cdot \nabla u^{m,\nu}(s, y) dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \leq C O_m(t) \frac{m^{1+\beta} t^{1+\frac{\gamma}{2}}}{2+\gamma} \nu^{\frac{\gamma}{2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}. \end{aligned}$$

4.7 Compactness arguments

4.7.1 Mild vanishing viscous

In order to pass to the limit $m \rightarrow +\infty$, $\nu \rightarrow 0$, according to the vanishing condition (3.2), we consider a subsequence given by the usual Arzelà-Ascoli theorem. However, this former result is available for uniform continuous function in a compact space. From the lack of continuity, uniformly on ν , in space of $u^{m,\nu}(t, \cdot)$ (only γ -Hölder continuous, $\gamma < 1$), we are stuck at a convergence in a compact subset of \mathbb{R}^d . For instance, the analysis performed in [Hon22] to get rid of the compactness convergence criterion for quasi-linear equations does not work here as there is no hope to obtain any strong formulation of the PDE (1.1).

We do not succeed to obtain any positive regularity on t , and so we cannot exploit uniform continuity in time to get a convergence of a sub-sequence of $u^{m,\nu}$ in $[0, T] \times \mathbb{R}^d$. Thus the convergence at any given time in all compacts set of the *mild vanishing viscous* solution in Theorem 2.

Nevertheless, we still can include a truncation procedure into a weak formulation in order to obtain a convergence in a distributional meaning and not a point-wise one as for the *mild vanishing viscous* solution.

4.7.2 Truncation procedure

The method is highly inspired by the one in [Hon22], we also consider a smooth cut-off $\vartheta_{y,R} \in \mathcal{D}$ supported in a ball $B_d(y, R) = \{x \in \mathbb{R}^d; |x - y| \leq R\}$, $y \in \mathbb{R}^d$ and defined by

$$\vartheta_{y,R}(x) = \vartheta_y\left(\frac{x}{R}\right), \quad (4.64)$$

where $\vartheta_y : \mathbb{R}^d \rightarrow [0, 1]^d$ is function lying in $C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ s.t.

$$\vartheta_y(x) = \begin{cases} x, & \text{if } |x - y| < 1, \\ 0, & \text{if } |x - y| > 2. \end{cases}$$

The corresponding truncated function is, for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$u_{y,R}^{m,\nu}(t, x) := u^{m,\nu}(t, \vartheta_{y,R}(x)). \quad (4.65)$$

We highlight the particular case

$$u_{x,R}^{m,\nu}(t, x) = u^{m,\nu}(t, x). \quad (4.66)$$

The above truncation solution (4.65) naturally appears when we write a weak formulation of the parabolic equation (2.12).

4.7.3 Weak solution of the parabolic approximating equation

For any smooth function φ_R supported on $B_d(0, R)$, a d-ball of radius $R > 0$ and center $(0, \dots, 0) \in \mathbb{R}^d$. We consider a weak formulation of the parabolic solution $u^{m,\nu}$, for any $(t, x) \in [0, T] \times \mathbb{R}^d$:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \left\{ \partial_t \varphi_R(s, y) u^{m,\nu}(s, y) + \varphi_R(s, y) \langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle + \nu \Delta \varphi_R(s, y) u^{m,\nu}(s, y) \right\} dy ds \\ &= \int_{\mathbb{R}^d} \varphi_R(0, y) g_m(y) dy - \int_{\mathbb{R}^d} \varphi_R(t, y) u^{m,\nu}(t, y) dy + \int_0^t \int_{\mathbb{R}^d} \varphi_R(s, y) f_m(s, y) dy ds, \end{aligned}$$

where in l.h.s. the limit of the first order term, $\langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle$, has *a priori* no point-wise limit neither in term of the usual distributional meaning of Schwartz. Indeed, as already enunciated in Section 3.1, the usual distribution theory does not provide any interpretation of a product of distributions, to get any limit result we have to thoroughly use the PDE.

By the cut-off definition, we equivalently have

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \left\{ \partial_t \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) + \varphi_R(s, y) \langle b_m(s, y), \nabla u_{0,R}^{m,\nu}(s, y) \rangle + \nu \Delta \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) \right\} dy ds \\ &= \int_{\mathbb{R}^d} \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) dy - \int_{\mathbb{R}^d} \varphi_R(0, y) g_m(y) dy + \int_0^t \int_{\mathbb{R}^d} \varphi_R(s, y) f_m(s, y) dy ds. \end{aligned} \quad (4.67)$$

Now, from compact argument developed in Section 4.7.1, we have that $u_{0,R}^{m,\nu}(s, \cdot)$ converges in $C_b^\gamma(K, \mathbb{R}^d)$, $K = B_d(0, R)$, towards a function $u_{0,R}(s, \cdot)$ when $(m, \nu) \rightarrow (+\infty, 0)$ and the condition (3.2) is satisfied. In other words, $u_{0,R}(s, \cdot)$ is a *mild vanishing viscous* solution of (1.1).

4.7.4 Mild-weak solution of the transport equation

To get a *mild-weak* solution we have to pass to the limit in the weak formulation (4.67) of the mollified parabolic equation (2.12). In equation (4.67), up to a sub-sequence selection, except for the first order

term $\varphi_R(s, y)\langle b(s, y), \nabla u_{0,R}^{m,\nu}(s, y) \rangle$, each contribution obviously has the good converge property by the Arzelà-Ascoli theorem. In particular, from (4.4), we have

$$\nu \int_0^t \int_{\mathbb{R}^d} \Delta \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) dy ds \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(3.2)} 0. \quad (4.68)$$

To deal with the *drift* part, we write by integration by parts

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle b_m(t, y), \nabla u_{0,R}^{m,\nu}(t, y) \rangle \varphi_R(t, y) dy \\ &= \int_{\mathbb{R}^d} \nabla \varphi_R(t, y) \cdot b_m(t, y) u_{0,R}^{m,\nu}(t, y) dy + \int_{\mathbb{R}^d} \varphi_R(t, y) \nabla \cdot b_m(t, y) u_{0,R}^{m,\nu}(t, y) dy \\ &=: B_1 + B_2. \end{aligned} \quad (4.69)$$

By the Besov duality property, see Proposition 1, we can write

$$|B_1| \leq \|b_m\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)}, \quad (4.70)$$

and

$$|B_2| \leq \|\nabla \cdot b_m\|_{L^\infty(B_{\infty,\infty}^{-1-\beta})} \|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^{1+\beta})}. \quad (4.71)$$

Control of $\|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)}$

Let us prove that $\|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)}$ is controlled uniformly in (m, ν) . By the thermic representation of the Besov norms (2.4), we have

$$\begin{aligned} \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)} &= \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^1} + \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^\beta)} \\ &= \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^1} + \int_0^1 \frac{1}{v} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) u_{0,R}^{m,\nu}(t, y) \nabla \varphi_R(t, y) dy \right| dz dv. \end{aligned}$$

The first contribution in the r.h.s. above is obviously bounded uniformly in (m, ν) by

$$\|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^1} \leq \|\nabla \varphi_R\|_{L^1} \|u_{0,R}^{m,\nu}\|_{L^\infty} \leq C \|\nabla \varphi_R\|_{L^1} (T \|f\|_{L^\infty} + \|g\|_{L^\infty}),$$

by uniform estimate (4.4).

For the second one, we need to deeply use the already known regularity of $u_{0,R}^{m,\nu}$. By cancellation and by triangular inequality, we obtain

$$\begin{aligned} \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^\beta)} &\leq \int_0^1 \frac{1}{v} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) \cdot \{[u_{0,R}^{m,\nu}(t, y) - u_{0,R}^{m,\nu}(t, z)] \nabla \varphi_R(t, y) \right. \\ &\quad \left. + u_{0,R}^{m,\nu}(t, z) [\nabla \varphi(t, y) - \nabla \varphi(t, z)] \right\} dy \Big| dz dv \\ &\leq \int_0^1 \frac{1}{v} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) \cdot \{[u_{0,R}^{m,\nu}(t, y) - u_{0,R}^{m,\nu}(t, z)] \nabla \varphi_R(t, y) \right. \\ &\quad \left. + u_{0,R}^{m,\nu}(t, z) (y-z) \cdot \int_0^1 D^2 \varphi_R(t, z + \mu(y-z)) d\mu \right\} dy \Big| dz dv, \end{aligned}$$

by Taylor expansion. Next, with the exponential absorbing property (2.9),

$$\begin{aligned} & \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^\beta)} \\ &\leq C \|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) |y-z|^\gamma |\nabla \varphi_R(t, y)| dy dz dv \\ &\quad + C \|u_{0,R}^{m,\nu}\|_{L^\infty} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) |y-z| \times \left| \int_0^1 D^2 \varphi_R(t, z + \mu(y-z)) d\mu \right| dy dz dv \\ &\leq C \|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)} \|\nabla \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\beta}{2}} dv + C \|u_{0,R}^{m,\nu}\|_{L^\infty} \|D^2 \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{1-\beta}{2}} dv, \end{aligned}$$

which is finite if $\beta < \gamma$.

Control of $\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^{1+\beta})}$

The analysis is similar as before, replacing β by $1 + \beta$ and $\nabla\varphi_R$ by φ_R :

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^{1+\beta})} = \|\varphi_R u_{0,R}^{m,\nu}\|_{L^1} + \|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^{1+\beta})}.$$

We readily get

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^1} \leq \|\varphi_R\|_{L^1} \|u_{0,R}^{m,\nu}\|_{L^\infty} \leq \|\varphi_R\|_{L^1} (T\|f\|_{L^\infty} + \|g\|_{L^\infty}),$$

and

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^{1+\beta})} \leq C \|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)} \|\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\beta-1}{2}} dv + C \|u_{0,R}^{m,\nu}\|_{L^\infty} \|\nabla\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{-\beta}{2}} dv,$$

this is finite if $1 + \beta < \gamma \iff \beta < -1 + \gamma$. Let us carefully notice that if there is the incompressible assumption $\nabla \cdot b = 0$, then $B_2 = 0$ and this former constraint disappears. Thus the different cases considered in Theorem 2.

The Bolzano-Weierstrass theorem then yields the result.

4.7.5 weak solution

The difficulty for the usual weak solution, here, is to prove that, up to a subsequence extraction,

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{\mathbb{R}^d} \int_0^T \langle b_m(t, y), \nabla u_{0,R}^{m,\nu}(t, y) \rangle \varphi_R(t, y) dy dt &= \int_0^T \int_{\mathbb{R}^d} \langle b(t, y), \nabla u_{0,R}(t, y) \rangle \varphi_R(t, y) dy dt \\ &= \int_0^T \int_{\mathbb{R}^d} \langle b(t, y), \nabla u(t, y) \rangle \varphi_R(t, y) dy dt, \end{aligned} \quad (4.72)$$

where $b \in L^\infty([0, T]; \tilde{B}_{\infty,\infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}))$ is the drift of the initial Cauchy problem (1.1) and coincides with the limit of b_m in $L^\infty([0, T]; B_{\infty,\infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}))$ for any $0 < \varepsilon$ when $m \rightarrow \infty$; also $u(s, \cdot) \in C_b^\gamma(K, \mathbb{R})$, $K = B_d(0, R)$, is the limit of $u^{m,\nu}(s, \cdot)$, up to a subsequence extraction possibly depending on the current time s , in $C_b^{\gamma-\tilde{\varepsilon}}(\mathbb{R}^d, \mathbb{R})$ for any $0 < \tilde{\varepsilon} < \gamma$, see Section 4.7.1.

Let us recall that $\tilde{B}_{\infty,\infty}^{-\beta}$ is the closure space of C_b^∞ in $B_{\infty,\infty}^{-\beta}$, from Appendix Section D, we still can take the regular sequence $(b_m)_{m \geq 1}$ defined in (4.2) to approximate b in $L^\infty([0, T]; B_{\infty,\infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}))$, for any $0 < \varepsilon$; whereas the considered solution $u(s, \cdot)$ may depend on the choice of mollification. In other words, we have:

$$\lim_{m \rightarrow \infty} \|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} \leq C \lim_{m \rightarrow \infty} \|\nabla \cdot b_m - \nabla \cdot b\|_{L^\infty(B_{\infty,\infty}^{-1-\beta-\varepsilon})} = 0. \quad (4.73)$$

For all $m > 0$, $t \in [0, T]$, we write by integration by parts that

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \langle b_m(t, y), \nabla u_{0,R}^{m,\nu}(t, y) \rangle \varphi(t, y) dy - \int_{\mathbb{R}^d} \langle b(t, y), \nabla u_{0,R}(t, y) \rangle \varphi(t, y) dy \right| \\ & \leq \left| \int_{\mathbb{R}^d} [b_m - b](t, y) u^{m,\nu}(t, y) \cdot \nabla \varphi_R(t, y) dy \right| + \left| \int_{\mathbb{R}^d} b(t, y) [u - u^{m,\nu}](t, y) \cdot \nabla \varphi_R(t, y) dy \right| \\ & \quad + \left| \int_{\mathbb{R}^d} \nabla \cdot [b_m - b](t, y) u^{m,\nu}(t, y) \varphi_R(t, y) dy \right| + \left| \int_{\mathbb{R}^d} \nabla \cdot b(t, y) [u - u^{m,\nu}](t, y) \varphi_R(t, y) dy \right| \\ & =: \tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_4. \end{aligned} \quad (4.74)$$

To deal with the first contribution, we aim to use (4.73). We have by the Besov duality result of Proposition 1:

$$\tilde{B}_1 \leq \|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} \|u^{m,\nu} \nabla \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}, \quad (4.75)$$

it then remains to control $\|u^{m,\nu} \nabla \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}$. Similar computations as in Section 4.7.4 yields that $\|u^{m,\nu} \nabla \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}$ is finite if $\beta < \gamma - \varepsilon$.

Also for the third term, which is null if $\nabla \cdot b = 0$, we have that

$$\tilde{B}_3 \leq \|\nabla \cdot b_m - \nabla \cdot b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} \|u^{m,\nu} \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}, \quad (4.76)$$

is finite if $\beta < -1 + \gamma - \varepsilon$.

Hence, from (4.73), we obtain

$$\tilde{B}_1 \xrightarrow[(m,\nu) \rightarrow (+\infty,0)]{(3.2)} 0, \quad \text{and} \quad \tilde{B}_3 \xrightarrow[(m,\nu) \rightarrow (+\infty,0)]{(3.2)} 0. \quad (4.77)$$

Now, let us handle with the second term in (4.74). We aim here to use the convergence of $u^{m,\nu}(s, \cdot)$ towards $u(s, \cdot)$ in the ball $B_d(0, R)$.

Again by the Besov duality result of Proposition 1, we have:

$$\begin{aligned} \tilde{B}_2 &= \left| \int_{\mathbb{R}^d} b(t, y) [u - u^{m,\nu}](t, y) \cdot \nabla \varphi_R(t, y) dy \right| \\ &\leq \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \| (u^{m,\nu} - u) \nabla \varphi_R \|_{L^\infty(B_{1,1}^\beta)}. \end{aligned} \quad (4.78)$$

By Arzelà-Ascoli theorem, we have

$$\|\nabla \varphi_R(u^{m,\nu} - u)\|_{L^\infty} \leq \|\nabla \varphi_R\|_{L^\infty} \| (u^{m,\nu} - u) \|_{L^\infty} \xrightarrow[(m,\nu) \rightarrow (+\infty,0)]{(3.2)} 0,$$

For the homogenous part of the Besov norm, we also mimic the analysis in the previous section replacing $u^{m,\nu}$ by $(u^{m,\nu} - u)$, for any $\varepsilon \in (0, \gamma)$:

$$\begin{aligned} &\| (u^{m,\nu} - u) \nabla \varphi_R \|_{L^\infty(\dot{B}_{1,1}^\beta)} \\ &\leq C \| (u^{m,\nu} - u) \|_{L^\infty(C^{\gamma-\varepsilon})} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) |y-z|^{\gamma-\varepsilon} |\nabla \varphi_R(t, y)| dy dz dv \\ &\quad + C \| (u^{m,\nu} - u) \|_{L^\infty} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) |y-z| \times \left| \int_0^1 D^2 \varphi_R(t, z + \mu(y-z)) d\mu \right| dy dz dv \\ &\leq C \| (u^{m,\nu} - u) \|_{L^\infty(C^{\gamma-\varepsilon})} \|\nabla \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\varepsilon-\beta}{2}} dv + C \| (u^{m,\nu} - u) \|_{L^\infty} \|D^2 \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{1-\beta}{2}} dv, \end{aligned}$$

which is finite as soon as $\beta < \gamma - \varepsilon < 0$, also by Arzelà-Ascoli theorem, we have the converging result $\| (u^{m,\nu} - u) \|_{L^\infty(C^{\gamma-\varepsilon})} \xrightarrow[(m,\nu) \rightarrow (+\infty,0)]{(3.2)} 0$. Therefore, we even get

$$\tilde{B}_2 \xrightarrow[(m,\nu) \rightarrow (+\infty,0)]{(3.2)} 0. \quad (4.79)$$

The last contribution \tilde{B}_4 , null if $\nabla \cdot b = 0$, is similar replacing $\nabla \varphi_R$ by φ_R and β by $\beta + 1$. Namely, we have

$$\begin{aligned} \tilde{B}_4 &\leq \|\nabla b\|_{L^\infty(B_{\infty,\infty}^{-1-\beta})} \| (u^{m,\nu} - u) \varphi_R \|_{L^\infty(B_{1,1}^{1+\beta})} \\ &= \|\nabla b\|_{L^\infty(B_{\infty,\infty}^{-1-\beta})} (\| (u^{m,\nu} - u) \varphi_R \|_{L^1} + \| (u^{m,\nu} - u) \varphi_R \|_{L^\infty(\dot{B}_{1,1}^{1+\beta})}), \end{aligned} \quad (4.80)$$

with

$$\|(u^{m,\nu} - u)\varphi_R\|_{L^1} \leq \|u^{m,\nu} - u\|_{L^\infty} \|\varphi_R\|_{L^1},$$

and

$$\begin{aligned} & \|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(\dot{B}_{1,1}^{1+\beta})} \\ &= \int_0^1 \frac{1}{v} v^{1-\frac{1+\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y)(u^{m,\nu} - u)(t,y)\varphi_R(t,y)dy \right| dz dv \\ &\leq C\|u^{m,\nu} - u\|_{L^\infty(C^{\gamma-\varepsilon})} \|\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\varepsilon-1-\beta}{2}} dv + C\|u^{m,\nu} - u\|_{L^\infty} \|D^2\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{-\beta}{2}} dv, \end{aligned}$$

which is finite if $\beta < -1 + \gamma - \varepsilon$; and by Arzelà-Ascoli theorem, we deduce

$$\tilde{B}_4 \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(3.2)} 0. \quad (4.81)$$

Hence, from (4.74), (4.77), (4.79) and (4.81) we conclude that the limit result (4.72) is true.

4.7.6 Control of $\|\partial_t u(t, \cdot)\|_{B_{\infty, \infty}^{-1+\gamma}}$

If $b \in L^\infty([0, T]; C^\alpha(\mathbb{R}^d, \mathbb{R}^d))$, $0 < 1 - \gamma < \alpha$, we derive an upper-bound of $\|\partial_t u(t, \cdot)\|_{B_{\infty, \infty}^{-1+\gamma}}$ by the equation (1.1) and by para-product result. But first of all, let us precise how point-wisely we have, with the viscous condition (3.2),

$$\lim_{(m,\nu) \rightarrow (+\infty, 0)} \nu \Delta u^{m,\nu}(t, \cdot) = 0. \quad (4.82)$$

In Appendix Section A.2, we establish that

$$\|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq \left(m^{2-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + Ctm^{2+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} O_m(t) \right) \exp(tm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}). \quad (4.83)$$

Hence, for

$$\nu \ll \left(m^{2-\gamma} (T\|f\|_{L^\infty} + [g]_\gamma) + CTm^{2+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} O_m(t) \right)^{-1} \exp(-Tm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}),$$

we deduce (4.82).

We can take the limit of equation (2.12), up to sub-sequence selection defined in the Arzelà-Ascoli theorem, for any $t \in (0, T]$

$$\lim_{(m,\nu) \rightarrow (+\infty, 0)} \partial_t u^{m,\nu}(t, \cdot) = \lim_{(m,\nu) \rightarrow (+\infty, 0)} \langle b_m(t, \cdot), \nabla u^{m,\nu}(t, \cdot) \rangle + f(t, \cdot). \quad (4.84)$$

But from paraproduct (3.5), we know that $\langle b_m(t, \cdot), \nabla u^{m,\nu}(t, \cdot) \rangle \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$, the result then follows.

5 Inviscid Burgers' equation

The controls (3.1) of the vanishing viscous solution(s) of the PDE (1.1) being independent on the first order term b , we can expect to obtain some fixed-point argument to consider that b being the solution u itself in dimension 1[§].

This Cauchy problem thus defined is called the inviscid Burgers' equation,

$$\begin{cases} \partial_t u(t, x) + u(t, x)\partial_x u(t, x) = f(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases} \quad (5.1)$$

Replacing b by u in the different definitions of solution in Section 2.3, we establish the second result of this paper.

[§]The analysis can be adapted for a more general dimension $d \geq 1$ up to some reformulation of $\langle u(t, x), \nabla u(t, x) \rangle$.

Theorem 3 (Existence of Hölder solution of the inviscid Burgers' equation). *For $\gamma \in (0, 1)$ be given. For all $f \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}, \mathbb{R})$, there is a mild vanishing viscosity solution $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}, \mathbb{R}))$ of (5.1) satisfying*

$$\begin{aligned} \|u\|_{L^\infty(C^\gamma)} &\leq T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma, \\ \|u\|_{L^\infty} &\leq T\|f\|_{L^\infty} + \|g\|_{L^\infty}, \end{aligned} \quad (5.2)$$

the condition on the vanishing viscosity is for a given constant C depending only on (γ, d) ,

$$\begin{aligned} \nu &\ll \left(m^{2-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma}+2} \|f\|_{L^\infty(C^\gamma)} (T\|f\|_{L^\infty} + \|g\|_{L^\infty}) \right)^{-\frac{4}{1-\gamma^2}} \exp\left(-4 \frac{m(T\|f\|_{L^\infty} + \|g\|_{L^\infty})T}{1-\gamma^2}\right), \\ \nu &\ll \left(Cm^2(T\|f\|_{L^\infty} + \|g\|_{L^\infty}) T^{\frac{2-\gamma^2+\gamma}{2\gamma}} (m^{1-\gamma}(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma))(1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\ &\quad \times \exp\left(-\frac{8mT(T\|f\|_{L^\infty} + \|g\|_{L^\infty})}{\gamma(1-\gamma)}\right). \end{aligned} \quad (5.3)$$

If $\gamma > \frac{1}{2}$ then the considered mild vanishing viscosity solution is also a mild-weak and a weak solution also if

$$\nu \ll (T\|f\|_{L^\infty} + [g]_\gamma)^{-1} \left(m^{2-\gamma} + CTm^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \right)^{-1} \exp(-2Tm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}), \quad (5.4)$$

then $\partial_t u(t, \cdot) \in B_{\infty,\infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$.

Let us remark that the last additional condition in Theorem 2 is satisfied if the considered *a priori* regularity of the solution is strong enough (*a priori* not the first condition as u is not incompressible, except if u is also solution of the Euler equation), as we have $-\gamma < -1 + \gamma \iff \gamma > \frac{1}{2}$.

Furthermore, the condition on the viscosity changes here comparing with Theorem 2, replacing $\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}$ by an upper-bound of $\|u\|_{L^\infty}$ by Feynman-Kac formula.

Remark 7. *Without considering the regularity condition $\gamma > \frac{1}{2}$ to get a weak solution, we may have pathologic situation. Specifically, let us consider steady-state the non-linear problem*

$$u(x)u'(x) = \frac{1}{2} \operatorname{sgn}(x),$$

whose $x \mapsto \sqrt{|x|}$ is solution which is as expected $\frac{1}{2}$ -Hölder continuous. In other words, if $\gamma = \frac{1}{2}$, we can find γ -Hölder steady-state solution of the inviscid Burgers' equation with source function being in $B_{\infty,\infty}^0(\mathbb{R}, \mathbb{R})$ but C_b^∞ almost everywhere and being the limit of a C_b^∞ function, e.g. \tanh .

Remark 8. *With our current approach, we cannot provide any Lipschitz control of a solution of the inviscid Burgers' equation (5.1) for the same reason as for the transport equation 1.1. This is not surprising by the well-known blowing-up of the gradient of a solution of the inviscid Burgers' equation (5.1).*

Actually, from the mild vanishing viscous solution, we see that for any $t \in [0, T]$, the solution $u(t, \cdot)$ given by the limit of a sub-sequence of $u^{m,\nu}(t, \cdot)$ depends on the mollification choice, moreover the sub-sequence choice also depends on the current time t . In other words, we can expect that $u(t, \cdot)$ can avoid the time of blowing-up thanks to a different choice of sub-sequence at each considered time.

Sketch of the proof of Theorem 3. To establish this result, we consider the mollified version of Burgers' equation for all $m \in \mathbb{R}_+$ and $\nu > 0$:

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + u^{m,\nu}(t, x) \partial_x u^{m,\nu}(t, x) - \nu \partial_{xx}^2 u^{m,\nu}(t, x) = f(t, x), & (t, x) \in [0, T] \times \mathbb{R}, \\ u_m(0, x) = g(x), & x \in \mathbb{R}, \end{cases} \quad (5.5)$$

where

$$u_m^{m,\nu}(t, x) := \int_{\mathbb{R}^d} \rho_m(y) u^{m,\nu}(t, y) dy. \quad (5.6)$$

It is direct from Theorem 3 in [Hon22] that there is a smooth solution of (5.5).

We can then perform the same computations as for the transport equation, where $\beta = 0$ and we change in the viscous condition $\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}$ by an upper-bound of $\|u\|_{L^\infty}$, namely by $T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma$ given by Feynman-Kac formula. Finally, we can take the limit, thanks to a compact argument, of a suitable sub-sequence yields the result. \square

A A priori controls for the parabolic approximation

A.1 Gradient estimates

Let us precise the control previously used:

$$\|\nabla u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq \left(T\|f^m\|_{L^\infty(C^1)} + \|g^m\|_{C^1} \right) \exp\left(Cm^{1+\beta}t\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \right) =: O_m(t). \quad (A.1)$$

We directly have from Duhamel formula:

$$\begin{aligned} & |\nabla u^{m,\nu}(t, x)| \\ & \leq \left(t\|\nabla f^m\|_{L^\infty} + \|\nabla g^m\|_{L^\infty} \right) \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] \cdot \nabla u^{m,\nu}(s, y) dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \leq \left(t\|\nabla f^m\|_{L^\infty} + \|\nabla g^m\|_{L^\infty} \right) \\ & \quad + C\|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \Big|_{(\tau,\xi)=(t,x)}. \end{aligned}$$

By absorbing property (2.9), we derive

$$\begin{aligned} & |\nabla u^{m,\nu}(t, x)| \\ & \leq \left(t\|\nabla f^m\|_{L^\infty} + \|\nabla g^m\|_{L^\infty} \right) \\ & \quad + Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \int_{\mathbb{R}^d} \bar{p}^{\tau,\xi}(s, t, x, y) \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \Big|_{(\tau,\xi)=(t,x)} \\ & \leq \left(t\|\nabla f^m\|_{L^\infty} + \|\nabla g^m\|_{L^\infty} \right) + Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} ds. \quad (A.2) \end{aligned}$$

The Gronwall lemma yields the result.

A.2 Hessian estimates

We perform a similar argument, but for the second derivatives we have to put a second derivative on $[b_m(s, \theta_t(\xi)) - b_m(s, y)] \cdot \nabla u(s, y)$. Indeed, if we twice differentiate $\hat{p}^{\tau,\xi}(s, t, x, y)$ there is no possibility to smoothen the blowing up in the contribution of ν by Hölder control (or even Lipschitz).

We obtain by Leibniz rules

$$\begin{aligned}
& |\nabla^2 u^{m,\nu}(t, x)| \\
& \leq \left(t \|\nabla^2 f^m\|_{L^\infty} + \|\nabla^2 g^m\|_{L^\infty} \right) + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla b(s, y) \cdot \nabla u(s, y) dy ds \right|_{\xi=x} \\
& \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] \nabla^2 u(s, y) dy ds \right|_{\xi=x} \\
& \leq m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \\
& \quad + C \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla b_m \otimes \nabla u^{m,\nu}(s, \cdot)\|_{L^\infty(C^1)} dy ds \\
& \quad + C \|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds.
\end{aligned}$$

Next, with Leibniz rules and absorbing property (2.9),

$$\begin{aligned}
& |\nabla^2 u^{m,\nu}(t, x)| \\
& \leq m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \int_{\mathbb{R}^d} \bar{p}^{\tau,\xi}(s, t, x, y) \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \\
& \quad + C m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty} ds \\
& \leq m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + C t m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla u^{m,\nu}\|_{L^\infty} \\
& \quad + C m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty} ds. \tag{A.3}
\end{aligned}$$

We finally get by the Gronwäll lemma and by identity (A.1)

$$\begin{aligned}
& \|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty} \tag{A.4} \\
& \leq \left(m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + C t m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} O_m(t) \right) \exp(t m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}). \tag{A.5}
\end{aligned}$$

We can also write by exponential absorbing property:

$$\|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq C m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(C t m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}) =: O_m^{(2)}(t). \tag{A.6}$$

We insist on the fact that the above inequality does not depend on ν .

A.3 Third derivatives estimates

In this section, we detail how to control the third derivatives of $u^{m,\nu}$. We use the same method as for the Hessian, the additional derivative is also put on $[b_m(s, \theta_t(\xi)) - b_m(s, y)] \cdot \nabla u(s, y)$.

$$\begin{aligned}
|\nabla^3 u^{m,\nu}(t, x)| &\leq \left(t \|\nabla^3 f^m\|_{L^\infty} + \|\nabla^3 g^m\|_{L^\infty} \right) + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla^2 b(s, y) \cdot \nabla u(s, y) dy ds \right|_{\xi=x} \\
&\quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla b(s, y) \cdot \nabla^2 u(s, y) dy ds \right|_{\xi=x} \\
&\quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] \nabla^3 u(s, y) dy ds \right|_{\xi=x} \\
&\leq C m^{3-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \\
&\quad + C \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \hat{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \\
&\quad \times (\|\nabla^2 b_m \cdot \nabla u^{m,\nu}(s, \cdot)\|_{L^\infty(C^1)} + \|\nabla b_m \cdot \nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty(C^1)}) dy ds \\
&\quad + C \|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \hat{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla^3 u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \\
&\leq C m^{3-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + C t m^{3+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla u^{m,\nu}\|_{L^\infty} \\
&\quad + C t m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla^2 u^{m,\nu}\|_{L^\infty} + C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla^3 u^{m,\nu}(s, \cdot)\|_{L^\infty} ds.
\end{aligned}$$

Eventually, by Gronwäll lemma, identities (A.1) and (A.4)

$$\|\nabla^3 u^{m,\nu}(t, \cdot)\|_{L^\infty} \tag{A.7}$$

$$\begin{aligned}
&\leq C \left(m^{3-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + t m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m O_m(t) + m O_m^{(2)}(t)) \right) \exp(t m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}) \\
&\leq C m^{3-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(C t m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}), \tag{A.8}
\end{aligned}$$

by exponential absorbing property.

A.4 Flow controls

This section is devoted to the regularity of the flow

$$\theta_{s,\tau}^m(x) := x + \int_s^\tau b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x)) d\tilde{s}.$$

Lemma 2. For any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$:

$$\sup_{\tilde{s} \in [0, \tau]} |\theta_{s,\tau}^m(x) - \theta_{s,\tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau).$$

By defintion, we have

$$\begin{aligned}
|\theta_{s,\tau}^m(x) - \theta_{s,\tau}^m(x')| &\leq |x - x'| + \left| \int_s^\tau b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x)) - b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x')) d\tilde{s} \right| \\
&\leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_s^\tau |\theta_{\tilde{s},\tau}^m(x) - \theta_{\tilde{s},\tau}^m(x')| d\tilde{s},
\end{aligned}$$

which is not the suitable inequality to apply directly Grönwall's lemma. To do so, we use a sup formulation, namely for any $r \leq \tau$, we can write similarly to above

$$\begin{aligned} \sup_{s \in [0, r]} |\theta_{s, \tau}^m(x) - \theta_{s, \tau}^m(x')| &\leq |x - x'| + \int_0^r |b_m(\tilde{s}, \theta_{\tilde{s}, \tau}^m(x)) - b_m(s, \theta_{\tilde{s}, \tau}^m(x'))| ds \\ &\leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_0^r |\theta_{\tilde{s}, \tau}^m(x) - \theta_{\tilde{s}, \tau}^m(x')| d\tilde{s} \\ &\leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_0^r \sup_{\tilde{s} \in [0, \tilde{s}]} |\theta_{\tilde{s}, \tau}^m(x) - \theta_{\tilde{s}, \tau}^m(x')| d\tilde{s}. \end{aligned}$$

We are now in position to use Grönwall's lemma, for $r = \tau$

$$\sup_{\tilde{s} \in [0, \tau]} |\theta_{\tilde{s}, \tau}^m(x) - \theta_{\tilde{s}, \tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau).$$

B Convergence of the mollified distribution

Proposition 4. *For any $\psi \in C^\gamma(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0, 1]$, we have for all $\vartheta \in \mathbb{N}_0^d$ and $\theta \in \mathbb{N}_0$ s.t. $|\vartheta| = \theta$, that $h_{m-2} \star D^\vartheta \psi \in C_b^\infty$ converges towards $D^\vartheta \psi$ in $\dot{B}_{\infty, \infty}^{-\theta}$ as $m \rightarrow +\infty$. More precisely, we have:*

$$\|h_{m-2} \star D^\vartheta \psi - D^\vartheta \psi\|_{\dot{B}_{\infty, \infty}^{-\theta}} \leq C[\psi]_\gamma m^{-\gamma}, \quad (\text{B.1})$$

and $D^\vartheta \psi \in \dot{B}_{\infty, \infty}^{\gamma-\theta}$. In particular, if $|\vartheta| = 0$, $h_{m-2} \star \psi \in C_b^\infty$ converges towards ψ in L^∞ as $m \rightarrow +\infty$, and:

$$\|h_{m-2} \star \psi - \psi\|_{L^\infty} \leq C[\psi]_\gamma m^{-\gamma}. \quad (\text{B.2})$$

Remark 9. *Actually, $\gamma \notin \{0, 1\}$ is not a restrictive condition as changing ϑ into $\tilde{\vartheta} \in \mathbb{N}_0^d$ such that $|\tilde{\vartheta}| = |\vartheta| + 1$ yields the same result.*

In particular, Proposition 4 is available for the Dirac distribution $\delta \in \dot{B}_{\infty, \infty}^{-d}$ regarded as the distributional derivative of the sign function (also regarded as the derivative of the absolute value), and for any derivative of the Dirac distribution by the same argument.

Proof of Proposition 4. Let us write $\varphi = D^\vartheta \psi$, with $\psi \in C^\gamma$

$$\begin{aligned} \|h_{m-2} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} &= \|D^\vartheta [h_{m-2} \star \psi - \psi]\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\vartheta}{2}} \|\partial_v h_v \star D^\vartheta [h_{m-2} \star \psi - \psi]\|_{L^\infty} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\vartheta}{2}} \|\partial_v D^\vartheta h_v \star [h_{m-2} \star \psi - \psi]\|_{L^\infty}, \end{aligned}$$

by integration by parts in convolutions. Next, we can explicitly write,

$$\begin{aligned} \|h_{m-2} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} &= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{1-\frac{-\vartheta}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m-2}(y-x) [\psi(x) - \psi(y)] dx dy \right| \\ &\leq C[\psi]_\gamma \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m-2}(y-x) |x-y|^\gamma dx dy \\ &\leq C[\psi]_\gamma m^{-\gamma} \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m-2}(y-x) dx dy, \end{aligned}$$

by exponential absorbing property (2.9). Integrating in space finally yields

$$\|h_{m-2} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} \leq C[\psi]_\gamma m^{-\gamma}.$$

Inequality (B.2) is direct with similar arguments. □

Corollary 5. For any $\psi \in \dot{B}_{\infty,\infty}^{\gamma}(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0, 1]$ we have, for all $\vartheta \in \mathbb{N}_0^d$ and $\theta \in \mathbb{N}_0$ s.t. $|\vartheta| = \theta$, that $h_{m^{-2}} \star D^{\vartheta} \psi \in C_b^{\infty}(\mathbb{R}^d, \mathbb{R})$ converges towards $D^{\vartheta} \psi$ in $\dot{B}_{\infty,\infty}^{-\theta+\gamma-\varepsilon}(\mathbb{R}^d, \mathbb{R})$, for any $\varepsilon \in (0, 1)$ s.t. $\theta - \gamma + \varepsilon > 0$, as $m \rightarrow +\infty$. More precisely, we have:

$$\|h_{m^{-2}} \star D^{\vartheta} \psi - D^{\vartheta} \psi\|_{\dot{B}_{\infty,\infty}^{-\theta+\gamma-\varepsilon}} \leq C[\psi]_{\gamma} m^{-\varepsilon}. \quad (\text{B.3})$$

Proof of Corollary 5. We still use the thermic representation, and by convolution property we have:

$$\begin{aligned} & \|h_{m^{-2}} \star \varphi - \varphi\|_{\dot{B}_{\infty,\infty}^{-\theta+\gamma-\varepsilon}} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^{\vartheta} [h_{m^{-2}} \star \psi - \psi]\|_{L^{\infty}(\mathbb{R}^d)} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v D^{\vartheta} h_v \star [h_{m^{-2}} \star \psi - \psi]\|_{L^{\infty}(\mathbb{R}^d)} \\ &= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v D^{\vartheta} h_v(z-y) h_{m^{-2}}(y-x) [\psi(x) - \psi(y)] dx dy \right|. \end{aligned}$$

For a given $v \in (0, +\infty)$, we compare the regular contribution v with the mollification contribution. In other words, we consider two possibilities.

- If $m^{-2} < v$, then:

$$\begin{aligned} & v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^{\vartheta} [h_{m^{-2}} \star \psi - \psi]\|_{L^{\infty}(\mathbb{R}^d)} \\ &= \sup_{z \in \mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v D^{\vartheta} h_v(z-y) h_{m^{-2}}(y-x) [\psi(x) - \psi(y)] dx dy \right| \\ &\leq C[\psi]_{\gamma} v^{\frac{-\gamma+\varepsilon}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x) |x-y|^{\gamma} dx dy \\ &\leq C[\psi]_{\gamma} v^{\frac{-\gamma+\varepsilon}{2}} m^{-\gamma} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x) dx dy \\ &\leq C[\psi]_{\gamma} m^{-\varepsilon}. \end{aligned} \quad (\text{B.4})$$

- If $m^{-2} \geq v$, then:

$$\begin{aligned} & v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^{\vartheta} [h_{m^{-2}} \star \psi - \psi]\|_{L^{\infty}(\mathbb{R}^d)} \\ &= \sup_{z \in \mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v h_v(z-y) D^{\vartheta} h_{m^{-2}}(y-x) [\psi(x) - \psi(y)] dx dy \right| \\ &\leq C[\psi]_{\gamma} m^{\frac{\theta}{2}} v^{\frac{\theta-\gamma+\varepsilon}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{c^{-1}m^{-1}}(y-x) |x-y|^{\gamma} dx dy \\ &\leq C[\psi]_{\gamma} m^{\theta-\gamma} v^{\frac{\theta-\gamma+\varepsilon}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x) dx dy \\ &\leq C[\psi]_{\gamma} m^{-\varepsilon}. \end{aligned} \quad (\text{B.5})$$

The result follows from (B.4) and (B.5). \square

Proposition 4 and Corollary 5 are more precise forms of the well known convergence in the distributional sense.

Proposition 6. For any $\psi \in \dot{B}_{\infty,\infty}^{\gamma}(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0, 1]$ we have for any $\vartheta \in \mathbb{N}_0^d$ that $h_{m^{-2}} \star D^{\vartheta} \psi \in C_b^{\infty}(\mathbb{R}^d, \mathbb{R})$ converges towards $D^{\vartheta} \psi$ in distributional sense as $m \rightarrow +\infty$. More precisely, we have for any $\eta \in C_0^{\infty}(\mathbb{R}^d, \mathbb{R})$:

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \eta(x-y) [h_{m^{-2}} \star D^{\vartheta} \psi(y) - D^{\vartheta} \psi(y)] dy \right| \leq C[\psi]_{\gamma} m^{-\gamma}. \quad (\text{B.6})$$

Remark 10. We precise that η is not supposed to be a Gaussian kernel, as in Proposition 4.

Proof. We directly write by convolution property:

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \eta(x-y) [h_{m^{-2}} \star D^\vartheta \psi(y) - D^\vartheta \psi(y)] dy \right| &= \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} D^\vartheta \eta(x-y) [h_{m^{-2}} \star \psi(y) - \psi(y)] dy \right| \\ &\leq C m^{-\gamma} [\psi]_\gamma \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} |D^\vartheta \eta(x-y)| dy \right| \\ &\leq C m^{-\gamma} [\psi]_\gamma, \end{aligned}$$

the penultimate inequality is consequence of inequality (B.2). \square

C Properties of derivatives of Besov distributions

Proposition 7. For any $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\nabla \varphi \in \dot{B}_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d)$, $\gamma \in (0, 1)$, there is a constant $C > 1$ such that:

$$\|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} \leq C \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma}.$$

Proof of Proposition 7. We first write by the thermic representation of the Besov norm and by integration by parts,

$$\begin{aligned} \|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} &= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{1-\frac{-1+\gamma}{2}} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) \nabla \varphi(y) dy \right| \\ &= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{\frac{3-\gamma}{2}} \left| \int_{\mathbb{R}^d} \nabla \cdot \partial_v h_v(z-y) [\varphi(y) - \varphi(z)] dy \right|, \end{aligned}$$

by absorbing property (2.9) we derive

$$\begin{aligned} \|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} &\leq C [\varphi]_\gamma \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{\frac{3-\gamma}{2}} \int_{\mathbb{R}^d} v^{-\frac{3}{2}} h_{C^{-1}v}(z-y) |y-z|^\gamma dy \\ &\leq C [\varphi]_\gamma \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) dy \\ &\leq C [\varphi]_\gamma. \end{aligned}$$

\square

We also derive the corresponding inequality for the inhomogeneous case.

Corollary 8. For any $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\nabla \varphi \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d)$, $\gamma \in (0, 1)$, there is a constant $c > 1$ such that:

$$\|\nabla \varphi\|_{B_{\infty, \infty}^{-1+\gamma}} \leq c \|\varphi\|_{B_{\infty, \infty}^\gamma}.$$

Proof of Corollary 8. From inequality (2.7), we have $\|\nabla \varphi\|_{B_{\infty, \infty}^{-1+\gamma}} \leq \frac{C}{1-\gamma} \|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}}$. Moreover, it is direct that

$$\begin{aligned} \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma} &= \sup_{v \in \mathbb{R}_+} v^{(1-\frac{\gamma}{2})} \|\partial_v h_v \star \varphi\|_{L^\infty} \leq \|\varphi\|_{\dot{B}_{\infty, \infty}^\alpha} + \sup_{v>1} v^{(1-\frac{\alpha}{2})} \|\partial_v h_v \star \varphi\|_{L^\infty} \\ &\leq \|\varphi\|_{\dot{B}_{\infty, \infty}^\alpha} + \|\varphi\|_{L^\infty} \\ &= \|\varphi\|_{B_{\infty, \infty}^\gamma}. \end{aligned}$$

In other words, we deduce by Proposition 7,

$$\|\nabla \varphi\|_{B_{\infty, \infty}^{-1+\gamma}} \leq \frac{C}{1-\gamma} \|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} \leq \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma} \leq C \|\varphi\|_{B_{\infty, \infty}^\gamma}.$$

\square

Remark 11. In all generality, the reverse inequality of the above results are not true. For example, any constant function lies in $B_{\infty,\infty}^\gamma$ but its derivative is 0, hence the $B_{\infty,\infty}^{-1+\gamma}$ norm is null, and the corresponding Besov norm equivalence obviously fails to be true.

For an equivalence version of this result, we need to consider extra Besov norms, see for instance [KMM07] identity (3.54) for a Triebel-Lizorkin spaces version.

D On the freedom of the mollification choice

In this section, we detail why if there is a sequence of smooth function converging toward a $B_{\infty,\infty}^{-\beta}$ distribution then the mollification procedure (4.2) converges also toward the distribution. In other words, if there is a sequence $(\bar{b}_n)_{n \geq 1}$ lying in $L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$ such that

$$\lim_{n \rightarrow \infty} \|\bar{b}_n - b\|_{L^\infty(B_{\infty,\infty}^{-\beta+\varepsilon})} = 0, \quad (\text{D.1})$$

for any $0 < \varepsilon$, then

$$\lim_{m \rightarrow \infty} \|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = 0, \quad (\text{D.2})$$

where b_m is defined in (4.2) by:

$$b_m(t, x) = b(t, \cdot) \star \rho_m(x),$$

with for any $z \in \mathbb{R}^d$, $\rho_m(z) := m^d \rho(mz)$ for $\rho(z) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{2}}$. Indeed, we readily write by triangular inequality:

$$\|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta+\varepsilon})} \leq \|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} + \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} + \|\bar{b}_n - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})}. \quad (\text{D.3})$$

The first term in the r.h.s. above write:

$$\|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = \|\rho_m \star (b - \bar{b}_n)\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})}.$$

Hence, we obtain

$$\|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = \sup_{v \in [0,1]} v^{\frac{\beta-\varepsilon}{2}} \|\rho_m \star h_v \star (b - \bar{b}_n)\|_{L^\infty} \leq \|\bar{b}_n - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})}, \quad (\text{D.4})$$

by triangular inequality.

Also, for the second term in (D.3), let us deal with the corresponding homogeneous norm,

$$\|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = \|\varphi(D)(\rho_m \star \bar{b}_n - \bar{b}_n)\|_{L^\infty} + \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(\dot{B}_{\infty,\infty}^{-\beta-\varepsilon})}.$$

It is direct that

$$\|\varphi(D)(\rho_m \star \bar{b}_n - \bar{b}_n)\|_{L^\infty} \leq C \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty} \leq C m^{-1} \|D\bar{b}_n\|_{L^\infty}. \quad (\text{D.5})$$

Next,

$$\begin{aligned} & \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(\dot{B}_{\infty,\infty}^{-\beta-\varepsilon})} \\ &= \sup_{v \in [0,1], t \in [0,T], z \in \mathbb{R}^d} v^{\frac{\beta+\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_v(z-y) \rho_m(y-x) [\bar{b}_n(t,x) - \bar{b}_n(t,y)] dx dy \right| \\ &\leq \|D\bar{b}_n\|_{L^\infty} \sup_{v \in [0,1], t \in [0,T], z \in \mathbb{R}^d} v^{\frac{\beta+\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) \rho_m(y-x) |x-y| dx dy \right| \\ &\leq C \|D\bar{b}_n\|_{L^\infty} m^{-1} \sup_{v \in [0,1], t \in [0,T], z \in \mathbb{R}^d} v^{\frac{\beta+\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) \rho_{C^{-1}m}(y-x) dx dy \right| \\ &= C \|D\bar{b}_n\|_{L^\infty} m^{-1}. \end{aligned} \quad (\text{D.6})$$

Let us choose $m \gg \|D\bar{b}_n\|_{L^\infty}$ which yields that $\lim_{m,n \rightarrow \infty} \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = 0$.

Finally, gathering identities (D.1), (D.3), (D.4), (D.5) and (D.6) yields the limit property (D.2).

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