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Hölder estimates for parabolic and transport equations with Besov coefficients

Igor Honoré*

March 29, 2023

Abstract

For the parabolic and the transport equations whose coefficient of the first term, denoted by b , can be in a negative Besov space, we provide a control of the solution in the Hölder space C^γ , $\gamma \in (0, 1)$, regardless of b and matching with the source functions regularity. We define some kind of solutions which do not require b to be Lipschitz continuous.

For a “very weak” solution, called in the paper *mild vanishing viscous*, in C^γ , there is no regularity constraint on b .

If b lies in a $\tilde{\gamma}$ -Hölder space, $\tilde{\gamma} > 1 - \gamma$, we establish that there is a weak solution in a γ -Hölder space.

If b is supposed to be divergence free, we obtain the same result for b having a negative regularity in space, precisely in $L^\infty(B_{\infty, \infty}^{-\beta})$ for $\beta < \gamma$.

Finally, if $\tilde{\gamma} > \frac{1}{1+\gamma}$ with a *vanishing viscous* condition, then the selected solution is unique. In this case, there is somehow a *regularisation* by turbulence (corresponding to the Reynolds number going to $+\infty$); the *vanishing viscosity* overwhelms the potential blowing up of the rough coefficients.

Importantly, as a by-product of our analysis, we are able to give a meaning of a product of distributions. For b lying in a C^γ , we obtain the same condition as for the usual Bony’s para-product; but in a weaker solution framework, the product is defined beyond the para-product condition and even with no constraint at all in the *mild vanishing viscous* context. We also obtain that the time averaging of the distributions product is γ -Hölder continuous. These new results happens because, in the considered product, one of the distribution is the gradient of a solution of a Partial Differential Equation.

Thanks to our analysis, we also get a Hölder control of a solution of the inviscid Burgers’ equation. Under some regularity and vanishing viscous constraint, the solution is a weak solution and is unique in a certain sense. The vanishing viscous procedure allows to avoid the well-known critical time of the solution built by characteristics.

Keywords: Regularisation by turbulence, Transport equations, Parabolic equations, Besov spaces, Product of distributions, Para-product, Inviscid Burgers’ equation.

MSC: Primary: 35K40, 35J75; Secondary: 35Q35, 46E35

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1 Introduction

1.1 Statement of the problem

For a given $d \in \mathbb{N}$, we consider the following d -dimensional Cauchy problem:

$$\begin{cases} \partial_t u(t, x) + \langle b(t, x), \nabla u(t, x) \rangle = f(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0, x) = g(x), & x \in \mathbb{R}^d. \end{cases} \quad (1.1)$$

For a finite $\beta \in \mathbb{R}$, we suppose that the transport coefficient, $b(t, \cdot)$, for any $t \in \mathbb{R}_+$, lies in the non-homogeneous Besov Hölder space $B_{\infty, \infty}^{-\beta}$; when $\beta > 0$, this can be regarded as a Hölder space with negative regularity.

We first establish in this article that there is a solution u which is Hölder continuous. To give a meaning of the product $\langle b(t, x), \nabla u(t, x) \rangle$, we introduce a kind of vanishing viscosity solution. We indeed consider the usual second order parabolic equations whose second order term ν , called viscosity, goes to 0,

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + \langle b_m(t, x), \nabla u^{m, \nu}(t, x) \rangle - \nu \Delta u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

the function b_m is a mollified version of the distributional valued b , also g_m and f_m stand respectively for a mollified version of g and f . The uniform control in L^∞ is direct by a probabilistic representation of the solution, see [Hon22], or by maximum principle, this result can also be regarded as a by-product of our analysis.

We write the solution as a perturbation of a PDE with constant components. These constants correspond to the first order term b_m taking at a *freezing point* through the corresponding flow, as done in [CDRHM18]. However, to estimate the Hölder norm, we have to distinguish two regimes, as usual in a parabolic context, the *diagonal* and the *off-diagonal* ones. In each regime, the choice of *freezing points* changes in order to get a negligible first order contribution when $\nu \rightarrow 0$, or with a time decomposition trick.

In our analysis, some singularities are overwhelmed by a small ν , we call this phenomenon a *regularisation by turbulence*. As for the fluid mechanics, we are able to define an associated Reynolds number which goes to $+\infty$ when $\nu \rightarrow 0$, corresponding to a turbulence regime, see for instance [FMRT01].

One of the crucial consequences of our analysis is that, we obtain a first general meaning of a classical product of distributions. Indeed, we succeed in giving a meaning of $\langle b(t, x), \nabla u(t, x) \rangle$ where b and ∇u have negative regularity. This is written as a weak limit of a sub-sequence of a smooth parabolic approximation. The price to pay in this representation is that we do not have usual uniqueness of the limit in this rough case.

Thanks to the techniques developed for the transport equation (1.1), we succeed to extend our analysis to parabolic equation

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + \langle b_m(t, x), \nabla u^{m, \nu}(t, x) \rangle - \nu \Delta u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.3)$$

where the condition on the *vanishing viscosity* is enhanced by a time cutting trick.

Importantly, this point of view allows us to establish uniqueness for the solution of transport equation (1.1), this is somehow due to the *regularisation by turbulence*.

Because the *a priori* estimates are independent on b , we are able to use a fixed-point argument to handle with the inviscid Burgers' equation

$$\begin{cases} \partial_t u(t, x) + u(t, x) \partial_x u(t, x) = f(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u(0, x) = g(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.4)$$

in dimension 1, see *e.g.* [BK21]. Importantly, again thanks by *regularisation by turbulence*, we deduce some kind of uniqueness of the solution u of (1.4). Precisely, on the one hand there is uniqueness of selection whatever the way to mollify, and on the other hand there is uniqueness whatever the choice of vanishing viscosity; nevertheless we do not succeed to get both uniquenesses for the same solution (which should imply usual uniqueness).

1.2 Existing results

1.2.1 On the transport equation

The Lipschitz framework is classical *via* the characteristic method. Indeed if $b \in L^\infty([0, T]; C^1(\mathbb{R}^d, \mathbb{R}^d))$, considering the ODE $\dot{X}_t = b(t, X_t)$, thanks to the Cauchy–Lipschitz theorem, there is a unique solution $u \in L^\infty([0, T]; C^1(\mathbb{R}^d, \mathbb{R}))$ of the transport equation (1.1). Out of this regular context, the analysis has to be more involved.

For instance, a meaning of the equation (1.1), when the coefficients are in a suitable Sobolev space, can be given by a renormalisation procedure developed by DiPerna and Lions [DPL89]. If $b \in L^1([0, T]; W_{loc}^{1,1}(\mathbb{R}^d, \mathbb{R}))$ and $\operatorname{div}(b) \in L^1([0, T]; L^\infty(\mathbb{R}^d, \mathbb{R}))$, they establish that the Cauchy problem (1.1) is well-posed in L^∞ . When, b is only supposed to have bounded variations in space, Ambrosio [Amb04] extends this result for $b \in L^1([0, T]; BV_{loc}(\mathbb{R}^d, \mathbb{R}))$, $\operatorname{div}(b)_- \in L^1([0, T]; L^\infty(\mathbb{R}^d, \mathbb{R}))$.

For other references on transport equation in the non Lipschitz case, see for instance to [MS18], [Xia19].

If b is only Hölder continuous then the Cauchy problem (1.1) is not well-posed any-longer, see the well-known counter-example

$$b(x) = \frac{1}{1-\gamma} \operatorname{sign}(x) (|x| \wedge R)^\gamma, \quad \gamma \in (0, 1). \quad (1.5)$$

With a multiplicative noise, Flandoli Gubinelli and Priola [FGP10], see also [FGP12] and [MO17], establish that the following Stochastic Partial Differential Equation

$$\begin{cases} d_t u + \langle b, \nabla u \rangle dt + \nabla u \circ dW_t = 0, \\ u(0, \cdot) = u_0(\cdot), \end{cases} \quad (1.6)$$

with $b \in L^\infty([0, T], C_b^\alpha)$ and $\operatorname{div}(b) \in L^p$ is well-posed. Here, the symbol \circ corresponds to the stochastic Stratonovich integral. This is a typical consequence of the regularisation by the noise, see also [FF13], [AF11], [Cat16].

This stochastic approach seems to be hopeless to get uniqueness by zero limit noise selection, indeed from [AF09] : if b is defined as in (1.5), then the following equation

$$\partial_t u^\varepsilon + b \cdot \partial_x u^\varepsilon = \varepsilon \nabla u^\varepsilon \circ dW_t,$$

has a weak convergence of the corresponding probability P^ε towards $\frac{\delta_{u_1} + \delta_{u_2}}{2}$, when $\varepsilon \rightarrow 0$, where u_1 and u_2 are two different solutions of the associated transport equation. Other counter-examples are stated in [Dep03].

In [CCS20], the authors show that there is $b \in L_{loc}^p$, $p \in [1, \frac{4}{3}]$, such that

$$\partial_t u^m + b_m \cdot \nabla u^m = 0,$$

where b_m is a regularisation of b , for $d = 3$, and s.t. there is no uniqueness of bounded distributional solutions when $m \rightarrow +\infty$. In [DLG22], the conclusion is the same for a compactly supported divergence-free vector field $b \in L^\infty$. To put it another way, there is no smooth selection principle by regularisation.

Finally, let us mention [CCS22], where the authors exhibit a first order coefficients b Hölder continuous such that the vanishing viscosity procedure from a parabolic approximation yields to several different solutions, uniqueness fails to be true in L^2 . In spite of this counter-example, there is no contradiction with our uniqueness result, stated in Theorem 3, as we only consider uniqueness in a Hölder space, defined further. The non-uniqueness seems to occur only in non-smooth functional spaces. Then we are able to affirm that there is a selection principle by vanishing viscosity, which is a positive answer to the question (Q3) in [CCS20].

We propose in this article a new approach to handle with the determinist transport equation (1.1). We consider *vanishing viscous solution*, see e.g. [Eva98], which is different to the *viscous solution* introduced by Crandall and Lions [CL83] for the Hamilton-Jacobi equation. We do not consider upper or lower solution, but a smooth well-posed parabolic equation and we take the limit, up to sub-sequence selection, of the mollification parameter and of the viscosity.

For the best author's knowledge, the notion of vanishing viscosity has been already used for several classes of evolution PDEs, e.g. hyperbolic ones in [BB05], but it was not developed to establish the regularity control of the solution of a general transport equation. Finally, with this method, we also deduce existence and uniqueness of a Hölder continuous solution of the inviscid Burgers' equation.

1.2.2 On the parabolic equation

Historically, the first Schauder estimates associated with the parabolic equation (1.2) with Hölder and bounded coefficients was proved by Friedman thanks to a *parametrix* approach, [Fri64]. Let us mention also the major reference of the parabolic equation studies by Ladyzenskaja, Solonnikov and Ural'ceva [LSU68]. Some equivalent results can be found in a wide literature, see for instance Krylov [Kry96] and Lieberman [Lie96].

The first article handle with unbounded coefficient in (1.2) is due to Krylov and Priola [KP10]. In parabolic and elliptic framework, they show the parabolic bootstrap through Schauder estimates. Let us also mention [KKL75], for a first partial result in a unbounded context; when coefficients are “merely” measurable in time is handled in [Lor11].

In a degenerate framework, with some Hörmander conditions, Lunardi establishes, in her work [Lun97], Schauder estimates for a linear b . For a fully non-linear Hölder continuous drift, b , Chaudru de Raynal et al. [CDRHM18] get the corresponding controls, see also [Pri09]. To extend our result to a degenerate chain, the principal points would be to control the regularity gain of the flow θ associated with b through the chain which is not direct when the non-degenerate components of the drift are distributional valued. This regularity control is crucial as the *proxy* density does depend on this flow.

When b is a tempered distribution, in particular when $b \in B_{\infty, \infty}^{-\beta}$ with $\beta < \frac{2}{3}$ some controls of the solution of (1.2) are established in [DD16] and [CC18], in order to build a “polymer measure”, an important object to study some stochastic partial differential equations such as the Kardar-Parisi-Zhang equation.

In [Dan07] and [BCD11], when b lies in a more general Besov space, $B_{p,q}^{-\beta}$, $p, q \neq +\infty$, some *a priori* estimates in Besov space is established for the solution of some parabolic equations. In particular, these controls are crucial in some approach to handle with the Navier-Stokes equation.

Considering more general Besov or Triebel-Lizorkin spaces would allow us to deal with other usual spaces in the analysis of PDEs. Let us recall some correspondences with some “common” functional spaces: the Lebesgue space is identified by a Triebel-Lizorkin space $L^p = F_{p,2}^0$, $1 < p < +\infty$, also we can write the functional space equality for the Sobolev space and Bessel potential space $W^{k,p} = L_k^p = F_{p,2}^k$, $1 < p < +\infty$, $k \in \mathbb{N}$, and for the Morrey-Campanato space, see the theorem page 50 in [Tri83], it is known that $C_p^\alpha = F_{p,\infty}^\alpha$, $0 < p \leq +\infty$, $\alpha > d(\frac{1}{p} - 1)_+$. These extensions require to carefully introduce estimates of some new integrals, instead of point-wise controls furnished in our analysis.

1.3 Some *a priori* attempts

1.3.1 Peano’s heuristic

The Peano counter-example yields a heuristic of the expected minimum regularity of b . The threshold comes from a regularisation by noise argument, for other explanation see [Fla11], and [CdRM17] in degenerate framework. Similarly to (1.5), let us consider $b(x) = \text{sign}(x)(|x| \wedge R)^\gamma$, $\gamma \in \mathbb{R}$, and the associated flow,

$$\frac{dX_t}{dt} = \text{sign}(X_t)|X_t|^\alpha, \quad X_0 = 0. \quad (1.7)$$

There are infinite solutions written, for any $t^* \in [0, T]$, by :

$$X_t = \pm c_\alpha (t - t^*)^{\frac{1}{1-\alpha}} \mathbb{1}_{[t^*, T]}(t). \quad (1.8)$$

The associated stochastic problem is

$$d\tilde{X}_t = \text{sign}(\tilde{X}_t)|\tilde{X}_t|^\alpha dt + \nu dW_t,$$

where $(W_t)_{t \geq 0}$ is a Brownian motion. Parabolic equation (1.2) is the determinist counterpart of this SDE, where the solution is given by a stochastic representation, the Feynman-Kac formula.

There is critical time when the noise overwhelms the singular drift, see [DF14], after this time the SDE solution fluctuates around a solution of the ODE (1.7).

As a consequence, we aim to compare the time scaling between the Brownian motion, i.e. $t^{\frac{1}{2}}$ (the variance being equal to t), with a solution of the ODE given in (1.8), i.e. $t^{\frac{1}{1-\alpha}}$.

To take advantage of the regularisation by noise before the critical point, the condition is

$$t^{\frac{1}{1-\alpha}} < t^{\frac{1}{2}},$$

which yields for a small time,

$$\frac{1}{1-\alpha} > \frac{1}{2} \Leftrightarrow \alpha > -1.$$

In other words, for $\beta = \alpha = -1 + \tilde{\gamma}$, $\tilde{\gamma} > 0$, the expected minimum regularity of the drift is $b(t, \cdot) \in B_{\infty, \infty}^{-1+\tilde{\gamma}} = C^{-1+\tilde{\gamma}}$.

There are numerous articles dealing with SDE and the associated parabolic equation with very rough drift, but in all of them there is a “macro” distance with the above Peano heuristic.

The case $\tilde{\gamma} > 1/3$ is dealt in [DD16] in dimension 1 and [CC18] for the multidimensional version; the authors thoroughly use rough path and para-control to build a polymer measure which allows to consider a solution to the KPZ equation.

Another notion of solution of SDE, called virtual, is introduced in [FIR17], where the constraint is $\tilde{\gamma} > 1/2$. Let us notice that under this constraint, $\tilde{\gamma} < 1/2$, there is no hope to obtain strong solution of the SDE, see for instance the counter-examples presented in [BC01], [Bar82].

1.3.2 On the limitation of the para-product

The last constraint, $\tilde{\gamma} > \frac{1}{2}$, appears naturally with *a priori* computations with usual tools. Indeed, from Duhamel formula the solution of the parabolic equation (1.3) writes

$$u^{m,\nu}(t, x) = \tilde{P}g_m(x) + \tilde{G}f_m(t, x) + \int_0^t ds \int_{\mathbb{R}^d} \tilde{p}(s, t, x, y) \langle b_m(y), \nabla u^{m,\nu}(s, y) \rangle dy,$$

where

$$\tilde{p}(s, t, x, y) := \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4\nu(t-s)}\right), \quad (1.9)$$

stands for the the standard heat kernel, also

$$\forall(t, x) \in (0, T] \times \mathbb{R}^d, \quad \tilde{G}f_m(t, x) := \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, t, x, y) f_m(s, y) dy ds, \quad (1.10)$$

is the corresponding Green operator, and

$$\tilde{P}g_m(t, x) := \int_{\mathbb{R}^d} \tilde{p}(0, t, x, y) g_m(y) dy, \quad (1.11)$$

the associated semi-group.

Let us suppose that $\nabla u^{m,\nu}(s, \cdot) \in C^\delta$, $\delta \geq 0$, therefore from the above Duhamel's formula, we should have:

$$x \mapsto \nabla \int_t^T ds \int_{\mathbb{R}^d} \tilde{p}(t, s, x, y) \langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle dy \in C^\delta.$$

However, from the para-product result, derived by Bony's microlocal analysis [Bon81], see also [GIP15], if $\nabla u^{m,\nu}(s, \cdot) \in C^\delta$ and $b_m(s, \cdot) \in C^{-\beta}$, such that $\delta - \beta > 0 \Leftrightarrow \beta < \delta$ then $\langle b_m(\cdot), \nabla u^{m,\nu}(s, \cdot) \rangle \in C^{-\beta}$. Hence, with some common computation of the heat kernel, we obtain that

$$\nabla \int_t^T ds \int_{\mathbb{R}^d} \tilde{p}(t, s, x, y) \langle b_m(y), \nabla u^{m,\nu}(s, y) \rangle dy \approx (T-t)^{\frac{1-\beta}{2}}.$$

Also, thanks the typical equivalence of the space-time with a parabolic scaling in the analysis of the parabolic bootstrap, see for instance [CDRHM18], where $(T-t)^2 \approx |x-x'|$, we deduce

$$x \mapsto \nabla \int_t^T ds \int_{\mathbb{R}^d} \tilde{p}(t, s, x, y) \langle b_m(y), \nabla u^{m,\nu}(s, y) \rangle dy \in C^{1-\beta}.$$

Then we readily derive that $\beta \leq 1 - \delta$, namely $\delta \leq 1 - \beta$. Combining with the para-product constraint $\beta < \delta$ yields $\beta \leq 1 - \beta$ and so $\beta < 1/2$ which is exactly the same regularity constraint of [FIR17].

This heuristic shows up the difficulty to use the usual para-product in such a rough framework. To handle with regularity $\beta > 1/2$, we then have to capitalise on other techniques. Specifically, we thoroughly exploit the fact that $u^{m,\nu}$ is solution of the parabolic equation (1.3), which allows to consider cases out of the Bony's para-product scope.

The paper is organised as following. The notations and the definitions used are gathered in Section 2. We bestow a User's guide in Section 3, where we exhibit the highlights of the article. A first result about the transport equation is stated in Section 4. We provide a discussion on the consequence in term of distributions product in Section 4.2. The complete analysis is detailed in Section 5. Our theorem on the parabolic equation and as well as the complete proof, are featured in Section 6. We adapt this second result to the transport equation in Section 7, which is a substantial improvement of

the first result on the transport equation. We provide the statement and the proof of the regularity of a solution to the inviscid Burgers' equation in Section 8.

We gather in Appendix some results about regularity controls on the solution of linear parabolic of second order in Sections A, B and C; a control for the non-linear Burgers' case is proved in Section D. Also some property of the Besov spaces are developed in Sections E-G. Precisely, in Section E, we establish that the space C_b^∞ functions are dense in the space of multi-differentiated Hölder continuous functions. Some inequalities over the norm of Besov-Hölder distributions with their derivative are established in Section F. Eventually, in Section G, we detail why the limit of regularised distribution in Besov-Hölder space does not depend on the choice of mollification procedure.

2 Notations and Definitions

From now on, we denote by $C > 0$ and $c > 1$ generic constants that may change from line to line but only depends on known parameters such as γ, d . Importantly, these constants do not depend on β .

We also write, for $\varepsilon, \tilde{\varepsilon} > 0$, the usual notation of asymptotic domination:

$$\varepsilon \ll \tilde{\varepsilon}, \text{ if } \frac{\varepsilon}{\tilde{\varepsilon}} \longrightarrow 0. \quad (2.1)$$

We also write $\xrightarrow[(\varepsilon, \tilde{\varepsilon}) \rightarrow (0, 0)]{(2.1)} 0$, the limit, up to some subsequence selection, under the condition (2.1).

2.1 Tensor and Differential notations

For any $z \in \mathbb{R}^d$, we use the decomposition $z = z_1 e_1 + \dots + z_d e_d$, where (e_1, \dots, e_d) is the canonical base of \mathbb{R}^d .

We usually use the notation ∂_t for the derivative in time $t \in [0, T]$ also ∂_{z_k} , $k \in \mathbb{N}$, is the derivative in the variable z_k .

The gradient in space is denoted by ∇ , in other words $\nabla = \partial_{z_1} e_1 + \dots + \partial_{z_d} e_d$.

The divergence write $\nabla \cdot = \text{div}$ and is defined for any \mathbb{R} -function $f : \mathbb{R}^d \mapsto \mathbb{R}$ by $\nabla \cdot f = \sum_{k=1}^d \partial_{z_k} f$.

From now on, the symbol “ \cdot ” between two tensors is the usual tensor contraction. For example, if $M \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ and $N \in \mathbb{R}^d$ then $M \cdot N$ is a $d \times d$ matrix. If the two considered tensors are vectors then “ \cdot ” matches with the scalar product which is also denoted by $\langle \cdot, \cdot \rangle$.

For any $\mathbb{R}^d \mapsto \mathbb{R}$, we define the Hessian matrix $D_z^2 f = (\partial_{z_i} \partial_{z_j} f)_{1 \leq i, j \leq d}$, and the usual Laplacian operator $\Delta f = \sum_{1 \leq i \leq d} \partial_{z_i}^2 f$.

More generally, for any $k \in \mathbb{N}$, $D_z^k f$ denotes the order k tensor $(\partial_{z_{i_1}} \dots \partial_{z_{i_k}} f)_{(i_1, \dots, i_k) \in \llbracket 1, d \rrbracket^k}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, we write $D_z^\alpha f = \partial_{z_1}^{\alpha_1} \dots \partial_{z_d}^{\alpha_d} f$, in particular if, for $i \in \llbracket 1, d \rrbracket$, $\alpha_i = 0$, there is no derivative in z_i in the expression of $D_z^\alpha f$.

We also denote for any $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, the order of this multi-index by $|\alpha| = \sum_{i=1}^m \alpha_i$.

2.2 Associated Hölder, Besov spaces

In this section, we provide some useful notations and functional spaces.

2.2.1 Hölder spaces

For any, $\beta \in (0, 1)$, $\|\cdot\|_{C^\beta(\mathbb{R}^m, \mathbb{R}^\ell)}$, $m \in \{1, d\}$, $\ell \in \{1, d, d \otimes d\}^*$ is the usual homogeneous Hölder norm, see e.g. Lunardi [Lun95] or Krylov [Kry96]. Precisely, for all $\psi \in C^\beta(\mathbb{R}^m, \mathbb{R}^\ell)$, we set the semi-norm:

$$\|\psi\|_{C^\beta} = [\psi]_\beta := \sup_{(x, y) \in (\mathbb{R}^m)^2, x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|^\beta}, \quad (2.2)$$

*we write $\mathbb{R}^{d \otimes d}$ for $\mathbb{R}^d \otimes \mathbb{R}^d$ the space of square matrices of size d .

the notation $|\cdot|$ is the Euclidean norm on the considered space. We denote by:

$$C_b^\delta(\mathbb{R}^m, \mathbb{R}^\ell) := \{\psi \in C^\delta(\mathbb{R}^m, \mathbb{R}^\ell) : \|\psi\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^\ell)} < +\infty\},$$

the associated subspace with bounded elements (non-homogeneous Hölder space). The corresponding Hölder norm is defined by:

$$\|\psi\|_{C_b^\delta(\mathbb{R}^m, \mathbb{R}^\ell)} := \|\psi\|_{C^\delta(\mathbb{R}^m, \mathbb{R}^\ell)} + \|\psi\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^\ell)}. \quad (2.3)$$

For the sake of notational simplicity, from now on we write:

$$\|\psi\|_{L^\infty} := \|\psi\|_{L^\infty(\mathbb{R}^d, \mathbb{R}^\ell)}, \quad \|\psi\|_{C^\delta} := \|\psi\|_{C^\delta(\mathbb{R}^d, \mathbb{R}^\ell)}, \quad \|\psi\|_{C_b^\delta} := \|\psi\|_{C_b^\delta(\mathbb{R}^d, \mathbb{R}^\ell)}.$$

For time dependent functions, $\varphi_1 \in L^\infty([0, T], C_b^\delta(\mathbb{R}^m, \mathbb{R}^\ell))$ and $\varphi_2 \in L^\infty([0, T], C^\delta(\mathbb{R}^m, \mathbb{R}^\ell))$, we define the norms:

$$\begin{aligned} \|\varphi_1\|_{L^\infty(C_b^\delta)} &:= \sup_{t \in [0, T]} \|\varphi_1(t, \cdot)\|_{C_b^\delta(\mathbb{R}^m, \mathbb{R}^\ell)}, \\ \|\varphi_2\|_{L^\infty(C^\delta)} &:= \sup_{t \in [0, T]} \|\varphi_2(t, \cdot)\|_{C^\delta(\mathbb{R}^m, \mathbb{R}^\ell)}, \\ \|\varphi_2\|_{L^\infty} &:= \sup_{t \in [0, T]} \|\varphi_2(t, \cdot)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^\ell)}. \end{aligned}$$

The test functions for some weak formulations of different solutions will be in $C_0^\infty(\mathbb{R}^d, \mathbb{R})$, which corresponds to the space of smooth functions infinitely differentiable with bounded derivatives and with a compact support.

2.2.2 Thermic characterization of the Besov space

We define the Besov spaces thanks to a thermic characteristic, see Triebel [Tri83] Section 2.6.4. For all $\alpha \in \mathbb{R}$, $q \in (0, +\infty]$, $p \in (0, \infty]$,

$$\|f\|_{B_{p,q}^\alpha} := \|\varphi(D)f\|_{L^p(\mathbb{R}^d)} + \|f\|_{\dot{B}_{p,q}^\alpha}, \quad \text{with } \|f\|_{\dot{B}_{p,q}^\alpha} := \left(\int_0^1 \frac{dv}{v} v^{(m-\frac{\alpha}{2})q} \|\partial_v^m h_v \star f\|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}, \quad (2.4)$$

where we define the heat kernel

$$h_v(z) := \frac{1}{(2\pi v)^{d/2}} \exp\left(-\frac{|z|^2}{2v}\right), \quad (2.5)$$

and $\varphi(D)f := (\varphi \hat{f})^\vee$ with $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(0) \neq 0$, \hat{f} and $(\varphi \hat{f})^\vee$ respectively denote the Fourier transform of f and the inverse Fourier transform of $\varphi \hat{f}$. Note that, when $\alpha > d(\frac{1}{p} - 1)_+ = d \max(0, \frac{1}{p} - 1)$ then in (2.4), it possible to replace $\|\varphi(D)f\|_{L^p(\mathbb{R}^d)}$ by $\|f\|_{L^p(\mathbb{R}^d)}$.

When $p = q = +\infty$, we naturally write:

$$\|f\|_{\dot{B}_{p,q}^\alpha} = \sup_{v \in [0,1]} v^{m-\frac{\alpha}{2}} \|\partial_v^m h_v \star f\|_{L^\infty(\mathbb{R}^d)},$$

and if $\alpha > d(\frac{1}{p} - 1)_+$,

$$\|f\|_{B_{p,q}^\alpha} = \|f\|_{L^\infty} + \sup_{v \in [0,1]} v^{m-\frac{\alpha}{2}} \|\partial_v^m h_v \star f\|_{L^\infty(\mathbb{R}^d)}.$$

We carefully point out that the homogeneous term $\|f\|_{\dot{B}_{p,q}^\alpha}$ does not define a norm associated to a Banach space. To consider the whole homogeneous Besov space we have to consider $v \in \mathbb{R}_+$ in the definition (2.4), for $\alpha < 0$, see e.g. Theorem 2.34 in [BCD11]. Somehow, for the inhomogeneous

norm defined in (2.4), the contribution of the heat kernel convolution for $v > 1$ is “hidden” in the inhomogeneous term $\|\varphi(D)f\|_{L^p(\mathbb{R}^d)}$.

For $\alpha > 0$, the homogeneous and respectively inhomogeneous Hölder spaces match with Besov space, namely $C^\alpha = \dot{B}_{\infty,\infty}^\alpha$ and $C_b^\alpha = B_{\infty,\infty}^\alpha$, see [Tri83] for details.

Our analysis tackles with inhomogeneous Besov spaces, in order to extend our analysis to the homogeneous ones some sophisticated changes should be performed as the homogeneous Besov spaces are *a priori* not Banach spaces; and we should consider the realisation of the space of homogeneous Besov spaces as a space of distributions defined quotiented by polynomials, see e.g. Proposition 3.8 in [LR02] to make it a Banach space.

If $\alpha < 0$ then it is known that $\dot{B}_{p,q}^\alpha \subset B_{p,q}^\alpha$, i.e. there is a constant $C > 0$ such that, for any $\alpha < 0$,

$$\|\cdot\|_{B_{p,q}^\alpha} \leq -\frac{C}{\alpha} \|\cdot\|_{\dot{B}_{p,q}^\alpha}. \quad (2.6)$$

We also introduce the distributions that can be approached by a mollification procedure. We put a *tilde* in order to mean that we consider the closure of C_b^∞ in the considered space[†]. Namely, for any $(\alpha, p, q) \in \mathbb{R} \times (1, +\infty] \times (1, +\infty]$ we define:

$$\tilde{B}_{p,q}^\alpha := \mathbf{cl}_{B_{p,q}^\alpha}(C_b^\infty), \quad \tilde{\dot{B}}_{p,q}^\alpha := \mathbf{cl}_{\dot{B}_{p,q}^\alpha}(C_b^\infty). \quad (2.7)$$

Remark 1. *If there is $\psi \in C^\gamma$, $\gamma \in (0, 1)$, such that $b = D^\alpha\psi$, $\alpha \in \mathbb{N}_0^d$, so $b \in \dot{B}_{\infty,\infty}^{-\beta}$ with $\beta = -|\alpha| + \gamma$, and in Appendix Section E, we show that $b \in \tilde{\dot{B}}_{\infty,\infty}^{-\beta}$.*

The last constraint on b , being the derivative of a Hölder function, is quiet natural when we consider the structure theorem of the tempered distributions \mathcal{S}' , see Theorem 8.3.1 in [Fri98]. We recall indeed that any $b \in \mathcal{S}'$ writes $b = D^\alpha\psi$ where $\alpha \in \mathbb{N}_0^d$ and ψ is a continuous function with polynomial growth.

Out of the Besov-Hölder space, namely if $1 \leq p, q < +\infty$, then $\tilde{B}_{p,q}^\alpha = B_{p,q}^\alpha$ and $\tilde{\dot{B}}_{p,q}^\alpha = \dot{B}_{p,q}^\alpha$, see Theorem 4.1.3 in [AH96], Proposition 2.27 and Proposition 2.74 in [BCD11]. For more Besov properties, we also mention [Pee76] and [Jaw77].

We might consider the non-homogeneous low-frequency cut-off in the Littlewood-Paley characterisation instead of usual mollification by convolution, as performed in the current paper, and adapt Lemma 2.73 in [BCD11] for the space $B_{\infty,\infty}^{-\beta}$.

2.2.3 Besov duality

In our analysis, we thoroughly use the Besov duality. The full proof of the duality of Besov spaces is established for example in Proposition 3.6 in [LR02] thanks to a Littlewood-Paley decomposition.

Proposition 1. *For all $1 \leq p, q \leq +\infty$ and $\alpha \in \mathbb{R}$, we have for all $\varphi, \psi \in \mathcal{S}'$:*

$$\left| \int_{\mathbb{R}^d} \varphi(y)\psi(y)dy \right| \leq C_{d,p,q,\alpha} \|\varphi\|_{B_{p,q}^\alpha} \|\psi\|_{B_{p',q'}^{-\alpha}},$$

with $1 \leq p', q' \leq +\infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Sketch of the proof. Let us suppose that w.l.o.g. that $1 < p, q < +\infty$ (the analysis is identical if we suppose that $1 < p', q' < +\infty$). It is known that $B_{p,q}^\alpha(\mathbb{R}^d, \mathbb{R})$ and $B_{p',q'}^{-\alpha}(\mathbb{R}^d, \mathbb{R})$ are in duality (Proposition 3.6 in [LR02]). Precisely, $B_{p,q}^\alpha$ is the dual of the closure of the Schwartz class \mathcal{S} in $B_{p',q'}^{-\alpha}$. But \mathcal{S} is dense in $B_{p,q}^\alpha$ (see for instance 4.1.3. in [AH96]).

□

The homogeneous counterpart of this result requires additional assumptions on the considered distributions, see for instance Proposition 2.29 in [BCD11].

[†]As in Proposition 3.6 in [LR02] for the closure Schwartz space, but we do not need the mollified versions of the considered distributions to be rapidly decreasing functions.

2.2.4 Usual tools for the Gaussian function

One of the reason to use the thermic representation of the Besov space comes from a well-known and important result about the Gaussian function: for any $\delta > 0$, there is $C_\delta = C_\delta(\delta) > 1$ such that:

$$\forall x \in \mathbb{R}^d, |x|^\delta e^{-|x|^2} \leq C_\delta e^{-C_\delta^{-1}|x|^2}. \quad (2.8)$$

Furthermore, we often use the cancellation principle: for all $f \in C^\gamma$, $\gamma \in (0, 1)$, $x \in \mathbb{R}^d$ and $\sigma > 0$

$$\nabla_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} f(y) dy = \int_{\mathbb{R}^d} \nabla_x e^{-\frac{|x-y|^2}{2\sigma}} [f(y) - f(x)] dy, \quad (2.9)$$

as the Gaussian function, up to a renormalisation by a multiplicative constant, is a probabilistic distribution, hence $\nabla_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} dy = 0$. Hence, we obtain,

$$\begin{aligned} (2\pi\sigma)^{\frac{d}{2}} \left| \nabla_x \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} f(y) dy \right| &\leq (2\pi\sigma)^{\frac{d}{2}} [f]_\gamma \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\sigma}} \frac{|y-x|}{\sigma} |y-x|^\gamma dy \\ &\leq C_\gamma (2\pi\sigma)^{\frac{d}{2}} [f]_\gamma \sigma^{\frac{\gamma-1}{2}} \int_{\mathbb{R}^d} e^{-C_\gamma^{-1} \frac{|x-y|^2}{2\sigma}} dy \\ &= C_\gamma [f]_\gamma \sigma^{\frac{\gamma-1}{2}}. \end{aligned}$$

The penultimate identity comes from the absorbing property (2.8).

2.3 Different definitions of solution to the transport equation problem

As said in the introduction, we need to carefully defined the suitable notion of solution, no strong solution can be in a negative Besov space or even in a Hölder space implying a product of distributions which is obviously not point-wisely defined.

Definition 1 (mild vanishing viscous). *A function u is said to be a mild vanishing viscous solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if for a sequence $(b_m)_{m \in \mathbb{N}}$ in $L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$ such that there is $\beta \in \mathbb{R}$,*

$$\forall \varepsilon > 0, \lim_{m \rightarrow +\infty} \|b_m - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} = 0, \quad (2.10)$$

for any $t \in [0, T]$, there exists a sub-sequence of $(u^{m, \nu}(t, \cdot))_{(m, \nu) \in \mathbb{R}_+^2}$ lying in $C_b^\gamma(\mathbb{R}^d, \mathbb{R})$ converging in the space $C_b^{\gamma-\tilde{\varepsilon}}(K, \mathbb{R})$, $0 < \tilde{\varepsilon} < \gamma$, for any compact subset $K \subset \mathbb{R}^d$, when $\nu \rightarrow 0$ and $m \rightarrow +\infty$ towards $u(t, \cdot) \in C_b^\gamma(K, \mathbb{R})$ and satisfying, for all $m \in \mathbb{N}$ and $\nu \in \mathbb{R}_+$,

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + \langle b_m(t, x), \nabla u^{m, \nu}(t, x) \rangle - \nu \Delta u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (2.11)$$

where $(f_m, g_m) \xrightarrow{m \rightarrow +\infty} (f, g)$ in $L^\infty([0, T]; C^\gamma(\mathbb{R}^d, \mathbb{R}))$.

We point out that such a sequence $(b_m)_{m \geq 0}$ exists if $b \in L^\infty([0, T]; \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$; or in particular if b is the derivative of a bounded Hölder continuous function but in this former case the limit result (2.10) has to be in the homogeneous space $L^\infty([0, T]; \dot{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, see Appendix Section E, and identity (2.10) is implied by (2.6).

Moreover, it is important to notice that the choice of sub-sequence may depend on the current time t . We do not succeed in getting uniform continuity in time t (only boundedness in L^∞), thus the impossibility to apply a suitable compact argument, we need to consider the problem at a fixed time. Nevertheless, for the sake of simplicity we write for the sub-sequence $u^{m, \nu}(t, \cdot)$ instead of a notation of the kind $u^{m_t, \nu_t}(t, \cdot)$.

Remark 2. We could consider another formulation of mild vanishing viscous solution, where the considered function is

$$w^{m,\nu}(t, x) = \int_0^t u^{m,\nu}(s, x) ds. \quad (2.12)$$

From estimates stated in Theorem 1 below, we have that $(t, x) \mapsto w^{m,\nu}(t, x)$ lies, uniformly in (m, ν) , in $C_b^1([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$, by the Arzelà-Ascoli theorem, we obtain a convergence in all compacts $[0, T] \times K$ of $[0, T] \times \mathbb{R}^d$ towards a function $w \in C_b^1([0, T]; C_b^\gamma(K, \mathbb{R}))$.

Let us define an alternative form of solution which is a mixed version between *mild* and *weak* solution.

Definition 2 (mild-weak solution). A function u is a mild-weak solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if u is a mild vanishing viscous solution, i.e. there is a smooth function sequence $(b_m)_{m \in \mathbb{R}_+} \in L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$ such that there is $\beta \in \mathbb{R}$,

$$\forall \varepsilon > 0, \lim_{m \rightarrow +\infty} \|b_m - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} = 0, \quad (2.13)$$

there exists a sub-sequence of $(u^{m,\nu})_{(m,\nu) \in \mathbb{R}_+^2}$ in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ strong solution of (2.11) converging, for any compact $K \subset \mathbb{R}^d$, in $L^\infty([0, T]; C_b^{\gamma-\varepsilon}(\mathbb{R}^d, \mathbb{R}))$, $0 < \varepsilon < \gamma$, towards the Hölder continuous $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and such that for any function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$, we have, up to a sub-sequence selection, for any $t \in [0, T]$,

$$\begin{aligned} \lim_{m \rightarrow \infty, \nu \rightarrow 0} \int_{\mathbb{R}^d} \left\{ \varphi(t, y) u^{m,\nu}(t, y) + \int_0^t \left\{ -\partial_t \varphi(s, y) u^{m,\nu}(s, y) + \langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle \varphi(s, y) \right\} ds \right\} dy \\ = \int_{\mathbb{R}^d} \varphi(0, y) g(y) dy + \int_{\mathbb{R}^d} \int_0^t \varphi(s, y) f(s, y) ds dy. \end{aligned} \quad (2.14)$$

The distributional formulation allows to give a sense to the potential irregularities of b and of $\nabla u^{\nu, m}$ when $\nu \rightarrow 0$, $m \rightarrow +\infty$, and to consider the whole space $[0, T] \times \mathbb{R}^d$, the cut-off of \mathbb{R}^d required to use the Arzelà-Ascoli theorem is included in the test function φ .

We define now the usual *weak* solution.

Definition 3 (weak solution). A function u is a weak solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if u is a mild vanishing viscous solution and for any function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$:

$$\begin{aligned} \int_{\mathbb{R}^d} \left\{ \varphi(t, y) u(t, y) + \int_0^t \left\{ -\partial_t \varphi(s, y) u(s, y) + \langle b(s, y), \nabla u(s, y) \rangle \varphi(s, y) \right\} ds \right\} dy \\ = \int_{\mathbb{R}^d} \varphi(0, y) g(y) dy + \int_{\mathbb{R}^d} \int_0^t \varphi(s, y) f(s, y) ds dy. \end{aligned} \quad (2.15)$$

Remark 3. We cannot hope to define classical solution in our irregular context. Indeed, even if roughly speaking $\partial_t u + \langle b, \nabla u \rangle$ is supposed to lie in $L^\infty(C_b^\gamma)$, we cannot a priori define point-wisely the classic scalar product between b and ∇u . If b has a blow up at a point $x_0 \in \mathbb{R}^d$ then, as u is solution of (2.11), $\lim_{x \rightarrow x_0} \int_0^t \langle b(s, x), \nabla u(s, x) \rangle ds$ is necessary finite for any $t \in [0, T]$ but we cannot give a meaning of $\langle b(t, x_0), \nabla u(t, x_0) \rangle$ in a point-wise sense. Roughly speaking, to handle distributional drift we have to stay in a distributional formulation of the solution.

3 User's guide

3.1 A first approach for the transport equation

In Section 4, we introduce a linear *proxy* of parabolic equation (1.2),

$$\begin{cases} \partial_t v^{m,\nu}(t, x) + \langle b_m(t, \theta_{t,\tau}(\xi)), \nabla v^{m,\nu}(t, x) \rangle - \nu \Delta v^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ v^{m,\nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (3.16)$$

where $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$ are the *freezing* parameters, and $\theta_{t,\tau}^m$ is the flow associated with b_m , namely,

$$\dot{\theta}_{t,\tau}^m(\xi) = -b_m(t, \theta_{t,\tau}^m(\xi)), \quad \theta_{\tau,\tau}^m(\xi) = \xi.$$

From these notations, we readily write the associated Duhamel formula,

$$v^{m,\nu}(t, x) = \hat{P}^{\tau,\xi} g_m(t, x) + \hat{G}^{\tau,\xi} f_m(t, x), \quad (3.17)$$

where $\hat{P}^{\tau,\xi}$ and $\hat{G}^{\tau,\xi}$ stand respectively for the semi-group and the Green operator associated with (3.16), the kernel being a perturbed heat kernel and detail in Section 5.1.3 below.

Obviously, v does depend on the parameters (τ, ξ) ; but this is not the case of the initial PDE (1.2), where we use the above Duhamel identity, based on (3.17):

$$u^{m,\nu}(t, x) = \hat{P}^{\tau,\xi} g_m(t, x) + \hat{G}^{\tau,\xi} f_m(t, x) + \hat{G}^{\tau,\xi} (b_\Delta^m[\tau, \xi] \cdot \nabla u^{m,\nu})(t, x). \quad (3.18)$$

The core of the first analysis for the transport equation is to make negligible the remainder term $\hat{G}^{\tau,\xi} (b_\Delta^m[\tau, \xi] \cdot \nabla u^{m,\nu})(t, x)$ thanks to the vanishing viscosity ν . This aim is quiet easy for the L^∞ norm, by choosing the parameters $(\tau, \xi) = (t, x)$; nevertheless for the Hölder estimates the analysis is much more involved. Indeed, to handles with $u^{m,\nu}(t, x) - u^{m,\nu}(t, x')$, we need to consider two pairs of parameters (τ, ξ) and (τ', ξ') , whose suitable choice depends on the current considered regime: *-diagonal* when x and x' are close w.r.t. the current time integral[‡] *-off-diagonal* when x and x' are fare from each other. Like for the *cut locus* trick (inspired in Riemannian manifold when there are multiple minimizing geodesics) introduced in [CDRHM18].

This regime separation implies to handle with extra contributions, see Sections 5.3.1, 5.3.3 and 5.3.5, which are treated in Sections 5.3.2 and 5.3.4.

At the end of the day, we derive a cumbersome upper-bound

$$\begin{aligned} & \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\ & \leq t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + C \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{2+\beta} O_m(2t) \left(\nu^{\frac{1-\gamma^2}{4}} t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} + \nu^{\frac{\gamma(1-\gamma)}{4}} t^{\frac{1}{2} + \frac{4-\gamma^2}{2\gamma}} \right) \\ & \quad + \frac{m^{1+\beta}}{1+\gamma} C \nu^{\frac{\gamma(1-\gamma)}{4}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \\ & \quad + \frac{\nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t) \\ & \quad + \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1} m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(\|b_m\|_{L^\infty(C^1)} t), \end{aligned} \quad (3.19)$$

with for any $t \in (0, T]$,

$$O_m(t) := \left(T \|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp\left(C m^{1+\beta} t \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \right),$$

see Lemma 2 further.

[‡]Again see Section 5.1.3 further for the full details

Thus the Hölder control, uniformly in (β, m, ν) , as soon as ν goes to 0 exponentially faster than $m \rightarrow +\infty$, and $\gamma < 1$.

The drawback of this approach, only based on a *regularisation by turbulence*, is that this vanishing viscosity constraint prevents us to get any balance with m , in order to deduce any uniqueness; also there is no possibility to obtain Hölder controls for the parabolic equation with non-negligible viscosity ν .

3.2 A new method for the parabolic equation

That is why we develop, in Section 6, an extension of this vanishing viscosity analysis. Indeed, instead of taking advantage of the negligible contributions of ν , we look at small time interval sizes whose machinery is similar. To be more specific, we decompose the Cauchy problem in small time intervals $\cup_{k \in [0, n-1]} [\tau_k, \tau_{k+1}] = [0, T]$, with $\tau_k = \frac{kT}{n}$, $n \in \mathbb{N}$.

For $k \in [0, n-1]$, and for any $[\tau_k, \tau_{k+1}]$, we define $u_{k+1}(t, \cdot) = u(t, \cdot)$, which satisfies the following Cauchy problem.

If $k \in [1, n-1]$

$$\begin{cases} \partial_t u_{k+1}^{m, \nu}(t, x) + \langle b_m, \nabla u_{k+1}^{m, \nu} \rangle(t, x) = \nu \Delta u_{k+1}^{m, \nu}(t, x) + f_m(t, x), & t \in (\tau_k, \tau_{k+1}] \\ u_{k+1}^{m, \nu}(\tau_k, x) = u_k^{m, \nu}(\tau_k, x) (= u^{m, \nu}(\tau_k, x)), \end{cases}$$

if $k = 0$

$$\begin{cases} \partial_t u_1^{m, \nu}(t, x) + \langle b_m, \nabla u_1^{m, \nu} \rangle(t, x) = \nu \Delta u_1^{m, \nu}(t, x) + f_m(t, x), & t \in (0, \tau_1) \\ u_1^{m, \nu}(0, x) = g_m(x) (= u^{m, \nu}(0, x)). \end{cases}$$

By mimicking the first computations done for the transport equation, we again obtain a heavy control

$$\begin{aligned} & \|u_{k+1}^{m, \nu}(t, \cdot)\|_{C^\gamma} \\ & \leq [u_k^{m, \nu}(\tau_k, \cdot)]_\gamma + \int_{\tau_k}^t \|f(s, \cdot)\|_{C^\gamma} ds \\ & + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} \int_{\tau_k}^t ds \\ & + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} (t - \tau_k) \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & + C\nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) (t - \tau_k)^{\frac{(1-\gamma)}{2}+1} \int_{\tau_k}^t ds, \end{aligned} \quad (3.20)$$

with

$$O_m^{(2)}(t) := \|\nabla^2 u^{m, \nu}(t, \cdot)\|_{L^\infty} \leq Cm^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(Cm^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}),$$

see Lemma 2. The upper-bound in (3.20) above does not match with (3.19), because we have to change the considered *diagonal/off-diagonal* regimes (due to the new paradigm of small time interval size instead of small ν).

Hence, iterating this inequality yields

$$\begin{aligned} & \|u^{m, \nu}(t, \cdot)\|_{C^\gamma} \\ & \leq \|g\|_{C^\gamma} + \int_0^t \|f(s, \cdot)\|_{C^\gamma} ds \\ & + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} \int_0^t ds \\ & + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} t \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & + C\nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) \left(\frac{T}{n}\right)^{\frac{(1-\gamma)}{2}+1} \int_0^t ds. \end{aligned}$$

Because, the solution $u^{m,\nu}$ does not depend on the number of time decomposition, we are able to pass to the limit $n \rightarrow +\infty$, and the control meet the Hölder estimates for transport equation of Theorem 1 below, without any vanishing viscosity.

With this time decomposition, we also derive the same control as for the upper-bound derive from the probabilistic Feynman-Kac formula or from the classic maximum principle, see Section 6.6.

3.3 A second approach for the transport equation

Thanks to this new time trick, we are able to enhance our first result on the transport equation, i.e. we obtain the same controls without the exponential convergence constraint on ν , see Theorem 3 stated in Section 7. This alleviation of assumption allows to handle with uniqueness, in a positive regularity framework, see Section 7.2. The above time cutting trick combined with the phenomenon of *regularisation by turbulence* yields uniqueness.

3.4 Application to the inviscid Burgers' equation

Eventually, by a fixed-point argument and by the previous analysis, we deduce existence and uniqueness of a Hölder continuous solution of the Burger's equation in Theorem 4 in Section 8. We carefully detail therein the proof of uniqueness, which is little bit more fussy than for the transport equation because of the regularised first order term b , which matches with the regularised solution itself $u_m^{m,\nu}$, is not *a priori* supposed to converge towards a unique limit u .

4 A first result on the transport equation

Our first statement is based on a *cut locus* trick combined with *vanishing viscous* properties which allows to get a first result about the transport equation (1.1). In order to get rid of this *vanishing viscous* case, namely to consider a parabolic equation (1.2) and to pass to the limit of the mollification parameter m , we need an other refinement based on a time decomposition specified in Section 6.3 further.

4.1 Statement

When b lies in a Hölder-Besov space, we succeed in obtaining the same regularity of the solution as for f and g . The type of solution strongly depends on the regularity of b .

Theorem 1 (Existence of solution to rough transport equation). *For $\gamma \in (0, 1)$, $\beta \in \mathbb{R}^*$ be given. For all $b \in L^\infty([0, T], \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, $f \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}^d, \mathbb{R})$, there is a mild vanishing viscosity solution $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of (1.1) satisfying*

$$\begin{aligned} \|u\|_{L^\infty(C^\gamma)} &\leq T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma, \\ \|u\|_{L^\infty} &\leq T\|f\|_{L^\infty} + \|g\|_{L^\infty}, \end{aligned} \quad (4.1)$$

if the conditions on the vanishing viscosity $0 < \nu < T^{-1}$, for a given constant $C > 0$ depending only on (γ, d) ,

$$\begin{aligned} \nu &\ll (m^{2+\beta-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma}+2} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})})^{-\frac{4}{1-\gamma^2}} \exp\left(-C \frac{m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} T}{1-\gamma^2}\right), \\ \nu &\ll \left(C m^{2+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} T^{\frac{2-\gamma^2+\gamma}{2\gamma}} (m^{1-\gamma} (T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)) (1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\ &\quad \times \exp\left(-\frac{8m^{1+\beta} T \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}}{\gamma(1-\gamma)}\right), \end{aligned} \quad (4.2)$$

are satisfied. Moreover, with additional conditions on b , we have:

i) **Incompressibility.** If $\beta < \gamma$ and $\nabla \cdot b = 0$ then the solution u is also a mild-weak and a weak solution.

ii) **Positive regularity.** If $\beta < -1 + \gamma$, namely if $b \in L^\infty([0, T]; C_b^{\tilde{\gamma}}(\mathbb{R}^d, \mathbb{R}^d))$, $\tilde{\gamma} > 1 - \gamma$, then the solution u is also a mild-weak and a weak solution, moreover if

$$\nu \|\nabla^2 g_m\|_{L^\infty} + \nu^{\frac{\tilde{\gamma}}{2}} m^{1-\tilde{\gamma}} \ll 1, \quad (4.3)$$

then, for any $t \in (0, T]$, $\partial_t u(t, \cdot) \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$.

Remark 4. For the mild vanishing viscous solution there is no restriction on b , roughly speaking there is an “infinite regularisation by turbulence” over the coefficients.

While for the usual weak solution, there is no more such infinite regularisation effect. The **Incompressibility** framework, i.e. $\nabla \cdot b = 0$, allows to still consider a negative regularity of b . Such divergence free condition for non-smooth distributions already exists for instance for Leray’s solution of Navier-Stokes equation [Ler34].

In the last case, i.e. **Positive regularity**, the considered drift b is supposed to be Hölder continuous in space, in particular lying in $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^d, \mathbb{R}^d))$, $\alpha > 1 - \gamma$ which is the Bony’s para-product assumption, see Section 4.2 below for more details.

Importantly, the above controls (4.1) do not depend on the drift b . This is crucial to consider very rough coefficients as well as non-linear equation such as the inviscid Burgers’ equation studied in Section 8. This seems to be paradoxical with the usual gradient control of the transport equation by characteristic method, we carefully explain the reasons in Section 5.3.7 further.

Remark 5. The exponential criterion in (4.2) relies on an a priori control by $\|u^{m, \nu}\|_{L^\infty}$, see Section 5.2.7 for more details. This condition prevents us to hope any balance between m and ν required to get usual uniqueness. Indeed, when we expand the computations, we see only polynomial dependency on (m, ν) in the upper-bounds. But the contribution on ν goes in the wrong way, and cannot be overwhelmed by polynomial converging terms in m , because at the best $m \sim |\ln(\nu)|$ from (4.2).

Even for b lying in a Hölder space, namely with a positive regularity, we cannot avoid an exponential criterion like in (4.2) by the current analysis, see again Section 5.2.7.

We fail to obtain a uniqueness of a viscous selection principle with this first analysis. Indeed in the a priori controls of (2.11) we have to suppose that ν goes to 0 much faster than m towards $+\infty$. This constraint prevents us to take advantage of the convergence of b_m towards b to balance the blow up in the viscosity ν occurring in the computation to get uniqueness.

4.2 On the product of distributions

The sense of some particular products of distributions is very challenging, and is related with many long-standing problems. For instance, Hairer in [Hai14] introduce a regularity structure theory which after some renormalisation allows to handle with products of distribution, and to give a meaning of stochastic partial differential equation such as KPZ [Hai13]. However, such renormalisation leads to blowing-up constants which is not the case in Theorem 1; the price that we have to pay is the potential non-uniqueness of the limit.

From the different formulations above, we define different meanings of the product $\langle b, \nabla u \rangle$. First of all, let us remark that by rough a priori controls, see Lemma 2 below,

$$\begin{aligned} [\Delta u^{m, \nu}(t, \cdot)]_\gamma &\leq 2^{1-\gamma} \|\nabla^2 u^{m, \nu}(t, \cdot)\|_{L^\infty}^{1-\gamma} \|\nabla^3 u^{m, \nu}(t, \cdot)\|_{L^\infty}^\gamma \\ &\leq C m^{2+\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(C m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}). \end{aligned}$$

Hence, if $\nu \ll C m^{-(2+\gamma)} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)^{-1} \exp(-C t m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})})$, implied by (4.2), then, up to subsequence choice, $\nu \Delta u^{m, \nu}(t, \cdot) \xrightarrow{(m, \nu) \rightarrow (+\infty, 0)} 0$ in $C_b^\gamma(\mathbb{R}^d, \mathbb{R})$.

Also, for a given $t \in [0, T]$, we see from the definition of *mild vanishing viscous* solution, up to subsequence choice according to the condition (4.2), that

$$\lim_{(m, \nu) \rightarrow (+\infty, 0)} \int_0^t \langle b_m(s, \cdot), \nabla u^{m, \nu}(s, \cdot) \rangle ds = g - u(t, \cdot) - \int_0^t f(s, \cdot) ds \in C_b^\gamma(K, \mathbb{R}), \quad (4.4)$$

for any compact $K \subset \mathbb{R}^d$. We highly point out that b lies in any arbitrary negative regularity in space $L^\infty([0, T]; B_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, $\beta \in \mathbb{R}_+$, and $\nabla u(s, \cdot) \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$.

In other words, thanks to the time averaging, we get a new para-product condition. Indeed, in general from Bony's microlocal analysis [Bon81], for all $\varphi \in B_{\infty, \infty}^{\alpha_1}$ and $\psi \in B_{\infty, \infty}^{\alpha_2}$, we have

$$\phi\psi \in B_{\infty, \infty}^{\alpha_1 \wedge \alpha_2}, \text{ if } \alpha_1 + \alpha_2 > 0. \quad (4.5)$$

However, the uniqueness of the limit in (4.4) seems to be false in general.

Also in the weak formulation, from Theorem 1, we obtain, if $\nabla \cdot b = 0$ and $\beta < \gamma$ (regularity condition weaker than (4.5)), a distributional meaning of $\langle b(s, \cdot), \nabla u(s, \cdot) \rangle$, but we still do not know in this case if the limit is unique.

Finally, we point out that in the **Positive regularity** framework, we meet the Bony's para-product condition (4.5). Indeed, for any $t \in [0, T]$, $b(t, \cdot) \in B_{\infty, \infty}^\alpha$, $\alpha = -\beta$, and $\nabla u(t, \cdot) \in B_{\infty, \infty}^{-1+\gamma}$ with $\alpha - 1 + \gamma > 0$. We are able to quantify the regularity, $\langle b(t, \cdot), \nabla u(t, \cdot) \rangle \in B_{\infty, \infty}^{-1+\gamma}$ by para-product detailed further in Section 5.6. Moreover, the time averaging version $\int_0^t \langle b(s, \cdot), \nabla u(s, \cdot) \rangle ds$ in the sense of (4.4) is γ -Hölder. We remark, as $\alpha \wedge (-1 + \gamma) = -1 + \gamma$ then there is a +1 gain of regularity comparing with the usual para-product result.

5 Proof of Theorem 1

5.1 Parabolic approximation procedure

Let us first smoothen the drift and the source functions of the parabolic approximation,

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + \langle b_m, \nabla u^{m, \nu} \rangle(t, x) - \nu \Delta u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (5.1)$$

where the mollified functions are defined by

$$\begin{aligned} b_m(t, x) &:= \int_{\mathbb{R}^d} \rho_m(x - y) b(t, y) dy, \\ f_m(t, x) &:= \int_{\mathbb{R}^d} \rho_m(x - y) f(t, y) dy, \\ g_m(t, x) &:= \int_{\mathbb{R}^d} \rho_m(x - y) g(y) dy, \end{aligned} \quad (5.2)$$

for $\rho_m(\cdot) := m^d \rho(m \cdot)$ where ρ is a non-negative smooth function ρ_m , such that $\int_{\mathbb{R}^d} \rho_m(x - y) dy = 1$. In particular, we choose $\rho = h_1$, the heat kernel defined in 2.4. In Appendix Sections G and E, we see that the limit of b_m does not depend on the choice of the mollification procedure, whereas the limit of $u^{m, \nu}$ potentially does.

In our analysis, we use some point-wise controls of the mollified functions or distributions whose blowing-up in the regularisation parameter m is stated below.

Lemma 1. *For all $m > 1$, and $\beta > 0$, if $b \in L^\infty([0, T], B_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, we have for any $(t, x) \in [0, T] \times \mathbb{R}^d$:*

$$\begin{aligned} |b_m(t, x)| &\leq C m^\beta \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}, \\ |\nabla b_m(t, x)| &\leq C m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}, \end{aligned} \quad (5.3)$$

where ∇b_m stands for the Jacobian matrix of b_m ; also if $\beta \leq 0$,

$$\begin{aligned} |b_m(t, x)| &\leq \|b\|_{L^\infty}, \\ |\nabla b_m(t, x)| &\leq C m^{1+\beta} \|b\|_{L^\infty(C^{-\beta})}. \end{aligned}$$

Proof of Lemma 1. From the mollification definition (5.2), we see, from (2.5), that $\rho_m = h_{m^{-2}}$, and from our scaling choice, we get for any $m > 1$

$$\begin{aligned} |b_m(t, x)| &= \left| \int_{\mathbb{R}^d} h_{m^{-2}}(x-y) b(t, y) dy \right| \\ &\leq m^\beta \sup_{\tilde{m}^{-2} \in [0,1], x \in \mathbb{R}^d} \tilde{m}^{-\beta} \left| \int_{\mathbb{R}^d} h_{\tilde{m}^{-2}}(x-y) b(t, y) dy \right|. \end{aligned}$$

We readily get by the thermic definition of the Besov norm (2.4):

$$|b_m(t, x)| \leq m^\beta \|b\|_{L^\infty(\dot{B}_{\infty, \infty}^{-\beta})} \leq m^\beta \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}.$$

For the second inequality, it is known that for any $t \in [0, T]$, $\nabla b(t, \cdot) \in B_{\infty, \infty}^{-1-\beta}$, see Theorem 9 of Chapter 3 in [Pee76], in particular if $\beta \in (0, -1)$ see Corollary 6 in Appendix Section F. Hence,

$$\begin{aligned} |\nabla_x b_m(t, x)| &\leq m^{1+\beta} \sup_{\tilde{m}^{-2} \in [0,1], \tilde{x} \in \mathbb{R}^d} \tilde{m}^{-\beta} \left| \int_{\mathbb{R}^d} h_{\tilde{m}^{-2}}(\tilde{x}-y) \nabla_y b(t, y) dy \right| \\ &= m^{1+\beta} \|\nabla b\|_{L^\infty(\dot{B}_{\infty, \infty}^{-1-\beta})}. \end{aligned}$$

We deduce that there is a constant $C = C(d) > 0$ such that:

$$|\nabla_x b_m(t, x)| \leq C m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})},$$

see Corollary 6 in Appendix Section F.

The two last inequalities, i.e. for the case $\beta \leq 0$, are standard. \square

5.1.1 L^∞ control

We have directly by the Feynman-Kac formulation the uniform control, see for example from the analysis performed in [Hon22], or from maximum principle for linear parabolic equation see e.g. [Lie96].

$$\|u^{m, \nu}\|_{L^\infty} \leq T \|f\|_{L^\infty} + \|g\|_{L^\infty}. \quad (5.4)$$

For the sake of completeness, we provide in Section 6.6 a way to get exactly the same upper-bound when $\nu \rightarrow 0$.

5.1.2 Some a priori controls

It is well-known that the unique solution $u^{m, \nu}$ of (5.1) is smooth, see [Fri64]. Some *a priori* controls, potentially blowing up with m and ν , are required in our analysis.

Lemma 2. *For $u^{m, \nu}$ strong solution of (5.1), we have*

$$\|\nabla u^{m, \nu}(t, \cdot)\|_{L^\infty} \leq \left(T \|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp \left(C m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \right) =: O_m(t), \quad (5.5)$$

and

$$\|\nabla^2 u^{m, \nu}(t, \cdot)\|_{L^\infty} \leq C m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(C m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}) =: O_m^{(2)}(t), \quad (5.6)$$

also

$$\|\nabla^3 u^{m, \nu}(t, \cdot)\|_{L^\infty} \leq C m^{3-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(C m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}). \quad (5.7)$$

The proof is postponed in Appendix Section A.

5.1.3 Proxy choice

To derive the other estimates from Duhamel formulation, we approximate the Cauchy problem around the flow associated to the smooth function b_m , which is unique by Cauchy-Lipschitz theorem. Namely, let us consider the unique function defined for any *freezing* point $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$ by,

$$\theta_{s,\tau}^m(\xi) := \xi + \int_s^\tau b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s}, \quad s \in [0, \tau]. \quad (5.8)$$

In other words, for any $t \in [0, \tau]$,

$$\dot{\theta}_{t,\tau}^m(\xi) = -b_m(t, \theta_{t,\tau}^m(\xi)), \quad \theta_{\tau,\tau}^m(\xi) = \xi.$$

We again rewrite the system of linear parabolic PDEs (5.2),

$$\partial_t u^{m,\nu}(t, x) + b_m(t, \theta_{t,\tau}^m(\xi)) \cdot \nabla u^{m,\nu}(t, x) - \nu \Delta u^{m,\nu}(t, x) = b_\Delta^m[\tau, \xi](t, x) \cdot \nabla u^{m,\nu}(t, x) + f_m(t, x), \quad (5.9)$$

where we define

$$b_\Delta^m[\tau, \xi](t, x) := b_m(t, \theta_{t,\tau}^m(\xi)) - b_m(t, x). \quad (5.10)$$

For such a fixed freezing point $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$, we use the corresponding Duhamel formula:

$$u^{m,\nu}(t, x) = \hat{P}^{\tau,\xi} g_m(t, x) + \hat{G}^{\tau,\xi} f_m(t, x) + \hat{G}^{\tau,\xi} (b_\Delta^m[\tau, \xi] \cdot \nabla u^{m,\nu})(t, x), \quad (5.11)$$

where we define, for any $f \in C_0^{1,2}((0, T] \times \mathbb{R}^d, \mathbb{R})$, the Green operator associated with the perturbed parabolic equation with constant coefficients (5.9),

$$\forall (t, x) \in (0, T] \times \mathbb{R}^d, \quad \hat{G}^{\tau,\xi} f_m(t, x) := \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) f(s, y) dy ds, \quad (5.12)$$

and for any $g \in C_0^2(\mathbb{R}^d, \mathbb{R})$, the associated semi-group

$$\hat{P}^{\tau,\xi} g_m(t, x) := \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(0, t, x, y) g_m(y) dy, \quad (5.13)$$

where the perturbed heat kernel is

$$\hat{p}^{\tau,\xi}(s, t, x, y) := \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{\left|x + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right). \quad (5.14)$$

We carefully point out that, from definition (5.8), if $(\tau, \xi) = (t, x)$,

$$\hat{p}^{t,x}(s, t, x, y) = \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{|\theta_{s,t}^m(x) - y|^2}{4\nu(t-s)}\right).$$

We have for each $\alpha \in \mathbb{N}^d$ that there is a constant $C_\alpha > 1$ s.t.

$$\begin{aligned} |D^\alpha \hat{p}^{\tau,\xi}(s, t, x, y)| &\leq \frac{C_\alpha [\nu(t-s)]^{-\frac{|\alpha|}{2}}}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-C_\alpha^{-1} \frac{\left|x + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right) \\ &=: C [\nu(t-s)]^{-\frac{|\alpha|}{2}} \hat{p}^{\tau,\xi}(s, t, x, y), \end{aligned} \quad (5.15)$$

and also, after the derivatives we can choose $(\tau, \xi) = (t, x)$, and $\gamma \in [0, 1]$,

$$\begin{aligned} |D^\alpha \hat{p}^{t,x}(s, t, x, y)| \times \left|y - x - \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s}\right|^\gamma &= |D^\alpha \hat{p}^{t,x}(s, t, x, y)| \times |y - \theta_{s,t}^m(x)|^\gamma \\ &\leq C [\nu(t-s)]^{-\frac{|\alpha|}{2} + \frac{\gamma}{2}} \hat{p}^{t,x}(s, t, x, y), \end{aligned} \quad (5.16)$$

from absorbing property (2.8). It also clear, for any $0 \leq s < t$, that

$$\partial_t \hat{p}^{\tau, \xi}(s, t, x, y) = \nu \Delta \hat{p}^{\tau, \xi}(s, t, x, y) - \langle b_m(t, \theta_{t, \tau}^m(\xi)), \nabla \hat{p}^{\tau, \xi}(s, t, x, y) \rangle, \quad (5.17)$$

which naturally implies that the function $u^{m, \nu}$ defined in (5.11) is indeed solution to (5.1) and to (5.9).

Finally, we will marginally use the “pure” heat kernel already defined in (5.18),

$$\tilde{p}(s, t, x, y) = \hat{p}^{\tau, \xi}\left(s, t, x - \int_s^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau}^m(\xi)) d\tilde{s}, y\right) = \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4\nu(t-s)}\right), \quad (5.18)$$

recalling the corresponding Green operator

$$\forall (t, x) \in (0, T] \times \mathbb{R}^d, \quad \tilde{G}f_m(t, x) := \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, t, x, y) f_m(s, y) dy ds, \quad (5.19)$$

and the associated semi-group

$$\tilde{P}g_m(t, x) := \int_{\mathbb{R}^d} \tilde{p}(0, t, x, y) g_m(y) dy. \quad (5.20)$$

5.2 Control of Hölder modulus

For any $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, we choose the associated *freezing* points (τ, ξ) and (τ', ξ') . Like in [CDRHM18], we take $\tau = \tau'$. The previous Duhamel formula (5.11) yields

$$\begin{aligned} & |u^{m, \nu}(t, x) - u^{m, \nu}(t, x')| \\ & \leq |\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x')| + |\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x')| \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) [b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) dy ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(s, t, x, y) [b_m(s, \theta_{s, \tau'}^m(\xi')) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) dy ds \right| \\ & =: |\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x')| + |\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x')| + |R^{\tau, \xi, \xi'}(t, x, x')|. \end{aligned} \quad (5.21)$$

However, our analysis need different choices of *freezing* point which yields extra contributions in the above inequality, the final Duhamel like identity is stated in (5.62) further.

By integration by parts, we are able to rewrite the remainder term by

$$\begin{aligned} & R^{\tau, \xi, \xi'}(t, x, x') \\ & = \left\{ \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau, \xi}(s, t, x, y) \cdot [b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)] [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))] dy ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau, \xi'}(s, t, x', y) \cdot [b_m(s, \theta_{s, \tau}^m(\xi')) - b_m(s, y)] [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))] dy ds \right\} \\ & \quad + \left\{ \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) \nabla \cdot b_m(s, y) [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))] dy ds \right. \\ & \quad \left. - \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(s, t, x', y) \nabla \cdot b_m(s, y) [u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))] dy ds \right\} \\ & =: R_1^{\tau, \xi, \xi'}(t, x, x') + R_2^{\tau, \xi, \xi'}(t, x, x'). \end{aligned} \quad (5.22)$$

Actually, this integration by parts is not essential, we could use a point-wise *a priori* control of $\nabla u^{m, \nu}$ performed in Section 6. However we aim to track as sharp as possible the required *a priori* controls in the upper-bounds. To be more specific, we unsuccessfully tried to only upper-bound by $\|u^{m, \nu}\|_{L^\infty(C_b^1)}$ instead of $\|u^{m, \nu}\|_{L^\infty(C_b^1)}$. This last value is finite but increases exponentially with m , see Lemma 2. This exponential blowing-up yields the limit criterion (4.2) of (m, ν) , and prevents us to get any balance between m and ν to conclude with uniqueness of solution.

5.2.1 Main terms

For the main contributions associated with f and g , we choose the *freezing* parameters to be $\tau = t$ and $\xi = \xi' = x$, we choose the same parameters as in the *diagonal* regime specified further.

Semi-group

We readily derive by change of variables:

$$\begin{aligned}
& \left| \hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi} g_m(t, x') \right|_{\tau=t, \xi=\xi'=x} \\
&= \left| \int_{\mathbb{R}^d} [\hat{p}^{\tau, \xi}(0, t, x, y) - \hat{p}^{\tau, \xi'}(0, t, x', y)] g_m(s, y) dy \right|_{\tau=t, \xi=\xi'=x} \\
&= \left| \int_{\mathbb{R}^d} [\hat{p}^{t, x}(0, t, 0, y)] [g_m(s, x + y) - g_m(s, x' + y)] dy \right| \\
&\leq [g]_{\gamma} |x - x'|^{\gamma}.
\end{aligned} \tag{5.23}$$

Green operator

We also get by change of variables:

$$\begin{aligned}
& \left| \hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi} f_m(t, x') \right|_{\tau=t, \xi=\xi'=x} \\
&= \left| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{t, x}(s, t, x, y) - \hat{p}^{t, x'}(s, t, x', y)] f_m(s, y) dy ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{t, x}(s, t, 0, y)] [f_m(s, x - y) - f_m(s, x' - y)] ds dy \right| \\
&\leq \|f\|_{L^\infty(C^\gamma)} |x - x'|^{\gamma} t.
\end{aligned} \tag{5.24}$$

5.2.2 Remainder term

To analyse the Hölder modulus of the remainder term, the core of the *a priori* controls, we separate the *diagonal* regime from the *off-diagonal* one, as performed in [CDRHM18]. This strategy is natural in view with the vanishing viscous solution selected by a parabolic approximation.

However, in the vanishing viscosity context, we have to carefully track the dependency on ν which yields, for our first approach, to consider a unusual criterion of *diagonal* / *off-diagonal* regime, unlike for the standard parabolic scaling.

Specifically, for any $x, x' \in \mathbb{R}^d$ and for given parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, to be tailored further, we call *off-diagonal* regime the case $|x - x'| > \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s > t_0$ with

$$t_0 := t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}, \tag{5.25}$$

called the *cut locus* time; on the contrary the *diagonal* regime holds when $|x - x'| \leq \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s \leq t_0$.

The point t_0 can be regarded as a *cut-locus* point where we “catch” the shortest path from $u^{m, \nu}(t, x)$ to $u^{m, \nu}(t, x')$ if $t \in [0, t_0]$ and we choose another way if $t_0 < t$. We carefully point out that this procedure yields an extra contribution in (5.21), this is detailed in Sections 5.3, 5.3.2.

We specify in Section 5.3.3 below why it is possible to choose different *freezing* parameters for the remainder term according to the current regime; meanwhile the semi-group and the Green operator dealt in Section 5.2.1 has somehow to stay in the *diagonal* regime.

5.2.3 Diagonal regime

If $|x - x'| \leq \nu^{\alpha_1}(t - s)^{\alpha_2} \Leftrightarrow s \leq t_0 = t - \nu^{-\frac{\alpha_1}{\alpha_2}}|x - x'|^{\frac{1}{\alpha_2}}$, the points x and x' are supposed to be closed from each other, then we pick $\xi = \xi' = x$, also $\tau = t$, then we define the associated space

$$A(x, x', \nu, t)(s) := \{|x - x'| \leq \nu^{\alpha_1}(t - s)^{\alpha_2}\}, \quad (5.26)$$

with the indicator function

$$\mathbb{1}_{A(x, x', \nu, t)}(s) := \begin{cases} 1 & \text{if } |x - x'| \leq \nu^{\alpha_1}(t - s)^{\alpha_2}, \\ 0 & \text{if } |x - x'| > \nu^{\alpha_1}(t - s)^{\alpha_2}, \end{cases}$$

also the associated remainder terms

$$\begin{aligned} R_{1,A}^{\tau, \xi, \xi'}(t, x, x') &:= \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) \\ &\quad \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau, \xi}(s, t, x, y) \cdot (b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))) \right. \\ &\quad \left. - \nabla \hat{p}^{\tau', \xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s, \tau}^m(\xi')) - b_m(s, y)) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))) \right] dy ds, \end{aligned}$$

and,

$$\begin{aligned} R_{2,A}^{\tau, \xi, \xi'}(t, x, x') &:= \int_0^t \mathbb{1}_{A(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \left[\hat{p}^{\tau, \xi}(s, t, x, y) \nabla \cdot b_m(s, y) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))) \right. \\ &\quad \left. - \hat{p}^{\tau', \xi'}(s, t, x', y) \nabla \cdot b_m(s, y) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))) \right] dy ds. \end{aligned} \quad (5.27)$$

Let us remark that, if $t_0 \geq 0$, we equivalently write the above terms by

$$\begin{aligned} R_{1,A}^{\tau, \xi, \xi'}(t, x, x') &= \int_0^{t_0} \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau, \xi}(s, t, x, y) \cdot (b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))) \right. \\ &\quad \left. - \nabla \hat{p}^{\tau', \xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s, \tau}^m(\xi')) - b_m(s, y)) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))) \right] dy ds \\ &=: \nabla \cdot \hat{G}_{0, t_0}^{\tau, \xi} \left\{ (b_m(\cdot, \theta_{\cdot, \tau}^m(\xi)) - b_m) (u^{m, \nu} - u^{m, \nu}(\cdot, \theta_{\cdot, \tau}^m(\xi))) \right\} (t, x) \\ &\quad - \nabla \cdot \hat{G}_{t_0, t}^{\tau', \xi'} \left\{ (b_m(\cdot, \theta_{\cdot, \tau}^m(\xi')) - b_m) (u^{m, \nu} - u^{m, \nu}(\cdot, \theta_{\cdot, \tau}^m(\xi'))) \right\} (t, x'), \end{aligned} \quad (5.28)$$

and by

$$\begin{aligned} &R_{2,A}^{\tau, \xi, \xi'}(t, x, x') \\ &= \int_0^{t_0} \int_{\mathbb{R}^d} \left[\hat{p}^{\tau, \xi}(s, t, x, y) \nabla \cdot b_m(s, y) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi))) \right. \\ &\quad \left. - \hat{p}^{\tau', \xi'}(s, t, x', y) \nabla \cdot b_m(s, y) (u^{m, \nu}(s, y) - u^{m, \nu}(s, \theta_{s, \tau}^m(\xi'))) \right] dy ds \\ &=: \hat{G}_{0, t_0}^{\tau, \xi} \left\{ \nabla \cdot b_m (u^{m, \nu} - u^{m, \nu}(\cdot, \theta_{\cdot, \tau}^m(\xi))) \right\} (t, x) - \hat{G}_{t_0, t}^{\tau', \xi'} \left\{ \nabla \cdot b_m (u^{m, \nu} - u^{m, \nu}(\cdot, \theta_{\cdot, \tau}^m(\xi'))) \right\} (t, x'). \end{aligned} \quad (5.29)$$

Remainder term $R_{1,A}^{\tau, \xi, \xi'}(t, x, x')$

By change of variables[§], we directly get

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\
&= \left| \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \right. \\
&\quad \cdot \left\{ (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, x + y)) (u^{m,\nu}(s, x + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\
&\quad \left. \left. - (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, x' + y)) (u^{m,\nu}(s, x' + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x},
\end{aligned}$$

by parity of the Gaussian density, specifying that

$$\hat{p}^{\tau,\xi}(s, t, 0, y) = \frac{1}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-\frac{\left|\int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right),$$

and expanding the terms above, gives us

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\
&= \left| \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot \left\{ b_m(s, \theta_{s,\tau}^m(\xi)) (u^{m,\nu}(s, x + y) - u^{m,\nu}(s, x' + y)) \right. \right. \\
&\quad \left. \left. + (b_m(s, x + y) - b_m(s, x' + y)) u^{m,\nu}(s, \theta_{s,\tau}^m(\xi)) \right. \right. \\
&\quad \left. \left. + b_m(s, x' + y) u^{m,\nu}(s, x' + y) - b_m(s, x + y) u^{m,\nu}(s, x + y) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x},
\end{aligned}$$

and putting together the corresponding contributions

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\
&= \left| \int_0^t \mathbf{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot \left\{ b_m(s, \theta_{s,\tau}^m(\xi)) (u^{m,\nu}(s, x + y) - u^{m,\nu}(s, x' + y)) \right. \right. \\
&\quad \left. \left. + (b_m(s, x + y) - b_m(s, x' + y)) u^{m,\nu}(s, \theta_{s,\tau}^m(\xi)) \right. \right. \\
&\quad \left. \left. + (b_m(s, x' + y) - b_m(s, x + y)) u^{m,\nu}(s, x' + y) \right. \right. \\
&\quad \left. \left. - b_m(s, x + y) (u^{m,\nu}(s, x + y) - u^{m,\nu}(s, x' + y)) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x}.
\end{aligned}$$

We finally get

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t,\xi=\xi'=x} \\
&= \left| \int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{A(x,x',\nu,t)}(s) \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \right. \\
&\quad \cdot \left\{ (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, x + y)) (u^{m,\nu}(s, x + y) - u^{m,\nu}(s, x' + y)) \right. \\
&\quad \left. \left. + (b_m(s, x' + y) - b_m(s, x + y)) (u^{m,\nu}(s, x' + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right\} dy ds \right|_{\tau=t,\xi=\xi'=x}, \tag{5.30}
\end{aligned}$$

[§]Specifically, we choose the new variable $y' = y - x$.

hence by gradient control (5.15),

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=\xi'=x} \\
& \leq C|x-x'| \|b_m\|_{L^\infty(C^1)} \|u^{m,\nu}\|_{L^\infty(C^1)} \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \\
& \quad \times \int_{\mathbb{R}^d} (\nu(t-s))^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s,t,0,y) \times (|\theta_{s,\tau}^m(x) - x - y| + |\theta_{s,\tau}^m(x) - x' - y|) dy ds \\
& \leq C|x-x'| \|b_m\|_{L^\infty(C^1)} \|u^{m,\nu}\|_{L^\infty(C^1)} \\
& \quad \times \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} (\nu(t-s))^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s,t,0,y) (|\theta_{s,\tau}^m(x) - x - y| + |x' - x|) dy ds \Big|_{\tau=t,\xi=\xi'=x},
\end{aligned} \tag{5.31}$$

where we recall from (5.15) that

$$\bar{p}^{\tau,\xi}(s,t,0,y) = \frac{C}{(4\pi\nu(t-s))^{\frac{d}{2}}} \exp\left(-C^{-1} \frac{|\int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\xi)) d\tilde{s} - y|^2}{4\nu(t-s)}\right).$$

and by Lemma 2:

$$\|u^{m,\nu}(t, \cdot)\|_{C^1} \leq \left(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\right) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right) = O_m(t). \tag{5.32}$$

For $(\tau, \xi) = (t, x)$, recalling that $\int_s^t b_m(\tilde{s}, \theta_{\tilde{s},t}(x)) d\tilde{s} = \theta_{s,t}^m(x) - x$, and by absorption property (2.8), we deduce

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=\xi'=x} \\
& \leq C|x-x'| \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} O_m(t) \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \left(1 + [\nu(t-s)]^{-\frac{1}{2}} |x-x'|\right) ds \\
& \leq C|x-x'|^\gamma \nu^{\alpha_1(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} O_m(t) \int_0^t (t-s)^{(1-\gamma)\alpha_2} \left(1 + \nu^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}} \nu^{\alpha_1}(t-s)^{\alpha_2}\right) ds,
\end{aligned}$$

because in the current *diagonal* regime, $|x-x'|^{1-\gamma} \leq \nu^{\alpha_1(1-\gamma)}(t-s)^{\alpha_2(1-\gamma)}$.

If $(1-\gamma)\alpha_2 > -1$ and $(1-\gamma)\alpha_2 - \frac{1}{2} + \alpha_2 > -1 \Leftrightarrow \alpha_2 > -\frac{1}{2(2-\gamma)} > -\frac{1}{1-\gamma} > -1$, the above time integral is finite, and after integration,

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t,x,x')|_{\tau=t,\xi=\xi'=x} \\
& \leq C|x-x'|^\gamma \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} O_m(t) \left(\nu^{\alpha_1(1-\gamma)} t^{1+(1-\gamma)\alpha_2} + \nu^{-\frac{1}{2}+\alpha_1(2-\gamma)} t^{\frac{1}{2}+\alpha_2(2-\gamma)}\right).
\end{aligned} \tag{5.33}$$

To consider vanishing viscous, it is necessary to have $\alpha_1(1-\gamma) > 0$ and $-\frac{1}{2} + \alpha_1(2-\gamma) > 0 \Leftrightarrow \alpha_1 > \frac{1}{2(2-\gamma)} > 0$ for $\gamma < 1$.

To sum up, we consider the constraints

$$\begin{aligned}
\alpha_1 & > \frac{1}{2(2-\gamma)}, \\
\alpha_2 & > -\frac{1}{2(2-\gamma)}.
\end{aligned} \tag{5.34}$$

Remainder term $R_{2,A}^{\tau,\xi,\xi'}(t,x,x')$

Also by change of variables and by similar computations as for the first remainder term $R_{1,A}^{\tau,\xi,\xi'}(t, x, x')$:

$$\begin{aligned}
& |R_{2,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
&= \left| \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \nabla \cdot b_m(s, x+y) (u^{m,\nu}(s, x+y) - u^{m,\nu}(s, \theta_{s,t}^m(x))) dy ds \right. \\
&\quad \left. - \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \nabla \cdot b_m(s, x'+y) (u^{m,\nu}(s, x'+y) - u^{m,\nu}(s, \theta_{s,t}^m(x))) dy ds \right| \\
&= \left| \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \cdot \left\{ \nabla \cdot b_m(s, x+y) (u^{m,\nu}(s, x+y) - u^{m,\nu}(s, x'+y)) \right. \right. \\
&\quad \left. \left. + (\nabla \cdot b_m(s, x+y) - \nabla \cdot b_m(s, x'+y)) (u^{m,\nu}(s, x+y) - u^{m,\nu}(s, \theta_{s,t}^m(x))) \right\} dy ds \right| \\
&\leq C|x-x'|(\|\nabla b_m\|_{L^\infty} + \|\nabla b_m\|_{L^\infty(C^1)})\|u^{m,\nu}\|_{L^\infty(C^1)} \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) ds.
\end{aligned}$$

Next, we obtain from (2.8) and (5.32),

$$\begin{aligned}
& |R_{2,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
&\leq C|x-x'| \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t) \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) (1 + [\nu(t-s)])^{\frac{1}{2}} ds.
\end{aligned}$$

By assumption of the *diagonal* regime, we get

$$\begin{aligned}
& |R_{2,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
&\leq C|x-x'|^\gamma \nu^{\alpha_1(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t) \int_0^t (t-s)^{\alpha_2(1-\gamma)} (1 + [\nu(t-s)])^{\frac{1}{2}} ds \\
&\leq C|x-x'|^\gamma \nu^{\alpha_1(1-\gamma)} t^{1+\alpha_2(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t), \tag{5.35}
\end{aligned}$$

for $\nu t \leq \nu T \leq 1$. The above inequality satisfies the vanishing viscosity analysis if $\alpha_1 > 0$ and if $\alpha_2 > \frac{-1}{1-\gamma}$ which is already required for the first remainder (because $\frac{1}{2(2-\gamma)} > 0$ and $\frac{-1}{2(2-\gamma)} > \frac{-1}{1-\gamma}$).

5.2.4 Off-diagonal regime

If $|x-x'| > \nu^{\alpha_1} (t-s)^{\alpha_2} \Leftrightarrow s > t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x-x'|^{\frac{1}{\alpha_2}}$, we recall the corresponding *cut locus* time

$$t_0 = t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x-x'|^{\frac{1}{\alpha_2}}.$$

In this case, we choose as *freezing* parameters $\xi = x$ and $\xi' = x'$. In the *off-diagonal* regime, the associated space is

$$A^c(x, x', \nu, t)(s) := \{|x-x'| > \nu^{\alpha_1} (t-s)^{\alpha_2}\}, \tag{5.36}$$

the indicator function

$$\mathbb{1}_{A^c(x,x',\nu,t)}(s) = 1 - \mathbb{1}_{A(x,x',\nu,t)}(s) = \begin{cases} 1 & \text{if } |x-x'| > \nu^{\alpha_1} (t-s)^{\alpha_2}, \\ 0 & \text{if } |x-x'| \leq \nu^{\alpha_1} (t-s)^{\alpha_2}, \end{cases}$$

and the associated remainder terms are

$$\begin{aligned}
& R_{1,A^c}^{\tau,\xi,\xi'}(t, x, x') \\
&= R_1^{\tau,\xi,\xi'}(t, x, x') - R_{1,A}^{\tau,\xi,\xi'}(t, x, x') \\
&= \int_0^t \mathbb{1}_{A^c(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau,\xi}(s, t, x, y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) \right. \\
&\quad \left. - \nabla \hat{p}^{\tau',\xi'}(s, t, x', y) \cdot (b_m(s, \theta_{s,\tau}^m(\xi')) - b_m(s, y)) (u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi'))) \right] dy ds,
\end{aligned}$$

and

$$\begin{aligned}
R_{2,A^c}^{\tau,\xi,\xi'}(t,x,x') &= R_2^{\tau,\xi,\xi'}(t,x,x') - R_{2,A}^{\tau,\xi,\xi'}(t,x,x') \\
&= \int_0^t \mathbb{1}_{A^c(x,x',\nu,t)}(s) \int_{\mathbb{R}^d} \left[\hat{p}^{\tau,\xi}(s,t,x,y) \nabla \cdot b_m(s,y) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi))) \right. \\
&\quad \left. - \hat{p}^{\tau,\xi'}(s,t,x',y) \nabla \cdot b_m(s,y) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi'))) \right] dy ds.
\end{aligned}$$

Like in the *diagonal* regime, we also rewrite the above remainder terms

$$\begin{aligned}
&R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x') \\
&= \int_{t_0}^t \int_{\mathbb{R}^d} \left[\nabla \hat{p}^{\tau,\xi}(s,t,x,y) \cdot (b_m(s,\theta_{s,\tau}^m(\xi)) - b_m(s,y)) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi))) \right. \\
&\quad \left. - \nabla \hat{p}^{\tau,\xi'}(s,t,x',y) \cdot (b_m(s,\theta_{s,\tau}^m(\xi')) - b_m(s,y)) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi'))) \right] dy ds \\
&=: \nabla \cdot \hat{G}_{t_0,t}^{\tau,\xi} \left\{ (b_m(\cdot, \theta_{\cdot,\tau}^m(\xi)) - b_m) (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi))) \right\} (t,x) \\
&\quad - \nabla \cdot \hat{G}_{t_0,t}^{\tau,\xi'} \left\{ (b_m(\cdot, \theta_{\cdot,\tau}^m(\xi')) - b_m) (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi'))) \right\} (t,x'), \tag{5.37}
\end{aligned}$$

and

$$\begin{aligned}
&R_{2,A^c}^{\tau,\xi,\xi'}(t,x,x') \\
&= \int_{t_0}^t \int_{\mathbb{R}^d} \left[\hat{p}^{\tau,\xi}(s,t,x,y) \nabla \cdot b_m(s,y) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi))) \right. \\
&\quad \left. - \hat{p}^{\tau,\xi'}(s,t,x',y) \nabla \cdot b_m(s,y) (u^{m,\nu}(s,y) - u^{m,\nu}(s,\theta_{s,\tau}^m(\xi'))) \right] dy ds \tag{5.38} \\
&=: \hat{G}_{t_0,t}^{\tau,\xi} \left\{ \nabla \cdot b_m (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi))) \right\} (t,x) - \hat{G}_{t_0,t}^{\tau,\xi'} \left\{ \nabla \cdot b_m (u^{m,\nu} - u^{m,\nu}(\cdot, \theta_{\cdot,\tau}^m(\xi'))) \right\} (t,x').
\end{aligned}$$

Remainder term $R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x')$

By triangular inequality

$$\begin{aligned}
&\left| R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x') \right|_{\tau=t,\xi=x,\xi'=x'} \\
&\leq 2C \|b_m\|_{L^\infty(C^1)} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \sup_{x \in \mathbb{R}^d} \int_{t_0}^t \int_{\mathbb{R}^d} (\nu(t-s))^{-\frac{1}{2}} \bar{p}^{t,x}(s,t,x,y) |y - \theta_{s,\tau}^m(\xi)|^{1+\gamma} dy ds.
\end{aligned}$$

Next, by absorbing property of the exponential (2.8) we get

$$\begin{aligned}
\left| R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x') \right|_{\tau=t,\xi=x,\xi'=x'} &\leq Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \int_{t_0}^t (\nu(t-s))^{\frac{\gamma}{2}} ds \\
&\leq \frac{m^{1+\beta}}{2+\gamma} C\nu^{\frac{\gamma}{2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} (t-t_0)^{1+\frac{\gamma}{2}}.
\end{aligned}$$

Eventually, by definition of the *cut locus* time (5.25):

$$\begin{aligned}
\left| R_{1,A^c}^{\tau,\xi,\xi'}(t,x,x') \right|_{\tau=t,\xi=x,\xi'=x'} &\leq \frac{m^{1+\beta}}{1+\gamma} C\nu^{\frac{\gamma}{2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} (\nu^{-\frac{\alpha_1}{\alpha_2}} |x-x'|^{\frac{1}{\alpha_2}})^{\frac{2+\gamma}{2}} \\
&\leq \frac{m^{1+\beta}}{1+\gamma} C\nu^{\frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} |x-x'|^{\frac{2+\gamma}{2\alpha_2}}. \tag{5.39}
\end{aligned}$$

Hence, we have a second constraint to be satisfied:

$$\begin{aligned}\gamma &= \frac{2+\gamma}{2\alpha_2}, \\ 0 &< \frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}.\end{aligned}\tag{5.40}$$

In particular, the parameter α_2 is then determined by $\alpha_2 = \frac{2+\gamma}{2\gamma}$.

Remainder term $R_{2,A^c}^{\tau,\xi,\xi'}(t, x, x')$

We also derive by triangular inequality,

$$\begin{aligned}& |R_{2,A^c}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=x, \xi'=x'} \\ & \leq \left| \int_0^t \mathbb{1}_{A^c(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x}(s, t, 0, y) \nabla \cdot b_m(s, x+y) (u^{m,\nu}(s, x+y) - u^{m,\nu}(s, \theta_{s,t}^m(x))) dy ds \right| \\ & \quad + \left| \int_0^t \mathbb{1}_{A^c(x, x', \nu, t)}(s) \int_{\mathbb{R}^d} \hat{p}^{t,x'}(s, t, 0, y) \nabla \cdot b_m(s, x'+y) (u^{m,\nu}(s, x'+y) - u^{m,\nu}(s, \theta_{s,t}^m(x'))) dy ds \right| \\ & \leq 2C \|\nabla \cdot b_m\|_{L^\infty} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \int_{t_0}^t [\nu(t-s)]^{\frac{\gamma}{2}} ds,\end{aligned}$$

by similar absorbing arguments previously performed; and finally

$$|R_{2,A^c}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=x, \xi'=x'} \leq \frac{m^{1+\beta}}{1+\gamma} C \nu^{\frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} |x - x'|^{\frac{2+\gamma}{2\alpha_2}}, \tag{5.41}$$

which yields the same constraints as (5.40).

5.2.5 Sum up on the constraints on α_1, α_2

Recalling from (5.40)

$$\begin{aligned}\alpha_2 &= \frac{2+\gamma}{2\gamma}, \\ 0 &< \frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2},\end{aligned}\tag{5.42}$$

and from (5.34), that we recall

$$\begin{aligned}\alpha_1 &> \frac{1}{2(2-\gamma)}, \\ \alpha_2 &> -\frac{1}{2(2-\gamma)},\end{aligned}\tag{5.43}$$

the second inequality above is satisfied by the choice $\alpha_2 = \frac{2+\gamma}{2\gamma} > 0 > -\frac{1}{2(2-\gamma)}$.

Next, combining (5.42) with (5.43):

$$\frac{\gamma}{2(2-\gamma)} < \alpha_1 \gamma < \frac{\gamma}{2}.\tag{5.44}$$

This is possible if

$$1 < 2 - \gamma \iff \gamma < 1,$$

which means that there is no possibility, with our strategy to obtain any Lipschitz control.

Furthermore, we also need to suppose that

$$\alpha_1 < \frac{1}{2}.\tag{5.45}$$

5.2.6 Choice of α_1

Finally, we calibrate α_1 such that the “worst” contribution of ν in the *diagonal* regime in (5.33) matches with the *off-diagonal* one in (5.39), namely

$$-\frac{1}{2} + \alpha_1(2 - \gamma) = \frac{\gamma}{2} - \frac{\alpha_1(2 + \gamma)}{2\alpha_2},$$

as $\frac{\gamma}{2} - \frac{\alpha_1(2+\gamma)}{2\alpha_2} = \frac{\gamma}{2} - \alpha_1\gamma$, we deduce

$$\alpha_1 = \frac{1 + \gamma}{4}. \quad (5.46)$$

We point out that the constraint (5.45) is indeed satisfied.

Let us detail that, with the choice (5.46), the conditions in (5.44) are satisfied, namely that $\frac{\gamma}{2(2-\gamma)} < \frac{\gamma(1+\gamma)}{4} < \frac{\gamma}{2}$, the second inequality is direct as $\frac{1+\gamma}{2} < 1$ for any $\gamma < 1$. In order to prove the first inequality, we equivalently need to have $\Gamma(\gamma) := (2 - \gamma)(1 + \gamma) > 2$. Differentiating this function readily gives $\Gamma'(\gamma) = -1 - \gamma + 2 - \gamma = 1 - 2\gamma$, hence $\inf_{\gamma \in (0,1)} \Gamma(\gamma) = \min_{\gamma \in \{0,1\}} \Gamma(\gamma) = 2$.

Remark 6. *The condition to get a diagonal regime is then $|x - x'| \leq \nu^{\frac{1+\gamma}{4}}(t - s)^{\frac{2+\gamma}{2\gamma}}$, which differs from the usual parabolic scale where (α_1, α_2) is replaced by $(\frac{1}{2}, \frac{1}{2})$; but recalling that $\alpha_1 = \frac{1}{2}$ is not allowed in (5.45). We do not seek for any parabolic bootstrap of regularity, unlike [CDRHM18], our goal is above all to control as sharp as possible the dependency on ν .*

5.2.7 Comments on the necessity of using the norm $\|u^{m,\nu}\|_{L^\infty(C^1)}$

At this stage of the proof, we can justify why we upper-bound by the blowing-up term $\|u^{m,\nu}\|_{L^\infty(C^1)}$ in the *diagonal* regime, which is overwhelmed by the viscosity ν but prevents to get uniqueness, see Remark 5; instead of the well-controlled $\|u^{m,\nu}\|_{L^\infty(C^\gamma)}$.

Let us rewrite one of the term of $R_{1,A}^{\tau,\xi,\xi'}(t, x, x')$ in r.h.s. in (5.30),

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, 0, y) \cdot (b_m(s, x' + y) - b_m(s, x + y)) \right. \\ & \quad \left. (u^{m,\nu}(s, x' + y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))) dy ds \right|_{\tau=t, \xi=\xi'=x} \\ & \leq C|x - x'| \|b_m\|_{L^\infty(C^1)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \int_{\mathbb{R}^d} [\nu(t - s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, 0, y) \times |\theta_{s,\tau}^m(\xi) - x' - y|^\gamma dy ds \\ & \leq C|x - x'| \|b_m\|_{L^\infty(C^1)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left([\nu(t - s)]^{\frac{\gamma-1}{2}} + [\nu(t - s)]^{-\frac{1}{2}} |x - x'|^\gamma \right) ds \\ & \leq C|x - x'|^\gamma \nu^{\alpha_1(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \\ & \quad \times (t - s)^{(1-\gamma)\alpha_2} \left([\nu(t - s)]^{\frac{\gamma-1}{2}} + [\nu(t - s)]^{-\frac{1}{2}} \nu^{\alpha_1\gamma} (t - s)^{\alpha_2\gamma} \right) ds \\ & = C|x - x'|^\gamma \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} \\ & \quad \times \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left((t - s)^{(1-\gamma)\alpha_2 + \frac{\gamma-1}{2}} \nu^{\alpha_1(1-\gamma) + \frac{\gamma-1}{2}} + \nu^{\alpha_1 - \frac{1}{2}} (t - s)^{\alpha_2 - \frac{1}{2}} \right) ds. \end{aligned}$$

Then the required, assumption on parameters is for this control (from the second additive term above)

$$\alpha_1 > \frac{1}{2}, \quad \alpha_2 > -\frac{1}{2},$$

which combined with the constraint on the *off-diagonal* regime (5.40) is absurd.

We could consider the case $\alpha_1 = \frac{1}{2}$, but this case yields no viscosity contribution and makes the previous upper-bounds blowing up with m (except for the usual framework, i.e. for b Lipschitz continuous); there is no possibility to obtain a *regularisation by turbulence* for such a choice.

If we suppose that b is γ -Hölder in space, in order to avoid any blowing-up in m , from (5.31), we are able to write

$$\begin{aligned}
& |R_{1,A}^{\tau,\xi,\xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
& \leq C|x-x'|^\gamma \|b\|_{L^\infty(C^\gamma)} \int_0^t \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \\
& \quad \int_{\mathbb{R}^d} \mathbb{1}_{A(x,x',\nu,t)}(s) [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, 0, y) \times |\theta_{s,\tau}^m(x) - x' - y|^\gamma dy ds \\
& \leq C|x-x'|^\gamma \|b\|_{L^\infty(C^\gamma)} \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \left([\nu(t-s)]^{\frac{\gamma-1}{2}} + [\nu(t-s)]^{-\frac{1}{2}} |x-x'|^\gamma \right) ds \\
& \leq C|x-x'|^\gamma \|b\|_{L^\infty(C^\gamma)} \int_0^t \mathbb{1}_{A(x,x',\nu,t)}(s) \|u^{m,\nu}(s, \cdot)\|_{C^\gamma} \\
& \quad \left([\nu(t-s)]^{\frac{\gamma-1}{2}} + \nu^{-\frac{1}{2}} (t-s)^{-\frac{1}{2}} \nu^{\alpha_1 \gamma} (t-s)^{\alpha_2 \gamma} \right) ds,
\end{aligned}$$

which goes to $+\infty$ when $\nu \rightarrow 0$, except if $\gamma = 1$. In other words, we need to consider the norms $\|b_m\|_{L^\infty(C^1)}$ and $\|u^{m,\nu}\|_{L^\infty(C^1)}$ on the one hand to smoothen the blowing-up in ν and to get a suitable control by $|x-x'|$ which allows to overwhelm ν in the *diagonal* regime.

Again, we could rewrite the analysis performed before this current section without doing an integration by parts in (5.22), and by upper-bounding with $\|\nabla u^{m,\nu}\|_{L^\infty}$. We choose to keep this separation of the remainder term defined in (5.22) in order to track precisely where the regime helps us to overuse the suitable *a priori* regularity of $u^{m,\nu}$ in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$.

5.3 On the discontinuous choice of freezing parameters

Let us carefully point out that even if the solution $u^{m,\nu}$ **does not** depend on the corresponding freezing parameter ξ , the choice of ξ in this section **does** depend on the current time variable of integration s .

Therefore, like for the approach developed in [CDRHM18], the *cut locus* time yields an additional contribution.

5.3.1 Consequence for $u^{m,\nu}(t, x')$

Previously, in the Hölder norm controls, we considered two points $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$. Let us specify how to write the solution $u^{m,\nu}(t, x')$ with the different choices of freezing parameter ξ' depending on the time variable of integration s . To do so, we first rewrite the theoretical representation of the solution where the initial time is $r \in [0, T]$, and the initial function is replaced by $u^{m,\nu}(r, x')$,

$$u^{m,\nu}(t, x') = P_r^{m,\nu} u^{m,\nu}(r, x') + G_r^{m,\nu} f_m(t, x'), \quad (5.47)$$

where $P_{r,t}^{m,\nu}$ and $G_{r,t}^{m,\nu}$ stand respectively for the semi-group and the Green operator associated with the Cauchy problem

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + \langle b_m(t, x), \nabla u^{m,\nu}(t, x) \rangle - \nu \Delta u^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in (r, T] \times \mathbb{R}^d, \\ u^{m,\nu}(r, x) = u^{m,\nu}(r, x), & x \in \mathbb{R}^d. \end{cases} \quad (5.48)$$

We also write

$$u^{m,\nu}(t, x) = \hat{P}_r^{\tau,\xi} u^{m,\nu}(r, \cdot)(t, x) + \hat{G}_r^{\tau,\xi} f_m(t, x) + \hat{G}_r^{\tau,\xi} (b_\Delta^m[\tau, \xi] \cdot \nabla u^{m,\nu})(t, x), \quad (5.49)$$

where the operators are defined by

$$\forall(t, x) \in (0, T] \times \mathbb{R}^d, \hat{G}_r^{\tau, \xi} f_m(t, x) := \int_r^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) f(s, y) dy ds, \quad (5.50)$$

and

$$\hat{P}_r^{\tau, \xi} g_m(t, x) := \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(r, t, x, y) g_m(y) dy. \quad (5.51)$$

Let us recall the definition of the transition time

$$t_0 := t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}} = t - \nu^{-\frac{\gamma(1+\gamma)}{2(2+\gamma)}} |x - x'|^{\frac{2\gamma}{2+\gamma}}. \quad (5.52)$$

If $t_0 \leq 0 \Leftrightarrow t \leq \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}$, the *off-diagonal* regime is in force, then we pick $\xi' = x'$ and there is no intricate choice of the *freezing* parameter.

However, if $t_0 > 0 \Leftrightarrow t > \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}$, we need to be more subtle to handle with the dependency on s for the value choice of $\xi' \in \mathbb{R}^d$. From now on, we suppose that $t_0 > 0$.

We next differentiate (5.49) w.r.t. r

$$0 = \partial_r (\hat{P}_r^{\tau, \xi'} u^{m, \nu}(r, \cdot))(t, x') + \partial_r \hat{G}_r^{\tau, \xi'} f_m(t, x') + \partial_r \hat{G}_r^{\tau, \xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m, \nu})(t, x'). \quad (5.53)$$

We integrate the variable r between $[t_0, t]$ with the *proxy* parameter $\xi' \in \mathbb{R}^d$,

$$0 = u^{m, \nu}(t, x') - [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - G_{t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{t_0}^{\tau, \xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m, \nu})(t, x'),$$

which yields for $t \in [t_0, T]$

$$u^{m, \nu}(t, x') = [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') + G_{t_0}^{\tau, \xi'} f_m(t, x') + \hat{G}_{t_0}^{\tau, \xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m, \nu})(t, x'). \quad (5.54)$$

Next, we integrate in time between $[0, t_0]$ with a different *freezing* parameter $\tilde{\xi}' \in \mathbb{R}^d$,

$$\begin{aligned} 0 &= [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') - \hat{P}_0^{\tau, \tilde{\xi}'} g_m(x') + \hat{G}_{t_0}^{\tau, \tilde{\xi}'} f_m(t, x') - \hat{G}_0^{\tau, \tilde{\xi}'} f_m(t, x') \\ &\quad + \hat{G}_{t_0}^{\tau, \tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m, \nu})(t, x') - \hat{G}_0^{\tau, \tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m, \nu})(t, x'). \end{aligned}$$

Hence,

$$\begin{aligned} u^{m, \nu}(t, x') &= [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') + G_{t_0}^{\tau, \xi'} f_m(t, x') + \hat{G}_{t_0}^{\tau, \xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m, \nu})(t, x') \\ &\quad - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') + \hat{P}_0^{\tau, \tilde{\xi}'} g_m(x') - \hat{G}_{t_0}^{\tau, \tilde{\xi}'} f_m(t, x') + \hat{G}_0^{\tau, \tilde{\xi}'} f_m(t, x') \\ &\quad - \hat{G}_{t_0}^{\tau, \tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m, \nu})(t, x') + \hat{G}_0^{\tau, \tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m, \nu})(t, x'). \end{aligned}$$

Defining

$$\forall(t', x) \in [0, t] \times \mathbb{R}^d, \hat{G}_{r, t'}^{\tau, \xi} f_m(t, x) := \int_r^{t'} \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) f(s, y) dy ds, \quad (5.55)$$

we write

$$\begin{aligned} u^{m, \nu}(t, x') &= [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') + G_{t_0}^{\tau, \xi'} f_m(t, x') \\ &\quad + \hat{G}_{t_0}^{\tau, \xi'} (b_\Delta^m[\tau, \xi'] \cdot \nabla u^{m, \nu})(t, x') + \hat{P}_0^{\tau, \tilde{\xi}'} g_m(x') + \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \\ &\quad + \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} (b_\Delta^m[\tau, \tilde{\xi}'] \cdot \nabla u^{m, \nu})(t, x'). \end{aligned} \quad (5.56)$$

There is an extra contribution $[P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x')$ due to the discontinuous *freezing* choice, the other terms match with the ones appearing in the above computations.

5.3.2 Extra contribution $[P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x')$

Thanks to a change of variables, we readily obtain

$$\begin{aligned} & [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \\ &= \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(t_0, t, x', y) u^{m, \nu}(t_0, y) dy - \int_{\mathbb{R}^d} \hat{p}^{\tau, \tilde{\xi}'}(t_0, t, x', y) u^{m, \nu}(t_0, y) dy \\ &= \int_{\mathbb{R}^d} \tilde{p}(t_0, t, x', y) \left[u^{m, \nu}(t_0, y + \int_{t_0}^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\xi')) d\tilde{s}) - u^{m, \nu}(t_0, y + \int_{t_0}^t b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\tilde{\xi}')) d\tilde{s}) \right] dy, \end{aligned}$$

recalling that $\tilde{p}(t_0, t, x', y)$ stands for the usual heat kernel defined in (5.18). Therefore,

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right| \\ & \leq \|u^{m, \nu}\|_{L^\infty(C^1)} \int_{t_0}^t |b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\xi')) - b_m(\tilde{s}, \theta_{\tilde{s}, \tau'}^m(\tilde{\xi}'))| d\tilde{s} \\ & \leq \|u^{m, \nu}\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_{t_0}^t |\theta_{\tilde{s}, \tau'}^m(\xi') - \theta_{\tilde{s}, \tau'}^m(\tilde{\xi}')| d\tilde{s}. \end{aligned}$$

Additionally, we get the *a priori* control for the flow.

Lemma 3. For all $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$ and $0 \leq s \leq \tau \leq T$:

$$\sup_{\tilde{s} \in [0, \tau]} |\theta_{\tilde{s}, \tau}^m(x) - \theta_{\tilde{s}, \tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau).$$

The proof is postponed in Appendix Section B.

We then deduce for $(\tau, \xi', \tilde{\xi}') = (t, x', x)$ that

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\ & \leq \|u^{m, \nu}\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_{t_0}^t |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} t) d\tilde{s} \\ & \leq \|u^{m, \nu}\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} |x - x'| (t - t_0) \exp(\|b_m\|_{L^\infty(C^1)} t) d\tilde{s}. \end{aligned}$$

Recalling that, see Lemma 2,

$$\|\nabla u^{m, \nu}(t, \cdot)\|_{L^\infty} \leq (T \|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}) \exp(Cm^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}) = O_m(t).$$

Therefore,

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\ & \leq m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} |x - x'| (t - t_0) \exp(\|b_m\|_{L^\infty(C^1)} t). \end{aligned} \quad (5.57)$$

Because $|x - x'| = \nu^{\alpha_1} (t - t_0)^{\alpha_1}$, we get

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\ & \leq |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{(1-\gamma)\alpha_1} (t - t_0)^{(1-\gamma)\alpha_2 + 1} \exp(\|b_m\|_{L^\infty(C^1)} t). \end{aligned} \quad (5.58)$$

As we have supposed that $t_0 \geq 0$, it follows that

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\ & \leq t^{(1-\gamma)\alpha_2 + 1} |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{(1-\gamma)\alpha_1} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t). \end{aligned} \quad (5.59)$$

Taking, the choice (5.40) and (5.46),

$$(\alpha_1, \alpha_2) = \left(\frac{1+\gamma}{4}, \frac{2+\gamma}{2\gamma} \right),$$

we finally derive,

$$\begin{aligned} & \left| [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\ & \leq \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1} |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & = \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1} |x - x'|^\gamma m^{1+\beta} O_m(2t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}. \end{aligned} \quad (5.60)$$

From definition of $O_m(t)$ in (5.32), we obtain the vanishing condition

$$\nu \ll (m^{1+\beta-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 1})^{-\frac{4}{1-\gamma^2}} \exp\left(-C \frac{m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} T}{1-\gamma^2}\right). \quad (5.61)$$

5.3.3 Justification of the *freezing* point change

For any $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and $(\tau, \xi, \xi', \tilde{\xi}') \in [0, T] \times \mathbb{R}^d$, we write from (5.56),

$$\begin{aligned} u^{m, \nu}(t, x) - u^{m, \nu}(t, x') &= \left[\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \tilde{\xi}'} g_m(t, x') \right] + \left[\hat{G}_{t_0, t}^{\tau, \xi} f_m(t, x) - \hat{G}_{t_0, t}^{\tau, \xi'} f_m(t, x') \right] \\ &+ \left[\hat{G}_{0, t_0}^{\tau, \xi} f_m(t, x) - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\ &+ \left[\tilde{G}_{t_0, t}^{\tau, \xi} [b_\Delta^m[\tau, \xi] \cdot \nabla u^{m, \nu}](t, x) - \tilde{G}_{t_0, t}^{\tau, \xi'} [b_\Delta^{m'}[\tau, \xi'] \cdot \nabla u^{m, \nu}](t, x') \right] \\ &+ \left[\tilde{G}_{0, t_0}^{\tau, \xi} [b_\Delta^m[\tau, \xi] \cdot \nabla u^{m, \nu}](t, x) - \tilde{G}_{0, t_0}^{\tau, \xi'} [b_\Delta^{m'}[\tau, \xi'] \cdot \nabla u^{m, \nu}](t, x') \right] \\ &+ [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x'). \end{aligned}$$

Because the l.h.s. of the first equality does not depend on (ξ, ξ') , we can get the infimum over these *freezing* points, namely

$$\begin{aligned} & u^{m, \nu}(t, x) - u^{m, \nu}(t, x') \\ &= \inf_{\xi, \xi', \tilde{\xi}' \in \mathbb{R}} \left\{ \left[\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \tilde{\xi}'} g_m(t, x') \right] + \left[\hat{G}_{t_0, t}^{\tau, \xi} f_m(t, x) - \hat{G}_{t_0, t}^{\tau, \xi'} f_m(t, x') \right] \right. \\ &+ \left[\hat{G}_{0, t_0}^{\tau, \xi} f_m(t, x) - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right] \\ &+ \left[\tilde{G}_{t_0, t}^{\tau, \xi} [b_\Delta^m[\tau, \xi] \cdot \nabla u^{m, \nu}](t, x) - \tilde{G}_{t_0, t}^{\tau, \xi'} [b_\Delta^{m'}[\tau, \xi'] \cdot \nabla u^{m, \nu}](t, x') \right] \\ &+ \left[\tilde{G}_{0, t_0}^{\tau, \xi} [b_\Delta^m[\tau, \xi] \cdot \nabla u^{m, \nu}](t, x) - \tilde{G}_{0, t_0}^{\tau, \xi'} [b_\Delta^{m'}[\tau, \xi'] \cdot \nabla u^{m, \nu}](t, x') \right] \\ &\left. + [P_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right\}. \end{aligned}$$

However, we aim to control the source functions term only in the *diagonal* regime, see Section 5.3.5 below for details. We rewrite,

$$\begin{aligned} & \left[\hat{G}_{t_0,t}^{\tau,\xi} f_m(t,x) - \hat{G}_{t_0,t}^{\tau,\xi'} f_m(t,x') \right] + \left[\hat{G}_{0,t_0}^{\tau,\xi} f_m(t,x) - \hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') \right] \\ = & \left[\hat{G}^{\tau,\xi} f_m(t,x) - \hat{G}^{\tau,\xi'} f_m(t,x') \right] + \left[\hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{0,t_0}^{\tau,\xi} f_m(t,x') \right], \end{aligned}$$

and,

$$\begin{aligned} & u^{m,\nu}(t,x) - u^{m,\nu}(t,x') \\ = & \inf_{\xi,\xi',\tilde{\xi}' \in \mathbb{R}} \left\{ \left[\hat{P}^{\tau,\xi} g_m(t,x) - \hat{P}^{\tau,\tilde{\xi}'} g_m(t,x') \right] + \left[\hat{G}^{\tau,\xi} f_m(t,x) - \hat{G}^{\tau,\tilde{\xi}'} f_m(t,x') \right] \right. \\ & + \left[\hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') \right] \\ & + \left[\tilde{G}_{t_0,t}^{\tau,\xi} [b_{\Delta}^m[\tau,\xi] \cdot \nabla u^{m,\nu}](t,x) - \tilde{G}_{t_0,t}^{\tau,\xi'} [b_{\Delta}^{m'}[\tau,\xi'] \cdot \nabla u^{m,\nu}](t,x') \right] \\ & + \left[\tilde{G}_{0,t_0}^{\tau,\xi} [b_{\Delta}^m[\tau,\xi] \cdot \nabla u^{m,\nu}](t,x) - \tilde{G}_{0,t_0}^{\tau,\xi'} [b_{\Delta}^{m'}[\tau,\xi'] \cdot \nabla u^{m,\nu}](t,x') \right] \\ & \left. + [P_{t_0}^{\tau,\xi'} u^{m,\nu}(t_0,\cdot)](t,x') - [\hat{P}_{t_0}^{\tau,\tilde{\xi}'} u^{m,\nu}(t_0,\cdot)](t,x') \right\}, \end{aligned} \quad (5.62)$$

because, from definitions (5.28), (5.29), (5.37) and (5.38), we have $\hat{G}^{\tau,\tilde{\xi}'} f_m(t,x') = \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') + \hat{G}_{t_0,t}^{\tau,\tilde{\xi}'} f_m(t,x')$.

Hence, taking $(\tau,\xi,\xi',\tilde{\xi}') = (t,x,x',x)$ yields the previous terms already controlled plus a new extra contribution $\left[\hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') \right]$.

5.3.4 Control of the new extra contribution $\left[\hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') \right]$

This last extra term is tackled similarly as the first one, see Section 5.3.2.

$$\begin{aligned} & \left[\hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') \right] \\ = & \int_0^{t_0} \int_{\mathbb{R}^d} \tilde{p}(s,t,x',y) \left[f_m(s,y + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau'}^m(\xi')) d\tilde{s}) - f_m(s,y + \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\tilde{\xi}')) d\tilde{s}) \right] dy ds. \end{aligned}$$

We readily obtain,

$$\begin{aligned} & \left| \hat{G}_{0,t_0}^{\tau,\xi'} f_m(t,x') - \hat{G}_{0,t_0}^{\tau,\tilde{\xi}'} f_m(t,x') \right| \\ \leq & \|f_m\|_{L^\infty(C^1)} \int_0^{t_0} \left| \int_s^t b_m(\tilde{s}, \theta_{\tilde{s},\tau'}^m(\xi')) - b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(\tilde{\xi}')) d\tilde{s} \right| ds \\ \leq & \|f_m\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_0^{t_0} \int_s^t |\theta_{\tilde{s},\tau'}^m(\xi') - \theta_{\tilde{s},\tau}^m(\tilde{\xi}')| d\tilde{s} ds. \end{aligned}$$

Again, we use, for any $(x,x') \in \mathbb{R}^d \times \mathbb{R}^d$, the control of the flow,

$$\sup_{\tilde{s} \in [0,\tau]} |\theta_{\tilde{s},\tau}^m(x) - \theta_{\tilde{s},\tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau), \quad (5.63)$$

see Lemma 3.

We then deduce for $(\tau, \xi', \tilde{\xi}') = (t, x', x)$ that

$$\begin{aligned}
& \left| \hat{G}_{0,t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0,t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\
& \leq \|f_m\|_{L^\infty(C^1)} \|b_m\|_{L^\infty(C^1)} \int_0^{t_0} \int_s^t |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} t) d\tilde{s} ds \\
& \leq \frac{1}{2} m^{1-\gamma} \|f\|_{L^\infty(C^\gamma)} m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} |x - x'| t^2 \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t). \tag{5.64}
\end{aligned}$$

From the definition of t_0 in (5.25), $|x - x'| = \nu^{\alpha_1} (t - t_0)^{\alpha_1}$, we get

$$\begin{aligned}
& \left| \hat{G}_{0,t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0,t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right|_{(\tau, \xi', \tilde{\xi}') = (t, x', x)} \\
& \leq |x - x'|^\gamma \frac{\nu^{(1-\gamma)\alpha_1} t^{(1-\gamma)\alpha_2 + 2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) \\
& = |x - x'|^\gamma \frac{\nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t). \tag{5.65}
\end{aligned}$$

Hence, the vanishing constraint is

$$\nu \ll (m^{2+\beta-\gamma} T^{\frac{(1-\gamma)(2+\gamma)}{2\gamma} + 2} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})})^{-\frac{4}{1-\gamma^2}} \exp\left(-4 \frac{m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} T}{1-\gamma^2}\right). \tag{5.66}$$

5.3.5 Comments on the choice of *freezing* point for the source functions terms

It is crucial to fix the same *freezing* point for the terms associated with source functions. In our context, it may be unavoidable. To fully explain this choice, let us develop the computations associated with these terms for the same choice of ξ and ξ' as for A , the *off-diagonal* regime, i.e. $\xi = x$ and $\xi' = x'$.

To deal with the semi-group, we consider an analysis of the type (or equivalent controls),

$$\begin{aligned}
& \left| \hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x') \right|_{\tau=t, \xi=x, \xi'=x'} \\
& = \left| \int_{\mathbb{R}^d} [\hat{p}^{\tau, \xi}(0, t, x, y) - \hat{p}^{\tau, \xi'}(0, t, x', y)] g_m(s, y) dy \right|_{\tau=t, \xi=x, \xi'=x'} \\
& = \left| \int_{\mathbb{R}^d} \tilde{p}(0, t, 0, y) [g_m(t, \theta_{0,t}^m(x) + y) - g_m(t, \theta_{0,t}^m(x') + y)] dy \right| \\
& \leq [g]_\gamma |\theta_{0,t}^m(x) - \theta_{0,t}^m(x')|^\gamma.
\end{aligned}$$

Similarly, for the Green operator,

$$\begin{aligned}
& \left| \hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x') \right|_{\tau=t, \xi=x, \xi'=x'} \\
& = \left| \int_0^t \int_{\mathbb{R}^d} [\hat{p}^{\tau, \xi}(s, t, x, y) - \hat{p}^{\tau, \xi'}(s, t, x', y)] f_m(s, y) dy ds \right|_{\tau=t, \xi=x, \xi'=x'} \\
& = \left| \int_0^t \int_{\mathbb{R}^d} [\tilde{p}(s, t, 0, y) [f_m(s, \theta_{s,t}^m(x) + y) - f_m(s, \theta_{s,t}^m(x') + y)]] dy ds \right| \\
& \leq \|f\|_{L^\infty(C^\gamma)} \int_0^t |\theta_{s,t}^m(x) - \theta_{s,t}^m(x')|^\gamma ds.
\end{aligned}$$

In other words, we see in the both controls above that we only upper-bound by the flow associated with b_m which is *a priori* not controlled uniformly on m in a suitable spatial Hölder space, see (5.63).

5.3.6 Final Hölder control

We have from the final Duhamel formula (5.56) combined with the estimates of each contribution stated in (5.33), (5.35), (5.39), (5.41), (5.60), (5.65):

$$\begin{aligned}
& \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\
\leq & t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + C\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{1+\beta} O_m(t) \left(\nu^{\alpha_1(1-\gamma)} t^{1+(1-\gamma)\alpha_2} + \nu^{-\frac{1}{2}+\alpha_1(2-\gamma)} t^{\frac{1}{2}+\alpha_2(2-\gamma)} \right) \\
& + C\nu^{\alpha_1(1-\gamma)} t^{1+\alpha_2(1-\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (m^{1+\beta} + m^{2+\beta}) O_m(t) \\
& + \frac{m^{1+\beta}}{1+\gamma} C\nu^{\frac{\gamma}{2}-\frac{\alpha_1(2+\gamma)}{2\alpha_2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \\
& + t^{(1-\gamma)\alpha_2+1} m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \nu^{(1-\gamma)\alpha_1} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t) \\
& + \frac{\nu^{(1-\gamma)\alpha_1} t^{(1-\gamma)\alpha_2+2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t). \tag{5.67}
\end{aligned}$$

In the above inequality, taking

$$(\alpha_1, \alpha_2) = \left(\frac{1+\gamma}{4}, \frac{2+\gamma}{2\gamma} \right), \tag{5.68}$$

and gathering some contributions, give

$$\begin{aligned}
& \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\
\leq & t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + C\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} m^{2+\beta} O_m(2t) \left(\nu^{\frac{1-\gamma^2}{4}} t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} + \nu^{\frac{\gamma(1-\gamma)}{4}} t^{\frac{1}{2}+\frac{4-\gamma^2}{2\gamma}} \right) \\
& + \frac{m^{1+\beta}}{1+\gamma} C\nu^{\frac{\gamma(1-\gamma)}{4}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \\
& + \nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma}+1} m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(\|b_m\|_{L^\infty(C^1)} t) \\
& + \frac{\nu^{\frac{1-\gamma^2}{4}} t^{\frac{(1-\gamma)(2+\gamma)}{2\gamma}+2}}{2} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t), \tag{5.69}
\end{aligned}$$

where we recall that

$$\begin{aligned}
O_m(t) &= \left(t\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right) \\
&\leq C m^{1-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right).
\end{aligned}$$

Then, for $\nu \ll T$ and because $\frac{1-\gamma^2}{4} > \frac{\gamma(1-\gamma)}{4}$, we obtain

$$\begin{aligned}
& \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\
\leq & t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \\
& + C\nu^{\frac{\gamma(1-\gamma)}{4}} m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \left(t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} O_m(2t) (1 + \|f\|_{L^\infty(C^\gamma)}) + \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \right).
\end{aligned}$$

For a ‘‘small’’ $\kappa \in (0, 1)$, we choose

$$\nu \leq \kappa^{\frac{4}{\gamma(1-\gamma)}} \left(1 + C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \right)^{-\frac{4}{\gamma(1-\gamma)}}, \tag{5.70}$$

which yields by circular argument

$$\begin{aligned}
\|u^{m,\nu}(t, \cdot)\|_{C^\gamma} &\leq (1-\kappa)^{-1} \left(t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \right. \\
&\quad \left. + C\nu^{\frac{\gamma(1-\gamma)}{4}} m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} O_m(2t) (1 + \|f\|_{L^\infty(C^\gamma)}) \right).
\end{aligned}$$

Finally, we also impose a given “small” $\tilde{\kappa} \in (0, 1)$,

$$\begin{aligned}
\nu &\leq \tilde{\kappa}^{\frac{4}{\gamma(1-\gamma)}} \left(C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} T^{1+(1-\gamma)\frac{2+\gamma}{2\gamma}} O_m(2T) (1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\
&= \tilde{\kappa}^{\frac{4}{\gamma(1-\gamma)}} \left(C m^{2+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} T^{\frac{2-\gamma^2+\gamma}{2\gamma}} (m^{1-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma)) (1 + \|f\|_{L^\infty(C^\gamma)}) \right)^{-\frac{4}{\gamma(1-\gamma)}} \\
&\quad \times \exp \left(-\frac{8m^{1+\beta} T \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}}{\gamma(1-\gamma)} \right), \tag{5.71}
\end{aligned}$$

which obviously implies condition (5.70) for $\kappa = \tilde{\kappa}$.

Under assumptions (5.70) and (5.71), we deduce for any $(\kappa, \tilde{\kappa}) \in (0, 1)^2$

$$\|u^{m,\nu}\|_{L^\infty(C^\gamma)} \leq (1 - \kappa)^{-1} (T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma + \tilde{\kappa}), \tag{5.72}$$

the required Hölder control (4.1) is then established when $\kappa, \kappa' \rightarrow 0$ and $(m, \nu) \rightarrow (+\infty, 0)$ according to conditions (4.2), up to a convergence argument developed in Section 5.5 below.

Let also notice the limit, up to selection of sub-sequence, can permute with the considered Hölder norm by a Fatou property like, see for instance Theorem 2.25 in [BCD11].

5.3.7 Why the computations are compatible with the usual method of characteristics?

In this section, we explain some links between the upper-bounds in (4.1) stated in Theorem 1 with the ones obtained by method of characteristics for the transport equation.

If b is constant, then the solution of the transport equation writes

$$u(t, x) = u_0(x - bt) + \int_0^t f(s, x - b(t - s)) ds,$$

which obviously satisfies inequalities (4.1).

If b is linear, namely replacing b by bx for a constant $b \in \mathbb{R}$, we obtain that the associated flow, introduced in (5.8), is

$$\theta_{s,\tau}(\xi) = \xi + b \int_s^\tau \theta_{\tilde{s},\tau}(\xi) d\tilde{s}, \tag{5.73}$$

which is solved by

$$\theta_{s,\tau}(\xi) = \xi e^{(\tau-s)b}.$$

Nonetheless, in this case the estimate (5.69) is not available any-more; the *a priori* controls of the gradient stated in Lemma 1 do not work any-longer. That is to say that in a smooth framework, the coefficient b has to be bounded, that is why we consider inhomogeneous Besov norm which is L^∞ bounded for positive regularity.

If b is bounded and smooth, the solution of the transport equation is

$$u(t, x) = u_0(\theta_{0,t}^m(x)) + \int_0^t f(s, \theta_{s,t}^m(x)) ds.$$

The control in L^∞ meet with (4.1). But it is clear that

$$\|\nabla u\|_{L^\infty} = \|\nabla \theta_{0,t}^m\|_{L^\infty} \|\nabla u_0\|_{L^\infty} + \int_0^t \|\nabla \theta_{s,t}^m\|_{L^\infty} \|\nabla f\|_{L^\infty} ds,$$

also from the definition of the flow (5.8), and Grönwall's lemma yields

$$\|\nabla\theta_{s,t}^m\|_{L^\infty} \leq \exp\left(\int_s^\tau \|\nabla b_m(\tilde{s}, \cdot)\|_{L^\infty} d\tilde{s}\right).$$

Therefore,

$$\|\nabla u\|_{L^\infty} \leq \exp\left(\int_0^t \|\nabla b_m(\tilde{s}, \cdot)\|_{L^\infty} d\tilde{s}\right) (\|\nabla u_0\|_{L^\infty} + \int_0^t \|\nabla f\|_{L^\infty} ds),$$

which does not match with the Hölder control (4.1) for the case $\gamma = 1$ or Hölder control by interpolation.

This odd mismatch comes from the gap between the $\gamma < 1$ and the Lipschitz control which can be seen in (5.69). Indeed, therein, if $\gamma = 1$ then there is no more negligible contribution of $\exp\left(\int_0^t \|\nabla b_m(\tilde{s}, \cdot)\|_{L^\infty} d\tilde{s}\right)$, and the analysis fall into the usual estimates by the method of characteristics.

5.4 Another control of uniform norm

In this section, we provide another way to get uniform control without stochastic representation or usual maximum principle; these computations lead some dependency on m overwhelmed by ν and might be useful in other contexts.

For the L^∞ , control we choose the *freezing* point as for the *diagonal* regime, i.e. $(\tau, \xi) = (t, x)$. The terms associated with the source functions are dealt easily:

$$|\hat{P}^{\tau,\xi} g_m(t, x)|\Big|_{\tau=t, \xi=x} \leq \|g\|_{L^\infty}, \quad |\hat{G}^{\tau,\xi} f_m(t, x)|\Big|_{\tau=t, \xi=x} \leq t\|f\|_{L^\infty}.$$

The remainder terms are controlled in a very similar way as previously in Section 5.2.4

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) \cdot \nabla u^{m,\nu}(s, y) dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \leq 2C \|b_m\|_{L^\infty(C^1)} \|\nabla u^{m,\nu}\|_{L^\infty} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) |y - \theta_{s,\tau}^m(\xi)|^\gamma dy ds \Big|_{\tau=t, \xi=x} \\ & \leq C m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla u^{m,\nu}\|_{L^\infty} \int_0^t (\nu(t-s))^{\frac{\gamma}{2}} ds, \end{aligned}$$

by (2.8), and we conclude by (5.32) that

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau,\xi}(s, t, x, y) (b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)) \cdot \nabla u^{m,\nu}(s, y) dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \leq C O_m(t) \frac{m^{1+\beta} t^{1+\frac{\gamma}{2}}}{2+\gamma} \nu^{\frac{\gamma}{2}} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}. \end{aligned}$$

5.5 Compactness arguments

5.5.1 Mild vanishing viscous

In order to pass to the limit $m \rightarrow +\infty$, $\nu \rightarrow 0$, according to the vanishing condition (4.2), we consider a subsequence given by the usual Arzelà-Ascoli theorem. However, this former result is available for uniform continuous function in a compact space. From the lack of smoothness, uniformly on ν , in space of $u^{m,\nu}(t, \cdot)$ (only γ -Hölder continuous, $\gamma < 1$), we are stuck at a convergence in a compact subset of \mathbb{R}^d . For instance, the analysis performed in [Hon22] to get rid of the compactness convergence criterion for quasi-linear equations does not work here as there is no hope to obtain any strong formulation of the PDE (1.1).

We do not succeed to obtain any positive regularity on t , which would imply dependency on b like in the linear case (5.73); and so we cannot exploit uniform continuity in time to get a convergence of a sub-sequence of $u^{m,\nu}$ in $[0, T] \times \mathbb{R}^d$. Thus the convergence at any given time in all compacts set of the *mild vanishing viscous* solution in Theorem 1.

Nevertheless, we are still able to include a truncation procedure into a weak formulation in order to obtain a convergence in a distributional meaning and not a point-wise one as for the *mild vanishing viscous* solution.

5.5.2 Truncation procedure

The method is highly inspired by the one in [Hon22], we also consider a smooth cut-off $\vartheta_{y,R} \in \mathcal{D}$ supported in a ball $B_d(y, R) = \{x \in \mathbb{R}^d; |x - y| \leq R\}$, $y \in \mathbb{R}^d$ and defined by

$$\vartheta_{y,R}(x) = \vartheta_y\left(\frac{x}{R}\right), \quad (5.74)$$

where $\vartheta_y : \mathbb{R}^d \rightarrow [0, 1]^d$ is function lying in $C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ s.t.

$$\vartheta_y(x) = \begin{cases} x, & \text{if } |x - y| < 1, \\ 0, & \text{if } |x - y| > 2. \end{cases}$$

The corresponding truncated function is, for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$u_{y,R}^{m,\nu}(t, x) := u^{m,\nu}(t, \vartheta_{y,R}(x)). \quad (5.75)$$

We highlight the particular case

$$u_{x,R}^{m,\nu}(t, x) = u^{m,\nu}(t, x). \quad (5.76)$$

The above truncation solution (5.75) naturally appears when we write a weak formulation of the parabolic equation (2.11).

5.5.3 Weak solution of the parabolic approximating equation

For any smooth function φ_R supported on $B_d(0, R)$, a d -ball of radius $R > 0$ and center $(0, \dots, 0) \in \mathbb{R}^d$. We consider a weak formulation of the parabolic solution $u^{m,\nu}$, for any $(t, x) \in [0, T] \times \mathbb{R}^d$:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \left\{ -\partial_t \varphi_R(s, y) u^{m,\nu}(s, y) + \varphi_R(s, y) \langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle + \nu \Delta \varphi_R(s, y) u^{m,\nu}(s, y) \right\} dy ds \\ &= \int_{\mathbb{R}^d} \varphi_R(0, y) g_m(y) dy - \int_{\mathbb{R}^d} \varphi_R(t, y) u^{m,\nu}(t, y) dy + \int_0^t \int_{\mathbb{R}^d} \varphi_R(s, y) f_m(s, y) dy ds, \end{aligned}$$

where in l.h.s. the limit of the first order term, $\langle b_m(s, y), \nabla u^{m,\nu}(s, y) \rangle$, has *a priori* no point-wise limit neither in term of the usual distributional meaning of Schwartz. Indeed, as already enunciated in Section 4.2, the usual distribution theory does not provide any interpretation of a product of distributions, to get any limit result we have to thoroughly use the PDE.

By the cut-off definition, we equivalently have

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \left\{ -\partial_t \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) + \varphi_R(s, y) \langle b_m(s, y), \nabla u_{0,R}^{m,\nu}(s, y) \rangle + \nu \Delta \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) \right\} dy ds \\ &= \int_{\mathbb{R}^d} \varphi_R(0, y) g_m(y) dy - \int_{\mathbb{R}^d} \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) dy + \int_0^t \int_{\mathbb{R}^d} \varphi_R(s, y) f_m(s, y) dy ds. \end{aligned} \quad (5.77)$$

Now, from compact argument developed in Section 5.5.1, we have that $u_{0,R}^{m,\nu}(s, \cdot)$ converges in $C_b^\eta(K, \mathbb{R}^d)$, $K = B_d(0, R)$, towards a function $u_{0,R}(s, \cdot)$ when $(m, \nu) \rightarrow (+\infty, 0)$ and the condition (4.2) is satisfied. In other words, $u_{0,R}(s, \cdot)$ is a *mild vanishing viscous* solution of (1.1).

5.5.4 Mild-weak solution of the transport equation

To get a *mild-weak* solution we have to pass to the limit in the weak formulation (5.77) of the mollified parabolic equation (2.11). In equation (5.77), up to a sub-sequence selection, except for the first order term $\varphi_R(s, y)\langle b(s, y), \nabla u_{0,R}^{m,\nu}(s, y) \rangle$, each contribution obviously has the good converge property by the Arzelà-Ascoli theorem. In particular, from (5.4), we have

$$\nu \int_0^t \int_{\mathbb{R}^d} \Delta \varphi_R(s, y) u_{0,R}^{m,\nu}(s, y) dy ds \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(4.2)} 0. \quad (5.78)$$

To deal with the *drift* part, we write by integration by parts

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle b_m(t, y), \nabla u_{0,R}^{m,\nu}(t, y) \rangle \varphi_R(t, y) dy \\ = & \int_{\mathbb{R}^d} \nabla \varphi_R(t, y) \cdot b_m(t, y) u_{0,R}^{m,\nu}(t, y) dy + \int_{\mathbb{R}^d} \varphi_R(t, y) \nabla \cdot b_m(t, y) u_{0,R}^{m,\nu}(t, y) dy \\ =: & B_1 + B_2. \end{aligned} \quad (5.79)$$

By the Besov duality property, see Proposition 1, we write

$$|B_1| \leq \|b_m\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)}, \quad (5.80)$$

and

$$|B_2| \leq \|\nabla \cdot b_m\|_{L^\infty(B_{\infty,\infty}^{-1-\beta})} \|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^{1+\beta})}. \quad (5.81)$$

Control of $\|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)}$

Let us prove that $\|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)}$ is controlled uniformly in (m, ν) . By the thermic representation of the Besov norms (2.4), we have

$$\begin{aligned} & \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^\beta)} \\ = & \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^1} + \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^\beta)} \\ = & \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^1} + \int_0^1 \frac{1}{v} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) u_{0,R}^{m,\nu}(t, y) \nabla \varphi_R(t, y) dy \right| dz dv. \end{aligned}$$

The first contribution in the r.h.s. above is obviously bounded uniformly in (m, ν) by

$$\|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^1} \leq \|\nabla \varphi_R\|_{L^1} \|u_{0,R}^{m,\nu}\|_{L^\infty} \leq C \|\nabla \varphi_R\|_{L^1} (T \|f\|_{L^\infty} + \|g\|_{L^\infty}),$$

by uniform estimate (5.4).

For the second one, we need to deeply use the already known regularity of $u_{0,R}^{m,\nu}$. By cancellation and by triangular inequality, we obtain

$$\begin{aligned} \|\nabla \varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\dot{B}_{1,1}^\beta)} & \leq \int_0^1 \frac{1}{v} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) \cdot \{ [u_{0,R}^{m,\nu}(t, y) - u_{0,R}^{m,\nu}(t, z)] \nabla \varphi_R(t, y) \right. \\ & \quad \left. + u_{0,R}^{m,\nu}(t, z) [\nabla \varphi(t, y) - \nabla \varphi(t, z)] \right\} dy \Big| dz dv \\ & \leq \int_0^1 \frac{1}{v} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) \cdot \{ [u_{0,R}^{m,\nu}(t, y) - u_{0,R}^{m,\nu}(t, z)] \nabla \varphi_R(t, y) \right. \\ & \quad \left. + u_{0,R}^{m,\nu}(t, z) (y-z) \cdot \int_0^1 D^2 \varphi_R(t, z + \mu(y-z)) d\mu \right\} dy \Big| dz dv, \end{aligned}$$

by Taylor expansion. Next, with the exponential absorbing property (2.8),

$$\begin{aligned}
& \|\nabla\varphi u_{0,R}^{m,\nu}\|_{L^\infty(\tilde{B}_{1,1}^\beta)} \\
& \leq C\|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y)|y-z|^\gamma |\nabla\varphi_R(t,y)| dy dz dv \\
& \quad + C\|u_{0,R}^{m,\nu}\|_{L^\infty} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y)|y-z| \int_0^1 |D^2\varphi_R(t,z+\mu(y-z))| d\mu dy dz dv \\
& \leq C\|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)} \|\nabla\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\beta}{2}} dv + C\|u_{0,R}^{m,\nu}\|_{L^\infty} \|D^2\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{1-\beta}{2}} dv,
\end{aligned}$$

which is finite if $\beta < \gamma$.

Control of $\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^{1+\beta})}$

The analysis is similar as before, replacing β by $1 + \beta$ and $\nabla\varphi_R$ by φ_R :

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(B_{1,1}^{1+\beta})} = \|\varphi_R u_{0,R}^{m,\nu}\|_{L^1} + \|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\tilde{B}_{1,1}^{1+\beta})}.$$

We readily get

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^1} \leq \|\varphi_R\|_{L^1} \|u_{0,R}^{m,\nu}\|_{L^\infty} \leq \|\varphi_R\|_{L^1} (T\|f\|_{L^\infty} + \|g\|_{L^\infty}),$$

and

$$\|\varphi_R u_{0,R}^{m,\nu}\|_{L^\infty(\tilde{B}_{1,1}^{1+\beta})} \leq C\|u_{0,R}^{m,\nu}\|_{L^\infty(C^\gamma)} \|\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\beta-1}{2}} dv + C\|u_{0,R}^{m,\nu}\|_{L^\infty} \|\nabla\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{-\beta}{2}} dv,$$

this is finite if $1 + \beta < \gamma \iff \beta < -1 + \gamma$. Let us carefully notice that if there is the incompressible assumption $\nabla \cdot b = 0$, then $B_2 = 0$ and this former constraint disappears. Thus the different cases considered in Theorem 1.

The Bolzano-Weierstrass theorem then yields the result.

5.5.5 Weak solution

The difficulty for the usual weak solution, here, is to prove that, up to a subsequence extraction,

$$\begin{aligned}
\lim_{m \rightarrow \infty} \int_{\mathbb{R}^d} \int_0^T \langle b_m(t,y), \nabla u_{0,R}^{m,\nu}(t,y) \rangle \varphi_R(t,y) dy dt &= \int_0^T \int_{\mathbb{R}^d} \langle b(t,y), \nabla u_{0,R}(t,y) \rangle \varphi_R(t,y) dy dt \\
&= \int_0^T \int_{\mathbb{R}^d} \langle b(t,y), \nabla u(t,y) \rangle \varphi_R(t,y) dy dt,
\end{aligned} \tag{5.82}$$

where $b \in L^\infty([0, T]; \tilde{B}_{\infty,\infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$ is the drift of the initial Cauchy problem (1.1) and coincides with the limit of b_m in $L^\infty([0, T]; B_{\infty,\infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))$, for any $0 < \varepsilon$ when $m \rightarrow \infty$; also $u(s, \cdot) \in C_b^\gamma(K, \mathbb{R})$, $K = B_d(0, R)$, is the limit of $u^{m,\nu}(s, \cdot)$, up to a subsequence selection possibly depending on the current time s , in $C_b^{\gamma-\tilde{\varepsilon}}(\mathbb{R}^d, \mathbb{R})$ for any $0 < \tilde{\varepsilon} < \gamma$, see Section 5.5.1.

Let us recall that $\tilde{B}_{\infty,\infty}^{-\beta}$ is the closure space of C_b^∞ in $B_{\infty,\infty}^{-\beta}$, from Appendix Section G, we still can take the regular sequence $(b_m)_{m \geq 1}$ defined in (5.2) to approximate b in $L^\infty([0, T]; B_{\infty,\infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))$, for any $0 < \varepsilon$; whereas the considered solution $u(s, \cdot)$ may depend on the choice of mollification. In other words, we have:

$$\lim_{m \rightarrow \infty} \|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} + \lim_{m \rightarrow \infty} \|\nabla \cdot b_m - \nabla \cdot b\|_{L^\infty(B_{\infty,\infty}^{-1-\beta-\varepsilon})} = 0. \tag{5.83}$$

For all $m > 0$, $t \in [0, T]$, we write by integration by parts that

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} \langle b_m(t, y), \nabla u_{0,R}^{m,\nu}(t, y) \rangle \varphi_R(t, y) dy - \int_{\mathbb{R}^d} \langle b(t, y), \nabla u_{0,R}(t, y) \rangle \varphi_R(t, y) dy \right| \\
& \leq \left| \int_{\mathbb{R}^d} [b_m - b](t, y) u^{m,\nu}(t, y) \cdot \nabla \varphi_R(t, y) dy \right| + \left| \int_{\mathbb{R}^d} b(t, y) [u - u^{m,\nu}](t, y) \cdot \nabla \varphi_R(t, y) dy \right| \\
& \quad + \left| \int_{\mathbb{R}^d} \nabla \cdot [b_m - b](t, y) u^{m,\nu}(t, y) \varphi_R(t, y) dy \right| + \left| \int_{\mathbb{R}^d} \nabla \cdot b(t, y) [u - u^{m,\nu}](t, y) \varphi_R(t, y) dy \right| \\
& =: \tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_4.
\end{aligned} \tag{5.84}$$

To deal with the first contribution, we aim to use (5.83). We have by the Besov duality result of Proposition 1:

$$\tilde{B}_1 \leq \|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} \|u^{m,\nu} \nabla \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}, \tag{5.85}$$

it then remains to control $\|u^{m,\nu} \nabla \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}$. Similar computations as in Section 5.5.4 yields that $\|u^{m,\nu} \nabla \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}$ is finite if $\beta < \gamma - \varepsilon$.

Also for the third term, which is null if $\nabla \cdot b = 0$, we have that

$$\tilde{B}_3 \leq \|\nabla \cdot b_m - \nabla \cdot b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} \|u^{m,\nu} \varphi_R\|_{L^\infty(B_{1,1}^{\beta+\varepsilon})}, \tag{5.86}$$

is finite if $\beta < -1 + \gamma - \varepsilon$.

Hence, from (5.83), we obtain

$$\tilde{B}_1 \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(4.2)} 0, \quad \text{and} \quad \tilde{B}_3 \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(4.2)} 0. \tag{5.87}$$

Now, let us handle with the second term in (5.84). We aim here to use the convergence of $u^{m,\nu}(s, \cdot)$ towards $u(s, \cdot)$ in the ball $B_d(0, R)$.

Again by the Besov duality result of Proposition 1, we have:

$$\begin{aligned}
\tilde{B}_2 & = \left| \int_{\mathbb{R}^d} b(t, y) [u - u^{m,\nu}](t, y) \cdot \nabla \varphi_R(t, y) dy \right| \\
& \leq \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \| (u^{m,\nu} - u) \nabla \varphi_R \|_{L^\infty(B_{1,1}^\beta)}.
\end{aligned} \tag{5.88}$$

By Arzelà-Ascoli theorem, we have

$$\|\nabla \varphi_R (u^{m,\nu} - u)\|_{L^\infty} \leq \|\nabla \varphi_R\|_{L^\infty} \| (u^{m,\nu} - u) \|_{L^\infty} \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(4.2)} 0.$$

For the homogenous part of the Besov norm, we also mimic the analysis in the previous section replacing $u^{m,\nu}$ by $(u^{m,\nu} - u)$, for any $\varepsilon \in (0, \gamma)$:

$$\begin{aligned}
& \| (u^{m,\nu} - u) \nabla \varphi_R \|_{L^\infty(\dot{B}_{1,1}^\beta)} \\
& \leq C \| (u^{m,\nu} - u) \|_{L^\infty(C^{\gamma-\varepsilon})} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) |y-z|^{\gamma-\varepsilon} |\nabla \varphi_R(t, y)| dy dz dv \\
& \quad + C \| (u^{m,\nu} - u) \|_{L^\infty} \int_0^1 \frac{1}{v} v^{-\frac{\beta}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) |y-z| \int_0^1 |D^2 \varphi_R(t, z + \mu(y-z))| d\mu dy dz dv \\
& \leq C \| (u^{m,\nu} - u) \|_{L^\infty(C^{\gamma-\varepsilon})} \|\nabla \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\varepsilon-\beta}{2}} dv + C \| (u^{m,\nu} - u) \|_{L^\infty} \|D^2 \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{1-\beta}{2}} dv,
\end{aligned}$$

which is finite as soon as $\beta < \gamma - \varepsilon < 0$, also by Arzelà-Ascoli theorem, we have the converging result

$$\| (u^{m,\nu} - u) \|_{L^\infty(C^{\gamma-\varepsilon})} \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(4.2)} 0. \quad \text{Therefore, we even get}$$

$$\tilde{B}_2 \xrightarrow[(m,\nu) \rightarrow (+\infty, 0)]{(4.2)} 0. \tag{5.89}$$

The last contribution \tilde{B}_4 , null if $\nabla \cdot b = 0$, is similar replacing $\nabla \varphi_R$ by φ_R and β by $\beta + 1$. Namely, we have

$$\begin{aligned} \tilde{B}_4 &\leq \|\nabla b\|_{L^\infty(B_{\infty,\infty}^{-1-\beta})} \|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(B_{1,1}^{1+\beta})} \\ &= \|\nabla b\|_{L^\infty(B_{\infty,\infty}^{-1-\beta})} (\|(u^{m,\nu} - u)\varphi_R\|_{L^1} + \|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(\tilde{B}_{1,1}^{1+\beta})}), \end{aligned} \quad (5.90)$$

with

$$\|(u^{m,\nu} - u)\varphi_R\|_{L^1} \leq \|u^{m,\nu} - u\|_{L^\infty} \|\varphi_R\|_{L^1},$$

and

$$\begin{aligned} &\|(u^{m,\nu} - u)\varphi_R\|_{L^\infty(\tilde{B}_{1,1}^{1+\beta})} \\ &= \int_0^1 \frac{1}{v} v^{1-\frac{1+\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) (u^{m,\nu} - u)(t,y) \varphi_R(t,y) dy \right| dz dv \\ &\leq C \|u^{m,\nu} - u\|_{L^\infty(C^{\gamma-\varepsilon})} \|\varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{\gamma-\varepsilon-1-\beta}{2}} dv + C \|u^{m,\nu} - u\|_{L^\infty} \|D^2 \varphi_R\|_{L^1} \int_0^1 \frac{1}{v} v^{\frac{-\beta}{2}} dv, \end{aligned}$$

which is finite if $\beta < -1 + \gamma - \varepsilon$ and by Arzelà-Ascoli theorem, we deduce

$$\tilde{B}_4 \xrightarrow[(m,\nu) \rightarrow (+\infty,0)]{(4.2)} 0. \quad (5.91)$$

Hence, from (5.84), (5.87), (5.89) and (5.91) we deduce (5.82).

5.6 Control of $\|\partial_t u(t, \cdot)\|_{B_{\infty,\infty}^{-1+\gamma}}$

If $b \in L^\infty([0, T]; C^{\tilde{\gamma}}(\mathbb{R}^d, \mathbb{R}^d))$, $0 < 1 - \gamma < \tilde{\gamma}$, we derive an upper-bound of $\|\partial_t u(t, \cdot)\|_{B_{\infty,\infty}^{-1+\gamma}}$ by the equation (1.1) and by para-product result. But first of all, let us precise why we have, point-wisely, with the viscous condition (4.2),

$$\lim_{(m,\nu) \rightarrow (+\infty,0)} \nu \Delta u^{m,\nu}(t, \cdot) = 0. \quad (5.92)$$

Recalling that Lemma 2,

$$\|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq \left(m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + C t m^{2-\tilde{\gamma}} \|b\|_{L^\infty(B_{\infty,\infty}^{\tilde{\gamma}})} O_m(t) \right) \exp(t m^{1-\tilde{\gamma}} \|b\|_{L^\infty(B_{\infty,\infty}^{\tilde{\gamma}})}). \quad (5.93)$$

Hence, for

$$\nu \ll \left(m^{2-\gamma} (T \|f\|_{L^\infty} + [g]_\gamma) + C T m^{2-\tilde{\gamma}} \|b\|_{L^\infty(B_{\infty,\infty}^{\tilde{\gamma}})} O_m(t) \right)^{-1} \exp(-T m^{1-\tilde{\gamma}} \|b\|_{L^\infty(B_{\infty,\infty}^{\tilde{\gamma}})}),$$

we deduce (5.92).

We are able to take the limit of equation (2.11), up to sub-sequence selection defined in the Arzelà-Ascoli theorem, for any $t \in (0, T]$,

$$\lim_{(m,\nu) \rightarrow (+\infty,0)} \partial_t u^{m,\nu}(t, \cdot) = \lim_{(m,\nu) \rightarrow (+\infty,0)} \langle b_m(t, \cdot), \nabla u^{m,\nu}(t, \cdot) \rangle + f(t, \cdot). \quad (5.94)$$

But from para-product (4.5), we know that $\langle b_m(t, \cdot), \nabla u^{m,\nu}(t, \cdot) \rangle \in B_{\infty,\infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$, $\nabla u^{m,\nu}(t, \cdot)$ being in $B_{\infty,\infty}^{-1+\gamma}$, the result then follows.

6 Analysis of the parabolic equation

The previous analysis strongly relies on the vanishing result but seems to be incompatible with the parabolic equation. We developed, here, a new technique based on a time decomposition trick, detailed in Section 6.5 below.

6.1 The types of solution to the Cauchy problem

6.1.1 Parabolic equation

For $0 < \gamma < 1$, and for a given viscosity $\nu > 0$, we define some solutions of the parabolic equation.

Definition 4 (mild solution). *We say that u is a mild solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if there is a sequence[¶] $(b_m)_{m \in \mathbb{R}_+}$ in $L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$ such that there is $\beta \in \mathbb{R}$,*

$$\forall \varepsilon > 0, \quad \lim_{m \rightarrow +\infty} \|b_m - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta - \varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} = 0, \quad (6.1)$$

for any $t \in [0, T]$, there exists a sub-sequence of $u^{m, \nu}(t, \cdot)_{m \in \mathbb{R}_+}$ lying in $C_b^\gamma(\mathbb{R}^d, \mathbb{R})$ converging, for any compact subset $K \subset \mathbb{R}^d$, when $m \rightarrow +\infty$, towards $u(t, \cdot) \in C_b^\gamma(K, \mathbb{R})$ and satisfying, for any $m \in \mathbb{R}_+$,

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + \langle b_m(t, x), \nabla u^{m, \nu}(t, x) \rangle - \nu \Delta u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases} \quad (6.2)$$

where $(f_m, g_m) \xrightarrow{m \rightarrow +\infty} (f, g)$ in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$.

Let us now, recall the notion of usual *weak* solution.

Definition 5 (weak solution). *A function u is a weak solution in $L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of equation (1.1) if u is a mild solution such that for any function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$:*

$$\begin{aligned} \int_{\mathbb{R}^d} \left\{ \varphi(t, y) u(t, y) + \int_0^t \left\{ -\partial_t \varphi(s, y) u(s, y) + \langle b(s, y), \nabla u(s, y) \rangle \varphi(s, y) + \nu u(s, y) \Delta \varphi(s, y) \right\} ds \right\} dy \\ = \int_{\mathbb{R}^d} \varphi(0, y) g(y) dy + \int_{\mathbb{R}^d} \int_0^t \varphi(s, y) f(s, y) ds dy. \end{aligned} \quad (6.3)$$

Again, this formulation allows us to give a usual distributional meaning of the product $\langle b, \nabla u \rangle$.

6.2 Main results on the parabolic equation

When b lies in Hölder-Besov space, we succeed in obtaining the same regularity of the solution as for f and g . The type of solution strongly depends on the regularity of b .

Theorem 2 (Rough parabolic equation in Hölder spaces). *For $\beta \in \mathbb{R}^*$ and $0 < \gamma < 1$, be given. For all $b \in L^\infty([0, T], \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, $f \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}^d, \mathbb{R})$, there is a mild solution $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of (1.1) satisfying*

$$\begin{aligned} \|u\|_{L^\infty(C^\gamma)} &\leq T \|f\|_{L^\infty(C^\gamma)} + \|g\|_{(C^\gamma)}, \\ \|u\|_{L^\infty} &\leq T \|f\|_{L^\infty} + \|g\|_{L^\infty}. \end{aligned} \quad (6.4)$$

Moreover, if $\beta < \gamma$ and $\nabla \cdot b = 0$ then the solution u is also a weak solution.

We do not consider the positive regular case, i.e. $\beta < 0$, whose control is well-known, the Schauder estimates are even in force, see [Fri64].

Remark 7. *The control (6.4) exactly matches with the Hölder estimates of the solution of the heat equation, independently of the dimension, and above all of b . As consequence, the mild solution has no condition on β , there is to say we define a solution beyond the Peano condition, $\beta > -1$, developed in Section 1.3.1.*

[¶]We have such a sequence $(b_m)_{m \geq 0}$, if $b \in L^\infty([0, T]; \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}))$.

In the incompressible case, $\nabla \cdot b$, the condition $\beta < \gamma < 1$ corresponds to the Peano's heuristic[‡] in Section 1.3.1. However, we fail to obtain uniqueness of the solution out of the usual Hölder continuous case (i.e. $\beta < 0$) handled in [Fri64], for more information see Remark 9 further.

The case $\beta = 0$ can be considered, replacing the condition $b \in L^\infty([0, T], \dot{B}_{\infty, \infty}^0(\mathbb{R}^d, \mathbb{R}^d))$ by $b \in L^\infty([0, T], L^\infty(\mathbb{R}^d, \mathbb{R}^d))$. Considering the Besov space $B_{\infty, \infty}^0$ would yield some refinements involving some logarithm corrections.

Again, the case $\beta < 0$ is the usual framework, see for instance [Fri64], [Kry96].

An interesting extension to this result would be to adapt our work for $f = b$ corresponding to the Zvonking transform which is crucial to deal with stochastic differential equations. The difficulty is that the consider Gaussian kernel proxy, in our analysis, does depend on b and the parameter of mollification, so the control of the source function become unclear in a rough framework.

6.3 Proof of Theorem 2

The beginning of the analysis for the parabolic case is similar as for the transport equation performed in Section 5, except that the goal of the regularisation is different. We aim here to raise the time contribution instead of the viscosity one. This allows us to conclude, thanks to a time cutting trick, without any vanishing viscosity.

Let us recall the Duhamel formula (5.21),

$$\begin{aligned}
& |u^{m, \nu}(t, x) - u^{m, \nu}(t, x')| \\
& \leq |\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x')| + |\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x')| \\
& \quad + \left| \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, x, y) [b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) dy ds \right. \\
& \quad \left. - \int_0^t \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi'}(s, t, x, y) [b_m(s, \theta_{s, \tau'}^m(\xi')) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) dy ds \right| \\
& =: |\hat{G}^{\tau, \xi} f_m(t, x) - \hat{G}^{\tau, \xi'} f_m(t, x')| + |\hat{P}^{\tau, \xi} g_m(t, x) - \hat{P}^{\tau, \xi'} g_m(t, x')| + |R^{\tau, \xi, \xi'}(t, x, x')|. \quad (6.5)
\end{aligned}$$

Here, we again want to calibrate properly the parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ in the *off-diagonal* regime, $|x - x'| > \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s > t_0$ where we recall that the *cut locus* time is defined by

$$t_0 = t - \nu^{-\frac{\alpha_1}{\alpha_2}} |x - x'|^{\frac{1}{\alpha_2}}; \quad (6.6)$$

also the *diagonal* regime happens if $|x - x'| \leq \nu^{\alpha_1} (t - s)^{\alpha_2} \Leftrightarrow s \leq t_0$.

We specify that the parameters (α_1, α_2) will be different as the ones tailored in Section 5.

6.3.1 Diagonal regime

Let us define the remainder term without integration by parts, unlike in identity (5.22),

$$\begin{aligned}
& R_A^{\tau, \xi, \xi'}(t, x, x') \\
& := \int_0^{t_0} \int_{\mathbb{R}^d} \left[\hat{p}^{\tau, \xi}(s, t, x, y) [b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) \right. \\
& \quad \left. - \hat{p}^{\tau, \xi'}(s, t, x', y) [b_m(s, \theta_{s, \tau'}^m(\xi')) - b_m(s, y)] \cdot \nabla u^{m, \nu}(s, y) \right] dy ds \\
& =: \hat{G}_{0, t_0}^{\tau, \xi} \left\{ (b_m(\cdot, \theta_{\cdot, \tau}^m(\xi)) - b_m) \cdot \nabla u^{m, \nu} \right\}(t, x) - \hat{G}_{0, t_0}^{\tau, \xi'} \left\{ (b_m(\cdot, \theta_{\cdot, \tau'}^m(\xi')) - b_m) \cdot \nabla u^{m, \nu} \right\}(t, x'). \quad (6.7)
\end{aligned}$$

Unlike in Section 5.2, we do not proceed with an integration by parts which seems to useless as detailed therein.

[‡]The true condition is $\beta < 1$, but this is the case, here, as γ which can be arbitrarily close to 1.

A simple change of variable gives

$$\begin{aligned} & |R_A^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ &= \left| \int_0^{t_0} \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, 0, y) \left\{ (b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, x + y)) \cdot \nabla u^{m, \nu}(s, x + y) \right. \right. \\ & \quad \left. \left. - (b_m(s, \theta_{s, \tau}^m(\xi')) - b_m(s, x' + y)) \cdot \nabla u^{m, \nu}(s, x' + y) \right\} dy ds \right|_{\tau=t, \xi=\xi'=x}, \end{aligned}$$

also

$$\begin{aligned} & |R_A^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ &= \left| \int_0^{t_0} \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, 0, y) \left\{ b_m(s, \theta_{s, \tau}^m(x)) \cdot (\nabla u^{m, \nu}(s, x + y) - \nabla u^{m, \nu}(s, x' + y)) \right. \right. \\ & \quad \left. \left. + b_m(s, x' + y) \cdot \nabla u^{m, \nu}(s, x' + y) - b_m(s, x + y) \cdot \nabla u^{m, \nu}(s, x + y) \right\} dy ds \right|_{\tau=t, \xi=\xi'=x}, \end{aligned}$$

and finally

$$\begin{aligned} & |R_A^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ &= \left| \int_0^{t_0} \int_{\mathbb{R}^d} \hat{p}^{\tau, \xi}(s, t, 0, y) \left\{ b_m(s, \theta_{s, \tau}^m(x)) \cdot (\nabla u^{m, \nu}(s, x + y) - \nabla u^{m, \nu}(s, x' + y)) \right. \right. \\ & \quad \left. \left. + (b_m(s, x' + y) - b_m(s, x + y)) \cdot \nabla u^{m, \nu}(s, x' + y) \right. \right. \\ & \quad \left. \left. - b_m(s, x + y) \cdot (\nabla u^{m, \nu}(s, x + y) - \nabla u^{m, \nu}(s, x' + y)) \right\} dy ds \right|_{\tau=t, \xi=\xi'=x}. \end{aligned}$$

Hence, we readily derive that

$$|R_A^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \leq C|x - x'| \|b_m\|_{L^\infty(C_b^1)} \|\nabla u^{m, \nu}\|_{L^\infty(C_b^1)} \int_0^{t_0} ds. \quad (6.8)$$

Recalling that we have to use the *a priori* controls of the gradient and the Hessian in Lemma 2,

$$\begin{aligned} \|u^{m, \nu}(t, \cdot)\|_{C^1} &\leq \left(T \|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp \left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \right) = O_m(t), \\ \|\nabla^2 u^{m, \nu}(t, \cdot)\|_{L^\infty} &\leq C m^{2-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(C m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}) = O_m^{(2)}(t). \end{aligned} \quad (6.9)$$

From the definition of the *diagonal* regime, $|x - x'| \leq |x - x'|^\gamma \nu^{\alpha_1(1-\gamma)}(t - s)^{(1-\gamma)\alpha_2}$, we deduce

$$\begin{aligned} & |R_A^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\ &\leq C|x - x'|^\gamma \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} m^{1+\beta} (O_m(t) + O_m^{(2)}(t)) \int_0^{t_0} \nu^{\alpha_1(1-\gamma)}(t - s)^{(1-\gamma)\alpha_2} ds. \end{aligned} \quad (6.10)$$

As announced, we aim to get positive time contribution inside the time integral, namely from the previous control, we suppose that

$$(1 - \gamma)\alpha_2 > 0 \Leftrightarrow \gamma < 1 \text{ and } \alpha_2 > 0. \quad (6.11)$$

We again see the impossibility to consider the Lipschitz case, $\gamma = 1$.

6.3.2 Off-diagonal regime

In this case, x and x' are supposed to be “far away” from each other, then we pick $\xi = x$ and $\xi' = x'$. The associated remainder terms is defined by

$$\begin{aligned}
& R_{A^c}^{\tau, \xi, \xi'}(t, x, x') \\
:= & R^{\tau, \xi, \xi'}(t, x, x') - R_A^{\tau, \xi, \xi'}(t, x, x') \\
= & \int_{t_0}^t \int_{\mathbb{R}^d} \left[\hat{p}^{\tau, \xi}(s, t, x, y) (b_m(s, \theta_{s, \tau}^m(\xi)) - b_m(s, y)) \cdot \nabla u^{m, \nu}(s, y) \right. \\
& \left. - \hat{p}^{\tau, \xi'}(s, t, x', y) (b_m(s, \theta_{s, \tau}^m(\xi')) - b_m(s, y)) \cdot \nabla u^{m, \nu}(s, y) \right] dy ds \\
=: & \hat{G}_{t_0, t}^{\tau, \xi} \left\{ (b_m(\cdot, \theta_{\cdot, \tau}^m(\xi)) - b_m) \cdot \nabla u^{m, \nu} \right\} (t, x) - \hat{G}_{t_0, t}^{\tau, \xi'} \left\{ (b_m(\cdot, \theta_{\cdot, \tau}^m(\xi')) - b_m) \cdot \nabla u^{m, \nu} \right\} (t, x').
\end{aligned} \tag{6.12}$$

The analysis is direct by triangular inequality

$$\begin{aligned}
& |R_{A^c}^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=x, \xi'=x'} \\
\leq & 2C \|b_m\|_{L^\infty(C^1)} \|u^{m, \nu}\|_{L^\infty(C^1)} \sup_{x \in \mathbb{R}^d} \int_{t_0}^t \int_{\mathbb{R}^d} \bar{p}^{t, x}(s, t, x, y) |y - \theta_{s, t}^m(x)| dy ds,
\end{aligned}$$

the absorbing property of the exponential (2.8) gives

$$\begin{aligned}
& |R_{A^c}^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=x, \xi'=x'} \\
\leq & Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} O_m(t) \int_{t_0}^t (\nu(t-s))^{\frac{1}{2}} ds \\
\leq & Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} O_m(t) \int_{t_0}^t \nu^{\frac{1}{2}}(t-s)^{\frac{1-\varepsilon}{2}} \nu^{-\frac{\alpha_1 \varepsilon}{2\alpha_2}} |x-x'|^{\frac{\varepsilon}{2\alpha_2}} ds,
\end{aligned} \tag{6.13}$$

for a given $0 < \varepsilon < 1$, and because the *off-diagonal* regime is in force, $|x-x'| \geq \nu^{\alpha_1}(t-s)^{\alpha_2}$.

The solution $u^{m, \nu}$ is supposed to be γ -Hölder then

$$\gamma = \frac{\varepsilon}{2\alpha_2} \Rightarrow \varepsilon = 2\alpha_2\gamma > 0.$$

Also, we have to suppose that $\varepsilon < 1$ (for positive time contribution purpose) which implies

$$\alpha_2 < \frac{1}{2\gamma}. \tag{6.14}$$

Let us consider equality between the two regimes from (6.10) and (6.13), i.e.

$$(1-\gamma)\alpha_2 = \frac{1-2\alpha_2\gamma}{2} \Leftrightarrow \alpha_2 = \frac{1}{2} < \frac{1}{2\gamma}, \tag{6.15}$$

which is exactly the parabolic scale, and also

$$\varepsilon = \gamma.$$

This is another way to see the difficulty to get a Lipschitz control, in our framework, because ε has to be strictly lower than 1.

We then conclude

$$\begin{aligned}
& |R^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \\
\leq & Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} |x-x'|^\gamma (O_m(t) + O_m^{(2)}(t)) \int_0^t (\nu^{\alpha_1(1-\gamma)} + \nu^{\frac{1}{2}-\alpha_1\gamma})(t-s)^{\frac{1-\gamma}{2}} ds.
\end{aligned}$$

Let us also consider the equality of the exponent of ν ,

$$\alpha_1(1 - \gamma) = \frac{1}{2} - \alpha_1\gamma \Leftrightarrow \alpha_1 = \frac{1}{2}.$$

We retrieve the usual parabolic scale for α_1 and α_2 . We deduce,

$$|R^{\tau, \xi, \xi'}(t, x, x')|_{\tau=t, \xi=\xi'=x} \leq Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} |x - x'|^\gamma (O_m(t) + O_m^{(2)}(t)) \int_0^t \nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}} ds. \quad (6.16)$$

6.4 Adapting the controls of the extra contributions

We have to change the parameters (α_1, α_2) into (5.59) and (5.65), which gives

$$\begin{aligned} & \left| [\hat{P}_{t_0}^{\tau, \xi'} u^{m, \nu}(t_0, \cdot)](t, x') - [\hat{P}_{t_0}^{\tau, \tilde{\xi}'} u^{m, \nu}(t_0, \cdot)](t, x') \right|_{(\tau, \xi', \tilde{\xi}')=(t, x', x)} \\ & \leq |x - x'|^\gamma m^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{(1-\gamma)}{2}+1} \exp(\|b_m\|_{L^\infty(C^1)} t), \end{aligned} \quad (6.17)$$

and

$$\begin{aligned} & \left| \hat{G}_{0, t_0}^{\tau, \xi'} f_m(t, x') - \hat{G}_{0, t_0}^{\tau, \tilde{\xi}'} f_m(t, x') \right|_{(\tau, \xi', \tilde{\xi}')=(t, x', x)} \\ & \leq |x - x'|^\gamma m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) \int_0^{t_0} \int_s^t \frac{\nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{(1-\gamma)}{2}}}{2} d\tilde{s} ds. \end{aligned} \quad (6.18)$$

6.4.1 Gathering the controls

To put in a nutshell, gathering all the previous estimates (5.23), (5.24), (6.16), (6.17), (6.18) into (5.62)

$$\begin{aligned} & \|u^{m, \nu}(t, \cdot)\|_{C^\gamma} \\ & \leq t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \\ & \quad + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \int_0^t \nu^{\frac{1-\gamma}{2}}(t-s)^{\frac{1-\gamma}{2}} ds \\ & \quad + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{(1-\gamma)}{2}+1} \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & \quad + m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) \int_0^t \int_s^t \frac{\nu^{\frac{1-\gamma}{2}}(t-t_0)^{\frac{(1-\gamma)}{2}}}{2} d\tilde{s} ds, \end{aligned} \quad (6.19)$$

with

$$\begin{aligned} O_m(t) & = \left(t \|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}\right) \\ & \leq C \left(m^{1-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \right) \exp\left(m^{1+\beta} t \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})}\right). \end{aligned}$$

Now, the idea is to make negligible the terms involving positive time contribution in the time integral.

6.5 The right control the Hölder modulus: the time cutting trick

We cut the Cauchy problem in small intervals of $[0, T]$, and we see, from computations below, that the first order term $b_\Delta^m \cdot \nabla u^{m, \nu}$ and the extra terms are negligible when the size of the time intervals goes to 0.

Let $n \in \mathbb{N}$, and for each $k \in \llbracket 0, n \rrbracket$,

$$\tau_k := \frac{k}{n}T,$$

we also denote, for all $x \in \mathbb{R}^3$, $k \in \llbracket 0, n-1 \rrbracket$ and $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{cases} u_{k+1}^{m,\nu}(t, x) & := u^{m,\nu}(t, x), \\ u_1^{m,\nu}(0, x) & := g_m(x). \end{cases}$$

The associated Cauchy problems write

If $k \in \llbracket 1, n-1 \rrbracket$

$$\begin{cases} \partial_t u_{k+1}^{m,\nu}(t, x) + \langle b_m, \nabla u_{k+1}^{m,\nu} \rangle(t, x) = \nu \Delta u_{k+1}^{m,\nu}(t, x) + f_m(t, x), & t \in (\tau_k, \tau_{k+1}] \\ u_{k+1}^{m,\nu}(\tau_k, x) = u_k^{m,\nu}(\tau_k, x), \end{cases}$$

if $k = 0$

$$\begin{cases} \partial_t u_1^{m,\nu}(t, x) + \langle b_m, \nabla u_1^{m,\nu} \rangle(t, x) = \nu \Delta u_1^{m,\nu}(t, x) + f_m(t, x), & t \in (0, \tau_1), \\ u_1^{m,\nu}(0, x) = g_m(x). \end{cases}$$

Using the Duhamel formulation (5.11) around the consider heat like equation, we get for any $t \in (\tau_k, \tau_{k+1}]$, if $k \in \llbracket 1, n-1 \rrbracket$,

$$u_{k+1}^{m,\nu}(t, x) = \hat{P}_{\tau_k}^{\tau, \xi} u_k^{m,\nu}(t, x) + \hat{G}_{\tau_k}^{\tau, \xi} f_m(t, x) + \hat{G}_{\tau_k}^{\tau, \xi} (\langle b_\Delta^m[\tau, \xi], \nabla u_k^{m,\nu} \rangle)(t, x), \quad (6.20)$$

and if $k = 0$,

$$u_1^{m,\nu}(t, x) = \hat{P}_0^{\tau, \xi} g_m(t, x) + \hat{G}_0^{\tau, \xi} f_m(t, x) + \hat{G}_0^{\tau, \xi} (\langle b_\Delta^m[\tau, \xi], \nabla u_1^{m,\nu} \rangle)(t, x).$$

Next, we use the corresponding Hölder control (6.19), including the extra contributions coming from the *cut locus* argument,

$$\begin{aligned} & \|u_{k+1}^{m,\nu}(t, \cdot)\|_{C^\gamma} \\ & \leq \int_{\tau_k}^t \|f(s, \cdot)\|_{C^\gamma} ds + [u_k^{m,\nu}(\tau_k, \cdot)]_\gamma \\ & \quad + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \int_{\tau_k}^t \nu^{\frac{1-\gamma}{2}} (t-s)^{\frac{1-\gamma}{2}} ds \\ & \quad + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} (t-t_0)^{\frac{(1-\gamma)}{2}+1} \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & \quad + m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) \int_{\tau_k}^t \int_{\tau_k}^t \frac{\nu^{\frac{1-\gamma}{2}} (t-t_0)^{\frac{(1-\gamma)}{2}}}{2} d\tilde{s} ds. \end{aligned} \quad (6.21)$$

Let us carefully point out that we can suppose that $(t - t_0) \leq (t - \tau_k)$, otherwise there is no off-diagonal regime and so no need to use the *cut locus* technique with the *extra* terms. Hence, recalling that $\tau_{k+1} - \tau_k = \frac{T}{n}$,

$$\begin{aligned} & \|u_{k+1}^{m,\nu}(t, \cdot)\|_{C^\gamma} \\ & \leq \int_{\tau_k}^t \|f(s, \cdot)\|_{C^\gamma} ds + [u_k^{m,\nu}(\tau_k, \cdot)]_\gamma \\ & \quad + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} \int_{\tau_k}^t ds \\ & \quad + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} (t - \tau_k) \exp(\|b_m\|_{L^\infty(C^1)} t) \\ & \quad + C \nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty, \infty}^{-\beta})} t) (t - \tau_k)^{\frac{(1-\gamma)}{2}+1} \int_{\tau_k}^t ds. \end{aligned} \quad (6.22)$$

For any $t \in [0, T]$, let $\kappa_n(t) = \lfloor \frac{nt}{T} \rfloor \in \llbracket 0, n \rrbracket$, such that $\tau_{\kappa_n(t)} \leq t < \tau_{\kappa_n(t)+1}$, and iterating the above inequality,

$$\begin{aligned}
& \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\
\leq & \|g\|_{C^\gamma} + \int_{t_{\kappa_n(t)}}^t \|f(s, \cdot)\|_{C^\gamma} ds \\
& + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} \int_{t_{\kappa_n(t)}}^t ds \\
& + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} (t - t_{\kappa_n(t)}) \exp(\|b_m\|_{L^\infty(C^1)} t) \\
& + C\nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t) (t - t_{\kappa_n(t)})^{\frac{(1-\gamma)}{2}+1} \int_{\tau_k}^t ds \\
& + \sum_{k=0}^{\kappa(t)-1} \left\{ \int_{\tau_k}^{\tau_{k+1}} \|f(s, \cdot)\|_{C^\gamma} ds \right. \\
& + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} \int_{\tau_k}^{\tau_{k+1}} ds \\
& + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} (\tau_{k+1} - \tau_k) \exp(\|b_m\|_{L^\infty(C^1)} t) \\
& \left. + C\nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t) (\tau_{k+1} - \tau_k)^{\frac{(1-\gamma)}{2}+1} \int_{\tau_k}^{\tau_{k+1}} ds \right\}.
\end{aligned}$$

By Chasles equality and by telescopic sum,

$$\begin{aligned}
& \|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \\
\leq & \|g\|_{C^\gamma} + \int_0^t \|f(s, \cdot)\|_{C^\gamma} ds \\
& + Cm^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (O_m(t) + O_m^{(2)}(t)) \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} \int_0^t ds \\
& + Cm^{1+\beta} O_m(t) \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \nu^{\frac{1-\gamma}{2}} \left(\frac{T}{n}\right)^{\frac{1-\gamma}{2}} t \exp(\|b_m\|_{L^\infty(C^1)} t) \\
& + C\nu^{\frac{1-\gamma}{2}} m^{2+\beta-\gamma} \|f\|_{L^\infty(C^\gamma)} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \exp(m^{1+\beta} \|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} t) \left(\frac{T}{n}\right)^{\frac{(1-\gamma)}{2}+1} \int_0^t ds.
\end{aligned}$$

The l.h.s. does not depend on n , then we are able to pass to the limit $n \rightarrow +\infty$,

$$\|u^{m,\nu}(t, \cdot)\|_{C^\gamma} \leq \|g\|_{C^\gamma} + \int_0^t \|f(s, \cdot)\|_{C^\gamma} ds. \quad (6.23)$$

6.6 Another control of uniform norm

By a similar way as for the Hölder control performed in the previous section, by Duhamel formula (6.20), after choosing $(\tau, \xi) = (t, x)$, we get

$$\|u_{k+1}^{m,\nu}(t, \cdot)\|_{L^\infty} \leq \|u_k^{m,\nu}(t_k, \cdot)\|_{L^\infty} + \int_{t_k}^t \|f(s, \cdot)\|_{L^\infty} ds + C \|b_m\|_{L^\infty(C^1)} \|\nabla u_k^{m,\nu}\|_{L^\infty} \int_{t_k}^t (t-s)^{\frac{1}{2}} ds.$$

Iterating this inequality,

$$\|u^{m,\nu}\|_{L^\infty} \leq \|g\|_{L^\infty} + \int_0^T \|f(s, \cdot)\|_{L^\infty} ds + C \frac{T^{\frac{3}{2}}}{n^{\frac{1}{2}}} \|b_m\|_{L^\infty(C^1)} \|\nabla u^{m,\nu}\|_{L^\infty}.$$

We obviously deduce when $n \rightarrow +\infty$, the following estimate,

$$\|u^{m,\nu}\|_{L^\infty} \leq \|g\|_{L^\infty} + \int_0^T \|f(s, \cdot)\|_{L^\infty} ds. \quad (6.24)$$

In fine, the time cutting trick allows to retrieve the powerful uniform estimate given by Feynman-Kac formula or usual maximum principle method.

7 Second result for the transport equation

From the previous analysis based on a time cutting trick, we are able to derive uniqueness of selection principle for the transport equation, defined below.

Definition 6 (Uniqueness). *There is a unique solution of (1.1) if for two solutions $u^{m,\nu}$ and $\bar{u}^{m,\bar{\nu}}$ of*

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + \langle b_m(t, x), \nabla u^{m,\nu}(t, x) \rangle - \nu \Delta u^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ u^{m,\nu}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases}$$

and respectively

$$\begin{cases} \partial_t \bar{u}^{m,\bar{\nu}}(t, x) + \langle \bar{b}_m(t, x), \nabla \bar{u}^{m,\bar{\nu}}(t, x) \rangle - \bar{\nu} \Delta \bar{u}^{m,\bar{\nu}}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\ \bar{u}^{m,\bar{\nu}}(0, x) = g_m(x), & x \in \mathbb{R}^d, \end{cases}$$

for $\nu, \bar{\nu} > 0$, with

$$\forall \varepsilon > 0, \lim_{m \rightarrow +\infty} \|b_m - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} + \lim_{n \rightarrow +\infty} \|\bar{b}_n - b\|_{L^\infty([0, T]; B_{\infty, \infty}^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} = 0, \quad (7.1)$$

converging, up to sub-sequence selection, towards two Hölder continuous solutions u, \bar{u} , when $(m, \nu, \bar{\nu}) \rightarrow (+\infty, 0, 0)$, then $u = \bar{u}$.

7.1 Statement for the transport equation

Theorem 3 (Rough transport equation in Hölder spaces). *For $\beta \in \mathbb{R}^*$ and $0 < \gamma < 1$ be given. For all $b \in L^\infty([0, T]; \tilde{B}_{\infty, \infty}^{-\beta}(\mathbb{R}^d, \mathbb{R}^d))$, $f \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}^d, \mathbb{R})$, there is a mild vanishing viscosity solution $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}^d, \mathbb{R}))$ of (1.1) satisfying*

$$\begin{aligned} \|u\|_{L^\infty(C^\gamma)} &\leq T \|f\|_{L^\infty(C^\gamma)} + \|g\|_{(C^\gamma)}. \\ \|u\|_{L^\infty} &\leq T \|f\|_{L^\infty} + \|g\|_{L^\infty}. \end{aligned} \quad (7.2)$$

i) **Incompressibility.** *If $\beta < \gamma$ and $\nabla \cdot b = 0$ then the solution u is also a weak and a mild-weak solution.*

ii) **Positive regularity.** *If $\beta < -1 + \gamma$, namely if $b \in L^\infty([0, T]; C_b^{\tilde{\gamma}}(\mathbb{R}^d, \mathbb{R}^d))$, $\tilde{\gamma} = -\beta > -1 + \gamma$, the solution u is also a weak solution and if we also suppose that*

$$\nu \|\nabla^2 g_m\|_{L^\infty} + \nu^{\frac{\gamma}{2}} m^{1-\tilde{\gamma}} \ll 1, \quad (7.3)$$

then $\partial_t u(t, \cdot) \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$, for any $t \in (0, T]$.

iii) **Greater regularity.** *If $\tilde{\gamma} = -\beta > \frac{1}{1+\gamma}$, and if*

$$\nu \|\nabla^2 g_m\|_{L^\infty} + m^{-\tilde{\gamma}+1-\tilde{\gamma}} \nu^{\frac{\gamma-1}{2}} + m^{1-\tilde{\gamma}} \nu^{\frac{\gamma}{2}} \ll 1, \quad (7.4)$$

the solution is unique**.

**Namely the limit solution does not depend on the choice of sequence (m, ν) and on the way to mollify b .

Remark 8. *There is still no constraint on b for the mild vanishing viscous solution, we observe an “infinite regularisation”. The two other types of solutions requires more constraints on b .*

*We even obtain uniqueness for the **Greater regularity** case, we use a kind of regularisation by turbulence which makes negligible the second order term. The condition on g means that the initial function has to be smooth enough such that $\nu \|\nabla^2 g_m\|_{L^\infty}$ is negligible, obvious if $g \in C^2(\mathbb{R}^d, \mathbb{R})$. Again, this a positive answer to the question (Q3) in [CCS20], but we fail to consider uniqueness in a rough (negative regularity) case, see Remark 9 further.*

In this Hölder case, the product $\langle b, \nabla u \rangle$ falls into the usual case of the para-product by the Bony’s microlocal analysis [Bon81]. In the negative regularity case, we again obtain a challenging result on the distribution product.

Except for the control of the time derivative and for uniqueness, there is no condition on the vanishing viscosity unlike in Theorem 1. This is due to the possibility to use the time decomposition trick which allows to get estimates independent on b .

7.2 Uniqueness for transport equation

Let us insist that uniqueness of *vanishing viscous* solution does not mean uniqueness of usual solution. Indeed, this question arises for the uniqueness of the limits of any sub-sequence of $(u^{m,\nu})_{m,\nu \geq 0}$ and for the non-dependency of the limit on the regularisation procedure; also the smooth selection principle established here does not depend on the choice of the vanishing sequence (ν) .

To get any point-wise uniqueness, we naturally suppose that $\beta < 0$ which means that b is $\tilde{\gamma}$ -Hölder continuous in space, $\tilde{\gamma} = -\beta$.

Let us suppose that there are two *vanishing viscous* solutions u and \bar{u} of (1.1) satisfying estimates (7.2). We then consider the associated mollified version $(u^{m,\nu})_{m \geq 0}$ and $(\bar{u}^{m,\bar{\nu}})_{m \geq 0}$ solutions, for any $x \in \mathbb{R}^d$, to

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + \langle b_m(t, x), \nabla u^{m,\nu}(t, x) \rangle - \bar{\nu} \Delta u^{m,\nu}(t, x) = f_m(t, x), & t \in [0, T], \\ u^{m,\nu}(0, x) = g_m(x), \end{cases} \quad (7.5)$$

where b_m is a mollified version of b as in (5.2) by a convolution with the Gaussian mollifier ρ_m , and

$$\begin{cases} \partial_t \bar{u}^{m,\bar{\nu}}(t, x) + \langle \bar{b}_m(t, x), \nabla \bar{u}^{m,\bar{\nu}}(t, x) \rangle - \bar{\nu} \Delta \bar{u}^{m,\bar{\nu}}(t, x) = f_m(t, x), & t \in [0, T], \\ \bar{u}_n(0, x) = g_m(x), \end{cases} \quad (7.6)$$

where \bar{b}_m is a mollified version of b which is potentially defined differently as in (5.2), and such that

$$\forall 0 < \varepsilon < 1, \quad \lim_{n \rightarrow +\infty} \|\bar{b}_m - b\|_{L^\infty([0, T]; C^{-\beta-\varepsilon}(\mathbb{R}^d, \mathbb{R}^d))} = 0. \quad (7.7)$$

From the linearity of the equations, we then derive that $U_m := u^{m,\nu} - \bar{u}^{m,\bar{\nu}}$ solves the following Cauchy problem for any $(t, x) \in [0, T] \times \mathbb{R}^d$:

$$\begin{cases} \partial_t U_m(t, x) + \langle \bar{b}_m, \nabla U_m \rangle(t, x) - \bar{\nu} \Delta U_m(t, x) = (\nu - \bar{\nu}) \Delta u^{m,\nu}(t, x) - \langle [b_m - \bar{b}_m], \nabla u^{m,\nu} \rangle(t, x), \\ U_m(0, x) = 0. \end{cases} \quad (7.8)$$

By uniform control (5.4), we directly derive that

$$\begin{aligned} \|U_m(t, \cdot)\|_{L^\infty} &\leq \int_0^t \|\langle [b_m - \bar{b}_m], \nabla u^{m,\nu} \rangle(s, \cdot)\|_{L^\infty} ds + \int_0^t (\nu - \bar{\nu}) \|\Delta u^{m,\nu}(s, \cdot)\|_{L^\infty} ds \\ &\leq T \|b_m - \bar{b}_m\|_{L^\infty} \|\nabla u^{m,\nu}\|_{L^\infty} + T |\nu - \bar{\nu}| \|\Delta u^{m,\nu}\|_{L^\infty}. \end{aligned} \quad (7.9)$$

It is clear that if b is $\tilde{\gamma}$ -Hölder continuous then $\|b_m - \bar{b}_m\|_{L^\infty} \leq C m^{-\tilde{\gamma}}$. To take advantage of the convergence of $\|b_m - \bar{b}_m\|_{L^\infty}$ towards 0, we need to use other a priori controls.

Lemma 4. *If $u^{m,\nu}$ is solution of (7.5), then, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, we get the gradient estimate*

$$|\nabla u^{m,\nu}(t, x)| \leq m^{1-\gamma} \left(t \|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma} \right) + C \|b_m\|_{L^\infty(C^1)} \left(t \|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma} \right) \nu^{\frac{\gamma-1}{2}} t^{\frac{\gamma+1}{2}},$$

and the Hessian estimate

$$\begin{aligned} & |\nabla^2 u^{m,\nu}(t, x)| \\ & \leq \left(C \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) + C \|b_m\|_{L^\infty(C^1)} \left(t \|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}}. \end{aligned}$$

The proof is deferred in Appendix Section C.

Hence,

$$\begin{aligned} & \|U_m(t, \cdot)\|_{L^\infty} \\ & \leq T m^{-\tilde{\gamma}} \left(m^{1-\gamma} (t \|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma}) + C m^{1-\tilde{\gamma}} \|b\|_{L^\infty(C^\gamma)} (t \|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma}) \nu^{\frac{\gamma-1}{2}} t^{\frac{\gamma+1}{2}} \right) \\ & \quad + T |\nu - \bar{\nu}| \left((C \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty}) + C \|b_m\|_{L^\infty(C^1)} (t \|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma}) \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \right). \end{aligned}$$

From this estimate, to prove uniqueness, we need to consider

$$m^{1-\gamma-\tilde{\gamma}} + \nu \|\nabla^2 g_m\|_{L^\infty} + m^{-\tilde{\gamma}+1-\tilde{\gamma}} \nu^{\frac{\gamma-1}{2}} + m^{1-\tilde{\gamma}} \nu^{\frac{\gamma}{2}} \ll 1, \quad (7.10)$$

the first and the third term are indeed negligible if $\tilde{\gamma} > \max(1 - \gamma, \frac{1}{2})$.

We also write for the two last terms,

$$\nu^{\frac{\gamma-1}{2}} \ll m^{-1+2\tilde{\gamma}}, \text{ and } m^{1-\tilde{\gamma}} \ll \nu^{-\frac{\gamma}{2}}.$$

Combining the two terms, we get

$$m^{1-\tilde{\gamma}} \ll \nu^{-\frac{\gamma}{2}} \ll \left(m^{\frac{2(-1+2\tilde{\gamma})}{1-\gamma}} \right)^{\frac{\gamma}{2}} = m^{\frac{\gamma(-1+2\tilde{\gamma})}{1-\gamma}}.$$

Hence, we have to suppose that

$$1 - \tilde{\gamma} < \frac{\gamma(-1 + 2\tilde{\gamma})}{1 - \gamma}.$$

This is equivalent to

$$1 - \tilde{\gamma} - \gamma + \gamma\tilde{\gamma} < \gamma(-1 + 2\tilde{\gamma}),$$

and

$$1 < \tilde{\gamma}\gamma + \tilde{\gamma} = \tilde{\gamma}(1 + \gamma),$$

thus the condition $\tilde{\gamma} > \frac{1}{1+\gamma} > \max(1 - \gamma, \frac{1}{2})$ of Theorem 3.

Remark 9. *We fail to get uniqueness for negative Besov regularity of b . Indeed, from the analysis performed in Section 6.5, and because we do not differentiate U_m in order to take unsuccessfully take advantage of Grönwall's lemma, we need to consider the term by Besov duality*

$$\begin{aligned} \|\nabla \tilde{p}(s, t, x, \cdot) U_m(s, \cdot)\|_{\dot{B}_{1,1}^\beta} &= \int_0^1 v^{-1} v^{1-\frac{\beta}{2}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v h_v(z - y) \nabla \tilde{p}(s, t, x, y) U_m(s, y) dy \right| dz dv \\ &\leq C \|U_m(s, \cdot)\|_{L^\infty} \int_0^1 v^{-1} v^{-\frac{\beta}{2}} [\nu(t - s)]^{-\frac{1}{2}} dv, \end{aligned}$$

which is finite if only $\beta < -1$, that means that b has to be Lipschitz continuous.

7.3 Control of $\|\partial_t u(t, \cdot)\|_{B_{\infty, \infty}^{-1+\gamma}}$

Like in Section 5.6, we need to get

$$\lim_{(m, \nu) \rightarrow (+\infty, 0)} \nu \Delta u^{m, \nu}(t, \cdot) = 0.$$

Because, now, we do not have to suppose any exponential convergence of ν unlike in (4.3) in Theorem 1, we use the *a priori* estimates of Lemma 4,

$$\begin{aligned} & |\nabla^2 u^{m, \nu}(t, x)| \\ & \leq \left(C \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) + C \|b_m\|_{L^\infty(C^1)} \left(t \|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}}. \end{aligned}$$

In other words, for

$$\nu \|\nabla^2 g_m\|_{L^\infty} + m^{1-\tilde{\gamma}} \nu^{\frac{\gamma}{2}} \ll 1,$$

and by para-product like in Section 5.6 $\langle b_m(t, \cdot), \nabla u^{m, \nu}(t, \cdot) \rangle \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$, we deduce $\partial_t u^{m, \nu}(t, \cdot) \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$.

8 Inviscid Burgers' equation

The controls (4.1) of the vanishing viscous solution of the PDE (1.1) being independent on the first order term b , we can expect to obtain some fixed-point argument to consider that b being the solution u itself in dimension 1^{††}.

Such a Cauchy problem thus defined is called the inviscid Burgers' equation,

$$\begin{cases} \partial_t u(t, x) + u(t, x) \partial_x u(t, x) = f(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases} \quad (8.1)$$

For more information about the Burgers' equation and the corresponding turbulence phenomenon, we refer to the recent book [BK21].

8.1 Statement about the Inviscid Burgers' equation

We obtain a different notion of uniqueness for this equation because the convergence of the mollified first order term, being the solution itself, is more intricate comparing with the transport equation case.

Definition 7 (Turbulent uniqueness). *There is a turbulent unique solution if there are two solutions $u^{m, \nu}$ and $u^{m, \bar{\nu}}$ of*

$$\begin{cases} \partial_t u^{m, \nu}(t, x) + u^{m, \nu}(t, x) \partial_x u^{m, \nu}(t, x) - \nu \partial_{xx}^2 u^{m, \nu}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}, \\ u^{m, \nu}(0, x) = g_m(x), & x \in \mathbb{R}, \end{cases} \quad (8.2)$$

and respectively

$$\begin{cases} \partial_t u^{m, \bar{\nu}}(t, x) + u^{m, \bar{\nu}}(t, x) \partial_x u^{m, \bar{\nu}}(t, x) - \bar{\nu} \partial_{xx}^2 u^{m, \bar{\nu}}(t, x) = f_m(t, x), & (t, x) \in (0, T] \times \mathbb{R}, \\ u^{m, \bar{\nu}}(0, x) = g_m(x), & x \in \mathbb{R}, \end{cases} \quad (8.3)$$

for $\nu, \bar{\nu} > 0$, and for any $(t, x) \in [0, T] \times \mathbb{R}$, where $u_n^{m, \nu}$ and $u_n^{m, \bar{\nu}}$ stand respectively for a mollified version of $u^{m, \nu}$ and $u^{m, \bar{\nu}}$, such that, for any $(t, x) \in [0, T] \times \mathbb{R}$,

$$\forall \varepsilon > 0, \lim_{n \rightarrow +\infty} \|u_n^{m, \nu} - u^{m, \nu}\|_{L^\infty([0, T]; C_b^{\gamma-\varepsilon})} = \lim_{n \rightarrow +\infty} \|u_n^{m, \bar{\nu}} - u^{m, \bar{\nu}}\|_{L^\infty([0, T]; C_b^{\gamma-\varepsilon})} = 0,$$

^{††}It is possible to adapt the analysis for a more general dimension $d \geq 1$ by a reformulation of the product $\langle u(t, x), \nabla u(t, x) \rangle$.

and,

$$|u_m^{m,\nu}(t,x) - u_m^{m,\bar{\nu}}(t,x)| \leq |u^{m,\bar{\nu}}(t,x) - u^{m,\nu}(t,x)|, \quad (8.4)$$

converging, up to sub-sequence selection, towards two Hölder continuous solutions u, \bar{u} , when $(m, \nu, \bar{\nu}) \rightarrow (+\infty, 0, 0)$ then $u = \bar{u}$.

The last condition (8.4) simply means that the mollification procedure behaves like a convolution with a smooth kernel (like performed in (5.2)).

Let us insist that the difference with the uniqueness introduced in Definition 6 is that the considered regularisation procedure for the first order terms $u^{m,\nu}$ and $u^{m,\bar{\nu}}$ is the same in equations (8.2) and (8.3); this explains the definition (8.4).

In particular, uniqueness introduced in Definition 6 (for the transport equation) yields turbulent uniqueness. We detail in Remark 12 why we have to handle with such a turbulent uniqueness or the viscous uniqueness, defined below, instead of the classical uniqueness^{‡‡} for the inviscid Burgers' equation.

Definition 8 (Viscous uniqueness). *There is a viscous unique solution if there are two solutions $u^{m,\nu}$ and $u^{m,\bar{\nu}}$ of*

$$\begin{cases} \partial_t u^{m,\nu}(t,x) + u_m^{m,\nu}(t,x) \partial_x u^{m,\nu}(t,x) - \nu \partial_{xx}^2 u^{m,\nu}(t,x) = f_m(t,x), & (t,x) \in (0,T] \times \mathbb{R}, \\ u^{m,\nu}(0,x) = g_m(x), & x \in \mathbb{R}, \end{cases} \quad (8.5)$$

and respectively

$$\begin{cases} \partial_t \bar{u}^{m,\nu}(t,x) + \bar{u}_m^{m,\nu}(t,x) \partial_x \bar{u}^{m,\nu}(t,x) - \nu \partial_{xx}^2 \bar{u}^{m,\nu}(t,x) = f_m(t,x), & (t,x) \in (0,T] \times \mathbb{R}, \\ \bar{u}^{m,\nu}(0,x) = g_m(x), & x \in \mathbb{R}, \end{cases} \quad (8.6)$$

for $\nu > 0$, with

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow +\infty} \|u_n^{m,\nu} - u^{m,\nu}\|_{L^\infty([0,T]; C_b^{\gamma-\varepsilon})} = \lim_{n \rightarrow +\infty} \|\bar{u}_n^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty([0,T]; C_b^{\gamma-\varepsilon})} = 0,$$

converging, up to sub-sequence selection, towards two Hölder continuous solutions u, \bar{u} , when $(m, \nu, \bar{\nu}) \rightarrow (+\infty, 0, 0)$ then $u = \bar{u}$.

Importantly, the above equations have the same viscosity $\nu > 0$, but the mollification procedure of the first order coefficient may be different. This uniqueness definition is, somehow, the complementary of the *turbulent* uniqueness in the usual uniqueness introduced in Definition 6.

Replacing b by u in the different definitions of solution in Section 6.1, we establish the last result of this paper.

Theorem 4 (Existence and uniqueness of Hölder solution of the inviscid Burgers' equation). *For $\gamma \in (0, 1)$ be given. For all $f \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}, \mathbb{R}))$ and $g \in C_b^\gamma(\mathbb{R}, \mathbb{R})$, there is a mild vanishing viscosity solution $u \in L^\infty([0, T]; C_b^\gamma(\mathbb{R}, \mathbb{R}))$ of (8.1) satisfying*

$$\begin{aligned} \|u\|_{L^\infty(C^\gamma)} &\leq T \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma, \\ \|u\|_{L^\infty} &\leq T \|f\|_{L^\infty} + \|g\|_{L^\infty}. \end{aligned} \quad (8.7)$$

i) Good regularity. *If $\gamma > \frac{1}{2}$ then the considered mild vanishing viscosity solution is also a mild-weak and a weak solution, and if*

$$\nu \|\nabla^2 g_m\|_{L^\infty} + \nu^{\frac{1}{2}} m^{1-\gamma} \ll 1, \quad (8.8)$$

then $\partial_t u(t, \cdot) \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d, \mathbb{R})$, $\forall t \in (0, T]$.

^{‡‡}Definition 6.

ii) **Fast vanishing viscosity.** If, for a constant $C > 0$ big enough,

$$\exp\left(CT(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}) \exp\left(Cmt(T\|f\|_{L^\infty} + \|g\|_{L^\infty})\right)\right) \left((1+m^{1-\gamma})\nu^{\frac{\gamma}{2}} + \nu\|\nabla^2 g_m\|_{L^\infty}\right) \ll 1, \quad (8.9)$$

then the solution is turbulent unique.

iii) **Slow vanishing viscosity.** If $\|\partial_x f\|_{L^\infty} + \|\partial_x g\|_{L^\infty} < +\infty$ and, for a constant $C > 0$ big enough,

$$\exp\left(CT\left(T\|\partial_x f\|_{L^\infty} + \|\partial_x g\|_{L^\infty}\right) \exp\left(CT\nu^{-1}(T\|f\|_{L^\infty} + \|g\|_{L^\infty})^2\right)\right) m^{-\gamma} \ll 1, \quad (8.10)$$

then the solution is viscous unique.

Let us remark that the condition ii) in Theorem 3 is satisfied if the considered *a priori* regularity of the solution is strong enough (*a priori* not the condition i) in Theorem 3, as u is not incompressible, except if u is also solution of the Euler equation), as we have $-\gamma < -1 + \gamma \iff \gamma > \frac{1}{2}$.

Remark 10. Without considering the regularity condition $\gamma > \frac{1}{2}$ to get a weak solution, we may have pathologic situation. Specifically, let us consider the steady-state non-linear problem

$$u(x)u'(x) = \frac{1}{2}\text{sgn}(x), \quad (8.11)$$

whose $x \mapsto \sqrt{|x|}$ is solution^{§§} which is as expected $\frac{1}{2}$ -Hölder continuous. In other words, if $\gamma = \frac{1}{2}$, we can explicitly find a γ -Hölder steady-state solution of the inviscid Burgers' equation with source function being in $B_{\infty,\infty}^0(\mathbb{R}, \mathbb{R})$ but C_b^∞ almost everywhere and being the limit of a C_b^∞ function, e.g. \tanh .

Remark 11. With our current approach, we fail to provide any Lipschitz control of a solution of the inviscid Burgers' equation (8.1) for the same reason as for the transport equation (1.1). This is not surprising by the well-known blowing-up of the gradient of a solution of the inviscid Burgers' equation (8.1).

Remark 12. The conditions (8.9) and (8.10) are not compatible, that is to say there is no uniqueness in the sense of Definition 6, being the combination of turbulent and viscous uniqueness. Condition (8.9) to get turbulent uniqueness means that ν goes to 0 exponentially faster than m goes to $+\infty$; whereas condition (8.10) to get viscous uniqueness implies that m goes to $+\infty$ exponentially faster than ν goes to 0.

We insist that turbulent uniqueness or viscous uniqueness of a such smooth solution of the inviscid Burgers' equation is not a contradiction with the usual counter-example built thanks to a characteristics, because we only consider solution selected by a vanishing viscosity approximation for a given mollification procedure. Somehow, this selection principle allows to avoid the blow-up time appearing in the characteristic building for a given vanishing viscous path.

Actually, from the mild vanishing viscous solution, we see that for any $t \in [0, T]$, the solution $u(t, \cdot)$ given by the limit of a sub-sequence of $u^{m,\nu}(t, \cdot)$ depends on the mollification choice, moreover the sub-sequence choice also depends on the current time t . In other words, $u(t, \cdot)$ seems to avoid the time of blowing-up thanks to a different choice of sub-sequence at each current time.

8.2 Proof of Theorem 4

To establish this result, we consider the mollified version of Burgers' equation, for all $m \in \mathbb{R}_+$ and $\nu > 0$,

$$\begin{cases} \partial_t u^{m,\nu}(t, x) + u^{m,\nu}(t, x)\partial_x u^{m,\nu}(t, x) - \nu\partial_{xx}^2 u^{m,\nu}(t, x) = f_m(t, x), & (t, x) \in [0, T] \times \mathbb{R}, \\ u_m(0, x) = g_m(x), & x \in \mathbb{R}, \end{cases} \quad (8.12)$$

^{§§}Also the function $x \mapsto -\sqrt{|x|}$, but this non-uniqueness should be related with some ergodic properties.

where $u_m^{m,\nu}$ stands for a mollified version of $u^{m,\nu}$, such that

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow +\infty} \|u_n^{m,\nu} - u^{m,\nu}\|_{L^\infty([0,T]; C_b^{\gamma-\varepsilon})} = 0,$$

It is direct from Theorem 3 in [Hon22] that there is a smooth solution of (8.12).

We then perform the same computations as for the transport equation, where $-\beta = \gamma$ and we change in the viscous condition $\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}$ by an upper-bound of $\|u\|_{L^\infty}$, namely by $T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma$ given by Feynman-Kac formula or from the time decomposition trick, see Section 6.6. Finally, we are able to take the limit, thanks to a compact argument, of a suitable sub-sequence yields the result, all the computations readily derives from the analysis of Theorems 1, 2 and 3.

8.2.1 Turbulent uniqueness

To establish uniqueness, let us consider a regularised Burgers' equation with another viscosity $\bar{\nu}$,

$$\begin{cases} \partial_t u^{m,\bar{\nu}}(t,x) + u^{m,\bar{\nu}}(t,x) \partial_x u^{m,\bar{\nu}}(t,x) - \bar{\nu} \partial_{xx}^2 u^{m,\bar{\nu}}(t,x) = f_m(t,x), & (t,x) \in [0,T] \times \mathbb{R}, \\ u^{m,\bar{\nu}}(0,x) = g_m(x), & x \in \mathbb{R}. \end{cases} \quad (8.13)$$

We highlight that $u_m^{m,\bar{\nu}}$ is also a regularisation version of $u^{m,\bar{\nu}}$ such that (8.4) is in force and,

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow +\infty} \|\bar{u}_n^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty([0,T]; C_b^{\gamma-\varepsilon})} = 0,$$

Like for $u^{m,\nu}$, we suppose that there is a constant $C > 0$ such that for any $(m, \bar{\nu}) \in \mathbb{R}_+^2$,

$$\|u^{m,\bar{\nu}}\|_{L^\infty(C_b^\gamma)} \leq C(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma),$$

under some specific asymptotic conditions on $(m, \bar{\nu})$. We still write $U_m := u^{m,\nu} - u^{m,\bar{\nu}}$ which is solution of

$$\partial_t U_m(t,x) + u^{m,\nu}(t,x) \partial_x U_m(t,x) - \bar{\nu} \Delta U_m(t,x) = -[u^{m,\nu} - u^{m,\bar{\nu}}](t,x) \partial_x u^{m,\bar{\nu}}(t,x) + (\nu - \bar{\nu}) \Delta u^{m,\nu}(t,x), \quad (8.14)$$

with $U_m(0,x) = 0$.

Adapting inequality (7.9) yields

$$\|U_m(t, \cdot)\|_{L^\infty} \leq \int_0^t \|[u^{m,\nu} - u^{m,\bar{\nu}}](s, \cdot) \partial_x u^{m,\bar{\nu}}(s, \cdot)\|_{L^\infty} ds + t|\nu - \bar{\nu}| \|\Delta u^{m,\nu}\|_{L^\infty}.$$

We then get by triangular inequality and by (8.4), we can suppose w.l.o.g. that $\bar{\nu} \leq \nu$ (if not we switch roles of ν and $\bar{\nu}$),

$$\|U_m(t, \cdot)\|_{L^\infty} \leq \bar{C} \int_0^t \|U_m(s, \cdot)\|_{L^\infty} \|\partial_x u^{m,\bar{\nu}}\|_{L^\infty} ds + t\nu \|\Delta u^{m,\nu}\|_{L^\infty}.$$

Recalling the Hessian estimate of Lemma 2,

$$\begin{aligned} & |\nabla^2 u^{m,\nu}(t,x)| \\ & \leq \left(C\nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) + C\|u^{m,\nu}\|_{L^\infty(C^1)} \left(t\|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \\ & \leq \left(C\nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) \\ & \quad + Cm^{1-\gamma} \|u^{m,\nu}\|_{L^\infty(C^\gamma)} \left(t\|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}}. \end{aligned}$$

The *a priori* control of $\|u^{m,\nu}\|_{L^\infty(C^\gamma)}$ obtained in (6.23) yields

$$\begin{aligned} |\nabla^2 u^{m,\nu}(t,x)| &\leq \left(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) \\ &\quad + Cm^{1-\gamma} \left(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \right) \left(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|U_m(t,\cdot)\|_{L^\infty} &\leq C \int_0^t \|U_m(s,\cdot)\|_{L^\infty} \|\partial_x u_m^{m,\nu}\|_{L^\infty} ds + T\nu \left(C\nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) \\ &\quad + Cm^{1-\gamma} \left(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \right) \left(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1}t^{\frac{\gamma}{2}}. \end{aligned}$$

By Grönwall's inequality, we derive

$$\begin{aligned} \|U_m(t,\cdot)\|_{L^\infty} &\leq e^{CT\|\partial_x u_m^{m,\nu}\|_{L^\infty}} T \left(C\nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \nu\|\nabla^2 g_m\|_{L^\infty} \right) \\ &\quad + Cm^{1-\gamma} \left(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \right) \left(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}. \end{aligned}$$

Recalling the gradient estimate from Lemma 2

$$\begin{aligned} \|\nabla u^{m,\nu}(t,\cdot)\|_{L^\infty} &\leq \left(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp \left(Cmt\|u^{m,\nu}\|_{L^\infty} \right) \\ &\leq \left(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1} \right) \exp \left(Cmt(T\|f\|_{L^\infty} + \|g\|_{L^\infty}) \right). \end{aligned}$$

Hence, we get

$$\begin{aligned} &\|U_m(t,\cdot)\|_{L^\infty} \tag{8.15} \\ &\leq T \exp \left(CT(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}) \exp \left(Cmt(T\|f\|_{L^\infty} + \|g\|_{L^\infty}) \right) \right) \\ &\quad \left(C\nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}}\|f\|_{L^\infty(C^\gamma)} + \nu\|\nabla^2 g_m\|_{L^\infty} + Cm^{1-\gamma} \left(T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma \right) \left(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}}t^{\frac{\gamma}{2}} \right), \end{aligned}$$

which goes to 0 if

$$\exp \left(CT(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}) \exp \left(Cmt(T\|f\|_{L^\infty} + \|g\|_{L^\infty}) \right) \right) \left((1+m^{1-\gamma})\nu^{\frac{\gamma}{2}} + \nu\|\nabla^2 g_m\|_{L^\infty} \right) \ll 1.$$

Turbulent uniqueness of *vanishing viscous* solution is then established.

8.2.2 Viscous uniqueness

Let us consider another regularised Burgers' equation with different mollification procedure but with the same viscosity $\nu > 0$,

$$\begin{cases} \partial_t \bar{u}^{m,\nu}(t,x) + \bar{u}_m^{m,\nu}(t,x) \partial_x \bar{u}^{m,\nu}(t,x) - \nu \partial_{xx}^2 \bar{u}^{m,\nu}(t,x) = f_m(t,x), & (t,x) \in [0,T] \times \mathbb{R}, \\ \bar{u}^{m,\nu}(0,x) = g_m(x), & x \in \mathbb{R}, \end{cases} \tag{8.16}$$

with $\nu > 0$, and where $\bar{u}_m^{m,\nu}$ is a regularisation of $\bar{u}^{m,\nu}$ (not necessarily a defined by a convolution) such that

$$\sup_{(m,\nu) \in \mathbb{R}_+^2} \|\bar{u}^{m,\nu}\|_{L^\infty(C_b^\gamma)} \leq T\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma < +\infty,$$

and for any $0 < \varepsilon < 1$,

$$\lim_{n \rightarrow +\infty} \|\bar{u}_n^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty} = \lim_{n \rightarrow +\infty} \|\bar{u}_n^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty(C_b^{\gamma-\varepsilon})} = 0. \quad (8.17)$$

We still write $U_m := u^{m,\nu} - \bar{u}^{m,\nu}$ which is solution of

$$\begin{cases} \partial_t U_m(t, x) + u_m^{m,\nu}(t, x) \partial_x U_m(t, x) - \nu \Delta U_m(t, x) = -[u_m^{m,\nu} - \bar{u}_m^{m,\nu}](t, x) \partial_x \bar{u}_m^{m,\nu}(t, x), \\ U_m(0, x) = 0. \end{cases} \quad (8.18)$$

We again adapt inequality (7.9) yields

$$\|U_m(t, \cdot)\|_{L^\infty} \leq \int_0^t \|(u_m^{m,\nu} - \bar{u}_m^{m,\nu}) \partial_x \bar{u}_m^{m,\nu}(s, \cdot)\|_{L^\infty} ds.$$

Let us denotes

$$\bar{u}_{m,m}^{m,\nu}(t, x) := \int_{\mathbb{R}} \rho_m(x - y) \bar{u}^{m,\nu}(t, y) dy, \quad (8.19)$$

such that

$$\|\bar{u}_{m,m}^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty} \leq C m^{-\gamma} \|\bar{u}^{m,\nu}\|_{L^\infty(C^\gamma)},$$

and

$$\|u_m^{m,\nu} - \bar{u}_{m,m}^{m,\nu}\|_{L^\infty} \leq \|u^{m,\nu} - \bar{u}^{m,\nu}\|_{L^\infty} = \|U_m(t, \cdot)\|_{L^\infty}.$$

We then get by triangular inequality,

$$\begin{aligned} \|U_m(t, \cdot)\|_{L^\infty} &\leq \int_0^t \left(\|(u_m^{m,\nu} - \bar{u}_{m,m}^{m,\nu})(s, \cdot)\|_{L^\infty} + \|(\bar{u}_{m,m}^{m,\nu} - \bar{u}^{m,\nu})(s, \cdot)\|_{L^\infty} \right) \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty} ds \\ &\leq \bar{C} \int_0^t \left(C m^{-\gamma} \|\bar{u}^{m,\nu}\|_{L^\infty(C^\gamma)} + \|U_m(s, \cdot)\|_{L^\infty} \right) \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty} ds \\ &\leq \bar{C} \int_0^t \left(C m^{-\gamma} (T \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + \|U_m(s, \cdot)\|_{L^\infty} \right) \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty} ds. \end{aligned}$$

Also by Grönwall's inequality, we derive

$$\|U_m(t, \cdot)\|_{L^\infty} \leq e^{CT \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty}} T C m^{-\gamma} (T \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty}.$$

From exponential absorbing property,

$$\begin{aligned} \|U_m(t, \cdot)\|_{L^\infty} &\leq e^{CT \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty}} T C m^{-\gamma} (T \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty} \\ &\leq e^{CT \|\partial_x \bar{u}_m^{m,\nu}\|_{L^\infty}} m^{-\gamma} (T \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma). \end{aligned} \quad (8.20)$$

We need a new *estimate* of the gradient to avoid any blowing up terms in m which cannot be balanced by the $m^{-\gamma}$ in front of the exponential; the singularity has to be in ν .

Lemma 5. *For $u^{m,\nu}$ solution to (8.12), we have*

$$\|\partial_x u^{m,\nu}\|_{L^\infty} \leq 2 \left(T \|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty} \right) \exp \left(C^2 \nu^{-1} \pi (T \|f\|_{L^\infty} + \|g\|_{L^\infty})^2 T \right).$$

The proof is postponed in Appendix Section D.

We deduce from (8.20),

$$\begin{aligned} \|U_m(t, \cdot)\|_{L^\infty} &\leq \exp \left(CT^2 \left(T \|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty} \right) \exp \left(C^2 \nu^{-1} \pi (T \|f\|_{L^\infty} + \|g\|_{L^\infty})^2 T \right) \right) \\ &\quad \times m^{-\gamma} (T \|f\|_{L^\infty(C^\gamma)} + [g]_\gamma), \end{aligned}$$

which goes to 0 under condition (8.10).

A Proof of Lemma 2

A.1 Gradient estimates

Let us precise the control previously used:

$$\|\nabla u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq \left(T\|f_m\|_{L^\infty(C^1)} + \|g_m\|_{C^1}\right) \exp\left(Cm^{1+\beta}t\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}\right) =: O_m(t). \quad (\text{A.1})$$

We directly have from Duhamel formula (5.11):

$$\begin{aligned} & |\nabla u^{m,\nu}(t, x)| \\ & \leq \left(t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty}\right) \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] \cdot \nabla u^{m,\nu}(s, y) dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \leq \left(t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty}\right) \\ & \quad + C\|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \Big|_{(\tau,\xi)=(t,x)}. \end{aligned}$$

By absorbing property (2.8), we derive

$$\begin{aligned} |\nabla u^{m,\nu}(t, x)| & \leq \left(t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty}\right) \\ & \quad + Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \int_{\mathbb{R}^d} \bar{p}^{\tau,\xi}(s, t, x, y) \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \Big|_{(\tau,\xi)=(t,x)} \\ & \leq \left(t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty}\right) + Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} ds. \end{aligned} \quad (\text{A.2})$$

Finally, Grönwall's lemma yields the result.

This useful *a priori* control allows to avoid any blow-up when $\nu \rightarrow 0$. However, to be able to prove uniqueness, we also need another estimate stated in Lemma 4 and proved in Section C.

A.2 Hessian estimates

We perform a similar argument, but for the second derivatives we have to put a second derivatives on $[b_m(s, \theta_t(\xi)) - b_m(s, y)] \cdot \nabla u(s, y)$. Indeed, if we twice differentiate $\hat{p}^{\tau,\xi}(s, t, x, y)$ there is no possibility to smoothen the blowing up the contribution of ν by Hölder control (or even Lipschitz).

We obtain by Leibniz rules

$$\begin{aligned} & |\nabla^2 u^{m,\nu}(t, x)| \\ & \leq \left(t\|\nabla^2 f_m\|_{L^\infty} + \|\nabla^2 g_m\|_{L^\infty}\right) + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla b(s, y) \cdot \nabla u(s, y) dy ds \right|_{\xi=x} \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] \nabla^2 u(s, y) dy ds \right|_{\xi=x} \\ & \leq m^{2-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \\ & \quad + C \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla b_m \otimes \nabla u^{m,\nu}(s, \cdot)\|_{L^\infty(C^1)} dy ds \\ & \quad + C\|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds. \end{aligned}$$

Next, with Leibniz rules and absorbing property (2.8),

$$\begin{aligned}
& |\nabla^2 u^{m,\nu}(t, x)| \\
& \leq m^{2-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + Cm^{2+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \int_{\mathbb{R}^d} \bar{p}^{\tau,\xi}(s, t, x, y) \|\nabla u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \\
& \quad + Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty} ds \\
& \leq m^{2-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + Ctm^{2+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla u^{m,\nu}\|_{L^\infty} \\
& \quad + Cm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty} ds. \tag{A.3}
\end{aligned}$$

We finally get by Grönwall's lemma and by identity (A.1)

$$\begin{aligned}
& \|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty} \\
& \leq \left(m^{2-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + Ctm^{2+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} O_m(t) \right) \exp(tm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}). \tag{A.4}
\end{aligned}$$

We also write by exponential absorbing property:

$$\|\nabla^2 u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq Cm^{2-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(Cm^{1+\beta}t\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}) =: O_m^{(2)}(t). \tag{A.5}$$

We insist on the fact that the above inequality does not depend on ν .

A.3 Third derivatives estimates

In this section, we detail how to control the third derivatives of $u^{m,\nu}$. We use the same method as for the Hessian, the additional derivative is also put on $[b_m(s, \theta_t(\xi)) - b_m(s, y)] \cdot \nabla u(s, y)$.

$$\begin{aligned}
& |\nabla^3 u^{m,\nu}(t, x)| \\
& \leq \left(t\|\nabla^3 f_m\|_{L^\infty} + \|\nabla^3 g_m\|_{L^\infty} \right) + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla^2 b(s, y) \cdot \nabla u(s, y) dy ds \right|_{\xi=x} \\
& \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla b(s, y) \cdot \nabla^2 u(s, y) dy ds \right|_{\xi=x} \\
& \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] \nabla^3 u(s, y) dy ds \right|_{\xi=x} \\
& \leq Cm^{3-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \\
& \quad + C \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \\
& \quad \times (\|\nabla^2 b_m \cdot \nabla u^{m,\nu}(s, \cdot)\|_{L^\infty(C^1)} + \|\nabla b_m \cdot \nabla^2 u^{m,\nu}(s, \cdot)\|_{L^\infty(C^1)}) dy ds \\
& \quad + C\|b_m\|_{L^\infty(C^1)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \bar{p}^{\tau,\xi}(s, t, x, y) |\theta_{s,\tau}^m(\xi) - y| \|\nabla^3 u^{m,\nu}(s, \cdot)\|_{L^\infty} dy ds \\
& \leq Cm^{3-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + Ctm^{3+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla u^{m,\nu}\|_{L^\infty} \\
& \quad + Ctm^{2+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \|\nabla^2 u^{m,\nu}\|_{L^\infty} + Cm^{2+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} \int_0^t \|\nabla^3 u^{m,\nu}(s, \cdot)\|_{L^\infty} ds.
\end{aligned}$$

Eventually, by Grönwall's lemma, identities (A.1) and (A.4) imply

$$\begin{aligned} & \|\nabla^3 u^{m,\nu}(t, \cdot)\|_{L^\infty} \\ & \leq C \left(m^{3-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) + tm^{2+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})} (mO_m(t) + mO_m^{(2)}(t)) \right) \\ & \quad \times \exp(Ctm^{1+\beta}\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}) \end{aligned} \tag{A.6}$$

$$\leq Cm^{3-\gamma} (t\|f\|_{L^\infty(C^\gamma)} + [g]_\gamma) \exp(Cm^{1+\beta}t\|b\|_{L^\infty(B_{\infty,\infty}^{-\beta})}), \tag{A.7}$$

by exponential absorbing property.

B Proof of Lemma 3

This section is devoted to the regularity of the flow

$$\theta_{s,\tau}^m(x) := x + \int_s^\tau b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x)) d\tilde{s}.$$

By definition, we have

$$\begin{aligned} |\theta_{s,\tau}^m(x) - \theta_{s,\tau}^m(x')| & \leq |x - x'| + \left| \int_s^\tau b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x)) - b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x')) d\tilde{s} \right| \\ & \leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_s^\tau |\theta_{\tilde{s},\tau}^m(x) - \theta_{\tilde{s},\tau}^m(x')| d\tilde{s}, \end{aligned}$$

which is not the suitable inequality to apply directly Grönwall's lemma. To do so, we use a sup formulation, namely for any $r \leq \tau$, we write similarly to above

$$\begin{aligned} \sup_{s \in [0,r]} |\theta_{s,\tau}^m(x) - \theta_{s,\tau}^m(x')| & \leq |x - x'| + \int_0^r |b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x)) - b_m(\tilde{s}, \theta_{\tilde{s},\tau}^m(x'))| d\tilde{s} \\ & \leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_0^r |\theta_{\tilde{s},\tau}^m(x) - \theta_{\tilde{s},\tau}^m(x')| d\tilde{s} \\ & \leq |x - x'| + \|b_m\|_{L^\infty(C^1)} \int_0^r \sup_{\tilde{s} \in [0,\tilde{s}]} |\theta_{\tilde{s},\tau}^m(x) - \theta_{\tilde{s},\tau}^m(x')| d\tilde{s}. \end{aligned}$$

We are now in position to use Grönwall's lemma, for $r = \tau$

$$\sup_{\tilde{s} \in [0,\tau]} |\theta_{\tilde{s},\tau}^m(x) - \theta_{\tilde{s},\tau}^m(x')| \leq |x - x'| \exp(\|b_m\|_{L^\infty(C^1)} \tau).$$

C Proof of Lemma 4

C.1 Gradient estimates

By integration by parts

$$\begin{aligned} & |\nabla u^{m,\nu}(t, x)| \\ & \leq t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty} \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla^2 \hat{p}^{\tau,\xi}(s, t, x, y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s, y)] [u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))] dy ds \right|_{(\tau,\xi)=(t,x)} \\ & \quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \nabla b_m(s, y) [u^{m,\nu}(s, y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))] dy ds \right|_{(\tau,\xi)=(t,x)}, \end{aligned}$$

and by exponential absorbing property,

$$\begin{aligned}
|\nabla u^{m,\nu}(t,x)| &\leq t\|\nabla f_m\|_{L^\infty} + \|\nabla g_m\|_{L^\infty} \\
&\quad + C\|b_m\|_{L^\infty(C^1)}\|u^{m,\nu}\|_{L^\infty(C^\gamma)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{\frac{\gamma-1}{2}} \tilde{p}^{\tau,\xi}(s,t,x,y) dy ds \Big|_{(\tau,\xi)=(t,x)} \\
&\leq m^{1-\gamma} \left(t\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C^\gamma} \right) + C\|b_m\|_{L^\infty(C^1)}\|u^{m,\nu}\|_{L^\infty(C^\gamma)} \nu^{\frac{\gamma-1}{2}} t^{\frac{\gamma+1}{2}}.
\end{aligned} \tag{C.8}$$

The last identity comes from the Hölder estimates stated in Theorem 2.

C.2 Hessian estimates

Like in Section C.1, we integrate by parts

$$\begin{aligned}
|\nabla^2 u^{m,\nu}(t,x)| &\leq \left(C\nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) \\
&\quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla^3 \tilde{p}^{\tau,\xi}(s,t,x,y) [b_m(s, \theta_{s,\tau}^m(\xi)) - b_m(s,y)] \right. \\
&\quad \left. [u^{m,\nu}(s,y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))] dy ds \right|_{(\tau,\xi)=(t,x)} \\
&\quad + \left| \int_0^t \int_{\mathbb{R}^d} \nabla^2 \tilde{p}^{\tau,\xi}(s,t,x,y) \nabla b_m(s,y) [u^{m,\nu}(s,y) - u^{m,\nu}(s, \theta_{s,\tau}^m(\xi))] dy ds \right|_{(\tau,\xi)=(t,x)},
\end{aligned}$$

and by exponential absorbing property after choosing the freezing parameters,

$$\begin{aligned}
&|\nabla^2 u^{m,\nu}(t,x)| \\
&\leq \left(C\nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) \\
&\quad + C\|b_m\|_{L^\infty(C^1)}\|u^{m,\nu}\|_{L^\infty(C^\gamma)} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{\frac{\gamma}{2}-1} \tilde{p}^{\tau,\xi}(s,t,x,y) dy ds \Big|_{(\tau,\xi)=(t,x)} \\
&\leq \left(C\nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}} \|f\|_{L^\infty(C^\gamma)} + \|\nabla^2 g_m\|_{L^\infty} \right) + C\|b_m\|_{L^\infty(C^1)} \left(t\|f_m\|_{L^\infty(C^\gamma)} + \|g_m\|_{C^\gamma} \right) \nu^{\frac{\gamma}{2}-1} t^{\frac{\gamma}{2}}.
\end{aligned} \tag{C.9}$$

D Proof of Lemma 5

From the usual Duhamel formula around the heat equation,

$$\begin{aligned}
|\partial_x u^{m,\nu}(t,x)| &\leq \left(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty} \right) + \left| \int_0^t \int_{\mathbb{R}^d} \partial_x \tilde{p}(s,t,x,y) u_m^{m,\nu}(s,y) \partial_x u^{m,\nu}(s,y) dy ds \right| \\
&\leq \left(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty} \right) \\
&\quad + C\|u^{m,\nu}\|_{L^\infty} \int_0^t \int_{\mathbb{R}^d} [\nu(t-s)]^{-\frac{1}{2}} \tilde{p}(s,t,x,y) \|\partial_x u^{m,\nu}(s,\cdot)\|_{L^\infty} dy ds.
\end{aligned}$$

By the well known L^∞ control, see Section 6.6, we obtain

$$\begin{aligned}
&|\partial_x u^{m,\nu}(t,x)| \\
&\leq \left(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty} \right) + C(T\|f\|_{L^\infty} + \|g\|_{L^\infty}) \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|\partial_x u^{m,\nu}(s,\cdot)\|_{L^\infty} ds.
\end{aligned} \tag{D.1}$$

Here, it is not possible to directly use Grönwall's lemma due to the “ t ” is in the integral. We have to consider Grönwall-Henry's lemma, cf. [Hen81] chapter 7 Lemma 7.1.1. .

Lemma 6 (Lemma of Grönwall-Henry). *Let $T > 0$, a positive a constant $K > 0$ and a non-negative function α such that the function $\varphi : [0, T] \rightarrow \mathbb{R}_+$ satisfying for any $0 < t < Ts$*

$$\varphi(t) \leq \alpha(t) + K \int_0^t (t-s)^{-1+\beta} \varphi(s) ds, \quad (\text{D.2})$$

then

$$\varphi(t) \leq \alpha(t) + \theta \int_0^t E'_\beta(\theta(t-s)) \alpha(s) ds, \quad (\text{D.3})$$

with for any $r \in [0, T]$,

$$\begin{aligned} \theta &= (K\Gamma(\beta))^{\frac{1}{\beta}}, \\ E_\beta(r) &= \sum_{n=0}^{+\infty} \frac{r^{n\beta}}{\Gamma(n\beta + 1)}, \\ E'_\beta(r) &= \partial_r E_\beta(r) = \sum_{n=0}^{+\infty} n\beta \frac{r^{n\beta-1}}{\Gamma(n\beta + 1)}, \end{aligned}$$

and if α is a non-decreasing function, then, for any $0 < t < T$,

$$\varphi(t) \leq \alpha(t) E_\beta(\theta t). \quad (\text{D.4})$$

The last inequality (D.4) is readily derived from (D.3).

The only needed case, here, is $\beta = \frac{1}{2}$, for the sake of completeness, we detail the useful proof of exercise 1 in [Hen81].

Lemma 7. *For any $t > 0$,*

$$e^t \leq E_{1/2}(t) \leq 2e^t.$$

Proof of Lemma 7. Let us recall that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, then we get, for any $r \geq 0$, by differentiating,

$$\begin{aligned} \partial_r E_{1/2}(r) &= r^{-1} \sum_{n=1}^{+\infty} \frac{n}{2} \frac{r^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \\ &= r^{-1} \sum_{n=1}^{+\infty} \frac{r^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \\ &= r^{-1} \left(\frac{r^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \sum_{n=2}^{+\infty} \frac{r^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \right) \\ &= (\pi r)^{-\frac{1}{2}} + E_{1/2}(r). \end{aligned}$$

This differential equation also writes

$$\partial_r (e^{-r} E_{1/2}(r)) = -e^{-r} E_{1/2}(r) + e^{-r} \partial_r E_{1/2}(r) = e^{-r} (\pi r)^{-\frac{1}{2}},$$

and by integrating, for any $t \geq 0$,

$$E_{1/2}(t) = e^t + e^t \pi^{-\frac{1}{2}} \int_0^t r^{-\frac{1}{2}} e^{-r} dr.$$

The lower bound of the lemma is direct, the upper-bound comes from

$$E_{1/2}(t) \leq e^t + e^t \pi^{-\frac{1}{2}} \int_0^{+\infty} r^{-\frac{1}{2}} e^{-r} dr = e^t + e^t \pi^{-\frac{1}{2}} \Gamma(\frac{1}{2}) = 2e^t. \quad (\text{D.5})$$

□

Coming back to inequality (D.1), we identify the notations in Lemma 6, $\beta = \frac{1}{2}$, and $\theta = C^2\nu^{-1}\pi(T\|f\|_{L^\infty} + \|g\|_{L^\infty})^2$, then from Lemmas 6 and 7,

$$\|\partial_x u^{m,\nu}(t, \cdot)\|_{L^\infty} \leq 2\left(t\|\partial_x f_m\|_{L^\infty} + \|\partial_x g_m\|_{L^\infty}\right) \exp\left(C^2\nu^{-1}\pi(T\|f\|_{L^\infty} + \|g\|_{L^\infty})^2 t\right).$$

E Convergence of the mollified distribution

Proposition 2. *For any $\psi \in C^\gamma(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0, 1]$, we have for all $\vartheta \in \mathbb{N}_0^d$ and $\theta \in \mathbb{N}_0$ s.t. $|\vartheta| = \theta$, that $h_{m-2} \star D^\vartheta \psi \in C_b^\infty$ converges towards $D^\vartheta \psi$ in $\dot{B}_{\infty, \infty}^{-\theta}$ as $m \rightarrow +\infty$. More precisely, we have:*

$$\|h_{m-2} \star D^\vartheta \psi - D^\vartheta \psi\|_{\dot{B}_{\infty, \infty}^{-\theta}} \leq C[\psi]_\gamma m^{-\gamma}, \quad (\text{E.1})$$

and $D^\vartheta \psi \in \dot{B}_{\infty, \infty}^{\gamma-\theta}$. In particular, if $|\vartheta| = 0$, $h_{m-2} \star \psi \in C_b^\infty$ converges towards ψ in L^∞ as $m \rightarrow +\infty$, and:

$$\|h_{m-2} \star \psi - \psi\|_{L^\infty} \leq C[\psi]_\gamma m^{-\gamma}. \quad (\text{E.2})$$

Remark 13. *Actually, $\gamma \notin \{0, 1\}$ is not a restrictive condition as changing ϑ into $\tilde{\vartheta} \in \mathbb{N}_0^d$ such that $|\tilde{\vartheta}| = |\vartheta| + 1$ yields the same result.*

In particular, Proposition 2 is available for the Dirac distribution $\delta \in \dot{B}_{\infty, \infty}^{-d}$ regarded as the distributional derivative of the sign function (also regarded as the derivative of the absolute value), and for any derivative of the Dirac distribution by the same argument.

Proof of Proposition 2. Let us write $\varphi = D^\vartheta \psi$, with $\psi \in C^\gamma$

$$\begin{aligned} \|h_{m-2} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} &= \|D^\vartheta [h_{m-2} \star \psi - \psi]\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{\vartheta}{2}} \|\partial_v h_v \star D^\vartheta [h_{m-2} \star \psi - \psi]\|_{L^\infty} \\ &= \sup_{v \in \mathbb{R}_+} v^{1-\frac{\vartheta}{2}} \|\partial_v D^\vartheta h_v \star [h_{m-2} \star \psi - \psi]\|_{L^\infty}, \end{aligned}$$

by integration by parts in convolutions. Next, we can explicitly write,

$$\begin{aligned} &\|h_{m-2} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} \\ &= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{1-\frac{\vartheta}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m-2}(y-x) [\psi(x) - \psi(y)] dx dy \right| \\ &\leq C[\psi]_\gamma \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c-1v}(z-y) h_{m-2}(y-x) |x-y|^\gamma dx dy \\ &\leq C[\psi]_\gamma m^{-\gamma} \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c-1v}(z-y) h_{m-2}(y-x) dx dy, \end{aligned}$$

by exponential absorbing property (2.8). Integrating in space finally yields

$$\|h_{m-2} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\vartheta}} \leq C[\psi]_\gamma m^{-\gamma}.$$

Inequality (E.2) is direct with similar arguments. \square

Corollary 5. *For any $\psi \in \dot{B}_{\infty, \infty}^\gamma(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0, 1]$ we have, for all $\vartheta \in \mathbb{N}_0^d$ and $\theta \in \mathbb{N}_0$ s.t. $|\vartheta| = \theta$, that $h_{m-2} \star D^\vartheta \psi \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$ converges towards $D^\vartheta \psi$ in $\dot{B}_{\infty, \infty}^{-\theta+\gamma-\varepsilon}(\mathbb{R}^d, \mathbb{R})$, for any $\varepsilon \in (0, 1)$ s.t. $\theta - \gamma + \varepsilon > 0$, as $m \rightarrow +\infty$. More precisely, we have:*

$$\|h_{m-2} \star D^\vartheta \psi - D^\vartheta \psi\|_{\dot{B}_{\infty, \infty}^{-\theta+\gamma-\varepsilon}} \leq C[\psi]_\gamma m^{-\varepsilon}. \quad (\text{E.3})$$

Proof of Corollary 5. We still use the thermic representation, and by convolution property we have:

$$\begin{aligned}
& \|h_{m^{-2}} \star \varphi - \varphi\|_{\dot{B}_{\infty, \infty}^{-\theta+\gamma-\varepsilon}} \\
&= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\theta+\gamma-\theta}{2}} \|\partial_v h_v \star D^\vartheta [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty(\mathbb{R}^d)} \\
&= \sup_{v \in \mathbb{R}_+} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v D^\vartheta h_v \star [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty(\mathbb{R}^d)} \\
&= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m^{-2}}(y-x) [\psi(x) - \psi(y)] dx dy \right|.
\end{aligned}$$

For a given $v \in (0, +\infty)$, we compare the regular contribution v with the mollification contribution. In other words, we consider two possibilities.

- If $m^{-2} < v$, then:

$$\begin{aligned}
& v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^\vartheta [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty(\mathbb{R}^d)} \\
&= \sup_{z \in \mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v D^\vartheta h_v(z-y) h_{m^{-2}}(y-x) [\psi(x) - \psi(y)] dx dy \right| \\
&\leq C[\psi]_\gamma v^{\frac{-\gamma+\varepsilon}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x) |x-y|^\gamma dx dy \\
&\leq C[\psi]_\gamma v^{\frac{-\gamma+\varepsilon}{2}} m^{-\gamma} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x) dx dy \\
&\leq C[\psi]_\gamma m^{-\varepsilon}.
\end{aligned} \tag{E.4}$$

- If $m^{-2} \geq v$, then:

$$\begin{aligned}
& v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \|\partial_v h_v \star D^\vartheta [h_{m^{-2}} \star \psi - \psi]\|_{L^\infty(\mathbb{R}^d)} \\
&= \sup_{z \in \mathbb{R}^d} v^{1-\frac{-\theta+\gamma-\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_v h_v(z-y) D^\vartheta h_{m^{-2}}(y-x) [\psi(x) - \psi(y)] dx dy \right| \\
&\leq C[\psi]_\gamma m^{\frac{\theta}{2}} v^{\frac{\theta-\gamma+\varepsilon}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{c^{-1}m^{-1}}(y-x) |x-y|^\gamma dx dy \\
&\leq C[\psi]_\gamma m^{\theta-\gamma} v^{\frac{\theta-\gamma+\varepsilon}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{c^{-1}v}(z-y) h_{m^{-2}}(y-x) dx dy \\
&\leq C[\psi]_\gamma m^{-\varepsilon}.
\end{aligned} \tag{E.5}$$

The result follows from (E.4) and (E.5). \square

Proposition 2 and Corollary 5 are more precise forms of the well known convergence in the distributional sense.

Proposition 3. For any $\psi \in \dot{B}_{\infty, \infty}^\gamma(\mathbb{R}^d, \mathbb{R})$, $\gamma \in (0, 1]$ we have for any $\vartheta \in \mathbb{N}_0^d$ that $h_{m^{-2}} \star D^\vartheta \psi \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$ converges towards $D^\vartheta \psi$ in distributional sense as $m \rightarrow +\infty$. More precisely, we have for any $\eta \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$:

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \eta(x-y) [h_{m^{-2}} \star D^\vartheta \psi(y) - D^\vartheta \psi(y)] dy \right| \leq C[\psi]_\gamma m^{-\gamma}. \tag{E.6}$$

Remark 14. We precise that η is not supposed to be a Gaussian kernel, as in Proposition 2.

Proof. We directly write by convolution property:

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \eta(x-y) [h_{m^{-2}} \star D^\vartheta \psi(y) - D^\vartheta \psi(y)] dy \right| \\
&= \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} D^\vartheta \eta(x-y) [h_{m^{-2}} \star \psi(y) - \psi(y)] dy \right| \\
&\leq C m^{-\gamma} [\psi]_\gamma \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} |D^\vartheta \eta(x-y)| dy \right| \\
&\leq C m^{-\gamma} [\psi]_\gamma,
\end{aligned}$$

the penultimate inequality is consequence of inequality (E.2). \square

F Properties of derivatives of Besov distributions

Proposition 4. *For any $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\nabla \varphi \in \dot{B}_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d)$, $\gamma \in (0, 1)$, there is a constant $C > 1$ such that:*

$$\|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} \leq C \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma}.$$

Proof of Proposition 4. We first write by the thermic representation of the Besov norm and by integration by parts,

$$\begin{aligned}
\|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} &= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{1-\frac{-1+\gamma}{2}} \left| \int_{\mathbb{R}^d} \partial_v h_v(z-y) \nabla \varphi(y) dy \right| \\
&= \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{\frac{3-\gamma}{2}} \left| \int_{\mathbb{R}^d} \nabla \cdot \partial_v h_v(z-y) [\varphi(y) - \varphi(z)] dy \right|,
\end{aligned}$$

by absorbing property (2.8) we derive

$$\begin{aligned}
\|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} &\leq C [\varphi]_\gamma \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} v^{\frac{3-\gamma}{2}} \int_{\mathbb{R}^d} v^{-\frac{3}{2}} h_{C^{-1}v}(z-y) |y-z|^\gamma dy \\
&\leq C [\varphi]_\gamma \sup_{v \in \mathbb{R}_+, z \in \mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) dy \\
&\leq C [\varphi]_\gamma.
\end{aligned}$$

\square

We also derive the corresponding inequality for the inhomogeneous case.

Corollary 6. *For any $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\nabla \varphi \in B_{\infty, \infty}^{-1+\gamma}(\mathbb{R}^d)$, $\gamma \in (0, 1)$, there is a constant $c > 1$ such that:*

$$\|\nabla \varphi\|_{B_{\infty, \infty}^{-1+\gamma}} \leq c \|\varphi\|_{B_{\infty, \infty}^\gamma}.$$

Proof of Corollary 6. From inequality (2.6), we have $\|\nabla \varphi\|_{B_{\infty, \infty}^{-1+\gamma}} \leq \frac{C}{1-\gamma} \|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}}$. Moreover, it is direct that

$$\begin{aligned}
\|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma} &= \sup_{v \in \mathbb{R}_+} v^{(1-\frac{\gamma}{2})} \|\partial_v h_v \star \varphi\|_{L^\infty} \leq \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma} + \sup_{v > 1} v^{(1-\frac{\gamma}{2})} \|\partial_v h_v \star \varphi\|_{L^\infty} \\
&\leq \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma} + \|\varphi\|_{L^\infty} \\
&= \|\varphi\|_{B_{\infty, \infty}^\gamma}.
\end{aligned}$$

In other words, we deduce by Proposition 4,

$$\|\nabla \varphi\|_{B_{\infty, \infty}^{-1+\gamma}} \leq \frac{C}{1-\gamma} \|\nabla \varphi\|_{\dot{B}_{\infty, \infty}^{-1+\gamma}} \leq \|\varphi\|_{\dot{B}_{\infty, \infty}^\gamma} \leq C \|\varphi\|_{B_{\infty, \infty}^\gamma}.$$

\square

Remark 15. In all generality, the reverse inequality of the above results are not true. For example, any constant function lies in $B_{\infty,\infty}^{\gamma}$ but its derivative is 0, hence the $B_{\infty,\infty}^{-1+\gamma}$ norm is null, and the corresponding Besov norm equivalence obviously fails to be true.

For an equivalence version of this result, we need to consider extra Besov norms, see for instance [KMM07] identity (3.54) for a Triebel-Lizorkin spaces version.

G On the freedom of the mollification choice

In this section, we detail why if there is a sequence of smooth function converging toward a $B_{\infty,\infty}^{-\beta}$ distribution then the mollification procedure (5.2) converges also toward the distribution. In other words, if there is a sequence $(\bar{b}_n)_{n \geq 1}$ lying in $L^\infty([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$ such that

$$\lim_{n \rightarrow \infty} \|\bar{b}_n - b\|_{L^\infty(B_{\infty,\infty}^{-\beta+\varepsilon})} = 0, \quad (\text{G.1})$$

for any $0 < \varepsilon$, then

$$\lim_{m \rightarrow \infty} \|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = 0, \quad (\text{G.2})$$

where b_m is defined in (5.2) by:

$$b_m(t, x) = b(t, \cdot) \star \rho_m(x),$$

with for any $z \in \mathbb{R}^d$, $\rho_m(z) := m^d \rho(mz)$ for $\rho(z) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{2}}$. Indeed, we readily write by triangular inequality:

$$\|b_m - b\|_{L^\infty(B_{\infty,\infty}^{-\beta+\varepsilon})} \leq \|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} + \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} + \|\bar{b}_n - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})}. \quad (\text{G.3})$$

The first term in the r.h.s. above write:

$$\|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = \|\rho_m \star (b - \bar{b}_n)\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})}.$$

Hence, we obtain

$$\|b_m - \rho_m \star \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = \sup_{v \in [0,1]} v^{\frac{\beta-\varepsilon}{2}} \|\rho_m \star h_v \star (b - \bar{b}_n)\|_{L^\infty} \leq \|\bar{b}_n - b\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})}, \quad (\text{G.4})$$

by triangular inequality.

Also, for the second term in (G.3), let us deal with the corresponding homogeneous norm,

$$\|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = \|\varphi(D)(\rho_m \star \bar{b}_n - \bar{b}_n)\|_{L^\infty} + \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(\dot{B}_{\infty,\infty}^{-\beta-\varepsilon})}.$$

It is direct that

$$\|\varphi(D)(\rho_m \star \bar{b}_n - \bar{b}_n)\|_{L^\infty} \leq C \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty} \leq C m^{-1} \|D\bar{b}_n\|_{L^\infty}. \quad (\text{G.5})$$

Next,

$$\begin{aligned} & \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(\dot{B}_{\infty,\infty}^{-\beta-\varepsilon})} \\ &= \sup_{v \in [0,1], t \in [0,T], z \in \mathbb{R}^d} v^{\frac{\beta+\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_v(z-y) \rho_m(y-x) [\bar{b}_n(t,x) - \bar{b}_n(t,y)] dx dy \right| \\ &\leq \|D\bar{b}_n\|_{L^\infty} \sup_{v \in [0,1], t \in [0,T], z \in \mathbb{R}^d} v^{\frac{\beta+\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) \rho_m(y-x) |x-y| dx dy \right| \\ &\leq C \|D\bar{b}_n\|_{L^\infty} m^{-1} \sup_{v \in [0,1], t \in [0,T], z \in \mathbb{R}^d} v^{\frac{\beta+\varepsilon}{2}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_{C^{-1}v}(z-y) \rho_{C^{-1}m}(y-x) dx dy \right| \\ &= C \|D\bar{b}_n\|_{L^\infty} m^{-1}. \end{aligned} \quad (\text{G.6})$$

Let us choose $m \gg \|D\bar{b}_n\|_{L^\infty}$ which yields that $\lim_{m,n \rightarrow \infty} \|\rho_m \star \bar{b}_n - \bar{b}_n\|_{L^\infty(B_{\infty,\infty}^{-\beta-\varepsilon})} = 0$.

Finally, gathering identities (G.1), (G.3), (G.4), (G.5) and (G.6) yields the limit property (G.2).

References

- [AF09] S. Attanasio and F. Flandoli. Zero-noise solutions of linear transport equations without uniqueness: an example. *C. R. Math. Acad. Sci. Paris*, 347(13-14):753–756, 2009.
- [AF11] S. Attanasio and F. Flandoli. Renormalized solutions for stochastic transport equations and the regularization by bilinear multiplication noise. *Comm. Partial Differential Equations*, 36(8):1455–1474, 2011.
- [AH96] D. R. Adams and L. I. Hedberg. *Function spaces and Potential Theory*. Springer, 1996.
- [Amb04] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.
- [Bar82] M. T. Barlow. One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc. (2)*, 26(2):335–347, 1982.
- [BB05] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.
- [BC01] R. F. Bass and Z.-Q. Chen. Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields*, 121(3):422–446, 2001.
- [BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.
- [BK21] A. Boritchev and S. Kuksin. *One-dimensional turbulence and the stochastic Burgers equation*, volume 255 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, [2021] ©2021.
- [Bon81] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.
- [Cat16] R. Catellier. Rough linear transport equation with an irregular drift. *Stoch. Partial Differ. Equ. Anal. Comput.*, 4(3):477–534, 2016.
- [CC18] G. Cannizzaro and K. Chouk. Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.*, 46(3):1710–1763, 2018.
- [CCS20] G. Ciampa, G. Crippa, and S. Spirito. Smooth approximation is not a selection principle for the transport equation with rough vector field. *Calc. Var. Partial Differential Equations*, 59(1):Paper No. 13, 21, 2020.
- [CCS22] M. Colombo, G. Crippa, and M. Sorella. Anomalous dissipation and lack of selection in the obukhov-corrsin theory of scalar turbulence. 2022. working paper or preprint.
- [CDRHM18] P.-É. Chaudru De Raynal, I. Honoré, and S. Menozzi. Sharp Schauder Estimates for some Degenerate Kolmogorov Equations. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, October 2018. accepted for publication.
- [CdRM17] P.-É. Chaudru de Raynal and S. Menozzi. Regularization effects of a noise propagating through a chain of differential equations: an almost sharp result. October 2017. to appear in trans. american math. society.
- [CL83] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

- [Dan07] R. Danchin. Uniform estimates for transport-diffusion equations. *J. Hyperbolic Differ. Equ.*, 4(1):1–17, 2007.
- [DD16] F. Delarue and R. Diel. Rough paths and 1d SDE with a time dependent distributional drift: application to polymers. *Probab. Theory Related Fields*, 165(1-2):1–63, 2016.
- [Dep03] N. Depauw. Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d’un hyperplan. *C. R. Math. Acad. Sci. Paris*, 337(4):249–252, 2003.
- [DF14] F. Delarue and F. Flandoli. The transition point in the zero noise limit for a 1d Peano example. *Discrete and Continuous Dynamical Systems*, 34(10):4071–4083, April 2014.
- [DLG22] C. De Lellis and V. Giri. Smoothing does not give a selection principle for transport equations with bounded autonomous fields. *Ann. Math. Qué.*, 46(1):27–39, 2022.
- [DPL89] R. Di Perna and P. L. Lions. Ordinary Differential Equations, transport theory and Sobolev Spaces. *Inv. Math.*, 98:511–547, 1989.
- [Eva98] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [FF13] E. Fedrizzi and F. Flandoli. Noise prevents singularities in linear transport equations. *J. Funct. Anal.*, 264(6):1329–1354, 2013.
- [FGP10] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Inventiones Mathematicae*, 180(1):1–53, 2010.
- [FGP12] F. Flandoli, M. Gubinelli, and E. Priola. Remarks on the stochastic transport equation with Hölder drift. *Rend. Semin. Mat. Univ. Politec. Torino*, 70(1):53–73, 2012.
- [FIR17] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Trans. Amer. Math. Soc.*, 369(3):1665–1688, 2017.
- [Fla11] F. Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.
- [FMRT01] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.
- [Fri64] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, 1964.
- [Fri98] F. G. Friedlander. *Introduction to the theory of distributions*. Cambridge University Press, Cambridge, second edition, 1998. With additional material by M. Joshi.
- [GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.
- [Hai13] M. Hairer. Solving the KPZ equation. *Ann. of Math. (2)*, 178(2):559–664, 2013.
- [Hai14] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.
- [Hen81] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.
- [Hon22] I. Honoré. Parabolic bootstrap for some non-linear equations. 2022. working paper or preprint.

- [Jaw77] B. Jawerth. Some observations on Besov and Lizorkin-Triebel spaces. *Math. Scand.*, 40(1):94–104, 1977.
- [KKL75] S. N. Kružkov, A. Kastro, and M. Lopes. Schauder type estimates, and theorems on the existence of the solution of fundamental problems for linear and nonlinear parabolic equations. *Dokl. Akad. Nauk SSSR*, 220:277–280, 1975.
- [KMM07] N. Kalton, S. Mayboroda, and M. Mitrea. Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations. In *Interpolation theory and applications*, volume 445 of *Contemp. Math.*, pages 121–177. Amer. Math. Soc., Providence, RI, 2007.
- [KP10] N. V. Krylov and E. Priola. Elliptic and parabolic second-order PDEs with growing coefficients. *Comm. Partial Differential Equations*, 35(1):1–22, 2010.
- [Kry96] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*. Graduate Studies in Mathematics 12. AMS, 1996.
- [Ler34] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934.
- [Lie96] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [Lor11] L. Lorenzi. Optimal Hölder regularity for nonautonomous Kolmogorov equations. *Discrete Contin. Dyn. Syst. Ser. S*, 4(1):169–191, 2011.
- [LR02] P.-G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. CRC Press, 2002.
- [LSU68] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural’ceva. *Linear and quasi-linear equations of parabolic type*. Vol.23 Trans. Math. Monog., AMS, Providence, 1968.
- [Lun95] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1995. [2013 reprint of the 1995 original] [MR1329547].
- [Lun97] A. Lunardi. Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in \mathbb{R}^n . *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 24(1):133–164, 1997.
- [MO17] D. A. C. Mollinedo and C. Olivera. Well-posedness of the stochastic transport equation with unbounded drift. *Bull. Braz. Math. Soc. (N.S.)*, 48(4):663–677, 2017.
- [MS18] S. Modena and L. Jr. Székelyhidi. Non-uniqueness for the transport equation with Sobolev vector fields. *Ann. PDE*, 4(2):Paper No. 18, 38, 2018.
- [Pee76] J. Peetre. *New thoughts on Besov spaces*. Mathematics Department, Duke University, Durham, N.C., 1976. Duke University Mathematics Series, No. 1.
- [Pri09] E. Priola. Global Schauder estimates for a class of degenerate Kolmogorov equations. *Studia Math.*, 194(2):117–153, 2009.
- [Tri83] H. Triebel. *Theory of function spaces, II*. Birkhauser, 1983.
- [Xia19] J. Xiao. The transport equation in the scaling invariant Besov or Essén-Janson-Peng-Xiao space. *J. Differential Equations*, 266(11):7124–7151, 2019.