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An aspect of the turnpike property. Long time horizon behavior

Martin Gugat ^{*} Jan Sokolowski [†]

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Abstract

The turnpike phenomenon concerns the structure of optimal controls and the optimal state of dynamic optimal control problems for long time horizons. The focus is regularly on the study of the interior of the time interval. Classical turnpike results state how the solution of the dynamic optimal control problems approaches the solution of the corresponding static optimal control problem in the interior of the time interval.

In this paper we look at a new aspect of the turnpike phenomenon. We show that for problems without explicit terminal condition, for large time horizons in the last part of the time interval for large T the optimal state approaches a certain limit trajectory that is independent of the terminal time exponentially fast. Similarly also in the initial part of the time interval for large T the optimal state approaches a certain limit state exponentially fast.

1 Introduction

The turnpike phenomenon concerns properties of solutions to dynamic optimal control problems for long time horizons. Usually the studies of the turnpike phenomenon focus on results about the behaviour of the optimal trajectories in the interior of the time interval. These results provide assumptions that imply

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that for long time horizons, in the interior of the time interval the solutions to the dynamic optimal control problems are approximated by the solutions to the corresponding static optimal control problem.

In this paper we focus on a different aspect, namely the limit trajectories for large time horizons on the initial part and the terminal part of the time interval. We consider dynamic optimal control problems with free terminal state for long time horizons. We show the existence of a limit trajectory on the last part of the time interval. It turns out, that with increasing time horizon T , the terminal arc approaches the limit trajectory that is independent of the time horizon exponentially fast. The corresponding limit states at the time $T - t$ only depend on the distance t to the terminal time T . Hence for long time horizons the optimal state of the dynamic problem in the last part of the time interval approaches a limit trajectory that is independent of the terminal time T . Moreover, also this convergence is exponentially fast.

We also show that on the first part of the time interval $[0, T]$, a limit trajectory that is independent of T does exist such that the optimal state of the dynamic problem approaches exponentially fast the limit trajectory with growing T .

Since there is a substantial amount of literature on the turnpike phenomenon, here we only give a short review. An early reference is [17]. A monograph on the turnpike phenomenon is [23] and an overview on discrete-time and continuous-time turnpike properties in optimal control is given in [4]. The survey [6] with a particular focus on the control of distributed parameter systems contains also additional references. Measure and integral turnpike properties have been studied in [19]. The turnpike property for systems that are governed by semilinear partial differential equations is studied in [7]. The relation of the turnpike property and the receding-horizon method has been studied in [2]. Manifold turnpikes are studied in [3].

To explain the point that we want to make further we start with a simple example.

Example. Let real numbers $y_d \neq 0$, $T > 0$, $\gamma > 0$ and y_0 be given. Consider the dynamic optimal control problem

$$\begin{cases} \min_{u \in L^2(0, T)} \int_0^T |y(\tau) - y_d|^2 + \gamma |u(\tau)|^2 d\tau \\ \text{subject to } y(0) = y_0, \quad y'(t) = y(t) + u(t). \end{cases}$$

The optimal state for the corresponding static optimal control problem

$$\min_{u \in \mathbb{R}} |y - y_d|^2 + \gamma |u|^2 \quad \text{subject to } 0 = y + u.$$

is given by $y^{(\sigma)} = \frac{1}{1+\gamma} y_d$. Define $\omega = \left(1 + \frac{1}{\gamma}\right)^{1/2}$.

The optimality conditions imply that the optimal state \hat{y}_T is given by

$$\hat{y}_T(t) = y^{(\sigma)} + y^{(\sigma)} \frac{1}{\omega - \tanh(\omega T)} \frac{\sinh(\omega t)}{\cosh(\omega T)}$$

$$+ \left[y_0 - y^{(\sigma)} \right] \frac{(\omega + 1) \exp(\omega(t - T)) + (\omega - 1) \exp(\omega(T - t))}{(\omega - 1) \exp(\omega T) + (\omega + 1) \exp(-\omega T)}.$$

Simple calculations show

$$\lim_{T \rightarrow \infty} \hat{y}_T(t) = y^{(\sigma)} + \left[y_0 - y^{(\sigma)} \right] \exp(-\omega t)$$

and

$$\lim_{T \rightarrow \infty} \hat{y}_T(T - t) = y^{(\sigma)} + \frac{1}{\omega - 1} y^{(\sigma)} \exp(-\omega t).$$

For $t > 0$ define the initial limit trajectory $y_{init}(t) = y^{(\sigma)} + \left[y_0 - y^{(\sigma)} \right] \exp(-\omega t)$ that is independent of T . Then $\lim_{T \rightarrow \infty} \hat{y}_T(t) = y_{init}(t)$. If $y_0 = y^{(\sigma)}$, we have $y_{init}(t) = y^{(\sigma)}$.

Define the terminal limit trajectory $y_{term}(t) = y^{(\sigma)} + \frac{1}{\omega - 1} y^{(\sigma)} \exp(-\omega t)$ that also does not depend on T . Then $\lim_{T \rightarrow \infty} \hat{y}_T(T - t) = y_{term}(t)$. The limit trajectory y_{term} can only be constant if $y^{(\sigma)} = 0$. We have

$$\lim_{T \rightarrow \infty} \hat{y}_T(T) = y_{term}(0).$$

Moreover both limits are reached exponentially fast.

Let $t_0 > 0$ be given. For $t \in (t_0, T - t_0)$ and T sufficiently large we have the inequality

$$|\hat{y}_T(t) - y^{(\sigma)}| \leq 2 \left[\left| y_{init}(0) - y^{(\sigma)} \right| + \left| y_{term}(0) - y^{(\sigma)} \right| \right] \exp(-\omega t_0)$$

which implies that in the interior of the time interval, the distance between the static optimal state and the dynamic optimal state decays exponentially fast. Note that by choosing t_0 sufficiently large we can make the upper bound $\exp(-\omega t_0)$ arbitrarily small.

This situation indicates that from a practical point of view it makes sense to use a feedback rule in the first part of the time interval to control the system to the optimal static state $y^{(\sigma)}$. If the feedback steers the system close to y_{init} this approach causes almost no loss of optimality compared with the problem for large time horizons.

Define $H_T(t) = \frac{1}{2} |\hat{y}_T(t) - y^{(\sigma)}|^2$. We will show that H_T has the meaning of a Lyapunov function for the first part of the time interval where it is decreasing exponentially fast. In contrast to the typical situation with Lyapunov functions, in the last part of the time interval it is increasing. Since $\hat{y}_T'' = \omega^2 (\hat{y}_T - y^{(\sigma)})$ we have

$$H_T''(t) = (\hat{y}_T'(t))^2 + (\hat{y}_T(t) - y^{(\sigma)}) \hat{y}_T''(t) = (\hat{y}_T'(t))^2 + 2\omega^2 H_T(t).$$

Hence H_T is convex. We have $H_T'(0) = (y_0 - y^{(\sigma)}) \hat{y}_T'(0)$ and

$$\lim_{T \rightarrow \infty} \hat{y}_T'(0) = -\omega(y_0 - y^{(\sigma)}) = y_{init}'(0).$$

If T is sufficiently large and $y_0 \neq y^{(\sigma)}$ we obtain $H_T'(0) < 0$. This implies that in the first part of $[0, T]$, H is strictly decreasing.

If $t_T^* > 0$ is such that $\hat{y}'_T(t_T^*) = 0$, we have $H'_T(t_T^*) = 0$. At such a point $t_T^* > 0$, the function $H \geq 0$ attains its minimal value. For $t > t_T^*$, the value of $H(t)$ can increase again and if $y^{(\sigma)} \neq 0$ this is what happens in the last part of the time-interval where $t \mapsto \hat{y}_T(T-t)$ approaches $t \mapsto y_{term}(t)$ for $T \rightarrow \infty$. In fact, since

$$\lim_{T \rightarrow \infty} \hat{y}'_T(T) = \frac{\omega}{\omega - 1} y^{(\sigma)} = -y'_{term}(0)$$

in this case we have

$$\lim_{T \rightarrow \infty} H'_T(T) = \lim_{T \rightarrow \infty} (\hat{y}_T(T) - y^{(\sigma)}) \hat{y}'_T(t) = -(y_{term}(0) - y^{(\sigma)}) y'_{term}(0) > 0.$$

The aim of this paper is to show that the situation that occurs in our example is typical and can also be found in a general framework.

The structure of the paper. This paper has the following structure. In Section 2 we define the optimal control problem that we investigate. We consider a system that is governed by a semi-group of contractions. We define a linear-quadratic optimal control problem.

In Section 3 we derive the optimality systems both for the dynamic optimal control problem and the corresponding static optimal control problem. Then we consider the difference between the dynamic and the static optimal solutions and derive a system with an initial condition and a terminal condition that is satisfied by this difference.

In Section 4 we show the existence of two limit trajectories for the boundaries of the time interval. We have an initial limit trajectories for fixed times $t > 0$ and a terminal limit trajectories for times $T-t$ with a fixed distance to the terminal time T . We derive representations of the long-time horizon limit terminal state, the long-time horizon limit trajectory for the last part of the time interval and the initial limit trajectory. We provide a representation in terms of the optimal adjoint state for the static problem and show the convergence towards the limit trajectory is exponentially fast.

In Section 5 we present examples with the transport equation and the wave equation to illustrate our findings. At the end of the paper in Section 6 we point out possible directions of future research.

2 The dynamic optimal control problem

Let U and X denote complex Hilbert spaces and \mathbb{T}_t a strongly continuous semi-group of contractions on X with generator A . Our setting is as in [20], Chapter 4. We consider a system that is governed by the differential equation

$$z'(t) = Az(t) + Bu(t) \tag{1}$$

where B is an admissible control operator that is defined in U and controls $u \in L^2_{loc}([0, \infty), U)$. Then for all $t \geq 0$ we can represent the state in the form

$$z(t) = z(0) + \int_0^t [Az(s) + Bu(s)] ds$$

(see Proposition 4.2.5 in [20]). We can also write

$$z(t) = \mathbb{T}_t z(0) + \int_0^t \mathbb{T}_{t-s} B u(s) ds.$$

Note that since B is an admissible control operator, for $t \geq 0$ the operator

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds$$

is bounded and $\|\Phi_t\| \leq \|\Phi_T\|$.

We assume that A is *skew-adjoint*, that is for all v, w in the domain of A we have

$$\langle Aw, v \rangle = \langle w, -Av \rangle. \quad (2)$$

Proposition 3.7.2 in [20] states that (2) is equivalent to the statement that both A and $-A$ are m -dissipative. Hence the Lumer-Phillips Theorem implies that A is the generator of a semi-group of contractions.

Let an initial state $z_0 \in X$ and a desired state $z_d \in X$ be given. Let $T > 0$ be given. For $u \in L_{loc}^2([0, T], U)$ we define the objective functional

$$J_T(u) = \int_0^T \|z(s) - z_d\|_X^2 + \|u(s)\|_U^2 ds$$

where the state z is defined as the solution of the initial value problem

$$z(0) = z_0, \quad z'(t) = Az(t) + Bu(t). \quad (3)$$

We consider the dynamic optimal control problem

$$\mathbf{P}(T) : \min_{u \in L_{loc}^2([0, T], U)} J_T(u). \quad (4)$$

Note that since the objective functional is strongly convex, $\mathbf{P}(T)$ can have at most one solution. Moreover, our assumptions allow to prove the existence of a solution of $\mathbf{P}(T)$ using the direct method of the calculus of variations.

3 Optimality conditions

In this section we study the necessary optimality conditions for the dynamic optimal control problem $\mathbf{P}(T)$ and the corresponding static optimal control problem. We also derive a differential equation for the difference of the dynamic optimal state/control pair and the static optimal state/control pair. This system is completed by an initial condition for the state and a terminal condition for the adjoint state.

3.1 Necessary optimality conditions for the dynamic optimal control problem

First we derive the necessary optimality conditions for the dynamic optimal control problem $\mathbf{P}(T)$. We introduce a variation δu of the control and the corresponding variation δz of the generated state. Then we have the initial condition $\delta z(0) = 0$ and the differential equation $\delta z' = A \delta z + B \delta u$. We introduce an adjoint state μ with the same regularity as z and obtain the optimality system

$$z(0) = z_0, \mu(T) = 0 \quad (5)$$

$$z' = Az + Bu \quad (6)$$

$$\mu' = -A^* \mu + z - z_d \quad (7)$$

$$u = B^* \mu. \quad (8)$$

We state the necessary optimality conditions in the following Lemma:

Lemma 1 *The solution of $\mathbf{P}(T)$ satisfies the necessary and sufficient optimality conditions (5)–(8) with the adjoint state μ that satisfies (7) in the sense*

$$\mu(t) = \int_T^t [A\mu(s) + (z(s) - z_d)] ds.$$

Proof. For all control variations δu , for the objective value we have

$$\begin{aligned} J_T(u + \delta u) &= \int_0^T \|z(s) + \delta z(s) - z_d\|_X^2 + \|u(s) + \delta u(s)\|_U^2 ds \\ &= J_T(u) + 2 \int_0^T \langle z(s) - z_d, \delta z(s) \rangle_X + \langle u, \delta u \rangle_U ds \\ &\quad + \int_0^T \|\delta z(s)\|_X^2 + \|\delta u(s)\|_U^2 ds \\ &\geq J_T(u) + 2 \int_0^T \langle z(s) - z_d, \delta z(s) \rangle_X + \langle u, \delta u \rangle_U ds \\ &\quad + 2 \int_0^T \langle \delta z'(s) - A\delta z(s) - B\delta u(s), \mu(s) \rangle_X ds \end{aligned}$$

Hence using (5), $\delta z(0) = 0$ and integration by parts we obtain the inequality

$$\begin{aligned} &J_T(u + \delta u) - J_T(u) \\ &\geq 2 \int_0^T \langle z - z_d - A^* \mu - \mu', \delta z \rangle_X + \langle u - B^* \mu, \delta u \rangle_U ds. \end{aligned}$$

Hence the inequality

$$J_T(u + \delta u) - J_T(u) \geq 0 \quad (9)$$

can only hold for all control variations δu , if (7) and (8) hold.

Note that if the optimality system holds, we also obtain the inequality (9), hence the optimality conditions are also sufficient. \square

3.2 Necessary optimality conditions for the static optimal control problem

The static optimal control problem corresponding to $\mathbf{P}(T)$ is obtained by cancelling the initial condition and the time dependence. We have the static optimal control problem

$$\mathbf{S} : \min_{u \in U} \|z - z_d\|_X^2 + \|u\|_U^2 \text{ subject to } 0 = Az + Bu. \quad (10)$$

Again the existence of a solution follows with the direct method of the calculus of variations and the uniqueness follows from the strict convexity of the objective functional. We denote the unique solution of \mathbf{S} by $(z^{(\sigma)}, u^\sigma)$.

We introduce a variation δu of the control and the corresponding variation δz of the generated state. Then we have $0 = A\delta z + B\delta u$. We introduce an adjoint state $\mu \in X$ and obtain the optimality system

$$0 = Az + Bu \quad (11)$$

$$0 = -A^*\mu + z - z_d \quad (12)$$

$$u = B^*\mu. \quad (13)$$

We state the necessary optimality conditions in the following Lemma:

Lemma 2 *The solution of \mathbf{S} satisfies the necessary and sufficient optimality conditions (11)–(13) with the adjoint state μ .*

Proof. For all control variations δu , for the objective value we have

$$\begin{aligned} & \|z + \delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 \\ = & \|z - z_d\|_X^2 + \|u\|_U^2 + 2\langle z - z_d, \delta z \rangle_X + \langle u, \delta u \rangle_U \\ + & \|\delta z\|_X^2 + \|\delta u\|_U^2 \\ \geq & \|z - z_d\|_X^2 + \|u\|_U^2 + 2\langle z - z_d, \delta z \rangle_X + \langle u, \delta u \rangle_U \\ + & 2\langle -A\delta z - B\delta u, \mu \rangle_X \end{aligned}$$

Hence we obtain the inequality

$$\begin{aligned} & \|z + \delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 - \|z - z_d\|_X^2 - \|u\|_U^2 \\ & \geq 2\langle z - z_d - A^*\mu, \delta z \rangle_X + 2\langle u - B^*\mu, \delta u \rangle_U. \end{aligned}$$

Hence the inequality

$$\|z + \delta z - z_d\|_X^2 + \|u + \delta u\|_U^2 \geq \|z - z_d\|_X^2 + \|u\|_U^2 \quad (14)$$

can only hold for all control variations δu , if (12) and (13) hold.

Note that if the optimality system (11)–(13) holds, we also obtain the inequality (14), hence the optimality conditions are also sufficient. \square

3.3 A system for the difference between the static and the dynamic solution

Now we derive a system for $\hat{z}_T - z^{(\sigma)}$ and $\hat{\mu}_T - \mu^{(\sigma)}$ where $(\hat{z}_T, \hat{\mu}_T)$ is the optimal pair of state and adjoint state and \hat{u}_T denotes the optimal control for $\mathbf{P}(T)$ and $(z^{(\sigma)}, \mu^{(\sigma)})$ is the optimal state and optimal adjoint state for \mathbf{S} and $u^{(\sigma)}$ denotes the optimal control for \mathbf{S} . Due to (5)-(8) and (11)-(13) we have

$$(\hat{z}_T - z^{(\sigma)})(0) = z_0 - z^{(\sigma)}, \hat{\mu}_T(T) - \mu^{(\sigma)} = -\mu^{(\sigma)}, \quad (15)$$

$$(\hat{z}_T - z^{(\sigma)})' = A(\hat{z}_T - z^{(\sigma)}) + B(\hat{u}_T - u^{(\sigma)}), \quad (16)$$

$$(\hat{\mu}_T - \mu^{(\sigma)})' = -A^*(\hat{\mu}_T - \mu^{(\sigma)}) + (\hat{z}_T - z^{(\sigma)}), \quad (17)$$

$$\hat{u}_T - u^{(\sigma)} = B^*(\hat{\mu}_T - \mu^{(\sigma)}). \quad (18)$$

Due to (2), we can derive an explicit representation of the general solution of the differential equation (16), (17) that takes into account (18). In addition, we introduce the condition

$$BB^* > 0 \quad (19)$$

in order to derive some convergence results.

Lemma 3 *Assume that A and BB^* commute and that (19) holds. Due to our assumptions, the operator*

$$\Lambda = (BB^*)^{1/2} \quad (20)$$

is well-defined as the square root of a positive self-adjoint operator (see [1], [18], [21]). The solution of

$$(z - z^{(\sigma)})' = A(z - z^{(\sigma)}) + \Lambda^2(\mu - \mu^{(\sigma)}), \quad (21)$$

$$(\mu - \mu^{(\sigma)})' = (z - z^{(\sigma)}) - A^*(\mu - \mu^{(\sigma)}) \quad (22)$$

can be written in the form

$$(z - z^{(\sigma)})(t) = \Lambda \exp(A + \Lambda t) l_+ + \Lambda \exp(A - \Lambda t) l_-, \quad (23)$$

$$(\mu - \mu^{(\sigma)})(t) = \exp(A + \Lambda t) l_+ + \exp(A - \Lambda t) l_-. \quad (24)$$

where l_+ and $l_- \in X$ are uniquely determined by (15).

We use the notation \hat{z}_T for the optimal state of $\mathbf{P}(T)$. We have

$$\lim_{T \rightarrow \infty} \hat{z}_T(T) = z^{(\sigma)} - \Lambda \mu^{(\sigma)}. \quad (25)$$

Proof. Let $l_+ \in X$ and $l_- \in X$ be given. For $t \in [0, T]$, we define

$$\beta^\pm(t) = \exp((A \pm \Lambda)t) l_\pm, \quad \alpha^\pm(t) = \pm \Lambda \exp((A \pm \Lambda)t) l_\pm.$$

Note that our assumptions imply that Λ and A commute. Hence we have

$$\begin{aligned} \partial_t \alpha^\pm(t) &= (\pm \Lambda A + \Lambda^2) \exp((A \pm \Lambda)t) l_\pm \\ &= A \alpha^\pm(t) + \Lambda^2 \exp(A \pm \Lambda t) l_\pm \\ &= A \alpha^\pm(t) + \Lambda^2 \beta^\pm(t) \end{aligned}$$

and

$$\partial_t \beta^\pm(t) = (A \pm \Lambda) \exp((A \pm \Lambda)t) l_\pm = \alpha^\pm(t) + A\beta^\pm(t).$$

Thus $(\alpha_+(t), \beta_+(t))$ and $(\alpha_-(t), \beta_-(t))$ are solutions of the system of differential equations (21), (22). If we insert the sum $(\alpha_+(t) + \alpha_-(t), \beta_+(t) + \beta_-(t))$ in the initial and terminal conditions (15) we obtain the following system of linear equations for (l_+, l_-) .

$$\alpha_+(0) + \alpha_-(0) = \Lambda l_+ - \Lambda l_- = z_0 - z^{(\sigma)}, \quad (26)$$

$$\beta_+(T) + \beta_-(T) = \exp((A + \Lambda)T) l_+ + \exp((A - \Lambda)T) l_- = -\mu^{(\sigma)}. \quad (27)$$

Equation (26) implies

$$\Lambda(l_+ - l_-) = z_0 - z^{(\sigma)}. \quad (28)$$

We have $\exp(-(A + \Lambda)T) \exp((A - \Lambda)T) = \exp((-A - \Lambda + A - \Lambda)T) = \exp(-2\Lambda T)$. Hence (27) implies

$$\Lambda l_+ + \exp(-2\Lambda T) \Lambda l_- = -\Lambda \exp(-(A + \Lambda)T) \mu^{(\sigma)}.$$

With (28) this yields

$$(I + \exp(-2\Lambda T)) \Lambda l_+ - \exp(-2\Lambda T)(z_0 - z^{(\sigma)}) = -\Lambda \exp(-(A + \Lambda)T) \mu^{(\sigma)}.$$

Thus we obtain

$$\Lambda l_+ = (I + \exp(-2\Lambda T))^{-1} \left[\exp(-2\Lambda T)(z_0 - z^{(\sigma)}) - \Lambda \exp(-(A + \Lambda)T) \mu^{(\sigma)} \right].$$

In the sequel, we use the notation $l_\pm(T)$ for l_\pm as a function of T because we are interested in the convergence for $T \rightarrow \infty$. On account of (2) and (19) this implies in particular that we have $\lim_{T \rightarrow \infty} \|\Lambda l_+(T)\| = 0$. Equation (28) implies

$$\Lambda l_-(T) = \Lambda l_+(T) - (z_0 - z^{(\sigma)}).$$

Hence we have $\lim_{T \rightarrow \infty} \Lambda l_-(T) = -(z_0 - z^{(\sigma)})$. We introduce the notation \hat{z}_T for the optimal state corresponding to the time horizon T . For $t \in [0, T]$ we set

$$\hat{z}_T^{(+)}(t) = \Lambda \exp((A + \Lambda)t) l_+(T), \quad \hat{z}_T^{(-)}(t) = \Lambda \exp((A - \Lambda)t) l_-(T). \quad (29)$$

Then we have

$$\hat{z}_T(T) - z^{(\sigma)} = \hat{z}_T^{(+)}(T) - \hat{z}_T^{(-)}(T). \quad (30)$$

Note that $\lim_{T \rightarrow \infty} \hat{z}_T^{(-)}(T) = 0$. Due to the Neumann series we have

$$\mathcal{N}(T) = (I + \exp(-2\Lambda T))^{-1} = \sum_{k=0}^{\infty} (-1)^k \exp(-2\Lambda kT).$$

We also use the notation

$$\Lambda l_+^{(1)}(T) = \mathcal{N}(T) \exp(-2\Lambda T)(z_0 - z^{(\sigma)}), \quad (31)$$

$$\Lambda l_+^{(2)}(T) = \mathcal{N}(T)\Lambda \exp(-(A + \Lambda)T)\mu^{(\sigma)}. \quad (32)$$

We have

$$\Lambda l_+(T) = \Lambda l_+^{(1)}(T) - \Lambda l_+^{(2)}(T) \quad (33)$$

and

$$z_T^{(+)}(T) = \exp((A + \Lambda)T)\Lambda l_+^{(1)}(T) - \exp((A + \Lambda)T)\Lambda l_+^{(2)}(T).$$

Since Λ and A commute, we have

$$\exp((A + \Lambda)T)\mathcal{N}(T)\exp(-2\Lambda T) = \exp((A - \Lambda)T)\mathcal{N}(T).$$

For the first term of $z_T^{(+)}(T)$ we obtain

$$\exp((A + \Lambda)T)\Lambda l_+^{(1)}(T) = \exp((A - \Lambda)T)\mathcal{N}(T)(z_0 - z^{(\sigma)}).$$

Hence for the limit we have

$$\lim_{T \rightarrow \infty} \exp((A + \Lambda)T)\Lambda l_+^{(1)}(T) = 0.$$

Since Λ and A commute, we have

$$\exp((A + \Lambda)T)\mathcal{N}(T)\exp(-(A + \Lambda)T) = \mathcal{N}(T).$$

Thus for the next term in our representation of $z_T^{(+)}(T)$ we have

$$\exp((A + \Lambda)T)\Lambda l_+^{(2)}(T) = \Lambda \mathcal{N}(T)\mu^{(\sigma)}.$$

This yields the limit

$$\lim_{T \rightarrow \infty} \exp((A + \Lambda)T)\Lambda l_+^{(2)}(T) = \Lambda \mu^{(\sigma)}$$

which implies

$$\lim_{T \rightarrow \infty} \hat{z}_T(T) - z^{(\sigma)} = -\Lambda \mu^{(\sigma)}.$$

Now the assertion follows. \square

4 The existence of two limit trajectories for the boundaries of the time interval

Now we state our result on the limit trajectory on the last part of the time interval. Let $t_0 > 0$ be fixed and $T > t_0$. We show that more generally than in Lemma 3, in the last part of the time interval $[0, T]$ (that is in $[T - t_0, T]$), the optimal trajectory approaches exponentially fast a limit-trajectory that is independent of the time horizon and where the value at the time $T - t$ only depends on t .

Theorem 1 Assume that A and $B B^*$ commute and that (19) holds. Let Λ be defined as in (20). Let $m > 0$ denote a constant such that

$$\|\Lambda x\|_X \geq m\|x\|_X. \quad (34)$$

Assume that

$$T \geq \frac{1}{m}. \quad (35)$$

For $s \in [0, T]$, define the limit trajectory

$$\zeta(s) = z^{(\sigma)} - \Lambda \exp((A + \Lambda)(s))\mu^{(\sigma)}.$$

For $t_0 > 0$ and all $t \in [T - t_0, T]$ we have

$$\|z_T(t) - \zeta(t - T)\|_X \leq 2 \frac{[1 + \exp(mt_0)]}{\exp(mT)} \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]. \quad (36)$$

This means that for $t \in [T - t_0, T]$ the optimal state $\hat{z}_T(t)$ approaches the limit trajectory $\zeta(T - t)$ exponentially fast with respect to T . Note that the point $\zeta(T - t)$ only depends on the distance of the terminal time T .

Proof. We use the same notation as in the proof of Lemma 3, in particular let $\Lambda l_+(T)$ be defined as in (33). This implies the inequality

$$\|\Lambda l_+(T)\|_X \leq \|\mathcal{N}(T)\| \|\exp(-\Lambda T)\| \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right].$$

Due to (34) we have

$$\|\exp(-\Lambda T)\| \leq \exp(-mT).$$

Since for the operator norm of $\mathcal{N}(T)$ we have

$$\|\mathcal{N}(T)\| \leq \frac{1}{1 - \exp(-2mT)},$$

we obtain the inequality

$$\|\Lambda l_+(T)\| \leq \frac{\exp(-mT)}{1 - \exp(-2mT)} \left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]. \quad (37)$$

This implies

$$\begin{aligned} \|\Lambda l_-(T)\|_X &\leq \|\Lambda l_+(T)\|_X + \|(z_0 - z^{(\sigma)})\|_X \\ &\leq \|(z_0 - z^{(\sigma)})\|_X + \frac{\exp(-mT)}{1 - \exp(-2mT)} \left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]. \end{aligned} \quad (38)$$

Note that (35) implies $\frac{\exp(-mT)}{1 - \exp(-2mT)} \leq 1$. Similar as in (30), for all $t \in [0, T]$ we have

$$\hat{z}_T(t) - z^{(\sigma)} = \hat{z}_T^{(+)}(t) - \hat{z}_T^{(-)}(t). \quad (39)$$

For all $t \in [T - t_0, T]$ have the inequality

$$\begin{aligned} \|\hat{z}_T^{(-)}(t)\|_X &\leq \exp(-m(T - t_0))\|\Lambda l_-(T)\|_X \\ &\leq 2 \exp(-m(T - t_0)) \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]. \end{aligned} \quad (40)$$

We have

$$z_T^{(+)}(t) = \exp((A + \Lambda)t)\Lambda l_+^{(1)}(T) - \exp((A + \Lambda)t)\Lambda l_+^{(2)}(T). \quad (41)$$

Thus for the first term of $z_T^{(+)}(t)$ we obtain

$$\exp((A + \Lambda)t)\Lambda l_+^{(1)}(T) = \exp(At) \exp(\Lambda(t - 2T))\mathcal{N}(T)(z_0 - z^{(\sigma)}).$$

Hence for all $t \in [0, T]$ we have the inequality

$$\|\exp((A + \Lambda)t)\Lambda l_+^{(1)}(T)\|_X \leq 2 \exp(-mT)\|z_0 - z^{(\sigma)}\|_X. \quad (42)$$

Now we consider the next term in our representation of $z_T^{(+)}(t)$. Since Λ and A commute, we have

$$\exp((A + \Lambda)t)\mathcal{N}(T) \exp(-(A + \Lambda)T) = \exp((A + \Lambda)(t - T))\mathcal{N}(T).$$

Thus for the next term in our representation of $z_T^{(+)}(t)$ we have

$$\exp((A + \Lambda)t)\Lambda l_+^{(2)}(T) = \Lambda \exp((A + \Lambda)(t - T))\mathcal{N}(T)\mu^{(\sigma)}. \quad (43)$$

Due to the Neumann series for the corresponding operator norm we have

$$\|\mathcal{N}(T) - I\| \leq \frac{\exp(-2mT)}{1 - \exp(-2mT)}.$$

Hence for all $t \in [0, T]$ we have the inequality

$$\begin{aligned} &\| - \exp((A + \Lambda)t)\Lambda l_+^{(2)}(T) + \Lambda \exp((A + \Lambda)(t - T))\mu^{(\sigma)}\|_X \\ &\leq \frac{\exp(-2mT)}{1 - \exp(-2mT)} \|\Lambda \mu^{(\sigma)}\|_X. \end{aligned}$$

Thus with (41) and (42) we obtain the inequality

$$\begin{aligned} &\|z_T^{(+)}(t) + \Lambda \exp((A + \Lambda)(t - T))\mu^{(\sigma)}\|_X \\ &\leq 2 \exp(-mT)\|z_0 - z^{(\sigma)}\|_X + \frac{\exp(-2mT)}{1 - \exp(-2mT)} \|\Lambda \mu^{(\sigma)}\|_X. \end{aligned}$$

With (40) for all $t \in [T - t_0, T]$ this implies

$$\|\hat{z}_T(t) - z^{(\sigma)} + \exp((A + \Lambda)(t - T))\Lambda \mu^{(\sigma)}\|_X \leq$$

$$2(1 + \exp(m t_0)) \exp(-m T) \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]$$

and (36) follows. \square

Now we state our second main result. We show that also on the first part of the time interval $[0, T]$ of the form $[0, t_0]$, where $t_0 > 0$ is fixed for large time horizons T the optimal trajectory approaches exponentially fast a limit-trajectory that is independent of the time horizon T .

Theorem 2 *Assume that A and $B B^*$ commute and that (19) holds. Let Λ be defined as in (20). Let $m > 0$ denote a constant such that (34) holds. Assume that (35) holds. For $s \in [0, T]$, define the limit trajectory*

$$\Xi(s) = z^{(\sigma)} + \exp((A - \Lambda)s)(z_0 - z^{(\sigma)}).$$

With the notation \hat{z}_T for the optimal state corresponding to the time horizon T for all $t \in [0, t_0]$ we have

$$\|\hat{z}_T(t) - \Xi(t)\|_X \leq \frac{3 + 2 \exp(m t_0)}{\exp(m T)} \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]. \quad (44)$$

This means that for $t \in [0, t_0]$ the optimal state $\hat{z}_T(t)$ approaches the limit trajectory $\Xi(t)$ exponentially fast with respect to T . Note that the point $\Xi(t)$ only depends on the distance to the initial time zero.

Proof. We use the same notation as in the proof of Theorem 1. Note that (35) implies $\|\mathcal{N}(T)\| \leq 2$. We use the representation (39) of $\hat{z}_T(t) - z^{(\sigma)}$ with $\hat{z}_T^{(+)}(t)$ and $\hat{z}_T^{(-)}(t)$ defined in (29). For all $t \in [0, T]$ we have (42) and (43) implies

$$\|\exp((A + \Lambda)t) \Lambda l_+^{(2)}(T)\| \leq 2 \exp(-m(T - t_0)) \|\Lambda \mu^{(\sigma)}\|_X.$$

Thus we obtain the inequality

$$\|\hat{z}_T^{(+)}(t)\|_X \leq 2[1 + \exp(m t_0)] \exp(-m T) \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right].$$

We have

$$\begin{aligned} \hat{z}_T^{(-)}(t) + \exp((A - \Lambda)t)(z_0 - z^{(\sigma)}) &= \exp((A - \Lambda)t) \left[\Lambda l_-(T) + (z_0 - z^{(\sigma)}) \right] \\ &= \exp((A - \Lambda)t) \Lambda l_+(T). \end{aligned}$$

With (37) this yields the inequality

$$\|\hat{z}_T^{(-)}(t) + \exp((A - \Lambda)t)(z_0 - z^{(\sigma)})\|_X \leq \frac{\exp(-2m T)}{1 - \exp(-2m T)} \left[\|z_0 - z^{(\sigma)}\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right].$$

Thus we obtain

$$\|z_T(t) - z^{(\sigma)} - \exp((A - \Lambda)t)(z_0 - z^{(\sigma)})\|_X$$

$$\begin{aligned}
&= \|\hat{z}_T^{(+)}(t) - \hat{z}_T^{(-)}(t) - \exp((A - \Lambda)t)(z_0 - z^{(\sigma)})\|_X \\
&\leq \|\hat{z}_T^{(+)}(t)\| + \|\hat{z}_T^{(-)}(t) + \exp((A - \Lambda)t)(z_0 - z^{(\sigma)})\|_X \\
&\leq (3 + 2 \exp(m t_0)) \exp(-mT) \left[\|(z_0 - z^{(\sigma)})\|_X + \|\Lambda \mu^{(\sigma)}\|_X \right]
\end{aligned}$$

and (44) follows. \square

Remark 1 For the optimal adjoint state we have the representation

$$\begin{aligned}
\hat{\mu}_T(t) - \mu^{(\sigma)} &= \exp((A + \Lambda)t)l_+(T) + \exp((A - \Lambda)t)l_-(T) \\
&= \Lambda^{-1}\hat{z}_T^{(+)}(t) + \Lambda^{-1}\hat{z}_T^{(-)}(t).
\end{aligned}$$

Hence results analogous to Theorem 1 and Theorem 2 hold for the optimal adjoint state $\hat{\mu}_T$. We also find an initial limit trajectory and a terminal limit trajectory for the optimal adjoint states. So we see that the situation for the optimal adjoint state and the optimal control has the same structure. This is similar as for the classical turnpike property, see [5].

5 Examples

In this section we discuss the phenomenon of initial and terminal limit trajectories for large time horizons for two specific optimal control problems with distributed parameter systems. We consider the transport equation and the wave equation.

5.1 An Example with the transport equation

Let $L > 0$, $T > L$, $\gamma > 0$ and $y_d \in \mathbb{R}$ be given. Let $Q = [0, T] \times [0, L]$. Let an initial state $y_0 \in C([0, L])$ be given. Consider the optimal control problem

$$\begin{cases} \min_{u \in L^2(0, T)} \int_0^T \int_0^L |y(t, x) - y_d|^2 dx dt + \gamma \int_0^T |u(t)|^2 dt \\ \text{subject to } y(0, x) = y_0(x) \text{ for } x \in [0, L], y(t, 0) = u(t) \text{ for } t \in [0, T] \text{ a.e.,} \\ \text{and } y_t + y_x = 0 \text{ on } Q. \end{cases}$$

The state on the triangle $G = \{(t, x); x \in [0, L], 0 \leq t \leq x\}$ depends on the initial state only and is not influenced by the control. Hence the contribution from G to the integral over Q in the objective functions does not play a role for the optimal control.

The state is constant on the characteristic lines that have the form $t = x + c$ for a real constant c . The values of the state on the different characteristic lines corresponding to different values of c are independent of each other. Thus we can decompose the problem in the family of optimization problems corresponding to the different characteristic lines that appear on Q .

For a constant $c \in [0, T]$ let $L_{term}(c) \in [0, L]$ be such that the characteristic curve corresponding to c is contained in Q for all $x \in [0, L_{term}(c)]$. If $c \leq T - L$ we have $L_{term}(c) = L$ and if $c > T - L$ we have $L_{term}(c) = T - c$.

The optimization problem for the characteristic curve corresponding to c is

$$\begin{cases} \min_{u(c) \in \mathbb{R}} \int_0^{L_{term}(c)} |y(s+c, s) - y_d|^2 ds + \gamma |u(c)|^2 \\ \text{subject to } y(c, 0) = u(c) \text{ and } y(s+c, s) = y(c, 0) \text{ for all } s \in [0, L_{term}(c)]. \end{cases}$$

Since for all $s \in [0, L_{term}(c)]$ we have $y(s+c, s) = u(c)$ the solution is equal to the minimum of the polynomial

$$p_c(\nu) = L_{term}(c) |\nu - y_d|^2 + \gamma |\nu|^2$$

that is given by

$$\hat{u}_T(c) = \frac{1}{1 + \frac{\gamma}{L_{term}(c)}} y_d.$$

Hence for all $t \in [0, T-L]$ we have the optimal control

$$\hat{u}_T(t) = \frac{1}{1 + \frac{\gamma}{L}} y_d$$

and the optimal state

$$\hat{y}_T(t, x-t) = \hat{u}_T(t). \quad (45)$$

So we see that in this example for all $t \in [0, T-L]$ the state $\hat{y}_T(t, \cdot)$ is independent of T . Hence for all $t \in (L, T-L]$ and $z \in [0, L]$ we have

$$y_{init}(t, z) = \lim_{T \rightarrow \infty} \hat{y}_T(t, z) = \frac{1}{1 + \frac{\gamma}{L}} y_d.$$

For $t \in (T-L, T]$ the optimal control is

$$\hat{u}_T(t) = \frac{1}{1 + \frac{\gamma}{T-t}} y_d$$

and the optimal state is given by (45). So we see that $\hat{u}_T(t)$ only depends on $T-t$. For $t \in (0, L)$ and $z \in [0, t]$ for the limit for $T \rightarrow \infty$ we have

$$y_{term}(t, z) = \lim_{T \rightarrow \infty} \hat{y}_T(T-t, z) = \frac{1}{1 + \frac{\gamma}{t}} y_d.$$

Moreover, we have

$$y_{term}(0, 0) = \lim_{T \rightarrow \infty} \hat{y}_T(T, 0) = 0.$$

The static optimal control problem is

$$\begin{cases} \min_{u \in \mathbb{R}} \int_0^L |y(x) - y_d|^2 dx + \gamma |u|^2 \\ \text{subject to } y(0) = u \text{ and } y_x = 0 \text{ on } [0, L]. \end{cases}$$

Hence we obtain the static optimal state

$$y^{(\sigma)} = \frac{1}{1 + \frac{\gamma}{L}} y_d$$

and the optimal control $u^{(\sigma)} = y^{(\sigma)}$. Thus for all $t \in (L, T - L]$ and $z \in [0, L]$ for the limit state we have

$$y_{init}(t, z) = y^{(\sigma)}.$$

The state for $t \in [0, L]$ is also independent of T since the values on the triangle G are independent of T .

So we see that in this example, we obtain explicit representations of the optimal state that show that on the interior time interval $(L, T - L)$ we have $\hat{y}_T(t, x) = y^{(\sigma)}$ and $\hat{u}_T(t) = u^{(\sigma)}$. Thus in this example, the optimal static state is reached exactly in finite time. This is similar as in the finite-time turnpike phenomenon that is described in [12]. However, in [12] this situation is enforced by non-smooth tracking terms.

The structure of the solution in the example is similar as in the numerical examples in [9] where the boundary control of a linear hyperbolic 2×2 system with space dimension 1 is studied. A related application is studied in [14], namely optimal control problems with the two-dimensional transport equation that are used for optimal treatment planning in radiotherapy.

5.2 An example with the wave equation

Similar as in [11] as an example we consider an optimal control problem for a system that is governed by the wave equation. We can write the second order system as a system of first order differential equations with a skew-adjoint operator A , see [20]. In this example, different from [11] we consider the energy as a part of the objective functional.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a piecewise C^1 boundary. Let $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ be given. Let $T > 0$ and $\gamma > 0$ be given. Define $Q = [0, T] \times \Omega$ and $\Sigma = [0, T] \times \partial\Omega$. Let a desired state $y_d \in H^2(\Omega) \cap H_0^1(\Omega)$ be given.

Consider the optimal control problem

$$\begin{cases} \min_{u \in L^2(Q)} \int_Q [|y_t|^2 + \|\nabla(y - y_d)\|_{\mathbb{R}^N}^2 + \gamma|u|^2] \\ \text{subject to } y(0, x) = y_0(x), y_t(0, x) = 0 \text{ for } x \in \Omega \text{ a.e.}, \\ y(t, x) = 0 \text{ for } (t, x) \in \Sigma \text{ a.e.}, \\ \text{and } y_{tt} = \Delta y + u \text{ on } Q. \end{cases}$$

For results on the well-posedness of the corresponding initial boundary value problem see [16]. Theorem 1 and Theorem 2 imply that for the optimal state $t \mapsto \hat{y}_T(t)$ for $t > 0$ there exist an initial limit trajectory $t \mapsto \lim_{T \rightarrow \infty} \hat{y}_T(t)$ and a terminal limit trajectory $t \mapsto \lim_{T \rightarrow \infty} \hat{y}_T(T - t)$.

Assume that $(\varphi_j)_{j=1}^\infty$ is a complete orthonormal sequence of eigenfunctions of the Dirichlet-Laplacian $A_0 = -\Delta$, with the homogeneous Dirichlet boundary conditions on $\partial\Omega$. So for $j \in \{1, 2, 3, \dots\}$ we have $-\Delta\varphi_j = \lambda_j\varphi_j$ with $\lambda_j \geq \lambda_{\min} > 0$. Assume that

$$\frac{1}{4\gamma} < \lambda_{\min}. \quad (46)$$

We can find real coefficients β_j and η_j such that $\sum |\beta_j|^2 < \infty$, $\sum |\eta_j|^2 < \infty$, $y_d = \sum_{j=1}^{\infty} \beta_j \varphi_j$ and $y_0 = \sum_{j=1}^{\infty} \eta_j \varphi_j$. We can represent the state $y(t, x)$ in the form $y(t, x) = \sum_{j=1}^{\infty} \alpha_j(t) \varphi_j(x)$ and the objective functional in terms of the coefficients $\alpha_j(t)$. We obtain

$$\sum_{j=1}^{\infty} \int_0^T |\alpha_j'(t)|^2 + \lambda_j |\alpha_j(t) - \beta_j|^2 + \gamma |\alpha_j''(t) + \lambda_j \alpha_j(t)|^2 dt.$$

So the spectral representation yields a sequence of optimal control problems for $j \in \{1, 2, 3, \dots\}$:

$$\min_{\alpha_j \in H^2([0, T])} \int_0^T |\alpha_j'(t)|^2 + \lambda_j |\alpha_j(t) - \beta_j|^2 + \gamma |\alpha_j''(t) + \lambda_j \alpha_j(t)|^2 dt$$

subject to $\alpha_j(0) = \eta_j$, $\alpha_j'(0) = 0$.

For $t \in [0, T]$, the optimality conditions yield the differential equation

$$\alpha_j^{(4)}(t) + \left(2\lambda_j - \frac{1}{\gamma}\right) \alpha_j^{(2)}(t) + \lambda_j \left(\lambda_j + \frac{1}{\gamma}\right) \alpha_j(t) = \lambda_j \frac{1}{\gamma} \beta_j$$

with the boundary conditions

$$\alpha_j(0) = \eta_j, \alpha_j'(0) = 0, \alpha_j^{(3)}(T) = \left(\frac{1}{\gamma} - \lambda_j\right) \alpha_j'(T), \alpha_j''(T) = -\lambda_j \alpha_j(T).$$

Define the characteristic polynomial $p_j(z) = z^4 + \left(2\lambda_j - \frac{1}{\gamma}\right) z^2 + \lambda_j \left(\lambda_j + \frac{1}{\gamma}\right)$. Due to (46) we have $p_j(z) = (z - \omega_j)(z + \omega_j)(z - \bar{\omega}_j)(z + \bar{\omega}_j)$ where $\pm\omega_j$ and $\pm\bar{\omega}_j$ denote the roots of p_j . We have $|\omega_j|^4 = \lambda_j \left(\lambda_j + \frac{1}{\gamma}\right)$, $Re(\omega_j^2) = -\left(\lambda_j - \frac{1}{2\gamma}\right)$ and $|Im(\omega_j^2)| = \frac{1}{\gamma} \sqrt{2\gamma \lambda_j - \frac{1}{4}}$. Moreover we have

$$|Re(\omega_j)| \geq \frac{1}{2\sqrt{\gamma}}. \quad (47)$$

For the sake of simplicity we consider the case $y_d = 0$, that is $\beta_j = 0$.

We obtain the representation for $t \mapsto \alpha_j(T; t)$ with fixed T :

$$\begin{aligned} \alpha_j(T; t) = & A_j(T) [(\bar{\omega}_j^2 + \lambda_j) \cosh(\omega_j(T-t)) - (\omega_j^2 + \lambda_j) \cosh(\bar{\omega}_j(T-t))] \\ & + B_j(T) \left[\bar{\omega}_j \left(\bar{\omega}_j^2 + \lambda_j - \frac{1}{\gamma} \right) \sinh(\omega_j(T-t)) - \omega_j \left(\omega_j^2 + \lambda_j - \frac{1}{\gamma} \right) \sinh(\bar{\omega}_j(T-t)) \right] \end{aligned}$$

where the coefficients $T \mapsto A_j(T)$, $T \mapsto B_j(T)$ have to be chosen such that $\alpha_j(T; 0) = \eta_j$ and $\alpha_j'(T; 0) = 0$. Note that the corresponding Wronski-matrix is regular. Define the denominator

$$\begin{aligned} N_j(T) = & |\omega_j|^2 \left[\frac{4\lambda_j}{\gamma} + \left(\frac{4\lambda_j}{\gamma} - \frac{1}{\gamma^2} \right) \cosh(\omega_j T) \cosh(\bar{\omega}_j T) \right] \\ & + \lambda_j \left(\frac{4\lambda_j}{\gamma} + \frac{1}{\gamma^2} \right) \sinh(\omega_j T) \sinh(\bar{\omega}_j T). \end{aligned}$$

Condition (46) implies $N_j(T) > 0$. We have

$$A_j(T) = -\eta_j \frac{|\omega_j|^2 \left[\left(\bar{\omega}_j^2 + \lambda_j - \frac{1}{\gamma} \right) \cosh(\omega_j T) - \left(\omega_j^2 + \lambda_j - \frac{1}{\gamma} \right) \cosh(\bar{\omega}_j T) \right]}{N_j(T)},$$

$$B_j(T) = -\eta_j \frac{-\omega_j(\bar{\omega}_j^2 + \lambda_j) \sinh(\omega_j T) + \bar{\omega}_j(\omega_j^2 + \lambda_j) \sinh(\bar{\omega}_j T)}{N_j(T)}.$$

The representations of $A_j(T)$ and $B_j(T)$ allow for the asymptotic analysis for $T \rightarrow \infty$. Due to (47) we have

$$\lim_{T \rightarrow \infty} A_j(T) = 0, \quad \lim_{T \rightarrow \infty} B_j(T) = 0.$$

For $t > 0$ this yields

$$\lim_{T \rightarrow \infty} \alpha_j(T; T - t) = 0.$$

For $y_d = 0$ the optimal static state is zero and it is attained with the minimal control cost zero. Hence it does not reduce the control cost and thus the value of the objective functional if towards the end of the time interval the control deviates from the optimal static control. Thus the optimal control stabilizes the state towards zero until the end of the time interval $[0, T]$. Note that the values $|\alpha_j^{(T)}(T - t)|$ decay exponentially fast with growing T with the order $\exp(-|Re(\omega_j)|T)$ and hence at least with the order $\exp(-\frac{1}{2\sqrt{\gamma}}T)$.

6 Conclusions

We have shown that under non-restrictive assumptions, for large time horizons the optimal states approach limit trajectories both in the first part of the time interval and in the last part of the time interval. The limit trajectories are independent of the time horizon T .

We have shown the results for linear quadratic optimal control problems with distributed parameter systems. We expect that our results can be generalized in several directions. A generalization to the case of convex objective functionals similar as in [8] is possible. In [8] also additional control constraints and state constraints are considered that are independent of the time horizons. Constraints of this type can also be integrated in the turnpike analysis of initial- and terminal limit trajectories.

The generalization of the results to the case of semilinear systems is also a topic of future research. If the nonlinear source term is of a dissipative nature, the proofs should have a similar structure but the linearization of the nonlinear source term appears in the adjoint equation in the necessary optimality conditions. Moreover, additional smallness assumptions for the initial state in appropriate function spaces are in general require to guarantee that the system is well-posed. Assumptions of this type are also required in the case of quasilinear systems. However, in order to guarantee that the solutions remain regular

also for large time horizons, additional constraints are necessary. In [10] state constraints with respect to the C^1 -norm are included in the optimal control problem. The constraints assure that the system state is a classical solution of the partial differential equation and no shocks can occur.

Also a generalization to games is of interest, We refer to [22] for the study of the classical turnpike phenomenon for discrete time games and to [13] for the definition of a networked boundary control game with the wave equation. The optimal control of the wave equation with measure valued controls has been studied in [15]. It is an interesting question for future research whether also for this problem a similar turnpike structure occurs for large time horizons.

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