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STABILITY FOR THE SOBOLEV INEQUALITY WITH EXPLICIT CONSTANTS

JEAN DOLBEAULT, MARIA J. ESTEBAN, ALESSIO FIGALLI, RUPERT L. FRANK,
AND MICHAEL LOSS

ABSTRACT. A quantitative version of the Bianchi–Egnell inequality concerning the stability of the Sobolev inequality is proved with explicit constants. For the proof we study a flow that interpolates continuously between a function and its symmetric decreasing rearrangement.

1. INTRODUCTION AND MAIN RESULT

In [9] Brezis and Lieb posed the question whether it is possible to bound the ‘Sobolev deficit’

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2$$

from below in terms of some natural distance from the manifold of optimizers. Here $d \geq 3$, $2^* = 2d/(d-2)$ is the ‘Sobolev exponent’, and

$$S_d = \frac{1}{4} d(d-2) |\mathbb{S}^d|^{2/d}$$

is the sharp Sobolev constant. The function f belongs to the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d)$, that is, it is in $L^1_{\text{loc}}(\mathbb{R}^d)$, its distributional gradient is a square-summable function and it vanishes at infinity in the sense that $|\{x \in \mathbb{R}^d : |f(x)| > \epsilon\}| < \infty$ for all $\epsilon > 0$. Here $|A|$ denotes the Lebesgue measure of a measurable set A . Throughout this paper we deal with

(J. Dolbeault, M.J. Esteban) CEREMADE (CNRS UMR No. 7534), PSL UNIVERSITY, UNIVERSITÉ PARIS-DAUPHINE, PLACE DE LATTRE DE TASSIGNY, 75775 PARIS 16, FRANCE.

E-mails: dolbeaul@ceremade.dauphine.fr, esteban@ceremade.dauphine.fr

(A. Figalli) MATHEMATICS DEPARTMENT, ETH ZÜRICH, RAMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND. *E-mail:* alessio.figalli@math.ethz.ch

(R.L. Frank) DEPARTMENT OF MATHEMATICS, LMU MUNICH, THERESIENSTR. 39, 80333 MÜNCHEN, GERMANY, AND MUNICH CENTER FOR QUANTUM SCIENCE AND TECHNOLOGY, SCHELLINGSTR. 4, 80799 MÜNCHEN, GERMANY, AND MATHEMATICS 253-37, CALTECH, PASADENA, CA 91125, USA. *E-mail:* r.frank@lmu.de

(M. Loss) SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY ATLANTA, GA 30332, UNITED STATES OF AMERICA. *E-mail:* loss@math.gatech.edu

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real-valued functions. With minor additional effort our arguments can be extended to the case of complex-valued functions.

Rodemich [39], Aubin [3] and Talenti [43] (see also [41]) proved that the Sobolev deficit is non-negative. Moreover, it was shown by Lieb [35], Gidas, Ni and Nirenberg [34] and Caffarelli, Gidas and Spruck [14] that the deficit vanishes if and only if the function f is of the form

$$f(x) = c(a + |x - b|^2)^{-\frac{d-2}{2}}, \quad (1)$$

where $a \in (0, \infty)$, $b \in \mathbb{R}^d$ and $c \in \mathbb{C}$ are constants. These functions are often called ‘Aubin–Talenti functions’. Let \mathcal{M} denote the $(d+2)$ -dimensional manifold of functions of the form (1).

The question of Brezis and Lieb was answered by Bianchi and Egnell [4]: there is a strictly positive constant c_{BE} such that for any $f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$

$$\mathcal{E}(f) := \frac{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2} \geq c_{\text{BE}}. \quad (2)$$

We denote by c_{BE} the optimal, that is, largest possible constant in (2).

Lions [37] has shown that, if the Sobolev deficit is small for some function f , then f has to be close to the manifold \mathcal{M} of Sobolev optimizers. The closeness is in the strongest possible sense, namely with respect to the norm in $\dot{H}^1(\mathbb{R}^d)$. The Bianchi–Egnell inequality (2) makes the qualitative result of Lions quantitative. In particular, it shows that the distance to the manifold vanishes at least like the square root of the Sobolev deficit. Such ‘stability’ estimates have been established in other contexts as well, e.g., for the isoperimetric inequality or for classical inequalities in real and harmonic analysis. In fact, stability has attracted a lot of attention in recent years and we refer to [33, 21, 28, 22, 18, 16, 19, 27, 20, 42, 31, 32, 29, 7] and the references within for a list of works in this direction. In several of them the strategy of Bianchi and Egnell or its generalizations play an important role.

An interesting point about (2) and other inequalities obtained by this method is that nothing seems to be known about the optimal value of the constant c_{BE} except for the fact that it is strictly positive. The proof in [4] proceeds by a spectral estimate combined with a compactness argument and hence cannot give any information about c_{BE} . Explicit quantitative estimates are known only for a distance to \mathcal{M} measured by a weaker norm than (2), functions of $\dot{H}^1(\mathbb{R}^d)$ satisfying additional constraints or superquadratic estimates of the distance which degenerate in a neighbourhood of \mathcal{M} and much more is known for subcritical interpolation inequalities than for Sobolev-type inequalities: see [6, 26, 25, 24, 7, 30] for some references.

It is the aim of this article to address the question of proving (2) with an explicit lower bound on c_{BE} . To formulate the result, let us introduce the notations

$$\nu(\delta) := \sqrt{\frac{\delta}{1-\delta}} \quad (3)$$

for any $\delta \in (0, 1)$ and, with $q = 2^*$ the Sobolev exponent,

$$\begin{aligned} \mathfrak{m}(\nu) &:= \frac{4}{d+4} - \frac{2}{q} \nu^{q-2} && \text{if } d \geq 6, \\ \mathfrak{m}(\nu) &:= \frac{4}{d+4} - \frac{1}{3}(q-1)(q-2)\nu - \frac{2}{q} \nu^{q-2} && \text{if } d = 4, 5, \\ \mathfrak{m}(\nu) &:= \frac{4}{7} - \frac{20}{3}\nu - 5\nu^2 - 2\nu^3 - \frac{1}{3}\nu^4 && \text{if } d = 3. \end{aligned} \quad (4)$$

With

$$\kappa_d := \sup_{0 < \delta < 1} \delta \mathfrak{m}(\nu(\delta)), \quad (5)$$

here is our theorem.

Theorem 1. *Let $d \geq 3$ and $f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$. Then*

$$\mathcal{E}(f) \geq \frac{1}{2} \kappa_d,$$

where $\mathfrak{m}(\nu)$ and $\nu(\delta)$ are defined by (4) and (3). If additionally $f \geq 0$, then we have

$$\mathcal{E}(f) \geq \kappa_d.$$

Thus, the optimal constant in (2) satisfies

$$c_{\text{BE}} \geq \frac{1}{2} \kappa_d.$$

We refer to Appendix B for considerations on the numerical values of our estimates. Concerning its behavior in high dimensions we will prove that

$$\frac{1}{2} \kappa_d \gtrsim \frac{2^d}{e^2 d^{1+\frac{d}{2}}} \quad \text{as } d \rightarrow +\infty. \quad (6)$$

Moreover, we have the *upper* bound

$$c_{\text{BE}} \leq \frac{4}{d+4} \quad (7)$$

for any $d \geq 3$, see Proposition 20.

A question that remains open is what the optimal value of the constant c_{BE} in (2) is and whether it is attained for an extremal function. After a first version of this paper has appeared on the arXiv we learned that T. König has shown that the inequality in (7) is strict in any dimension $d \geq 3$. This is reminiscent of the planar isoperimetric inequality, where the constant in the quantitative isoperimetric inequality with Frankel asymmetry is strictly smaller than the constant in the corresponding spectral gap inequality. In that case one can prove the existence of an optimizing function; see [5]. For further studies under an additional convexity assumption, see [15, 2, 23].

Let us describe the strategy of the proof of Theorem 1. It consists of two parts. In a first part we prove a lower bound on $\mathcal{E}(f)$ for *non-negative functions* f . In a second part we show that this implies a lower bound for general functions f . The two parts are independent of each other in the sense that the second part does not rely on the method by which in the first part a lower bound was obtained.

Let us begin by describing the strategy of proving a bound for non-negative functions. Superficially, the proof is analogous to that by Bianchi and Egnell [4], namely, one splits the problem into two regions, one where f is close to the manifold of Sobolev optimizers and the other where it is far away. These regions are defined in terms of the quantity $\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2 / \|\nabla f\|_{L^2(\mathbb{R}^d)}^2$, specifically by requiring that this quantity is either less or equal than δ , or (strictly) bigger than δ . Here $\delta > 0$ is a free parameter that we will optimize over at the end. Note that, since $\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2 \leq \|\nabla f\|_{L^2(\mathbb{R}^d)}^2$, we may always assume that $\delta \leq 1$ and even $\delta < 1$.

In Section 3, for $\delta \in (0, 1)$ small enough, we estimate $\mu(\delta)$ such that

$$\mathcal{E}(f) \geq \mu(\delta) \quad (8)$$

whenever

$$0 < \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2 \leq \delta \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \quad (9)$$

and prove that $\mu(\delta)$ is positive. The argument in this regime is similar to that of Bianchi and Egnell and is based on a spectral gap inequality. We make the qualitative expansions of [4] quantitative and obtain for all $\delta \in (0, 1)$ a remainder bound smaller than an explicit, strictly positive constant.

In Section 2, we obtain a lower bound on $\mathcal{E}(f)$ in case $\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2 > \delta \|\nabla f\|_{L^2(\mathbb{R}^d)}^2$ for an arbitrary *non-negative function* f . To handle this regime we use an idea taken from a paper by Christ [20] in which he establishes a quantitative error term for the Riesz rearrangement inequality. The idea, in a rough outline, is to construct a continuous family of rearrangements f_τ , $0 \leq \tau < \infty$, such that $f_0 = f$, $\|f_\tau\|_{L^{2^*}(\mathbb{R}^d)} = \|f\|_{L^{2^*}(\mathbb{R}^d)}$, $\tau \mapsto \|\nabla f_\tau\|_{L^2(\mathbb{R}^d)}$ is non-increasing and $\inf_{g \in \mathcal{M}} \|\nabla(f_\tau - g)\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0$ as $\tau \rightarrow \infty$. Clearly

$$\mathcal{E}(f) \geq \frac{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2}{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2} = 1 - S_d \frac{\|f\|_{L^{2^*}(\mathbb{R}^d)}^2}{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2} \geq \frac{\|\nabla f_\tau\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f_\tau\|_{L^{2^*}(\mathbb{R}^d)}^2}{\|\nabla f_\tau\|_{L^2(\mathbb{R}^d)}^2}.$$

Starting with $\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2 > \delta \|\nabla f\|_{L^2(\mathbb{R}^d)}^2$, one would like to run the flow until at a certain point τ_0 one has

$$\inf_{g \in \mathcal{M}} \|\nabla(f_{\tau_0} - g)\|_{L^2(\mathbb{R}^d)}^2 = \delta \|\nabla f_{\tau_0}\|_{L^2(\mathbb{R}^d)}^2 \quad (10)$$

and, using (8), one would conclude that

$$\mathcal{E}(f) \geq \frac{\|\nabla f_{\tau_0}\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f_{\tau_0}\|_{L^{2^*}(\mathbb{R}^d)}^2}{\|\nabla f_{\tau_0}\|_{L^2(\mathbb{R}^d)}^2} = \delta \frac{\|\nabla f_{\tau_0}\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f_{\tau_0}\|_{L^{2^*}(\mathbb{R}^d)}^2}{\inf_{g \in \mathcal{M}} \|\nabla(f_{\tau_0} - g)\|_{L^2(\mathbb{R}^d)}^2} \geq \delta \mu(\delta).$$

The details of this argument are more involved than presented here, mostly because the function $\tau \mapsto \|\nabla f_\tau\|_{L^2(\mathbb{R}^d)}$ need not be continuous, so the existence of a τ_0 as in (10) is not guaranteed.

Collecting the results of Sections 2 and 3 provides us with bounds on $\mathcal{E}(f)$ for *non-negative functions* f : see Proposition 19. This concludes the first part of the proof.

In the second part of the proof, in Section 4, we prove a lower bound on $\mathcal{E}(f)$ for *sign-changing functions* f . The proof exploits a concavity property inherent in the problem and shows that only one of the positive and negative parts is really relevant for the problem at hand. More precisely, if, say $\|f_-\|_{L^{2^*}(\mathbb{R}^d)}^2 \leq \|f_+\|_{L^{2^*}(\mathbb{R}^d)}^2$, then $\|\nabla f_-\|_{L^2(\mathbb{R}^d)}$ is bounded by a constant times the Sobolev deficit. This allows one to essentially reduce the problem to the non-negative function f_+ .

Additional results on the symmetric decreasing rearrangement and the numerical values of our estimates are given in two appendices, A and B, at the end of the paper.

In order to make notations lighter, we will write $\|\cdot\|_q = \|\cdot\|_{L^q(\mathbb{R}^d)}$ whenever the space is \mathbb{R}^d with Lebesgue measure.

2. COMPETING SYMMETRIES, THE SEQUENCE AND THE FLOW

In a first step one uses ‘competing symmetries’ to move a *non-negative* initial function f close, but not exactly to the desired location (9). This is done by building a sequence $(f_n)_{n \in \mathbb{N}}$ and considering in Lemma 5 two alternatives, (a) and (b). In case (a), the whole sequence stays outside the neighbourhood (9) as well as its limit. In case (b), in a further step one uses a continuous rearrangement (flow) to achieve (10) or, actually, a substitute for it.

2.1. Competing symmetries. The functional $\mathcal{E}(f)$ is conformally invariant in the sense that if $C : \mathbb{R}^d \cup \{\infty\} \rightarrow \mathbb{R}^d \cup \{\infty\}$ is a conformal map, the function

$$f_C(x) = |\det DC(x)|^{1/2^*} f(C(x))$$

satisfies

$$\mathcal{E}(f_C) = \mathcal{E}(f).$$

In order to verify this, we recall that any conformal map is a composition of scalings, translations, rotations and inversions. For scalings, translations and rotations in \mathbb{R}^d the claimed invariance is easy to see. The additional map to consider is the inversion $I(x) = \frac{x}{|x|^2}$ and a straightforward change of variables shows that

$$\|\nabla f_I\|_2^2 = \|\nabla f\|_2^2, \quad \|f_I\|_{2^*}^2 = \|f\|_{2^*}^2.$$

The equality

$$\inf_{g \in \mathcal{M}} \|\nabla(f_I - g)\|_2^2 = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

follows from

$$\inf_{g \in \mathcal{M}} \|\nabla(f_I - g)\|_2^2 = \inf_{g \in \mathcal{M}} \|\nabla(f - g_I)\|_2^2 = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

since $I^2 = I$ and $g \rightarrow g_I$ maps the set \mathcal{M} to itself in a one-to-one and onto fashion.

Another and perhaps easier way to see the conformal invariance is to pull the problem up to the sphere via the stereographic projection. We denote by $s = (s_1, s_2, \dots, s_{d+1})$ the coordinates in \mathbb{R}^{d+1} . Then the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ can be parametrized in terms of stereographic coordinates by

$$s_j = \frac{2x_j}{1+|x|^2}, \quad j = 1, \dots, d, \quad s_{d+1} = \frac{1-|x|^2}{1+|x|^2}.$$

We set

$$F(s) = \left(\frac{1+|x|^2}{2} \right)^{\frac{d-2}{2}} f(x) \tag{11}$$

and find by a straightforward computation that

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \|f\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2} = \frac{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \frac{1}{4} d(d-2) \|F\|_{L^2(\mathbb{S}^d)}^2 - S_d \|F\|_{L^{2^*}(\mathbb{S}^d)}^2}{\inf_{G \in \mathcal{M}} \left\{ \|\nabla F - \nabla G\|_{L^2(\mathbb{S}^d)}^2 + \frac{1}{4} d(d-2) \|F - G\|_{L^2(\mathbb{S}^d)}^2 \right\}},$$

where G is a function of the form

$$G(s) = c(a + b \cdot s)^{-\frac{d-2}{2}},$$

and $a > 0$, $b \in \mathbb{R}^d$ and $c \in \mathbb{C}$ are constants, and \mathcal{M} is the corresponding set of functions. On the sphere the inversion I takes the form of the reflection $(s_1, \dots, s_d, s_{d+1}) \rightarrow (s_1, \dots, s_d, -s_{d+1})$.

A second ingredient for the construction of the flow is the technique of ‘competing symmetries’, invented in [17]. Consider any non-negative function $f \in \dot{H}^1(\mathbb{R}^d)$ and its counterpart $F \in H^1(\mathbb{S}^d)$ given by (11). Set

$$(UF)(s) = F(s_1, s_2, \dots, s_{d+1}, -s_d),$$

which corresponds to a rotation by $\pi/2$ that maps the ‘north pole’ axis $(0, 0, \dots, 1)$ to $(0, \dots, 1, 0)$. Reversing (11) the function that corresponds to UF on \mathbb{R}^d is given by

$$(Uf)(x) = \left(\frac{2}{|x - e_d|^2} \right)^{\frac{d-2}{2}} f \left(\frac{x_1}{|x - e_d|^2}, \dots, \frac{x_{d-1}}{|x - e_d|^2}, \frac{|x|^2 - 1}{|x - e_d|^2} \right), \quad (12)$$

where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. It follows that

$$\mathcal{E}(Uf) = \mathcal{E}(f).$$

The operation U is obviously linear, invertible and an isometry on $L^{2^*}(\mathbb{R}^d)$.

We also consider the symmetric decreasing rearrangement

$$\mathcal{R}f(x) = f^*(x).$$

The most important properties are that f and f^* are equimeasurable and that $\|\nabla f^*\|_2 \leq \|\nabla f\|_2$. For elementary properties of rearrangements the reader may consult [36]. Being equimeasurable, this map is also an isometry on $L^{2^*}(\mathbb{R}^d)$. It is when using the decreasing rearrangement that we use the fact that f is a non-negative function. For functions that change sign one conventionally defines their rearrangement as the rearrangement of their absolute value. Passing from a function to its absolute value does not alter the numerator of $\mathcal{E}(f)$ but may decrease the denominator so that other arguments are needed.

On \mathbb{R}^d , let

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2d}} \left(\frac{2}{1 + |x|^2} \right)^{\frac{d-2}{2}}. \quad (13)$$

Note that $\|g_*\|_{2^*} = 1$ because it is obtained as the stereographic projection of the constant function on \mathbb{S}^d with 2^* -norm equal to 1. The following theorem was proved in [17].

Theorem 2. *Let $f \in L^{2^*}(\mathbb{R}^d)$ be a non-negative function. Consider the sequence $(f_n)_{n \in \mathbb{N}}$ of functions*

$$f_n = (\mathcal{R}U)^n f \quad \forall n \in \mathbb{N}. \quad (14)$$

Then

$$\lim_{n \rightarrow \infty} \|f_n - h_f\|_{2^*} = 0$$

where $h_f = \|f\|_{2^*} g_* \in \mathcal{M}$. Moreover, if $f \in \dot{H}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_2^2)_{n \in \mathbb{N}}$ is a non-increasing sequence.

It does not seem clear whether the functional $\mathcal{E}(f)$ decreases or increases under rearrangement. The next lemma helps to explain this point. Define \mathcal{M}_1 to be the set of the elements in \mathcal{M} with 2^* -norm equal to 1.

Lemma 3. *For any $f \in \dot{H}^1(\mathbb{R}^d)$, we have*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (f, g^{2^*-1})^2.$$

Here (\cdot, \cdot) is the $L^2(\mathbb{R}^d)$ inner product or, more precisely, the duality pairing between $L^{2^*}(\mathbb{R}^d)$ and $L^{(2^*)'}(\mathbb{R}^d)$.

Proof. Let g be any Aubin–Talenti function with $\|g\|_{2^*} = 1$. The function g is an optimizer of the Sobolev inequality, i.e., $\|\nabla g\|_2^2 = S_d \|g\|_{2^*}^2 = S_d$ and is a solution of the Sobolev equation

$$-\Delta g = S_d \frac{g^{2^*-1}}{\|g\|_{2^*}^{2^*-2}} = S_d g^{2^*-1}.$$

Hence for any non-negative constant c we find

$$\|\nabla(f - cg)\|_2^2 = \|\nabla f\|_2^2 - 2c(\nabla f, \nabla g) + c^2 \|\nabla g\|_2^2 = \|\nabla f\|_2^2 - 2c S_d(f, g^{2^*-1}) + S_d c^2$$

and minimizing with respect to c we find the lower bound $\|\nabla f\|_2^2 - S_d(f, g^{2^*-1})^2$, which proves the lemma. \square

Under the decreasing rearrangement, the term $\|\nabla f\|_2^2$ does not increase whereas the term $\sup_{g \in \mathcal{M}_1}(f, g^{2^*-1})^2$ increases. To see this, note that the supremum is attained at some Aubin–Talenti function of the form (1) that is a strictly symmetric function about the point $b \in \mathbb{R}^d$. Replacing f by its symmetric decreasing rearrangement about that point increases $(f, g^{2^*-1})^2$, in fact strictly unless f is already symmetric decreasing about the point b . Thus, while the numerator in $\mathcal{E}(f)$ decreases under rearrangements so does the denominator and there are no direct conclusions to be drawn from this. The next lemma summarizes what we have shown.

Lemma 4. *For the sequence $(f_n)_{n \in \mathbb{N}}$ in Theorem 2 we have that $n \mapsto \sup_{g \in \mathcal{M}_1}(f_n, g^{2^*-1})^2$ is strictly increasing, $n \mapsto \inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2$ is strictly decreasing and*

$$\lim_{n \rightarrow \infty} \inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 - S_d \|h_f\|_2^2 = \lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 - S_d \|f\|_2^2.$$

Proof. From

$$\inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \|\nabla f_n\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (f_n, g^{2^*-1})^2$$

we see that the first term converges since $(\|\nabla f_n\|_2^2)_{n \in \mathbb{N}}$ is a non-increasing sequence. For the second term, which is strictly increasing, we have by Hölder’s inequality

$$\sup_{g \in \mathcal{M}_1} (f_n, g^{2^*-1})^2 \leq \|f_n\|_{2^*}^2 = \|f\|_{2^*}^2$$

and since g_* as defined in (13) is in \mathcal{M}_1 we have

$$\liminf_{n \rightarrow \infty} \sup_{g \in \mathcal{M}_1} (f_n, g^{2^*-1})^2 \geq \liminf_{n \rightarrow \infty} (f_n, g_*)^2 = \|f\|_{2^*}^2$$

by Theorem 2. \square

Lemma 5. *Assume that $0 \leq f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ satisfies*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \geq \delta \|\nabla f\|_2^2$$

and let $(f_n)_{n \in \mathbb{N}}$ be the sequence defined by (14). Then one of the following alternatives holds:

(a) *for all $n = 0, 1, 2, \dots$ we have*

$$\inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 \geq \delta \|\nabla f_n\|_2^2$$

(b) *there is a natural number n_0 such that*

$$\inf_{g \in \mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_2^2 \geq \delta \|\nabla f_{n_0}\|_2^2$$

and

$$\inf_{g \in \mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|_2^2 < \delta \|\nabla f_{n_0+1}\|_2^2.$$

Proof. Assume that alternative (a) does not hold. Then there is a largest value $n_0 \geq 0$ such that $\inf_{g \in \mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_2^2 \geq \delta \|\nabla f_{n_0}\|_2^2$. \square

Lemma 6. *Assume that $0 \leq f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ satisfies*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \geq \delta \|\nabla f\|_2^2$$

and suppose that in Lemma 5 alternative (a) holds for the sequence $(f_n)_{n \in \mathbb{N}}$ defined by (14). Then

$$\mathcal{E}(f) \geq \delta.$$

Proof. We have

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \|f\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2} \geq \frac{\|\nabla f\|_2^2 - S_d \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \geq \frac{\|\nabla f_n\|_2^2 - S_d \|f\|_{2^*}^2}{\|\nabla f_n\|_2^2}, \quad (15)$$

where the second inequality is a consequence of $\|\nabla f_n\|_2^2 \leq \|\nabla f\|_2^2$ for all $n = 0, 1, 2, \dots$ proved in Theorem 2. By the assumption that alternative (a) holds and by Lemma 4, we learn that

$$\lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 \leq \frac{1}{\delta} \lim_{n \rightarrow \infty} \inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta} \left(\lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 - S_d \|f\|_{2^*}^2 \right).$$

Since

$$\lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 - S_d \|f\|_{2^*}^2 \geq \delta \lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 \geq \delta S_d \lim_{n \rightarrow \infty} \|f_n\|_{2^*}^2 = \delta S_d \|f\|_{2^*}^2 > 0,$$

we can take the limit as $n \rightarrow \infty$ on the right side of (15) and compute the limit of the quotient as the quotient of the limits. This proves the lemma. \square

2.2. Continuous rearrangement. Next, we analyze the case where the alternative (b) in Lemma 5 holds. Let us introduce

$$\mu(\delta) := \inf \left\{ \mathcal{E}(f) : 0 \leq f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}, \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \leq \delta \|\nabla f\|_2^2 \right\}. \quad (16)$$

Lemma 7. *For any $\delta \in (0, 1]$, we have $\mu(\delta) \leq 1$.*

Proof. By Lemma 3, we have

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (f, g^{2^*-1})^2$$

and it follows from Hölder's inequality that

$$\sup_{g \in \mathcal{M}_1} (f, g^{2^*-1})^2 \leq \|f\|_{2^*}^2.$$

Thus, the denominator in $\mathcal{E}(f)$ that enters the definition of $\mu(\delta)$ is at least as large as the numerator, so the quotient is at most 1. \square

Our goal in this subsection is to prove the following lower bound.

Lemma 8. *Assume that $0 \leq f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ satisfies*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \geq \delta \|\nabla f\|_2^2$$

for some $\delta \in (0, 1)$ and suppose that in Lemma 5 alternative (b) holds for the sequence $(f_n)_{n \in \mathbb{N}}$ of Theorem 2 defined by (14). Then, with $\mu(\delta)$ defined by (16), we have

$$\mathcal{E}(f) \geq \delta \mu(\delta).$$

For the proof of this lemma we introduce a continuous rearrangement flow, which interpolates between a function and its symmetric decreasing rearrangement. The basic ingredient for this flow is similar to a flow that Brock introduced [10, 11] and which interpolates between a function and its Steiner symmetrization with respect to a given hyperplane. Brock's construction, in turn, is based on ideas of Rogers [40] and Brascamp–Lieb–Luttinger [8]. Our flow is obtained by glueing together infinitely many copies of Brock's flows with respect to a sequence of judiciously chosen hyperplanes. A similar construction was performed by Bucur and Henrot [12]; see also [20].

More specifically, for a given hyperplane H , Brock's flow interpolates between a given function f and f^{*H} , the Steiner symmetrized function with respect to H . The family that interpolates between f and f^{*H} is denoted by f_τ^H , $\tau \in [0, \infty]$, and we have

$$f_0 = f, \quad f_\infty^H = f^{*H}.$$

Further, for any τ , f_τ^H and f are equimeasurable, i.e.,

$$|\{x \in \mathbb{R}^d : f_\tau^H(x) > t\}| = |\{x \in \mathbb{R}^d : f(x) > t\}| \quad \forall t > 0.$$

Moreover, if $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, then $\tau \mapsto f_\tau^H$ is continuous in $L^p(\mathbb{R}^d)$.

By choosing a sequence of hyperplanes we construct another flow $\tau \mapsto f_\tau$ that has the same properties but interpolates between f and f^* , the symmetric decreasing rearrangement. In Appendix A we explain this in more detail and prove the following properties that are important for our proof, assuming $f \in \dot{H}^1(\mathbb{R}^d)$. From the $L^{2^*}(\mathbb{R}^d)$ continuity of the flow we will deduce that

$$\lim_{\tau \rightarrow \tau_0} \sup_{g \in \mathcal{M}_1} (f_\tau, g)^2 = \sup_{g \in \mathcal{M}_1} (f_{\tau_0}, g)^2. \quad (17)$$

Concerning the gradient we prove the monotonicity

$$\|\nabla f_{\tau_2}\|_2 \leq \|\nabla f_{\tau_1}\|_2, \quad 0 \leq \tau_1 \leq \tau_2 \leq \infty,$$

and the right continuity

$$\lim_{\tau_2 \rightarrow \tau_1^+} \|\nabla f_{\tau_2}\|_2 = \|\nabla f_{\tau_1}\|_2, \quad 0 \leq \tau_1 < \infty. \quad (18)$$

Proof of Lemma 8. We begin by motivating and explaining the strategy of the proof. As before, we bound

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \|f\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2} \geq \frac{\|\nabla f\|_2^2 - S_d \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \geq \frac{\|\nabla f_{n_0}\|_2^2 - S_d \|f_{n_0}\|_{2^*}^2}{\|\nabla f_{n_0}\|_2^2}. \quad (19)$$

We could bound the right side further from below by replacing f_{n_0} by f_{n_0+1} . This bound, however, might be too crude for our purposes and we proceed differently. The move from f_{n_0} to f_{n_0+1} consists of two steps, namely first applying a conformal rotation and second applying symmetric decreasing rearrangement. The first step leaves all terms on the right side invariant and we do carry out this step. The second step leaves the 2^* -norm invariant, while the gradient term does not go up. In fact, the gradient term might go down too far. Therefore, we replace the application of the rearrangement by a continuous rearrangement flow. In order to make the notation less cumbersome we shall denote Uf_{n_0} by \mathbf{f}_0 where U denotes the conformal rotation (12). We denote by \mathbf{f}_τ , $0 \leq \tau \leq \infty$, the continuous rearrangement starting at \mathbf{f}_0 and let

$$\mathbf{f}_\infty = f_{n_0+1}. \quad (20)$$

Ideally, we would like to find $\tau_0 \in [0, \infty)$ such that

$$\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_{\tau_0} - \nabla g\|_2^2 = \delta \|\nabla \mathbf{f}_{\tau_0}\|_2^2.$$

Then the right side of (19) is equal to

$$1 - S_d \frac{\|\mathbf{f}_0\|_{2^*}^2}{\|\nabla \mathbf{f}_0\|_2^2} \geq 1 - S_d \frac{\|\mathbf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathbf{f}_{\tau_0}\|_2^2} = \delta \frac{\|\nabla \mathbf{f}_{\tau_0}\|_2^2 - S_d \|\mathbf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_{\tau_0} - \nabla g\|_2^2},$$

which can be bounded from below by $\delta \mu(\delta)$, since \mathbf{f}_{τ_0} is admissible in the infimum (16). This would prove the desired bound.

The problem with this argument is that the existence of such a $\tau_0 \in [0, \infty)$ is in general not clear, since neither of the terms $\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2$ and $\|\nabla \mathbf{f}_\tau\|_2^2$ needs to be continuous in τ . Nevertheless, we will be able to adapt the above argument to yield the same conclusion.

We now turn to the details of the argument. Recalling that

$$\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_0 - \nabla g\|_2^2 \geq \delta \|\nabla \mathbf{f}_0\|_2^2,$$

we define

$$\tau_0 := \inf \left\{ \tau \geq 0 : \inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2 < \delta \|\nabla \mathbf{f}_\tau\|_2^2 \right\}$$

with the convention that $\inf \emptyset = \infty$. If $\tau < \tau_0 \in (0, \infty]$, similarly as before, the right side of (19) is equal to

$$\frac{\|\nabla \mathbf{f}_0\|_2^2 - S_d \|\mathbf{f}_0\|_{2^*}^2}{\|\nabla \mathbf{f}_0\|_2^2} = 1 - S_d \frac{\|\mathbf{f}_0\|_{2^*}^2}{\|\nabla \mathbf{f}_0\|_2^2} \geq \frac{\|\nabla \mathbf{f}_\tau\|_2^2 - S_d \|\mathbf{f}_\tau\|_{2^*}^2}{\|\nabla \mathbf{f}_\tau\|_2^2} \geq \delta \frac{\|\nabla \mathbf{f}_\tau\|_2^2 - S_d \|\mathbf{f}_\tau\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2},$$

where the last inequality arises from $\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2 \geq \delta \|\nabla \mathbf{f}_\tau\|_2^2$ for any $\tau \in [0, \tau_0)$. Taking the limit inferior as $\tau \rightarrow \tau_0^-$, we obtain

$$\frac{\|\nabla \mathbf{f}_0\|_2^2 - S_d \|\mathbf{f}_0\|_{2^*}^2}{\|\nabla \mathbf{f}_0\|_2^2} \geq \delta \frac{\lim_{\tau \rightarrow \tau_0^-} \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \|\mathbf{f}_{\tau_0}\|_{2^*}^2}{\liminf_{\tau \rightarrow \tau_0^-} \inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2}. \quad (21)$$

Note that the denominator appearing here does not vanish. Indeed, we have

$$\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2 \geq \delta \|\nabla \mathbf{f}_\tau\|_2^2 \geq \delta S_d \|\mathbf{f}_\tau\|_{2^*}^2 = \delta S_d \|f\|_{2^*}^2 > 0 \quad \forall \tau \in [0, \tau_0)$$

and, as a consequence,

$$\liminf_{\tau \rightarrow \tau_0^-} \inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2 \geq \delta S_d \|f\|_{2^*}^2 > 0.$$

The same inequality (21) remains valid if $\tau_0 = 0$ and if we interpret $\lim_{\tau \rightarrow \tau_0^-}$ and $\liminf_{\tau \rightarrow \tau_0^-}$ as evaluating at $\tau_0 = 0$.

At this point we find it convenient to apply Lemma 3 and use the representation

$$\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2 = \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_\tau, g^{2^*-1})^2.$$

Using (17), that is, the continuity of $\tau \mapsto \sup_{g \in \mathcal{M}_1} (\mathbf{f}_\tau, g^{2^*-1})^2$, we see that

$$\liminf_{\tau \rightarrow \tau_0^-} \inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\tau - \nabla g\|_2^2 = \lim_{\tau \rightarrow \tau_0^-} \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2.$$

Thus, the relevant quotient is equal to

$$\frac{\lim_{\tau \rightarrow \tau_0^-} \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \|\mathbf{f}_{\tau_0}\|_{2^*}^2}{\lim_{\tau \rightarrow \tau_0^-} \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2}. \quad (22)$$

Our goal in the remainder of this proof is to show that this quotient is larger or equal than $\mu(\delta)$. We will use the fact that

$$\sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2 \leq \|\mathbf{f}_{\tau_0}\|_{2^*}^2, \quad (23)$$

which follows from Hölder's inequality. We also note that equality holds here if and only if $\mathbf{f}_{\tau_0} \in \mathcal{M}$.

Let us first handle the case where $\mathbf{f}_{\tau_0} \in \mathcal{M}$. Then by (2.2) and because of equality in (23), the quotient (22) is equal to 1, which by Lemma 7 can be further bounded from below by $\mu(\delta)$, leading to the claimed bound. This completes the proof in the case $\mathbf{f}_{\tau_0} \in \mathcal{M}$ and in what follows we assume

$$\mathbf{f}_{\tau_0} \notin \mathcal{M}.$$

As a consequence of this assumption and (23), we have

$$\|\nabla \mathbf{f}_{\tau_0}\|_2^2 > S_d \|\mathbf{f}_{\tau_0}\|_{2^*}^2 \geq S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2. \quad (24)$$

Next, we observe that for $\alpha > \beta$ the function $x \mapsto (x - \alpha)/(x - \beta)$ is monotone increasing on the interval (β, ∞) . This, together with the strict inequality in (24), implies that the quotient (22) can be bounded from below by

$$\frac{\lim_{\tau \rightarrow \tau_0^-} \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \|\mathbf{f}_{\tau_0}\|_{2^*}^2}{\lim_{\tau \rightarrow \tau_0^-} \|\nabla \mathbf{f}_\tau\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2} \geq \frac{\|\nabla \mathbf{f}_{\tau_0}\|_2^2 - S_d \|\mathbf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathbf{f}_{\tau_0}\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2}. \quad (25)$$

We now claim that

$$\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_{\tau_0} - \nabla g\|_2^2 \leq \delta \|\nabla \mathbf{f}_{\tau_0}\|_2^2. \quad (26)$$

Once this is proved, we can bound the right side of (25) from below by $\mu(\delta)$. This inequality is the claimed inequality after taking into account (21).

To prove (26), we first note that it is verified if $\tau_0 = \infty$. Indeed, $\mathbf{f}_\infty = \mathbf{f}_{n_0+1}$ by (20) and therefore, by assumption of alternative (b), $\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_\infty - \nabla g\|_2^2 < \delta \|\nabla \mathbf{f}_\infty\|_2^2$.

Now let $\tau_0 < \infty$. We argue by contradiction and assume that

$$\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_{\tau_0} - \nabla g\|_2^2 > \delta \|\nabla \mathbf{f}_{\tau_0}\|_2^2. \quad (27)$$

Because of this strict inequality and the definition of τ_0 there are $\sigma_k \in (\tau_0, \infty)$ for any $k \in \mathbb{N}$ with $\lim_{k \rightarrow \infty} \sigma_k = \tau_0$ such that $\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_{\sigma_k} - \nabla g\|_2^2 < \delta \|\nabla \mathbf{f}_{\sigma_k}\|_2^2$, that is,

$$\|\nabla \mathbf{f}_{\sigma_k}\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\sigma_k}, g^{2^*-1})^2 < \delta \|\nabla \mathbf{f}_{\sigma_k}\|_2^2 \quad \forall k \in \mathbb{N}.$$

Letting $k \rightarrow \infty$ and using (17) as well as the right continuity of $\|\nabla \mathbf{f}_\tau\|_2^2$, see (18), we deduce that

$$\|\nabla \mathbf{f}_{\tau_0}\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} (\mathbf{f}_{\tau_0}, g^{2^*-1})^2 \leq \delta \|\nabla \mathbf{f}_{\tau_0}\|_2^2.$$

This is the same as $\inf_{g \in \mathcal{M}} \|\nabla \mathbf{f}_{\tau_0} - \nabla g\|_2^2 \leq \delta \|\nabla \mathbf{f}_{\tau_0}\|_2^2$ and contradicts (27). This proves (26) and completes the proof of the lemma. \square

Remark 9. *The above argument would be simpler if $\tau \mapsto \|\nabla f_\tau\|_2^2$ were continuous for an appropriate choice of hyperplanes (see Appendix A) in the definition of the flow. Since the flow is weakly continuous in $\dot{H}^1(\mathbb{R}^d)$, continuity of the norm is equivalent to (strong) continuity of the flow in $\dot{H}^1(\mathbb{R}^d)$. Thus, for continuity of the norm for an appropriate choice of hyperplanes, it is necessary that there is such a choice for which the Steiner symmetrizations approximate f^* in $\dot{H}^1(\mathbb{R}^d)$. According to a theorem of Burchard [13] this holds if and only if f is co-area regular, i.e. if and only if the distribution function*

$$h \mapsto |\{x \in \mathbb{R}^d : f(x) > h, \nabla f(x) = 0\}|$$

has no absolutely continuous component. As shown by Almgren and Lieb [1], both co-area regular and co-area irregular functions are dense for $d \geq 2$.

2.3. Summary. Let us summarize the result of this section.

Corollary 10. *Take $\delta \in (0, 1)$ and assume that $0 \leq f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ satisfies*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \geq \delta \|\nabla f\|_2^2.$$

Then, with $\mu(\delta)$ defined by (16), we have

$$\mathcal{E}(f) \geq \delta \mu(\delta).$$

Proof. By Lemma 5 either alternative (a) or (b) holds. In the first case, we apply Lemmas 6 and 7, and in the second case, we apply Lemma 8. \square

3. ANALYSIS CLOSE TO THE MANIFOLD OF OPTIMIZERS

In this section we analyze the case of $\delta > 0$ small and bound $\mu(\delta)$ from below by an explicit function of δ .

3.1. Expansions with remainder terms. Our goal in this subsection is to prove the following proposition.

Proposition 11. *Let X be a measure space and $u, r \in L^q(X)$ for some $q \geq 2$ with $u \geq 0$ and $u + r \geq 0$. Assume also that $\int_X u^{q-1} r \, dx = 0$.*

- *If $2 \leq q \leq 3$, then*

$$\|u + r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2} r^2 \, dx + \frac{2}{q} \int_X r_+^q \, dx \right).$$

- *If $3 \leq q \leq 4$, then*

$$\begin{aligned} \|u + r\|_q^2 &\leq \|u\|_q^2 \\ &+ \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2} r^2 \, dx + \frac{1}{3} (q-1)(q-2) \int_X u^{q-3} r^3 \, dx + \frac{2}{q} \int_X |r|^q \, dx \right). \end{aligned}$$

- *If $q = 6$, then*

$$\begin{aligned} \|u + r\|_q^2 &\leq \|u\|_q^2 + \|u\|_q^{2-q} \left(5 \int_X u^{q-2} r^2 \, dx + \frac{20}{3} \int_X u^{q-3} r^3 \, dx \right. \\ &\quad \left. + 5 \int_X u^{q-4} r^4 \, dx + 2 \int_X u^{q-5} r^5 \, dx + \frac{1}{3} \int_X r^6 \, dx \right). \end{aligned}$$

Similar bounds can also be derived for $q \in (4, \infty) \setminus \{6\}$. They become increasingly more complicated as q passes an integer. We restrict ourselves to the case $q = 6$, which is the only case in $(4, \infty)$ that we need, as it corresponds to the Sobolev exponent in dimension $d = 3$. For the proof of the proposition, we need two lemmas, which we discuss next.

Lemma 12. *We have the upper bounds following expansions.*

- If $2 \leq q \leq 3$, then, for all $x \geq -1$,

$$(1+x)^q \leq 1 + qx + \frac{1}{2}q(q-1)x^2 + x_+^q.$$

- If $3 \leq q \leq 4$, then, for all $x \geq -1$,

$$(1+x)^q \leq 1 + qx + \frac{1}{2}q(q-1)x^2 + \frac{1}{6}q(q-1)(q-2)x^3 + |x|^q.$$

- If $q = 6$, then, for all $x \geq -1$,

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$$

Proof. We begin with the case $2 \leq q \leq 3$ and set

$$f(x) := (1+x)^q - 1 - qx - \frac{1}{2}q(q-1)x^2 - x_+^q.$$

For any $x \geq -1$, we compute

$$f'(x) = q((1+x)^{q-1} - 1 - (q-1)x - x_+^{q-1}),$$

$$f''(x) = q(q-1)((1+x)^{q-2} - 1 - x_+^{q-2}).$$

For $-1 \leq x \leq 0$ we clearly have $(1+x)^{q-2} - 1 - x_+^{q-2} = (1-|x|)^{q-2} - 1 \leq 0$. For $x \geq 0$ we have, by a well-known elementary inequality, $(1+x)^{q-2} - 1 - x_+^{q-2} = (1+x)^{q-2} - 1 - x^{q-2} \leq 0$. To summarize, f is concave on $[-1, \infty)$. We conclude that, for all $x \geq -1$,

$$f(x) \leq f(0) - f'(0)x.$$

Since $f(0) = f'(0) = 0$, this is the claimed inequality.

We now turn to the case $3 \leq q \leq 4$ and set this time

$$f(x) := (1+x)^q - 1 - qx - \frac{1}{2}q(q-1)x^2 - \frac{1}{6}q(q-1)(q-2)x^3 - |x|^q.$$

Again, we compute

$$f'(x) = q((1+x)^{q-1} - 1 - (q-1)x - \frac{1}{2}(q-1)(q-2)x^2 - |x|^{q-2}x),$$

$$f''(x) = q(q-1)((1+x)^{q-2} - 1 - (q-2)x - |x|^{q-2}).$$

Since again $f(0) = f'(0) = 0$, the claimed inequality will follow if we can show concavity of f on $[-1, \infty)$, that is, $g \leq 0$ on $[-1, \infty)$ where

$$g(x) := (1+x)^{q-2} - 1 - (q-2)x - |x|^{q-2}.$$

We compute

$$g'(x) = (q-2)((1+x)^{q-3} - 1 - |x|^{q-4}x),$$

$$g''(x) = (q-2)(q-3)((1+x)^{q-4} - |x|^{q-4}).$$

We discuss g separately on $[-1, 0]$ and on $(0, \infty)$.

- We begin with the second case. For $x > 0$ we have, by the same elementary inequality as before, $(1+x)^{q-3} - 1 - x^{q-3} < 0$. Thus, $g' < 0$ on $(0, \infty)$. Since $g(0) = 0$, we deduce $g < 0$ on $(0, \infty)$.

- Now let us consider the interval $[-1, 0]$. We see that $g'' > 0$ on $(-1, -1/2)$ and $g'' < 0$ on $(-1/2, 0)$. Therefore g' is increasing on $(-1, -1/2)$ and decreasing on $(-1/2, 0)$. Since $g'(-1) = g'(0) = 0$, we conclude that $g' > 0$ on $(-1, 0)$ and therefore g is increasing on $(-1, 0)$. Since $g(0) = 0$ we conclude that $g < 0$ on $[-1, 0)$, as claimed.

If $q = 6$ we simply expand $(1+x)^6$. This completes the proof of the lemma. \square

We will also use the following elementary lemma.

Lemma 13. *If $q \geq 2$, then, for all $x \geq 0$,*

$$(1+x)^{\frac{2}{q}} \leq 1 + \frac{2}{q}x.$$

Proof of Proposition 11. We only give the proof in the case $2 \leq q \leq 3$. We have, by Lemma 12, almost everywhere on X ,

$$(u+r)^q \leq u^q + q u^{q-1} r + \frac{1}{2} q (q-1) u^{q-2} r^2 + r_+^q.$$

Integrating this and using the assumed orthogonality condition, we obtain

$$\int_X (u+r)^q dx \leq \int_X u^q dx + \frac{1}{2} q (q-1) \int_X u^{q-2} r^2 dx + \int_X r_+^q dx.$$

Applying Lemma 13, we obtain

$$\left(\int_X (u+r)^q dx \right)^{\frac{2}{q}} \leq \left(\int_X u^q dx \right)^{\frac{2}{q}} + \left(\int_X u^q dx \right)^{\frac{2-q}{q}} \left((q-1) \int_X u^{q-2} r^2 dx + \frac{2}{q} \int_X r_+^q dx \right).$$

This is the claimed inequality for $2 \leq q \leq 3$. The proof in the remaining cases proceeds similarly. \square

3.2. Application to the Sobolev functional. Throughout this subsection, we assume that $d \geq 3$ and we set

$$q = 2^* = \frac{2d}{d-2}.$$

We recall that we denote by S_d the optimal constant in the Sobolev inequality $\|\nabla f\|_2^2 \geq S_d \|f\|_q^2$ and by \mathcal{M} the set of all optimizers in this inequality.

Proposition 14. *Let $q = 2^*$, $0 \leq f \in \dot{H}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that*

$$\|\nabla f - \nabla u\|_2 = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2.$$

Set $r := f - u$ and $\sigma := \|r\|_q / \|u\|_q$.

- If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d (q-1) \|u\|_q^{2-q} u^{q-2} r^2 \right) dx - \frac{2}{q} \|\nabla r\|_2^2 \sigma^{q-2}.$$

- If $d = 4, 5$, we have

$$\begin{aligned} & \|\nabla f\|_2^2 - S_d \|f\|_q^2 \\ & \geq \int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d (q-1) \|u\|_q^{2-q} u^{q-2} r^2 \right) dx - \|\nabla r\|_2^2 \left(\frac{1}{3} (q-1) (q-2) \sigma + \frac{2}{q} \sigma^{q-2} \right). \end{aligned}$$

- If $d = 3$, we have

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \int_{\mathbb{R}^d} \left(|\nabla r|^2 - 5 S_3 \|u\|_6^{-4} u^4 r^2 \right) dx - \|\nabla r\|_2^2 \left(\frac{20}{3} \sigma + 5 \sigma^2 + 2 \sigma^3 + \frac{1}{3} \sigma^4 \right).$$

Proof. This follows directly from Proposition 11 in the previous section. We note that the orthogonality conditions

$$(\nabla r, \nabla u) = (\nabla f - \nabla u, \nabla u) = 0 \quad \text{and} \quad (r, u^{q-1}) = (f - u, u^{q-1}) = 0 \quad (28)$$

are satisfied because of the choice of u . Indeed, since \mathcal{M} is closed under multiplication by a scalar, we find that

$$0 = \frac{d}{d\alpha} \|\nabla f - \alpha \nabla u\|_2^2 \Big|_{\alpha=1} = 2(\nabla f, \nabla u) - 2\|\nabla u\|_2^2 = 2(\nabla r, \nabla u) = 2c_u(r, u^{q-1}),$$

where, in the last equality, we used the equation $-\Delta u = c_u u^{q-1}$ with $c_u := \|\nabla u\|_2^2 / \|u\|_q^q$.

Finally, we use the Sobolev inequality $S_d \|r\|_q^2 \leq \|\nabla r\|_2^2$ for the term multiplying the quantity involving σ . \square

Let us recall a spectral gap inequality which appears, for instance, in Rey's paper [38, Appendix D] slightly before the work of Bianchi and Egnell [4].

Lemma 15. *Let $d \geq 3$, $q = 2^*$, $f \in \dot{H}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that $\|\nabla f - \nabla u\|_2 = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$. Then $r := f - u$ satisfies*

$$\int_{\mathbb{R}^d} (|\nabla r|^2 - S_d(q-1) \|u\|_q^{2-q} |u|^{q-2} r^2) dx \geq \frac{4}{d+4} \int_{\mathbb{R}^d} |\nabla r|^2 dx.$$

Proof. By translation and dilation invariance, we may assume that $u(x) = c(1 + |x|^2)^{-(d-2)/2}$ for some constant $c > 0$. Then, by inverse stereographic projection and the discussion in Subsection 2.1, the question becomes to prove the inequality

$$\int_{\mathbb{S}^d} (|\nabla R|^2 - d|R|^2) d\omega \geq \frac{4}{d+4} \int_{\mathbb{S}^d} (|\nabla R|^2 + \frac{1}{4}d(d-2)|R|^2) d\omega \quad (29)$$

for all R that are orthogonal to spherical harmonics of degrees $\ell \leq 1$. Diagonalizing the Laplace–Beltrami operator, the inequality becomes

$$\ell(\ell + d - 1) - d \geq \frac{4}{d+4} \left(\ell(\ell + d - 1) + \frac{1}{4}d(d-2) \right) \quad \text{for all } \ell \geq 2.$$

This is elementary to check. \square

Remark 16. *If we look at the quadratic form $r \mapsto \int_{\mathbb{R}^d} (|\nabla r|^2 - S_d(q-1) \|u\|_q^{2-q} |u|^{q-2} r^2) dx$, one may wonder why no essential spectrum should be taken into account. This is indeed an issue (see for instance [7, Proposition 1.16]) if the form is defined on $L^2(\mathbb{R}^d)$, but not anymore if we consider the operator $-u^{2-q}\Delta - S_d(q-1) \|u\|_q^{2-q}$ on $L^2(\mathbb{R}^d, u^{q-2} dx)$ as the image of $H^1(\mathbb{S}^d)$ through the stereographic projection is compactly embedded in $L^2(\mathbb{R}^d, u^{q-2} dx)$, so that only discrete spectrum has to be taken into account. With $r \in \dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d, u^{q-2} dx)$, the spectral gap computation in the proof of Lemma 15 is justified as already noted in [4, Appendix].*

Now we insert the spectral gap inequality in the expansion and obtain the following.

Corollary 17. *Let $q = 2^*$ and $0 \leq f \in \dot{H}^1(\mathbb{R}^d)$. Set $\mathcal{D}(f) := \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$ and $\tau := \mathcal{D}(f) / (\|\nabla f\|_2^2 - \mathcal{D}(f)^2)^{1/2}$.*

- *If $d \geq 6$, we have*

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \left(\frac{4}{d+4} - \frac{2}{q} \tau^{q-2} \right) \mathcal{D}(f)^2.$$

- If $d = 4, 5$, we have

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \left(\frac{4}{d+4} - \frac{1}{3}(q-1)(q-2)\tau - \frac{2}{q}\tau^{q-2}\right) \mathcal{D}(f)^2.$$

- If $d = 3$, we have

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \left(\frac{4}{7} - \frac{20}{3}\tau - 5\tau^2 - 2\tau^3 - \frac{1}{3}\tau^4\right) \mathcal{D}(f)^2.$$

Proof. Let $u \in \mathcal{M}$ be such that $r = f - u$ satisfies $\|\nabla r\| = \mathcal{D}(f)$. Set $\sigma := \|r\|_q/\|u\|_q$. Then, by combining Proposition 14 and Lemma 15, we obtain the following bounds.

- If $d \geq 6$,

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \left(\frac{4}{d+4} - \frac{2}{q}\sigma^{q-2}\right) \mathcal{D}(f)^2.$$

- If $d = 4, 5$,

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \left(\frac{4}{d+4} - \frac{1}{3}(q-1)(q-2)\sigma - \frac{2}{q}\sigma^{q-2}\right) \mathcal{D}(f)^2.$$

- If $d = 3$,

$$\|\nabla f\|_2^2 - S_d \|f\|_q^2 \geq \left(\frac{4}{7} - \frac{20}{3}\sigma - 5\sigma^2 - 2\sigma^3 - \frac{1}{3}\sigma^4\right) \mathcal{D}(f)^2.$$

We slightly weaken these inequalities, but convert them into a more explicit form. Set

$$\rho := \mathcal{D}(f)/\|\nabla f\|_2.$$

We recall that $(\nabla f, \nabla u) = \|\nabla u\|_2^2$ was shown in (28). This implies

$$\rho^2 \|\nabla f\|_2^2 = \mathcal{D}(f)^2 = \|\nabla f - \nabla u\|_2^2 = \|\nabla f\|_2^2 - \|\nabla u\|_2^2,$$

that is, $\|\nabla u\|_2 = \sqrt{1 - \rho^2} \|\nabla f\|_2$. As a consequence,

$$\begin{aligned} \sigma &= \|f - u\|_q/\|u\|_q \leq \|\nabla f - \nabla u\|_2/\|\nabla u\|_2 \\ &= (1 - \rho^2)^{-1/2} \|\nabla f - \nabla u\|_2/\|\nabla f\|_2 = (1 - \rho^2)^{-1/2} \rho = \tau. \end{aligned}$$

Thus we can replace σ by τ in the above bounds and obtain the assertion of the corollary. \square

Recall that, according to (16),

$$\mu(\delta) = \inf \left\{ \frac{\|\nabla f\|_2^2 - S_d \|f\|_q^2}{\mathcal{D}(f)^2} : 0 \leq f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}, \mathcal{D}(f)^2 \leq \delta \|\nabla f\|_2^2 \right\}.$$

With the notation $\nu(\delta)$ and $\mathfrak{m}(\nu)$ introduced in (3) and (4) we reformulate Corollary 17 as follows.

Corollary 18. *With the above notations, we have $\mu(\delta) \geq \mathfrak{m}(\nu(\delta))$.*

Proof. The result follows from Corollary 17 if we note that

$$\tau = \frac{\mathcal{D}(f)/\|\nabla f\|_2}{\sqrt{1 - \mathcal{D}(f)^2/\|\nabla f\|_2^2}} \leq \nu(\delta).$$

\square

Since $\lim_{\nu \rightarrow 0^+} \mathfrak{m}(\nu) = 4/(d+4) > 0$ for any $d \geq 3$, it is clear that κ_d defined by (5) is a finite, positive real number that depends only on d .

4. PROOF OF THE MAIN RESULT

We recall that c_{BE} denotes the optimal constant in (2). Similarly, we denote by $c_{\text{BE}}^{\text{pos}}$ the optimal constant in (2) when restricted to non-negative functions f . Thus, $c_{\text{BE}}^{\text{pos}} \geq c_{\text{BE}}$. We do not know whether these two constants coincide or not. The main result in this section will be to prove a lower bound on c_{BE} in terms of $c_{\text{BE}}^{\text{pos}}$.

Before doing this, however, we collect the results of Sections 2 and 3 in an explicit stability result for *non-negative functions* that goes as follows.

Proposition 19. *Let $d \geq 3$. Then*

$$c_{\text{BE}}^{\text{pos}} \geq \kappa_d,$$

where κ_d is given by (5) with $\mathbf{m}(\nu)$ defined in (4) and $\nu(\delta)$ defined in (3). Explicitly, if $f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ is a non-negative function, then $\mathcal{E}(f)$ in (2) is bounded below by κ_d .

Proof. The proof of Proposition 19 is a consequence of Corollaries 10 and 18. \square

A standard consequence of the spectral analysis of Lemma 15 is an upper estimate.

Proposition 20. *For any $d \geq 3$, we have the upper bound*

$$c_{\text{BE}} \leq c_{\text{BE}}^{\text{pos}} \leq \frac{4}{d+4}.$$

Proof. This bound is derived using the fact that the constant $4/(d+4)$ in inequality (29) is optimal and attained if and only if R is a spherical harmonic of degree two. \square

Now we turn our attention to the proof of Theorem 1 and consider *sign-changing functions*.

Proposition 21. *For any $d \geq 3$,*

$$c_{\text{BE}} \geq \min \left\{ \frac{1}{2} c_{\text{BE}}^{\text{pos}}, 1 - 2^{-\frac{2}{d}} \right\}.$$

Proof of Proposition 21. To simplify the notation, given a function $v \in \dot{H}^1(\mathbb{R}^d)$, we define the deficit

$$D(v) := \|\nabla v\|_{L^2(\mathbb{R}^d)}^2 - S_d \|v\|_{L^{2^*}(\mathbb{R}^d)}^2.$$

Also, we set $\alpha_d := \frac{2}{2^*} = 1 - \frac{2}{d} < 1$,

$$h(m) := m^{\alpha_d} + (1-m)^{\alpha_d} - 1, \quad \text{and} \quad h_d := h\left(\frac{1}{2}\right) = 2^{1-\alpha_d} - 1 = 2^{\frac{2}{d}} - 1.$$

Let us consider a function $u \in \dot{H}^1(\mathbb{R}^d)$. With no loss of generality we can assume that $\|u\|_{L^{2^*}(\mathbb{R}^d)} = 1$. Let u_{\pm} denote the positive and negative parts, set

$$m := \|u_{-}\|_{L^{2^*}(\mathbb{R}^d)}^{2^*},$$

and assume (without loss of generality) that

$$m \in [0, 1/2]. \tag{30}$$

Note that $\|u_{+}\|_{L^{2^*}(\mathbb{R}^d)}^{2^*} = 1 - m$ and $\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 = \|\nabla u_{-}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla u_{+}\|_{L^2(\mathbb{R}^d)}^2$. Hence, we have

$$D(u) = \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - S_d = D(u_{+}) + D(u_{-}) + S_d h(m). \tag{31}$$

Since the function $m \mapsto h(m)$ is monotone increasing and concave on $[0, 1/2]$, it holds

$$2h_d m \leq h(m). \tag{32}$$

Also, if we set $\xi_d := 2(1 - 2^{-\alpha_d})$, the function $f(m) := (1 - m)^{\alpha_d} - 1 + \xi_d m$ is such that $f(0) = f(1/2) = 0$ and $f''(m) \leq 0$, so that $f(m) \geq 0$ for all $m \in [0, 1/2]$. Hence, by (30), we have

$$(1 - m)^{\alpha_d} \geq 1 - \xi_d m,$$

which, by the definition of $h(m)$, yields

$$h(m) \geq m^{\alpha_d} - \xi_d m.$$

Combining this bound with (32), this gives

$$\left(1 + \frac{\xi_d}{2h_d}\right) h(m) \geq m^{\alpha_d}.$$

Therefore, recalling (31) and noticing that $D(u_-) + S_d m^{\alpha_d} = \|\nabla u_-\|_{L^2(\mathbb{R}^d)}^2$, we get

$$D(u) \geq D(u_+) + D(u_-) + S_d \frac{2h_d}{2h_d + \xi_d} m^{\alpha_d} \geq D(u_+) + \frac{2h_d}{2h_d + \xi_d} \|\nabla u_-\|_{L^2(\mathbb{R}^d)}^2.$$

By definition, we have

$$D(u_+) \geq c_{\text{BE}}^{\text{pos}} \inf_{g \in \mathcal{M}} \|\nabla u_+ - \nabla g\|_{L^2(\mathbb{R}^d)}^2.$$

As a consequence, if $g_+ \in \mathcal{M}$ is optimal for u_+ , we obtain

$$\begin{aligned} D(u) &\geq c_{\text{BE}}^{\text{pos}} \|\nabla u_+ - \nabla g_+\|_{L^2(\mathbb{R}^d)}^2 + \frac{2h_d}{2h_d + \xi_d} \|\nabla u_-\|_{L^2(\mathbb{R}^d)}^2 \\ &\geq \min \left\{ c_{\text{BE}}^{\text{pos}}, \frac{2h_d}{2h_d + \xi_d} \right\} \left(\|\nabla u_+ - \nabla g_+\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla u_-\|_{L^2(\mathbb{R}^d)}^2 \right) \\ &\geq \frac{1}{2} \min \left\{ c_{\text{BE}}^{\text{pos}}, \frac{2h_d}{2h_d + \xi_d} \right\} \|\nabla u - \nabla g_+\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Noticing that $2h_d + \xi_d = 2 \cdot 2^{\frac{2}{d}} - 2 + 2 - 2^{1-\alpha_d} = 2^{\frac{2}{d}}$ we get

$$\frac{h_d}{2h_d + \xi_d} = 2^{-\frac{2}{d}} \left(2^{\frac{2}{d}} - 1 \right) = 1 - 2^{-\frac{2}{d}},$$

which concludes the proof. \square

Corollary 22. *For any $d \geq 3$ we have*

$$c_{\text{BE}} \geq \frac{1}{2} \kappa_d$$

with κ_d given by (5).

Theorem 1 is now a straightforward consequence of Proposition 19 and Corollary 22.

Proof. In dimensions $d \leq 6$ one can verify that $\frac{2}{d+4} \leq 1 - 2^{-2/d}$. By Proposition 20, this implies $\frac{1}{2} c_{\text{BE}}^{\text{pos}} \leq 1 - 2^{-2/d}$, so Proposition 21 gives $c_{\text{BE}} \geq \frac{1}{2} c_{\text{BE}}^{\text{pos}}$.

In general dimension $d \geq 3$, we use $\nu(\delta) > \sqrt{\delta}$ and $\mathfrak{m}(\nu) \leq \frac{4}{d+4} - \frac{2}{q} \nu^{q-2}$ to get

$$\delta \mathfrak{m}(\nu(\delta)) \leq \frac{4}{d+4} \delta - \frac{2}{q} \delta^{\frac{q}{2}} =: f_d(\delta). \quad (33)$$

Maximizing $\delta \mapsto f_d(\delta)$ shows that

$$\kappa_d \leq f_d \left(\left(\frac{4}{d+4} \right)^{\frac{1}{q-2}} \right) = \frac{2^{d+1}}{d(d+4)^{d/2}} =: \kappa_d^\infty.$$

For any $d \geq 4$, we have $d + 4 \geq 2^3$ while for $d = 3$, we notice that $\sqrt{3087} = d(d + 4)^{d/2} \geq 2^{3d/2} = \sqrt{512}$. Hence $d(d + 4)^{d/2} \geq 2^{3d/2}$ and

$$\frac{1}{2} \kappa_d \leq \frac{1}{2} \kappa_d^\infty = \frac{2^d}{d(d + 4)^{d/2}} \leq 2^{-\frac{d}{2}} < 1 - 2^{-\frac{2}{d}} \quad (34)$$

for any $d \geq 3$. The last (strict) inequality in (34) follows from the case $x = 2/d$ in the elementary inequality

$$h(x) := 2^{-x} + 2^{-1/x} < 1 \quad \forall x \in (0, 1).$$

To prove the latter, one can for instance notice that $\lim_{x \rightarrow 0^+} h(x) = h(1) = 1$, $\lim_{x \rightarrow 0^+} h'(x) = -\log 2 < 0$ and that the equality $h'(x) = 0$ means that $2^{-1/x} = x^2 2^{-x}$, so that, for this value of x , $h(x) = (1 + x^2) 2^{-x} \leq 1$ (to see that, note that the function $x \mapsto (1 + x^2) 2^{-x}$ has value 1 at $x \in \{0, 1\}$ and its second derivative is nonnegative is positive in $[0, 1]$). If for some $x \in (0, 1)$, h would take values larger or equal than 1, then we would achieve a contradiction at the maximum point. This proves the last inequality in (34).

By combining (34) with Propositions 19 and 21 we obtain the claimed lower bound in the corollary. \square

Let us investigate the limit as $d \rightarrow +\infty$. From inequality (33) it follows that the value of $\delta > 0$ that realizes the maximum of $\delta \mapsto \delta \mathbf{m}(\nu(\delta))$ is less than the first positive root of $\delta \mapsto f_d(\delta)$ as defined in (33), *i.e.*,

$$\delta \leq \left(\frac{4d}{(d-2)(d+4)} \right)^{\frac{d}{2}-1} \sim 2^{d-2} d^{-\frac{d}{2}} e^{-1} \quad \text{as } d \rightarrow +\infty,$$

and therefore becomes small as $d \rightarrow +\infty$. As a consequence, $\nu(\delta) \sim \sqrt{\delta}$, and asymptotically (33) is saturated to leading order. By the maximization of f_d as in the proof of Corollary 22 we infer that

$$\kappa_d \sim \kappa_d^\infty \sim \frac{2^{d+1}}{e^2 d^{1+\frac{d}{2}}} \quad \text{as } d \rightarrow +\infty. \quad (35)$$

Inequality (6) now follows from Corollary 22. \square

APPENDIX A. SOME REMARKS ABOUT CONTINUOUS REARRANGEMENT

In this appendix we review the continuous rearrangement of Brock and of Bucur–Henrot and prove some of its properties.

Brock's continuous rearrangement is based on the following operation for functions of one real variable that are finite union of disjoint characteristic functions $\sum_{k=1}^N \chi_{(-a_k, a_k)}(x - b_k)$. Replace this function by $\sum_{k=1}^N \chi_{(-a_k, a_k)}(x - e^{-t} b_k)$ where t varies from 0 to ∞ . As t increases, the intervals start moving closer and as soon as any two intervals touch one stops the process and redefines the set of intervals by joining the two that touched. Then one restarts the process and keeps repeating it until all of them are joined into one. The movement stops once this interval is centered at the origin. By the outer regularity of Lebesgue measure the level sets of a measurable function can be approximated by open sets and, since in one dimension this is a countable union of open intervals, one can further approximate the level set by a finite number of open disjoint intervals for which one uses the sliding argument explained above.

As mentioned before, this procedure can be generalized to higher dimensions by considering Steiner symmetrization with respect to a hyperplane. One considers any hyperplane H through the origin and then rearranges the function symmetrically about the hyperplane along each line perpendicular to H , resulting in a function denoted by f^{*H} . For more information

see [36]. In this fashion one obtains a continuous rearrangement $f \rightarrow \mathbf{f}_\tau^H, \tau \in [0, \infty]$, which was studied in detail by Brock [10, 11]. We shall refer to the statements in those papers.

To pass from Steiner symmetrization to the symmetric decreasing rearrangement we consider a sequence of continuous Steiner symmetrizations and chain them with a new continuous parameter à la Bucur–Henrot. Inspired by [12, 20], we proceed as follows. Given a function $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$ there is a sequence $(H_n)_{n \in \mathbb{N}}$ of hyperplanes such that, defining recursively with $f_0 = f$,

$$f_n := f_{n-1}^{*H_n}, \quad n = 1, 2, \dots,$$

we have

$$f_n \rightarrow f^* \quad \text{in } L^p(\mathbb{R}^d) \quad \text{as } n \rightarrow \infty.$$

In fact, it is shown in [45, Theorem 4.3] that this holds for ‘almost every’ (in an appropriate sense) choice of hyperplanes. It is also of interest that this sequence can actually be chosen in a universal fashion (that is, independent of f and p); see [44, Theorem 5.2].

Given f and the sequence $(f_n)_{n \in \mathbb{N}}$ as above, we set for any $n = 0, 1, 2, \dots$

$$\phi_n(\tau) := e^{\frac{\tau-n}{n+1-\tau}} - 1, \quad \tau \in [n, n+1],$$

and define

$$\mathbf{f}_\tau := \mathbf{f}_{n, \phi_n(\tau)}, \tag{36}$$

where the right side denotes Brock’s continuous Steiner symmetrization with respect to the hyperplane H_n with parameter $\phi_n(\tau)$ applied to f_n . As τ runs from n to $n+1$, $\phi_n(\tau)$ runs from 0 to ∞ , so \mathbf{f}_τ is well defined even for $\tau \in \mathbb{N}_0$.

From the properties of Brock’s flow, see, in particular, [11, Lemma 4.1], we obtain the following properties for our flow.

Proposition 23. *Let $d \geq 1$, $1 \leq p < \infty$ and let $0 \leq f \in L^p(\mathbb{R}^d)$. Then, for any $\tau \in [0, \infty]$, the function \mathbf{f}_τ defined by (36) is in $L^p(\mathbb{R}^d)$ and $\|\mathbf{f}_\tau\|_p = \|f\|_p$. Moreover, for any $\tau \in [0, \infty]$ and any sequence $(\tau_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \tau_n = \tau$,*

$$\lim_{n \rightarrow \infty} \|\mathbf{f}_{\tau_n} - \mathbf{f}_\tau\|_p = 0.$$

The following fact is important for us.

Lemma 24. *Let $d \geq 3$ and $0 \leq f \in L^{2^*}(\mathbb{R}^d)$. The function*

$$\tau \mapsto \sup_{u \in \mathcal{M}_1} (\mathbf{f}_\tau, u^{2^*-1})^2$$

with \mathbf{f}_τ defined by (36) is continuous.

Proof. We use the fact, shown in Proposition 23, that

$$\lim_{\tau_1 \rightarrow \tau_2} \|\mathbf{f}_{\tau_1} - \mathbf{f}_{\tau_2}\|_{2^*} = 0.$$

Fix $\varepsilon > 0$. There exists $u_1 \in \mathcal{M}_1$ such that $\sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_1}, u^{2^*-1})| \leq |(\mathbf{f}_{\tau_1}, u_1^{2^*-1})| + \varepsilon$ and hence

$$\begin{aligned} \sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_1}, u^{2^*-1})| - \sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_2}, u^{2^*-1})| &\leq |(\mathbf{f}_{\tau_1}, u_1^{2^*-1})| + \varepsilon - |(\mathbf{f}_{\tau_2}, u_1^{2^*-1})| \\ &\leq |(\mathbf{f}_{\tau_1}, u_1^{2^*-1}) - (\mathbf{f}_{\tau_2}, u_1^{2^*-1})| + \varepsilon, \end{aligned}$$

which by Hölder’s inequality is bounded above by

$$\|\mathbf{f}_{\tau_1} - \mathbf{f}_{\tau_2}\|_{2^*} \|u_1^{2^*-1}\|_q + \varepsilon = \|\mathbf{f}_{\tau_1} - \mathbf{f}_{\tau_2}\|_{2^*} + \varepsilon$$

with $q = \frac{2^*}{2^*-1}$. Hence

$$\limsup_{\tau_2 \rightarrow \tau_1} \left(\sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_1}, u^{2^*-1})| - \sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_2}, u^{2^*-1})| \right) \leq \varepsilon.$$

There exists $u_2 \in \mathcal{M}_1$ such that $\sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_2}, u^{2^*-1})| \leq |(\mathbf{f}_{\tau_2}, u_2^{2^*-1})| + \varepsilon$ and hence

$$\sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_1}, u^{2^*-1})| - \sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_2}, u^{2^*-1})| \geq |(\mathbf{f}_{\tau_1}, u_2^{2^*-1})| - |(\mathbf{f}_{\tau_2}, u_2^{2^*-1})| - \varepsilon,$$

which is greater or equal to

$$- |(\mathbf{f}_{\tau_1}, u_2^{2^*-1}) - (\mathbf{f}_{\tau_2}, u_2^{2^*-1})| - \varepsilon \geq -\|\mathbf{f}_{\tau_1} - \mathbf{f}_{\tau_2}\|_{2^*} - \varepsilon.$$

Hence

$$\liminf_{\tau_2 \rightarrow \tau_1} \left(\sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_1}, u^{2^*-1})| - \sup_{u \in \mathcal{M}_1} |(\mathbf{f}_{\tau_2}, u^{2^*-1})| \right) \geq -\varepsilon.$$

This proves the claimed continuity. \square

We now consider the behavior of the gradient under the rearrangement flow. The following proposition is closely related to [11, Theorems 3.2 and 4.1], but there inhomogeneous Sobolev spaces are considered, which leads to some minor changes. For the sake of simplicity we provide the details.

Proposition 25. *Let $0 \leq f \in \dot{H}^1(\mathbb{R}^d)$. Then \mathbf{f}_τ defined by (36) is in $\dot{H}^1(\mathbb{R}^d)$ and $\tau \mapsto \|\nabla \mathbf{f}_\tau\|_2$ is a non-increasing, right-continuous function.*

Proof. By construction, it suffices to prove these properties for Brock's flow. Since the latter has the semigroup property $(\mathbf{f}_\sigma)_\tau = \mathbf{f}_{\sigma+\tau}$ for all $\sigma, \tau \geq 0$, it suffices to prove monotonicity and right-continuity at $\tau = 0$.

We begin with the proof of monotonicity, which we first prove under the additional assumption that $f \in L^2(\mathbb{R}^d)$. This is shown in [11, Theorem 3.2], but we give an alternative proof. We proceed as in the proof of [36, Lemma 1.17]. Extending [10, Corollary 2] to the sequence of Steiner symmetrizations we find for three non-negative functions f, g, h that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{f}_\tau(x) g_\tau(x-y) h_\tau(y) dx dy \geq \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x) g(x-y) h(y) dx dy.$$

If we choose $g(x-y)$ to be the standard heat kernel, i.e., $g(x-y) = e^{\Delta t}(x-y)$, then $g_\tau(x-y) = g(x-y)$ and hence

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{f}_\tau(x) e^{\Delta t}(x-y) \mathbf{f}_\tau(y) dx dy \geq \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x) e^{\Delta t}(x-y) f(y) dx dy.$$

Since $\|\mathbf{f}_\tau\|_2 = \|f\|_2$ by the equimeasurability of rearrangement,

$$\frac{1}{t} (\|\mathbf{f}_\tau\|_2^2 - (\mathbf{f}_\tau, e^{\Delta t} \mathbf{f}_\tau)) \leq \frac{1}{t} (\|f\|_2^2 - (f, e^{\Delta t} f))$$

and letting $t \rightarrow 0$ yields the first claim under the additional assumption $f \in L^2(\mathbb{R}^d)$.

For general $0 \leq f \in \dot{H}^1(\mathbb{R}^d)$ we apply the above argument to the functions $(f - \epsilon)_+$, $\epsilon > 0$. They belong to $L^2(\mathbb{R}^d)$ since f vanishes at infinity and belongs to $L^{2^*}(\mathbb{R}^d)$. We obtain

$$\|\nabla((f - \epsilon)_\tau)_+\|_2 \leq \|\nabla(f - \epsilon)_+\|_2 \leq \|\nabla f\|_2. \quad (37)$$

We claim that $\mathbf{f}_\tau \in \dot{H}^1(\mathbb{R}^d)$ and $\nabla((f - \epsilon)_\tau)_+ \rightharpoonup \nabla \mathbf{f}_\tau$ in $L^2(\mathbb{R}^d)$ as $\epsilon \rightarrow 0^+$. Once this is shown, the claimed inequality follows from (37) by the weak lower semicontinuity of the L^2 norm.

To prove the claimed weak convergence, note that by (37), $\nabla((f - \epsilon)_+)_\tau$ is bounded in $L^2(\mathbb{R}^d)$ as $\epsilon \rightarrow 0^+$ and therefore has a weak limit point. Let $F \in L^2(\mathbb{R}^d)$ be any such limit point. Since $(f - \epsilon)_+ \rightarrow f$ in $L^{2^*}(\mathbb{R}^d)$, the nonexpansivity of the rearrangement [10, Lemma 3] implies that $((f - \epsilon)_+)_\tau \rightarrow f_\tau$ in $L^{2^*}(\mathbb{R}^d)$. Thus, for any $\Phi \in C_c^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (\nabla \cdot \Phi) f_\tau dx \leftarrow \int_{\mathbb{R}^d} (\nabla \cdot \Phi) ((f - \epsilon)_+)_\tau dx = - \int_{\mathbb{R}^d} \Phi \cdot \nabla ((f - \epsilon)_+)_\tau dx \rightarrow - \int_{\mathbb{R}^d} \Phi \cdot F dx$$

as $\epsilon \rightarrow 0^+$. This proves that f_τ is weakly differentiable with $\nabla f_\tau = F$. In particular, $f_\tau \in \dot{H}^1(\mathbb{R}^d)$ (note that f_τ vanishes at infinity since f does and since these functions are equimeasurable) and the limit point F is unique. This concludes the proof of the first part of the proposition.

Let us now show the right-continuity at $\tau = 0$. It follows from Proposition 23 that $f_\tau \rightarrow f$ in $L^{2^*}(\mathbb{R}^d)$ as $\tau \rightarrow 0^+$. This implies that $\nabla f_\tau \rightarrow \nabla f$ in $L^2(\mathbb{R}^d)$ as $\tau \rightarrow 0^+$. (Indeed, the argument is similar to the one used in the first part of the proof. The family ∇f_τ is bounded in $L^2(\mathbb{R}^d)$ as $\tau \rightarrow 0^+$ and, if F denotes any weak limit point in $L^2(\mathbb{R}^d)$, then the convergence in $L^{2^*}(\mathbb{R}^d)$ and the definition of weak derivatives implies that $F = \nabla f$.) By weak lower semicontinuity, we deduce that

$$\|\nabla f\|_2 \leq \liminf_{\tau \rightarrow 0^+} \|\nabla f_\tau\|_2.$$

This, together with the reverse inequality, which was established in the first part of the proof, proves the claimed right continuity. \square

We note that the proposition remains valid for $0 \leq f \in \dot{W}^{1,p}(\mathbb{R}^d)$ with $1 \leq p < d$. If $p \neq 2$, the monotonicity for the gradient for $f \in W^{1,p}(\mathbb{R}^d)$ is proved in [11, Theorem 3.2]. The remaining arguments above carry over to $p \neq 2$.

APPENDIX B. NUMERICAL VALUES

Although rather small and not algebraically computable, the estimate κ_d in Proposition 19 and Corollary 22 has a finite value for any $d \geq 3$. Numerically, the value is found by inverting (3) and optimizing $\nu \mapsto \nu^2 (1 + \nu^2)^{-1} m(\nu)$ on $(0, +\infty)$. This function takes the value 0 at $\nu = 0$, is increasing on a neighbourhood of $\nu = 0^+$, and has a unique positive maximum point. For our values, see Figure 1 and Table 1.

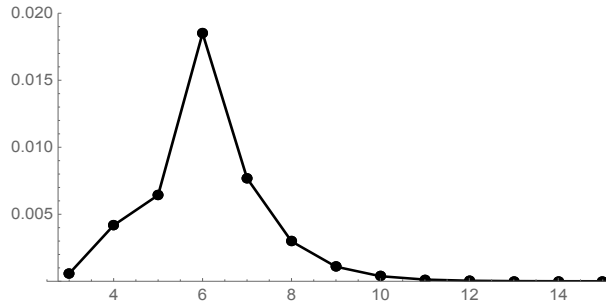


FIGURE 1. Plot of $d \mapsto \kappa_d(d)$ for $d = 3, 4, \dots, 15$.

d	3	4	5	6	7	8	15
κ_d	5.7142×10^{-4}	4.1779×10^{-3}	6.4259×10^{-3}	1.8507×10^{-2}	7.6876×10^{-3}	2.9776×10^{-3}	1.1213×10^{-6}
κ_d^∞	0.28797	0.125	0.052675	2.1333×10^{-2}	8.2845×10^{-3}	3.0864×10^{-3}	1.1213×10^{-6}

TABLE 1. Numerical values of the constant κ_d in Proposition 19 and Corollary 22 and of its upper bound κ_d^∞ , for some values of d . See (5) and (4) for the definitions of κ_d and κ_d^∞ .

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