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To cite this version:

Jean Dolbeault, Maria J. Esteban, Rupert L Frank, Michael Loss. A remark on an inequality by Bianchi and Egnell. 2022. hal-03780031v1

HAL Id: hal-03780031

https://hal.science/hal-03780031v1

Preprint submitted on 18 Sep 2022 (v1), last revised 21 Jul 2023 (v4)

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A REMARK ON AN INEQUALITY BY BIANCHI AND EGNELL

JEAN DOLBEAULT, MARIA J. ESTEBAN, RUPERT L. FRANK, AND MICHAEL LOSS

Abstract. A quantitative version of the Bianchi–Egnell inequality concerning the stability of the Sobolev inequality is proved for non-negative functions. For the proof we study a flow that interpolates continuously between a function and its symmetric decreasing rearrangement.

1. introduction and main result

In [9] Brezis and Lieb posed the question whether it is possible to bound the 'Sobolev deficit'

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

from below in terms of some natural distance from the manifold of optimizers. Here $d \geq 3$, $2^* = 2\,d/(d-2)$ is the 'Sobolev exponent', and

$$S_d = \tfrac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{2/d}$$

is the sharp Sobolev constant. The function f belongs to the homogeneous Sobolev space $\dot{\mathrm{H}}^1(\mathbb{R}^d)$, that is, it is in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, its distributional gradient is a square-summable function and it vanishes at infinity in the sense that $|\{x \in \mathbb{R}^d : |f(x)| > \epsilon\}| < \infty$ for all $\epsilon > 0$. Here $|A|$ denotes the Lebesgue measure of a measurable set A. Throughout this paper we deal with real-valued functions. With minor additional effort our arguments can be extended to the case of complex-valued functions.

Rodemich [39], Aubin [3] and Talenti [43] (see also [41]) proved that the Sobolev deficit is non-negative. Moreover, it was shown by Lieb [35], Gidas, Ni and Nirenberg [34] and Caffarelli, Gidas and Spruck [14] that the deficit vanishes if and only if the function f is of the form

$$f(x) = c\,\big(a + |x-b|^2\big)^{-\frac{d-2}{2}}\ , \tag{1}$$

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Date: September 18, 2022. *File:* stability.tex.
2020 *Mathematics Subject Classification.* Primary: 49J40; Secondary: 26D10, 58E35.
Key words and phrases. Sobolev inequality, stability, rearrangement, Steiner symmetrization.
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Acknowledgements. Partial support through US National Science Foundation grants DMS-1954995 (R.L.F.) and DMS-2154340 (M.L.), as well as through the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Germanys Excellence Strategy EXC-2111-390814868 (R.L.F.) and the French National Research Agency (ANR) projects EFI (ANR-17-CE40-0030, J.D.) and molQED (ANR-17-CE29-0004, M.J.E.) is acknowledged.

where $a \in (0,\infty)$, $b \in \mathbb{R}^d$ and $c \in \mathbb{C}$ are constants. These functions are often called 'Aubin–Talenti functions'. Let $\mathcal{M}$ denote the $(d+2)$-dimensional manifold of functions of the form (1).

The question of Brezis and Lieb was answered by Bianchi and Egnell [4]: there is a strictly positive constant $c_{\rm BE}$ such that for any $f \in \dot{\rm H}^1(\mathbb{R}^d) \setminus \mathcal{M}$

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{{\rm L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{{\rm L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{{\rm L}^2(\mathbb{R}^d)}} \ge c_{\rm BE}\,. \tag{2}$$

Lions [37] has shown that, if the Sobolev deficit is small for some function f, then f has to be close to the manifold $\mathcal{M}$ of Sobolev optimizers. The closeness is in the strongest possible sense, namely with respect to the norm in $\dot{H}^1(\mathbb{R}^d)$. The Bianchi–Egnell inequality (2) makes the qualitative result of Lions quantitative. In particular, it shows that the distance to the manifold vanishes quadratically in the Sobolev deficit. Such 'stability' estimates have been established in other contexts as well, e.g., for the isoperimetric inequality or for classical inequalities in real and harmonic analysis. In fact, stability has attracted a lot of attention in recent years and we refer to [33, 21, 28, 22, 18, 16, 19, 27, 20, 42, 31, 32, 29, 7] for an incomplete list of works in this direction. In several of them the strategy of Bianchi and Egnell or its generalizations play an important role.

An interesting point about (2) and other inequalities obtained by this method is that nothing seems to be known about the optimal value of the constant $c_{\rm BE}$ except for the fact that it is strictly positive. The proof in [4] proceeds by a spectral estimate combined with a compactness argument and hence cannot give any information about $c_{\rm BE}$. Explicit quantitative estimates are known only for a distance to $\mathcal{M}$ measured by a weaker norm than (2), functions of $\dot{\rm H}^1(\mathbb{R}^d)$ satisfying additional constraints or superquadratic estimates of the distance which degenerate in a neighbourhood of $\mathcal{M}$ and much more is known for subcritical interpolation inequalities than for Sobolev-type inequalities: see [6, 26, 25, 24, 7, 30] for some references.

It is the aim of this note to address the question of proving (2) with an explicit lower bound on $c_{\rm BE}$ for any *non-negative function* $f \in \dot{\rm H}^1(\mathbb{R}^d)\setminus\mathcal{M}$. We did not succeed in proving a bound for sign-changing functions f and, in particular, we did not succeed in proving the natural conjecture that $c_{\rm BE}$ is the infimum over non-negative functions in $\dot{\rm H}^1(\mathbb{R}^d)\setminus\mathcal{M}$. Whether this is true or not is an open problem. We shall assume, henceforth, that we consider the functional $\mathcal{E}(f)$ for non-negative functions only.

Let us introduce the notation

$$\nu(\delta) := \sqrt{\frac{\delta}{1-\delta}} \tag{3}$$

for any $\delta \in (0,1)$ and assume that $q = 2^*$ denotes the Sobolev exponent. Here is our theorem.

Theorem 1.1. *Let $d \ge 3$, $q = 2\,d/(d-2)$ and $f \in \dot{\rm H}^1(\mathbb{R}^d)\setminus\mathcal{M}$ be any non-negative function. Then $\mathcal{E}(f)$ in (2) is bounded below with a bound given by*

$$\mathcal{E}(f) \ge \sup_{0<\delta<1} \delta\,\mu(\delta)$$

where $\mu(\delta) \ge \mathsf{m}\big(\nu(\delta)\big)$ with $\delta \mapsto \nu(\delta)$ as in (3) and m explicitly defined by

$$\begin{aligned} \mathsf{m}(\nu) &:= \tfrac{4}{d+4} - \tfrac{2}{q}\,\nu^{q-2} && \text{if} \quad d \ge 6\,,\\ \mathsf{m}(\nu) &:= \tfrac{4}{d+4} - \tfrac{1}{3}\,(q-1)\,(q-2)\,\nu - \tfrac{2}{q}\,\nu^{q-2} && \text{if} \quad d = 4,5\,,\\ \mathsf{m}(\nu) &:= \tfrac{4}{7} - \tfrac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \tfrac{1}{3}\,\nu^4 && \text{if} \quad d = 3\,. \end{aligned} \tag{4}$$

We refer to Appendix B for considerations on the numerical values of our lower bound on $\mathcal{E}(f)$. We also note the *upper bound*

$$\inf_{0\le f\in \dot H^1(\mathbb{R}^d)\setminus\mathcal{M}} \mathcal{E}(f) \le \frac{4}{d+4}\,; \tag{5}$$

see (29).

Apart from the extension to sign-changing functions, another question that remains open is what the optimal value of the constant $c_{\rm BE}$ in (2) is and whether it is attained for an extremal function. In particular, is it only attained in the limit of functions approaching the manifold $\mathcal{M}$ and therefore given by a spectral gap? This would correspond to equality in (5). This is *not* the case for the corresponding question in the case of the planar isoperimetric inequality, where the constant is strictly smaller than the constant in the corresponding spectral gap inequality and where one can prove the existence of an optimizing function; see [5]. For further studies under an additional convexity assumption, see [15, 2, 23].

Let us describe the strategy of the proof of Theorem 1.1. Superficially, it is analogous to that by Bianchi and Egnell [4] in the sense that one splits the problem into two regions, one where f is close to the manifold of Sobolev optimizers and the other where it is far away. These regions are defined in terms of the quantity $\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}/\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, specifically by requiring that this quantity is either less or equal than δ, or (strictly) bigger than δ. Here $\delta>0$ is a free parameter that we will optimize over at the end. Note that, since $\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we may always assume that $\delta\le 1$ and even $\delta<1$.

In Section 3, for $\delta\in(0,1)$ small enough, we estimate $\mu(\delta)$ such that

$$\mathcal{E}(f)\ge \mu(\delta) \tag{6}$$

whenever

$$0<\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \tag{7}$$

and prove that $\mu(\delta)$ is positive. The argument in this regime is similar to that of Bianchi and Egnell and is based on a spectral gap inequality. We make the qualitative expansions of [4] quantitative and obtain for all $\delta\in(0,1)$ a remainder bound smaller than an explicit, strictly positive constant.

In Section 2, we obtain a lower bound on $\mathcal{E}(f)$ in case $\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. To handle this regime we use an idea taken from a paper by Christ [20] in which he establishes a quantitative error term for the Riesz rearrangement inequality. The idea, in a rough outline, is to construct a continuous family of rearrangements f_τ, $0\le\tau<\infty$, such that $f_0=f$, $\|f_\tau\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, $\tau\mapsto\|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)}$ is non-increasing and $\inf_{g\in\mathcal{M}}\|\nabla(f_\tau-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\to 0$ as $\tau\to\infty$. Clearly

$$\mathcal{E}(f)\ge \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} = 1-S_d\,\frac{\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \ge \frac{\|\nabla f_\tau\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-S_d\,\|f_\tau\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f_\tau\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,.$$

Starting with $\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, one would like to run the flow until at a certain point τ_0 one has

$$\inf_{g\in\mathcal{M}}\|\nabla(f_{\tau_0}-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = \delta\,\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \tag{8}$$

and, using (6), one would conclude that

$$\mathcal{E}(f) \ge \frac{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f_{\tau_0}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} = \delta\,\frac{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f_{\tau_0}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla(f_{\tau_0}-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \ge \delta\,\mu(\delta)\,.$$

The details of this argument are more involved than presented here, mostly because the function $\tau \mapsto \|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)}$ need not be continuous, so the existence of a τ_0 as in (8) is not guaranteed. The details will be given in Section 2.

Additional results on the symmetric decreasing rearrangement and numerical values of the constant κ in Theorem 1.1 are given in two appendices, A and B at the end of the paper. In order to make notations lighter, we will write $\|\cdot\|_q = \|\cdot\|_{\mathrm{L}^q(\mathbb{R}^d)}$ whenever the space is $\mathbb{R}^d$ with Lebesgue's measure.

2. Competing symmetries, the sequence and the flow

In a first step one uses 'competing symmetries' to move the initial function f close, but not exactly to the desired location (7). This is done by building a sequence $(f_n)_{n\in\mathbb{N}}$ and considering in Lemma 2.4 two alternatives, (a) and (b). In case (a), the whole sequence stays outside the neighbourhood (7) as well as its limit: this is anyway enough to prove the result of Theorem 1.1. In case (b), in a further step one uses a continuous rearrangement (flow) to achieve (8) or, actually, a substitute for it.

2.1. **Competing symmetries.** The functional $\mathcal{E}(f)$ is conformally invariant in the sense that if $C:\mathbb{R}^d\cup\{\infty\}\to\mathbb{R}^d\cup\{\infty\}$ is a conformal map, the function

$$f_C(x) = |\det DC(x)|^{1/2^*} f\big(C(x)\big)$$

satisfies

$$\mathcal{E}(f_C) = \mathcal{E}(f)\,.$$

In order to verify this, we recall that any conformal map is a composition of scalings, translations, rotations and inversions. For scalings, translations and rotations in $\mathbb{R}^d$ the claimed invariance is easy to see. The additional map to consider is the inversion $I(x) = \frac{x}{|x|^2}$ and a straightforward change of variables shows that

$$\|\nabla f_I\|_2^2 = \|\nabla f\|_2^2\,,\quad \|f_I\|_{2^*}^2 = \|f\|_{2^*}^2\,.$$

The equality

$$\inf_{g\in\mathcal{M}}\|\nabla(f_I-g)\|_2^2 = \inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2$$

follows from

$$\inf_{g\in\mathcal{M}}\|\nabla(f_I-g)\|_2^2 = \inf_{g\in\mathcal{M}}\|\nabla(f-g_I)\|_2^2 = \inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2$$

since $I^2=I$ and $g\to g_I$ maps the set $\mathcal{M}$ to itself in a one-to-one and onto fashion.

Another and perhaps easier way to see the conformal invariance is to pull the problem up to the sphere via the stereographic projection. We denote by $s=(s_1,s_2,\dots,s_{d+1})$ the coordinates in $\mathbb{R}^{d+1}$. Then the unit sphere $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ can be parametrized in terms of stereographic coordinates by

$$s_j = \frac{2\,x_j}{1+|x|^2}\,,\quad j=1,\dots,d\,,\quad s_{d+1} = \frac{1-|x|^2}{1+|x|^2}\,.$$

We set

$$F(s) = \left(\frac{1+|x|^2}{2}\right)^{\frac{d-2}{2}} f(x) \tag{9}$$

and find by a straightforward computation that

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \|f\|_{2^*}^2}{\inf_{g\in\mathscr{M}} \|\nabla f - \nabla g\|_2^2} = \frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\, d\,(d-2)\, \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - S_d \|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}}{\inf_{G\in\mathscr{M}} \left\{ \|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\, d\,(d-2)\, \|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right\}},$$

where G is a function of the form

$$G(s) = c\,\big(a + b\cdot s\big)^{-\frac{d-2}{2}},$$

and $a > 0$, $b \in \mathbb{R}^d$ and $c \in \mathbb{C}$ are constants, and $\mathscr{M}$ is the corresponding set of functions. On the sphere the inversion I takes the form of the reflection $(s_1, \dots, s_d, s_{d+1}) \to (s_1, \dots, s_d, -s_{d+1})$.

A second ingredient for the construction of the flow is the technique of 'competing symmetries', invented in [17]. Consider any non-negative function $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and its counterpart $F \in \mathrm{H}^1(\mathbb{S}^d)$ given by (9). Set

$$(UF)(s) = F(s_1, s_2, \dots, s_{d+1}, -s_d)\,,$$

which corresponds to a rotation by $\pi/2$ that maps the 'north pole' axis $(0, 0, \dots, 1)$ to $(0, \dots, 1, 0)$. Reversing (9) the function that corresponds to UF on $\mathbb{R}^d$ is given by

$$(Uf)(x) = \left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2}, \dots, \frac{x_{d-1}}{|x-e_d|^2}, \frac{|x|^2-1}{|x-e_d|^2}\right), \tag{10}$$

where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. It follows that

$$\mathcal{E}(Uf) = \mathcal{E}(f)\,.$$

The operation U is obviously linear, invertible and an isometry on $\mathrm{L}^{2^*}(\mathbb{R}^d)$.

We also consider the symmetric decreasing rearrangement

$$\mathcal{R}f(x) = f^*(x)\,.$$

The most important properties are that f and f^* are equimeasurable and that $\|\nabla f^*\|_2 \le \|\nabla f\|_2$. For elementary properties of rearrangements the reader may consult [36]. Being equimeasurable, this map is also an isometry on $\mathrm{L}^{2^*}(\mathbb{R}^d)$. It is when using the decreasing rearrangement that we use the fact that f is a non-negative function. For functions that change sign one conventionally defines their rearrangement as the rearrangement of their absolute value. Passing from a function to its absolute value does not alter the numerator of $\mathcal{E}(f)$, but we have not been able to quantify its influence on the denominator.

On $\mathbb{R}^d$, let

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}. \tag{11}$$

Note that $\|g_*\|_{2^*} = 1$ because it is obtained as the stereographic projection of the constant function on $\mathbb{S}^d$ with 2^*-norm equal to 1. The following theorem was proved in [17].

Theorem 2.1. *Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions*

$$f_n = (\mathcal{R}U)^n f \quad \forall\, n \in \mathbb{N}\,. \tag{12}$$

Then

$$\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$$

where $h_f = \|f\|_{2^*}\, g_* \in \mathcal{M}$. Moreover, if $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a non-increasing sequence.

It does not seem clear whether the functional $\mathcal{E}(f)$ decreases or increases under rearrangement. The next lemma helps to explain this point. Define $\mathcal{M}_1$ to be the set of the elements in $\mathcal{M}$ with 2^*-norm equal to 1.

Lemma 2.2. *For any $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, we have*

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(f, g^{2^*-1}\right)^2 .$$

Here $(\cdot,\cdot)$ is the $\mathrm{L}^2(\mathbb{R}^d)$ inner product or, more precisely, the duality pairing between $\mathrm{L}^{2^*}(\mathbb{R}^d)$ and $\mathrm{L}^{(2^*)'}(\mathbb{R}^d)$.

Proof. Let g be any Aubin–Talenti function with $\|g\|_{2^*} = 1$. The function g is an optimizer of the Sobolev inequality, i.e., $\|\nabla g\|_2^2 = S_d\, \|g\|_{2^*}^2 = S_d$ and is a solution of the Sobolev equation

$$-\Delta g = S_d\, \frac{g^{2^*-1}}{\|g\|_{2^*}^{2^*-2}} = S_d\, g^{2^*-1}\,.$$

Hence for any non-negative constant c we find

$$\|\nabla(f - c\,g)\|_2^2 = \|\nabla f\|_2^2 - 2\,c\,(\nabla f, \nabla g) + c^2\, \|\nabla g\|_2 = \|\nabla f\|_2^2 - 2\,c\,S_d\, \left(f, g^{2^*-1}\right) + S_d\, c^2$$

and minimizing with respect to c we find the lower bound $\|\nabla f\|_2^2 - S_d\, \left(f, g^{2^*-1}\right)^2$, which proves the lemma. □

Under the decreasing rearrangement, the term $\|\nabla f\|_2^2$ does not increase whereas the term $\sup_{g\in\mathcal{M}_1} \left(f, g^{2^*-1}\right)^2$ increases. To see this, note that the supremum is attained at some Aubin–Talenti function of the form (1) that is a strictly symmetric function about the point $b \in \mathbb{R}^d$. Replacing f by its symmetric decreasing rearrangement about that point increases $\left(f, g^{2^*-1}\right)^2$, in fact strictly unless f is already symmetric decreasing about the point b. Thus, while the numerator in $\mathcal{E}(f)$ decreases under rearrangements so does the denominator and there are no direct conclusions to be drawn from this. The next lemma summarizes what we have shown.

Lemma 2.3. *For the sequence $(f_n)_{n\in\mathbb{N}}$ in Theorem 2.1 we have that $n \mapsto \sup_{g\in\mathcal{M}_1} \left(f_n, g^{2^*-1}\right)^2$ is strictly increasing, $\lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_{2^*}^2$ is strictly decreasing and*

$$\lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|h_f\|_{2^*}^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|f\|_{2^*}^2\,.$$

Proof. From

$$\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \|\nabla f_n\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(f_n, g^{2^*-1}\right)^2$$

we see that the first term converges since $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a non-increasing sequence. For the second term, which is strictly increasing, we have by Hölder's inequality

$$\sup_{g\in\mathcal{M}_1}\left(f_n, g^{2^*-1}\right)^2 \le \|f_n\|_{2^*}^2 = \|f\|_{2^*}^2$$

and since g_* as defined in (11) is in $\mathcal{M}_1$ we have

$$\liminf_{n\to\infty}\sup_{g\in\mathcal{M}_1}\left(f_n, g^{2^*-1}\right)^2 \ge \liminf_{n\to\infty}(f_n, g_*)^2 = \|f\|_{2^*}^2$$

by Theorem 2.1. □

Lemma 2.4. *Assume that* $0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\setminus\mathcal{M}$ *satisfies*

$$\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2 \ge \delta\,\|\nabla f\|_2^2$$

and let $(f_n)_{n\in\mathbb{N}}$ *be the sequence defined by* (12)*. Then one of the following alternatives holds:*

(a) *for all* $n = 0, 1, 2\dots$ *we have*

$$\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|_2^2 \ge \delta\,\|\nabla f_n\|_2^2$$

(b) *there is a natural number* n_0 *such that*

$$\inf_{g\in\mathcal{M}}\|\nabla f_{n_0}-\nabla g\|_2^2 \ge \delta\,\|\nabla f_{n_0}\|_2^2$$

and

$$\inf_{g\in\mathcal{M}}\|\nabla f_{n_0+1}-\nabla g\|_2^2 < \delta\,\|\nabla f_{n_0+1}\|_2^2\,.$$

Proof. Assume that alternative (a) does not hold. Then there is a largest value $n_0 \ge 0$ such that $\inf_{g\in\mathcal{M}}\|\nabla f_{n_0}-\nabla g\|_2^2 \ge \delta\,\|\nabla f_{n_0}\|_2^2$. □

Lemma 2.5. *Assume that* $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\setminus\mathcal{M}$ *satisfies*

$$\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2 \ge \delta\,\|\nabla f\|_2^2$$

and suppose that in Lemma 2.4 alternative (a) *holds for the sequence* $(f_n)_{n\in\mathbb{N}}$ *defined by* (12)*. Then*

$$\mathcal{E}(f) \ge \delta\,.$$

Proof. We have

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f_n\|_2^2}\,, \tag{13}$$

where the second inequality is a consequence of $\|\nabla f_n\|_2^2 \le \|\nabla f\|_2^2$ for all $n = 0, 1, 2,\dots$ proved in Theorem 2.1. By the assumption that alternative (a) holds and by Lemma 2.3, we learn that

$$\lim_{n\to\infty}\|\nabla f_n\|_2^2 \le \frac{1}{\delta}\lim_{n\to\infty}\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|_2^2 = \frac{1}{\delta}\left(\lim_{n\to\infty}\|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2\right)\,.$$

Since

$$\lim_{n\to\infty}\|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2 \ge \delta\lim_{n\to\infty}\|\nabla f_n\|_2^2 \ge \delta\,S_d\lim_{n\to\infty}\|f_n\|_{2^*}^2 = \delta\,S_d\,\|f\|_{2^*}^2 > 0\,,$$

we can take the limit as $n\to\infty$ on the right side of (13) and compute the limit of the quotient as the quotient of the limits. This proves the lemma. □

2.2. **Continuous rearrangement.** Next, we analyze the case where the alternative (b) in Lemma 2.4 holds. Let us introduce

$$\mu(\delta) := \inf \left\{ \mathcal{E}(f) : 0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M} \,, \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \le \delta \, \|\nabla f\|_2^2 \right\} . \tag{14}$$

Lemma 2.6. *For any $\delta \in (0,1]$, we have $\mu(\delta) \le 1$.*

Proof. By Lemma 2.2, we have

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g \in \mathcal{M}_1} \left(f, g^{2^*-1}\right)^2$$

and it follows from Hölder's inequality that

$$\sup_{g \in \mathcal{M}_1} \left(f, g^{2^*-1}\right)^2 \le \|f\|_{2^*}^2 \,.$$

Thus, the denominator in $\mathcal{E}(f)$ that enters the definition of $\mu(\delta)$ is at least as large as the numerator, so the quotient is at most 1. □

Our goal in this subsection is to prove the following lower bound.

Lemma 2.7. *Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$ satisfies*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2 \ge \delta \, \|\nabla f\|_2^2$$

for some $\delta \in (0,1)$ and suppose that in Lemma 2.4 alternative (b) *holds for the sequence $(f_n)_{n \in \mathbb{N}}$ of Theorem 2.1 defined by* (12)*. Then, with $\mu(\delta)$ defined by* (14)*, we have*

$$\mathcal{E}(f) \ge \delta \, \mu(\delta) \,.$$

For the proof of this lemma we introduce a continuous rearrangement flow, which interpolates between a function and its symmetric decreasing rearrangement. The basic ingredient for this flow is similar to a flow that Brock introduced [10, 11] and which interpolates between a function and its Steiner symmetrization with respect to a given hyperplane. Brock's construction, in turn, is based on ideas of Rogers [40] and Brascamp–Lieb–Luttinger [8]. Our flow is obtained by glueing together infinitely many copies of Brock's flows with respect to a sequence of judiciously chosen hyperplanes. A similar construction was performed by Bucur and Henrot [12]; see also [20].

More specifically, for a given hyperplane H, Brock's flow interpolates between a given function f and f^{*H}, the Steiner symmetrized function with respect to H. The family that interpolates between f and f^{*H} is denoted by $f_\tau^H, \tau \in [0,\infty]$, and we have

$$f_0 = f \,, \quad f_\infty^H = f^{*H} \,.$$

Further, for any τ, f_τ^H and f are equimeasurable, i.e.,

$$\left|\left\{x \in \mathbb{R}^d : f_\tau^H(x) > t\right\}\right| = \left|\left\{x \in \mathbb{R}^d : f(x) > t\right\}\right| \quad \forall\, t > 0 \,.$$

Moreover, if $f \in \mathrm{L}^p(\mathbb{R}^d)$ for some $1 \le p < \infty$, then $\tau \mapsto f_\tau^H$ is continuous in $\mathrm{L}^p(\mathbb{R}^d)$.

By choosing a sequence of hyperplanes we construct another flow $\tau \mapsto f_\tau$ that has the same properties but interpolates between f and f^*, the symmetric decreasing rearrangement. In Appendix A we explain this in more detail and prove the following properties that are important for our proof, assuming $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$. From the $\mathrm{L}^{2^*}(\mathbb{R}^d)$ continuity of the flow we will deduce that

$$\lim_{\tau \to \tau_0} \sup_{g \in \mathcal{M}_1} (f_\tau, g)^2 = \sup_{g \in \mathcal{M}_1} (f_{\tau_0}, g)^2 \,. \tag{15}$$

Concerning the gradient we prove the monotonicity

$$\|\nabla f_{\tau_2}\|_2 \leq \|\nabla f_{\tau_1}\|_2 \,, \quad 0 \leq \tau_1 \leq \tau_2 \leq \infty \,, \tag{16}$$

and the right continuity

$$\lim_{\tau_2 \to \tau_1^+} \|\nabla f_{\tau_2}\|_2 = \|\nabla f_{\tau_1}\|_2 \,, \quad 0 \leq \tau_1 < \infty \,. \tag{17}$$

Proof of Lemma 2.7. We begin by motivating and explaining the strategy of the proof. As before, we bound

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2} \geq \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \geq \frac{\|\nabla f_{n_0}\|_2^2 - S_d \, \|f_{n_0}\|_{2^*}^2}{\|\nabla f_{n_0}\|_2^2} \,. \tag{18}$$

We could bound the right side further from below by replacing f_{n_0} by f_{n_0+1}. This bound, however, might be too crude for our purposes and we proceed differently. The move from f_{n_0} to f_{n_0+1} consists of two steps, namely first applying a conformal rotation and second applying symmetric decreasing rearrangement. The first step leaves all terms on the right side invariant and we do carry out this step. The second step leaves the 2^*-norm invariant, while the gradient term does not go up. In fact, the gradient term might go down too far. Therefore, we replace the application of the rearrangement by a continuous rearrangement flow. In order to make the notation less cumbersome we shall denote $U f_{n_0}$ by f_0 where U denotes the conformal rotation (10). We denote by f_τ, $0 \leq \tau \leq \infty$, the continuous rearrangement starting at f_0 and let

$$\mathsf{f}_\infty = f_{n_0+1} \,. \tag{19}$$

Ideally, we would like to find $\tau_0 \in [0, \infty)$ such that

$$\inf_{g \in \mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 = \delta \, \|\nabla \mathsf{f}_{\tau_0}\|_2^2 \,.$$

Then the right side of (18) is equal to

$$1 - S_d \, \frac{\|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \geq 1 - S_d \, \frac{\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2} = \delta \, \frac{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d \, \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2} \,,$$

which can be bounded from below by $\delta \, \mu(\delta)$, since f_{τ_0} is admissible in the infimum (14). This would prove the desired bound.

The problem with this argument is that the existence of such a $\tau_0 \in [0, \infty)$ is in general not clear, since neither of the terms $\inf_{g \in \mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2$ and $\|\nabla \mathsf{f}_\tau\|_2^2$ needs to be continuous in τ. Nevertheless, we will be able to adapt the above argument to yield the same conclusion.

We now turn to the details of the argument. Recalling that

$$\inf_{g \in \mathcal{M}} \|\nabla \mathsf{f}_0 - \nabla g\|_2^2 \geq \delta \, \|\nabla \mathsf{f}_0\|_2^2 \,,$$

we define

$$\tau_0 := \inf \left\{ \tau \geq 0 : \inf_{g \in \mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2 < \delta \, \|\nabla \mathsf{f}_\tau\|_2^2 \right\}$$

with the convention that $\inf \emptyset = \infty$. If $\tau < \tau_0 \in (0, \infty]$, similarly as before, the right side of (18) is equal to

$$\frac{\|\nabla \mathsf{f}_0\|_2^2 - S_d \, \|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} = 1 - S_d \, \frac{\|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \geq \frac{\|\nabla \mathsf{f}_\tau\|_2^2 - S_d \, \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_\tau\|_2^2} \geq \delta \, \frac{\|\nabla \mathsf{f}_\tau\|_2^2 - S_d \, \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g \in \mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2} \,,$$

where the last inequality arises from $\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2 \ge \delta \|\nabla \mathsf{f}_\tau\|_2^2$ for any $\tau \in [0,\tau_0)$. Taking the limit inferior as $\tau \to \tau_0^-$, we obtain

$$\frac{\|\nabla \mathsf{f}_0\|_2^2 - S_d \|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \ge \delta \, \frac{\lim_{\tau\to\tau_0^-} \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\liminf_{\tau\to\tau_0^-} \inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2} \,. \tag{20}$$

Note that the denominator appearing here does not vanish. Indeed, we have

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2 \ge \delta \|\nabla \mathsf{f}_\tau\|_2^2 \ge \delta \, S_d \|\mathsf{f}_\tau\|_{2^*}^2 = \delta \, S_d \|f\|_{2^*}^2 > 0 \quad \forall \, \tau \in [0,\tau_0)$$

and, as a consequence,

$$\liminf_{\tau\to\tau_0^-} \inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2 \ge \delta \, S_d \|f\|_{2^*}^2 > 0 \,.$$

The same inequality (20) remains valid if $\tau_0 = 0$ and if we interpret $\lim_{\tau\to\tau_0^-}$ and $\liminf_{\tau\to\tau_0^-}$ as evaluating at $\tau_0 = 0$.

At this point we find it convenient to apply Lemma 2.2 and use the representation

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2 = \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_\tau, g^{2^*-1}\right)^2 \,.$$

Using (15), that is, the continuity of $\tau \mapsto \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_\tau, g^{2^*-1}\right)^2$, we see that

$$\liminf_{\tau\to\tau_0^-} \inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\tau - \nabla g\|_2^2 = \lim_{\tau\to\tau_0^-} \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2 \,.$$

Thus, the relevant quotient is equal to

$$\frac{\lim_{\tau\to\tau_0^-} \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\lim_{\tau\to\tau_0^-} \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2} \,. \tag{21}$$

Our goal in the remainder of this proof is to show that this quotient is larger or equal than $\mu(\delta)$. We will use the fact that

$$\sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2 \le \|f_{\tau_0}\|_{2^*}^2 \,, \tag{22}$$

which follows from Hölder's inequality. We also note that equality holds here if and only if $\mathsf{f}_{\tau_0} \in \mathcal{M}$.

Let us first handle the case where $\mathsf{f}_{\tau_0} \in \mathcal{M}$. Then by (2.2) and because of equality in (22), the quotient (21) is equal to 1, which by Lemma 2.6 can be further bounded from below by $\mu(\delta)$, leading to the claimed bound. This completes the proof in the case $\mathsf{f}_{\tau_0} \in \mathcal{M}$ and in what follows we assume

$$\mathsf{f}_{\tau_0} \notin \mathcal{M} \,.$$

As a consequence of this assumption and (22), we have

$$\|\nabla \mathsf{f}_{\tau_0}\|_2^2 > S_d \|\mathsf{f}_{\tau_0}\|_{2^*}^2 \ge S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2 \,. \tag{23}$$

Next, we observe that for $\alpha > \beta$ the function $x \mapsto (x-\alpha)/(x-\beta)$ is monotone increasing on the interval (β,∞). This, together with the strict inequality in (23), implies that the quotient (21) can be bounded from below by

$$\frac{\lim_{\tau\to\tau_0^-} \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\lim_{\tau\to\tau_0^-} \|\nabla \mathsf{f}_\tau\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2} \ge \frac{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d \|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2} \,. \tag{24}$$

We now claim that

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 \le \delta \, \|\nabla \mathsf{f}_{\tau_0}\|_2^2 \,. \tag{25}$$

Once this is proved, we can bound the right side of (24) from below by $\mu(\delta)$. This inequality is the claimed inequality after taking into account (20).

To prove (25), we first note that it is verified if $\tau_0 = \infty$. Indeed, $\mathsf{f}_\infty = f_{n_0+1}$ by (19) and therefore, by assumption of alternative (b), $\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_\infty - \nabla g\|_2^2 < \delta \, \|\nabla \mathsf{f}_\infty\|_2^2$.

Now let $\tau_0 < \infty$. We argue by contradiction and assume that

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 > \delta \, \|\nabla \mathsf{f}_{\tau_0}\|_2^2 \,. \tag{26}$$

Because of this strict inequality and the definition of τ_0 there are $\sigma_k \in (\tau_0, \infty)$ for any $k \in \mathbb{N}$ with $\lim_{k\to\infty} \sigma_k = \tau_0$ such that $\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\sigma_k} - \nabla g\|_2^2 < \delta \, \|\nabla \mathsf{f}_{\sigma_k}\|_2^2$, that is,

$$\|\nabla \mathsf{f}_{\sigma_k}\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\sigma_k}, g^{2^*-1}\right)^2 < \delta \, \|\nabla \mathsf{f}_{\sigma_k}\|_2^2 \quad \forall\, k \in \mathbb{N} \,.$$

Letting $k \to \infty$ and using (15) as well as the right continuity of $\|\nabla \mathsf{f}_\tau\|_2^2$, see (17), we deduce that

$$\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(\mathsf{f}_{\tau_0}, g^{2^*-1}\right)^2 \le \delta \, \|\nabla \mathsf{f}_{\tau_0}\|_2^2 \,.$$

This is the same as $\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 \le \delta \, \|\nabla \mathsf{f}_{\tau_0}\|_2^2$ and contradicts (26). This proves (25) and completes the proof of the lemma. □

Remark 2.8. *The above argument would be simpler if $\tau \mapsto \|\nabla \mathsf{f}_\tau\|_2^2$ were continuous for an appropriate choice of hyperplanes (see Appendix A) in the definition of the flow. Since the flow is weakly continuous in $\dot{\mathrm{H}}^1(\mathbb{R}^d)$, continuity of the norm is equivalent to (strong) continuity of the flow in $\dot{\mathrm{H}}^1(\mathbb{R}^d)$. Thus, for continuity of the norm for an appropriate choice of hyperplanes, it is necessary that there is such a choice for which the Steiner symmetrizations approximate f^* in $\dot{\mathrm{H}}^1(\mathbb{R}^d)$. According to a theorem of Burchard* [13] *this holds if and only if f is co-area regular, i.e, if and only if the distribution function*

$$h \mapsto |\{x \in \mathbb{R}^d : f(x) > h,\, \nabla f(x) = 0\}|$$

has no absolutely continuous component. As shown by Almgren and Lieb [1]*, both co-area regular and co-area irregular functions are dense for $d \ge 2$.*

2.3. **Summary.** Let us summarize the result of this section.

Corollary 2.9. *Take $\delta \in (0,1)$ and assume that $0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$ satisfies*

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 \ge \delta \, \|\nabla f\|_2^2 \,.$$

Then, with $\mu(\delta)$ defined by (14)*, we have*

$$\mathcal{E}(f) \ge \delta \, \mu(\delta) \,.$$

Indeed, by Lemma 2.4 either alternative (a) or (b) holds. In first case, we apply Lemmas 2.5 and 2.6, and in the second case, we apply Lemma 2.7. In the next section we will bound $\mu(\delta)$ from below by an explicit function of δ.

3. Analysis close to the manifold of optimizers

3.1. **Expansions with remainder terms.** Our goal in this subsection is to prove the following proposition.

Proposition 3.1. *Let X be a measure space and $u, r \in \mathrm{L}^q(X)$ for some $q \ge 2$ with $u \ge 0$ and $u + r \ge 0$. Assume also that $\int_X u^{q-1}\, r\, dx = 0$.*

- *If $2 \le q \le 3$, then*
$$\|u+r\|_q^2 \le \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2}\, r^2\, dx + \frac{2}{q} \int_X r_+^q\, dx \right).$$
- *If $3 \le q \le 4$, then*
$$\|u+r\|_q^2 \le \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2}\, r^2\, dx + \tfrac{1}{3}\,(q-1)\,(q-2) \int_X u^{q-3}\, r^3\, dx + \tfrac{2}{q} \int_X |r|^q\, dx \right).$$
- *If $q = 6$, then*
$$\|u+r\|_q^2 \le \|u\|_q^2 + \|u\|_q^{2-q} \Big(5 \int_X u^{q-2}\, r^2\, dx + \tfrac{20}{3} \int_X u^{q-3}\, r^3\, dx + 5 \int_X u^{q-4}\, r^4\, dx + 2 \int_X u^{q-5}\, r^5\, dx + \tfrac{1}{3} \int_X r^6\, dx \Big).$$

Similar bounds can also be derived for $q \in (4, \infty) \setminus \{6\}$. They become increasingly more complicated as q passes an integer. We restrict ourselves to the case $q = 6$, which is the only case in $(4, \infty)$ that we need, as it corresponds to the Sobolev exponent in dimension $d = 3$. For the proof of the proposition, we need two lemmas, which we discuss next.

Lemma 3.2. *We have the upper bounds following expansions.*

- *If $2 \le q \le 3$, then, for all $x \ge -1$,*
$$(1+x)^q \le 1 + q\,x + \tfrac{1}{2}\,q\,(q-1)\,x^2 + x_+^q\,.$$
- *If $3 \le q \le 4$, then, for all $x \ge -1$,*
$$(1+x)^q \le 1 + q\,x + \tfrac{1}{2}\,q\,(q-1)\,x^2 + \tfrac{1}{6}\,q\,(q-1)\,(q-2)\,x^3 + |x|^q\,.$$
- *If $q = 6$, then, for all $x \ge -1$,*
$$(1+x)^6 = 1 + 6\,x + 15\,x^2 + 20\,x^3 + 15\,x^4 + 6\,x^5 + x^6\,.$$

Proof. We begin with the case $2 \le q \le 3$ and set
$$f(x) := (1+x)^q - 1 - q\,x - \tfrac{1}{2}\,q\,(q-1)\,x^2 - x_+^q\,.$$
For any $x \ge -1$, we compute
$$f'(x) = q\left((1+x)^{q-1} - 1 - (q-1)\,x - x_+^{q-1} \right),$$
$$f''(x) = q\,(q-1)\left((1+x)^{q-2} - 1 - x_+^{q-2} \right).$$
For $-1 \le x \le 0$ we clearly have $(1+x)^{q-2} - 1 - x_+^{q-2} = (1-|x|)^{q-2} - 1 \le 0$. For $x \ge 0$ we have, by a well-known elementary inequality, $(1+x)^{q-2} - 1 - x_+^{q-2} = (1+x)^{q-2} - 1 - x^{q-2} \le 0$. To summarize, f is concave on $[-1, \infty)$. We conclude that, for all $x \ge -1$,
$$f(x) \le f(0) - f'(0)\,x\,.$$

Since $f(0) = f'(0) = 0$, this is the claimed inequality.

We now turn to the case $3 \le q \le 4$ and set this time

$$f(x) := (1+x)^q - 1 - q\,x - \tfrac{1}{2}\,q\,(q-1)\,x^2 - \tfrac{1}{6}\,q\,(q-1)\,(q-2)\,x^3 - |x|^q\,.$$

Again, we compute

$$\begin{aligned} f'(x) &= q\left((1+x)^{q-1} - 1 - (q-1)\,x - \tfrac{1}{2}\,(q-1)\,(q-2)\,x^2 - |x|^{q-2}\,x\right), \\ f''(x) &= q\,(q-1)\left((1+x)^{q-2} - 1 - (q-2)\,x - |x|^{q-2}\right). \end{aligned}$$

Since again $f(0) = f'(0) = 0$, the claimed inequality will follow if we can show concavity of f on $[-1,\infty)$, that is, $g \le 0$ on $[-1,\infty)$ where

$$g(x) := (1+x)^{q-2} - 1 - (q-2)\,x - |x|^{q-2}\,.$$

We compute

$$\begin{aligned} g'(x) &= (q-2)\left((1+x)^{q-3} - 1 - |x|^{q-4}\,x\right), \\ g''(x) &= (q-2)\,(q-3)\left((1+x)^{q-4} - |x|^{q-4}\right). \end{aligned}$$

We discuss g separately on $[-1,0]$ and on $(0,\infty)$.

- We begin with the second case. For $x > 0$ we have, by the same elementary inequality as before, $(1+x)^{q-3} - 1 - x^{q-3} < 0$. Thus, $g' < 0$ on $(0,\infty)$. Since $g(0) = 0$, we deduce $g < 0$ on $(0,\infty)$.
- Now let us consider the interval $[-1,0]$. We see that $g'' > 0$ on $(-1,-1/2)$ and $g'' < 0$ on $(-1/2,0)$. Therefore g' is increasing on $(-1,-1/2)$ and decreasing on $(-1/2,0)$. Since $g'(-1) = g'(0) = 0$, we conclude that $g' > 0$ on $(-1,0)$ and therefore g is increasing on $(-1,0)$. Since $g(0) = 0$ we conclude that $g < 0$ on $[-1,0)$, as claimed.

If $q = 6$ we simply expand $(1+x)^6$. This completes the proof of the lemma. □

We will also use the following elementary lemma.

Lemma 3.3. *If $q \ge 2$, then, for all $x \ge 0$,*

$$(1+x)^{\frac{2}{q}} \le 1 + \tfrac{2}{q}\,x\,.$$

Proof of Proposition 3.1. We only give the proof in the case $2 \le q \le 3$. We have, by Lemma 3.2, almost everywhere on X,

$$(u+r)^q \le u^q + q\,u^{q-1}\,r + \tfrac{1}{2}\,q\,(q-1)\,u^{q-2}\,r^2 + r_+^q\,.$$

Integrating this and using the assumed orthogonality condition, we obtain

$$\int_X (u+r)^q\,dx \le \int_X u^q\,dx + \tfrac{1}{2}\,q\,(q-1)\int_X u^{q-2}\,r^2\,dx + \int_X r_+^q\,dx\,.$$

Applying Lemma 3.3, we obtain

$$\left(\int_X (u+r)^q\,dx\right)^{\frac{2}{q}} \le \left(\int_X u^q\,dx\right)^{\frac{2}{q}} + \left(\int_X u^q\,dx\right)^{\frac{2-q}{q}} \left((q-1)\int_X u^{q-2}\,r^2\,dx + \tfrac{2}{q}\int_X r_+^q\,dx\right).$$

This is the claimed inequality for $2 \le q \le 3$. The proof in the remaining cases proceeds similarly. □

3.2. **Application to the Sobolev functional.** Throughout this subsection, we assume that $d \geq 3$ and we set

$$q = 2^* = \frac{2\,d}{d-2}\,.$$

We recall that we denote by S_d the optimal constant in the Sobolev inequality $\|\nabla f\|_2^2 \geq S_d\,\|f\|_q^2$ and by $\mathcal{M}$ the set of all optimizers in this inequality.

Proposition 3.4. *Let $q = 2^*$, $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that*

$$\|\nabla f - \nabla u\|_2 = \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2\,.$$

Set $r := f - u$ and $\sigma := \|r\|_q/\|u\|_q$.

- *If $d \geq 6$, we have*

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \int_{\mathbb{R}^d} \Big(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,u^{q-2}\,r^2\Big)\,dx - \tfrac{2}{q}\,\|\nabla r\|_2^2\,\sigma^{q-2}\,.$$

- *If $d = 4$, 5, we have*

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \int_{\mathbb{R}^d} \big(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,u^{q-2}\,r^2\big)\,dx - \|\nabla r\|_2^2\,\Big(\tfrac{1}{3}\,(q-1)\,(q-2)\,\sigma + \tfrac{2}{q}\,\sigma^{q-2}\Big)\,.$$

- *If $d = 3$, we have*

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \int_{\mathbb{R}^d} \big(|\nabla r|^2 - 5\,S_3\,\|u\|_6^{-4}\,u^4\,r^2\big)\,dx - \|\nabla r\|_2^2\,\big(\tfrac{20}{3}\,\sigma + 5\,\sigma^2 + 2\,\sigma^3 + \tfrac{1}{3}\,\sigma^4\big)\,.$$

Proof. This follows directly from Proposition 3.1 in the previous section. We note that the orthogonality conditions

$$(\nabla r, \nabla u) = (\nabla f - \nabla u, \nabla u) = 0 \quad \text{and} \quad (r, u^{q-1}) = (f - u, u^{q-1}) = 0 \tag{27}$$

are satisfied because of the choice of u. Indeed, since $\mathcal{M}$ is closed under multiplication by a scalar, we find that

$$0 = \tfrac{d}{d\alpha}\|\nabla f - \alpha\,\nabla u\|_2^2\,\big|_{\alpha=1} = 2\,(\nabla f, \nabla u) - 2\,\|\nabla u\|_2^2 = 2\,(\nabla r, \nabla u) = 2\,c_u\,(r, u^{q-1})\,,$$

where, in the last equality, we used the equation $-\Delta u = c_u\,u^{q-1}$ with $c_u := \|\nabla u\|_2^2/\|u\|_q^q$.

Finally, we use the Sobolev inequality $S_d\,\|r\|_q^2 \leq \|\nabla r\|_2^2$ for the term multiplying the quantity involving σ. □

Let us recall a spectral gap inequality which appears, for instance, in Rey's paper [38, Appendix D] slightly before the work of Bianchi and Egnell [4].

Lemma 3.5. *Let $d \geq 3$, $q = 2^*$, $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that $\|\nabla f - \nabla u\| = \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|$. Then $r := f - u$ satisfies*

$$\int_{\mathbb{R}^d} \big(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,|u|^{q-2}\,r^2\big)\,dx \geq \frac{4}{d+4}\int_{\mathbb{R}^d} |\nabla r|^2\,dx\,.$$

Proof. By translation and dilation invariance, we may assume that $u(x) = c\,\big(1 + |x|^2\big)^{-(d-2)/2}$ for some constant $c > 0$. Then, by inverse stereographic projection and the discussion in Subsection 2.1, the question becomes to prove the inequality

$$\int_{\mathbb{S}^d} \big(|\nabla R|^2 - d\,|R|^2\big)\,d\omega \geq \frac{4}{d+4}\int_{\mathbb{S}^d} \big(|\nabla R|^2 + \tfrac{1}{4}\,d\,(d-2)\,|R|^2\big)\,d\omega \tag{28}$$

for all R that are orthogonal to spherical harmonics of degrees $\ell \le 1$. Diagonalizing the Laplace–Beltrami operator, the inequality becomes

$$\ell\,(\ell+d-1)-d \ge \frac{4}{d+4}\,\big(\ell\,(\ell+d-1)+\tfrac14\,d\,(d-2)\big) \quad \text{for all}\,\ell\ge 2\,.$$

This is elementary to check. □

Remark 3.6. *If we look at the quadratic form* $r \mapsto \int_{\mathbb{R}^d}\big(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,|u|^{q-2}\,r^2\big)\,dx$, *one may wonder why no essential spectrum should be taken into account. This is indeed an issue (see for instance* [7, Proposition 1.16]*) if the form is defined on* $\mathrm{L}^2(\mathbb{R}^d)$, *but not anymore if we consider the operator* $-\,u^{2-q}\,\Delta - S_d\,(q-1)\,\|u\|_q^{2-q}$ *on* $\mathrm{L}^2(\mathbb{R}^d,\,u^{q-2}\,dx)$ *as the image of* $\mathrm{H}^1(\mathbb{S}^d)$ *through the stereographic projection is compactly embedded in* $\mathrm{L}^2(\mathbb{R}^d,\,u^{q-2}\,dx)$, *so that only discrete spectrum has to be taken into account. With* $r\in\dot{\mathrm{H}}^1(\mathbb{R}^d)\hookrightarrow \mathrm{L}^{2^*}(\mathbb{R}^d)\hookrightarrow \mathrm{L}^2(\mathbb{R}^d,\,u^{q-2}\,dx)$, *the spectral gap computation in the proof of Lemma 3.5 is justified as already noted in* [4, Appendix].

Now we insert the spectral gap inequality in the expansion and obtain the following.

Corollary 3.7. *Let* $q=2^*$ *and* $0\le f\in\dot{\mathrm{H}}^1(\mathbb{R}^d)$. *Set* $\mathcal{D}(f):=\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2$ *and* $\tau:=\mathcal{D}(f)/(\|\nabla f\|_2^2-\mathcal{D}(f)^2)^{1/2}$.

- *If* $d\ge 6$, *we have*
$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \ge \big(\tfrac{4}{d+4}-\tfrac{2}{q}\,\tau^{q-2}\big)\,\mathcal{D}(f)^2\,.$$
- *If* $d=4,\,5$, *we have*
$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \ge \big(\tfrac{4}{d+4}-\tfrac{1}{3}\,(q-1)\,(q-2)\,\tau-\tfrac{2}{q}\,\tau^{q-2}\big)\,\mathcal{D}(f)^2\,.$$
- *If* $d=3$, *we have*
$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \ge \big(\tfrac{4}{7}-\tfrac{20}{3}\,\tau-5\,\tau^2-2\,\tau^3-\tfrac{1}{3}\,\tau^4\big)\,\mathcal{D}(f)^2\,.$$

Proof. Let $u\in\mathcal{M}$ be such that $r=f-u$ satisfies $\|\nabla r\|=\mathcal{D}(f)$. Set $\sigma:=\|r\|_q/\|u\|_q$. Then, by combining Proposition 3.4 and Lemma 3.5, we obtain the following bounds.

- If $d\ge 6$,
$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \ge \big(\tfrac{4}{d+4}-\tfrac{2}{q}\,\sigma^{q-2}\big)\,\mathcal{D}(f)^2\,.$$
- If $d=4,\,5$,
$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \ge \big(\tfrac{4}{d+4}-\tfrac{1}{3}\,(q-1)\,(q-2)\,\sigma-\tfrac{2}{q}\,\sigma^{q-2}\big)\,\mathcal{D}(f)^2\,.$$
- If $d=3$,
$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \ge \big(\tfrac{4}{7}-\tfrac{20}{3}\,\sigma-5\,\sigma^2-2\,\sigma^3-\tfrac{1}{3}\,\sigma^4\big)\,\mathcal{D}(f)^2\,.$$

We slightly weaken these inequalities, but convert them into a more explicit form. Set

$$\rho:=\mathcal{D}(f)/\|\nabla f\|_2\,.$$

We recall that $(\nabla f,\nabla u)=\|\nabla u\|_2^2$ was shown in (27). This implies

$$\rho^2\,\|\nabla f\|_2^2=\mathcal{D}(f)^2=\|\nabla f-\nabla u\|_2^2=\|\nabla f\|_2^2-\|\nabla u\|_2^2\,,$$

that is, $\|\nabla u\|_2=\sqrt{1-\rho^2}\,\|\nabla f\|_2$. As a consequence,

$$\begin{aligned}\sigma=\|f-u\|_q/\|u\|_q &\le \|\nabla f-\nabla u\|_2/\|\nabla u\|_2\\ &= (1-\rho^2)^{-1/2}\,\|\nabla f-\nabla u\|_2/\|\nabla f\|_2=(1-\rho^2)^{-1/2}\,\rho=\tau\,.\end{aligned}$$

Thus we can replace σ by τ in the above bounds and obtain the assertion of the corollary. □

We can reformulate this corollary using the notation $\mathcal{D}(f)$, $\delta \mapsto \nu(\delta)$ as in (3), m explicitly defined by (4) and, according to (14),

$$\mu(\delta) = \inf \left\{ \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_q^2}{\mathcal{D}(f)^2} \, : \, 0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M} \, , \, \mathcal{D}(f)^2 \le \delta \, \|\nabla f\|_2^2 \right\} .$$

Corollary 3.8. *With the above notations, we have* $\mu(\delta) \ge \mathsf{m}\big(\nu(\delta)\big)$.

This follows immediately from the previous corollary if we note that

$$\tau = \frac{\mathcal{D}(f)/\|\nabla f\|}{\sqrt{1 - \mathcal{D}(f)^2/\|\nabla f\|_2^2}} \le \nu(\delta) \, .$$

Combining Corollaries 2.9 and 3.8 completes the proof of Theorem 1.1. □

A standard consequence of the above analysis is the estimate

$$\inf_{0 \le f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}} \mathcal{E}(f) \le \frac{4}{d+4} \, . \tag{29}$$

Indeed, this bound is derived using the fact that the constant $4/(d+4)$ in inequality (28) is optimal and attained if (and only if) R is a spherical harmonic of degree two.

Appendix A. Some remarks about continuous rearrangement

In this appendix we review the continuous rearrangement of Brock and of Bucur–Henrot and prove some of its properties.

Brock's continuous rearrangement is based on the following operation for functions of one real variable that are finite union of disjoint characteristic functions $\sum_{k=1}^N \chi_{(-a_k,a_k)}(x - b_k)$. Replace this function by $\sum_{k=1}^N \chi_{(-a_k,a_k)}\big(x - e^{-t} \, b_k\big)$ where t varies from 0 to ∞. As t increases, the intervals start moving closer and as soon as any two intervals touch one stops the process and redefines the set of intervals by joining the two that touched. Then one restarts the process and keeps repeating it until all of them are joined into one. The movement stops once this interval is centered at the origin. By the outer regularity of Lebesgue measure the level sets of a measurable function can be approximated by open sets and, since in one dimension this is a countable union of open intervals, one can further approximate the level set by a finite number of open disjoint intervals for which one uses the sliding argument explained above.

As mentioned before, this procedure can be generalized to higher dimensions by considering Steiner symmetrization with respect to a hyperplane. One considers any hyperplane H through the origin and then rearranges the function symmetrically about the hyperplane along each line perpendicular to H, resulting in a function denoted by f^{*H}. For more information see [36]. In this fashion one obtains a continuous rearrangement $f \to \mathsf{f}_\tau^H, \tau \in [0, \infty]$, which was studied in detail by Brock [10, 11]. We shall refer to the statements in those papers.

To pass from Steiner symmetrization to the symmetric decreasing rearrangement we consider a sequence of continuous Steiner symmetrizations and chain them with a new continous parameter à la Bucur–Henrot. Inspired by [12, 20], we proceed as follows. Given a function $f \in \mathrm{L}^p(\mathbb{R}^d)$ for some $1 \le p < \infty$ there is a sequence $(H_n)_{n \in \mathbb{N}}$ of hyperplanes such that, defining recursively with $f_0 = f$,

$$f_n := f_{n-1}^{*H_n} \, , \quad n = 1, 2, \dots \, ,$$

we have

$$f_n \to f^* \quad \text{in } \mathrm{L}^p(\mathbb{R}^d) \quad \text{as} \quad n \to \infty\,.$$

In fact, it is shown in [45, Theorem 4.3] that this holds for 'almost every' (in an appropriate sense) choice of hyperplanes. It is also of interest that this sequence can actually be chosen in a universal fashion (that is, independent of f and p); see [44, Theorem 5.2].

Given f and the sequence $(f_n)_{n\in\mathbb{N}}$ as above, we set for any $n = 0, 1, 2, \dots$

$$\phi_n(\tau) := e^{\frac{\tau-n}{n+1-\tau}} - 1\,, \quad \tau \in [n, n+1]\,,$$

and define

$$\mathsf{f}_\tau := \mathsf{f}_{n,\phi_n(\tau)}\,, \tag{30}$$

where the right side denotes Brock's continuous Steiner symmetrization with respect to the hyperplane H_n with parameter $\phi_n(\tau)$ applied to f_n. As τ runs from n to $n+1$, $\phi_n(\tau)$ runs from 0 to ∞, so f_τ is well defined even for $\tau \in \mathbb{N}_0$.

From the properties of Brock's flow, see, in particular, [11, Lemma 4.1], we obtain the following properties for our flow.

Proposition A.1. *Let $d \geq 1$, $1 \leq p < \infty$ and let $0 \leq f \in \mathrm{L}^p(\mathbb{R}^d)$. Then, for any $\tau \in [0,\infty]$, the function f_τ defined by (30) is in $\mathrm{L}^p(\mathbb{R}^d)$ and $\|\mathsf{f}_\tau\|_p = \|\mathsf{f}\|_p$. Moreover, for any $\tau \in [0,\infty]$ and any sequence $(\tau_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \tau_n = \tau$,*

$$\lim_{n\to\infty} \|\mathsf{f}_{\tau_n} - \mathsf{f}_\tau\|_p = 0\,.$$

The following fact is important for us.

Lemma A.2. *Let $d \geq 3$ and $0 \leq f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$. The function*

$$\tau \mapsto \sup_{u\in\mathcal{M}_1} \left(\mathsf{f}_\tau, u^{2^*-1}\right)^2$$

with f_τ defined by (30) is continuous.

Proof. We use the fact, shown in Proposition A.1, that

$$\lim_{\tau_1\to\tau_2} \|\mathsf{f}_{\tau_1} - \mathsf{f}_{\tau_2}\|_{2^*} = 0\,.$$

Fix $\varepsilon > 0$. There exists $u_1 \in \mathcal{M}_1$ such that $\sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_1}, u^{2^*-1}\right)\right| \leq \left|\left(\mathsf{f}_{\tau_1}, u_1^{2^*-1}\right)\right| + \varepsilon$ and hence

$$\begin{aligned}\sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_1}, u^{2^*-1}\right)\right| - \sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_2}, u^{2^*-1}\right)\right| &\leq \left|\left(\mathsf{f}_{\tau_1}, u_1^{2^*-1}\right)\right| + \varepsilon - \left|\left(\mathsf{f}_{\tau_2}, u_1^{2^*-1}\right)\right| \\ &\leq \left|\left(\mathsf{f}_{\tau_1}, u_1^{2^*-1}\right) - \left(\mathsf{f}_{\tau_2}, u_1^{2^*-1}\right)\right| + \varepsilon\,,\end{aligned}$$

which by Hölder's inequality is bounded above by

$$\|\mathsf{f}_{\tau_1} - \mathsf{f}_{\tau_2}\|_{2^*} \, \|u_1^{2^*-1}\|_q + \varepsilon = \|\mathsf{f}_{\tau_1} - \mathsf{f}_{\tau_2}\|_{2^*} + \varepsilon$$

with $q = \frac{2^*}{2^*-1}$. Hence

$$\limsup_{\tau_2\to\tau_1} \left(\sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_1}, u^{2^*-1}\right)\right| - \sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_2}, u^{2^*-1}\right)\right| \right) \leq \varepsilon\,.$$

There exists $u_2 \in \mathcal{M}_1$ such that $\sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_2}, u^{2^*-1}\right)\right| \leq \left|\left(\mathsf{f}_{\tau_2}, u_2^{2^*-1}\right)\right| + \varepsilon$ and hence

$$\sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_1}, u^{2^*-1}\right)\right| - \sup_{u\in\mathcal{M}_1} \left|\left(\mathsf{f}_{\tau_2}, u^{2^*-1}\right)\right| \geq \left|\left(\mathsf{f}_{\tau_1}, u_2^{2^*-1}\right)\right| - \left|\left(\mathsf{f}_{\tau_2}, u_2^{2^*-1}\right)\right| - \varepsilon\,,$$

which is greater or equal to

$$-\left|\left(\mathsf{f}_{\tau_1}, u_2^{2^*-1}\right) - \left(\mathsf{f}_{\tau_2}, u_2^{2^*-1}\right)\right| - \varepsilon \geq -\|\mathsf{f}_{\tau_1} - \mathsf{f}_{\tau_2}\|_{2^*} - \varepsilon\,.$$

Hence

$$\liminf_{\tau_2\to\tau_1}\left(\sup_{u\in\mathcal{M}_1}\left|\left(\mathsf{f}_{\tau_1}, u^{2^*-1}\right)\right| - \sup_{u\in\mathcal{M}_1}\left|\left(\mathsf{f}_{\tau_2}, u^{2^*-1}\right)\right|\right) \geq -\varepsilon\,.$$

This proves the claimed continuity. □

We now consider the behavior of the gradient under the rearrangement flow. The following proposition is closely related to [11, Theorems 3.2 and 4.1], but there inhomogeneous Sobolev spaces are considered, which leads to some minor changes. For the sake of simplicity we provide the details.

Proposition A.3. *Let $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$. Then f_τ defined by (30) is in $\dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $\tau \mapsto \|\nabla \mathsf{f}_\tau\|_2$ is a non-increasing, right-continuous function.*

Proof. By construction, it suffices to prove these properties for Brock's flow. Since the latter has the semigroup property $(\mathsf{f}_\sigma)_\tau = \mathsf{f}_{\sigma+\tau}$ for all σ, $\tau \geq 0$, it suffices to prove monotonicity and right-continuity at $\tau = 0$.

We begin with the proof of monotonicity, which we first prove under the additional assumption that $f \in \mathrm{L}^2(\mathbb{R}^d)$. This is shown in [11, Theorem 3.2], but we give an alternative proof. We proceed as in the proof of [36, Lemma 1.17]. Extending [10, Corollary 2] to the sequence of Steiner symmetrizations we find for three non-negative functions f, g, h that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathsf{f}_\tau(x)\, g_\tau(x-y)\, h_\tau(y)\, dx\, dy \geq \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\, g(x-y)\, h(y)\, dx\, dy\,.$$

If we choose $g(x-y)$ to be the standard heat kernel, i.e., $g(x-y) = e^{\Delta t}(x-y)$, then $g_\tau(x-y) = g(x-y)$ and hence

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathsf{f}_\tau(x)\, e^{\Delta t}(x-y)\, \mathsf{f}_\tau(y)\, dx\, dy \geq \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\, e^{\Delta t}(x-y)\, f(y)\, dx\, dy\,.$$

Since $\|\mathsf{f}_\tau\|_2 = \|f\|_2$ by the equimeasurability of rearrangement,

$$\frac{1}{t}\left(\|\mathsf{f}_\tau\|_2^2 - \left(\mathsf{f}_\tau, e^{\Delta t}\,\mathsf{f}_\tau\right)\right) \leq \frac{1}{t}\left(\|f\|_2^2 - \left(f, e^{\Delta t} f\right)\right)$$

and letting $t \to 0$ yields the first claim under the additional assumption $f \in \mathrm{L}^2(\mathbb{R}^d)$.

For general $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ we apply the above argument to the functions $(f-\epsilon)_+$, $\epsilon > 0$. They belong to $\mathrm{L}^2(\mathbb{R}^d)$ since f vanishes at infinity and belongs to $\mathrm{L}^{2^*}(\mathbb{R}^d)$. We obtain

$$\left\|\nabla\left((f-\epsilon)_+\right)_\tau\right\|_2 \leq \|\nabla(f-\epsilon)_+\|_2 \leq \|\nabla f\|_2\,. \tag{31}$$

We claim that $\mathsf{f}_\tau \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $\nabla\left((f-\epsilon)_+\right)_\tau \rightharpoonup \nabla \mathsf{f}_\tau$ in $\mathrm{L}^2(\mathbb{R}^d)$ as $\epsilon \to 0^+$. Once this is shown, the claimed inequality follows from (31) by the weak lower semicontinuity of the L^2 norm.

To prove the claimed weak convergence, note that by (31), $\nabla\left((f-\epsilon)_+\right)_\tau$ is bounded in $\mathrm{L}^2(\mathbb{R}^d)$ as $\epsilon \to 0^+$ and therefore has a weak limit point. Let $F \in \mathrm{L}^2(\mathbb{R}^d)$ be any such limit point. Since $(f-\epsilon)_+ \to f$ in $\mathrm{L}^{2^*}(\mathbb{R}^d)$, the nonexpansivity of the rearrangement [10, Lemma 3] implies that $\left((f-\epsilon)_+\right)_\tau \to \mathsf{f}_\tau$ in $\mathrm{L}^{2^*}(\mathbb{R}^d)$. Thus, for any $\Phi \in C_c^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (\nabla\cdot\Phi)\, \mathsf{f}_\tau\, dx \leftarrow \int_{\mathbb{R}^d} (\nabla\cdot\Phi)\left((f-\epsilon)_+\right)_\tau dx = -\int_{\mathbb{R}^d} \Phi\cdot\nabla\left((f-\epsilon)_+\right)_\tau dx \to -\int_{\mathbb{R}^d} \Phi\cdot F\, dx$$

as $\epsilon \to 0^+$. This proves that f_τ is weakly differentiable with $\nabla \mathsf{f}_\tau = F$. In particular, $\mathsf{f}_\tau \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ (note that f_τ vanishes at infinity since f does and since these functions are equimeasurable) and the limit point F is unique. This concludes the proof of the first part of the proposition.

Let us now show the right-continuity at $\tau = 0$. It follows from Proposition A.1 that $\mathsf{f}_\tau \to f$ in $\mathrm{L}^{2^*}(\mathbb{R}^d)$ as $\tau \to 0^+$. This implies that $\nabla \mathsf{f}_\tau \rightharpoonup \nabla f$ in $\mathrm{L}^2(\mathbb{R}^d)$ as $\tau \to 0^+$. (Indeed, the argument is similar to the one used in the first part of the proof. The family $\nabla \mathsf{f}_\tau$ is bounded in $\mathrm{L}^2(\mathbb{R}^d)$ as $\tau \to 0^+$ and, if F denotes any weak limit point in $\mathrm{L}^2(\mathbb{R}^d)$, then the convergence in $\mathrm{L}^{2^*}(\mathbb{R}^d)$ and the definition of weak derivatives implies that $F = \nabla f$.) By weak lower semicontinuity, we deduce that

$$\|\nabla f\|_2 \leq \liminf_{\tau \to 0^+} \|\nabla \mathsf{f}_\tau\|_2 \,.$$

This, together with the reverse inequality, which was established in the first part of the proof, proves the claimed right continuity. □

We note that the proposition remains valid for $0 \leq f \in \dot{W}^{1,p}(\mathbb{R}^d)$ with $1 \leq p < d$. If $p \neq 2$, the monotonicity for the gradient for $f \in W^{1,p}(\mathbb{R}^d)$ is proved in [11, Theorem 3.2]. The remaining arguments above carry over to $p \neq 2$.

Appendix B. Numerical values

Although rather small and not algebraically computable, the estimate

$$\kappa := \sup_{0<\delta<1} \delta\, \mu(\delta)$$

in Theorem 1.1 has a finite value for any $d \geq 3$. Numerically, the value is found by inverting (3) and optimizing $\nu \mapsto \nu^2 \left(1+\nu^2\right)^{-1} \mathsf{m}(\nu)$ on $(0, +\infty)$. This function takes the value 0 at $\nu = 0$, is increasing on a neighbourhood of $\nu = 0^+$, and has a unique positive maximum point. For our values, see Figure 1 and Table 1.

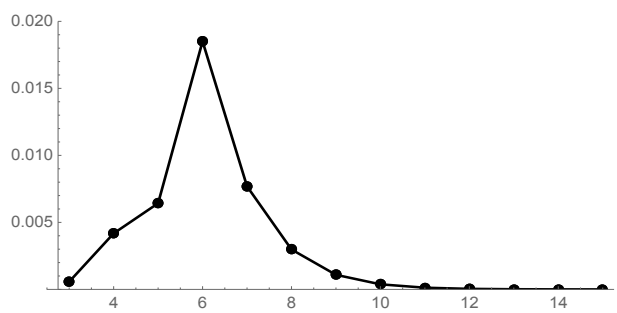

Figure 1. Plot of $d \mapsto \kappa(d)$ for $d = 3, 4, \ldots, 15$.

d	3	4	5	6	7	8	15
κ	5.7142×10^{-4}	4.1779×10^{-3}	6.4259×10^{-3}	1.8507×10^{-2}	7.6876×10^{-3}	2.9776×10^{-3}	1.1213×10^{-6}

Table 1. Numerical values of the estimate of Theorem 1.1 for some values of d.

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