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The Value Does Not Exist!

A Motivation for Extremal Analysis [1]

Jean-Pierre Aubin[2] **and Hélène Frankowska**[3]

1 *[A Tribute to Alain Bensoussan]*

Jean-Pierre Aubin met Alain Bensoussan during the summer of 1969 at the Mathematics Research Center (MRC) at the University of Wisconsin at Madison and learned that, he too, was a doctoral student of Jacques-Louis Lions. The same year, Jacques-Louis Lions, knowing with indulgence the perverse taste for economics and humanities of the first author in those troubled times following the wars in Algeria and Vietnam and the rise of individualism which blew up in 1968, wrote to him about the creation of the Paris-Dauphine University dedicated to economics and management, which was desperately looking for mathematicians. Alain was a probabilist specialized in optimal control of stochastic PDEs and was also interested in these questions: together, we joined this university in the fall of 1969 and founded the department and the Research Center for Decision Mathematics (CEREMADE), of which Alain was the first director, before being the president of INRIA and later the president of CNES, the French spacial agency. These activities did not slow down his research activities that he pursed in the United States and Hong Kong.

Hélène Frankowska, a former student of Czesław Olech at Warsaw University and specialist of optimal control, joined Paris-Dauphine University ten years later and thus collaborated with us ever since.

Control theory, game theory, economics, management, and financial investigations have been our common playground all these years.

The pages that follow thus bear Alain's imprint.

Jean-Pierre Aubin and Hélène Frankowska

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Synopsis

Standard mathematical economics studies the production, exchange, and consumption of goods "provided with units of measurement,*" as in physics, in order to be enumerated, quantified, added, etc. Therefore, "baskets of goods," which should describe subsets of goods, are mathematically represented as* commodity vectors of a vector space*, linear combination of units of goods, evaluated by* prices*, which are* linear *numerical functions. Therefore, in this sense, mathematical economics is a branch of physics.*
However, economics, and many other domains of life sciences, investigate also what will be called entities*, defining* elements deprived of units of measure*, which thus cannot be enumerated.*

1. *Denoting by X the set of entities $x \in X$* deprived of units of measurement*, a "basket of goods" is actually a* subset *$K \subset X$ of the set entities, i.e., an element of the* "hyperset" *$\mathcal{P}(X)$, the family of subsets of X, and no longer a commodity vector of the vector space of commodities;*

2. *Entities can be "gathered" instead of being "added";*

3. *Entities can still be evaluated by a* family *of functions $A : x \in X \mapsto A(x) \in \mathbb{R}$ regarded as a "valuators," in lieu and place of linear "prices" evaluating the units of economic goods.*

4. *Subsets of entities can be evaluated by an* "interval of values" between two extremal ones*, the minimum and the maximum, instead of the sum of values of units of goods weighted by their quantities.*

Life sciences dealing with intertwined relations among many combinations of entities, hypersets offer metaphors of "Lamarckian complexity" that keeps us away from binary relations, graphs of functions, and set-valued maps, to focus our attention on "multinary relations" *between families of hypersets. Even deprived of units of measurement, these "proletarian" entities still enjoy enough properties for this pauperization to be mathematically lethal.*
This is the object of this extremal manifesto*:* in economics and other domains of life sciences, vector spaces should yield their imperial status of "state space" to hypersets and linear prices to hypervaluators*.*
We no longer have to add goods which can only be gathered, prices do not have to be linear, although it costs some effort to deprive oneself of the powerful and luxurious charms of convex and linear functional analysis motivated by physics.
These sacrifices concern only economics and other fields of life science, since physicists deal with experimental observations of objects endowed with unit of measurement by adequate processes of measurement. They can happily live in vector spaces without any guilt. This is not the case of life scientists, who have mainly history to support and validate their observations, with, sometimes, the privilege to statistically measure the frequency of some of them.

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The title of this study is borrowed from that of *La vie n'existe pas !*, by *Ernest Kahane*[4]: We hope that we have not betrayed its spirit: this title seems peremptory whereas it is only provocative, more so to convince the reader of the contrary!

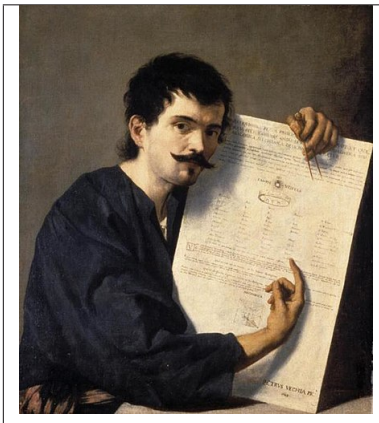
The concept of value is polysemous, ranging from moral value (courage) to venal value (wealth), from the value of happiness to that of work, from the value of numbers to that of the shareholders, from the value of work to that of mathematics, and so many others, for no list being exhaustive. To the point where, more and more, one wishes to *measure* by numbers what *can neither be enumerated nor measured* by lack of units or measurement, unless using dubious palliatives such as grades for evaluating a student's homework, or nowadays "commodification" processes for assigning numerical values to assess symphonies, paintings, books, etc. Therefore we cannot use vector spaces to acclimate these entities in vector spaces to elaborate mathematical metaphors.

But what about ideas, or any other entity deprived of units of measurement? Nothing forbids to evaluate them by *arbitrary numerical functions* assigning subjectively numerical values to entities. In this case, we may hope to transfer the properties of the set of real numbers on the set of these entities, by adding values of ideas even though we cannot add these ideas: we are no longer compelled to consider mathematical economics as a branch of mathematical physics, Which it is definitively not the case.

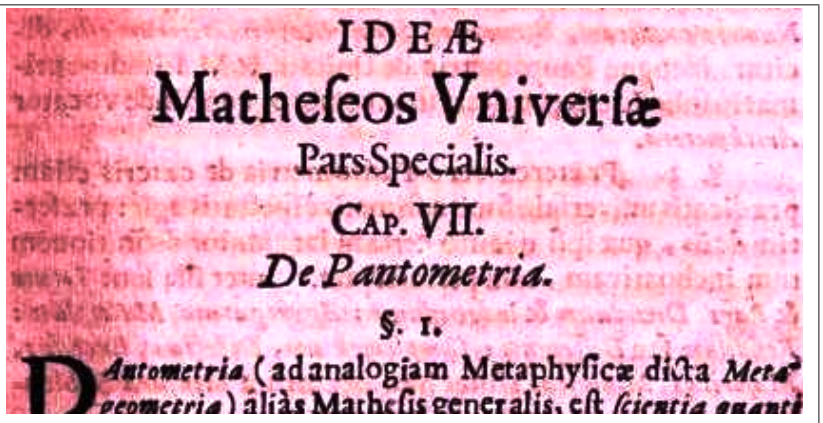
In particular, there is an actual tendency to evaluate anything, notably by euros, dollars, yen, etc., and, doing so, to measure the dollar value of some human beings by dollar values five hundred times greater than that of millions of others.

This drift triggered the desire to know more and to denounce what can be called the *pantometric drug* to follow the dream of *Erhard Weigel* who introduced the concept "Pantometria" in his book *Idea matheseos universae cum speciminibus inventionum mathematicarum* in 1669.

2 *[**Weigel's Pantometria**]*



painting by Pietro della Vecchia *(1649)* .



IDEÆ
Matheſeos Vniverſæ
Pars Specialis.
CAP. VII.
De Pantometria.
§. I.
Pantometria (ad analogiam Metaphyſicæ dicta Meta-
geometria) aliàs Matheſis generalis, eſt ſcientia quanti

From 1653 until his death, Weigel was professor of mathematics at Jena University. He had the privilege of teaching the young Gottfried Leibniz in 1663, and other notable students. He also worked to make science more widely accessible to the public, and what would today be considered a popularizer of science.

[4] *Life Does Not Exist!*, by the famous biologist, father of the no less famous mathematician *Jean-Pierre Kahane*.

For 3.7 billion years, at each date, each *living organism*

1. *transforms* our planet and its biotope by continuously *producing*, *exchanging*, and *consuming* some of the resources through complex photosynthetic and metabolic mechanisms using solar energy, carbon, and water mainly along the trophic chains;

2. during *temporal windows* of finite duration separating their *birth* from their *death*;

3. but *surviving* in their lineage by reproduction mechanisms, asexual as well as sexual, creating similar organisms in their main line, but different in some details.

This exchange processes between production and consumption have been described by the concept of *catallaxy*[5], a term coined by *Richard Whately* in his *Introductory Lectures on Political Economy* published in 1831, in which he wrote: "*[...] the name I should have preferred as the most descriptive, and on the whole least objectionable, is that of catallactics, or the* Science of Exchanges."

He presumably presupposed exchanges of entities[6] produced (processed) for consumption, pre- and post- exchange activities *proper to human beings*. For, even though all living organisms are catallactic, human beings have appropriated a major place because *they have invented the* Word *and the* Number[7], for better and for worse. Human beings are in some sense "talkative and digital living beings" possessing the unique mathematical capability of counting and playing with numbers.

It is curious that the classical Greeks, who have personified so many abstract concepts by the gods of their beautiful and bountiful mythology, have avoided reifying the *Word* and the *Number* by some of the Olympians[8]. The "Logos" (reason, discourse) has no mythological status, much less the "Arithmos". They were not to pay the attention they deserve in the poetic and metaphorical mythology of their compatriots. However, *Pythagoras* made mathematics a sort of sect of which he was the high priest, surrounded by the faithful esoteric and exoteric disciples, the apostles of this sect which became ... the Church of mathematics.

The facetious course of the sun "around the earth" is perceived by all living organisms, which have progressively invented biological clocks to adapt their metabolic activities to the days, seasons, and years to grow and multiply. They learned to adapt to the regularity of these repetitive, yet stammering, periods, to adapt to the irreversible arrow of time while expecting in vain eternal returns to repeat themselves. Phylogenesis has indeed taken all its time to synchronize with the *biological clocks* of living organisms the *perceptions* of the cyclic nature of the succession of days, months, and years and of their duration. The durations of the temporal windows of this "passing time" are biologically perceived by their biological clocks of each living organism. They are resonating with the perception of our planet's rotation around this star.

Time is born in the cognitive system of living organisms as soon as they recognize, memorize, and "remember" their recorded perception-action pairs to recall one of them when necessary and implement it.

---

[5] Derived from the Greek verb *katallássein*, to exchange, to be compared to the term *bargaining* used by *Adam Smith*. Catallaxy became popular when *Ludwig von Mises* used it, followed by *Friedrich Hayek*. Catallaxy and catallactics are synonyms as economy and economics are.

[6] Among these entities exchanged by human beings, ideas, theories, scientific or not, etc., can be exchanged: we are also kinds of conveyors of ideas, emitted by others, transmitted to others, and, sometimes, betrayed to discover new ones.

[7] "In the beginning was the Word, and the Word was with God, and the Word was God", wrote John the Apostle. He should have added the Number. As for *Aesop*, slave of the philosopher *Xanthus*, who gave him the order to buy at the market the best, and next, the worst. He could have added fingers to the tongues he brought back to obey his master, for nothing is *at the same time better and worse* than speech *and* digital activities.

[8] Even though *Theophrastus* and other classical Greek authors mention *Palamedes*, the inventor of letters and numbers, probably derived from Egyptian and Phoenician myths. He was forgotten by *Homer* (probably because he was even more cunning than his rival Ulysses, who had avenged himself for his vexations by conceiving an odious and mortal conspiracy), according to *Françoise Chenet*.

The perception of the *evolution* of entities, stars, climatic, geologic and tectonic events, living beings, etc., questioned human beings as soon as they were able to count and recount, and exchange the gathered information, share it and use it. This is the main motivation of this study.

As soon as they invented the *language* to describe and exchange their perception of the world, they "understood" it by relating pairs of entities by "*metaphors*[9]". One can also imagine that after contemplating the succession of days and nights, seasons and years, human beings conceived dates and counted them, as they counted the number of steps to cover distances and compare them. They may have recorded them using pebbles or incisions in wood, inventing the calendar and the writing on various substrates.

They not only invented *numbers and identified sets of entities, but* used *those handy numbers to "evaluate" entities by numbers regarded as their* values*, opportune means of summarizing complex entities.*

For that purpose, they had to conceive *units of measurements* of some entities whenever it was possible and to *count* the number of units: they conceived *numerical functions* such as measures of physical entities and prices. This innovation has completely overturned the economic exchanges, confining its analysis to a small number of physical entities endowed with units of measurement in order to be

1. *enumerated*, by counting the number of the unit of measurement of each good of the basket of goods;
2. and next, *evaluated* by assessing the units of measurement of each good by another unit of measurement that transcend them, a *monetary unit*.

By selecting only measurable goods from among all the entities, economics, as well as other domains of life sciences, was mathematically reduced to a branch of physics.

However, there are many more entities that are deprived of units of value, from ideas to theorems, even though everyone is still tempted to evaluate them with numbers. This falls outside the fields of mathematics motivated solely by physics, the science of measurable quantities.

We thus have to evaluate what is not measurable: catallaxy offers a motivation to do so.

1 Catallaxy

1.1 Environment

Catallaxy concerns the interactions between an *environment* X and *organisms* transforming its elements and subsets, producing, exchanging, and consuming some of them. Among them, human beings, who added to this environment the set of monetary values they used to fluidize exchanges and to hoard them to insure their future: catallaxy became economy, both *micro*, concerning goods and commodities, and *macro*, studying prices and means of payment.

[9]This is how *consciousness* appeared, if we follow *Julian Jaynes* in his book *The Origin of Consciousness in the Breakdown of the Bicameral Mind*, who defines and justifies consciousness in a very appealing way as the part of *the language* used to build metaphors in order to understand a phenomenon in the environment, be it physical, biological, social or cultural. Mathematically, metaphors can be translated as *binary relations* on a product of two sets, idempotent, transitive but not necessarily symmetrical to translate implications from one term of the metaphor to the other.

In this study, an *environment* X is an arbitrary set, *deprived of any mathematical structure*, made of elements which are regarded as *entities*[10].

Actually, what matters are not only these entities $x \in X$ ranging over an arbitrary set, but also

1. "*baskets of entities*[11] $K \subset X$," which are subsets made of entities. These subsets range over what is called *hyperset*[12] $\mathcal{P}(X)$, the family of subsets K of X. More generally, we define recursively the hypersets $\mathcal{P}_r(X) := \mathcal{P}(\mathcal{P}_{r-1}(X))$ of order $r \in \mathbb{N}$ as the hyperset of the hyperset of order $r-1$ and by $\sigma_r : \mathcal{P}_{r-1}(X) \mapsto \mathcal{P}_r(X)$ the canonical injections.

2. *transforms* described on temporal windows[13] $[\Delta, \Omega]$ of duration $\Omega - \Delta > 0$, that allow distinguishing an *input* $x_\Delta \in X$ at initial date Δ from the *output* $x_\Omega \in X$ at terminal date $\Omega > \Delta$ of the transform, summarized by the *input-output pair of entities* (x_Δ, x_Ω).

3. combinations of transforms.

Notwithstanding the set X may be deprived of any mathematical structures, its hyperset $\mathcal{P}(X)$ is always endowed with the *Boolean operations*: *unions*, *set-differences*, and other Boolean operations on hypersets, discovered in 1847 by *George Boole*. They *play the same role*[14] *as the addition and subtraction* of goods, granted with measurement units and commodity vectors. Even though *entities cannot be added, they, at least, may be "gathered" or "aggregated."*

Less popular than vector spaces, *hypersets*[15], families of subsets of an underlying space, have been extensively investigated ever since the discovery and emergence of "general set theory" by *Georg Cantor* and *Richard Dedekind* in the 1870s.

Extremal analysis offers another approach which does not require an *a priori* metric structure on the set, but only the choice of a separable space of valuators.

---

[10] Or *monads*, from Greek *monos*, "alone," since it does not specify the nature of individuated things. This term was used by *Pythagoras*, and later, by *Giordano Bruno*, *Gottfried Leibniz*, *Ernst Haeckel*, and too many other ones to be quoted. It is also employed in mathematics by *Roger Godement* in 1958, *Alexandre Grothendieck*, and *Pierre Cartier*.

Even though it denotes precisely what we mean by entity, it could be judged as too pedantic. Entities (or elements) denote in this study what are not vectors ranging over vector spaces, which, for their part, can be added and multiplied by numbers.

[11] Instead of basket of goods preempted by economists to describe commodity vectors, which we shall also call *resources*.

[12] We make the choice to adopt the terminology "hyperset" to call the set of its subsets and not the term *"power" space* as it is often done, to avoid a *polysemic conflict* with the concept of power space $\mathbb{V}^{\mathbb{X}} := \prod_{x \in \mathbb{X}} \mathbb{V}_x$, where $\mathbb{V}_x := \mathbb{V}$ for all $x \in \mathbb{X}$, as well as the definition by *Bourbaki* of power of a set as its cardinal as well as $\mathcal{P}(X)$.

[13] Temporal windows are described by two numbers (Δ, Ω) of initial date Δ and terminal date $\Omega \geqslant \Delta$ or the pair $(\Omega, \Omega - \Delta)$ a terminal date Ω and the positive *duration* $\Omega - \Delta$ which offer an adequate mathematical description of *time*, a polysemous word meaning at least dates and duration.

[14] Which obeys the iron law of formal logic defining what *truth* is, to which mathematicians must comply under penalty of infamous error.

[15] The spaces underlying hypersets can be vector spaces, in which case they enjoy more properties. See for instance [1, Akian, Gaubert & Kolokoltsov], [5, Aubin & Dordan], the book *Espaces topologiques, fonctions multivoques*, [15, Berge], *Set-valued analysis*, [8, Aubin & Frankowska], *Variational Analysis*, [19, Rockafellar & Wets], among many other books.
Mutational and Morphological Analysis: Tools for Shape Regulation and Morphogenesis, [2, Aubin], and *Mutational Analysis*, [18, Lorenz], introduce a mutational structure to define a differential calculus on metric spaces, and, in particular, on the hyperset of compact subsets of a metric space. They govern the evolution of subsets in the same way that differential equations govern the evolution of vectors.
The story began with *Jean Céa* (see Céa J. (1971) *Optimisation : théorie et algorithmes*, Dunod) followed by *Jean-Paul Zolésio* and *Michel Delfour*, by defining *shape derivatives*, see [17, Delfour M. and Zolesio] *Shapes and geometries: analysis, differential calculus and optimization.*

Any subset X being the largest element of its hyperset $\mathcal{P}(X)$ can be *compartmentalized* as the union of its subsets. Each of them is naturally associated with its complement, separated by their *comon boundary*, the site of the exchanges between the inside and the outside of elements produced by metabolic and catalytic mechanisms. They offer propitious mathematical metaphors of evolutionary mechanisms of *living organisms*, constrained by homeostatic or viability constraints.

As living organisms are complex, their mathematical transcriptions can be expected to be as complex to take into account their *reticular properties*, since organisms are made of organs, themselves made of lower level organs, etc.

Therefore, any "organism" can be mathematically depicted by a family of ℓ subsets $K_i \subset X$, $i \in \mathbb{I}_\ell := \{1, \dots \ell\}$, ranging from pairwise disjoint subsets, as autonomous physical object, to nested chains $K_1 \subset \dots \subset K_i \subset \dots \subset K_\ell$, as genes and mitochondria in biological cells of organisms, among all the $\begin{pmatrix} \ell \\ k \end{pmatrix} := \dfrac{\ell!}{k!(\ell-k)!}$ intersections $K_i \backslash K_j = K_i \cap \complement K_j$ known as *koinès*[16] contained in k subsets K_i and outside of the other ℓ subsets K_j in the family. They go from *autonomous* subsets (that are not contained in any strict superset) to its *root* (the intersection of all the sets of the family). Naturally, some of these subsets may be empty, which also provides structural information.

This offers a mathematical metaphor of "nested structures" prevalent in biology and sociology, that *Lamarck* had emphasized in order to define a concept of *hierarchical complexity*, different from the concept of *connectionist complexity* suggested by *Darwin* and his followers.

Can we do more than just use the boolean structures of hypersets?

Yes, since the field $\mathbb{R}$ of real numbers is equipped with the richest and prototypical mathematical structures, both algebraic, reticular, and topological, the temptation is great to convey its properties to any set X as well as its hypersets by associating with it the power spaces $X^{\mathbb{R}}$ and $\mathbb{R}^X$: that is lucky since we recognize

1. the *plain set* $X^{\mathbb{R}}$ of *evolutions* associating with dates $t \in \mathbb{R} \mapsto X$ the elements $x(t) \in X$ (or *morphological evolutions* $t \in \mathbb{R} \mapsto K(t) \in \mathcal{P}(X)$) at date t, since one of the characteristics of living organisms is to evolve in an intertwined manner,

2. the *vector space*[17] $\mathbb{R}^X$ of *valuators* $A : x \in X \mapsto A(x) \in \mathbb{R}$ *evaluating* elements $x \in X$ (and subsets $K \subset X$ by $K \in \mathcal{P}(X) \mapsto \sup_{x \in K} A(x) \in \mathbb{R} \cup \{+\infty\}$) as well;

which are the main themes of *extremal analysis*: some *mathematical structures of the set X and its hypersets $\mathcal{P}(X)$ will be derived from the choice of a vector space $\mathcal{S}(X) \subset \mathbb{R}^X$ of valuators* allowing us to transfer many results of functional analysis valid in vector spaces.

1.2 Evaluation

Ever since humans invented numbers, they were tempted to associate with the entities they observed some numbers known under the concept of value, so polysemous that it lost any value!

It is at this stage that the distinction of physical elements, supplied with unit of measures, and economic entities deprived of them occurs: it entails a dichotomy among *prices* for the first ones, *valuators* for the others.

[16] Terms borrowed from linguistics designating in Greek the "common" of a dialect common to the Hellenic world.

[17] Even though we cannot add entities of $x \in X$, we can still add their values $A_1(x) \in \mathbb{R}$ and $A_2(x) \in \mathbb{R}$ and define the sum $A_1 + A_2 : x \in X \mapsto (A_1 + A_2)(x) := A_1(x) + A_2(x) \in \mathbb{R}$.
Generally speaking, any power space Y^X inherits the properties of the set Y.

1.2.1 Prices of Commodities with Units of Measure

Any transformation of the entities $K \subset X \mapsto L \subset Y$ by an organism takes "time," actually, "duration," to allow entities to *evolve* along the dates of a temporal window because organisms are *actors* (climatic, geologic, biologic ones, living organisms and among them, human beings, etc.). It *transforms* the entities $x \in X$ or resources $K \in \mathcal{P}(X)$ to another resource $L \in \mathcal{P}(Y)$, by *producing*, *exchanging*, and *consuming*, or *destroying* them.

Whenever actors are human beings, the matters pertaining to evaluation are raised. In this way numbers were associated with entities and soon, replaced them in exchange processes: they became *evaluators* by assigning numerical values to entities, and, for that purpose, *designed* numerical functions that we call *valuators* and *agreed* to use.

To do so, they had to *enumerate* entities, and thus, to restrict evaluations to entities endowed with *units of measurement*: they became physicists, and, in their wake, economists.

Historically, ℓ elements $i \in \{1, \ldots, \ell\}$ of a *finite set* could be evaluated by their *quantities* $x_i \in \mathbb{R}$ as soon as *measurement units* e_i were agreed upon, so that the quantification process was invented and practiced as a first evaluation procedure.

The space of the *quantifications* $i \in \{1, \ldots, \ell\} \mapsto x_i \in \mathbb{R}$ became a theme of investigations before it was called a *vector space* $\mathbb{R}^\ell$ since *Pierre de Fermat* (for $\ell = 3$) and *René Descartes* (for $\ell = 2$) around 1636. What was decided at the time was to focus on "*adding the objects by adding their quantities*," so that these objects became mathematically known as vectors, that economists called *commodities*, for vectors of ℓ goods described by

$$x \;:=\; \sum_{i=1}^{\ell} x_i e^i \;=:\; (x_1, \ldots, x_\ell) \;\in\; \mathbb{R}^\ell \tag{1}$$

and thus, ranging over the *finite dimensional vector space* $\mathbb{R}^\ell$.

Later arose the need to simplify and accelerate the exchange of cumbersome objects by "measuring the measurement units" with an artificial consensual unit, the mother of all units of measurement: the *numéraire*. It was used to evaluate the ℓ measurement units e^i of the goods i by the map $p : i \in \{1, \ldots, \ell\} \mapsto p^i := p(e^i) \in \mathbb{R}$, which were extended to linear forms $p \in \mathbb{R}^{\ell\star}$ by their value, the real number

$$\langle p, x \rangle \;:=\; p(x) \;:=\; \sum_{i=1}^{\ell} p^i x_i \;=\; \sum_{i=1}^{\ell} p(e^i) x_i \;\in\; \mathbb{R}.$$

These linear valuators $p : x \in \mathbb{R}^\ell \mapsto \mathbb{R}$ are called *prices*. When it comes to goods endowed with units of measurement, economic actors could restrict their exchanges of commodity vectors ranging in vector spaces and evaluate their monetary values with linear prices.

They had to create a monetary institution providing the community of actors an amount $m = \sum_{i=1}^{\ell} m_i \in \mathbb{R}$ of means of payments $m_i \in \mathbb{R}$ to fluidize the exchange of commodities $x_i \in X$ by the exchange m_i's.

Each society thus creates its own numéraire[18] *in which their citizens must trust* for the money to exist.

The central bank is supposed to represent the interests of society by creating the adequate amount of means of payment, production, exchange, and consumption in terms of goods and prices chosen by the economic actors. *Confidence* of members of the society in this numéraire, this ultimate abstract and mysterious ingredient, escapes

[18] Many monetary systems emerge in situations where confidence in a central currency is eroding. They are known by many names, far more than fifty shades of *grey markets*: vouchers, local currencies, such as the *argentino*, which appeared during the Argentinean crisis of the 2000s, Breton *galleco*, the most recent *horabank*, currency in hours of work, etc., not to mention the many Ponzi, or pyramid, schemes that crooks reinvent every day. The latest one is *bitcoin*, an online currency *anonymously* designed in 2009, assuming that servers are paying a security deposit through a mechanism that is not very explicit, if not obscure, and is used to launder dirty money.

mathematical description. Unfortunately, central banks result from the monetary creation of a number of financial institutions following their own objectives. So, awaiting the emergence of the "bancor[19]," the monetary unit of state monetary units, monetary crises are bound to emerge periodically.

1.2.2 Valuators of Entities without Units of Measure

Devoid of Units of Measurement, Happiness Is Priceless..., as well as most of *entities* perceived by our brains, from smiles to theorems. Even though they have their own *identity*, they are deprived of units of measurement and *cannot be enumerated*, even though each and everyone tries hard to provide them: *standard economic value by prices does not exist for such entities.*

Is the hypothesis that units of measurement exist necessary to define a concept of value ?

Not necessarily, since it suffices to summarize entities by values without assuming that they have unit measurement to enumerate them. For instance, by considering numerical functions $A : x \in X \mapsto A(x)$ playing the role of "*valuators*" ... providing values to any entity x of the set X, and using them to describe their mutual relations and their various combinations: value functions are no longer prices since X is no longer a vector space of commodities of goods supplied with measurement units.

Turgot answered this question in his *Valeurs et monnaies* of 1769: he wrote that *"It is therefore impossible to express value in itself; and all that human language can state in this regard is that the value of one thing equals the value of another* and that *"price is always the statement of value"*, *"value is the price of the thing"*[20]. He added: "*There is no fundamental unit given by nature, there is only an arbitrary and conventional unit. Every commodity has the two essential properties of money to measure and represent any value, and, in this sense, every commodity is money*[21]."

We shall begin our journey by following this advice: we assume that the set X of entities is deprived of any mathematical structure. However, we may *use any numerical function* $A : x \in X \mapsto v = A(x) \in \mathbb{R}$ to assign a value to any entity. This is the reason why we call them *valuators*.

In other words, we *assume* that valuators range over the vector space $\mathcal{A}(X) := \mathbb{R}^X$ of "$\mathcal{A}$ll" numerical functions. Actually, we may require that this valuators satisfy only "$\mathcal{S}$ome" specific properties, which means that they must range over a separable (or separated[22]) vector subspace $\mathcal{S}(X) \subset \mathcal{A}(X)$ playing the role of a *parameter* according to the nature of the problem, even though this added flexibility opens a Pandora's box.

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[19] The monetary Messiah, the supranational currency that *Keynes* had proposed at the Bretton Woods summit in 1944 without success against the Gold Exchange Standard. *Harry White* representing the U. S. Treasury Department (and Soviet spy as well), successfully defended it to ensure the supremacy of the US Dollar which acts as the international reserve currency, playing in a weak sense the role of an "international bank" until 1971.

[20] "*Il est donc impossible d'exprimer la valeur en elle-même ; et tout ce que peut énoncer à cet égard le langage humain, c'est que la valeur d'une chose égale la valeur d'une autre* et que *le prix est toujours l'énonciation de la valeur, la valeur est le prix de la chose.*"

[21] "*La valeur n'a d'autre mesure que la valeur : il n'y a pas d'unité fondamentale donnée par la nature, il n'y a qu'une unité arbitraire et de convention. Toute marchandise a les deux propriétés essentielles de la monnaie de mesurer et de représenter toute valeur, et, dans ce sens, toute marchandise est monnaie.*"

[22] The vector space $\mathcal{S}(X)$ is *separable* (or separated) if we can associate with any pair of two different elements x and $y \neq x$ of X at least one valuator $A \in \mathcal{S}(x)$ assigning to them different values $A(x) \neq A(y)$, as *Turgot* suggested, and supply them with stronger topologies than the *pointwise convergence topology* stating that $A_n \in \mathcal{S}(X)$ converges pointwise to $A \in \mathcal{S}(X)$ if for any $x \in X$, $A_n(x)$ converges to $A(x)$ in $\mathbb{R}$.

Can we use valuators $A \in \mathcal{S}(X)$ to also evaluate subsets $K \in \mathcal{P}(X)$ by subsets of values? The answer to this rhetorical question is naturally positive, since it is impossible to resist to the temptation to *ascribe to* subsets $K := \bigcup_{x \in K} \{x\} \in \mathcal{P}(X)$ the *interval*

$$[A](K) \ := \ \left[\Delta^{\flat}(K)(A) \ := \ \inf_{x \in K} A(x), \Delta^{\sharp}(K)(A) \ := \ \sup_{x \in K} A(x) \right]$$

of values $A(x) \in \mathbb{R}$ between the smallest and the largest values of their elements, regarded as *hypervalues*. They provide the basic tools of *extremal analysis*. We observe that:

• When $K := \{x\}$ is reduced to a singleton, then $[A](\{x\}) = A(x)$, so that hypervalues of sets coincide with values of elements whenever they are reduced to them: *the minimal consistency requirement is satisfied*;

• The use of valuators allows us to add and multiply the values of entities even when we cannot add or multiply the entities;

• This approach provides an alternative to the more than a century-old allurement to measure subsets by one value only[23];

• *Hypersets share the same separable subspace of valuators $A \in \mathcal{S}(X)$, and thus, can be evaluated with the same valuators.*

• Valuators $A \in \mathcal{S}(X)$ not only provide *evaluation mechanisms* of elements of sets and subsets of hypersets, but play also a crucial role to define

1. *selection mechanisms* $\Sigma_K(A) \subset K$ of K of those elements $x^\star \in K$ *maximizing the value* $A(x^\star) = \sup_{x \in K} A(x)$ (if any);
2. the *normal cone* $N_K(x) := \left\{ A \in \mathcal{S}(X) \text{ such that } A(x) = \Delta^{\sharp}(K)(A) \right\} \subset \mathbb{R}^X$ to K at $x \in K$;
3. and the *tangent set* $T_K(x) := \{v \in X \text{ such that, } \forall\, A \in N_K(x), \ \ A(v) \ \leqslant \ 0\}$ to K at $x \in K$.

Actually, the tangents to graph of numerical functions at points of their graph have been defined in 1637 by *Pierre de Fermat*[24] and the normals by *René Descartes*. After myriad investigations, among which those of *Giuseppe Peano*, *Georges Bouligand*, and *Francesco Severi*, they culminated in the current *set-valued analysis* in vector spaces and *mutational analysis* in metric spaces.

Those simple definitions allow us to go one step further to build a differential calculus for any set X.

Such mathematical metaphors may better fit to subsume mathematically the concepts underlying *catallaxy*:

3 ***[Mathematical Metaphor of Catallaxy]***
A mathematical metaphor of any plain set X of entities catallaxy *is described by*

[23]This is done by using "*magnitudes*," which are hyperfunctions $\mu : K \in \mathcal{P}(X) \mapsto \mu(K) \in \mathbb{R}_+$ to evaluate subsets, required to satisfy the two following reasonable conditions:

1. $\mathbb{A}(\emptyset) \ = \ 0$ (the empty set has no value);
2. $\mathbb{A}$ is increasing: if $K \subset L$, then $0 \leqslant \mu(K) \leqslant \mu(L)$ (the larger the resource, the more valuable it is);

among which we find *measures* and *capacities* of all obediences, as well as fuzzy and cost sets, and all their variants, loss functions in game theory, both cooperative and non-cooperative, to name but a few of a very long list, so strong is the desire to evaluate elements and subsets. On top of it, they are chosen *a priori*, depriving us of a possibility to evaluate *a posteriori* elements in an interval, in order to answer additional and/or specific questions.

[24]After the example of the tangent to a circle by *Euclid* in book III of the Elements.

1. *the hyperset $\mathcal{P}(X)$ of subsets $K \subset X$ exchanged by "catallactic actors" supplied with the* Boolean structure *(union, intersection, set-differences, etc.);*

2. *the set $\mathbb{R}$ of scalars regarded as values $v \in \mathbb{R}$ and the family $\mathbb{I}$ of intervals interpreted as hypervalues $[v^\flat, v^\sharp] \subset \mathbb{R}$ (designed to fluidize exchanges);*

3. *any* separable vector space *$\mathcal{S}(X)$ of functions regarded as* valuators *$x \in X \mapsto A(x) \in \mathbb{R}$, supplied with the pointwise convergence topology;*

4. *the set of* hypervaluators *$K \in \mathcal{P}(X) \mapsto [\inf_{x\in K} A(x), \sup_{x\in K} A(x)] \subset \mathbb{R}$.*

Even though the set X of entities is deprived of any mathematical structure, it inherits some properties of the topological vector space structure *of the separable spaces $\mathcal{S}(X) \subset \mathbb{R}^X$ of valuators, which, in the last analysis, depend on the arithmetic and lattice properties of the real line $\mathbb{R}$.*

Once defined, it is enough to switch on this mathematical machinery to study the properties of hypervalues of resources and values of their elements to offer convincing mathematical metaphors to catallactic issues, focusing our attention on *evolutions* of subsets and their elements as well as their temporal and intertemporal *evaluation*.

2 The Overall Strategy: Transfer of Operations on Intervals of Values through Valuators

In order to benefit from the profusion of mathematical structures on the real line $\mathbb{R}$ and on its intervals, the simple idea is to *transfer* them to X and $\mathcal{P}(X)$ through the valuators $A \in \mathcal{S}(X)$ in the following simple, yet somewhat cumbersome, manner.

We associate with $x \in X$ its value $A(x) \in \mathbb{R}$ and with any subset $K \subset X$ their

$$\begin{cases} \textit{lower value}: & \Delta^\flat(K)(A) := \inf_{x\in K} A(x) \\ \textit{upper value}: & \Delta^\sharp(K)(A) := \sup_{x\in K} A(x) \\ \textit{interval value}: & [A](K) := [\Delta^\flat(K)(A), \Delta^\sharp(K)(A)] \\ \textit{crux}: & K^{\circledast} := \left\{A \in \mathcal{S}(X) \text{ such that } -\infty < \Delta^\flat(K)(A) \leqslant \Delta^\sharp(K)(A) < \infty\right\} \subset \mathcal{S}(X) \\ \textit{confinor}: & K^{\bowtie} := \displaystyle\bigcap_{A\in K^{\circledast}} A^{-1}([\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]) \supset K. \end{cases} \tag{2}$$

We observe that the confinor $K^{\bowtie}$ of K is equivalently defined by

$$K^{\bowtie} := \left\{x \in X \text{ such that; } \forall\, A \in \mathcal{S}(X),\ \ A(x) \leqslant \Delta^\sharp(K)(A)\right\}$$

because $\mathcal{S}(X)$ is a topological vector space.

The fact that a property in X has to be checked *for every* $A \in \mathcal{S}(X)$ may appear awkward and unwieldy at a first sight, but it is actually often used in mathematics[25]

[25] For instance, the set of all distributions $T \in \mathcal{D}^\star(\Delta)$ on an open subset $\Delta \subset \mathbb{R}^\ell$ plays the role of a separable space $\mathcal{S}(X)$ of valuators of indefinitely differential functions with compact support ranging over the space $\mathcal{D}(\Delta)$, playing the role of the underlying set X.

Both the concepts of crux and confinement hull of a subset $K \subset X$ depend on the choice of the separate vector space $\mathcal{S}(X)$ of valuators: the larger it is, the larger the crux $K^{\circledast}_{\mathcal{S}}$ and the smaller the confinement hull $K^{\bowtie}_{\mathcal{S}}$. The set difference $K^{\bowtie} \backslash K$ is *the set of elements which are evaluated on the same footing as elements of K without belonging to it.* Only confined subsets $K = K^{\bowtie}$ can be strictly observed or perceived.

The larger the set of valuators, the better the precision of the perception of the subset of entities.

Indeed, the size of a subset $\mathcal{E} \subset K^{\circledast}$ of valuators in the crux of a set $K \subset X$ measures the "quality" of its perception[26], since inclusions $\mathcal{E}_1 \subset \mathcal{E}_2 \subset K^{\circledast}$ imply inclusions

$$K \;\subset\; K^{\bowtie} \;\subset\; \bigcap_{A \in \mathcal{E}_2} A^{-1}([\Delta^{\flat}(K)(A), \Delta^{\sharp}(K)(A)]) \;\subset\; \bigcap_{A \in \mathcal{E}_1} ([\Delta^{\flat}(K)(A), \Delta^{\sharp}(K)(A)]). \tag{3}$$

The use of a single valuator is thus the coarsest and the most arbitrary.

In biology, for instance, *visual perception* involves at each instant as many valuators as the 95 million rod and cone cells of the human eyes' retina triggering the optic nerve sending the signals to the brain.

Even in economics, there are as many unit prices as goods endowed with units of measure, which are gathered to evaluate commodities through price vectors. However, after *Jeremy Bentham* and *John Stuart Mill* introduced the concept of *utility function* as a *unique* measure of "satisfaction" provided by a set of entities, *Léon Walras* chose this concept to develop what became a "rational choice theory" by assuming that any economic actor maximizes its utility function under financial constraints to define a demand function. He consulted *Henri Poincaré* who expressed strong doubts in a famous correspondence in 1900–1901. Even worse, macro economists introduced the gross national product[27] (GNP) to measure market value of all the goods and services produced in one year by labor and property supplied by the citizens of a country.

The *confinor* (or *confinement hull*) $K^{\bowtie}$ *plays a central role analogous to the closed convex sets* when X is a topological vector space and $\mathcal{S}(X) = X^{\star}$. In particular, they share the statement provided by the Hahn–Banach separation theorem, one of the three main theorems concerning topological vector spaces bearing the name of *Stefan Banach*[28]: any element $x \notin K^{\bowtie}$ outside a confined subset can be separated by at least one valuator $A \in \mathcal{S}(X)$ in the sense that

$$A(x) \;>\; \Delta^{\sharp}(K^{\bowtie})(A). \tag{4}$$

In the same vein, we can transfer properties of the set $\mathbb{I}$ of intervals of $\mathbb{R}$ using this strategy:

4 ***[Transfer of Properties of Intervals]*** *Let us consider any property denoted* Property$([v^{\flat}, v^{\sharp}])$ *of intervals* $[v^{\flat}, v^{\sharp}] \subset \mathbb{R}$. *In order to transfer it to X,* it is sufficient to require that this given property on subsets $K \in \mathcal{P}(X)$ is satisfied ***for all intervals*** $[\Delta^{\flat}(K)(A), \Delta^{\sharp}(K)(A)] \in \mathbb{I} \;\subset\; \mathcal{P}(\mathbb{R})$.

In other words, this can be informally depicted by the symbolic pseudo-formula

[26] This is how the influence of posts mailed by users of social networks is actually measured by the number of likes they get, i.e., the size of their valuators.

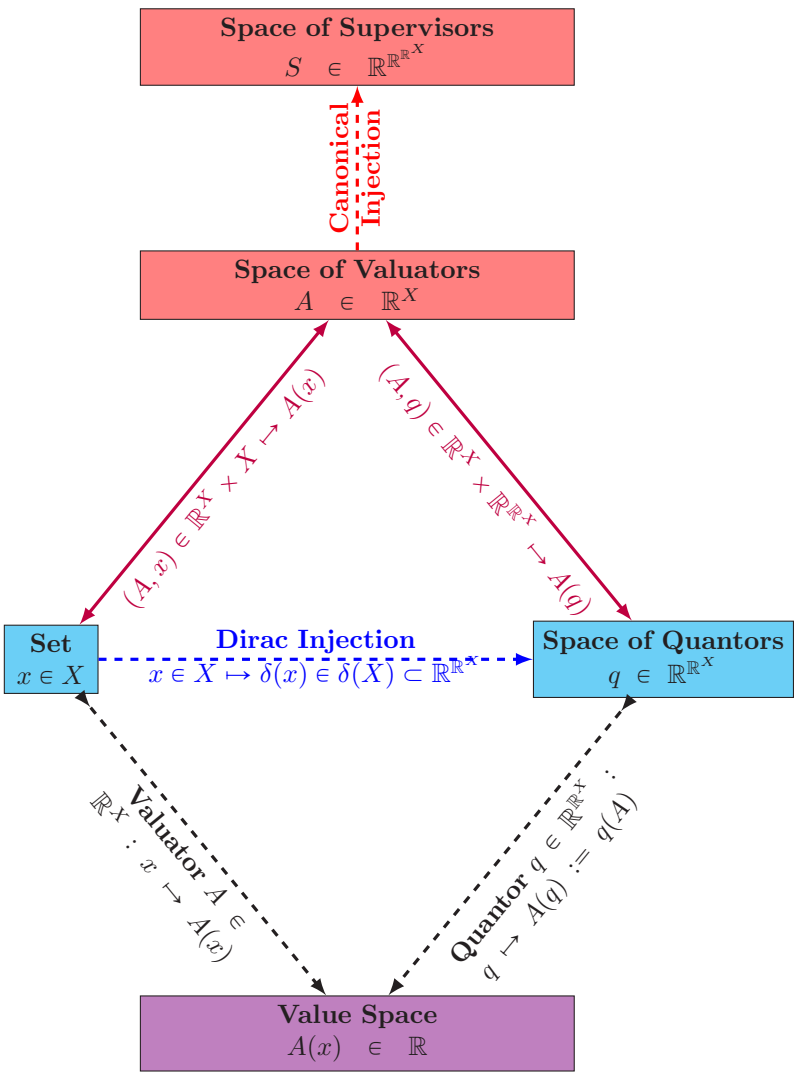
[27] Invented in 1934 by *Simon Kuznets*. Specialists of evolutionary biology have fallen into the same trap by explaining the emergence of organs through the optimization of an ad hoc criterion.

[28] For vector spaces, the separation property is a consequence of the barycentric property characterizing convexity. Confined subsets can also be characterized by a barycentric property: the analogy between convexity and confinement is quite deep.

$$\mathrm{Property}(K) \;:=\; \bigcap_{A\in K^{\circledast}} A^{-1}(\mathrm{Property}([\Delta^{\flat}(K)(A), \Delta^{\sharp}(K)(A)])) \tag{5}$$

This is summarized in the following diagram.

5 ***[The Inheritance Diagram]***



Set and Vector Spaces of Values, Valuators, Quantors, and Supervisors

The set $\mathbb{R}$ of values transmits inheritance to power spaces of the set $\mathbb{R}$ of values:

1. *the space $\mathbb{R}^X$ of* valuators, ***heir to the first degree*** *(of mathematical kinship, so to speak);*
2. *the space $\mathbb{R}^{\mathbb{R}^X}$ of* quantors, ***heir to the second degree****;*
3. *the space $\mathbb{R}^{\mathbb{R}^{\mathbb{R}^X}}$ of* supervisors, ***heir to the third degree***

It illustrates the fact that the set X itself is a ***presumptive heir****, in the sense that it can be canonically identified through the Dirac injection δ with a subset, its* helitope *$\delta(X) \subset \mathbb{R}^{\mathbb{R}^X}$, a subset of the space $\mathbb{R}^{\mathbb{R}^X}$ of quantors and, thus, can inherit in its turn of the set $\mathbb{R}$ of values through the* Dirac injection *associating with $x \in X$ the quantor*

$$q := \delta(x) : A \in \mathbb{R}^X \mapsto \delta(x)(A) := A(x) \in \mathbb{R}.$$

This *transfer technique of properties of the real line* $\mathbb{R}$ associated with the choice of a separable space $\mathcal{S}(X) \subset \mathbb{R}^X$ of valuators plays the same role that the classical transfer of those properties to vector spaces, for example, by transferring operations on $\mathbb{R}$ to vectors of $\mathbb{R}^\ell$ "components by components" in which we lavishly live since 1888 following their axiomatization by *Giuseppe Peano*. Contrary to all our predecessors who have constituted this intellectual monument that is functional analysis on vector spaces, all that remains to us is to accomplish the modest task of transferring and adapting some of these concepts and theorems that interrelate them to this new

"extremal" framework.

In other words, translating the individual "realities" of real numbers by $A^{-1}(\text{Property}([\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]))$ in (5), the intersection $\text{Property}(K) := \bigcap_{A\in K^\circledast} A^{-1}(\text{Property}([\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]))$ is what *is "common" to all the individual evaluations* as a way to transfer the "reality" of real numbers!

We observe at once that *the larger the separable space $\mathcal{S}(X)$ of valuators, the smaller the sets of transferable properties of* $\mathbb{R}$. This raises the question of the choice of spaces of valuators. For example,

1. The separation assumption on $\mathcal{S}(X)$ is equivalent to the "consistency assumption":
$$\forall\, x \in X,\ \{x\} = \bigcap_{A\in\mathcal{S}(X)} A^{-1}(A(x)). \tag{6}$$

2. On the algebraic side, we can define the *extremal sums* of subsets of plain sets by setting
$$K \oplus L := \bigcap_{A\in\mathcal{S}(X)} A^{-1}\left[\Delta^\flat(K)(A) + \Delta^\flat(L)(A), \Delta^\sharp(K)(A) + \Delta^\sharp(L)(A)\right] \tag{7}$$
and *extremal transitions* $\theta_h(x,u)$ from $x \in X$ in the direction $u \in X$ by :
$$\forall\, x \in X,\ \forall\, u \in X,\ \forall\, h > 0,\ \ \theta_h(x,u) := \bigcap_{A\in\mathcal{S}(X)} A^{-1}(A(x) + hA(u)) \in X, \tag{8}$$
the cornerstone of differential calculus.

3. On the *topological* side, we define the *mapwise topology* $X_\mathcal{S}$ on the set X, under which *all valuators* $A \in \mathcal{S}(X)$ *are continuous*: a sequence x_n converges to x if, for all $A \in \mathcal{S}(X)$, $A(x_n)$ converges to $A(x)$.

   The larger the space of valuators, the stronger the mapwise topology, the fewer compacts there are.

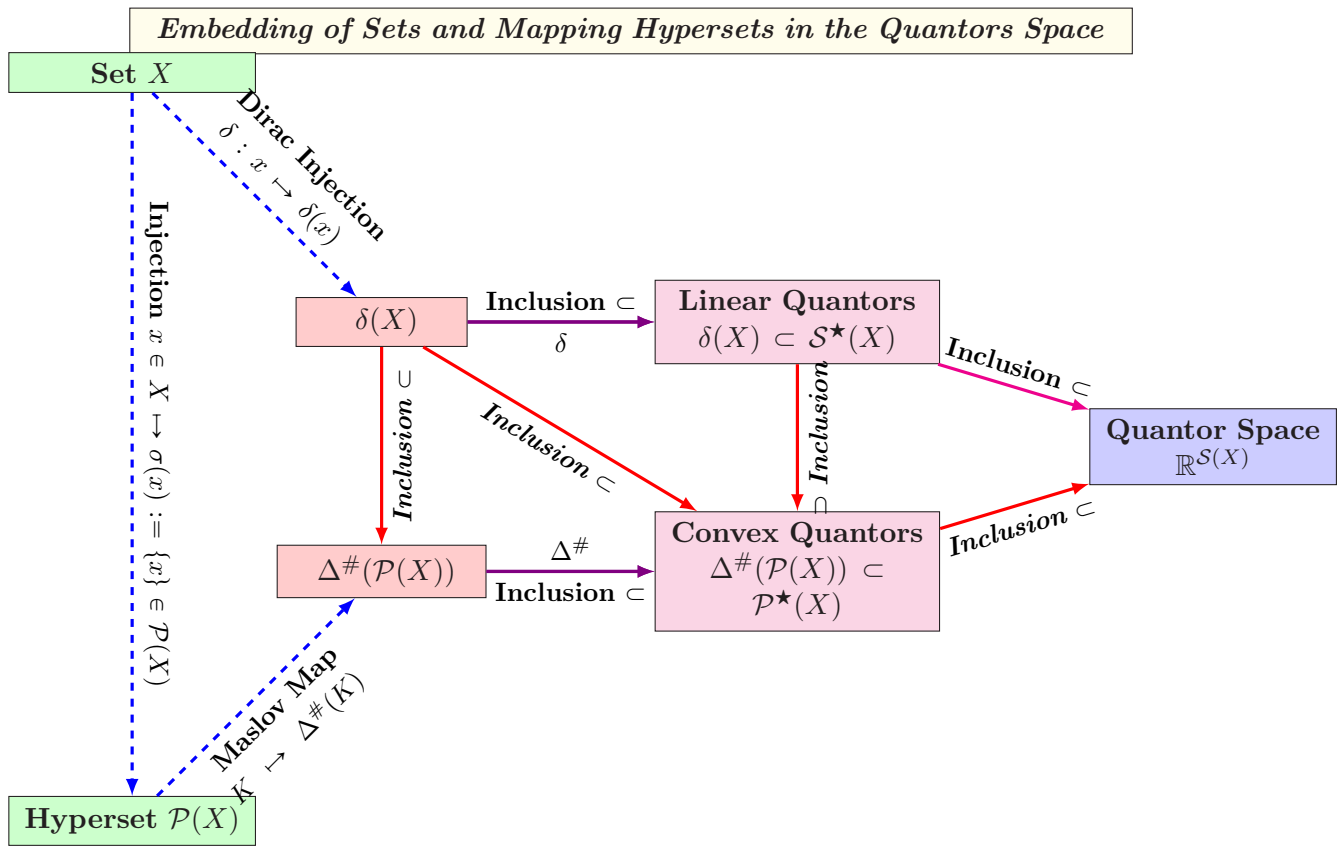
For any subset K and any valuator A, the set $A(K) := A\left(\bigcup_{x\in K}\{x\}\right) = \bigcup_{x\in K} A(x)$ of values of elements of K *is the subset of an interval*, but not necessarily equal to the interval $[\Delta^\flat(K)(A), \Delta^\sharp(K)(A)]$. It could be a union of a dusting of smaller disjoint intervals, excluding a reasonable analysis of $A(K)$.

We note that *adding* commodity vectors and *gathering* subsets share the same purpose but are very different, since $x + x = 2x \neq x$ if $x \neq 0$ would become $K \cup K = K = K \cap K$ for any $K \in \mathcal{P}(X)$, *the boolean idempotence trap*. No longer would hoarders increase subsets by replacing addition of vectors by unions of sets: none of them can be a subset of another for the union to be strictly larger. Gathering and adding elements refer to different algebras, the arithmetic and the max-plus ones, since, using valuators
$$\forall\, A \in \mathcal{S}(X),\ \forall\, x \in K,\ y \in K,\ \left\{\begin{array}{rcl} \max(A(x), A(y)) & \leqslant & \Delta^\#(K)(A) \\ A(x) + A(y) & \leqslant & 2\Delta^\#(K)(A). \end{array}\right. \tag{9}$$

To summarize, instead of supplying a set X with algebraic and/or topological structures and derive the sets of morphisms and/or continuous numerical functions as it is customary to proceed, we propose instead to start by equipping the set X with a vector space of valuators and derive the algebraic and/or topological structures on the set under which the valuators are morphisms and/or continuous.

Let us denote by $\mathcal{S}^\star(X)$ the algebraic dual of $\mathcal{S}(X)$, the vector space of Dirac measures $A \mapsto \langle \delta(x), A\rangle = A(x)$ when x ranges over X, and by $\mathcal{P}^\star(X)$ the set of Maslov measures $A \mapsto \Delta^\sharp(K)(A)$ when K ranges over $\mathcal{P}(X)$.

6 ***[Quantors and Supervisors]***



We can then prove a general *Minsup theorem* when the subset $K = K^{\bowtie}$ is a bounded confined subset (instead of bounded closed convex set). Among the many consequences of the *Minsup theorem*, we obtain the following result. If

1. $K = K^{\bowtie} \subset X$ is a *confined* subset *bounded* for the mapwise topology $X_{\mathcal{S}}$,
2. $\mathcal{E} \subset \mathcal{S}(X)$ is a *nonempty bounded closed convex* subset of valuators,

then there exist both $\widehat{x} \in K$ and $\widehat{A} \in \mathcal{E}$ such that

$$\inf_{A \in \mathcal{E}} A(\widehat{x}) \;=\; \widehat{A}(\widehat{x}) \;=\; \sup_{x \in K} \widehat{A}(x). \tag{10}$$

This closes the gap between *micro-catallactics* dealing with entities $\widehat{x} \in K$ maximizing $\widehat{A}(x) \leqslant \widehat{A}(\widehat{x})$ at $\widehat{x}$ and *macro-catallactics* concerned with valuators $\widehat{A} \in \mathcal{S}(X)$ minimizing budgetary constraints $\widehat{A}(\widehat{x}) \leqslant A(\widehat{x})$ at $\widehat{A}$.

3 Binary and Multinary Relations: Towards Relational Analysis

Once the general results are gathered for any set X and their hypersets $\mathcal{P}_r(\mathrm{X})$, we must turn our attention to the product $\prod_{i=1}^{\ell} X_i$ of ℓ sets X_i of sequences $(x_1, \ldots, x_\ell)$ of interrelated elements of each of the sets X_i.

When $\ell = 2$, we recover the *input-output relations* between the input $x_1 \in X_1$ of the pair $(x_1, x_2) \in \mathcal{R} \subset X_1 \times X_2$ and its output $x_2 \in X_2$, where $\mathcal{R}$ is called t*he graph of the binary relation*.

At the dawn of XXth century, the founders of set theory after *Georg Cantor*, *Giuseppe Peano*, *Felix Hausdorff*, *René Baire*, *Maurice Fréchet*, *Francesco Severi*, *Kazimierz Kuratowski*, *Georges Bouligand*, to name a few, used what became the concept of binary relations, the graphs of *order relations* and, more generally, of *set-valued maps*. They generalized the concept of maps introduced in 1370 by *Oresme*, who represented them in a graphical form which prefigured the first bar graphs and histograms of *André-Michel Guerry*.

Dealing with binary relation and single-valued maps was not a problem as long as motivations for taking into consideration the set-valued character of some maps were ignored at that time when only *well posed problems* in Hadamard's sense became the tablets of the mathematical law. *Ill posed problems, inverse problems*, and many other unorthodox problems were turning up in every domain of activity under a manifold of new names, whenever the existence of a solution may fail for some data, whenever uniqueness of the solution is at stake, whenever instability occurred. They were heretics and their proponents had long been excommunicated. We had to wait for the 1950s, when operational research, the mathematical formalization of Walras economic equilibrium, the rise of convex analysis after *Moritz Fenchel*, *Jean-Jacques Moreau*, and *Terry Rockafellar* laid its corner stone to investigate set-valued analysis and, now, *relational analysis*, among many other motivations. Since 1981, set-valued maps and relations could be differentiated (both graphical derivatives in vector spaces in 1980 and, in 1992, pointwise derivatives in metric spaces).

However, should we consider only binary relations ? It is curious to observe that the need for ternary relations was raised in linguistics since *Ferdinand de Saussure* and *Charles Peirce*, the founders of modern linguistics, invented ternary relations "*sign/signifier/signified/referent*" and "*sign/object/ interpretant*," respectively.

Relational analysis was born after meetings in New York during the Second World War among three exiled Europeans, the linguist *Roman Jakobson* who instilled the idea of structural linguistics to the anthropologist *Claude Lévi-Strauss*, who, in turn, induced the mathematician *André Weil*, the "father" of *Nicolas Bourbaki*, who wrote an appendix "*Sur l'étude algébrique de certains types de lois du mariage*'" to the first part of Claude Lévi-Strauss' "*Structures Elémentaires de la parenté*" that appeared in 1949. They shared the conviction that the properties of *relations* between sets of concepts or objects were at least as important and crucial to understand them as the properties of their elements: it was the birth of *structuralism*. A rare example of mathematics motivated by social sciences after the discovery of probabilities by *Blaise Pascal* and *Pierre de Fermat* and *Condorcet social choice functions*.

It is therefore hardly surprising that *Nicolas Bourbaki* introduced in 1956 the concept of *relation* in section 3.13 of *Theory of Sets* (*Théorie des ensembles*), in which he wrote *"[...] we have considered only the case of three sets merely to fix the ideas; analogous considerations hold for any finite number of sets*[29]*"*. However, "he" later abandoned the study of general relations which could be called *multinary* to concentrate on binary relations. "He" restricted "his" attention in the volumes that follow to input-output *single-valued maps* under the pretext that *set-valued maps are single-valued maps from sets to hypersets*, forgetting that, hypersets do not always faithfully

[29] "*On n'a envisagé que le cas de trois ensembles, uniquement pour fixer les idées; des considérations analogues valent pour plusieurs ensembles en nombre fini quelconque.*"

inherit structures of the underlying spaces. Even *category theory*[30] concentrates only on input-output morphisms and functors.

7 ***[Multinary Relations]***

In the case of product $\prod_{i=1}^{\ell} X_i$ *of* ℓ *sets* X_i*, we restrict the hyperset of the product to the product of hypersets and the space of valuators on the product to the product of the spaces of valuators:*

1. *the product* $\prod_{i=1}^{\ell} \mathcal{P}(X_i) \subset \mathcal{P}(\prod_{i=1}^{\ell} X_i)$ *of hypersets;*
2. *the product* $\prod_{i=1}^{\ell} \mathcal{S}_i(X_i) \subset \mathcal{S}(\prod_{i=1}^{\ell} X_i)$ *of spaces of valuators,*

for preserving this important structure of product of sets. In this case, to require that any element $x = (x_1, x_2, \ldots, x_j, \ldots, x_{\ell-1}, x_\ell)$ *belongs to a subset* $\mathcal{R} \subset \prod_{i=1}^{\ell} X_i$ imposes relations among the ℓ components x_j of $x \in \mathcal{R}$*, justifying the interpretation of subsets of* $\mathcal{R} \subset \prod_{i=1}^{\ell} X_i$ *as* n-ary relations.

One can also attach to the product $\prod_{i=1}^{\ell} X_i$ *sequences of quantifiers* $\exists$ *or* $\forall$ *(or none). In this case, this indicates that a property on the variables* $(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_\ell) \in \prod_{i=1}^{\ell} X_i$ *must be satisfied*

1. *by* at least one *variable* $x_i \in X_i$ *concerned with the quantifier* $\exists$ *, in which cases these variable are regarded as* controls *whenever there exists an actor controlling its evolution or* regulons *in the opposite case,*
2. *by* all *variables* $x_j \in X_j$ *concerned with the quantifier* X_j*, in which cases these variable are regarded as* tyches *or perturbations.*

In the latter case, the ordering of the spaces X_i *is important since* quantifiers do not commute*, except in exceptional situations requiring difficult proofs, such as in the minimax theorem in game theory.*

Since the order of the sets X_i in the product $\prod_{i=1}^{\ell} X_i$ is arbitrary, we can use any permutation $i \in \{1, \ldots, \ell\} \mapsto j = \sigma(i) \in \{1, \ldots, \ell\}$ among the $\ell!$ permutations (bijective applications) of $\{1, \ldots, \ell\}$. They induce the $\ell!$ relations $\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(\ell)}) \in \sigma(\mathcal{R})$. Happily for $\ell = 2$, there are only $2! = 2$ permutations $\sigma(1,2) = (1,2)$ and $\sigma^{-1}(1,2) = (2,1)$, the relation $\sigma(\mathcal{R}) = \mathcal{R}$ and its inverse $\mathcal{R}^{-1} := \sigma^{-1}(\mathcal{R})$.

4 The Ultimate Inputs: Dates and Durations

Living organisms perceive the *modifications* of their environment through their senses (and, for human beings, the instruments they invent) and *adapt* to them. Cyclic "evolution of celestial bodies," to start with. They record and remember their perceptions, deduce the existence of a "before" and an "after," and therefore, of time and its arrow of both irreversibility and cyclicity, a *helicoidal time*, so to speak.

Human beings could only *observe, verbalize, and enumerate* the dates of birth and death of living beings on the surface of our planet, and next *infer* at least that they *are born and die in finite time.*

[30] Introduced in 1942 by *Samuel Eilenberg* (rehabilitating earlier Polish prolegomena) and *Saunders Mac Lane* and developed later by *Alexandre Grothendieck* who introduced the concept of *topos*.

8 *[**The Zero and the Infinities**] Actually, instants (temporal windows of duration* 0*) and points (of diameter* 0*) have been measured yet: the smallest duration measured so far is the yoctosecond (*10^{-24} *seconds), and the smallest length is the attometer (*10^{-18} *meters).*
No more than $+\infty$ *and* $-\infty$ *has been measured in our universe, especially since physicists state that in both their "big bang" and "big crunch" theories, the universe also was born and will die in finite time.*
What is an obstacle for physicists is not an impediment for mathematicians, who did not hesitate to invent them, after so many clever trials, failures, and ephemeral success to be renewed again and again.

This is a motivation to return to the origins of differential calculus based on the normal cone to the graph $\mathrm{Graph}(x(\cdot)) \subset \mathbb{R} \times X$ of an evolution $x(\cdot) : t \in \mathbb{R} \mapsto x(t) \in X$ at $(t, x(t))$. By convention, we supply the space $\mathbb{R} \times X$ with the separable vector space $\mathbb{R} \times \mathcal{S}(X)$ of valuators

$$(-v_1, A) : (t, x) \in \mathbb{R} \times X \mapsto A(x) - v_1 t. \tag{11}$$

Any evolution $x(\cdot)$ is characterized by its *evolutionary graph* $\mathrm{Graph}(x(\cdot)) \subset \mathbb{R} \times X$, set of pairs $(t, x(t))$ when the dates t range over an interval of $\mathbb{R}$.

The *normal cone* $N_{\mathrm{Graph}(x(\cdot))}(t, x(t)) \subset \mathbb{R} \times \mathcal{S}(X)$ to the graph of an evolution $x(\cdot) : t \mapsto x(t) \in X$ at $(t, x(t)) \in \mathrm{Graph}(x(\cdot))$ is the set of pairs $(-v_1, A) \in \mathbb{R} \times \mathcal{S}(X)$ satisfying

$$\left\{ \begin{array}{l} ((-v_1, A)(t, x(t))) \ = \ A(x(t)) - v_1 t \\ \ = \ \sup_{(s,x(s))}((-v_1, A))(s, x(s)) \ \geqslant \ A(x(s)) - v_1 s. \end{array} \right. \tag{12}$$

Setting successively $s = t - h \ \leqslant \ t$ whenever $h > 0$ and $s = t + h \ \geqslant \ t$ whenever $h > 0$, we infer that

$$\left\{ \begin{array}{c} (-v_1, A) \in N_{\mathrm{Graph}(x(\cdot))}(t, x(t)) \\ \text{if and only if} \\ \overrightarrow{\nabla} A(x(t)) \ := \ \sup\limits_{h>0} \dfrac{A(x(t+h)) - A(x(t))}{h} \ \leqslant \ v_1 \ \leqslant \ \inf\limits_{h>0} \dfrac{A(x(t)) - A(x(t-h))}{h} \ =: \ \overleftarrow{\nabla} A(x(t)). \end{array} \right. \tag{13}$$

Observe also that $\overleftarrow{\nabla} A(x(t)) \ = \ -\overrightarrow{\nabla}(-A(x(t)))$.

9 *[**Extremal Coderivatives and Velocities of Evolutions**] We say that*

1. $\overrightarrow{\nabla} A(x(t))$ *and* $\overleftarrow{\nabla} A(x(t))$ *are, respectively, the* extremal prospective derivatives *and* extremal retrospective derivatives *of the evolution* $x(\cdot)$ valuated *by* $A \in \mathcal{S}(X)$ *at date* t*;*

2. *the interval* $\nabla A(x(t)) \ := \ \left[\overrightarrow{\nabla} A(x(t)), \overleftarrow{\nabla} A(x(t))\right]$ *is the* extremal interval derivative *of* $x(\cdot)$ *at* t valuated *by* A*;*

3. *the extremal* coderivative $\partial x(t) \subset \mathcal{S}(X)$ *and the* extremal derivative $\nabla x(t) \subset X$ *are, respectively, defined by*

$$\boxed{\begin{array}{lcl} \partial x(t) & := & \left\{ A \in \mathcal{S}(X) \text{ such that } \left[\overrightarrow{\nabla} A(x(t)), \overleftarrow{\nabla} A(x(t))\right] \neq \varnothing \right\} \ \subset \ \mathcal{S}(X) \\ \nabla x(t) & := & \displaystyle\bigcap_{A \in \partial x(t)} A^{-1}\left[\overrightarrow{\nabla} A(x(t)), \overleftarrow{\nabla} A(x(t))\right] \ \subset \ X \end{array}} \tag{14}$$

If there exists one valuator $A^\star \in \partial x(t)$ such that

$$\overrightarrow{\nabla} A^\star(x(t)) := \sup_{h>0} \frac{A^\star(x(t+h)) - A^\star(x(t))}{h} = \inf_{h>0} \frac{A^\star(x(t)) - A^\star(x(t-h))}{h} =: \overleftarrow{\nabla} A^\star(x(t)),$$

then $\nabla x(t)$ is reduced to a singleton. *In this case, we say that* $x(\cdot)$ *is* extremally differentiable *at date* t. *Whenever the coderivative* $\partial x(t) = \{A\} \neq 0 \in \mathcal{S}(X)$ *is reduced to one element only, the evolution is said to be* smoothly differentiable *at* $x(t)$.

5 The Ultimate Outputs: Values

A more appealing title for this section would have been "Time and Money" as a nod from beyond the grave to *Benjamin Franklin* who wrote to his son: *"Remember that Time is Money,"* since time is the ultimate input. As for money, it is only an example of value $v \in \mathbb{R}$ when it is endowed with a unit of measure, the *numéraire*[31], and an origin, 0.

Dates are the inputs[32] and values the outputs of the perception of the environment, two numbers that the human brains perceive, learn, and use (and abuse when they are under the hold of the *pantometric drug*) to frame the perception between them.

This suggests devotion of particular attention to the

1. *Evaluations* $v \in C(x) := [C^\flat(x), C^\sharp(x)] \subset \mathbb{R}$ of entities $x \in X$ obtained through *numerical relations* $C : X \mapsto \mathbb{I}$, described by their graphs $\mathrm{Graph}(C) \subset X \times \mathbb{I}$. In this case, we chose for spaces of valuators the spaces $\mathcal{S}(X) \times \mathbb{R}$.

2. *Evolutions* $t \in [t_0, t_1] \mapsto x(t) \in X$ of entities and *morphological evolutions* (nicknamed *tubes*) $t \in [t_0, t_1] \mapsto K(t) \in \mathcal{P}(X)$ of their subsets, as well as their "velocities" $\upsilon(t)$ adequately transferred and defined, described by their $\mathrm{Graph}(K(\cdot)) \subset \mathbb{R} \times X$. In this case, we chose for space of valuators the spaces $\mathbb{R} \times \mathcal{S}(X)$.

3. The Maupertuis' concept of *action* and the economic concept of *intertemporal evaluation* of an *evolution* defined by
$$\left[\int_{t_0}^{t_1} L^\flat(x(t), \upsilon(t))dt, \int_{t_0}^{t_1} L^\sharp(x(t), \upsilon(t))dt\right] \tag{15}$$
by associating with the kinetic pairs $(x(t), \upsilon(t))$ of entities and their velocities the evolutions $(c(t), c'(t)) \in [C^\flat(x(t), C^\sharp(x(t)))] \times [L^\flat(x(t), \upsilon(t)), L^\sharp(x(t), \upsilon(t))]$ and their integral over the temporal window.

4. The *inertia principle* pointed out later at least by *Alfred Wallace* (1823–1913) in 1858 in an essay that he sent to *Darwin* (1809–1882), who wrote: "*[...] and lastly, although each species must have passed through numerous transitional stages, it is probable that the periods, during which each underwent modification, though many and long as measured by years, have been short in comparison with the periods during which each remained in an unchanged condition.*" It amounts to say that the velocity remains constant as long as viability is not at stake, or that decisions should be taken at the propitious instant instead of minimizing optimality criteria which may not exist.

[31] Only used when X is a vector space of commodity vectors and the space of valuators is its dual $X^\star$.
[32] To which one can add durations and multi-criteria values if needed.

5. The "*micro-catallactic*" evolutions $t \mapsto K(t)$ of baskets goods with which we associate "*macro-catallactic*" evolutions $t \mapsto (A(t), v(t))$ of the valuator-value pairs related to subsets $K(t)$ by budgetary constraints

$$\forall\, t \in [t_0, t_1],\ \ [\Delta^\flat(K(t))(A(t)), \Delta^\sharp(K(t))(A(t))] \ \subset\ [r^\flat(t), r^\sharp(t)], \tag{16}$$

summarising inequalities $r^\flat(t) \leqslant \Delta^\flat(K(t))(A(t)) \leqslant \Delta^\sharp(K(t))(A(t)) \leqslant r^\sharp(t)$ between lower and upper incomes offered to catallactic actors, lower production costs, and upper consumption values of the subsets $K(t)$ of goods.

Knowing the evolutionary mechanism governing the evolution $t \mapsto K(t) \in \mathcal{P}(X)$ of baskets of goods produced, exchanged, and consumed by catallactic actors, from the evolutionary mechanism of the valuator-value pairs $(A(t), (r^\flat(t), r^\sharp(t)))$ governed by the actors and a "last-resort value provider" satisfying the above budgetary requirement.

6 The Choice of Subspaces of Valuators

Excluding the case when the set X is itself the product of several sets, as indicated above, the space X may be endowed with specific algebraic or topological structures. In this case, the question arises whether the structure on X defined by a separable space $\mathcal{S}(X)$ of valuators is consistent or not with this *a priori* given structure.

In most cases, choosing for space of valuators the morphisms for this structure allows us to compare the two structures. This is the reason why whenever X is a topological vector space, we choose for space of valuators the space $X^\star := \mathcal{L}(X, Y)$ of continuous linear functionals on X. However, the hyperset $\mathcal{P}(X)$ does not naturally inherit this vector structure, so that some of the results below must be adapted to study the evolutions of tubes into vector spaces or set-valued functions on vector spaces.

We mention other well known examples of sets X supplied with well studied structures and adapted subsets of valuators:

1. When X is a compact metric space, then the Banach space of continuous functions $f : X \to \mathbb{R}$ is a space of valuators;

2. When X is a metric space, the Banach space of Lipschitz functions $f : X \to \mathbb{R}$ is a natural candidate for a set of valuators[33];

3. When X is a measurable space, the Lebesgue spaces $L^p(X)$ are sets of valuators;

4. When $X \subset \mathbb{R}^\ell$ is an open subset, the space $\mathcal{E}(X)$ of indefinitely differentiable functions, as the Sobolev spaces, are natural spaces[34] of valuators in this context.

These well known specific spaces of valuators are adapted to the particular structure of X.

Another important structure which is currently extensively studied is the case of subsets X endowed with a *reproducing kernel*[35], a positive definite symmetric function playing the role of a (non bilinear) scalar product $k(x, y) : X \times X \mapsto \mathbb{R}$. The functions $K_y : x \in X \mapsto K_y(x) := k(x, y) \in \mathbb{R}$ indexed by the elements $y \in X$ generate a

[33] Their duals are, respectively, the spaces of *Radon measures* and *Wasserstein measures* with compact support.

[34] Its dual is the space of distribution with compact support.

[35] They were used by *Vapnik* and *Lerner* when they invented the "support vector machines" (Séparateurs à Vaste Marge in French) algorithms in 1963, shortened to "SVM". They were used recently in control theory (See for instance [10, Aubin-Frankowski], [11, Aubin-FrankowskiBensoussan], [12, Aubin-FrankowskiGaubert]).

pre-Hilbertian vector space of valuators K_y endowed with the scalar product $((K_y, K_z)) = k(y, z)$. Its completion $\mathcal{H}_k(X)$ is called the *reproducing kernel Hilbert space* in 1950 by *Nachman Aronszajn*. We can adapt this approach by replacing kernels $k : X \times X \mapsto X$ with functions $g : X \times X^* \mapsto \mathbb{R}$ with which we associate dual uniform spaces of valuators on X and X^*: they play the role of the duality product on vector spaces in duality. Reproducing kernels and gists allow the transfer of many properties of Hilbert and topological vector spaces, in particular, the separation theorems for separating disjoint confined subsets.

Concerning the choice of the space of values, we chose $\mathbb{R}$ since it is the oldest and richest set of numbers, the cornerstone of vector spaces and metric spaces, for instance. But at this level, too, it is possible to choose any space $\mathbb{V}$ of values and, in the same spirit, use valuators in the space $\mathbb{V}^X$ and transfer its structure to any plain set X.

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