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# From microscopic to macroscopic scale equations: mean field, hydrodynamic and graph limits

Thierry Paul\*      Emmanuel Trélat†

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## Abstract

Considering finite particle systems, we elaborate on various ways to pass to the limit as the number of agents tends to infinity, either by mean field limit, deriving the Vlasov equation, or by hydrodynamic or graph limit, obtaining the Euler equation. We provide convergence estimates. We also show how to pass from Liouville to Vlasov or to Euler by taking adequate moments. Our results encompass and generalize a number of known results of the literature. As a surprising consequence of our analysis, we show that sufficiently regular solutions of any linear PDE can be approximated by solutions of systems of  $N$  particles, to within  $1/\log \log(N)$ .

## Contents

### 1 Introduction

#### 1.1 Setting

Multi-agent collective models have regained an increasing interest over the last years, due in particular to their connection with mean field and graph limit equations. At the microscopic scale, such models consist of considering particles or agents evolving according to the dynamics

$$\dot{\xi}_i(t) = \frac{1}{N} \sum_{j=1}^N G_{ij}^N(t, \xi_i(t), \xi_j(t)), \quad i \in \{1, \dots, N\}, \quad (1)$$

for some (large) number of agents  $N \in \mathbb{N}^*$  where, for every  $i \in \{1, \dots, N\}$ ,  $\xi_i(t) \in \mathbb{R}^d$  (for some  $d \in \mathbb{N}^*$ ) stands for various parameters describing the behavior of the  $i^{\text{th}}$  agent and  $G_{ij}^N : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a mapping modeling the interaction between the  $i^{\text{th}}$  and  $j^{\text{th}}$  agents.

Dynamics of the form (??) are used in a wide range of very different problems, ranging from the study of flocking and swarming in biology, of modeling traffic flows, to dynamics evolution in social sciences (see, e.g., [?, ?, ?, ?, ?, ?, ?, ?, ?], just to mention a few of a vast literature).

Among classes of multi-agent systems, we point out the so-called opinion systems that have the striking property of exhibiting features nowadays grouped under the common denomination of self-organization: their large-time asymptotic behavior shows consensus phenomena, namely an

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alignment of all values  $\xi_i(t)$  to a single one. These models correspond to  $G_{ij}^N(t, \xi_i, \xi_j) = \sigma_{ij}(\xi_j - \xi_i)$ , i.e., their dynamics is

$$\dot{\xi}_i(t) = \frac{1}{N} \sum_{j=1}^N \sigma_{ij}(\xi_j(t) - \xi_i(t)) \quad (2)$$

where  $(\sigma_{ij})_{1 \leq i, j \leq N}$  is a  $N$ -by- $N$  matrix whose spectral properties may cause the above-mentioned asymptotic behavior. We refer to the recent [?] for a large set of references concerning the two systems (??) and (??), where also the case of time-dependent matrices  $\sigma$  is treated.

The large  $N$  limit of systems (??), (??) has been extensively studied over the last years. In [?] the author shows how to pass to the continuum limit in nonlocally coupled dynamical networks by using the concept of *graph limit*. This concept has also been used recently in [?] to obtain discrete-to-continuum convergence results with error estimates in the Wasserstein distance. We also mention the recent articles [?, ?, ?]. In a nutshell, the graph limit allows one to pass to the limit from the general system of agents (??) to an integro-differential equation by interpreting the right-hand side of (??) as a Riemann sum. Then, obtaining the limit equation is seen as passing to the limit in a Riemann sum and thus obtaining a continuous integral. This is what has been done in [?, ?, ?] for the opinion propagation model  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$ , leading to the graph limit equation

$$\partial_t y(t, x) = \int_{\Omega} \sigma(x, x')(y(t, x') - y(t, x)) dx'$$

where for instance  $\sigma(i, j) = \sigma_{ij}$ . This example is particularly paradigmatic of what we develop in the present paper.

Another important class of systems (??) concerns particles  $\xi_i = (p_i, q_i) \in \mathbb{R}^d \times \mathbb{R}^d$ , either Hamiltonian in which case  $G$  takes the form

$$G_H(t, \xi_i, \xi_j) = \left( \begin{array}{c} p_i \\ \nabla V(q_i - q_j) \end{array} \right) \quad (3)$$

for some potential  $V$ , or of Cucker-Smale type in which case we have

$$G_{CS}(t, \xi_i, \xi_j) = \left( \begin{array}{c} p_i \\ F(|q_i - q_j|)(p_i - p_j) \end{array} \right) \quad (4)$$

for some influence function  $F$ .

The main difference between general systems (??) and the particular systems (??) and (??) is that for the latter the mapping  $G_{ij}$  does not depend on  $i$  and  $j$ , that is on the “names” of the agents. A consequence is that the associated evolution equation preserves the *indistinguishability* of the particles, a feature often considered as fundamental for the large  $N$  limit of particle systems. One of the objectives of the present article is to show how to extend the standard mean field methods to the non-indistinguishable setting, simply by endowing to the index  $i$  the status of a parameter, treated as a new state variable of zero dynamics.

The systematic study of large  $N$  limit of particle systems has a long and glorious history, starting with Hartree (see [?]) in the late 20’s for quantum systems, and then Vlasov in the 40’s (see [?]) who derived the eponymous *kinetic equation* (here, we present it in a form suitable for our purposes), called *Vlasov equation*,

$$\partial_t \mu_t + \operatorname{div}_{\xi}(\mathcal{X}[\mu_t] \mu_t) = 0 \quad \text{with} \quad \mathcal{X}[\mu] = \int G(\xi, \xi') d\mu(\xi'). \quad (5)$$

There are two classical ways for deriving (??). A first consists of using the concept of empirical measure  $\mu_t^e(\xi) = \frac{1}{N} \sum_{i=1}^N \delta(\xi - \xi_i(t))$ , which is a solution for (??), and then of taking the *mean*

*field limit*. A second consists of using marginals of the solution of the *Liouville equation* associated to the particle system, namely the equation satisfied by the pushforward of probability measures on  $\mathbb{R}^{2Nd}$  under the flow generated by the particle system. In the latter case one shows that the first marginal of this pushforward, which is a probability measure on  $\mathbb{R}^{2d}$ , satisfies at the limit  $N \rightarrow +\infty$  the Vlasov equation (??). The last step of this process, called the *hydrodynamic limit*, starts from the observation that (??) preserves the structure  $\mu_t(\xi) = \mu_t(q, p) = \nu(t, q)\delta(p - y(t, q))$  leading to the so-called *Euler system* of equations satisfied by the pair  $(\nu, y)$ .

One of the main steps in the developments of the present article is to highlight that, after having derived the Vlasov equation associated to (??) thanks to the trick consisting of parametrizing the status of the index  $i$  as already mentioned, the associated Euler equation (not a system anymore because the extra dynamical variables  $i$  remain at rest and thus give no kinetic part in the Euler system) coincides with the *graph limit* equation associated to (??) – a nontrivial fact, even at the conceptual level as discussed in Section ?? of the paper.

This article is devoted to unifying and generalizing, to some extent, the classical ways to pass to the limit in families of particle systems. The mean field limit, even for distinguishable particles, leads to the Vlasov equation. The hydrodynamic limit leads to the Euler equation. The Liouville equation is a lift of the particle system in a space of probability measures. We analyze in detail the various relationships between particle system, Vlasov, Liouville and Euler, showing how to pass from one to another and deriving, under appropriate assumptions, some convergence estimates. While some of the results are classical (or straightforward extensions of known results), most of them are new and we hope that the overall study may serve to unify different viewpoints.

Let us point out a difference of methodology between the present paper and several previous works concerning the case of indistinguishable particles: we do not estimate propagation of couplings but we rather use direct estimates of the particles and Vlasov flows thanks to the use of a convenient Wasserstein distance. This allows us to obtain, as a by-product of our main results, quantitative mean-field limits for more general vector fields than in (??) and (??).

The last part of our paper deals somehow with the inverse path: given a general partial differential equation (PDE), is it possible to construct explicitly an agent system of the form (??) such that the corresponding graph limit equation coincides with the given PDE we started with? Surprisingly, this happens to be true in a very general setting and this is a consequence of the analysis done in the paper.

The question of whether some classes of PDEs are a “natural” limit of particle systems is classical in fluid mechanics and certainly dates back to Euler: it is classical that the Euler fluid equation can be seen, at least formally, as the limit of evolving “particles of fluids”. This has been formalized in the famous article [?] where Arnol’d interpreted the Euler equation as a geodesic equation in the space of diffeomorphisms, leading to a number of subsequent studies; we refer to [?] (see also the references therein) for a survey on how to “cook up” appropriate groups of diffeomorphisms (and thus, of particle systems) to generate classes of fluid PDEs, like Euler, Camassa-Holm, etc. We also refer to [?] for a recent survey on microscopic, mesoscopic and macroscopic scales for fluid dynamics.

But it is much less classical to show that other, more general PDEs can as well be obtained by passing to the limit in some particle systems. For transport equations, the topic has been extensively studied in [?, ?, ?]. Recently, thanks to the concept of graph limit elaborated in [?], it has been possible to show that heat-like equations can as well be obtained as limits of particle systems (see also [?, ?, ?, ?]). In [?], the authors provide a rigorous derivation from the kinetic Cucker-Smale model to the macroscopic pressureless Euler system by hydrodynamic limit, using entropy methods and deriving error estimates.

Actually, during the *Leçons Jacques-Louis Lions* given in our laboratory in the fall 2021 by Dejan Slepcev, we were intrigued by his way of deriving heat-like equations from unusual particle

systems, by taking not only the limit as the number  $N$  of agents tends to  $+\infty$ , but also another parameter  $\varepsilon$  tends to 0, at some precise scaling (see [?]). The role of  $\varepsilon$  is to smoothen the dynamics. His striking exposition has been for us a great source of inspiration and has motivated the last part of the present article.

In this last part, we provide for a large range of linear PDEs a natural and constructive way for associating an agent system to them. Shortly, as a particular case of our analysis, considering a general PDE

$$\partial_t y(t, x) = \sum_{|\alpha| \leq p} a_\alpha(x) \partial_x^\alpha y(t, x) = Ay(t, x), \quad (6)$$

we show that (??) is the graph limit of (for example) the particle system (??) with

$$G_{ij}(t, \xi, \xi') = G_\varepsilon(t, i, j, \xi, \xi') = \xi' \sum_{|\alpha| \leq p} a_\alpha(x) \partial_x^\alpha \frac{e^{-\frac{(x-x')^2}{2\varepsilon}}}{(\pi\varepsilon)^{\frac{1}{2}}}$$

in the limit  $N \gg \varepsilon^{-1} \rightarrow +\infty$ , with some appropriate scalings. We establish convergence estimates in Wasserstein distance in general, and in  $L^2$  norm under an additional (but general) semigroup assumption.

**Structure of the article.** Section ?? is devoted to studying the passage from microscopic to mesoscopic scale: in other words we show how to pass “from particle to Vlasov” by mean field limit. In Theorem ??, we establish existence, uniqueness and stability properties for the Vlasov equation (??) for distinguishable particles. We recall the classical Lagrangian and Eulerian viewpoints. For the latter, we elaborate on the Liouville equation associated with the particle system. Theorems ?? and ?? are devoted to establish Wasserstein estimates quantifying the discrepancy, as  $N \rightarrow +\infty$ , between the first marginal of the solution of the Liouville equation and the solution of the Vlasov equation; in other words, these results show how Vlasov can be recovered from Liouville by taking marginals and passing to the limit (propagation of chaos). We also give corollaries of the theorems in the indistinguishable case, thus recovering known results.

Section ?? is devoted to studying the passage from mesoscopic to macroscopic scale: in other words we show how to pass “from Vlasov to Euler” by hydrodynamic limit, mainly consisting of taking the moment of order 1. Proposition ?? in that section is concerned with the well known monokinetic approach, but we also investigate the moment of order 2, yielding some consensus results.

Section ?? is devoted to studying the passage from microscopic to macroscopic scale: in other words we show how to pass “from particle to Euler” by graph limit. Theorems ?? and ?? quantify some convergence estimates in  $L^\infty$  norm as  $N \rightarrow +\infty$ . We also discuss how Euler can be recovered from Liouville by taking adequate moments.

Section ?? provides a synthetic summarize of all relationships that we have unraveled. In particular, Figure ?? illustrates the various two-ways passages between particle (microscopic) systems, the Liouville (probabilistic) equation, the Vlasov (mesoscopic, mean field) equation, and the Euler (macroscopic, graph limit) equation. This section can even be read as a motivating preliminary before going ahead.

Finally, as announced, as a surprising byproduct built on the previous developments, we show in Section ?? that general linear PDEs can be obtained by passing to the limit in explicit particle systems, thanks to two asymptotic parameters.

In order to state all subsequent results, we recall in Section ?? hereafter some notations and concepts that we use throughout, in particular the Wasserstein distance and another distance obtained by disintegration of measures, and the concept of tagged partition that is classically used in Riemann integration theory.

We gather in Appendix ?? a number of useful results on the Wasserstein distance, empirical and (so-called semi-empirical measures. Appendix ?? is devoted to proving some of the main theorems.

## 1.2 General notations

Let  $E$  be a Polish space, endowed with a distance  $d_E$ .

**Hölder and Lipschitz mappings.** Let  $U$  be a subset of  $E$ . Let  $k \in \mathbb{N}^*$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^k$ . Given any  $\alpha \in (0, 1]$ , we denote by  $\mathcal{C}^{0,\alpha}(U, \mathbb{R}^k)$  the set of all continuous mappings  $g \in \mathcal{C}^0(U, \mathbb{R}^k)$  that are  $\alpha$ -Hölder continuous (with respect to the norm  $\|\cdot\|$ ), meaning that

$$\text{Hol}_\alpha(g) = \sup_{\substack{y, y' \in U \\ y \neq y'}} \frac{\|g(y) - g(y')\|}{d_E(y, y')^\alpha} < +\infty.$$

When  $\alpha = 1$ , we speak of a Lipschitz mapping and we denote  $\text{Lip}(g) = \text{Hol}_1(g)$ . When  $U$  is compact,  $\mathcal{C}^{0,\alpha}(U, \mathbb{R}^k)$  is a Banach space endowed with the norm

$$\|g\|_{\mathcal{C}^{0,\alpha}(U, \mathbb{R}^k)} = \max_{y \in U} \|g(y)\| + \text{Hol}_\alpha(g).$$

When  $k = 1$  and  $\alpha = 1$ , we denote  $\text{Lip}(U) = \mathcal{C}^{0,1}(U, \mathbb{R})$ .

**Probability Radon measures.** We denote by  $\mathcal{P}(E)$  the set of nonnegative probability Radon measures on  $E$ . We also consider  $\mathcal{P}_c(E)$ ,  $\mathcal{P}^{ac}(E)$ , where the subscript  $c$  means “with compact support” and the superscript  $ac$  means “absolutely continuous with respect to a Lebesgue measure” (in the case where  $E$  is equipped with a Lebesgue measure), and for every  $p \geq 1$  the set  $\mathcal{P}_p(E)$  stands for the set of all  $\mu \in \mathcal{P}(E)$  that have a finite moment of order  $p$ , i.e.,  $\int_E d_E(y_0, y)^p d\mu(y) < +\infty$  where  $y_0 \in E$  is arbitrary. Given any Borel mapping  $\phi : E \rightarrow F$  where  $F$  is another Polish space and given any  $\mu \in \mathcal{P}(E)$ , the image (or pushforward) of  $\mu$  under  $\phi$  is  $\phi_*\mu = \mu \circ \phi^{-1}$ .

We denote by  $\mathcal{C}^0(E)$  the set of continuous functions on  $E$  and by  $\mathcal{C}_c^0(E)$  the set of continuous functions of compact support on  $E$ . When  $E$  is a smooth manifold, we adopt similar notations for the set  $\mathcal{C}^\infty(E)$  of smooth functions on  $E$ . We recall that the topological dual  $(\mathcal{C}_c^0(E))'$  (resp.,  $(\mathcal{C}^0(E))'$ ) is the set of all Radon measures on  $E$  (resp., with compact support). Endowed with the total variation norm  $\|\cdot\|_{TV}$  which is the dual norm, it is a Banach space.

Throughout the paper,  $\delta_\star$  is the Dirac measure at  $\star$ .

**Wasserstein distance.** Given any  $p \geq 1$ , the Wasserstein distance  $W_p(\mu_1, \mu_2)$  of order  $p$  between two probability measures  $\mu_1, \mu_2 \in \mathcal{P}(E)$ , with respect to the distance  $d_E$ , is defined as the infimum of the Monge-Kantorovich cost  $\int_{E^2} d_E(y_1, y_2)^p d\Pi(y_1, y_2)$  over the set of probability measures  $\Pi \in \mathcal{P}(E^2)$  coupling  $\mu_1$  with  $\mu_2$ , i.e., whose marginals on the two copies of  $E$  are  $\mu_1$  and  $\mu_2$ :

$$W_p(\mu_1, \mu_2) = \inf \left\{ \left( \int_{E^2} d_E(y_1, y_2)^p d\Pi(y_1, y_2) \right)^{1/p} \mid \Pi \in \mathcal{P}(E^2), (\pi_1)_*\Pi = \mu_1, (\pi_2)_*\Pi = \mu_2 \right\} \quad (7)$$

where  $\pi_1 : E^2 \rightarrow E$  and  $\pi_2 : E^2 \rightarrow E$  are the canonical projections defined by  $\pi_1(y_1, y_2) = y_1$  and  $\pi_2(y_1, y_2) = y_2$  for all  $(y_1, y_2) \in E \times E$ . Equivalently,

$$W_p(\mu_1, \mu_2) = \inf \left\{ \left( \mathbb{E} d_E(Y_1, Y_2)^p \right)^{1/p} \mid \text{law}(Y_1) = \mu_1, \text{law}(Y_2) = \mu_2 \right\} \quad (8)$$

where the infimum is taken over all possible random variables  $Y_1$  and  $Y_2$  (defined on a same probability space, with values in  $E$ ) having the laws  $\mu_1$  and  $\mu_2$  respectively. Then,  $W_p$  is a distance on  $\mathcal{P}_p(E)$ , which metrizes the weak convergence in  $\mathcal{P}_p(E)$  in the following sense: given  $\mu \in \mathcal{P}_p(E)$  and given a sequence  $(\mu_j)_{j \in \mathbb{N}^*}$  in  $\mathcal{P}_p(E)$ , we have  $W_p(\mu_j, \mu) \rightarrow 0$  as  $j \rightarrow +\infty$  if and only if  $\int_E f d\mu_j \rightarrow \int_E f d\mu$  for every continuous bounded function  $f$  on  $E$  and  $\int_E d_E(y_0, y)^p d\mu_j(y) \rightarrow \int_E d_E(y_0, y)^p d\mu(y)$  as  $j \rightarrow +\infty$  for some (and thus any)  $y_0 \in E$  (see [?, Chapter 5, Section 5.2] or [?, Theorem 6.9]), if and only if  $\int_E f d\mu_j \rightarrow \int_E f d\mu$  for every continuous function  $f$  on  $E$  such that  $|f(y)| \leq C(1 + d_E(y_0, y)^p)$  for every  $y \in E$ , for some  $C > 0$  and some (and thus any)  $y_0 \in E$  (see [?, Theorem 7.12]). It can be noted that, given any subset  $K \subset E$  of finite diameter, we have

$$1 \leq p_1 \leq p_2 \Rightarrow W_{p_1}(\mu_1, \mu_2) \leq W_{p_2}(\mu_1, \mu_2) \leq \text{diam}_E(K)^{1-p_1/p_2} W_{p_1}(\mu_1, \mu_2)^{p_1/p_2} \quad (9)$$

for all  $\mu_1, \mu_2 \in \mathcal{P}_c(E)$  of compact support contained in  $K$  (see [?, Chapter 5]), where  $\text{diam}_E(K)$  is the supremum of all  $d_E(y, y')$  over all possible  $y, y' \in K$ .

For  $p = 1$ , the duality formula for the Kantorovich-Rubinstein distance (see [?, Chapter 5]) gives the equivalent definition

$$W_1(\mu_1, \mu_2) = \sup \left\{ \int_E f d(\mu_1 - \mu_2) \mid f \in \text{Lip}(E), \text{Lip}(f) \leq 1 \right\}, \quad (10)$$

valid for all  $\mu_1, \mu_2 \in \mathcal{P}_1(E)$ .

For  $p = +\infty$ , we set  $W_\infty(\mu_1, \mu_2) = \lim_{p \rightarrow +\infty} W_p(\mu_1, \mu_2)$  (see [?, Chapter 5, Section 5.5.1]).

Note that the infimum in (??), as well as in (??), is a minimum (i.e., there exists an optimal coupling) and that the supremum in (??) is a maximum (see [?, Chapters 4 and 5] or [?, Chapter 3, Section 3.1.1]).

**Disintegration.** In this paper, we are going to consider measures on  $\Omega \times \mathbb{R}^d$ , for  $d \in \mathbb{N}^*$  (and on  $\Omega^k \times (\mathbb{R}^d)^k$  for  $k \in \mathbb{N}^*$ ), where  $(\Omega, d_\Omega)$  is a complete metric space and  $\mathbb{R}^d$  is endowed with an arbitrary norm  $\|\cdot\|$ . We endow  $\Omega \times \mathbb{R}^d$  with the distance  $d_{\Omega \times \mathbb{R}^d} = d_\Omega + d_{\mathbb{R}^d}$  where  $d_{\mathbb{R}^d}$  is the distance on  $\mathbb{R}^d$  induced by the norm  $\|\cdot\|$ .

Denoting by  $\pi : \Omega \times \mathbb{R}^d \rightarrow \Omega$  the canonical projection, given any  $\mu \in \mathcal{P}(\Omega \times \mathbb{R}^d)$ , in the sequel we will always denote by  $\nu$  the nonnegative probability Radon measure on  $\Omega$  defined as the image (pushforward) of  $\mu$  under  $\pi$ ,

$$\nu = \pi_* \mu = \mu \circ \pi^{-1}, \quad (11)$$

that is also the marginal of  $\mu$  on  $\Omega$ . Note that, since  $\pi$  is continuous,  $\text{supp}(\nu) = \overline{\pi(\text{supp}(\mu))}$ . By disintegration of  $\mu$  with respect to  $\nu$ , there exists a family  $(\mu_x)_{x \in \Omega}$  of probability Radon measures on  $\mathbb{R}^d$  (uniquely defined  $\nu$ -almost everywhere) such that  $\mu = \int_\Omega \mu_x d\nu(x)$ , i.e.,

$$\int_{\Omega \times \mathbb{R}^d} h(x, \xi) d\mu(x, \xi) = \int_\Omega \int_{\mathbb{R}^d} h(x, \xi) d\mu_x(\xi) d\nu(x)$$

for every Borel measurable function  $h : \Omega \times \mathbb{R}^d \rightarrow [0, +\infty)$  (see, e.g., [?]). Moreover, we set  $\mu_x = 0$  whenever  $x \in \Omega \setminus \text{supp}(\nu)$ .

When  $\Omega$  is a smooth manifold, if  $\mu \in \mathcal{P}^{ac}(\Omega \times \mathbb{R}^d)$  with a density  $f \in L^1(\Omega \times \mathbb{R}^d)$ , i.e.,  $\frac{d\mu}{dx d\xi}(x, \xi) = f(x, \xi)$ , then  $\nu$  is absolutely continuous, of density  $\frac{d\nu}{dx}(x) = \int_{\mathbb{R}^d} f(x, \xi) d\xi$ , and for  $\nu$ -almost every  $x \in \Omega$  the probability measure  $\mu_x$  has the density  $\frac{d\mu_x}{d\xi}(\xi) = \frac{f(x, \xi)}{\int_{\mathbb{R}^d} f(x, \xi') d\xi'}$ .

Given any  $\mu^1, \mu^2 \in \mathcal{P}_1(\Omega \times \mathbb{R}^d)$  having the same marginal  $\nu$  on  $\Omega$ , we define

$$L_\nu^1 W_p(\mu^1, \mu^2) = \int_\Omega W_p(\mu_x^1, \mu_x^2) d\nu(x). \quad (12)$$

Obviously,  $L_\nu^1 W_p$  is a distance on the subset denoted  $\mathcal{P}_p^\nu(\Omega \times \mathbb{R}^d)$  of elements of  $\mathcal{P}_p(\Omega \times \mathbb{R}^d)$  having the same marginal  $\nu$ . Note that  $W_1(\mu^1, \mu^2) \leq L_\nu^1 W_1(\mu^1, \mu^2)$  for all  $\mu_1, \mu_2 \in \mathcal{P}_1^\nu(\Omega \times \mathbb{R}^d)$ .<sup>1</sup>

**Tagged partitions.** Let  $\nu \in \mathcal{P}(\Omega)$ . We say that  $(\mathcal{A}^N, X^N)_{N \in \mathbb{N}^*}$  is a family of *tagged partitions* of  $\Omega$  associated with  $\nu$  if  $\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$  is a  $N$ -tuple of disjoint subsets  $\Omega_i^N \subset \Omega$  such that

$$\Omega = \bigcup_{i=1}^N \Omega_i^N \quad \text{with} \quad \nu(\Omega_i^N) = \frac{1}{N} \quad \text{and} \quad \text{diam}_\Omega(\Omega_i^N) \leq \frac{C_\Omega}{N^r} \quad \forall i \in \{1, \dots, N\}, \quad (13)$$

for some  $C_\Omega > 0$  and  $r > 0$  not depending on  $N$ , and  $X^N = (x_1^N, \dots, x_N^N)$  is a  $N$ -tuple of points  $x_i^N \in \Omega_i^N$ . Here,  $\text{diam}_\Omega(\Omega_i^N)$  is the supremum of all  $d_\Omega(x, x')$  over all possible  $x, x' \in \Omega_i^N$ .

Families of tagged partitions always exist when  $\Omega$  is a compact  $n$ -dimensional smooth manifold having a boundary or not and  $\nu$  is a Lebesgue measure on  $\Omega$ , with  $r = 1/n$ . For instance, when  $\Omega = [0, 1]$ , we take  $\Omega_i^N = [a_i^N, a_{i+1}^N)$  for some subdivision  $0 = a_1^N < a_2^N < \dots < a_{N+1}^N = 1$  satisfying (??); when  $d\nu(x) = dx$ , a natural choice is  $a_i^N = \frac{i-1}{N}$ , and  $x_i^N = a_i^N$  or  $\frac{a_i^N + a_{i+1}^N}{2}$ , for every  $i \in \{1, \dots, N\}$  (and then  $C_\Omega = 1$  and  $r = 1$  in this case). When  $\Omega$  is a compact domain of  $\mathbb{R}^n$ , a family of tagged partitions is obtained by considering a family of meshes, as classically done in numerical analysis, with  $r = 1/n$ .

The concept of tagged partition is used in Riemann (and more generally, Henstock-Kurzweil) integration theory. We refer to [?] for (much more) general results. A real-valued function  $f$  on  $\Omega$ , of compact support, is said to be  $\nu$ -Riemann integrable if it is bounded,  $\nu$ -measurable, and if, for any family  $(\mathcal{A}^N, X^N)_{N \in \mathbb{N}^*}$  of tagged partitions, we have

$$\sum_{i=1}^N \int_{\Omega_i^N} |f(x) - f(x_i^N)| d\nu(x) = o(1) \quad (14)$$

and thus

$$\int_\Omega f d\nu = \frac{1}{N} \sum_{i=1}^N f(x_i^N) + o(1) \quad (15)$$

as  $N \rightarrow +\infty$ . A function  $f$  of essential compact support on  $\Omega$  is  $\nu$ -Riemann integrable if and only if  $f$  is bounded and continuous  $\nu$ -almost everywhere on  $\Omega$ .

## 2 From microscopic to mesoscopic scale

(“from particle to Vlasov”, mean field limit)

### 2.1 Particle system

Let  $d \in \mathbb{N}^*$  be fixed. Throughout the paper, we consider an arbitrary norm  $\|\cdot\|$  on  $\mathbb{R}^d$ . At the microscopic level, given any  $N \in \mathbb{N}^*$ , we consider a system of  $N$  interacting “particles” or “agents”  $\xi_i^N(t) \in \mathbb{R}^d$ , called the *particle system* (or *multiagent system*), of dynamics

$$\boxed{\dot{\xi}_i^N(t) = \frac{1}{N} \sum_{j=1}^N G_{ij}^N(t, \xi_i^N(t), \xi_j^N(t)), \quad i = 1, \dots, N} \quad (16)$$

<sup>1</sup>Indeed,  $\int_{\Omega \times \mathbb{R}^d} f d(\mu^1 - \mu^2) = \int_\Omega \int_{\mathbb{R}^d} f(x, \xi) d(\mu_x^1 - \mu_x^2) d\nu(x) \leq \int_\Omega \text{Lip}(f(x, \cdot)) W_1(\mu_x^1, \mu_x^2) d\nu(x)$  for every  $f \in \text{Lip}(\Omega \times \mathbb{R}^d)$ , and if  $\text{Lip}(f) \leq 1$  then  $\text{Lip}(f(x, \cdot)) \leq 1$  for every  $x \in \Omega$ . Then, take the supremum over all  $f$ .



where  $G_{ij}^N : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  stands for the interaction between the particles  $i$  and  $j$ . The dot stands for the time derivative. The usual case, widely treated in the existing literature, is when the particles are indistinguishable and the interaction mapping is the same for all pairs of particles and moreover does not depend on  $N$ , i.e.,  $G_{ij}^N = G$ . We show here that there is no difficulty to treat the more general situation where the particles are distinguishable and the interactions depend on the agents. In (??),  $G_{ij}^N$  depends on  $i, j, N$ .

Throughout the paper, we make the following crucial assumption:

(G) There exist a complete metric space  $(\Omega, d_\Omega)$  and a *continuous* mapping

$$\begin{aligned} G : \mathbb{R} \times \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d &\rightarrow \mathbb{R}^d \\ (t, x, x', \xi, \xi') &\mapsto G(t, x, x', \xi, \xi') \end{aligned}$$

locally Lipschitz with respect to  $(\xi, \xi')$  uniformly with respect to  $(t, x, x')$  on any compact subset of  $\mathbb{R} \times \Omega \times \Omega$ , such that, for every  $N \in \mathbb{N}^*$ , there exist  $x_1^N, \dots, x_N^N$  in  $\Omega$  such that

$$G(t, x_i^N, x_j^N, \xi, \xi') = G_{ij}^N(t, \xi, \xi') \quad \forall t \in \mathbb{R} \quad \forall \xi, \xi' \in \mathbb{R}^d \quad \forall i, j \in \{1, \dots, N\}. \quad (17)$$

Under Assumption ??, for every  $N \in \mathbb{N}^*$  the particle system (??) is equivalently written as

$$\boxed{\begin{aligned} \dot{x}_i^N(t) &= 0 \\ \dot{\xi}_i^N(t) &= \frac{1}{N} \sum_{j=1}^N G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)), \quad i = 1, \dots, N \end{aligned}} \quad (18)$$

The variables  $x_i^N \in \Omega$  are parameters, and a usual way to treat parameters in differential equations is to treat them as state variables whose dynamics are zero, whence the dynamics  $\dot{x}_i^N(t) = 0$  above. For each index  $i$ , the variable  $x_i^N$  can be seen as the “name” of the agent  $i$ , used to distinguish it from the others.

Assumption ?? (in particular, (??)) is a kind a continuous interpolation of the mappings  $G_{ij}^N$ . The continuity assumption includes the idea of the existence of a limit system as  $N \rightarrow +\infty$ . In some sense, this assumption is unavoidable: indeed, if  $G$  were not required to be continuous, then completely different systems (??) could be considered as  $N$  varies and then no limit (at least, in a strong sense) for large  $N$  could exist. Note anyway that, interestingly, the authors of [?] do not assume ??, but in order to pass to the mean field limit they make another assumption of uniform boundedness on their dynamics in order to have a weak star limit. However at the limit the distinguishability of particles is lost. In contrast, in our paper we want to obtain strong (mean field, hydrodynamic, graph) limits and to preserve distinguishability at the limit.

Note that Assumption ?? implies that the Lipschitz constants of the mappings  $G_{ij}^N$  are uniformly bounded (with respect to  $i, j, N$ ) on any compact.

In Assumption ??, the complete metric space  $\Omega$  used for the parameters  $x_i^N$  is arbitrary. For instance we can take  $\Omega = [0, 1]$ , but we allow for more general sets, in view of deriving on  $\Omega$  some interesting classes of PDEs (see Section ??).

The choice of the possible values of the  $x_i^N$  is not imposed in Assumption ?. If one wishes moreover to fix some precise points  $x_i^N$ , such as the natural ones  $x_i^N = \frac{i}{N}$  when  $\Omega = [0, 1]$ , often used in numerical analysis, then having (??) satisfied requires some compatibility conditions on the mappings  $G_{ij}^N$ .

Setting  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$ , the system (??) can also be written in the form

$$\boxed{\dot{\Xi}^N(t) = Y^N(t, X^N, \Xi^N(t))} \quad (19)$$

where  $\Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$ . Here and in what follows, the time-dependent vector field  $Y^N(t, X, \cdot)$  on  $(\mathbb{R}^d)^N$ , depending on the parameter  $X \in \Omega^N$ , is defined by

$$Y^N(t, X, \cdot) = (Y_1^N(t, X, \cdot), \dots, Y_N^N(t, X, \cdot)) \quad (20)$$

with

$$Y_i^N(t, X, \Xi) = \frac{1}{N} \sum_{j=1}^N G(t, x_i, x_j, \xi_i, \xi_j) \quad \forall i \in \{1, \dots, N\} \quad (21)$$

for all  $t \in \mathbb{R}$ ,  $X = (x_1, \dots, x_N) \in \Omega^N$  and  $\Xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^d)^N$ . We denote by  $(\Phi^N(t, X, \cdot))_{t \in I}$  ( $I \subset \mathbb{R}$ ) the local-in-time flow of diffeomorphisms of  $\mathbb{R}^{dN}$  generated by the time-dependent vector field  $Y^N(t, X, \cdot)$ : this flow, called the *particle flow*, is parametrized by  $X \in \Omega^N$ . We have  $\Xi^N(t) = \Phi^N(t, X^N, \Xi^N(0))$  for every  $t \in I$ .

**Lemma 1.** *[Uniform maximal time] For any compact subset  $K$  of  $\Omega \times \mathbb{R}^d$ , there exists a maximal time  $T_{\max}(K) \in (0, +\infty]$  such that, for any  $N \in \mathbb{N}^*$ , for any  $(X, \Xi(0)) \in K^N$ ,<sup>2</sup> the unique solution  $t \mapsto \Phi^N(t, X, \Xi(0))$  of (??), of initial condition  $(X, \Xi(0))$  at  $t = 0$ , is well defined on  $[0, T_{\max}(K))$ . Moreover, for any  $T \in [0, T_{\max}(K))$ , the set  $\Phi^N([0, T] \times K^N)$  is contained in a compact subset of  $\mathbb{R}^d$  depending on  $T$  but not on  $N$ .*

Lemma ?? shows that, given a compact set  $K$  of initial conditions, the time  $T_{\max}(K)$  is uniform with respect to  $N \in \mathbb{N}^*$ , and that, given any  $T \in (0, T_{\max}(K))$ , any solution of (??) on  $[0, T]$ , starting in  $K$  at  $t = 0$ , is contained in a compact set that depends on  $T$  but not on  $N$ .

Lemma ?? straightforwardly follows from the usual proof of the Picard-Lindelöf theorem by a fixed point argument (see [?, Chapter II]), using Assumption ?? and the fact that, for every  $T > 0$ , on  $[0, T] \times K^N$  the vector field  $Y^N$  is uniformly bounded with respect to  $N$  and is Lipschitz with respect to  $\Xi$  uniformly with respect to  $(t, X)$  on any compact, with a Lipschitz constant that is uniform with respect to  $N$ . Note that, for a given  $N \in \mathbb{N}^*$ , the maximal time of definition of the solution  $t \mapsto \Phi^N(t, X, \Xi(0))$  may be larger than  $T_{\max}(K)$ ; what is important in the lemma is the uniform bound below with respect to  $N$ .

Of course, if  $G$  is globally Lipschitz with respect to  $(\xi, \xi') \in \mathbb{R}^d \times \mathbb{R}^d$ , uniformly with respect to  $(t, x, x')$  on any compact subset of  $[0, +\infty) \times \Omega \times \Omega$ , then  $T_{\max}(K) = +\infty$  for any compact  $K \subset \Omega \times \mathbb{R}^d$ . But our framework is more general and allows for superlinearities.

Some examples covered by this general framework are in order.

**Example 1.** *Consider the linear Hegselmann–Krause first-order consensus system (see [?]), modeling for instance the propagation of opinions (studied in [?]), of dynamics*

$$\dot{\xi}_i^N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_{ij}^N (\xi_j^N(t) - \xi_i^N(t)), \quad i = 1, \dots, N, \quad (22)$$

with constant interaction coefficients  $\sigma_{ij}^N \geq 0$  (not necessarily symmetric). Assumption ?? requires that there exist  $\Omega$  and a continuous function  $\sigma$  on  $\Omega^2$  such that, for every  $N \in \mathbb{N}^*$ , there exist distinct points  $x_1^N, \dots, x_N^N$  in  $\Omega$  such that  $\sigma(x_i^N, x_j^N) = \sigma_{ij}^N$ . We have then  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$  for all  $(t, x, x', \xi, \xi') \in \mathbb{R} \times \Omega^2 \times (\mathbb{R}^d)^2$ . More general models can be considered.

<sup>2</sup>With a slight abuse of notation,  $(X, \Xi(0)) \in K^N$  means that  $(x_i, \xi_i(0)) \in K$  for every  $i \in \{1, \dots, N\}$ , where  $X = (x_1, \dots, x_N)$  and  $\Xi(0) = (\xi_1(0), \dots, \xi_N(0))$ .

**Example 2.** Setting  $d = 2r$  and  $\xi = (q, p) \in \mathbb{R}^r \times \mathbb{R}^r$ , the Cucker–Smale model (see [?]) is given by

$$\dot{q}_i^N(t) = p_i^N(t), \quad \dot{p}_i^N(t) = \frac{1}{N} \sum_{j=1}^N a(\|q_i^N(t) - q_j^N(t)\|)(p_j^N(t) - p_i^N(t)), \quad i = 1, \dots, N,$$

for some potential function  $a$  on  $\mathbb{R}$  of class  $\mathcal{C}^1$ . Assumption ?? is satisfied with  $G = (G_q, G_p)$  where  $G_q(t, x, x', \xi, \xi') = p$  and  $G_p(t, x, x', \xi, \xi') = a(\|q - q'\|)(p' - p)$  (not depending on  $(x, x')$ : particles are indistinguishable).

Many variants of that model are covered by our framework, for instance the potential may depend on  $i$  and  $j$  (distinguishable case), and other terms can be added to the dynamics of  $p_i$ , for instance self-propulsion and attraction-repulsion forces (like in [?]).

**Example 3.** Still with  $d = 2r$  and  $\xi = (q, p) \in \mathbb{R}^r \times \mathbb{R}^r$ , given any  $N \in \mathbb{N}^*$ , consider the Hamiltonian function

$$H^N(q_1, p_1, \dots, q_N, p_N) = \sum_{j=1}^N h_j^N(q_j, p_j) + \frac{1}{N} \sum_{j,k=1}^N h_{jk}^N(q_j, p_j, q_k, p_k)$$

for some  $C^1$  functions  $h_j^N$  and  $h_{jk}^N$ . The Hamiltonian system of  $N$  particles, given by  $\dot{q}_i = \frac{\partial H}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial H}{\partial q_i}$  for  $i \in \{1, \dots, N\}$ , can be written as (??) with

$$G_{ij}^N(t, \xi, \xi') = \begin{pmatrix} \partial_2 h_i^N(q, p) + \partial_2 h_{ij}^N(q, p, q', p') + \partial_4 h_{ji}^N(q', p', q, p) \\ -\partial_1 h_i^N(q, p) - \partial_1 h_{ij}^N(q, p, q', p') - \partial_3 h_{ji}^N(q', p', q, p) \end{pmatrix}$$

where  $\partial_k$  denotes the partial derivative with respect to the  $k^{\text{th}}$ -variable.

Having Assumption ?? satisfied requires at least that the Hamiltonians  $h_j^N$  and  $h_{jk}^N$  be uniformly (wrt  $j, k, N$ ) locally Lipschitz. Note that the Hamiltonian  $H^N$  involves sums of “single” and of “pairwise” Hamiltonians, but not of “triplewise” or more. Actually, many classical Hamiltonian systems of  $N$  particles are written as above with Hamiltonians not depending on  $j, k, N$ .

When  $N$  becomes larger and larger, we want to pass to the limit in some sense and replace the set of particles with a nonnegative Radon measure on  $\Omega \times \mathbb{R}^d$ . With this objective in mind, two classical viewpoints are the Lagrangian and the Eulerian one.

The *Lagrangian* viewpoint consists of keeping the trajectories of (??), taking the mean field limit by embedding trajectories with an empirical measure on  $\Omega \times \mathbb{R}^d$  to solutions of the *Vlasov equation* (or *continuity equation*) in  $\Omega \times \mathbb{R}^d$ . This is done in Section ??.

The *Eulerian* viewpoint consists of using the flow of diffeomorphisms of  $Y$  to propagate an initial measure on  $\Omega^N \times \mathbb{R}^{dN}$ , thus obtaining the *Liouville equation* in  $\Omega^N \times \mathbb{R}^{dN}$ . This is done in Section ??.

Hereafter, we elaborate in detail on these procedures and we then show in Section ?? how to recover the Vlasov equation from the Liouville equation by taking adequate marginals, obtaining convergence estimates in Wasserstein distance.

## 2.2 Lagrangian viewpoint: mean field limit and Vlasov equation

Within the Lagrangian viewpoint, the  $N$  particles at time  $t$  are embedded as Dirac masses to the space of Radon measures, and their corresponding average, the empirical measure, converges by the so-called mean field limit procedure, as  $N \rightarrow +\infty$ , to a probability Radon measure  $\mu(t)$

on  $\Omega \times \mathbb{R}^d$  satisfying the Vlasov equation. When  $\mu(t)$  is absolutely continuous with respect to a Lebesgue measure, its density  $f(t, x, \xi)$  is the density of particles having the parameter  $x$  at  $\xi$  at time  $t$ .

In the existing literature, it is often said that, in order to pass to the mean field limit, it is necessary that the particles be indistinguishable, and that the interaction mapping  $G$  must be the same for all pairs of particles. Hereafter, we show that such an assumption is unnecessary and that there is no difficulty to consider interactions depending on agents.

### 2.2.1 Mean field

Given any  $\mu \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , we define  $\nu = \pi_* \mu$  by (??) (marginal of  $\mu$  on  $\Omega$ ) and we define the *mean field*, also called *interaction kernel*, as the non-local time-dependent one-parameter (the parameter is  $x \in \Omega$ ) vector field on  $\mathbb{R}^d$  given by

$$\begin{aligned} \mathcal{X}[\mu](t, x, \xi) &= \int_{\Omega \times \mathbb{R}^d} G(t, x, x', \xi, \xi') d\mu(x', \xi') \\ &= \int_{\Omega} \int_{\mathbb{R}^d} G(t, x, x', \xi, \xi') d\mu_{x'}(\xi') d\nu(x') \quad \forall (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^d \end{aligned} \quad (23)$$

Note that  $\mathcal{X}[\mu](t, x, \xi)$  is the expectation of  $G(t, x, x', \xi, \xi')$  with respect to the measure  $\mu$ , performed with respect to  $(x', \xi') \in \Omega \times \mathbb{R}^d$  (see Appendix ?? for more details and consequences of that definition). Recall that  $G$  satisfies Assumption ??.

**Example 4.** In Example ?? (Hegselmann–Krause model), the mean field is

$$\mathcal{X}[\mu](t, x, \xi) = \int_{\Omega \times \mathbb{R}^d} \sigma(x, x')(\xi' - \xi) d\mu(x', \xi') \quad \forall (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^d,$$

and in Example ?? (Cucker–Smale model) it is given by

$$\mathcal{X}[\mu](t, x, \xi) = \left( \int_{\Omega \times \mathbb{R}^r \times \mathbb{R}^r} a(\|q - q'\|)(p' - p) d\mu(x', \xi') \right) \quad \forall (t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^{2r},$$

### 2.2.2 Vlasov equation

We consider the *Vlasov (or continuity) equation*

$$\partial_t \mu + \operatorname{div}_{\xi}(\mathcal{X}[\mu]\mu) = 0 \quad (24)$$

where the divergence<sup>3</sup> acts only with respect to  $\xi$ . It is a nonlocal transport equation because the velocity field  $\mathcal{X}[\mu]$  defined by (??) is nonlocal.

Given any interval  $I \subset \mathbb{R}$ , let  $\mathcal{C}^0(I, \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  be the Banach space of continuous mappings  $t \in I \mapsto \mu(t) \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , with  $\mathcal{P}_c(\Omega \times \mathbb{R}^d)$  endowed with the weak topology (metrized by the Wasserstein distance  $W_p$ , for any  $p \in [1, +\infty)$ , as recalled in Section ??).

We define  $\mathcal{C}_{\text{comp}}^0(I, \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  as the set of all  $\mu \in \mathcal{C}^0(I, \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  that are *equi-compactly supported* on any compact interval of  $I$ , meaning that for any  $t_1, t_2 \in I$ , there exists a compact subset  $K \subset \Omega \times \mathbb{R}^d$  such that  $\operatorname{supp}(\mu(t)) \subset K$  for every  $t \in [t_1, t_2]$ . There exist elements of  $\mathcal{C}^0(I, \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  that are not equi-compactly supported on any compact interval of  $I$  (for instance, if  $I = [0, T]$ , take  $\mu(t) = (1 - e^{-1/t})\delta_0 + e^{-1/t}\delta_{1/t}$ ).

<sup>3</sup>Recall that  $\operatorname{div}(\mathcal{X}\mu) = L_{\mathcal{X}}\mu$  (Lie derivative of the measure  $\mu$ ) is the measure defined by  $\langle L_{\mathcal{X}}\mu, f \rangle = -\langle \mu, L_{\mathcal{X}}f \rangle = -\int_{\mathbb{R}^d} \mathcal{X} \cdot \nabla f d\mu$  for every  $f \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ .

In view of obtaining existence and uniqueness of solutions of (??), we consider the following concept of solution. Assuming that  $0 \in I$ , by definition, a solution  $t \mapsto \mu(t)$  of (??) on  $I$  such that  $\mu(0) = \mu_0 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  is an element  $\mu \in \mathcal{C}_{\text{comp}}^0(I, \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  such that, denoting  $\mu_t = \mu(t)$ ,<sup>4</sup> for every  $g \in C_c^\infty(\Omega \times \mathbb{R}^d)$ , the function  $t \mapsto \int g d\mu_t$  is absolutely continuous on  $I$  and

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} g(x, \xi) d\mu_t(x, \xi) &= \int_{\Omega \times \mathbb{R}^d} g(x, \xi) d\mu_0(x, \xi) \\ &+ \int_0^t \int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} \langle \nabla_\xi g(x, \xi), G(\tau, x, x', \xi, \xi') \rangle d\mu_\tau(x', \xi') d\mu_\tau(x, \xi) d\tau \end{aligned} \quad (25)$$

for almost every  $t \in I$ .

**Remark 1.** Given any solution  $t \mapsto \mu(t)$  on  $[0, T)$  of the Vlasov equation (??), the total mass  $\mu(t)(\Omega \times \mathbb{R}^d)$  is constant with respect to  $t$ , i.e.,  $\mu(t)$  is a probability measure for every  $t \in [0, T)$ . Also, the marginal  $\nu = \pi_* \mu(t)$  does not depend on  $t$ , because the Vlasov equation can be written as  $\partial_t \mu + L_{\mathcal{X}[\mu]} \mu = 0$  with the Lie derivative acting with respect to the variable  $\xi$ , and we have  $\pi_* L_{\mathcal{X}[\mu]} = 0$ .

Disintegrating  $\mu_t = \mu(t)$  as  $\mu_t = \int_{\Omega} \mu_{t,x} d\nu(x)$  with respect to its marginal  $\nu = \pi_* \mu_t$  on  $\Omega$  (which does not depend on  $t$  by Remark ??), by uniqueness  $\nu$ -almost everywhere of the disintegration, (??) is equivalent to

$$\partial_t \mu_{t,x} + \text{div}_\xi (\mathcal{X}[\mu_t](t, x, \cdot) \mu_{t,x}) = 0 \quad (26)$$

for  $\nu$ -almost every  $x \in \Omega$ . Note that the time evolution of  $\mu_{t,x} = \varphi_{\mu_0}(t, x, \cdot) \mu_{0,x}$  depends on the whole  $\mu_0$  and not only on  $\mu_{0,x}$ , since  $\mathcal{X}[\mu_t]$  involves an integral over all possible  $x' \in \Omega$ .

**Theorem 1.** [Existence, uniqueness and stability properties for the Vlasov equation (??)]  
Recalling Assumption ??, let  $p \in [1, +\infty)$  be arbitrary.

(A) Given any  $\mu_0 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , setting  $T_0 = T_{\max}(\text{supp}(\mu_0))$  (given by Lemma ??), there exists a unique solution  $\mu \in \mathcal{C}_{\text{comp}}^0([0, T_0], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  of the Vlasov equation (??) (in the sense (??)) such that  $\mu(0) = \mu_0$ . Moreover,  $t \mapsto \mu(t)$  is locally Lipschitz with respect to  $t$  for the distance  $W_p$ , and we have

$$\mu(t) = \varphi_{\mu_0}(t) \mu_0, \quad (27)$$

which is a notation meaning that  $\mu_{t,x} = \varphi_{\mu_0}(t, x, \cdot) \mu_{0,x}$  for every  $t \in [0, T_0)$  and  $\nu$ -almost every  $x \in \Omega$ , and where  $t \mapsto \varphi_{\mu_0}(t, x, \cdot)$  is the unique solution (Vlasov flow) of

$$\partial_t \varphi_{\mu_0}(t, x, \cdot) = \mathcal{X}[\mu(t)](t, x, \cdot) \circ \varphi_{\mu_0}(t, x, \cdot) \quad (28)$$

such that  $\varphi_{\mu_0}(0, x, \cdot) = \text{id}_{\mathbb{R}^d}$  for  $\nu$ -almost every  $x \in \Omega$ . Moreover, if  $\mu_0 \in \mathcal{P}_c^{ac}(\Omega \times \mathbb{R}^d)$  then  $\mu(t) \in \mathcal{P}_c^{ac}(\Omega \times \mathbb{R}^d)$  for every  $t \in [0, T_0)$ . Furthermore:

(A<sub>1</sub>) Any solution of (??) depends continuously on its initial condition  $\mu(0) \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  for the weak topology in the following sense: given any compact subset  $K \subset \Omega \times \mathbb{R}^d$ , given any  $\mu(0) \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  such that  $\text{supp}(\mu(0)) \subset K$ , given any (equi-compactly supported) sequence of measures  $\mu^k(0) \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  such that  $\text{supp}(\mu^k(0)) \subset K$  for every  $k \in \mathbb{N}^*$ , if  $\mu^k(0)$  converges weakly to  $\mu(0)$  (equivalently,  $W_p(\mu^k(0), \mu(0)) \rightarrow 0$ ) as  $k \rightarrow +\infty$ , then  $\mu^k(t)$  converges weakly to  $\mu(t)$  (equivalently,  $W_p(\mu^k(t), \mu(t)) \rightarrow 0$ ) as  $k \rightarrow +\infty$ , uniformly on any compact interval of  $[0, T_{\max}(K))$ .

<sup>4</sup>Note that, seeing  $\mu$  as a measure on  $I \times \Omega \times \mathbb{R}^d$ , the marginal of  $\mu$  on  $I$  is the Lebesgue measure and the disintegration of  $\mu$  is  $\mu = \int_I \mu_t dt$ .

**(A<sub>2</sub>)** For all solutions  $\mu^1, \mu^2 \in \mathcal{C}_{\text{comp}}^0([0, T], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  of (??) (for some  $T > 0$ ) such that  $\mu^1(0), \mu^2(0) \in \mathcal{P}_c^\nu(\Omega \times \mathbb{R}^d)$  have the same marginal  $\nu$  on  $\Omega$ , setting<sup>5</sup>

$$S_{\mu^1, \mu^2}(\tau) = \text{supp}(\nu) \times (\varphi_{\mu_0^1}(\tau, \text{supp}(\mu_0^1) \cup \text{supp}(\mu_0^2)) \cup \text{supp}(\mu^2(\tau)))$$

and defining

$$C_{\mu^1, \mu^2}(t) = \exp\left(2 \int_0^t \text{ess sup}_{(x, \xi), (x', \xi') \in S_{\mu^1, \mu^2}(\tau)} \|(\partial_\xi G, \partial_{\xi'} G)(\tau, x, x', \xi, \xi')\| d\tau\right), \quad (29)$$

we have

$$L_\nu^1 W_p(\mu^1(t), \mu^2(t)) \leq C_{\mu^1, \mu^2}(t) L_\nu^1 W_p(\mu^1(0), \mu^2(0)) \quad \forall t \in [0, T] \quad (30)$$

(where  $L_\nu^1 W_p$  is defined by (??)).

**(B)** Assume moreover that  $G$  is locally Lipschitz with respect to  $(x, x', \xi, \xi')$  uniformly with respect to  $t$  on any compact interval. For all solutions  $\mu^1(\cdot), \mu^2(\cdot) \in \mathcal{C}_{\text{comp}}^0([0, T], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  of (??) (for some  $T > 0$ ), setting

$$S_{\mu^1, \mu^2}(\tau) = (\text{supp}(\nu_1) \cup \text{supp}(\nu_2)) \times (\varphi_{\mu_0^1}(\tau, \text{supp}(\mu_0^1) \cup \text{supp}(\mu_0^2)) \cup \text{supp}(\mu^2(\tau)))$$

and defining

$$C_{\mu^1, \mu^2}(t) = \exp\left(2 \int_0^t \text{Lip}(G(\tau, \cdot, \cdot, \cdot, \cdot))|_{S_{\mu^1, \mu^2}(\tau)^2} d\tau\right), \quad (31)$$

we have

$$W_p(\mu^1(t), \mu^2(t)) \leq C_{\mu^1, \mu^2}(t) W_p(\mu^1(0), \mu^2(0)) \quad \forall t \in [0, T]. \quad (32)$$

Theorem ?? is proved in Appendix ?. The statement ?? of Theorem ?? is a slight extension, with parameter  $x$ , of [?, Theorem 2.3] (see also [?, ?, ?]) where it is assumed that  $G$  is globally Lipschitz. Without parameter  $x$ , we recover the famous stability estimate obtained by Dobrushin in [?] (see Corollary ?? further). The statement ?? seems to be new. Note that, in ??, the initial measures  $\mu_1(0)$  and  $\mu_2(0)$  are required to have the same marginal (and thus, equivalently,  $\mu^1(t)$  and  $\mu^2(t)$  have the same marginal for any  $t$ ). On the contrary, in ?? and in ??, the measures under consideration are not assumed to have the same marginal. In ??, the weak convergence  $\mu^k(0) \rightharpoonup \mu(0)$  implies the weak convergence  $\nu^k \rightharpoonup \nu$  of marginals but it is wrong in general that  $\mu_x^k(0) \rightharpoonup \mu_x(0)$  for  $x \in \Omega$ .

In the statement ??, the assumption that  $G$  is locally Lipschitz with respect to  $(x, x', \xi, \xi')$  is much stronger than ??: in Example ?? (resp., Example ??) this requires  $\sigma$  (resp.,  $a$ ) to be locally Lipschitz. In general, requiring that  $G$  be locally Lipschitz with respect to  $(x, x')$  is not a natural assumption for the particle system (??). Note that, under this stronger assumption, the unique solution  $\mu(\cdot)$  in ?? is locally Lipschitz with respect to  $t$  for the Wasserstein distance  $W_1$ .

Finally, in the usual statements existing in the literature,  $G$  is assumed to be globally Lipschitz. Here, under the weaker assumption ??, we have a maximal time of definition of  $\mu$  depending on the compact support of  $\mu_0$ , according to Lemma ?. Note that, when  $G$  is bounded, we can consider in Theorem ?? measures that are not of compact support.

<sup>5</sup>Note that  $S_{\mu^1, \mu^2}(t)$  is compact, that  $\varphi_{\mu_0^1}(t, \text{supp}(\mu_0^1)) = \text{supp}(\mu_0^1)$  and  $\text{supp}(\mu^1(t)) \cup \text{supp}(\mu^2(t)) \subset S_{\mu^1, \mu^2}(t)$ .

**Particular case: the indistinguishable case.** We speak of the “indistinguishable case” whenever  $G$  does not depend on  $(x, x')$ . This is the classical case that has been much studied in the existing literature. The mean field is then

$$\mathcal{X}[\bar{\mu}](t, \xi) = \int_{\mathbb{R}^d} G(t, \xi, \xi') d\mu(\xi) \quad \forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^d \quad (33)$$

where  $\bar{\mu} \in \mathcal{P}(\mathbb{R}^d)$  is denoted with an upper bar, to avoid any confusion with measures  $\mu \in \mathcal{P}(\Omega \times \mathbb{R}^d)$ .

We have the following corollary of Theorem ??, already well known in the existing literature (famous Dobrushin estimate, see [?]).

**Corollary 1.** *Assume that  $G$  does not depend on  $(x, x')$ . Let  $p \in [1, +\infty)$  be arbitrary. Given any  $\bar{\mu}_0 \in \mathcal{P}_c(\mathbb{R}^d)$ , there exists a unique solution  $\bar{\mu} \in \mathcal{C}_{\text{comp}}^0([0, T_{\max}(\text{supp}(\bar{\mu}_0))], \mathcal{P}_c(\mathbb{R}^d))$  of the Vlasov equation (??) (without dependence on  $x$ ), locally Lipschitz with respect to  $t$  for the distance  $W_p$ , such that  $\bar{\mu}(0) = \bar{\mu}_0$ , and we have*

$$\bar{\mu}(t) = \varphi_{\mu_0}(t, \cdot) * \bar{\mu}_0$$

where  $t \mapsto \varphi_{\mu_0}(t, \cdot)$  is the unique solution of  $\partial_t \varphi_{\mu_0}(t, \cdot) = \mathcal{X}[\mu(t)](t, \cdot) \circ \varphi_{\mu_0}(t, \cdot)$  such that  $\varphi_{\mu_0}(0, \cdot) = \text{id}_{\mathbb{R}^d}$ . Moreover, if  $\bar{\mu}_0 \in \mathcal{P}_c^{\text{ac}}(\mathbb{R}^d)$  then  $\bar{\mu}(t) \in \mathcal{P}_c^{\text{ac}}(\mathbb{R}^d)$  for every  $t \in \mathbb{R}$ . Furthermore, we have

$$W_p(\bar{\mu}^1(t), \bar{\mu}^2(t)) \leq C_{\bar{\mu}^1, \bar{\mu}^2}(t) W_p(\bar{\mu}^1(0), \bar{\mu}^2(0)) \quad \forall t \in [0, T] \quad (34)$$

for all locally Lipschitz solutions  $\bar{\mu}^1(\cdot)$  and  $\bar{\mu}^2(\cdot)$  of (??) on  $[0, T]$  (for some  $T > 0$ ) such that  $\bar{\mu}^1(0), \bar{\mu}^2(0) \in \mathcal{P}_c(\mathbb{R}^d)$ . Here,  $C_{\bar{\mu}^1, \bar{\mu}^2}(t)$  is defined by (??) or (??) (without dependence on  $x$ ).

*Proof.* Let  $\bar{\nu}$  be an arbitrary probability measure on  $\Omega$ . Given any  $\bar{\mu} \in \mathcal{P}(\mathbb{R}^d)$ , we define  $\mu \in \mathcal{P}(\Omega \times \mathbb{R}^d)$  by  $\mu = \bar{\nu} \otimes \bar{\mu}$ : the marginal of  $\mu$  on  $\Omega$  is  $\bar{\nu}$  and the disintegration of  $\mu = \int_{\Omega} \mu_x d\bar{\nu}(x)$  with respect to  $\bar{\nu}$  is given by  $\mu_x = \bar{\mu}$  if  $x \in \text{supp}(\bar{\nu})$  and  $\mu_x = 0$  if  $x \in \Omega \setminus \text{supp}(\bar{\nu})$ .

This embedding allows us to recover Corollary ?? as a consequence of Theorem ??. Indeed, obviously,  $\bar{\mu}(\cdot)$  is solution of the Vlasov equation (??) without dependence on  $x$  if and only if  $\mu(\cdot) = \bar{\nu} \otimes \bar{\mu}(\cdot)$  is solution of the Vlasov equation (??). This gives the first part of the corollary.

To obtain (??), it suffices to take  $\bar{\nu} = \delta_{\bar{x}}$  for some  $\bar{x} \in \Omega$  and to note that  $W_p(\bar{\mu}^1, \bar{\mu}^2) = W_p(\bar{\nu} \otimes \bar{\mu}^1, \bar{\nu} \otimes \bar{\mu}^2)$ . Then, (??) follows from (??) or from (??).  $\square$

### 2.2.3 Relationship between the particle system and the Vlasov equation

For every  $N \in \mathbb{N}^*$ , given any  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$  and any  $\Xi^N = (\xi_1^N, \dots, \xi_N^N) \in \mathbb{R}^{dN}$ , we define the *empirical measure*  $\mu_{(X^N, \Xi^N)}^e \in \mathcal{P}(\Omega \times \mathbb{R}^d)$  corresponding to  $(X^N, \Xi^N)$  by

$$\mu_{(X^N, \Xi^N)}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N}. \quad (35)$$

The disintegration of  $\mu_{(X^N, \Xi^N)}^e$  with respect to its marginal  $\nu_{\Xi^N}^e = \pi_* \mu_{(X^N, \Xi^N)}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$  on  $\Omega$  (that is itself an empirical measure corresponding to  $X^N$ ) gives the family of conditional measures defined by  $(\mu_{(X^N, \Xi^N)}^e)_x = \delta_{\xi_i^N}$  if  $x = x_i^N$  and 0 otherwise.

The relationship between the particle system (??) and the Vlasov equation (??) is given by the result below. To simplify the notation, hereafter we denote  $\mu_N^e = \mu_{(X^N, \Xi^N)}^e$  and  $\nu_N^e = \nu_{\Xi^N}^e$ .

**Proposition 1.** *If  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$  is a solution on  $[0, T]$  (for some  $T > 0$ ) of the particle system (??) with parameter  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$ , then*

$$t \mapsto \mu_N^e(t) = \varphi_{\mu_N^e(0)}(t)_* \mu_N^e(0) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}$$

*is a locally Lipschitz solution of the Vlasov equation (??) on  $[0, T]$ . The converse is true if all  $x_i^N$  are distinct and all  $\xi_i^N(t)$  are distinct.*

*Actually,  $t \mapsto \Xi^N(t)$  is solution on  $[0, T]$  of (??) with parameter  $X^N$  if and only if*

$$\xi_i^N(t) = \varphi_{\mu_N^e(0)}(t, x_i^N, \xi_i^N(0)) \quad \forall t \in [0, T] \quad \forall i \in \{1, \dots, N\}. \quad (36)$$

*Proof.* The Vlasov equation (??) is written as  $\partial_t \mu + L_{\mathcal{X}[\mu]} \mu = 0$  with the Lie derivative acting with respect to the variable  $\xi$ . Hence, setting  $X^N(t) = (x_1^N(t), \dots, x_N^N(t))$  and  $\Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$ , the mapping  $t \mapsto \mu_N^e(t)$  is a locally Lipschitz solution of the Vlasov equation (??) if and only if, for any  $g \in \mathcal{C}_c^\infty(\Omega \times \mathbb{R}^d)$ , we have  $\langle \partial_t \mu_N^e + L_{\mathcal{X}[\mu_N^e]} \mu_N^e, g \rangle = 0$ , i.e.,

$$0 = \frac{1}{N} \sum_{i=1}^N \left( \frac{d}{dt} g(x_i^N(t), \xi_i^N(t)) - \partial_\xi g(x_i^N(t), \xi_i^N(t)) \cdot \frac{1}{N} \sum_{j=1}^N G(t, x_i^N(t), x_j^N(t), \xi_i^N(t), \xi_j^N(t)) \right)$$

which is satisfied if  $t \mapsto (X^N, \Xi^N(t))$  is solution of (??). If all  $x_i^N$  are distinct and all  $\xi_i^N(t)$  are distinct, the converse is obtained by taking  $g$  localized around  $(x_i^N, \xi_i^N(t))$ .

To obtain the second part of the proposition, we note that

$$\mathcal{X}[\mu_N^e](t, x, \xi) = \frac{1}{N} \sum_{j=1}^N G(t, x, x_j^N, \xi, \xi_j^N)$$

and thus  $\mathcal{X}[\mu_N^e](t, x_i^N, \xi_i^N) = Y_i(t, X^N, \Xi^N)$  for every  $i \in \{1, \dots, N\}$ . Therefore, (??) is equivalent to  $\dot{\xi}_i^N(t) = \mathcal{X}[\mu_N^e(t)](t, x_i^N, \xi_i^N(t))$  for every  $i \in \{1, \dots, N\}$ . Besides, by definition of  $t \mapsto \varphi_{\mu_N^e(0)}(t, x_i^N, \cdot)$  (given in ?? in Theorem ??), we have

$$\partial_t \varphi_{\mu_N^e(0)}(t, x_i^N, \xi_i^N(0)) = \mathcal{X}[\mu_N^e(t)](t, x_i^N, \varphi_{\mu_N^e(0)}(t, x_i^N, \xi_i^N(0)))$$

with  $\varphi_{\mu_N^e(0)}(0, x_i^N, \xi_i^N(0)) = \xi_i^N(0)$ . Then, (??) follows by Cauchy uniqueness.  $\square$

This shows that there is no difficulty to consider the mean field limit of a system of interacting particles in which the interactions depend on the agents, provided that there is a limit function  $G$  satisfying Assumption ???: one just has to define additional variables  $x_i^N$ , for  $i = 1, \dots, N$ , and couple the dynamics of the  $\xi_i^N$  to the inertial equation  $\dot{x}_i^N = 0$ . This idea originates from [?, Section 5.2]. It seems that, in spite of its simplicity, such a generalization has not been considered in the literature.

Therefore, as in the classical situation where the interaction mapping  $G$  does not depend on the agents, any solution of the system of particles (??) can be embedded to a solution of the Vlasov equation (??) by considering an empirical measure.

As a consequence of the statements ?? and ?? of Theorem ?? and of Proposition ??, we have the following corollary (the last part of which is already well known in the indistinguishable case).

**Corollary 2.** *Let  $K$  be a compact subset of  $\Omega \times \mathbb{R}^d$ . Let  $p \in [1, +\infty)$  be arbitrary. Let  $\mu_0 \in \mathcal{P}_c(K)$  and let  $t \mapsto \mu(t) = \varphi_{\mu_0}(t, \cdot, \cdot)_* \mu_0$  be the solution on  $[0, T_{\max}(K))$  of the Vlasov equation (??) such that  $\mu(0) = \mu_0$ . Besides, for every  $N \in \mathbb{N}^*$ , let  $(X^N, \Xi^N) \in K^N$  be such that the empirical measure*



$\mu_N^e(0) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(0)}$  converges weakly (equivalently, in Wasserstein distance  $W_p$ ) to  $\mu_0$  as  $N \rightarrow +\infty$  (see Appendix ?? for general results). For every  $N \in \mathbb{N}^*$ , let  $t \mapsto \Xi^N(t)$  be the solution on  $[0, T_{\max}(K))$  of the particle system (??) with parameter  $X^N$  such that  $\Xi^N(0) = \Xi_0^N$ .

Then, the empirical measure  $\mu_N^e(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}$  converges weakly (equivalently, in Wasserstein distance  $W_p$ ) to  $\mu(t)$  as  $N \rightarrow +\infty$ , uniformly with respect to  $t$  on any compact interval of  $[0, T_{\max}(K))$ .

If moreover  $G$  is locally Lipschitz with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact), then

$$W_p(\mu(t), \mu_N^e(t)) \leq C_{\mu, \mu_N^e}(t) W_p(\mu_0, \mu_N^e(0))$$

for every  $t \in [0, T_{\max}(K))$  (with  $C_{\mu, \mu_N^e}(t)$  defined by (??)).

Lemmas ?? and ?? in Appendix ?? provide general results ensuring that  $W_p(\mu_0, \mu_N^e(0)) \rightarrow 0$  as  $N \rightarrow +\infty$ , and Lemma ?? gives an estimate of convergence, at rate  $\frac{1}{N^{r/p}}$ , within the framework of tagged partitions.

**Remark 2.** Alternatively, instead of empirical measures, we may also consider *semi-empirical measures*: setting

$$(\mu_0)_{X^N}^{se} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \mu_{0, x_i^N},$$

the unique solution  $t \mapsto \tilde{\mu}^N(t) = \varphi_{(\mu_0)_{X^N}^{se}}(t, \cdot, \cdot)_* (\mu_0)_{X^N}^{se}$  of the Vlasov equation (??) such that  $\tilde{\mu}^N(0) = (\mu_0)_{X^N}^{se}$  is of the form  $\tilde{\mu}^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \tilde{\mu}_{t, x_i^N}^N$  (it differs from the semi-empirical measure  $\mu(t)_{X^N}^{se} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \mu_{t, x_i^N}$ ). Its marginal on  $\Omega$  is the empirical measure  $\nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$ .

Lemma ?? in Appendix ?? provides results on the convergence of  $W_p(\mu_0, (\mu_0)_{X^N}^{se})$  to 0, as well as estimates with a rate of convergence under appropriate assumptions.

### 2.3 Eulerian viewpoint: Liouville equation

The Eulerian viewpoint consists of propagating, for any parameter  $X \in \Omega^N$ , an initial probability measure in  $\mathbb{R}^{dN}$  under the flow of diffeomorphisms  $\Phi^N(t, X, \cdot)$  of  $\mathbb{R}^{dN}$  generated by the time-dependent vector field  $Y^N(t, X, \cdot)$  defined by (??).

Given  $N \in \mathbb{N}^*$  fixed, we consider the ( $N$ -body) *Liouville equation* associated with the time-dependent vector field  $Y^N$  defined by (??), depending on the parameter  $X^N \in \Omega^N$ , given by

$$\boxed{\partial_t \rho^N + \operatorname{div}_{\Xi}(Y^N \rho^N) = 0} \tag{37}$$

This is a usual transport equation on  $\mathbb{R}^{dN}$ , parametrized by  $X^N \in \Omega^N$ , where the divergence is considered with respect to  $\Xi = (\xi_1, \dots, \xi_N)$ , and we thus have the following standard result. Here, it is understood that  $\rho^N(t)$  is a probability Radon measure on  $(\Omega \times \mathbb{R}^d)^N \simeq \Omega^N \times \mathbb{R}^{dN}$ .

**Proposition 2.** *Let  $K$  be a compact subset of  $\Omega \times \mathbb{R}^d$ . Let  $\rho_0^N \in \mathcal{P}_c(\Omega^N \times \mathbb{R}^{dN})$  be such that all marginals of  $\rho_0^N$  on any copy of  $\Omega \times \mathbb{R}^d$  are supported in the same compact  $K$ . There exists a unique solution  $t \mapsto \rho^N(t)$  of the Liouville equation (??) in  $\mathcal{C}^0([0, T_{\max}(K)), \mathcal{P}_c(\Omega^N \times \mathbb{R}^{dN}))$ , locally Lipschitz with respect to  $t$  for the distance  $L_{\theta}^1 W_1$  (where  $\theta$  is defined below), such that  $\rho^N(0) = \rho_0^N$ , given by*

$$\rho^N(t) = \Phi^N(t)_* \rho_0^N \tag{38}$$

*i.e.,  $\rho^N(t)$  is the image (pushforward) of  $\rho_0^N$  under the particle flow.*

The notation (??) is slightly abusive. To explain it, let us make precise some notations and in particular the disintegration procedure. Given any measure  $\rho \in \mathcal{P}(\Omega^N \times \mathbb{R}^{dN})$ , denoting by  $\pi^{\otimes N} : \Omega^N \times \mathbb{R}^{dN} \rightarrow \Omega^N$  the canonical projection, we will always denote by  $\theta$  the probability Radon measure on  $\Omega^N$  given by  $\theta = (\pi^{\otimes N})_* \rho$  (image of  $\rho$  under  $\pi^{\otimes N}$ ), that is the marginal of  $\rho$  on  $\Omega^N$ . By disintegration of  $\rho$  with respect to  $\theta$ , there exists a family  $(\rho_X)_{X \in \Omega^N}$  of probability Radon measures on  $\mathbb{R}^{dN}$  such that  $\rho = \int_{\Omega^N} \rho_X d\theta(X)$ .

With these notations,  $\rho_t^N = \rho^N(t)$  is disintegrated as  $\rho_t^N = \int_{\Omega^N} \rho_{t,X}^N d\theta^N(X)$  with respect to its marginal  $\theta^N = (\pi^{\otimes N})_* \rho^N(t)$  on  $\Omega^N$ . The marginal  $\theta^N$  does not depend on  $t$  because (??) can be written as  $\partial_t \rho^N + L_{Y^N} \rho^N = 0$ , with the Lie derivative acting with respect to the variable  $\xi$ , and we have  $(\pi^{\otimes N})_* L_{Y^N} = 0$ . Finally, (??) means that

$$\rho_{t,X}^N = (\Phi_{t,X}^N)_* \rho_{0,X}^N$$

for every  $t \in [0, T_{\max}(K))$  and for  $\theta^N$ -almost every  $X \in \Omega^N$ .

**Remark 3.** If  $\rho_0^N = \delta_{X^N} \otimes \delta_{\Xi_0^N}$  for some  $(X^N, \Xi_0^N) \in K^N$  then  $\rho^N(t) = \delta_{X^N} \otimes \delta_{\Xi^N(t)}$  where  $t \mapsto \Xi^N(t)$  is the solution on  $[0, T_{\max}(K))$  of the particle system (??) with parameter  $X^N$  such that  $\Xi^N(0) = \Xi_0^N$ . In other words, the solutions of the particle system are naturally embedded as Dirac measures solutions of the Liouville system.

Hence, in some sense, the Liouville equation contains all possible solutions of the particle system. But it contains more: considering the particle system (??), instead of taking a *deterministic* initial condition  $\Xi^N(0) = \Xi_0^N \in \mathbb{R}^{dN}$ , one may want to take a *distribution* of initial conditions, for instance one may want to consider all possible initial conditions that are distributed around  $\Xi_0^N$  according to a Gaussian law, in order to take into account noise or uncertainties in the initial conditions. In such a way, the Liouville equation (??) has a *probabilistic* interpretation with respect to the particle system (??).

If the probability measure  $\rho^N(t)$  on  $\Omega^N \times \mathbb{R}^{dN}$  has a density  $f^N$ , then  $f^N(t, X, \Xi)$  represents the density of particles having the positions  $X = (x_1, \dots, x_N) \in \Omega^N$  and respective momenta  $\Xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^{dN}$ . This is in contrast with the mean field procedure that consists of taking the large  $N$  limit of the average over all particles but one. In the next section we show how to recover Vlasov from Liouville by taking marginals.

## 2.4 Recovering Vlasov from Liouville by taking marginals

Compared with  $\mu(t)$  that is a probability measure on  $\Omega \times \mathbb{R}^d$ ,  $\rho^N(t)$  is a probability measure on  $(\Omega \times \mathbb{R}^d)^N \simeq \Omega^N \times \mathbb{R}^{dN}$ . It is thus tempting to search for a relationship between  $\mu(t)$  and  $\rho^N(t)$  by taking marginals of  $\rho^N(t)$ . This is what has been done in [?] or in [?, ?] in the different context of quantum mechanics. Adapted to the present situation, the method developed in [?], which provides an explicit rate of convergence, consists of proving that the marginals of the solutions  $\rho(t)$  of (??) are close, in Wasserstein topology, to solutions  $\mu(t)$  of the Vlasov equation (??), as established hereafter.

As we are going to see, this can be done by taking adequate initial conditions  $\rho_0^N$  for the Liouville equation (??). We have to perform a symmetrization under permutations for the initial condition  $\rho_0^N$  and also for the corresponding solution  $\rho^N(t)$ , not only with respect to  $\Xi$  but also with respect to the parameter variable  $X$ . Note that the symmetrization is not preserved by the flow, so we have to consider the symmetrization  $\rho^N(t)^s$  at any time  $t$ .

Given any  $\rho \in \mathcal{P}(\Omega^N \times \mathbb{R}^{dN})$ , we define the measure  $\rho^s \in \mathcal{P}(\Omega^N \times \mathbb{R}^{dN})$ , called the symmetrization under permutations of  $\rho$  (see Appendix ??), by

$$\int_{\Omega^N \times \mathbb{R}^{dN}} f(X, \Xi) d\rho^s(X, \Xi) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{\Omega^N \times \mathbb{R}^{dN}} f(\sigma \cdot X, \sigma \cdot \Xi) d\rho(X, \Xi) \quad (39)$$

for every  $f \in \mathcal{C}_c^0(\Omega^N \times \mathbb{R}^{dN})$ , where  $\sigma \cdot X = (x_{\sigma(1)}, \dots, x_{\sigma(N)})$  and  $\sigma \cdot \Xi = (\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)})$  for all  $X \in \Omega^N$  and  $\Xi \in \mathbb{R}^{dN}$ , and where  $\mathfrak{S}_N$  is the group of permutations of  $N$  elements.

Now, given any  $k \in \{1, \dots, N\}$ , we denote by  $\rho_{N:k}^s$  the  $k^{\text{th}}$ -order marginal of  $\rho^s$  (not to be confused with the symmetrization under permutations of the marginal, which we do not use), which is, by definition, the image of  $\rho^s$  under the projection of  $\Omega^N \times \mathbb{R}^{dN}$  onto the product  $\Omega^k \times \mathbb{R}^{dk}$  of the  $k$  first copies of  $\Omega$  with the  $k$  first copies of  $\mathbb{R}^d$ .

Since we are going to compute Wasserstein distances in  $(\Omega \times \mathbb{R}^d)^k \simeq \Omega^k \times \mathbb{R}^{dk}$ , we have to choose a distance in that space. Recall that  $\Omega \times \mathbb{R}^d$  is equipped with the distance  $d_{\Omega \times \mathbb{R}^d} = d_{\Omega} + d_{\mathbb{R}^d}$  where  $d_{\mathbb{R}^d}$  is the distance on  $\mathbb{R}^d$  induced by the norm  $\|\cdot\|$  (which is arbitrary). Let  $q \in [1, +\infty]$  be arbitrary. Given any  $k \in \mathbb{N}^*$ , we endow  $(\Omega \times \mathbb{R}^d)^k$  with the  $\ell^q$  distance based on  $d_{\Omega \times \mathbb{R}^d}$ , defined by

$$\begin{aligned} d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}((X, \Xi), (X', \Xi')) &= \|(d_{\Omega \times \mathbb{R}^d}((x_1, \xi_1), (x'_1, \xi'_1)), \dots, d_{\Omega \times \mathbb{R}^d}((x_k, \xi_k), (x'_k, \xi'_k)))\|_{\ell^q} \\ &= \begin{cases} \left( \sum_{i=1}^k (d_{\Omega}(x_i, x'_i) + \|\xi_i - \xi'_i\|)^q \right)^{1/q} & \text{if } q \in [1, +\infty) \\ \max_{1 \leq i \leq k} (d_{\Omega}(x_i, x'_i) + \|\xi_i - \xi'_i\|) & \text{if } q = +\infty \end{cases} \end{aligned} \quad (40)$$

for all  $X = (x_1, \dots, x_k)$  and  $X' = (x'_1, \dots, x'_k)$  in  $\Omega^k$  and for all  $\Xi = (\xi_1, \dots, \xi_k)$  and  $\Xi' = (\xi'_1, \dots, \xi'_k)$  in  $\mathbb{R}^{dk}$ . Note that, when  $k = 1$ , we have  $d_{\Omega \times \mathbb{R}^d}^{[q]} = d_{\Omega \times \mathbb{R}^d}^{[1]} = d_{\Omega} + d_{\mathbb{R}^d}$ .

Given any  $p, q \in [1, +\infty]$ , we denote by  $W_p^{[q]}$  the Wasserstein distance  $W_p$  on  $\mathcal{P}(\Omega^k \times (\mathbb{R}^d)^k)$  with respect to the distance  $d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}$ .

We refer to the beginning of Appendix ?? and in particular to Remark ?? for comments on the importance of choosing a distance on the product space  $(\Omega \times \mathbb{R}^d)^k$  and for remarks on the Wasserstein distance  $W_p^{[q]}$ . In particular, by (??), we have  $W_p^{[q_2]} \leq W_p^{[q_1]} \leq k^{\frac{1}{q_1} - \frac{1}{q_2}} W_p^{[q_2]}$  if  $1 \leq q_1 \leq q_2 \leq +\infty$  for any  $p \in [1, +\infty]$ .

In this section, we establish two ways for recovering Vlasov from Liouville by taking marginals.

Let  $\mu_0 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , disintegrated as  $\mu_0 = \int_{\Omega} \mu_{0,x} d\nu(x)$  with respect to its marginal  $\nu = \pi_* \mu_0$  on  $\Omega$ . Setting  $T_0 = T_{\max}(\text{supp}(\mu_0))$  (as given by Lemma ??), we consider the unique solution  $t \mapsto \mu(t) = \varphi_{\mu_0}(t) \# \mu_0$  in  $\mathcal{C}_{\text{comp}}^0([0, T_0], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  of the Vlasov equation (??) such that  $\mu(0) = \mu_0$ , as given by Theorem ?. Recall that  $\mu_{t,x} = \varphi_{\mu_0}(t, x, \cdot) \# \mu_{0,x}$  for  $\nu$ -almost every  $x \in \Omega$ .

Hereafter, we propose two possible choices of  $\rho_0^N \in \mathcal{P}_c(\Omega^N \times \mathbb{R}^{dN})$ , generating by Proposition ?? the solution  $\rho^N(t) = \Phi^N(t) \# \rho_0^N$  of the Liouville equation (??) from which we recover at the larger  $N$  limit the solution  $\mu(t)$  of the Vlasov equation (??) by taking marginals.

In Theorem ??, we take  $\rho_0^N$  Dirac; in Theorem ??, we take  $\rho_0^N$  “semi-Dirac”. In both cases, we prove that  $\rho^N(t)_{N:k}^s$  converges to  $\mu(t)^{\otimes k}$  as  $N \rightarrow +\infty$  and we establish convergence estimates in Wasserstein distance  $W_p^{[q]}$ . The fact that the  $k^{\text{th}}$ -order marginal  $\rho^N(t)_{N:k}^s$  of the symmetrization of  $\rho^N(t)$ , which is absolutely not a tensor product at time  $t = 0$ , becomes however the tensor product  $\mu(t)^{\otimes k}$  at the limit  $N \rightarrow +\infty$ , is usually referred to as *propagation of chaos*.

#### 2.4.1 First way, with $\rho_0^N$ Dirac

Given any fixed  $N \in \mathbb{N}^*$ , let  $(X^N, \Xi_0^N) \in \text{supp}(\mu_0)^N \subset \Omega^N \times \mathbb{R}^{dN}$  be arbitrary. Typically we may want that the empirical measure  $\mu_{(X^N, \Xi_0^N)}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_{0,i}^N}$  converges to  $\mu_0$  in Wasserstein distance as  $N \rightarrow +\infty$  (see Appendix ?? for such conditions). Let  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$  be the solution on  $[0, T_0]$  of the particle system (??) such that  $\Xi^N(0) = \Xi_0^N$ . If  $\mu_{(X^N, \Xi_0^N)}^e$  converges

to  $\mu_0$  then, by Corollary ??, the empirical measure  $\mu_{(X^N, \Xi^N(t))}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}$  converges to  $\mu(t)$  in Wasserstein distance as  $N \rightarrow +\infty$ .

Defining  $\rho_0^N \in \mathcal{P}_c(\Omega^N \times \mathbb{R}^{dN})$  as the Dirac measure  $\rho_0^N = \delta_{X^N} \otimes \delta_{\Xi_0^N}$ , by Remark ??, the unique solution of the Liouville equation (??) such that  $\rho^N(0) = \rho_0^N$ , is given by the Dirac measure

$$\rho^N(t) = \Phi^N(t) * \rho_0^N = \delta_{X^N} \otimes \delta_{\Xi^N(t)} \quad \forall t \in [0, T_0].$$

It is then easy to see that  $\rho^N(t)_{N:1}^s = \mu_{(X^N, \Xi^N(t))}^e$  (see the proof of the theorem below). Therefore, if  $\mu_{(X^N, \Xi_0^N)}^e$  converges weakly to  $\mu_0$  then  $\rho^N(t)_{N:1}^s$  converges weakly to  $\mu(t)$  as  $N \rightarrow +\infty$ . The convergence is less obvious for the marginals of order  $k \geq 2$ .

Recall that  $G$  satisfies Assumption ??.

**Theorem 2.** *We have the following statements, for any  $p \in [1, +\infty)$  and  $q \in [1, +\infty]$ .*

- (A) *If  $\mu_{(X^N, \Xi_0^N)}^e$  converges weakly (equivalently, in Wasserstein distance  $W_p$ ) to  $\mu_0$  as  $N \rightarrow +\infty$ , then, for every  $k \in \mathbb{N}^*$ ,  $\rho^N(t)_{N:k}^s$  converges weakly (equivalently, in Wasserstein distance  $W_p^{[q]}$ ) to  $\mu(t)^{\otimes k}$  as  $N \rightarrow +\infty$ , uniformly with respect to  $t$  on any compact interval of  $[0, T_0]$ .*
- (B) *Assuming that  $G$  is locally Lipschitz with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact), setting*

$$S_\mu^N(\tau) = \text{supp}(\mu(\tau)) \cup \{(x_i^N, \xi_i^N(\tau)) \mid i \in \{1, \dots, N\}\} \quad (41)$$

and defining

$$C_\mu^N(t) = \exp\left(2 \int_0^t \text{Lip}(G(\tau, \cdot, \cdot, \cdot, \cdot) |_{S_\mu^N(\tau)^2}) d\tau\right), \quad (42)$$

for every  $N \in \mathbb{N}^*$  and for every  $t \in [0, T_0]$  we have  $\rho^N(t)_{N:1}^s = \mu_{(X^N, \Xi^N(t))}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}$  and

$$W_p(\rho^N(t)_{N:1}^s, \mu(t)) \leq C_\mu^N(t) W_p(\mu_{(X^N, \Xi_0^N)}^e, \mu_0) \quad (43)$$

and, for every  $k \in \mathbb{N}^*$  such that  $k^2 \leq 2N \ln(1 + \frac{1}{2^p})$ ,

$$\begin{aligned} W_p^{[q]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) &\leq 2k^{1/q} \left(\frac{k^2}{N}\right)^{1/p} (\text{diam}_\Omega(\text{supp}(\nu)) + \text{diam}_{\mathbb{R}^d}(\Xi^N(t))) \\ &\quad + k^{1/q} C_\mu^N(t) W_p(\mu_{(X^N, \Xi_0^N)}^e, \mu_0) \end{aligned} \quad (44)$$

where  $\text{diam}_{\mathbb{R}^d}(\Xi^N(t)) = \max_{1 \leq i, j \leq N} \|\xi_i^N(t) - \xi_j^N(t)\|$ .

Theorem ?? is proved in Appendix ??.

In (??),  $\text{diam}_\Omega(\text{supp}(\nu)) = \max_{x, x' \in \text{supp}(\nu)} d_\Omega(x, x')$ , and the Wasserstein distance  $W_p^{[q]}$  on  $\Omega^k \times (\mathbb{R}^d)^k$  is computed with respect to the distance  $d_{\Omega^k \times (\mathbb{R}^d)^k}^{[q]}$  defined by (??). Since  $W_p^{[q]} \leq k^{\frac{1}{q}} W_p^{[\infty]}$  (by (??)), the strongest inequality (??) is obtained when  $q = +\infty$ .

Lemmas ?? and ?? in Appendix ?? show that there always exists a sequence of empirical measures  $\mu_{(X^N, \Xi_0^N)}^e$  converging weakly to  $\mu_0$ . As alluded above, to obtain an interesting convergence estimate from Item ?? of this theorem, we apply Lemma ?? in Appendix ??, which yields the estimate  $W_p(\mu_{(X^N, \Xi_0^N)}^e, \mu_0) \leq \frac{1}{N^{r/p}} C_{\Omega \times \mathbb{R}^d}^{1/p} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu_0))^{1-1/p}$  under the assumption of the existence of a family of tagged partitions. As noted in this appendix, there exist plenty of results quantifying the convergence of empirical measures to a given measure (see, e.g., [?]). Lemma ?? is a rough result.

**Corollary 3.** *In the context of Item ?? of Theorem ??, we assume moreover that there exists a family of tagged partitions of  $\text{supp}(\mu_0)$  associated with  $\mu_0$  (see Section ??), i.e., for every  $N \in \mathbb{N}^*$  there exists a partition of  $\text{supp}(\mu_0) = \cup_{i=1}^N F_i^N$  such that all subsets  $F_i^N \subset \Omega \times \mathbb{R}^d$  are  $\mu_0$ -measurable, pairwise disjoint, satisfy  $\mu_0(F_i^N) = \frac{1}{N}$  and  $\text{diam}_{\Omega \times \mathbb{R}^d}(F_i^N) \leq C_{\Omega \times \mathbb{R}^d}/N^r$  for some  $C_{\Omega \times \mathbb{R}^d} > 0$  and  $r > 0$  not depending on  $N$ , and  $N$ -tuples  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$  and  $\Xi_0^N = (\xi_{0,1}^N, \dots, \xi_{0,N}^N) \in (\mathbb{R}^d)^N$  such that  $(x_i^N, \xi_{0,i}^N) \in F_i^N$  for every  $i \in \{1, \dots, N\}$ . Then, for every  $t \in [0, T_0)$ ,*

$$W_p(\rho^N(t)_{N:1}^s, \mu(t)) \leq \frac{1}{N^{r/p}} C_{\Omega \times \mathbb{R}^d}^{1/p} \text{diam}_E(\text{supp}(\mu_0))^{1-1/p} C_\mu^N(t)$$

and, for every  $k \in \mathbb{N}^*$  such that  $k^2 \leq 2N \ln(1 + \frac{1}{2^p})$ ,

$$W_p^{[\infty]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) \leq 2 \left( \frac{k^2}{N} \right)^{1/p} (\text{diam}_\Omega(\text{supp}(\nu)) + \text{diam}_{\mathbb{R}^d}(\Xi^N(t))) + \frac{C_{\Omega \times \mathbb{R}^d}^{1/p}}{N^{r/p}} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu_0))^{1-1/p} C_\mu^N(t). \quad (45)$$

When  $\Omega$  is a  $n$ -dimensional manifold (thus  $\dim(\Omega \times \mathbb{R}^d) = n + d$ ), we have  $r = 1/(n + d) < 1$ .

According to the estimate (??),  $\rho^N(t)_{N:k}^s$  converges to  $\mu(t)^{\otimes k}$  in Wasserstein distance  $W_p^{[\infty]}$  as  $N \rightarrow +\infty$ , uniformly with respect to  $t$  on compact intervals of  $[0, T_0)$ , at rate  $1/N^{r/p}$  if  $k \ll N^{(1-r)/2}$  and at rate  $k^{2/p}/N^{1/p}$  if  $N^{(1-r)/2} \ll k \ll N^{1/2}$ . The rate of convergence can be improved if one uses better results for convergence of empirical measures.

Note that the assumption of a family of tagged partitions in Corollary ?? essentially entails that  $\mu_0$  be absolutely continuous with respect to a Lebesgue measure on  $\Omega \times \mathbb{R}^d$ .

**Particular case: the indistinguishable case.** Recall that, in the indistinguishable case where  $G$  does not depend on  $(x, x')$ , the mean field is given by (??). We have the following corollary of Theorem ??.

**Corollary 4.** *Assume that  $G$  does not depend on  $(x, x')$ . Let  $\bar{\mu}_0 \in \mathcal{P}_c(\mathbb{R}^d)$  and let  $t \mapsto \bar{\mu}(t)$  be the unique solution on  $[0, T_0)$ , with  $T_0 = T_{\max}(\text{supp}(\bar{\mu}_0))$ , of the Vlasov equation (??) (without dependence on  $x$ ) such that  $\bar{\mu}(0) = \bar{\mu}_0$  (see Corollary ??). Besides, let  $\bar{\rho}_0^N = \delta_{\Xi_0^N}$  and let  $t \mapsto \bar{\rho}^N(t) = \delta_{\Xi^N(t)}$  be the unique solution on  $[0, T_0)$  of the Liouville equation (??) (without dependence on  $X$ ) such that  $\bar{\rho}^N(0) = \bar{\rho}_0^N$ . Then, for every  $t \in [0, T_0)$ ,*

$$W_p(\bar{\rho}^N(t)_{N:1}^s, \bar{\mu}(t)) \leq C_\mu^N(t) W_p(\bar{\mu}_{\Xi_0^N}^e, \bar{\mu}_0) \quad (46)$$

(and we note that  $\bar{\rho}^N(t)_{N:1}^s = \mu_{\Xi^N(t)}^e$ ), where  $\bar{\mu}_{\Xi^N(t)}^e = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_i^N(t)}$  (empirical measure) and where  $C_\mu^N(t)$  is defined by (??) (without dependence on  $x, x'$ ), and, for every  $k \in \{2, \dots, N\}$ ,

$$W_p^{[\infty]}(\bar{\rho}^N(t)_{N:k}^s, \bar{\mu}(t)^{\otimes k}) \leq 2 \left( \frac{k^2}{N} \right)^{1/p} \text{diam}_{\mathbb{R}^d}(\Xi^N(t)) + C_\mu^N(t) W_p(\bar{\mu}_{\Xi_0^N}^e, \bar{\mu}_0). \quad (47)$$

*Proof.* Following the proof of Corollary ?? and choosing  $\bar{\nu} = \delta_{\bar{x}}$  for some arbitrary  $\bar{x} \in \Omega$ , in the indistinguishable case  $\bar{\mu}(\cdot)$  is solution of the Vlasov equation (??) (without dependence on  $x$ ) if and only if  $\mu(\cdot) = \delta_{\bar{x}} \otimes \bar{\mu}(\cdot)$  is solution of the Vlasov equation (??). We now define  $X^N = (\bar{x}, \dots, \bar{x}) \in \Omega^N$ , and we take  $\rho_0^N = \delta_{X^N} \otimes \delta_{\Xi_0^N}$  as initial condition for the Liouville equation in Theorem ??, so that  $\rho^N(t) = \delta_{X^N} \otimes \bar{\rho}^N(t)$  where  $\bar{\rho}^N(t) = \delta_{\Xi^N(t)}$ . With these choices, we obviously have  $\rho^N(t)_{N:k}^s = \delta_{\bar{x}}^{\otimes k} \otimes \bar{\rho}^N(t)_{N:k}^s$ , and then (??) and (??) straightforwardly follows from (??) and (??), by applying Remark ?? in Appendix ??.  $\square$

### 2.4.2 Second way, with $\rho_0^N$ “semi-Dirac”

Given any fixed  $N \in \mathbf{N}^*$ , let  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$  be arbitrary. We set  $\delta_{X^N} = \delta_{x_1^N} \otimes \dots \otimes \delta_{x_N^N}$  and  $\rho_{0,X^N}^N = \mu_{0,x_1^N} \otimes \dots \otimes \mu_{0,x_N^N}$ . Defining  $\rho_0^N \in \mathcal{P}_c(\Omega^N \times \mathbb{R}^{dN})$  as the “semi-Dirac” measure  $\rho_0^N = \delta_{X^N} \otimes \rho_{0,X^N}^N$ , we consider the unique solution on  $[0, T_0]$  of the Liouville equation (??) such that  $\rho^N(0) = \rho_0^N$ , given by the “semi-Dirac” measure

$$\rho^N(t) = \Phi^N(t)_* \rho_0^N = \delta_{X^N} \otimes \Phi(t, X^N, \cdot)_* \rho_{0,X^N}^N = \delta_{X^N} \otimes \rho_{t,X^N}^N.$$

Note indeed that the marginal  $\theta^N = (\pi^{\otimes N})_* \rho_t^N$  of  $\rho_t^N = \rho^N(t)$  on  $\Omega^N$  is  $\theta^N = \delta_{X^N}$ , and that  $\rho_{t,X^N}^N = \Phi^N(t, X^N, \cdot)_* \rho_{0,X^N}^N$ .

As a preliminary remark, we claim that, at  $t = 0$ , we have

$$(\rho_0^N)_{N:1}^s = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \mu_{0,x_i^N} = (\mu_0)_{X^N}^{se} \quad (48)$$

(semi-empirical measure), which converges weakly to  $\mu_0$  as  $N \rightarrow +\infty$  under slight assumptions on  $\mu_0$ , by Lemma ?? in Appendix ?. More generally,  $(\rho_0^N)_{N:k}^s$  converges weakly to  $\mu_0^{\otimes k}$  (in the proof of the theorem hereafter, we give an explicit expression for  $(\rho_0^N)_{N:k}^s$ , using (??) in Appendix ?). In the theorem below, we establish that this convergence is propagated in time.

**Theorem 3.** *We assume that the norm  $\|\cdot\|$  on  $\mathbb{R}^d$  is induced by a scalar product on  $\mathbb{R}^d$ . Let  $p \in [1, 2]$  and  $q \in [1, +\infty]$  be arbitrary.*

- (A) *Assume that  $x \mapsto \mu_{0,x}$  is  $\nu$ -almost everywhere continuous for the Wasserstein distance  $W_p$ . Then, for every  $k \in \mathbf{N}^*$ ,  $\rho^N(t)_{N:k}^s$  converges weakly to  $\mu(t)^{\otimes k}$  (equivalently, in Wasserstein distance  $W_p^{[q]}$ ) as  $N \rightarrow +\infty$ , uniformly with respect to  $t$  on any compact interval of  $[0, T_0]$ .*
- (B) *Assuming that  $G$  is locally Lipschitz with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact), defining  $S_\mu^N(\tau)$  by (??) and*

$$C_\mu(t) = 11 \left( 1 + 70 \max_{0 \leq \tau \leq t} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu(\tau))) \right)^{1/2} \exp \left( 2t \max_{0 \leq \tau \leq t} \|G(\tau, \cdot, \cdot, \cdot)_{|S_\mu^N(\tau)^2}\|_{\mathcal{C}^{0,1}} \right), \quad (49)$$

we have, for every  $N \in \mathbf{N}^*$ , for every  $k \in \{1, \dots, N\}$  such that  $k^2 \leq 2N \ln(\frac{3}{2})$ ,

$$\begin{aligned} & W_p^{[q]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) \\ & \leq k^{1/q} C_\mu(t) \max \left( \left( \frac{k^2}{N} \right)^{1/p}, \frac{1}{N^{\frac{1}{q}-\frac{1}{2}}}, N^{1-\frac{1}{q}} \sqrt{W_1((\mu_0)_{X^N}^{se}, \mu_0)}, W_p((\mu_0)_{X^N}^{se}, \mu_0) \right) \end{aligned} \quad (50)$$

for every  $t \in [0, T_0]$  (for  $k = 1$ , without the first term in the above parenthesis).

Theorem ?? is proved in Appendix ?. Note that the  $p$ -Wasserstein distance at the left-hand side of (??) is considered with  $p \leq 2$ , because in the proof we use in an instrumental way a variance-type estimate, measuring the  $L^2$  discrepancy between the mean field and the particle vector field (see Appendix ?). Besides,  $q \in [1, +\infty]$  is arbitrary, but only the values  $q \in [1, 2)$  are meaningful. The strongest estimate inferred from (??) is when  $q = 1$ , i.e., when one takes the  $\ell^1$  distance on  $\Omega^k \times (\mathbb{R}^d)^k$ . This choice has no importance while  $k$  is small, but becomes important if one takes for instance  $k = N^{1/4}$ .

To obtain an interesting convergence result from this theorem, we apply the second item of Lemma ?? of Appendix ??, which yields  $W_1((\mu_0)_{X^N}^{se}, \mu_0) \leq \frac{(L+1)C_\Omega}{N^r}$  and  $W_p((\mu_0)_{X^N}^{se}, \mu_0) \leq \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu_0))^{1-1/p} ((L+1)C_\Omega/N^r)^{1/p}$  under a regularity assumption on  $\mu_0$ .

**Corollary 5.** *In the context of Item ?? of Theorem ??, we assume moreover that, for every  $N \in \mathbb{N}^*$ , there exists a tagged partition  $(\mathcal{A}^N, X^N)$  of  $\Omega$  associated with  $\nu$  satisfying (??) (see Section ??), and that  $x \mapsto \mu_{0,x}$  is Lipschitz for the Wasserstein distance  $W_1$ , i.e., that there exists  $L > 0$  such that  $W_1(\mu_{0,x}, \mu_{0,y}) \leq L d_\Omega(x, y)$  for  $\nu$ -almost all  $x, y \in \Omega$ . Then*

$$W_p^{[q]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) \leq k^{1/q}(L+1)C_\Omega C_\mu(t) \max\left(\left(\frac{k^2}{N}\right)^{1/p}, \frac{1}{N^{\frac{1}{q}-\frac{1}{2}}}, \frac{1}{N^{\frac{r}{2}+\frac{1}{q}-1}}, \frac{1}{N^{r/p}}\right) \quad (51)$$

for every  $t \in [0, T_0)$ .

When  $\Omega$  is a  $n$ -dimensional manifold, we have  $r = 1/n$ , hence, if we take  $q = 1$  and  $p = 1$ , the rate of convergence provided by (??) is  $\frac{k}{N^{1/2n}}$ .

Note that the assumption of a family of tagged partitions in Corollary ?? essentially entails that  $\nu$  be absolutely continuous with respect to a Lebesgue measure on  $\Omega$ .

**Particular case: the indistinguishable case.** We have the following corollary of Theorem ?? (still assuming that the norm  $\|\cdot\|$  on  $\mathbb{R}^d$  is induced by a scalar product on  $\mathbb{R}^d$ , that  $p \in [1, 2]$  and that  $q \in [1, +\infty)$ ).

**Corollary 6.** *Assume that  $G$  does not depend on  $(x, x')$ . Let  $\bar{\mu}_0 \in \mathcal{P}_c(\mathbb{R}^d)$  and let  $t \mapsto \bar{\mu}(t)$  be the unique solution on  $[0, T_0)$ , with  $T_0 = T_{\max}(\text{supp}(\bar{\mu}_0))$ , of the Vlasov equation (??) (without dependence on  $x$ ) such that  $\bar{\mu}(0) = \bar{\mu}_0$  (see Corollary ??). Besides, let  $\bar{\rho}_0^N = \bar{\mu}_0^{\otimes N}$  and let  $t \mapsto \bar{\rho}^N(t) = \Phi(t, \cdot)_* \bar{\rho}_0^N$  be the unique solution on  $[0, T_0)$  of the Liouville equation (??) (without dependence on  $X$ ) such that  $\bar{\rho}^N(0) = \bar{\rho}_0^N$ . Then, for every  $N \in \mathbb{N}^*$ , for every  $k \in \{1, \dots, N\}$ , we have*

$$W_p^{[q]}(\bar{\rho}^N(t)_{N:k}^s, \bar{\mu}(t)^{\otimes k}) \leq k^{1/q} C_{\bar{\mu}}(t) \max\left(\left(\frac{k^2}{N}\right)^{1/p}, \frac{1}{N^{\frac{1}{q}-\frac{1}{2}}}\right)$$

for every  $t \in [0, T_0)$ , where  $C_{\bar{\mu}}(t)$  is defined as in (??) (without dependence on  $x$ ).

*Proof.* The proof is the same as the one of Corollary ??: we take  $\bar{\nu} = \delta_{\bar{x}}$  for an arbitrary  $\bar{x} \in \Omega$ . Then  $(\mu_0)_{X^N}^{se} = \delta_{\bar{x}} \otimes \bar{\mu}_0$  and thus  $W_1((\mu_0)_{X^N}^{se}, \mu_0) = W_p((\mu_0)_{X^N}^{se}, \mu_0) = 0$ .  $\square$

**Remark 4.** Applying Corollary ?? to the kinetic plus potential Hamiltonian case where we have  $G(t, (q_i, p_i), (q_j, p_j)) = (p_i, \nabla V(q_i - q_j))$ , we recover [?, Theorem 3.1]. The corollary can also be applied to more general Hamiltonian systems, for example,  $G(t, (q_i, p_i), (q_j, p_j)) = (p_i, -\nabla(V(q_i - q_j) + (p_i - A(q_i))^2))$ , where a magnetic field associated to a vector potential  $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ; or to Cucker–Smale systems, for which  $G(t, (q_i, p_i), (q_j, p_j)) = (p_i, F(|q_i - q_j|)(p_i - p_j))$ , and generalizations introduced in [?].

### 3 From mesoscopic to macroscopic scale

(“from Vlasov to Euler”, hydrodynamic limit)

Given any  $\mu \in \mathcal{P}(\Omega \times \mathbb{R}^d)$ , disintegrated as  $\mu = \int_\Omega \mu_x d\nu(x)$ , the three macroscopic quantities that are usually considered in the *hydrodynamic limit* procedure are the three first moments of the measure  $\mu$  with respect to  $\xi$  (see, e.g., [?]), leading to define, for  $\nu$ -almost every  $x \in \Omega$ :

- the total mass  $\rho(x) \geq 0$  of  $\mu_x$  by

$$\rho(x) = \int_{\mathbb{R}^d} d\mu_x(\xi) = \mu_x(\mathbb{R}^d),$$

(moment of order 0) which is here assumed to be equal to 1 for  $\nu$ -almost every  $x \in \Omega$ ;

- the “speed”  $y(x) \in \mathbb{R}^d$  by

$$\rho(x)y(x) = \int_{\mathbb{R}^d} \xi d\mu_x(\xi),$$

(moment of order 1) which is also the expectation of any random law of probability distribution  $\mu_x$ ;

- and the “temperature”  $T(x) \geq 0$  by

$$d\rho(x)T(x) = \int_{\mathbb{R}^d} \|\xi - y(x)\|^2 d\mu_x(\xi)$$

(moment of order 2) which is a variance, or equivalently, if  $\|\cdot\|$  is the Euclidean norm, by

$$\frac{1}{2}\rho(x)\|y(x)\|^2 + \frac{d}{2}\rho(x)T(x) = \frac{1}{2} \int_{\mathbb{R}^d} \|\xi\|^2 d\mu_x(\xi).$$

Let  $t \mapsto \mu(t)$  be a fixed locally Lipschitz solution of the Vlasov equation (??) (recall that the mean field  $\mathcal{X}[\mu]$  is defined by (??)). According to Remark ??, its marginal  $\nu(t) = \nu$  on  $\Omega$  does not depend on  $t$ . Following the hydrodynamic limit procedure recalled above (see also, e.g. [?, ?, ?]), for every  $t \in \mathbb{R}$  and for  $\nu$ -almost every  $x \in \Omega$ , we define the three first moments  $\rho(t, x)$ ,  $y(t, x)$  and  $T(t, x)$  of  $\mu(t)$ . The moment  $\rho(t, x)$  of order 0 does not depend on  $t$  and is equal to 1 for  $\nu$ -almost every  $x \in \Omega$  and 0 otherwise. Let us study the moments of order one and two.

### 3.1 Moment of order 1: Euler equation

Given any solution  $t \mapsto \mu(t)$  of the Vlasov equation (??) on  $[0, T)$  (for some  $T > 0$ ), of marginal  $\nu$  on  $\Omega$ , using the disintegration of  $\mu$  with respect to  $\nu$  we define

$$\boxed{y(t, x) = \int_{\mathbb{R}^d} \xi d\mu_{t,x}(\xi)} \quad (52)$$

for  $\nu$ -almost every  $x \in \Omega$ , and  $y(t, x) = 0$  for every  $x \in \Omega \setminus \text{supp}(\nu)$ , for every  $t \in [0, T)$ . Using (??) (or, rather, (??)), we have

$$\partial_t y(t, x) = \langle \partial_t \mu_{t,x}, \xi \mapsto \xi \rangle = \langle \mu_{t,x}, L\mathcal{X}[\mu_t](t, x, \cdot)(\xi \mapsto \xi) \rangle = \int_{\mathbb{R}^d} \mathcal{X}[\mu_t](t, x, \xi) d\mu_{t,x}(\xi)$$

which is a kind of “mean” mean field, since the mean field is now averaged under  $\mu_{t,x}$ . Hence

$$\partial_t y(t, x) = \int_{\mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} G(t, x, x', \xi, \xi') d\mu_t(x', \xi') d\mu_{t,x}(\xi). \quad (53)$$

It is remarkable that, for some classes of functions  $G$ , we obtain a “closed” equation in  $y$ :

#### 3.1.1 Hegselmann–Krause model: linear Euler equation

**Proposition 3.** *In the Hegselmann–Krause (opinion propagation) model given in Example ??, we have  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$  and*

$$\boxed{\partial_t y(t, x) = (Ay(t))(x)} \quad (54)$$

(Euler equation), where  $A$  is the bounded operator on  $L^2_\nu(\Omega, \mathbb{R}^d)$  defined by

$$(Ay)(x) = \int_{\Omega} \sigma(x, x')(y(x') - y(x)) d\nu(x') \quad \forall y \in L^2_\nu(\Omega, \mathbb{R}^d) \quad (55)$$

and  $\nu$  is the marginal of  $\mu(t)$  on  $\Omega$  (not depending on  $t$ ).



*Proof.* Using the disintegration of the measure, we infer from (??) that

$$\begin{aligned} \partial_t y(t, x) &= \underbrace{\int_{\mathbf{R}^d} d\mu_{t,x}(\xi)}_{=1} \int_{\Omega} \sigma(x, x') \underbrace{\int_{\mathbf{R}^d} \xi' d\mu_{t,x'}(\xi')}_{=y(t,x')} d\nu(x') \\ &\quad - \underbrace{\int_{\mathbf{R}^d} \xi d\mu_{t,x}(\xi)}_{=y(t,x)} \int_{\Omega} \sigma(x, x') \underbrace{\int_{\mathbf{R}^d} d\mu_{t,x'}(\xi')}_{=1} d\nu(x') \end{aligned}$$

and the result follows.  $\square$

**Remark 5.** If  $\mu(0) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(0)} = \mu_{(X^N, \Xi_0^N)}^e$  as in Proposition ??, then  $\mu(t) = \mu_{(X^N, \Xi^N(t))}^e$  whose marginal on  $\Omega$  is  $\nu = \nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$  and whose disintegration with respect to  $\nu$  is  $\mu_{t,x} = \delta_{\xi_i^N(t)}$  if  $x = x_i^N$  for  $i \in \{1, \dots, N\}$  and 0 otherwise. In this case, in the context of Proposition ??, we have then  $y(t, x) = \xi_i^N(t)$  if  $x = x_i^N$  for  $i \in \{1, \dots, N\}$  and 0 otherwise, and the differential equation (??) exactly coincides with the particle system (??).

### 3.1.2 Open issue: how to obtain a closed equation?

In Section ??, in the Hegselmann-Krause model the operator  $A$  is linear. More generally, the proof of Proposition ?? shows that, when  $G$  is linear with respect to  $(\xi, \xi')$ , we obtain for  $y(t, x)$  a linear Euler equation: we obtain directly a “closed” equation.

This simple argument does not seem to work as soon as  $G$  is nonlinear with respect to  $(\xi, \xi')$ . An open question is to characterize the functions  $G$  so that, for any solution  $\mu$  of (??), the function  $y$  defined by (??) satisfies an “Euler equation”, possibly nonlinear,  $\partial_t y(t, \cdot) = A(y(t, \cdot))$ . We face here with the classical problem in kinetic theory of considering the three first moments of a solution  $\mu$  of the Vlasov equation, and searching how to close the moment system since a priori the equations depend on higher-order moments. Suitable closure assumptions are not known so far, in general (see [?] for interesting comments, see also Section ?? further). This is why it is usual to consider a monokinetic ansatz for  $\mu$ , as explained in the following section.

### 3.1.3 The $\nu$ -monokinetic case: general nonlinear Euler equation

In this section, we assume that  $\Omega$  is compact. Let us consider specific solutions  $\mu$  of the Vlasov equation (??), that are  $\nu$ -monokinetic, meaning that  $\mu$  is delta-valued in the  $\xi$  variable and has the marginal  $\nu$  on  $\Omega$ . Given any  $\nu \in \mathcal{P}(\Omega)$  and any measurable function  $y : \Omega \rightarrow \mathbb{R}^d$ , we define the  $\nu$ -monokinetic measure  $\mu_y^\nu$  on  $\Omega \times \mathbb{R}^d$  by

$$\mu_y^\nu = \nu \otimes \delta_{y(\cdot)}. \quad (56)$$

We have  $y(x) = \int_{\Omega} \xi d(\mu_y^\nu)_x(\xi)$  for  $\nu$ -almost every  $x \in \Omega$  (as in (??)), where the disintegration of  $\mu_y^\nu$  with respect to its marginal  $\nu$  on  $\Omega$  is given by the family of conditional measures defined by  $(\mu_y^\nu)_x = \delta_{y(x)}$ .

**Proposition 4.** *Recall that  $\Omega$  is assumed to be compact. Let  $\nu \in \mathcal{P}(\Omega)$ . Let  $T > 0$  and let  $t \mapsto y(t, \cdot) \in L_\nu^\infty(\Omega, \mathbb{R}^d)$  be a locally Lipschitz mapping on  $[0, T]$ .*

*The mapping  $t \mapsto \mu(t) = \mu_{y(t, \cdot)}^\nu \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , of marginal  $\nu$  on  $\Omega$ , is a ( $\nu$ -monokinetic) solution on  $[0, T]$  of the Vlasov equation (??) with the general mean field (??) if and only if the mapping  $t \mapsto y(t, \cdot) \in L_\nu^\infty(\Omega, \mathbb{R}^d)$  is a solution on  $[0, T]$  of the (nonlinear) Euler equation*

$$\boxed{\partial_t y(t, \cdot) = A(t, y(t, \cdot))} \quad (57)$$

where  $A : \mathbb{R} \times L_\nu^\infty(\Omega, \mathbb{R}^d) \rightarrow L_\nu^\infty(\Omega, \mathbb{R}^d)$  is the nonlinear operator (depending on  $\nu$ ) defined by

$$A(t, y)(x) = \int_{\Omega} G(t, x, x', y(x), y(x')) d\nu(x') \quad (58)$$

(recall that  $G$  satisfies Assumption ??) for every  $t \in \mathbb{R}$  and for every  $y \in L_\nu^\infty(\Omega, \mathbb{R}^d)$ .

*Proof.* When  $\mu_t = \mu_{y(t, \cdot)}^\nu$ , (??) gives  $\mathcal{X}[\mu_t](t, x, \xi) = \int_{\Omega} G(t, x, x', \xi, y(t, x')) d\nu(x')$ . The proof is then straightforward, and we can note that  $A(t, y)(x) = \mathcal{X}[\mu_y^\nu](t, x, y(x))$ .  $\square$

**Remark 6.** In turn, Proposition ?? gives the following result. Assume that  $\Omega$  is compact. Let  $\nu \in \mathcal{P}(\Omega)$  and let  $y^0 \in L_\nu^\infty(\Omega, \mathbb{R}^d)$ . We denote by  $K' = \text{ess.im}(y^0)$  its essential range (it is a compact subset of  $\mathbb{R}^d$ ) and we set  $K = \Omega \times K'$  (compact). Since the unique solution of the Vlasov equation (??) such that  $\mu(0) = \mu_{y^0}^\nu = \nu \otimes \delta_{y^0(\cdot)}$  is well defined on  $[0, T_{\max}(K))$  (by Theorem ??) and is given by  $\mu(t) = \mu_{y(t, \cdot)}^\nu$  (by Proposition ??), it follows that the nonlinear Euler equation (??) has a unique solution on  $[0, T_{\max}(K))$  such that  $y(0, \cdot) = y^0(\cdot)$ .

**Remark 7.** When  $\Omega$  is not compact, the above results remain true provided that there exists a compact subset  $\Omega_1$  of  $\Omega$  such that  $G(t, x, x', \xi, \xi') = 0$  for every  $x \in \Omega \setminus \Omega_1$  and all  $(t, x', \xi, \xi') \in \mathbb{R} \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d$ , and the initial condition  $y^0$  for the Euler equation satisfies  $y^0(x) = 0$  for  $\nu$ -almost every  $x \in \Omega \setminus \Omega_1$ . Indeed, in this case the solution of the Euler equation is supported in  $\Omega_1$ . Alternatively, we can also assume that  $\text{supp}(\nu) \subset \Omega_1$ .

When  $\mu_t$  is not of the form  $\mu_{y(t, \cdot)}^\nu$ ,  $t \mapsto y(t, \cdot)$  fails in general to satisfy a ‘‘closed’’ equation (i.e.,  $\partial_t y(t, \cdot)$  may not be expressible only in function of  $y(t, \cdot)$ ). Instead, there may be a full hierarchy of equations coupling all the moments of  $\mu_{t, x}$  (see Section ??). Anyway, when convergence to consensus holds, we expect that any solution  $\mu$  of (??) is asymptotically of the form  $\mu_{y(t, \cdot)}^\nu$ .

**Relationship between the particle system (??) and the Euler equation (??).** If the mapping  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$  is a locally Lipschitz solution of the particle system (??), then the mapping  $t \mapsto y(t, \cdot)$ , where  $y(t, x)$  is defined as the moment of order 1 (i.e., by (??)) of the empirical measure  $\mu_{(X^N, \Xi^N(t))}^e$  (whose marginal is  $\nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$ ), is a locally Lipschitz solution of the Euler equation (??) where the operator  $A$  is defined by (??) with the measure  $\nu_{X^N}^e$ . Note that, in this embedding from (??) into (??), we have  $y(t, x) = \xi_i^N(t)$  if  $x = x_i^N$  for  $i \in \{1, \dots, N\}$  and 0 otherwise, and  $\nu$  is purely atomic.

Conversely, if  $\nu = \nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$  and if the mapping  $t \mapsto y(t)$  is a locally Lipschitz solution of the Euler equation (??), then the mapping  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$ , with  $\xi_i^N(t) = y(t, x_i^N)$  for  $i \in \{1, \dots, N\}$ , is a locally Lipschitz solution of the particle system (??). Note that, however, we may have  $y(t, x) \neq 0$  for  $x \notin \{x_1^N, \dots, x_N^N\}$ . This is a kind of projection.

This general equivalence works because, when  $\nu = \nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$  and  $y(t, x_i^N) = \xi_i^N(t)$ , the  $\nu$ -monokinetic measure  $\mu_{y(t, \cdot)}^\nu$  coincides with the empirical measure  $\mu_{(X^N, \Xi^N(t))}^e$ . Indeed,

$$\mu_{y(t, \cdot)}^\nu = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{y(t, \cdot)} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{y(t, x_i^N)} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)} = \mu_{(X^N, \Xi^N(t))}^e.$$

### 3.2 Moment of order 2

In this section, we assume that  $\|\cdot\|$  is the Euclidean norm. We define

$$T(t, x) = \frac{1}{d} \int_{\mathbb{R}^d} \|\xi - y(t, x)\|^2 d\mu_{t, x}(\xi)$$

for  $\nu$ -almost every  $x \in \Omega$ . Note that  $T(t, x) = 0$  for  $\nu$ -almost every  $x \in \Omega \setminus \text{supp}(\nu)$ . Using (??) (or, rather, (??)) and noting that  $\int_{\mathbf{R}^d} \langle \xi - y(t, x), \partial_t y(t, x) \rangle_{\mathbf{R}^d} d\mu_{t,x}(\xi) = 0$ , we compute

$$\partial_t T(t, x) = \frac{2}{d} \int_{\mathbf{R}^d} \langle \xi - y(t, x), \mathcal{X}[\mu_t](t, x, \xi) \rangle_{\mathbf{R}^d} d\mu_{t,x}(\xi). \quad (59)$$

**Proposition 5.** *In the Hegselmann–Krause model given in Example ??, we have  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$  and*

$$\boxed{\partial_t T(t, x) = -2S(x)T(t, x)}$$

where  $S(x) = \int_{\Omega} \sigma(x, x') d\nu(x')$  for  $\nu$ -almost every  $x \in \Omega$ . Hence  $t \mapsto T(t, x) = T(0, x)e^{-2tS(x)}$  decreases exponentially to 0 as  $t \rightarrow +\infty$  for  $\nu$ -almost every  $x \in \Omega$  such that  $S(x) > 0$ .

*Proof.* We have  $\mathcal{X}[\mu_t](t, x, \xi) = \int_{\Omega} \int_{\mathbf{R}^d} \sigma(x, x')(\xi' - \xi) d\mu_{t,x'}(\xi') d\nu(x')$  in the Hegselmann–Krause model (see Example ??), and thus

$$\mathcal{X}[\mu_t](t, x, \xi) = -S(x) \int_{\mathbf{R}^d} (\xi - y(t, x)) d\mu_{t,x'}(\xi') + \int_{\Omega} \int_{\mathbf{R}^d} \sigma(x, x')(\xi' - y(t, x)) d\mu_{t,x'}(\xi') d\nu(x').$$

Since the second term does not depend on  $\xi$ , using again the fact that  $\int_{\mathbf{R}^d} (\xi - y(t, x)) d\mu_{t,x}(\xi) = 0$  by definition, and using (??), the result follows.  $\square$

**Remark 8.** We will see in Remark ?? in Section ?? that, in the Hegselmann–Krause model, all moments of order  $\geq 2$  satisfy the same differential equation, and thus, decrease exponentially to 0 as  $t \rightarrow +\infty$  as soon as  $S(x) > 0$  for  $\nu$ -almost every  $x \in \Omega$ . This shows that, under the latter assumption, the solution  $t \mapsto \mu(t)$  of the Vlasov equation (??) is such that  $\mu_{t,x}$  is exponentially close (in Wasserstein distance) to the Dirac measure  $\delta_{y(t,x)}$  as  $t \rightarrow +\infty$ .

In [?], convergence to consensus is proved for the Euler equation under the assumptions that  $d\nu(x) = dx$ , that  $S(x) \geq \delta > 0$  for almost every  $x \in \Omega$  and that the (infinite-dimensional) graph associated with  $\sigma$  be strongly connected. This remark shows that the result of [?] may be slightly generalized by relaxing the assumption on  $S$  to the assumption that  $S(x) > 0$  for  $\nu$ -almost every  $x \in \Omega$ .

For general functions  $G$ , the question to know whether  $T$  is the solution of some “closed” equation is open.

In the  $\nu$ -monokinetic case, i.e., assuming that  $\mu$  is of the form (??) and is a locally Lipschitz solution of (??), we have  $T(t, x) = 0$ . This is expected since  $T(t, x)$  is the variance and thus measures the distance to the average  $y(t, x)$ .

### 3.3 Generalization: coupled equations of moments

More generally, assuming  $d = 1$  to simplify, let us set, formally,

$$G(t, x, x', \xi, \xi') = G(t, x, x', y(t, x), y(t, x')) + \sum_{i+j \geq 1} g_{ij}(t, x, x')(\xi - y(t, x))^i (\xi' - y(t, x'))^j$$

where  $y(t, x) = \int_{\mathbf{R}} \xi d\mu_{t,x}(\xi)$  is the moment of order 1 of  $\mu_{t,x}$  (recall that the moment of order 0 is  $y_0(t, x) = \int_{\mathbf{R}} d\mu_{t,x}(\xi) = 1$ ). Defining the central moment of order  $i$  by

$$y_i(t, x) = \int_{\mathbf{R}} (\xi - y(t, x))^i d\mu_{t,x}(\xi) \quad \forall i \in \mathbf{N}$$

(note that  $y_0(t, x) = 1$  and  $y_1(t, x) = 0$ ), we have

$$\begin{aligned}\mathcal{X}[\mu_t](t, x, \xi) &= \int_{\Omega \times \mathbf{R}} G(t, x, x', \xi, \xi') d\mu_t(x', \xi') \\ &= \int_{\Omega} G(t, x, x', y(t, x), y(t, x')) d\nu(x') \\ &\quad + \sum_{i+j \geq 1} (\xi - y(t, x))^i \int_{\Omega} g_{ij}(t, x, x') y_j(t, x') d\nu(x')\end{aligned}$$

and thus, using (??),

$$\boxed{\mathcal{X}[\mu_t](x, \xi) = (A(t, y(t)))(x) + \sum_{i+j \geq 1} (\xi - y(t, x))^i \int_{\Omega} g_{ij}(t, x, x') y_j(t, x') d\nu(x')}$$

It is interesting to see that, in the above formal expansion of  $\mathcal{X}[\mu_t](x, \xi)$  using the centered moments, the first term is  $(A(t, y(t)))(x)$ .

Therefore, we have

$$\begin{aligned}\partial_t y(t, x) &= \int_{\mathbf{R}} \mathcal{X}[\mu_t](x, \xi) d\mu_{t,x}(\xi) \\ &= (A(t, y(t)))(x) + \sum_{i+j \geq 1} \left( \int_{\Omega} g_{ij}(t, x, x') y_j(t, x') d\nu(x') \right) y_i(t, x)\end{aligned}$$

(actually since  $y_1 = 0$  the above sum can be taken over all pairs  $(i, j)$  such that  $i + j \geq 2$ ) and, for every  $k \in \mathbb{N} \setminus \{0, 1\}$ ,

$$\begin{aligned}\partial_t y_k(t, x) &= \langle \mu_{t,x}, L_{\mathcal{X}[\mu_t]} \cdot (\xi \mapsto (\xi - y(t, x))^k) \rangle - \langle \mu_{t,x}, k(\xi - y(t, x))^{k-1} \partial_t y(t, x) \rangle \\ &= k \int_{\mathbf{R}} (\xi - y(t, x))^{k-1} (\mathcal{X}[\mu_t](x, \xi) - \partial_t y(t, x)) d\mu_{t,x}(\xi) \\ &= k \int_{\mathbf{R}} (\xi - y(t, x))^{k-1} \left( \mathcal{X}[\mu_t](x, \xi) - \int_{\mathbf{R}} \mathcal{X}[\mu_t](x, \xi') d\mu_{t,x}(\xi') \right) d\mu_{t,x}(\xi) \\ &= k \sum_{i+j \geq 1} \left( \int_{\Omega} g_{ij}(t, x, x') y_j(t, x') d\nu(x') \right) \int_{\mathbf{R}} (\xi - y(t, x))^{k-1} ((\xi - y(t, x))^i - y_i(t, x)) d\mu_{t,x}(\xi) \\ &= k \sum_{i+j \geq 1} \left( \int_{\Omega} g_{ij}(t, x, x') y_j(t, x') d\nu(x') \right) (y_{k-1+i}(t, x) - y_{k-1}(t, x) y_i(t, x))\end{aligned}$$

(actually since  $y_1 = 0$  the pair  $(i = 0, j = 1)$  does not occur in the above sum). In full generality, all equations of moments are coupled and we have no closed system.

Closing the hierarchy of equations satisfied by all the moments  $y_i(t, x)$ , for  $i \in \mathbb{N}^*$ , might be done by adding a small parameter  $\varepsilon$ . This is an open question.

**Remark 9.** In the Hegselmann–Krause model given in Example ??, we have  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$  and thus  $g_{ij} = 0$  if  $i + j \geq 2$  and  $g_{01} = -g_{10} = \sigma$ . We recover the facts that the equation in  $y$  is closed and that  $\partial_t y_2(t, x) = -2S(x)y_2(t, x)$ . Moreover, a straightforward computation shows that

$$\frac{1}{k} \partial_t y_k(t, x) = -S(x)y_k(t, x) \quad \forall k \in \mathbb{N} \setminus \{0, 1\},$$

thus generalizing the case  $k = 2$  studied in Proposition ??. Therefore,  $y_k(t, x) = y_k(0, x)e^{-tS(x)}$ .

## 4 From microscopic to macroscopic scale

(“from particle to Euler”, graph limit)

We have seen in the previous section that the passage from micro to macro is very general and that, instead of considering Riemann sums, one can observe that it results from the coincidence of the empirical measure with the  $\nu$ -monokinetic measure.

Anyway, in this section we are going to explore the point of view of Riemann sums, in order to derive error estimates mainly resulting from the discrepancy between an integral and a Riemann sum, building on the concept of graph limit introduced in [?].

For second-order systems like the celebrated Cucker–Smale system, the authors of [?] considered the three scales (micro, meso and macroscopic). Recently, in [?], the authors provide a rigorous derivation from the kinetic Cucker–Smale model to the macroscopic pressureless Euler system by hydrodynamic limit, using entropy methods and deriving error estimates.

By combining the various possibilities to derive the macroscopic quantities, we show how to obtain explicit error estimates for the direct derivation of the graph limit (i.e., the macroscopic model) from the microscopic model by first taking the mean field limit and obtaining the (kinetic) Vlasov equation, and then by taking the hydrodynamic limit. The price to pay is that we obtain estimates in weak topology (Wasserstein distance) instead of estimates in  $L^2$  or  $L^\infty$  norm as provided by the graph limit methods, but the gain is to have an explicit  $O(1/N)$  rate of convergence.

We have seen in Section ?? that it is not always possible to pass from the mesoscopic to the macroscopic scale, because the equation obtained for the moment of order 1 may not be closed. However, we have seen in Proposition ?? that, for the Hegselmann–Krause model, the equation in  $y$  is closed and is linear.

### 4.1 Convergence estimates for the graph limit

Throughout this section, we assume that  $\Omega$  is compact. Let  $\nu \in \mathcal{P}(\Omega)$ . We consider the general nonlinear Euler equation (??), with the nonlinear operator  $A$  defined by (??). Recall that  $G$  satisfies Assumption ??.

We also assume that there exists a family  $(\mathcal{A}^N, X^N)_{N \in \mathbb{N}^*}$  of tagged partitions associated with  $\nu$  satisfying (??) (see Section ??), with  $\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$  and  $X^N = (x_1^N, \dots, x_N^N)$ . We have the following two theorems.

**Theorem 4.** *Let  $y^0$  be a bounded and  $\nu$ -almost everywhere continuous function on  $\Omega$  (thus,  $\nu$ -Riemann integrable), with values in  $\mathbb{R}^d$ .*

*On the one part, we consider the unique solution  $t \mapsto y(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$  on  $[0, T_{\max}(K))$  of the (nonlinear) Euler equation (??) such that  $y(0, \cdot) = y^0(\cdot)$ , where  $K = \Omega \times \text{ess.im}(y^0)$  (compact) and  $\text{ess.im}(y^0) \subset \mathbb{R}^d$  is the essential range of  $y^0$ .*

*On the other, for any  $N \in \mathbb{N}^*$ , we consider the unique solution  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$  on  $[0, T_{\max}(K))$  of the particle system (??) such that  $\xi_i^N(0) = y^0(x_i^N)$  for every  $i \in \{1, \dots, N\}$ , and we set*

$$y^N(t, x) = \sum_{i=1}^N \xi_i^N(t) \mathbb{1}_{\Omega_i^N}(x) \quad \forall (t, x) \in \mathbb{R} \times \Omega \quad (60)$$

where  $\mathbb{1}_{\Omega_i^N}$  is the characteristic function of  $\Omega_i^N$ , defined by  $\mathbb{1}_{\Omega_i^N}(x) = 1$  if  $x \in \Omega_i^N$  and 0 otherwise.

- For every  $t \in [0, T_{\max}(K))$ ,  $y(t, \cdot)$  is bounded and continuous  $\nu$ -almost everywhere on  $\Omega$ , with the same continuity set as  $y^0$ , and

$$\|y(t, \cdot) - y^N(t, \cdot)\|_{L^\infty(\Omega, \mathbb{R}^d)} = o(1) \quad (61)$$

as  $N \rightarrow +\infty$ , where the remainder term  $o(1)$  is uniform with respect to  $t$  on any compact interval of  $[0, T_{\max}(K)]$ . In particular,

$$\max_{i \in \{1, \dots, N\}} \|y(t, x_i^N) - \xi_i^N(t)\| = o(1). \quad (62)$$

- Assume that there exists  $\alpha \in (0, 1]$  such that  $y^0 \in \mathcal{C}^{0, \alpha}(\Omega, \mathbb{R}^d)$  and  $G$  is locally  $\alpha$ -Hölder continuous with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact). Then, for every  $t \in [0, T_{\max}(K)]$ , we have  $y(t, \cdot) \in \mathcal{C}^{0, \alpha}(\Omega, \mathbb{R}^d)$  with

$$\text{Hol}_\alpha(y(t, \cdot)) \leq e^{tL_y(t)} (1 + \text{Hol}_\alpha(y(0, \cdot))) \quad (63)$$

and, for every  $N \in \mathbb{N}^*$ ,

$$\max_{i \in \{1, \dots, N\}} \|y(t, x_i^N) - \xi_i^N(t)\| \leq \frac{C_\Omega^\alpha}{N^{r\alpha}} (1 + \text{Hol}_\alpha(y^0)) e^{2tL_y^N(t)} \quad (64)$$

and actually,

$$\|y(t, \cdot) - y^N(t, \cdot)\|_{L^\infty(\Omega, \mathbb{R}^d)} \leq 2 \frac{C_\Omega^\alpha}{N^{r\alpha}} (1 + \text{Hol}_\alpha(y^0)) e^{2tL_y^N(t)} \quad (65)$$

where  $C_\Omega$  is given by (??). The constant  $L_y^N(t)$  in (??) and (??) is defined by

$$L_y^N(t) = \max_{0 \leq \tau \leq t} \text{Hol}_\alpha(G(\tau, \cdot, \cdot, \cdot, \cdot)_{|\Omega^2 \times S_y^N(\tau)^2}) + \max_{\substack{x, x' \in \Omega \\ 0 \leq \tau \leq t}} \text{Lip}(G(\tau, x, x', \cdot, \cdot)_{|S_y^N(\tau)^2}) \quad (66)$$

where  $S_y^N(\tau) \subset \mathbb{R}^d$  is the (compact) convex closure of all  $y(\tau, x)$  for  $x \in \Omega$  and all  $\xi_i^N(\tau)$  for  $i \in \{1, \dots, N\}$ . The constant  $L_y(t)$  in (??) is defined as  $L_y^N(t)$  but with  $S_y^N(\tau)$  replaced by  $S_y(\tau)$  that is the convex closure of all  $y(\tau, x)$  for  $x \in \Omega$ , i.e., like  $S_y^N(\tau)$  but without the  $\xi_i^N(\tau)$ . We have  $L_y(t) \leq L_y^N(t)$ .

Theorem ?? is proved in Appendix ?. Note that, by Lemma ??, given any  $T \in [0, T_{\max}(K)]$ , the sets  $S_y^N(t)$  and thus the scalars  $L_y^N(t)$  are uniformly bounded with respect to  $t \in [0, T]$  and to  $N \in \mathbb{N}^*$ .

**Remark 10.** Theorem ?? can be extended to the case where  $\Omega$  is not compact, under the following additional assumptions:

- the family of tagged partitions is such that the points  $x_i^N$  remain in a compact subset of  $\Omega$ ;
- the initial condition  $y^0$  is of compact essential support;
- the set  $S_y^N(\tau) \subset \Omega \times \mathbb{R}^d$  is defined as the compact closure of all  $(x, y(\tau, x))$  for  $x \in \text{ess supp}(y(\tau, \cdot))$  (essential support) and all  $(x_i^N, \xi_i^N(\tau))$  for  $i \in \{1, \dots, N\}$ .

The above assumptions imply that  $y(t, \cdot)$  is of compact essential support, for every  $t \geq 0$ , and that  $L_y^N(t)$  is well defined.

**Theorem 5.** Let  $K'$  be a compact subset of  $\mathbb{R}^d$ . Given any  $N \in \mathbb{N}^*$ , let  $\Xi_0^N \in (K')^N$ . We set  $K = \Omega \times K'$ .

On the one part, we consider the unique solution  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t)) \in \mathbb{R}^{dN}$  on  $[0, T_{\max}(K)]$  of the particle system (??) such that  $\Xi^N(0) = \Xi_0^N$ , and we define  $y^N(t, x)$  by (??).

On the other part, we consider the unique solution  $t \mapsto y_N(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$  on  $[0, T_{\max}(K))$  of the Euler equation (??) such that  $y_N(0, \cdot) = y^N(0, \cdot)$  (i.e.,  $y_N(0, x) = \xi_i^N(0)$  if  $x \in \Omega_i^N$ ). Then, for every  $t \in [0, T_{\max}(K))$ ,

$$\|y_N(t, \cdot) - y^N(t, \cdot)\|_{L^\infty(\Omega, \mathbb{R}^d)} = o(1) \quad (67)$$

as  $N \rightarrow +\infty$ , where the remainder term  $o(1)$  is uniform with respect to  $t$  on any compact interval of  $[0, T_{\max}(K))$ . If moreover  $G$  is locally  $\alpha$ -Hölder continuous with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact), then, for every  $N \in \mathbb{N}^*$  and every  $t \in [0, T_{\max}(K))$ ,

$$\|y_N(t, \cdot) - y^N(t, \cdot)\|_{L^\infty(\Omega, \mathbb{R}^d)} \leq 2 \frac{C_\Omega^\alpha}{N^{r\alpha}} e^{2tL_{y_N}^N(t)} \quad \forall t \geq 0, \quad (68)$$

where  $L_{y_N}^N(t)$  is defined by (??) (with  $y$  replaced by  $y_N$ ).

Theorem ?? is proved in Appendix ?. Note that, by Lemma ??, given any  $T \in [0, T_{\max}(K))$ , the scalars  $L_{y_N}^N(t)$  are uniformly bounded with respect to  $t \in [0, T]$  and to  $N \in \mathbb{N}^*$ .

Note that, in particular, taking  $x = x_i$  in (??), we have

$$\max_{i \in \{1, \dots, N\}} \|y_N(t, x_i^N) - \xi_i^N(t)\| \leq 2 \frac{C_\Omega^\alpha}{N^{r\alpha}} e^{2tL_{y_N}^N(t)},$$

which improves the estimates obtained in [?].

**Remark 11.** In Appendix ??, we provide estimates on the discrepancy between empirical measures and  $\nu$ -monokinetic measures. Lemma ?? of that appendix, combined with Theorem ?? and with the proof of that proposition, yields estimates on the discrepancy of the empirical measure  $\mu_{(X^N, \Xi^N(t))}^e$  with respect to the  $\nu$ -monokinetic measures  $\mu_{y(t, \cdot)}^\nu$  or  $\mu_{y^N(t, \cdot)}^\nu$ .

**Remark 12.** The proofs of Theorems ?? and ?? that we provide in Appendices ?? and ?? are direct, but actually one can also prove these propositions by applying Corollary ?? with  $\mu(t) = \mu_{y(t, \cdot)}^\nu = \nu \otimes \delta_{y(t, \cdot)}$  (the  $\nu$ -monokinetic measure) and use Lemma ?? of Appendix ??.

## 4.2 Additional remarks: from Liouville to Euler

In Section ?? we considered the direct passage from the particle system (??) to the Euler (graph limit) equation (??) through the system of ODEs defining the particle dynamics and in the previous Sections ?? and ?? we also reached the same Euler equation via the Vlasov equation in the pure mean field paradigm.

The Liouville equation (??) being the transport equation lifting the particle system (??), a natural question is to wonder whether there exists a direct way to pass from Liouville to Euler. Our objective in this section is to provide a quantity cooked up out of the solution  $\rho(\cdot)$  of the Liouville equation (??), converging to the solution of the Euler equation as  $N \rightarrow +\infty$ . The question may fill a gap in the general micro-meso-macroscopic landscapes.

Let us explain how this can be done. Considering a system of  $N$  particles, each of them living in a phase space  $\Omega \times \mathbb{R}^d$ , the meaning of the solution  $\rho(t) \in \mathcal{P}(\Omega \times \mathbb{R}^d)$  of the Liouville equation (??) is the following, when it has a density with respect to the Lebesgue measure: for any  $X = (x_1, \dots, x_N) \in \Omega^N$  and any  $\Xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^{dN}$ ,  $\rho(X, \Xi)$  is the joint probability that, for every  $i \in \{1, \dots, N\}$ , the  $i^{\text{th}}$  particle has position and momentum  $(x_i, \xi_i)$ . In Section ?? we have shown that, for appropriate initial conditions  $\rho(0)$ , we recover the mean field limit by taking the limit  $N \rightarrow +\infty$  of the average over all particles but one and then by taking marginals.

The Liouville paradigm enlarges the moment setting to a probabilistic one: every agent has a moment, but it hesitates randomly between several values that can be assigned to it. Of course the

monokinetic case through the Vlasov equation exhausts this random feature by assigning a single moment. But it is quite remarkable, and one has to say still mysterious for us, that, for the opinion propagation model outside monokineticity, the marginal of the full density, namely a probability “average over all particles but one” leads through, and *after* the large  $N$  limit, its first moment to the same limit as the fundamentally different “discrete to continuous” passage emblematic to the graph limit.

It is therefore interesting to remove this “after the large  $N$  limit” and pass directly from Liouville to Euler and answer the aforementioned question: what is the solution to the graph limit equation the large  $N$  limit of? Hereafter, we describe a way to pass directly from the Liouville equation (??) to the Euler equation (??), without having to consider the Vlasov equation. Given any  $\rho \in \mathcal{P}(\Omega^N \times \mathbb{R}^{dN})$ , we define the probability measure  $\mathcal{M}_1[\rho] \in \mathcal{P}(\Omega)$  by

$$\int_{\Omega} f d(\mathcal{M}_1[\rho]) = \int_{\Omega \times \mathbb{R}^d} f(x)\xi d\rho_{N:1}^s(x, \xi)$$

for every  $f \in \mathcal{C}_c^0(\Omega)$ .

**Lemma 2.** *As in Theorem ??, let  $y^0$  be a bounded and  $\nu$ -almost everywhere continuous function on  $\Omega$ , with values in  $\mathbb{R}^d$ , and let  $t \mapsto y(t, \cdot) \in L^\infty(\Omega, \mathbb{R}^d)$  be the unique solution on  $[0, T_{\max}(K))$  of the (nonlinear) Euler equation (??) such that  $y(0, \cdot) = y^0(\cdot)$ , where  $K = \Omega \times \text{ess.im}(y^0)$ .*

*Besides, for any  $N \in \mathbb{N}^*$ , let  $\Xi_0^N = (\xi_{0,1}^N, \dots, \xi_{0,N}^N) \in (\mathbb{R}^d)^N$  be such that  $\xi_{0,i}^N = y^0(x_i^N)$  for every  $i \in \{1, \dots, N\}$ .*

*As in Theorem ??, we set  $\rho_0^N = \delta_{X^N} \otimes \delta_{\Xi_0^N}$  and we consider the unique solution  $t \mapsto \rho^N(t) = \delta_{X^N} \otimes \delta_{\Xi^N(t)}$  of the Liouville equation (??) on  $[0, T_{\max}(K))$  such that  $\rho^N(0) = \rho_0^N$ .*

*Then  $\mathcal{M}_1[\rho^N(t)]$  converges weakly to  $y(t, \cdot)$   $\nu$  as  $N \rightarrow +\infty$ , uniformly with respect to  $t$  on compact intervals, with convergence estimates in Wasserstein distance  $W_1$  under additional regularity assumptions on  $G$  as in Theorem ??.*

*Proof.* In Theorem ??, we have proved that  $\rho^N(t)_{N:1}^s$  is asymptotically (weakly) close to  $\mu(t)$  as  $N \rightarrow +\infty$ . Since  $\mu(t) = \int_{\Omega} \mu_{t,x} d\nu(x)$  and  $y(t, x) = \int_{\Omega} \xi d\mu_{t,x}(\xi)$ , the lemma follows.  $\square$

## 5 Summary: relationships between various scales

In the previous sections, we have investigated the following three scales (recall that  $G$  satisfies Assumption ??):

- The microscopic model, which is the particle system

$$\dot{\xi}_i^N(t) = \frac{1}{N} \sum_{j=1}^N G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)), \quad i = 1, \dots, N. \quad (69)$$

When extending this system by setting  $\dot{x}_i^N(t) = 0$ , in some sense we perform an extension of the particle system to the phase space.

- The mesoscopic model, which is the (kinetic) Vlasov equation

$$\partial_t \mu + \text{div}_\xi(\mathcal{X}[\mu]\mu) = 0 \quad (70)$$

where  $\mathcal{X}[\mu](t, x, \xi) = \int_{\Omega \times \mathbb{R}^d} G(t, x, x', \xi, \xi') d\mu(x', \xi')$  for all  $(t, x, \xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^d$ , obtained by mean field limit.



- the macroscopic model, which is the Euler equation

$$\partial_t y(t, x) = (A(t, y(t)))(x) = \int_{\Omega} G(t, x, x', y(t, x), y(t, x')) d\nu(x') \quad (71)$$

where  $\nu \in \mathcal{P}(\Omega)$ , obtained by graph limit.

Additionally, we have also considered the Liouville equation, having a probabilistic interpretation,

$$\partial_t \rho^N + \operatorname{div}_{\Xi}(Y^N \rho^N) = 0 \quad (72)$$

where  $Y^N$  is the vector field in  $\mathbb{R}^{dN}$  representing the system of all particles.

Figure ?? illustrates the various relationships that we have investigated in the paper, and that we comment hereafter.

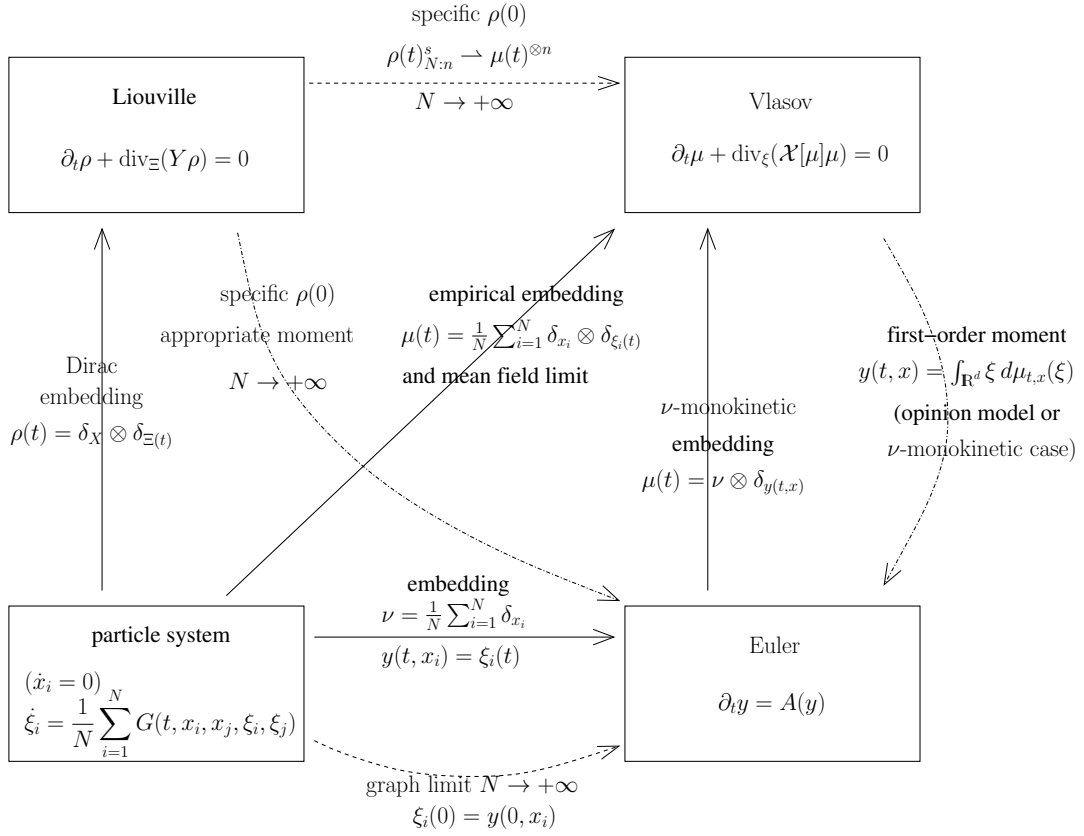


Figure 1: Relationships between particle (microscopic) system, Liouville (probabilistic) equation, Vlasov (mesoscopic, mean field) equation, Euler (macroscopic, graph limit) equation. We do not write the superscript  $N$  in the various formulas to keep a better readability.

**Particle to Liouville.** Any solution  $\Xi^N(\cdot)$  of the particle system (??) can be embedded as a Dirac measure  $\rho^N(\cdot) = \delta_{X^N} \otimes \delta_{\Xi^N(\cdot)}$  that is a solution of the Liouville equation (??).

**Particle to Vlasov.** By Proposition ??, any solution  $\Xi^N(\cdot)$  of the particle system (??) can be embedded to an empirical measure  $\mu(\cdot) = \mu_{(X^N, \Xi^N(\cdot))}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(\cdot)}$  that is a solution of the Vlasov equation (??). Conversely if an empirical measure  $\mu(\cdot) = \mu_{(X^N, \Xi^N(\cdot))}^e$  (with distinct points) is a solution of the Vlasov equation (??) then  $\Xi^N(\cdot)$  must be a solution of (??).

In this context, the mean field limit consists of taking the limit  $N \rightarrow +\infty$ .

**Particle to Euler.** Any solution  $\Xi^N(\cdot)$  of the particle system (??) can be embedded to a solution of the general nonlinear Euler equation (??) by setting  $\nu = \nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$ ,  $y(t, x) = \xi_i^N(t)$  if  $x = x_i^N$  for some  $i \in \{1, \dots, N\}$  and 0 otherwise. Conversely, if  $\nu = \nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$  and if  $t \mapsto y(t, \cdot)$  is a locally Lipschitz solution of the Euler equation (??), then  $\Xi^N(\cdot) = (\xi_1^N(\cdot), \dots, \xi_N^N(\cdot))$ , with  $\xi_i^N(\cdot) = y(\cdot, x_i^N)$  for every  $i \in \{1, \dots, N\}$ , is a locally Lipschitz solution of the particle system (??). Note however that  $y(t, x)$  may not be zero for  $x \in \Omega \setminus \{x_1^N, \dots, x_N^N\}$ .

Alternatively and much more interestingly, to pass from the microscopic to the macroscopic scale, by Theorems ?? and ??, one can take the graph limit of the particle system and thus obtain the Euler equation, with estimates of convergence as  $N \rightarrow +\infty$ .

**Liouville to Vlasov.** By Theorems ?? or ??, one can recover the solutions of the Vlasov equation (??) from those of the Liouville equation (??), for some appropriate initial conditions  $\rho(0)$ , by taking marginals and taking the limit  $N \rightarrow +\infty$ .

**Euler to Vlasov.** By Section ??, given any  $\nu \in \mathcal{P}(\Omega)$  and any solution  $t \mapsto y(t, \cdot)$  of the Euler equation (??), the  $\nu$ -monokinetic measure mapping  $t \mapsto \mu(t) = \mu_{y(t, \cdot)}^\nu = \nu \otimes \delta_{y(t, \cdot)}$  defined by (??) is a solution of the Vlasov equation (??). This embedding from the macroscopic to the mesoscopic scale is completely general and is valid for the general mean field  $\mathcal{X}[\mu]$  defined by (??) and for the general nonlinear operator  $A$  defined by (??).

**Vlasov to Euler.** Here, and only here, we assume, first, that we are in the Hegselmann–Krause model given in Example ??, i.e.,  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$ ; or more generally we assume that  $G$  is linear with respect to  $(\xi, \xi')$ . Proposition ?? says that, given any solution  $t \mapsto \mu(t)$  of the Vlasov equation (??), defining  $\nu = \pi_* \mu(t)$  (marginal of  $\mu(t)$ , which does not depend on  $t$ ), the moment mapping  $t \mapsto y(t, \cdot)$  of order 1, defined by  $y(t, x) = \int_{\mathbb{R}^d} \xi d\mu_{t, x}(\xi)$ , is a solution of the Euler equation (??) (which is linear in this case).

As discussed in Section ??, there is a second way, still not general, of passing from Vlasov to Euler, by assuming that the solution  $\mu(\cdot)$  of the Vlasov equation is  $\nu$ -monokinetic. In this case, its moment  $y$  of order 1 is solution of the nonlinear Euler equation (??).

This projection from the mesoscopic to the macroscopic scale is not general because, in general,  $y$  does not satisfy a closed equation.

**Liouville to Euler.** Lemma ?? in Section ?? shows how to pass from Liouville to Euler, for specific initial conditions  $\rho^N(0)$ , by taking an adequate moment of  $\rho^N(t)$  and then passing to the limit  $N \rightarrow +\infty$ .

Finally, all above relationships are general (i.e., valid for a general interaction mapping  $G$ ) except the transition from the mesoscopic (kinetic, mean field) model to the macroscopic (Euler) model, which is valid if  $G$  is linear with respect to  $(\xi, \xi')$  but fails in general. The graph limit procedure is of a different nature and rather relies on the usual limit in Riemann integration theory, as explained in Section ??.

Anyway, what is interesting in the above arguments is that it may not be relevant to place the mesoscopic level in-between the microscopic level and the macroscopic one.

## 6 Finite particle approximations of evolution equations

### 6.1 Setting

Considering the complete metric space  $(\Omega, d_\Omega)$  of Assumption ??, throughout this section, we assume either that:

(O<sub>1</sub>)  $\Omega$  is the compact closure of a bounded open subset of  $\mathbb{R}^n$  with a Lipschitz boundary,  $d_\Omega$  is the induced Euclidean distance, and  $\nu$  is the restriction to  $\Omega$  of the Lebesgue measure of  $\mathbb{R}^n$ ;

or that:

(O<sub>2</sub>)  $\Omega$  is a smooth compact Riemannian manifold of dimension  $n$ ,  $d_\Omega$  is its Riemannian distance, and  $\nu$  is the canonical Riemannian measure.

In the case ??,  $\Omega$  is usually called a Lipschitz compact domain of  $\mathbb{R}^n$  (of interior denoted by  $\mathring{\Omega}$ ). In the case ??, for example  $\Omega$  may be the sphere or the torus of dimension  $n$ .

Without loss of generality, we assume that the volume  $|\Omega|$  of  $\Omega$  is equal to 1, so that  $\nu$ , hereafter, is the probability Lebesgue measure on  $\Omega$  (with  $\frac{d\nu}{dx} = 1$  in local coordinates).

Under ?? or ??, there always exist families of tagged partitions associated with  $\nu$  satisfying (??) with  $r = 1/n$  (see Section ??), indexed by  $N \in \mathbb{N}^*$ .

In the sequel, we denote by  $\mathring{\Omega}$  the interior of  $\Omega$  (in the case ?? we have  $\mathring{\Omega} = \Omega$ ).

Let  $X$  be a Banach functional space containing  $\mathcal{C}_c^\infty(\mathring{\Omega}, \mathbb{R}^d)$  as a dense subspace (for instance,  $X = L^2(\Omega, \mathbb{R}^d)$ ). We consider the abstract linear evolution equation

$$\boxed{\partial_t y = Ay} \tag{73}$$

where, for all  $t \geq 0$  and  $\xi \in \mathbb{R}^d$ ,  $A$  is a linear operator on  $X$ , continuous from  $\mathcal{C}_c^\infty(\mathring{\Omega}, \mathbb{R}^d)$  to the distributional space  $\mathcal{C}_c^\infty(\mathring{\Omega}, \mathbb{R}^d)'$ . The latter general assumption ensures that  $A$  has a Schwartz kernel  $[A] \in \mathcal{C}_c^\infty(\mathring{\Omega} \times \Omega, \mathbb{R}^d)'$ , i.e.,  $Ay(x) = \int_\Omega [A](x, x')y(x')$  for every  $y \in \mathcal{C}_c^\infty(\mathring{\Omega}, \mathbb{R}^d)$  where the integral is understood as  $\langle [A](x, \cdot), y \rangle$  with the distribution bracket.

Assuming that  $t \mapsto y(t) \in X$  is a solution of (??), since  $y(t)$ , as an element of the functional space  $X$ , is a function on  $\Omega$ , in the sequel we denote indifferently  $y(t)(x) = y(t, x)$  for all  $t \geq 0$  and  $x \in \Omega$ .

Our objective is to prove that, under appropriate assumptions, regular solutions  $t \mapsto y(t)$  of (??) can be approximated by the solutions of a family of finite particle systems.

Particle approximations are well known for some classes of PDEs, like fluid equations: for fluid Euler or Navier-Stokes equations, one often speaks of “fluid particles”, in accordance with the classical Eulerian or Lagrangian viewpoints. In this section, we show that particle approximations can be achieved for general linear PDEs and even for abstract evolution systems like (??).

The idea relies on Theorem ?? in Section ??, which shows that the solutions of Euler equations (??), with  $A$  defined by (??) with a continuous interaction mapping  $G$ , can be approximated with the solutions of the family of particle systems (??) (indexed by  $N$ ) corresponding to  $G$ . A linear PDE, with an unbounded operator, cannot be written as the Euler equation (??) because, with a continuous mapping  $G$ , one cannot generate by (??) an unbounded operator  $A$ . This is why we are going to introduce, in addition to the parameter  $N$ , another (small) parameter  $\varepsilon$ , in order to approximate unbounded operators  $A$  by a family of bounded operators  $A_\varepsilon$  to which we can then apply the particle approximation result of Theorem ??.

**Preliminaries and strategy.** We have seen in Proposition ?? in Section ?? that, taking  $G(t, x, x', \xi, \xi') = \sigma(x, x')(\xi' - \xi)$  (Hegselmann–Krause model), we obtain the linear Euler equation (??) with  $A$  defined by (??).

Actually, when  $G(t, x, x', \xi, \xi') = \sigma(x, x')\xi'$ , setting  $y(x') = \int_{\mathbb{R}^d} \xi' d\mu_{x'}(\xi')$  as in Section ?? (moment of order 1 of  $\mu$ ), the mean field  $\mathcal{X}[\mu](t, x, \xi)$ , defined by (??), does not depend on  $(t, \xi)$  and is given by

$$\mathcal{X}[\mu](x) = \int_{\Omega} \sigma(x, x')y(x') d\nu(x') = (Ay)(x)$$

which thus defines the Hilbert-Schmidt operator  $A$  of kernel  $\sigma$  with respect to  $\nu$ . Following Section ??, if  $t \mapsto \mu(t)$  is solution of the Vlasov equation (??) then its moment of order one  $t \mapsto y(t, \cdot)$  is solution of the linear Euler equation  $\partial_t y = Ay$ .

The above operator  $A$  is bounded, but replacing  $\sigma$  with a general distributional Schwartz kernel  $[A]$  and having in mind the Schwartz kernel theorem, one is led to consider a general linear operator  $Ay(x) = \int_{\Omega} [A](x, x')y(x')$ . For instance if  $[A](x, x') = \delta'_x$ , the distributional derivative of the Dirac measure  $\delta_x$  at  $x$ , then  $A = -\partial_x$ . The differential equation  $\partial_t y = Ay$  is then the transport equation  $\partial_t y + \partial_x y = 0$ .

Let us use the above example as a paradigm to approximate arbitrary unbounded operators, by designing a sufficiently smooth approximation  $\sigma_\varepsilon$  of an arbitrary Schwartz kernel  $[A]$ . Following this idea, we set

$$G_\varepsilon(t, x, x', \xi, \xi') = \sigma_\varepsilon(x, x') \xi' \quad (74)$$

where  $\varepsilon > 0$  is a small parameter. Recalling that  $\nu$  is here the probability Lebesgue measure on  $\Omega$ , given any  $\varepsilon \in (0, 1]$  we consider the Euler equation corresponding to (??), given by

$$\partial_t y_\varepsilon(t) = A_\varepsilon y_\varepsilon(t) \quad (75)$$

where

$$A_\varepsilon f(x) = \int_{\Omega} \sigma_\varepsilon(x, x')f(x') dx' \quad \forall f \in \mathcal{C}^\infty(\mathring{\Omega}) \quad \forall x \in \Omega. \quad (76)$$

Here  $\sigma_\varepsilon$  is the density of the Schwartz kernel  $[A_\varepsilon]$  of  $A_\varepsilon$  with respect to the Lebesgue measure. In what follows we are going to design an adequate interaction function  $\sigma_\varepsilon$  such that  $A_\varepsilon$  defined by (??) converges to  $A$  defined by (??) as  $\varepsilon \rightarrow 0$ , in an appropriate sense. For instance, if  $\sigma_\varepsilon(x, \cdot)$  is a smooth function (not depending on  $(t, \xi)$ ) approximating the distributional derivative  $\delta'_x$  of the Dirac measure at  $x$ , then  $A_\varepsilon \rightarrow A = -\partial_x$  and hence at the limit  $\varepsilon \rightarrow 0$  we recover the transport equation  $\partial_t y + \partial_x y = 0$ .

Now, given any  $\varepsilon \in (0, 1]$ , let us introduce the particle approximation of (??). Let  $(\mathcal{A}^N, X^N)_{N \in \mathbb{N}^*}$  be a family of tagged partitions associated with  $\nu$  satisfying (??) with  $r = 1/n$  (see Section ??), with  $\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$  and  $X^N = (x_1^N, \dots, x_N^N)$ . We consider the particle system corresponding to (??), given for every  $N \in \mathbb{N}^*$  by

$$\xi_{\varepsilon, i}^N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_\varepsilon(x_i^N, x_j^N) \xi_{\varepsilon, j}^N(t) \quad \forall i \in \{1, \dots, N\}. \quad (77)$$

Denoting by  $t \mapsto \Xi_\varepsilon^N(t) = (\xi_{\varepsilon, 1}^N(t), \dots, \xi_{\varepsilon, N}^N(t))$  an arbitrary solution of (??) (well defined and smooth on  $\mathbb{R}$ ), we set

$$y_\varepsilon^N(t, x) = \sum_{i=1}^N \xi_{\varepsilon, i}^N(t) \mathbf{1}_{\Omega_i^N}(x) \quad \forall (t, x) \in \mathbb{R} \times \Omega. \quad (78)$$

Note that  $y_\varepsilon^N$  is  $\mathcal{C}^\infty$  in  $t$  and piecewise constant in  $x$ , and that  $y_\varepsilon^N(t, x_i^N) = \xi_{\varepsilon,i}^N(t)$  for every  $i \in \{1, \dots, N\}$ .

The particle system (??) is expected to provide a particle approximation of the evolution equation (??), in the sense that it is expected that solutions  $y$  of (??) are limits of  $y_\varepsilon^N$  as  $N \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ . However, since the particle system (??) does not have any (classical) limit as  $\varepsilon \rightarrow 0$ , in order to derive convergence estimates we will have to let  $N$  tend to  $+\infty$  and  $\varepsilon$  to 0 at some appropriate scale. Our strategy will be in two steps:

1. Design a family of bounded operators  $A_\varepsilon$  of the form (??) and, given a solution  $y$  of (??), derive a convergence estimate of solutions  $y_\varepsilon$  of the “ $\varepsilon$ -Euler” equation (??) to  $y$ . In the context of semigroups developed in Section ??, this essentially amounts to applying the Duhamel formula.
2. For any  $\varepsilon$  fixed, by Theorem ?? in Section ??, derive a convergence estimate as  $N \rightarrow +\infty$  of the particle approximation  $y_\varepsilon^N$  defined by (??) to  $y_\varepsilon$ , with constants keeping track of the dependence with respect to  $\varepsilon$  and  $N$ .

The convergence estimates of particle solutions of (??) to solutions of (??) are then obtained by the triangular inequality. They depend on  $\varepsilon$  and  $N$ , but as already alluded the estimates blow up when  $\varepsilon \rightarrow 0$  with  $N$  being fixed and thus the limits must be taken at some appropriate scaling.

**Structure of the section.** In Section ?? we give a general particle approximation result for the abstract evolution equation (??). The main result is Theorem ??.

Although very general, these results are however abstract and not constructive. In Section ??, we consider linear PDEs defined with classical differential operators, and thanks to convolution we design explicit finite particle approximations. In this section, the main result is Theorem ??.

## 6.2 A general abstract result within semigroup theory

In this section, we use semigroup theory to derive in a very simple way a general approximation theorem. We consider the abstract linear evolution equation

$$\partial_t y = Ay. \tag{79}$$

**Semigroup assumption.** We assume that the operator  $A$  on the Banach space  $X$  is defined on a dense domain  $D(A) \subset X$  and generates a  $C_0$  semigroup  $(e^{tA})_{t \geq 0}$  in  $X$ . In particular, there exist  $M \geq 1$  and  $\beta \in \mathbb{R}$  such that

$$\|e^{tA}\|_{L(X)} \leq M e^{\beta t} \quad \forall t \geq 0. \tag{80}$$

Given  $y^0 \in D(A)$ , there exists a unique solution  $y \in C^0([0, +\infty), D(A)) \cap C^1((0, +\infty), X)$  of (??) such that  $y(0) = y^0$ , which is  $y(t) = e^{tA}y^0$  (see [?]). In what follows we are going to approximate this solution  $y$  by finite particles.

**Bounded operator approximation.** We assume that there exists a family of bounded operators  $A_\varepsilon$  on  $X$ , indexed by  $\varepsilon \in (0, 1]$ , satisfying the following properties:

- Denoting by  $e^{tA_\varepsilon}$  the usual exponential of a bounded operator, we have

$$\|e^{tA_\varepsilon}\|_{L(X)} \leq M e^{\beta t} \quad \forall t \geq 0 \quad \forall \varepsilon \in (0, 1]. \tag{81}$$

- There exist a Banach subspace  $Z \subset X$ , dense in  $X$ ,  $C_A > 0$  and a continuous function  $\chi : [0, 1] \rightarrow [0, +\infty)$ , satisfying  $\chi(0) = 0$ , such that

$$\|(A_\varepsilon - A)z\|_X \leq C_A \chi(\varepsilon) \|z\|_Z \quad \forall z \in Z \quad \forall \varepsilon \in (0, 1]. \tag{82}$$

- For any  $\varepsilon \in (0, 1]$ , the kernel  $\sigma_\varepsilon$  of  $A_\varepsilon$  (defined by (??), i.e.,  $A_\varepsilon f(x) = \int_\Omega \sigma_\varepsilon(x, x') f(x') dx'$  for every  $f \in \mathcal{C}^\infty(\hat{\Omega})$  and every  $x \in \Omega$ ) is Lipschitz continuous on  $\Omega \times \Omega$ .

An example of bounded approximation operator  $A_\varepsilon$  satisfying (??), not explicit but fully general, is given by the *Yosida approximant*

$$A_\varepsilon = J_\varepsilon A \quad \text{where} \quad J_\varepsilon = (\text{id} - \varepsilon A)^{-1},$$

which indeed satisfies (??) (see [?, ?]). The assumption (??) entails two things: first, since it will be applied to the solution  $y$  of (??), it requires that  $y$  be sufficiently regular; second, (??) refers to convergence estimates, which are often proved by explicit approximation constructions (see also [?] for finite-dimensional approximations with error estimates), as we will do hereafter. For instance when  $X = L^2(\Omega, \mathbb{R}^d)$ , the Banach space  $Z$  may be a subspace of functions of  $X$  having a certain number of bounded derivatives. The third assumption is related to the regularity of  $A_\varepsilon$ ; for instance if  $A$  is a differential operator then it is satisfied by iterating the Yosida approximation, taking  $A_\varepsilon = J_\varepsilon^j A$  for  $j$  large enough. Of course, if  $A$  is unbounded then  $\|\sigma_\varepsilon\|_{L^\infty(\Omega^2)} + \text{Lip}(\sigma_\varepsilon) \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ . All in all, the first assumption (??) (which is classical in the Trotter-Kato theorem) is the most stringent; it is usually established in practice by means of dissipativity properties, and this is also what we will do in the explicit construction hereafter.

Given any  $\varepsilon \in (0, 1]$ , the “ $\varepsilon$ -Euler” equation (??) becomes

$$\partial_t y_\varepsilon(t) = A_\varepsilon y_\varepsilon(t) \tag{83}$$

and any solution of (??) is given by  $y_\varepsilon(t) = e^{tA_\varepsilon} y_\varepsilon(0)$ . Besides, for any  $N \in \mathbb{N}^*$ , the particle system (??) is autonomous and is written as

$$\dot{\xi}_{\varepsilon,i}^N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_\varepsilon(x_i^N, x_j^N) \xi_{\varepsilon,j}^N(t) \quad \forall i \in \{1, \dots, N\}. \tag{84}$$

**Theorem 6.** *Let  $T > 0$  be arbitrary. In addition to the above semigroup and bounded approximation operator assumptions, we assume that  $y^0 = y(0) \in \text{Lip}(\Omega, \mathbb{R}^d)$  and that  $y \in L^1([0, T], Z)$ . For any  $\varepsilon \in (0, 1]$  and any  $N \in \mathbb{N}^*$ , let  $t \mapsto \Xi_\varepsilon^N(t) = (\xi_{\varepsilon,1}^N(t), \dots, \xi_{\varepsilon,N}^N(t))$  be the unique solution of (??) such that  $\xi_{\varepsilon,i}^N(0) = y^0(x_i^N)$  for every  $i \in \{1, \dots, N\}$ , and let  $y_\varepsilon^N$  be defined by (??).*

- (i) *If there exists a continuous and dense embedding  $L^\infty(\Omega, \mathbb{R}^d) \hookrightarrow X$ , i.e., if there exists  $C_\infty > 0$  such that  $\|z\|_X \leq C_\infty \|z\|_{L^\infty}$  for any  $z \in L^\infty(\Omega, \mathbb{R}^d)$ , then*

$$\begin{aligned} \|y_\varepsilon^N - y\|_{\mathcal{C}^0([0,T],X)} &\leq C_A M e^{\beta T} \chi(\varepsilon) \|y\|_{L^1([0,T],Z)} \\ &\quad + \frac{2}{N^{1/n}} C_\infty C_\Omega (1 + \text{Lip}(y^0)) \exp\left(2T\|\sigma_\varepsilon\|_{L^\infty} + 2T\|y^0\|_{L^\infty} \text{Lip}(\sigma_\varepsilon) e^{2T\|\sigma_\varepsilon\|_{L^\infty}}\right) \end{aligned} \tag{85}$$

for every  $N \in \mathbb{N}^*$  and every  $\varepsilon \in (0, 1]$ .

- (ii) *If there exists a continuous and dense embedding  $X \hookrightarrow L^\infty(\Omega, \mathbb{R}^d)$ , i.e., if there exists  $C_\infty > 0$  such that  $\|z\|_{L^\infty} \leq C_\infty \|z\|_X$  for any  $z \in X$ , then*

$$\begin{aligned} \|y_\varepsilon^N - y\|_{L^\infty([0,T] \times \Omega)} &\leq C_\infty C_A M e^{\beta T} \chi(\varepsilon) \|y\|_{L^1([0,T],Z)} \\ &\quad + 2 \frac{C_\Omega}{N^{1/n}} (1 + \text{Lip}(y^0)) \exp\left(2T\|\sigma_\varepsilon\|_{L^\infty} + 2T\|y^0\|_{L^\infty} \text{Lip}(\sigma_\varepsilon) e^{2T\|\sigma_\varepsilon\|_{L^\infty}}\right) \end{aligned} \tag{86}$$

for every  $N \in \mathbb{N}^*$  and every  $\varepsilon \in (0, 1]$ .

*Proof.* Given any  $\varepsilon \in (0, 1]$ , the unique solution of (??) such that  $y_\varepsilon(0) = y^0$  is given by  $y_\varepsilon(t) = e^{tA_\varepsilon}y^0$ .

The proof, which is easy, is done in three steps: in the first, we establish that  $y_\varepsilon$  converges to  $y$ ; in the second, we apply Theorem ?? (see Section ??) to prove that  $y_\varepsilon$  is approximated by the solutions of the particle system (??); we conclude by the triangular inequality.

**First step: convergence of  $y_\varepsilon$  towards  $y$ .** Given any  $\varepsilon \in (0, 1]$ , writing that  $\partial_t(y_\varepsilon - y) = A_\varepsilon y_\varepsilon - Ay = A_\varepsilon(y_\varepsilon - y) + (A_\varepsilon - A)y$  and integrating, we obtain (Duhamel formula)

$$y_\varepsilon(t) - y(t) = \int_0^t e^{(t-\tau)A_\varepsilon} (A_\varepsilon - A)y(\tau) d\tau,$$

and using the Jensen inequality and the estimates (??) and (??), we infer that

$$\begin{aligned} \|y_\varepsilon(t) - y(t)\|_X &\leq \int_0^t \left\| e^{(t-\tau)A_\varepsilon} (A_\varepsilon - A)y(\tau) \right\|_X d\tau \\ &\leq \int_0^t M e^{\beta(t-\tau)} \|(A_\varepsilon - A)y(\tau)\|_X d\tau \leq C_A M e^{\beta t} \chi(\varepsilon) \|y\|_{L^1([0,t],Z)} \end{aligned} \quad (87)$$

for every  $t \in [0, T]$ .

**Second step: particle approximation.** Given any  $\varepsilon \in (0, 1]$ , since  $y_\varepsilon(0) = y^0 \in L^\infty(\Omega, \mathbb{R}^d)$  and since  $\partial_t y_\varepsilon(t, \cdot) = \int_\Omega \sigma_\varepsilon(\cdot, x') y_\varepsilon(t, x') dx'$  and  $\dot{\xi}_{\varepsilon,i}^N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_\varepsilon(x_i^N, x_j^N) \xi_{\varepsilon,j}^N(t)$ , estimating roughly and integrating we get that

$$\max(\|\Xi_\varepsilon^N(t)\|_\infty, \|y_\varepsilon(t)\|_{L^\infty(\Omega, \mathbb{R}^d)}) \leq e^{t\|\sigma_\varepsilon\|_{L^\infty(\Omega^2)}} \|y^0\|_{L^\infty(\Omega, \mathbb{R}^d)} \quad \forall t \geq 0. \quad (88)$$

Since  $y^0 \in \text{Lip}(\Omega, \mathbb{R}^d)$ , it follows from the estimates (??) and (??) of Theorem ?? in Section ?? that  $y_\varepsilon(t) \in \text{Lip}(\Omega, \mathbb{R}^d)$  and  $\text{Lip}(y_\varepsilon(t)) \leq e^{tL_\varepsilon(t)}(1 + \text{Lip}(y^0))$  for every  $t \geq 0$ , and

$$\|y_\varepsilon^N(t) - y_\varepsilon(t)\|_{L^\infty(\Omega, \mathbb{R}^d)} \leq \frac{2C_\Omega}{N^{1/n}} (1 + \text{Lip}(y^0)) e^{2tL_\varepsilon(t)} \quad \forall t \geq 0, \quad (89)$$

where, using (??) and the particular form of the mapping  $G_\varepsilon$  defined by (??),  $L_\varepsilon(t)$  is given by

$$L_\varepsilon(t) = \|\sigma_\varepsilon\|_{L^\infty} + \text{Lip}(\sigma_\varepsilon) e^{t\|\sigma_\varepsilon\|_{L^\infty}} \|y^0\|_{L^\infty}. \quad (90)$$

**Conclusion.** In case ??, we have  $\|y_\varepsilon^N(t) - y_\varepsilon(t, \cdot)\|_X \leq C_\infty \|y_\varepsilon^N(t) - y_\varepsilon(t)\|_{L^\infty}$ . Using the triangular inequality, we infer from (??) and (??) that, for every  $t \in [0, T]$  and every  $\varepsilon \in (0, 1]$ ,

$$\|y(t) - y_\varepsilon^N(t)\|_X \leq C_A M e^{\beta T} \chi(\varepsilon) \|y\|_{L^1([0,T],Z)} + \frac{2}{N^{1/n}} C_\infty C_\Omega (1 + \text{Lip}(y^0)) e^{2TL_\varepsilon(T)}$$

and (??) follows, using (??).

In case ??, we have  $\|y_\varepsilon(t) - y(t)\|_{L^\infty} \leq C_\infty \|y_\varepsilon(t) - y(t)\|_X$ . Using the triangular inequality, we infer from (??) and (??) that, for every  $t \in [0, T]$  and every  $\varepsilon \in (0, 1]$ ,

$$\|y(t) - y_\varepsilon^N(t)\|_{L^\infty} \leq C_\infty C_A M e^{\beta T} \chi(\varepsilon) \|y\|_{L^1([0,T],Z)} + \frac{2}{N^{1/n}} C_\Omega (1 + \text{Lip}(y^0)) e^{2TL_\varepsilon(T)}$$

and (??) follows, using (??). □

**Remark 13** (Comments on Theorem ??). To illustrate and understand the convergence estimates (??) and (??), let us assume that  $\chi(\varepsilon) \sim \varepsilon$  and that  $\|\sigma_\varepsilon\|_{L^\infty} + \text{Lip}(\sigma_\varepsilon) \sim \frac{1}{\varepsilon^k}$  for some  $k \in \mathbf{N}^*$  (this will be the case in the explicit construction hereafter), as  $\varepsilon \rightarrow 0$ . Then, ignoring constants, the right-hand side of (??) or (??) is of the order of

$$\varepsilon + \frac{1}{N^{1/n}} \exp\left(\frac{1}{\varepsilon^k} \exp\left(\frac{1}{\varepsilon^k}\right)\right).$$

In order to pass to the limit as  $N \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ , it is appropriate to choose parameters such that this term tends to 0. An optimization argument shows that the best choice for  $\varepsilon$  in function of  $N$  is

$$\varepsilon_N \sim \left(\frac{1}{\ln \ln N}\right)^{\frac{1}{k}}$$

as  $N \rightarrow +\infty$ , and in this case the estimate (??) (applied, typically, with  $X = L^2(\Omega, \mathbf{R}^d)$ ) gives

$$\|y_{\varepsilon_N}^N - y\|_{\mathcal{C}^0([0,T],X)} \leq \text{Cst} \left(\frac{\text{Cst}}{\ln \ln N}\right)^{\frac{1}{k}} (1 + o(1)).$$

Such estimates are reminiscent of those found in [?] concerning the linear Boltzmann and the (hydrodynamic limit) heat equation.

The above double exponential (or double logarithm) is a general estimate that can be improved under additional assumptions. Indeed, the estimate (??) is very rough and can be improved for example if the norm of  $e^{tA_\varepsilon}$  as a bounded linear operator on  $L^\infty(\Omega)$  is uniformly bounded with respect to  $\varepsilon$  on compact intervals of time.

### 6.3 Application to linear PDEs and explicit construction

In local coordinates  $x$  on  $\Omega$ , we denote  $D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$  where  $\partial_i$  is the partial derivative with respect to the  $i^{\text{th}}$  variable of  $x$  (which we do not denote by  $x_i$  because the notation is already used for the tagged partitions), where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$  and we set  $|\alpha| = \sum_{i=1}^n \alpha_i$ .

Let  $p \in \mathbf{N}^*$  be arbitrary. Throughout the section, we assume that  $X = L^2(\Omega, \mathbf{R}^d)$  and that

$$A = \sum_{|\alpha| \leq p} a_\alpha(\cdot) D^\alpha \quad \forall (t, \xi) \in \mathbf{R} \times \mathbf{R}^d, \quad (91)$$

i.e., we consider the linear partial differential equation

$$\partial_t y(t, x) = \sum_{|\alpha| \leq p} a_\alpha(x) D^\alpha y(t, x) \quad (92)$$

with some prescribed conditions at the boundary of  $\Omega$  when  $\Omega$  has a boundary. Here, for every  $\alpha \in \mathbf{N}^k$  such that  $|\alpha| \leq p$ ,  $a_\alpha(\cdot) \in W^{1,\infty}(\Omega)$ .

As an application of Theorem ??, our objective is to prove that, under appropriate assumptions, the solutions of (??) can be approximated by the solutions of a family of finite particle systems, of which we design an explicit construction.

For every  $k \in [1, +\infty]$ , we denote by  $W^{p,k}(\Omega)$  the Sobolev space of functions  $f$  on  $\Omega$  whose partial (distributional) derivatives up to order  $p$  are identified with functions of  $L^k(\Omega)$ , endowed with the norm

$$\|f\|_{W^{p,k}(\Omega)} = \max_{|\alpha| \leq p} \|D^\alpha f\|_{L^k(\Omega)}.$$

For  $k = 2$ , we denote  $H^p(\Omega) = W^{p,2}(\Omega)$ .



### 6.3.1 Main result

**Semigroup assumption.** We assume that the operator  $A$  on  $L^2(\Omega, \mathbb{R}^d)$  is defined on a domain  $D(A) \subset H^p(\Omega, \mathbb{R}^d)$ , dense in  $L^2(\Omega, \mathbb{R}^d)$ , which may encode some Dirichlet or Neumann like boundary conditions, maybe of higher order, and that there exists  $\beta \geq 0$  such that  $A - \beta \text{id}$  generates a  $C_0$  semigroup of contractions in  $L^2(\Omega, \mathbb{R}^d)$ . Hence we have  $M = 1$  in the inequality (??) and, by the Lumer-Phillips theorem,  $A - \beta \text{id}$  is  $m$ -dissipative (see [?]), which means that it is dissipative (i.e.,  $\langle (A - \beta \text{id})f, f \rangle_{L^2(\Omega)} \leq 0$  for every  $f \in D(A)$ ) and that  $\text{Ran}((\beta + 1) \text{id} - A) = ((\beta + 1) \text{id} - A)D(A) = L^2(\Omega)$ .

**Particle approximation.** Let  $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  be a nonnegative symmetric smooth function on  $\mathbb{R}^n$ , of compact support contained in the closed unit ball  $\bar{B}(0, 1)$ , such that  $\int_{\mathbb{R}^n} \eta(x) dx = 1$ . Here, symmetric means that  $\eta(x) = \eta(-x)$  for every  $x \in \mathbb{R}^n$ . We set  $C_\eta = \int_{\mathbb{R}^n} \|x\| \eta(x) dx$ . For example, we can take

$$\eta(x) = \begin{cases} c e^{1/(\|x\|^2 - 1)} & \text{if } \|x\| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where  $c > 0$  is a normalization constant. Given any  $\varepsilon \in (0, 1]$ , we denote by  $\eta_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  the (mollifier) function given by

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right) \quad \forall x \in \mathbb{R}^n$$

and we define  $\sigma_\varepsilon$  by

$$\sigma_\varepsilon(x, x') = \sum_{|\alpha| \leq p} \int_{\Omega} \eta_\varepsilon(x - z) a_\alpha(z) (D^\alpha \eta_\varepsilon)(z - x') dz \quad \forall x, x' \in \Omega \times \Omega. \quad (93)$$

We have  $\sigma_\varepsilon \in \mathcal{C}^\infty(\Omega \times \Omega)$  (it is smooth up to the boundary) and

$$\|\sigma_\varepsilon\|_{L^\infty(\Omega^2)} \leq \frac{C_L}{\varepsilon^{n+p}} \quad \text{and} \quad \text{Lip}(\sigma_\varepsilon) \leq \frac{C_L}{\varepsilon^{n+p+1}}$$

for some constant  $C_L > 0$  depending on  $\eta$  and on the Lipschitz constants of the coefficients  $a_\alpha$  but not depending on  $\varepsilon$ . Recall that  $G_\varepsilon(x, x', \xi') = \sigma_\varepsilon(x, x') \xi'$ .

**Theorem 7.** *Let  $T > 0$ . We assume that  $y \in L^1([0, T], W^{p+1, \infty}(\Omega, \mathbb{R}^d))$  is a solution of (??) such that  $y(0, \cdot) \in \text{Lip}(\Omega, \mathbb{R}^d)$ . For any  $\varepsilon \in (0, 1]$  and any  $N \in \mathbb{N}^*$ , let  $t \mapsto \Xi_\varepsilon^N(t) = (\xi_{\varepsilon, 1}^N(t), \dots, \xi_{\varepsilon, N}^N(t))$  be the unique solution of the particle system (??) such that  $\xi_{\varepsilon, i}^N(0) = y(0, x_i^N)$  for every  $i \in \{1, \dots, N\}$ , and let  $y_\varepsilon^N$  be defined by (??).*

*Then there exists  $C > 0$  such that, for every  $N \in \mathbb{N}^*$  and every  $\varepsilon \in (0, 1]$ ,*

$$\|y_\varepsilon^N - y\|_{\mathcal{C}^0([0, T], L^2(\Omega, \mathbb{R}^d))} \leq C \left( \varepsilon + \frac{1}{N^{1/n}} \exp\left(\frac{C}{\varepsilon^{n+p+1}} \exp\left(\frac{C}{\varepsilon^{n+p}}\right)\right) \right) \quad (94)$$

*except in case ?? when  $n = 1$ , in which case the first term  $\varepsilon$  in the parenthesis at the right-hand side of (??) must be replaced with  $\sqrt{\varepsilon}$ .*

*As a consequence, taking  $\varepsilon_N \sim \left(\frac{C}{\ln \ln N}\right)^{\frac{1}{n+p}}$  as  $N \rightarrow +\infty$  (see Remark ??), (??) gives*

$$\|y_{\varepsilon_N}^N - y\|_{\mathcal{C}^0([0, T], L^2(\Omega, \mathbb{R}^d))} \leq \left(\frac{C}{\ln \ln N}\right)^{\frac{1}{n+p}} \quad (95)$$

### 6.3.2 Proof of Theorem ??

We are going to apply the item ?? of Theorem ?? with  $X = L^2(\Omega, \mathbb{R}^d)$ ,  $Z = W^{p+1, \infty}(\Omega, \mathbb{R}^d)$ ,  $C_\infty = 1$  (because  $|\Omega| = 1$ ),  $M = 1$ .

Recall that the operator  $A_\varepsilon$  is defined by (??), i.e.,  $A_\varepsilon y(x) = \int_\Omega \sigma_\varepsilon(x, x') f(x') dx'$  for every  $f \in \mathcal{C}^\infty(\mathring{\Omega})$  and every  $x \in \Omega$ . We have to prove that the assumptions (??) and (??) are satisfied, and we are going to compute  $C_A$  and  $\chi(\varepsilon)$ . To do that, we first express  $A_\varepsilon$  using an unusual convolution that we introduce next.

**Definition and properties of a convolution operator.** Given any  $k \in [1, +\infty)$  and any  $f \in L^k(\Omega)$ , let us define and give some properties of the smooth approximation  $\eta_\varepsilon \star_\Omega f \in \mathcal{C}^\infty(\mathring{\Omega})$  of  $f$  for any  $\varepsilon \in (0, 1]$ .

In the case ??, i.e., when  $\Omega$  is a smooth compact Riemannian manifold of dimension  $n$ , using a smooth partition of unity over an atlas of  $\Omega$ , we can always write  $f = \sum_{i=1}^m f_i$  for some  $m \in \mathbb{N}^*$  and for some functions  $f_i \in L^k(\Omega)$  whose essential support is contained in a chart of the atlas. In each chart,  $\eta_\varepsilon \star f_i$  can thus be defined as the standard convolution in  $\mathbb{R}^n$  for every  $\varepsilon > 0$  sufficiently small. At the global level, this defines the function  $\eta_\varepsilon \star_\Omega f \in \mathcal{C}^\infty(\mathring{\Omega})$ .

In the case ??, i.e., when  $\Omega$  is the compact closure of a bounded open subset of  $\mathbb{R}^n$  with a Lipschitz boundary, we have to be careful with the boundary. Given any  $f \in L^2(\Omega)$ , for any  $\varepsilon \in (0, 1]$ , we define the function  $\eta_\varepsilon \star_\Omega f$  on  $\Omega$  by

$$\eta_\varepsilon \star_\Omega f(x) = \int_\Omega \eta_\varepsilon(x - x') f(x') dx' \quad \forall x \in \Omega$$

but we stress that this is not a usual convolution (the integral is performed on  $\Omega$  only) and thus the usual properties of the convolution cannot be used directly. This is why, hereafter, we relate this unusual convolution with the usual one, by extending functions on  $\Omega$  to  $\mathbb{R}^n$  by 0 outside of  $\Omega$ . Given any  $f \in L^2(\Omega)$ , we denote by  $\tilde{f} = f \mathbf{1}_\Omega$  the extension of  $f$  to  $\mathbb{R}^n$  by 0. For any  $\varepsilon \in (0, 1]$ , we consider the function  $\eta_\varepsilon \star \tilde{f} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  defined by the usual convolution

$$(\eta_\varepsilon \star \tilde{f})(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x - x') \tilde{f}(x') dx' = \int_\Omega \eta_\varepsilon(x - x') f(x') dx' \quad \forall x \in \mathbb{R}^n,$$

whose support satisfies  $\text{supp}(\eta_\varepsilon \star \tilde{f}) \subset \Omega + \overline{B}(0, \varepsilon)$ . We have

$$\eta_\varepsilon \star_\Omega f = (\eta_\varepsilon \star \tilde{f})|_\Omega,$$

i.e.,  $\eta_\varepsilon \star_\Omega f$  is the restriction of  $\eta_\varepsilon \star \tilde{f}$  to  $\Omega$ . Hence  $\eta_\varepsilon \star_\Omega f \in \mathcal{C}^\infty(\Omega)$ : it is smooth up to the boundary of the compact domain  $\Omega$ . We also have  $\eta_\varepsilon \star_\Omega f = (\tilde{f} \star \eta_\varepsilon)|_\Omega$ . Finally, as a consequence of the properties of the usual convolution, we have  $\eta_\varepsilon \star_\Omega f \rightarrow f$  in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ .

More generally, for every  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , the functions  $D^\alpha(\eta_\varepsilon \star_\Omega f)$ ,  $D^\alpha(\eta_\varepsilon) \star_\Omega f$  and  $\eta_\varepsilon \star_\Omega D^\alpha f$  (provided that  $D^\alpha f \in L^2(\Omega)$  in the latter case) are smooth on  $\Omega$  and are the restrictions to  $\Omega$  of the smooth functions  $D^\alpha(\eta_\varepsilon \star \tilde{f})$ ,  $D^\alpha(\eta_\varepsilon) \star \tilde{f}$  and  $\eta_\varepsilon \star D^\alpha \tilde{f}$  on  $\mathbb{R}^n$ , respectively.<sup>6</sup> In particular, the function  $A(\eta_\varepsilon \star_\Omega f)$  is the restriction to  $\Omega$  of  $A(\eta_\varepsilon \star \tilde{f})$ .

With these definitions, in both cases ?? and ??, for every  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  we have  $D^\alpha(\eta_\varepsilon \star_\Omega f) = (D^\alpha \eta_\varepsilon) \star_\Omega f = \eta_\varepsilon \star_\Omega D^\alpha f$  for every  $f \in L^2(\Omega)$  (such that  $D^\alpha f \in L^2(\Omega)$  for the last equality) and  $D^\alpha(\eta_\varepsilon \star_\Omega f) \rightarrow D^\alpha f$  in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  if  $D^\alpha f \in L^2(\Omega)$ . Note that the function  $\sigma_\varepsilon$  defined by (??) is also given by

$$\sigma_\varepsilon(x, x') = \left( \eta_\varepsilon \star_\Omega \sum_{|\alpha| \leq p} a_\alpha(\cdot) (D^\alpha \eta_\varepsilon)(\cdot - x') \right)(x) \quad \forall x, x' \in \Omega.$$

<sup>6</sup>There is a small subtlety for defining the latter one, removed by noting that  $D^\alpha \tilde{f} = \widetilde{D^\alpha f}$  a.e. if  $D^\alpha f \in L^2(\Omega)$ .

**Expressing  $A_\varepsilon$  with the convolution operator  $\eta_\varepsilon \star \Omega$ .**

**Lemma 3.** *Given any  $\varepsilon \in (0, 1]$  and any  $f \in \mathcal{C}^\infty(\mathring{\Omega})$ , we have<sup>7</sup>*

$$A_\varepsilon f = \eta_\varepsilon \star \Omega A(\eta_\varepsilon \star \Omega f) = \left( \eta_\varepsilon \star (A(\eta_\varepsilon \star \tilde{f}) \mathbb{1}_\Omega) \right) \Big|_\Omega.$$

*Proof.* For  $x \in \Omega$  fixed, we have, using that  $D^\alpha \eta_\varepsilon \star \Omega f = D^\alpha (\eta_\varepsilon \star \Omega f)$ ,

$$(A_\varepsilon f)(x) = \int_\Omega \sigma_\varepsilon(x, x') f(x') dx' = \left( \eta_\varepsilon \star \Omega \sum_{|\alpha| \leq p} a_\alpha D^\alpha \eta_\varepsilon \star \Omega f \right)(x) = \left( \eta_\varepsilon \star \Omega A(\eta_\varepsilon \star \Omega f) \right)(x),$$

thus giving the lemma. □

**Uniform stability property (??).** Thanks to Lemma ??, we can now establish (??).

**Lemma 4.** *For all  $f, g \in L^2(\Omega)$ , we have  $\langle \eta_\varepsilon \star \Omega g, f \rangle_{L^2(\Omega)} = \langle g, \eta_\varepsilon \star \Omega f \rangle_{L^2(\Omega)}$ .*

*Proof.* Using that  $\eta_\varepsilon \star \Omega g = (\eta_\varepsilon \star \tilde{g})|_\Omega$  and that  $\tilde{f} = 0$  on  $\mathbb{R}^n \setminus \Omega$ , we have  $\langle \eta_\varepsilon \star \Omega g, f \rangle_{L^2(\Omega)} = \langle \eta_\varepsilon \star \tilde{g}, \tilde{f} \rangle_{L^2(\mathbb{R}^n)}$ . Now, using the fact that  $\eta_\varepsilon$  is symmetric, i.e., that  $\eta_\varepsilon(z) = \eta_\varepsilon(-z)$  for any  $z \in \mathbb{R}^n$ , and that this property ensures that the usual convolution by  $\eta_\varepsilon$  is symmetric in  $L^2(\mathbb{R}^n)$ , we infer that  $\langle \eta_\varepsilon \star \Omega g, f \rangle_{L^2(\Omega)} = \langle \tilde{g}, \eta_\varepsilon \star \tilde{f} \rangle_{L^2(\mathbb{R}^n)}$ . But the latter term is equal to  $\langle g, \eta_\varepsilon \star \Omega f \rangle_{L^2(\Omega)}$ . The lemma is proved. □

**Lemma 5.** *Like the operator  $A - \beta \text{id}$ , the operator  $A_\varepsilon - \beta \text{id}$  is  $m$ -dissipative on  $L^2(\Omega, \mathbb{R}^d)$ , for any  $\varepsilon \in (0, 1]$ . As a consequence, we have*

$$\|e^{tA_\varepsilon}\|_{L(L^2(\Omega, \mathbb{R}^d))} \leq e^{\beta t} \quad \text{and} \quad \|e^{tA}\|_{L(L^2(\Omega, \mathbb{R}^d))} \leq e^{\beta t} \quad \forall t \geq 0 \quad \forall \varepsilon \in (0, 1].$$

*Therefore (??) is satisfied (with  $M = 1$ ).*

*Proof.* Given any  $f \in \mathcal{C}^\infty(\mathring{\Omega})$ , applying Lemma ?? to  $g = (A - \beta \text{id})(\eta_\varepsilon \star \Omega f)$ , we have

$$\langle (A_\varepsilon - \beta \text{id})f, f \rangle_{L^2(\Omega)} = \langle \eta_\varepsilon \star \Omega (A - \beta \text{id})(\eta_\varepsilon \star \Omega f), f \rangle_{L^2(\Omega)} = \langle (A - \beta \text{id})(\eta_\varepsilon \star \Omega f), \eta_\varepsilon \star \Omega f \rangle_{L^2(\Omega)} \leq 0$$

because  $A - \beta \text{id}$  is dissipative. Since  $A_\varepsilon$  is bounded, we have  $D(A_\varepsilon) = L^2(\Omega)$ , and thus its adjoint  $A_\varepsilon^*$  is bounded and  $D(A_\varepsilon^*) = L^2(\Omega)$ . Therefore, obviously,  $A_\varepsilon^* - \beta \text{id}$  is also dissipative. The conclusion now follows from the Lumer-Phillips theorem (see [?, Chapter II, Corollary 3.17] or [?, Chapter 1, Theorem 4.3]). □

**Remark 14.** Lemma ?? is the key step where we use the particular form  $A_\varepsilon f = \eta_\varepsilon \star \Omega A(\eta_\varepsilon \star \Omega f)$ , in order to ensure dissipativity. It would not work if we had chosen  $A_\varepsilon f = A(\eta_\varepsilon \star \Omega f)$ . Note that, as mentioned in Section ??, (??) is always satisfied when choosing the Yosida approximant  $A_\varepsilon = (\text{id} - \varepsilon A)^{-1} A$ . The interest of the above construction is that it is fully explicit.

**A first convergence property of  $A_\varepsilon$ .**

**Lemma 6.** *Given any  $f \in \mathcal{C}^\infty(\mathring{\Omega})$ , we have  $A_\varepsilon f \rightarrow Af$  in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ .*

<sup>7</sup>The function  $A(\eta_\varepsilon \star \Omega f)$  is the restriction of  $A(\eta_\varepsilon \star \tilde{f})$  to  $\Omega$  and, denoting by  $g_\varepsilon = A(\eta_\varepsilon \star \tilde{f}) \mathbb{1}_\Omega$  the extension to  $\mathbb{R}^n$  by 0 of the function  $A(\eta_\varepsilon \star \Omega f)$ , the function  $\eta_\varepsilon \star \Omega A(\eta_\varepsilon \star \Omega f)$  is the restriction to  $\Omega$  of the function  $\eta_\varepsilon \star g_\varepsilon$ .

*Proof.* By the triangular inequality, using the expression of  $A_\varepsilon$  given by Lemma ??,

$$\|A_\varepsilon f - Af\|_{L^2(\Omega)} \leq \|\eta_\varepsilon \star_\Omega (A(\eta_\varepsilon \star_\Omega f) - Af)\|_{L^2(\Omega)} + \|\eta_\varepsilon \star_\Omega Af - Af\|_{L^2(\Omega)}.$$

The second term at the right-hand side of that inequality converges to 0 as  $\varepsilon \rightarrow 0$ , because  $Af \in L^2(\Omega)$ . To handle the first term, we use the Young inequality  $\|g \star h\|_{L^r(\mathbb{R}^n)} \leq B_{p,q} \|g\|_{L^p(\mathbb{R}^n)} \|h\|_{L^q(\mathbb{R}^n)}$  with  $g = \eta_\varepsilon$ ,  $h = \tilde{r}_\varepsilon$  where  $r_\varepsilon = A(\eta_\varepsilon \star_\Omega f) - Af$ , and with  $r = 2$ ,  $p = 1$  and  $q = 2$ , obtaining

$$\|\eta_\varepsilon \star_\Omega r_\varepsilon\|_{L^2(\Omega)} = \|(\eta_\varepsilon \star \tilde{r}_\varepsilon)|_\Omega\|_{L^2(\mathbb{R}^n)} \leq \|\eta_\varepsilon \star \tilde{r}_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq B_{1,2} \|\eta_\varepsilon\|_{L^1(\Omega)} \|\tilde{r}_\varepsilon\|_{L^2(\mathbb{R}^n)} = B_{1,2} \|r_\varepsilon\|_{L^2(\Omega)}$$

because  $\|\eta_\varepsilon\|_{L^1(\Omega)} = 1$ , and we conclude that  $A_\varepsilon f \rightarrow Af$  in  $L^2(\Omega)$  by noticing that  $r_\varepsilon \rightarrow 0$  in  $L^2(\Omega)$  because

$$A(\eta_\varepsilon \star_\Omega f) = \sum_{|\alpha| \leq p} a_\alpha D^\alpha (\eta_\varepsilon \star_\Omega f) = \sum_{|\alpha| \leq p} a_\alpha \eta_\varepsilon \star_\Omega D^\alpha f \longrightarrow \sum_{|\alpha| \leq p} a_\alpha D^\alpha f = Af$$

in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ . □

In terms of Schwartz kernels, the kernel of  $A_\varepsilon$  is obtained by convoluting to the left and “to the right” (in some sense) the Schwartz kernel of  $A$  with  $\eta_\varepsilon$ , and that, for every  $x \in \Omega$  fixed, the function  $x' \mapsto \sigma_\varepsilon(x, x')$  converges in the distributional sense to the distribution  $\sum_{|\alpha| \leq p} (-1)^{|\alpha|} a_\alpha \delta_x^{(\alpha)}$  as  $\varepsilon \rightarrow 0$  (where  $\delta_x^{(\alpha)}$  is a distributional derivative of the Dirac  $\delta_x$  at  $x$ ), which is the Schwartz kernel  $[A](x, \cdot)$  of  $A$ .

The convergence property stated in Lemma ?? is not enough to get (?). We need to refine the analysis and establish some error estimates. We start by refining our analysis of the unusual convolution operator introduced previously.

**Convergence estimates for the convolution operator  $\eta_\varepsilon \star_\Omega$ .** Set  $C_a = \max_{|\alpha| \leq p} \|a_\alpha\|_{W^{1,\infty}(\Omega)}$ .

We introduce the following notation: in the case ??, for any  $\varepsilon \in (0, 1]$  we define the compact subset  $\Omega_\varepsilon$  of  $\mathring{\Omega}$  by

$$\Omega_\varepsilon = \{x \in \Omega \mid d_\Omega(x, \partial\Omega) \geq \varepsilon\} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega + B(0, \varepsilon)).$$

There exists a constant  $C_{\partial\Omega} > 0$  such that

$$|\Omega \setminus \Omega_\varepsilon| \leq C_{\partial\Omega} \varepsilon^n \quad \forall \varepsilon \in (0, 1].$$

In the case ?? we simply set  $\Omega_\varepsilon = \Omega$  and  $C_{\partial\Omega} = 0$ .

We will also need to use extension operators in the case ??: according to [?, Chap. VI, Sec. 3, Theorem 5] (see also [?, Chap. 12]), there exist  $C_E > 0$  and a linear continuous operator  $E$  mapping functions on  $\Omega$  to functions on  $\mathbb{R}^n$ , such that the restriction of  $Ef$  to  $\Omega$  coincides with  $f$  and  $\|Ef\|_{W^{j,k}(\mathbb{R}^n)} \leq C_E \|f\|_{W^{j,k}(\Omega)}$  for every  $f \in W^{j,k}(\Omega)$  and for every  $j \in \mathbb{N}$  and every  $k \in [1, +\infty]$  (Stein extension). In the case ??, accordingly, we set  $C_E = 1$ .

**Lemma 7.** *Given any  $\varepsilon \in (0, 1]$ , we have*

$$\begin{aligned} \|\eta_\varepsilon \star_\Omega f - f\|_{L^\infty(\Omega)} &\leq 2\|f\|_{L^\infty(\Omega)} \quad \forall f \in L^\infty(\Omega), \\ |\eta_\varepsilon \star_\Omega f(x) - f(x)| &\leq C_E C_\eta \varepsilon \|f\|_{W^{1,\infty}(\Omega)} \quad \forall x \in \Omega_\varepsilon \quad \forall f \in W^{1,\infty}(\Omega). \end{aligned}$$

*Proof.* The first inequality is obviously obtained by using that  $\|\eta_\varepsilon \star \tilde{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|\tilde{f}\|_{L^\infty(\mathbb{R}^n)}$  because  $\|\eta_\varepsilon\|_{L^1(\mathbb{R}^n)} = 1$ .

Let  $x \in \Omega_\varepsilon$  be arbitrary. Since  $\text{supp}(\eta_\varepsilon) \subset \bar{B}(0, \varepsilon)$  and thus  $\text{supp}(\eta_\varepsilon(x - \cdot)) \subset \bar{B}(x, \varepsilon) \subset \Omega$ , we have  $\int_\Omega \eta_\varepsilon(x - x') dx' = \int_{\mathbb{R}^n} \eta_\varepsilon(x - x') dx' = 1$ , hence  $f(x) = \int_\Omega \eta_\varepsilon(x - x') f(x') dx'$  and

$$|\eta_\varepsilon \star f(x) - f(x)| \leq \int_\Omega \eta_\varepsilon(x - x') |f(x') - f(x)| dx' \leq \int_{\mathbb{R}^n} \eta_\varepsilon(x - x') |Ef(x') - Ef(x)| dx'$$

where  $Ef$  is the Stein extension of  $f$ , defined above (actually the latter inequality is even an equality because  $\eta_\varepsilon(x - x') = 0$  for any  $x' \in \mathbb{R}^n \setminus \Omega$ , since  $x \in \Omega_\varepsilon$ ). It follows from the mean value theorem that

$$|Ef(x') - Ef(x)| \leq \|Ef\|_{W^{1,\infty}(\mathbb{R}^n)} \|x - x'\| \leq C_E \|f\|_{W^{1,\infty}(\Omega)} \|x - x'\|.$$

Hence

$$|\eta_\varepsilon \star f(x) - f(x)| \leq C_E \|f\|_{W^{1,\infty}(\Omega)} \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \eta\left(\frac{x - x'}{\varepsilon}\right) \|x - x'\| dx' = C_E C_\eta \varepsilon \|f\|_{W^{1,\infty}(\Omega)}$$

by using the change of variable  $x' = x - \varepsilon s$ .

Note that, in the above argument, we have used a  $W^{1,\infty}$  extension of  $f$  (and not the extension by 0, which is not in  $W^{1,\infty}(\mathbb{R}^n)$ ) in order to use the mean value theorem, because, in the case ??,  $\Omega$  may not be convex.  $\square$

**Convergence properties (??) of  $A_\varepsilon$ .** We are now in a position to establish (??).

**Lemma 8.** *Given any  $\varepsilon \in (0, 1]$ , we have*

$$\|(A_\varepsilon - A)f\|_{L^\infty(\Omega)} \leq 4n^{p+1} C_a \|f\|_{W^{p,\infty}(\Omega)} \quad \forall f \in W^{p,\infty}(\Omega), \quad (96)$$

$$|(A_\varepsilon - A)f(x)| \leq 2n^{p+1} C_E C_\eta C_a \varepsilon \|f\|_{W^{p+1,\infty}(\Omega)} \quad \forall x \in \Omega_\varepsilon \quad \forall f \in W^{p+1,\infty}(\Omega). \quad (97)$$

As a consequence,

$$\|(A_\varepsilon - A)f\|_{L^2(\Omega)} \leq 2n^{p+1} C_a \sqrt{C_E^2 C_\eta^2 \varepsilon^2 + 4C_{\partial\Omega} \varepsilon^n} \|f\|_{W^{p+1,\infty}(\Omega)} \quad \forall f \in W^{p+1,\infty}(\Omega), \quad (98)$$

and therefore (??) is satisfied with  $C_A = 2n^{p+1} C_a$  and  $\chi(\varepsilon) = \sqrt{C_E^2 C_\eta^2 \varepsilon^2 + 4C_{\partial\Omega} \varepsilon^n}$ .

Note that, in the case ??, we have  $C_{\partial\Omega} = 0$  and then the above estimate is in  $\varepsilon$ . Actually, in both cases ?? and ?? the estimate is in  $\varepsilon$  as  $\varepsilon \rightarrow 0$  except in the case ?? when moreover  $n = 1$ , in which case the estimate is in  $\sqrt{\varepsilon}$ .

*Proof.* For any  $\alpha \in \mathbb{N}^k$  such that  $|\alpha| \leq p$ , noting that  $D^\alpha(\eta_\varepsilon \star_\Omega f - f) = \eta_\varepsilon \star_\Omega D^\alpha f - D^\alpha f$ , we infer from Lemma ?? applied to  $D^\alpha f$  that

$$\|D^\alpha(\eta_\varepsilon \star_\Omega f - f)\|_{L^\infty(\Omega)} \leq 2\|f\|_{W^{p,\infty}(\Omega)}$$

$$|D^\alpha(\eta_\varepsilon \star_\Omega f - f)(x)| = |\eta_\varepsilon \star_\Omega (D^\alpha f)(x) - D^\alpha f(x)| \leq C_E C_\eta \varepsilon \|f\|_{W^{1+|\alpha|,\infty}(\Omega)} \quad \forall x \in \Omega_\varepsilon$$

and thus, using that  $A = \sum_{|\alpha| \leq p} a_\alpha D^\alpha$  and that  $\|a_\alpha\|_{L^\infty(\Omega)} \leq C_a$ , and since the number of  $\alpha \in \mathbb{N}^n$  such that  $|\alpha| \leq p$  is  $1 + n + \dots + n^p = \frac{n^{p+1} - 1}{n - 1} \leq n^{p+1}$ , we obtain

$$\|A(\eta_\varepsilon \star_\Omega f - f)\|_{L^\infty(\Omega)} \leq 2n^{p+1} C_a \|f\|_{W^{p,\infty}(\Omega)}$$

$$|A(\eta_\varepsilon \star_\Omega f - f)(x)| \leq n^{p+1} C_E C_\eta C_a \varepsilon \|f\|_{W^{p+1,\infty}(\Omega)} \quad \forall x \in \Omega_\varepsilon.$$

Besides, we infer from Lemma ?? applied to  $A(\eta_\varepsilon \star_\Omega f)$ , using that  $\|a_\alpha\|_{W^{1,\infty}(\Omega)} \leq C_a$ , that

$$\begin{aligned} \|\eta_\varepsilon \star_\Omega A(\eta_\varepsilon \star_\Omega f) - A(\eta_\varepsilon \star_\Omega f)\|_{L^\infty(\Omega)} &\leq 2\|A(\eta_\varepsilon \star_\Omega f)\|_{L^\infty(\Omega)} \\ &\leq 2n^{p+1} C_a \|\eta_\varepsilon \star_\Omega f\|_{W^{p,\infty}(\Omega)} \leq 2n^{p+1} C_a \|f\|_{W^{p,\infty}(\Omega)} \end{aligned}$$

and

$$\begin{aligned} |\eta_\varepsilon \star_\Omega A(\eta_\varepsilon \star_\Omega f)(x) - A(\eta_\varepsilon \star_\Omega f)(x)| &\leq C_E C_\eta \varepsilon \|A(\eta_\varepsilon \star_\Omega f)\|_{W^{1,\infty}(\Omega)} \\ &\leq n^{p+1} C_E C_\eta C_a \varepsilon \|\eta_\varepsilon \star_\Omega f\|_{W^{p+1,\infty}(\Omega)} \leq n^{p+1} C_E C_\eta C_a \varepsilon \|f\|_{W^{p+1,\infty}(\Omega)} \quad \forall x \in \Omega_\varepsilon \end{aligned}$$

where we have used that  $\|\eta_\varepsilon \star g\|_{L^\infty(\mathbb{R}^n)} \leq \|g\|_{L^\infty(\mathbb{R}^n)}$  for any  $g \in L^\infty(\mathbb{R}^n)$  (note that  $\|\eta_\varepsilon\|_{L^1(\mathbb{R}^n)} = 1$ ). Finally, by the triangular inequality, we have

$$|(A_\varepsilon - A)f(x)| \leq |\eta_\varepsilon \star_\Omega A(\eta_\varepsilon \star_\Omega f)(x) - A(\eta_\varepsilon \star_\Omega f)(x)| + |A(\eta_\varepsilon \star_\Omega f - f)(x)|$$

and the estimates (??) and (??) follow.

To establish (??), we write

$$\|(A_\varepsilon - A)f\|_{L^2(\Omega)}^2 = \int_{\Omega_\varepsilon} |(A_\varepsilon - A)f(x)|^2 dx + \int_{\Omega \setminus \Omega_\varepsilon} |(A_\varepsilon - A)f(x)|^2 dx.$$

Using (??), the first term is estimated by

$$\int_{\Omega_\varepsilon} |(A_\varepsilon - A)f(x)|^2 dx \leq 4n^{2(p+1)} C_E^2 C_\eta^2 C_a^2 \varepsilon^2 \|f\|_{W^{p+1,\infty}(\Omega)}^2,$$

and using (??), the second term is estimated by

$$\int_{\Omega \setminus \Omega_\varepsilon} |(A_\varepsilon - A)f(x)|^2 dx \leq \|(A_\varepsilon - A)f\|_{L^\infty(\Omega)}^2 |\Omega \setminus \Omega_\varepsilon| \leq 16n^{2(p+1)} C_a^2 \|f\|_{W^{p,\infty}(\Omega)}^2 C_{\partial\Omega} \varepsilon^n,$$

and the conclusion follows.  $\square$

## 6.4 Further remarks

In this section, we show that the particle approximation result stated in Theorem ?? can be extended to some cases where the operator does even not generate a semigroup, like the case of the backward heat equation  $\partial_t y = -\Delta y$ .

We have seen in Section ?? that the strategy to approximate a given solution  $y$  of (??) goes in two steps: first, find an adequate bounded approximation  $A_\varepsilon$  of  $A$ , and  $y_\varepsilon$  of  $y$ ; second, take the particle approximation  $y_\varepsilon^N$  of  $y_\varepsilon$ . The second step is an automatic consequence of Theorem ?? in Section ?? and is thus general. The first step has been performed in Sections ?? and ?? by applying the Duhamel formula, within the semigroup context, which required the instrumental uniform stability estimate (??): this is in such a way that, in the first step of the proof of Theorem ??, we have established the inequality (??), i.e.,

$$\|y_\varepsilon(t) - y(t)\|_X \leq C\chi(\varepsilon)\|y\|_{L^1([0,T],Z)} \quad \forall t \in [0, T] \quad (99)$$

for some  $C > 0$ . But this first step, requiring the demanding estimate (??), can be dropped if one is able to design a bounded approximation  $A_\varepsilon$  of  $A$  and an approximation  $y_\varepsilon$  of  $y$  such that the estimate (??) is satisfied. And indeed this can often be done, without requiring any semigroup property. Let us give some examples.

**Backward heat equation.** Consider the backward heat equation  $\partial_t y = -\Delta y$  and its approximation  $\partial_t y_\varepsilon = -\Delta_\varepsilon y_\varepsilon$  where  $\Delta_\varepsilon$  is a bounded approximation of  $\Delta$  as done in the previous sections. Assuming that  $y(0) = e^{t\Delta} f$  for some  $f \in L^2(\Omega, \mathbb{R}^d)$ , we have  $y(t) = e^{(T-t)\Delta} f$  for every  $t \in [0, T]$ . Now, we take  $y_\varepsilon(0) = e^{t\Delta_\varepsilon} f$  and we have as well  $y_\varepsilon(t) = e^{(T-t)\Delta_\varepsilon} f$  for every  $t \in [0, T]$ . Then, obviously, the Duhamel formula gives (??).

Of course, this works because we have considered a very regular initial condition. More generally, the argument works for operators with constant coefficients, taking Fourier transforms and considering initial conditions whose Fourier transform has a compact support.

**Variational inequalities.** There exists a wide existing literature on variational inequalities, with the objective of establishing the existence of a solution to a nonlinear equation  $\partial_t y = A(y)$  by approximating the nonlinear unbounded operator  $A$  with a bounded operator  $A_\varepsilon$ . The estimate (??) can then be obtained from energy considerations, or from Galerkin approximation considerations, etc. Most of known equations having a physical meaning enter in such a framework.

More generally, we think that our result can be extended to quasilinear or nonlinear evolution equations, in an appropriate nonlinear semigroup context. This is an open question on which we are currently working.

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## A Appendix

Let  $E$  be a Polish space, endowed with a distance  $d_E$ .

### A.1 Some general facts on the Wasserstein distance

**Choice of a distance on  $E^k$ .** Let  $q \in [1, +\infty]$  be arbitrarily fixed. Given any  $k \in \mathbb{N}^*$ , we endow  $E^k$  with the  $\ell^q$  distance based on  $d_E$ , defined by

$$d_{E^k}^{[q]}(y, y') = \|(d_E(y_1, y'_1), \dots, d_E(y_k, y'_k))\|_{\ell^q} = \begin{cases} \left( \sum_{i=1}^k d_E(y_i, y'_i)^q \right)^{1/q} & \text{if } 1 \leq q < +\infty \\ \max_{1 \leq i \leq k} d_E(y_i, y'_i) & \text{if } q = +\infty \end{cases} \quad (100)$$

for all  $y = (y_1, \dots, y_k)$  and  $y' = (y'_1, \dots, y'_k)$  in  $E^k$ .

Fixing such a choice has an impact on the computation of the Wasserstein distance  $W_p$  between two probability measures on  $E^k$ . Indeed, this means that the distance (??) is used in the definition (??) of  $W_p$ , and that, in the definition (??) of  $W_1$ , the Lipschitz constants must be computed with the distance (??). The lemma below is thus important to compute Lipschitz constants.

**Lemma 9.** *Let  $f \in \text{Lip}(E^k)$ . Then, for any  $y_2, \dots, y_k \in E$ , the mapping  $y_1 \mapsto f(y_1, y_2, \dots, y_k)$  is Lipschitz, of Lipschitz constant less than  $\text{Lip}(f)$ . We set  $\text{Lip}_{y_1}(f) = \max\{\text{Lip}(f(\cdot, y_2, \dots, y_k)) \mid y_2, \dots, y_k \in E\}$ . All other  $\text{Lip}_{y_i}(f)$  are defined similarly, for  $i = 2, \dots, k$ . We have*

$$\text{Lip}(f) = \|( \text{Lip}_{y_1}(f), \dots, \text{Lip}_{y_k}(f) )\|_{\ell^{q'}} = \begin{cases} \left( \sum_{i=1}^k \text{Lip}_{y_i}(f)^{q'} \right)^{1/q'} & \text{if } q' < +\infty \\ \max_{1 \leq i \leq k} \text{Lip}_{y_i}(f) & \text{if } q' = +\infty \end{cases}$$

where  $q' \in [1, +\infty]$  is defined by  $\frac{1}{q} + \frac{1}{q'} = 1$ .

*Proof.* It suffices to write

$$\begin{aligned} & |f(y_1, \dots, y_k) - f(y'_1, \dots, y'_k)| \\ & \leq |f(y_1, y_2, \dots, y_k) - f(y'_1, y_2, \dots, y_k)| + \dots + |f(y'_1, \dots, y'_{k-1}, y_k) - f(y'_1, \dots, y'_{k-1}, y'_k)| \\ & \leq \sum_{i=1}^k \text{Lip}_{y_i}(f) d_E(y_i, y'_i) \end{aligned}$$

and to use the Hölder inequality.  $\square$

**Remark 15.** The choice of a distance  $d_{E^k}^{[q]}$  on the tensor product  $E^k$  (i.e., the choice of  $q \in [1, +\infty]$ ) is far from being insignificant because, although all norms are equivalent in  $E^k$ , comparing them gives constants depending on  $k$ . The choice thus becomes particularly meaningful when  $k$  is large.

Another remark is that the definition (??) is based on the usual  $\ell^q$  norm, for  $q \in [1, +\infty]$ . Other choices are possible, but in order to keep many of the statements further the convexity of the norm is important.

**Notation  $W_p^{[q]}$ .** For all  $p, q \in [1, +\infty]$ , following Remark ??, hereafter we denote by  $W_p^{[q]}$  the Wasserstein distance  $W_p$  on  $\mathcal{P}(E^k)$  (defined by (??)) with respect to the distance  $d_{E^k}^{[q]}$  on  $E^k$ .

It follows from the usual inequalities for  $\ell^q$  norms in  $\mathbb{R}^k$  that  $q \mapsto d_{E^k}^{[q]}$  is decreasing and

$$1 \leq q_1 \leq q_2 \leq +\infty \Rightarrow d_{E^k}^{[q_2]} \leq d_{E^k}^{[q_1]} \leq k^{\frac{1}{q_1} - \frac{1}{q_2}} d_{E^k}^{[q_2]}$$

and thus

$$1 \leq q_1 \leq q_2 \leq +\infty \Rightarrow W_p^{[q_2]} \leq W_p^{[q_1]} \leq k^{\frac{1}{q_1} - \frac{1}{q_2}} W_p^{[q_2]} \quad (101)$$

for any  $p \in [1, +\infty]$ . These inequalities complement (??). For  $p$  fixed, in the family of distances  $W_p^{[q]}$ , for  $q \in [1, +\infty]$ , the  $\ell^1$  distance  $W_p^{[1]}$  is the weakest one. This is an important point because, in the existing literature, very often the  $\ell^2$  distance  $W_p^{[2]}$  is used, but in this work the use of  $q = 1$  is crucial for some parts.

In all subsections hereafter, we fix an arbitrary  $p \in [1, +\infty]$ . The case  $p = +\infty$  is obtained by taking the limit when it makes sense. We also fix an arbitrary  $q \in [1, +\infty]$ .

### A.1.1 Convexity

**Lemma 10** ( $(W_p)^p$  is convex). *Given any  $\mu_1, \mu_2, \mu'_1, \mu'_2 \in \mathcal{P}(E)$  and any  $\lambda \in [0, 1]$ , we have*

$$W_p(\lambda\mu_1 + (1-\lambda)\mu_2, \lambda\mu'_1 + (1-\lambda)\mu'_2)^p \leq \lambda W_p(\mu_1, \mu'_1)^p + (1-\lambda) W_p(\mu_2, \mu'_2)^p.$$

*Proof.* This result is a particular case of [?, Part I, Chapter 4, Theorem 4.8]. Let  $\Pi_i$  be an optimal coupling between  $\mu_i$  and  $\mu'_i$ , for  $i = 1, 2$ . Then  $\Pi = \lambda\Pi_1 + (1-\lambda)\Pi_2$  couples  $\lambda\mu_1 + (1-\lambda)\mu_2$  and  $\lambda\mu'_1 + (1-\lambda)\mu'_2$  (maybe not optimally). Hence

$$\begin{aligned} & W_p(\lambda\mu_1 + (1-\lambda)\mu_2, \lambda\mu'_1 + (1-\lambda)\mu'_2)^p \leq \int_E d_E(x, x')^p d\Pi(x, x') \\ & = \lambda \int_E d_E(x, x')^p d\Pi_1(x, x') + (1-\lambda) \int_E d_E(x, x')^p d\Pi_2(x, x') = \lambda W_p(\mu_1, \mu'_1)^p + (1-\lambda) W_p(\mu_2, \mu'_2)^p \end{aligned}$$

and the lemma follows.  $\square$



**Lemma 11.** Let  $\mu_1, \mu_2, \beta \in \mathcal{P}(E)$  and let  $\varepsilon \in (0, 1]$  be such that  $\mu_1 = (1 + \varepsilon)\mu_2 - \varepsilon\beta$ . Then

$$W_p(\mu_1, \mu_2) \leq \varepsilon^{1/p} W_p(\mu_1, \beta)$$

and, assuming that  $\varepsilon < 1$ ,

$$W_p(\mu_1, \mu_2) \leq \frac{\varepsilon^{1/p}}{1 - \varepsilon^{1/p}} W_p(\mu_2, \beta).$$

In the particular case  $p = 1$ , we have  $W_1(\mu_1, \mu_2) = \varepsilon W_1(\mu_2, \beta)$ .

*Proof.* We have  $\mu_2 = \frac{1}{1+\varepsilon}\mu_1 + \frac{\varepsilon}{1+\varepsilon}\beta$  (convex combination), and applying Lemma ?? we get  $W_p(\mu_1, \mu_2)^p \leq \frac{\varepsilon}{1+\varepsilon} W_p(\mu_1, \beta)^p \leq \varepsilon W_p(\mu_1, \beta)^p$ , and the first inequality follows. The second inequality is obtained by using the triangular inequality  $W_p(\mu_1, \beta) \leq W_p(\mu_1, \mu_2) + W_p(\mu_2, \beta)$ . When  $p = 1$ , given any  $f \in \mathcal{C}_c^0(E)$ , we have  $\int_E f d(\mu_1 - \mu_2) = \varepsilon \int_E f d(\mu_2 - \beta)$ , and taking (in two steps) the supremum over all  $f$  such that  $\text{Lip}(f) \leq 1$ , we get  $W_1(\mu_1, \mu_2) = \varepsilon W_1(\mu_2, \beta)$ .  $\square$

### A.1.2 Symmetrization

Let  $N \in \mathbb{N}^*$  be arbitrary. Given any  $\mu \in \mathcal{P}(E^N)$ , the measure  $\mu^s \in \mathcal{P}(E^N)$ , called the symmetrization under permutations of  $\mu$ , is defined by

$$\mu^s = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \sigma_* \mu \quad (102)$$

where the measure  $\sigma_* \mu$  is defined by  $\langle \sigma_* \mu, f \rangle = \langle \mu, \sigma^* f \rangle$  and  $(\sigma^* f)(y) = f(\sigma \cdot y)$ , with  $\sigma \cdot y = (y_{\sigma(1)}, \dots, y_{\sigma(N)})$  for every  $y \in E^N$  and for every  $\sigma \in \mathfrak{S}_N$ , where  $\mathfrak{S}_N$  is the group of permutations of  $N$  elements. Here,  $\langle \cdot, \cdot \rangle$  is the duality bracket. Equivalently,

$$\int_{E^N} f(y) d\mu^s(y) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{E^N} f(\sigma \cdot y) d\mu(y) \quad \forall f \in \mathcal{C}_c^0(E^N).$$

**Lemma 12.** Given any  $\mu_1, \mu_2 \in \mathcal{P}(E^N)$ , we have

$$W_p^{[q]}(\mu_1^s, \mu_2^s) \leq W_p^{[q]}(\mu_1, \mu_2).$$

In this lemma, the Wasserstein distance  $W_p$  is computed with respect to the  $\ell^q$  distance  $d_{E^N}^{[q]}$ .

*Proof.* This follows from Lemma ??, since  $\mu^s$  is written as the convex combination (??), noting that  $W_p^{[q]}(\sigma_* \mu_1, \sigma_* \mu_2) = W_p^{[q]}(\mu_1, \mu_2)$  for any  $\sigma \in \mathfrak{S}_N$  because the distance  $d_{E^N}^{[q]}$  defined by (??) is itself symmetric and because, for any  $\Pi$  coupling  $\mu_1$  and  $\mu_2$  and for any  $\sigma \in \mathfrak{S}_N$ ,  $(\sigma \otimes \sigma)_* \Pi$  couples  $\sigma_* \mu_1$  and  $\sigma_* \mu_2$ .  $\square$

### A.1.3 Marginals

Let  $N \in \mathbb{N}^*$  be arbitrary. Given any  $\mu \in \mathcal{P}(E^N)$  and any  $k \in \{1, \dots, N\}$ , the  $k^{\text{th}}$ -order marginal  $\mu_{N:k} \in \mathcal{P}(E^k)$  of  $\mu$  is the image of  $\mu$  under the canonical projection  $\pi_k : E^N = E^k \times E^{N-k} \rightarrow E^k$ .

**Lemma 13.** Given any  $\mu_1, \mu_2 \in \mathcal{P}(E^N)$  and any  $k \in \{1, \dots, N\}$ , we have

$$W_p^{[q]}((\mu_1)_{N:k}, (\mu_2)_{N:k}) \leq W_p^{[q]}(\mu_1, \mu_2). \quad (103)$$

The Wasserstein distance at the left-hand (resp., right-hand) side of (??) is computed with respect to the  $\ell^q$  distance  $d_{E^k}^{[q]}$  (resp.,  $d_{E^N}^{[q]}$ ). We will establish in Lemma ?? in Appendix ?? a stronger estimate when  $\mu_1$  and  $\mu_2$  are symmetric.

*Proof.* Let  $\Pi$  be an optimal coupling between  $\mu_1$  and  $\mu_2$ . Then, obviously,  $(\pi_k)_*\Pi$  couples (maybe not optimally)  $(\pi_k)_*\mu_1 = (\mu_1)_{N:k}$  and  $(\pi_k)_*\mu_2 = (\mu_2)_{N:k}$ . Therefore

$$\begin{aligned} W_p^{[q]}((\mu_1)_{N:k}, (\mu_2)_{N:k})^p &\leq \int_{E^k} d_{E^k}^{[q]}((y_1, \dots, y_k), (y'_1, \dots, y'_k))^p d(\pi_k)_*\Pi((y_1, \dots, y_k), (y'_1, \dots, y'_k)) \\ &\leq \int_{E^N} d_{E^k}^{[q]}(\pi_k(y), \pi_k(y'))^p d\Pi(y, y') \\ &\leq \int_{E^N} d_{E^N}^{[q]}(y, y')^p d\Pi(y, y') = W_p^{[q]}(\mu_1, \mu_2)^p \end{aligned}$$

where we have used that  $d_{E^k}^{[q]}(\pi_k(y), \pi_k(y')) \leq d_{E^N}^{[q]}(y, y')$ .  $\square$

#### A.1.4 Tensor product

Let  $k \in \mathbb{N}^*$ . For every  $i \in \{1, \dots, k\}$ , let  $E_i$  be a Polish space, endowed with a distance  $d_{E_i}$ . We endow the product space  $E_1 \times \dots \times E_k$  with the distance

$$d_{E_1 \times \dots \times E_k}^{[q]}(y, y') = \|(d_{E_1}(y_1, y'_1), \dots, d_{E_k}(y_k, y'_k))\|_{\ell^q} = \begin{cases} \left( \sum_{i=1}^k d_{E_i}(y_i, y'_i)^q \right)^{1/q} & \text{if } 1 \leq q < +\infty \\ \max_{1 \leq i \leq k} d_{E_i}(y_i, y'_i) & \text{if } q = +\infty \end{cases} \quad (104)$$

for all  $y = (y_1, \dots, y_k)$ ,  $y' = (y'_1, \dots, y'_k) \in E_1 \times \dots \times E_k$ .

**Lemma 14.** *Given any  $\mu_1, \mu'_1 \in \mathcal{P}(E_1)$ ,  $\dots$ ,  $\mu_k, \mu'_k \in \mathcal{P}(E_k)$ , we have, for every  $j \in \{1, \dots, k\}$ ,*

$$W_p(\mu_j, \mu'_j) \leq W_p^{[q]} \left( \bigotimes_{i=1}^k \mu_i, \bigotimes_{i=1}^k \mu'_i \right) \leq \max(k^{\frac{1}{q} - \frac{1}{p}}, 1) \left( \sum_{i=1}^k W_p(\mu_i, \mu'_i)^p \right)^{1/p} \quad (105)$$

and the right-hand side inequality in (??) is an equality if  $p = q$ .

Taking  $E_i = E$ ,  $d_{E_i} = d_E$ ,  $\mu_i = \mu$  and  $\mu'_i = \mu'$  for every  $i \in \{1, \dots, n\}$ , we have the slightly stronger inequality

$$W_p^{[q]}(\mu^{\otimes k}, (\mu')^{\otimes k}) \leq k^{1/q} W_p(\mu, \mu') \quad (106)$$

and the inequality is an equality if  $p = q$ .

Lemma ?? can be found in [?].

The Wasserstein distance  $W_p$  at the left-hand side of (??) is computed with respect to the distance  $d_{E_j}$ . The Wasserstein distance  $W_p^{[q]}$  in the middle of (??) is computed with respect to the distance  $d_{E_1 \times \dots \times E_k}^{[q]}$  defined by (??).

The Wasserstein distance  $W_p^{[q]}$  at the left-hand side of (??) is computed with respect to the distance  $d_{E^k}^{[q]}$  defined by (??). Recall that  $q \in [1, +\infty]$  has been chosen arbitrarily to define this distance. At the right-hand side of (??), if  $q = +\infty$  then  $k^{1/q} = 1$ .

**Remark 16.** As a particular case of (??), taking  $k = 2$  and  $\mu_2 = \mu'_2 = \mu$ , we have

$$W_p(\mu_1, \mu'_1) \leq W_p^{[q]}(\mu_1 \otimes \mu, \mu'_1 \otimes \mu) = W_p^{[q]}(\mu \otimes \mu_1, \mu \otimes \mu'_1) \leq \max(2^{\frac{1}{q} - \frac{1}{p}}, 1) W_p(\mu_1, \mu'_1).$$

In particular, if  $p \leq q$  then  $W_p(\mu_1, \mu'_1) = W_p^{[q]}(\mu_1 \otimes \mu, \mu'_1 \otimes \mu) = W_p^{[q]}(\mu \otimes \mu_1, \mu \otimes \mu'_1)$ .

*Proof.* We have  $W_p(\mu_j, \mu'_j) \leq W_p^{[q]} \left( \bigotimes_{i=1}^k \mu_i, \bigotimes_{i=1}^k \mu'_i \right)$  for every  $i \in \{1, \dots, k\}$ : this is proved like in Lemma ?? because  $\mu_j$  is the marginal on  $E_j$  of the measure  $\bigotimes_{i=1}^k \mu_i$  on  $E$ , and similarly for  $\mu'_j$ . Therefore the left-hand side inequality in (??) follows.

Let us now establish the right-hand side inequality in (??), for  $q < +\infty$ . For every  $i \in \{1, \dots, k\}$ , let  $\Pi_i$  be an optimal coupling between  $\mu_i$  and  $\mu'_i$ . Then, obviously,  $\Pi = \bigotimes_{i=1}^k \Pi_i$  couples (maybe not optimally)  $\bigotimes_{i=1}^k \mu_i$  and  $\bigotimes_{i=1}^k \mu'_i$ . Therefore

$$W_p^{[q]} \left( \bigotimes_{i=1}^k \mu_i, \bigotimes_{i=1}^k \mu'_i \right)^p \leq \int_{E_1 \times E_1} \cdots \int_{E_k \times E_k} \left( \sum_{i=1}^k d_{E_i}(y_i, y'_i)^q \right)^{p/q} d\Pi_k(y_k, y'_k) \cdots d\Pi_1(y_1, y'_1).$$

If  $p \geq q$ , using the convexity inequality  $(|a_1| + \cdots + |a_k|)^r \leq k^{r-1} (|a_1|^r + \cdots + |a_k|^r)$  for  $r \geq 1$  (with equality for  $r = 1$ ), we obtain

$$W_p^{[q]} \left( \bigotimes_{i=1}^k \mu_i, \bigotimes_{i=1}^k \mu'_i \right)^p \leq k^{\frac{p}{q}-1} \sum_{i=1}^k W_p(\mu_i, \mu'_i)^p$$

and the inequality is an equality if  $p = q$  because in this case  $\Pi$  is an optimal coupling. If  $p \leq q$ , using the inequality  $(|a_1| + \cdots + |a_k|)^{1/r} \leq |a_1|^{1/r} + \cdots + |a_k|^{1/r}$  for  $r \geq 1$ , we obtain

$$W_p^{[q]} \left( \bigotimes_{i=1}^k \mu_i, \bigotimes_{i=1}^k \mu'_i \right)^p \leq \sum_{i=1}^k W_p(\mu_i, \mu'_i)^p.$$

All in all, we have established (??).

To prove (??), using the definition (??) of  $W_p$ , we note that

$$W_p^{[q]}(\mu^{\otimes k}, (\mu')^{\otimes k})^p \leq \mathbb{E} \left( \sum_{i=1}^k d_E(Y, Y')^q \right)^{p/q} = k^{p/q} \mathbb{E} d_E(Y, Y')^p = k^{p/q} W_p(\mu, \mu')^p$$

where  $Y$  and  $Y'$  are random variables (with values in  $E$ ) of laws  $\mu$  and  $\mu'$ , such that  $W_p(\mu, \mu')^p = \mathbb{E} d_E(Y, Y')^p$ .  $\square$

### A.1.5 Diameter of the support

**Lemma 15.** *Given any  $\mu_1, \mu_2 \in \mathcal{P}_c(E)$ , we have*

$$W_p(\mu_1, \mu_2) \leq \text{diam}_E(\text{supp}(\mu_1) \cup \text{supp}(\mu_2)) = \max\{d_E(y, y') \mid y, y' \in \text{supp}(\mu_1) \cup \text{supp}(\mu_2)\}.$$

*Proof.* By (??), since  $W_p(\mu_1, \mu_2)^p$  is the infimum of  $\int_{E^2} d_E(y, y')^p d\Pi(y, y')$  over all probability measures  $\Pi$  on  $E^2$  coupling  $\mu_1$  and  $\mu_2$ , we have  $W_p(\mu_1, \mu_2) \leq \max\{d_E(y_1, y_2) \mid y_1 \in \text{supp}(\mu_1), y_2 \in \text{supp}(\mu_2)\}$ , and the result follows.  $\square$

### A.1.6 Propagation

In this section, we assume that  $E$  is a Banach space, endowed with a norm  $\|\cdot\|_E$ . Let also  $\Lambda$  (space of parameters) be a Polish space, endowed with a distance  $d_\Lambda$ . The space  $\Lambda \times E$  is endowed with the distance  $d_{\Lambda \times E} = d_\Lambda + d_E$ , where  $d_E$  is the distance on  $E$  induced by the norm  $\|\cdot\|_E$ .

**Lemma 16.** For  $i = 1, 2$ , let  $Y^i(t, \lambda, \cdot)$  be a continuous time-varying vector field on  $E$ , depending on the parameter  $\lambda \in \Lambda$ , locally Lipschitz with respect to  $(\lambda, y) \in \Lambda \times E$  uniformly with respect to  $t$  on any compact interval, generating a flow  $(\Phi^i(t, t_0, \lambda, \cdot))_{t \in \mathbb{R}}$  (assumed to be well defined for every  $t \in \mathbb{R}$ ) for any  $t_0 \in \mathbb{R}$ , that is,

$$\begin{aligned}\partial_t \Phi^i(t, t_0, \lambda, y) &= Y^i(t, \lambda, \Phi^i(t, t_0, \lambda, y)) \\ \Phi^i(t_0, t_0, \lambda, y) &= y\end{aligned}$$

for all  $t, t_0 \in \mathbb{R}$ ,  $y \in E$  and  $\lambda \in \Lambda$ . Given any  $t_0 \in \mathbb{R}$  and any  $\mu^1(t_0), \mu^2(t_0) \in \mathcal{P}_c(\Lambda \times E)$ , we set  $\mu_t^i = \mu^i(t) = \Phi^i(t, t_0)_* \mu^i(t_0)$  for every  $t \geq t_0$ , for  $i = 1, 2$ ; this notation means, denoting by  $\nu^i$  the (constant in time) marginal of  $\mu^i(t)$  on  $\Lambda$  and disintegrating  $\mu_t^i = \int_{\Lambda} \mu_{t,\lambda}^i d\nu^i(\lambda)$ , that  $\mu_{t,\lambda}^i = \Phi^i(t, t_0, \lambda, \cdot)_* \mu^i(t_0)$  for  $\nu^i$ -almost every  $\lambda \in \Lambda$ . For every  $p \in [1, +\infty)$ , we have

$$W_p(\mu^1(t), \mu^2(t)) \leq e^{(t-t_0)L([t_0,t])} W_p(\mu^1(t_0), \mu^2(t_0)) + M([t_0, t]) \frac{e^{(t-t_0)L([t_0,t])} - 1}{L([t_0, t])} \quad (107)$$

for every  $t \geq t_0$ , where<sup>8</sup>

$$L([t_0, t]) = \max_{t_0 \leq \tau \leq t} \text{Lip}(Y^1(\tau, \cdot, \cdot)|_{S(\tau)}), \quad (108)$$

$$S(t) = (\text{supp}(\nu^1) \cup \text{supp}(\nu^2)) \times \Phi^1(t, t_0, \text{supp}(\mu^1(t_0)) \cup \text{supp}(\mu^2(t_0))) \cup \text{supp}(\mu^2(t)),$$

$$M([t_0, t]) = \max\{\|Y^1(\tau, \lambda, y) - Y^2(\tau, \lambda, y)\|_E \mid t_0 \leq \tau \leq t, (\lambda, y) \in \text{supp}(\mu^2(\tau))\}. \quad (109)$$

Alternatively, the second term at the right-hand side of (??) can be replaced by

$$M_p([t_0, t]) (t - t_0)^{1/p} \left( \frac{e^{(t-t_0)p'L([t_0,t])} - 1}{p'L([t_0, t])} \right)^{1/p'} \quad (110)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$  and

$$M_p([t_0, t]) = \max_{t_0 \leq \tau \leq t} \left( \int_{\Lambda \times E} \|Y^1(\tau, \lambda, y) - Y^2(\tau, \lambda, y)\|_E^p d\mu_\tau^2(\lambda, y) \right)^{1/p}. \quad (111)$$

Some remarks are in order:

- In (??) (and in (??)), it is understood that if  $L([t_0, t]) = 0$  then  $\frac{e^{(t-t_0)L([t_0,t])} - 1}{L([t_0,t])}$  is replaced by  $t - t_0$ . Lemma ?? extends [?, Proposition 4] to the case with parameters and to the local Lipschitz case; also, the alternative (not usual) estimate with (??) is useful to derive some results of this paper.
- If  $Y^1 = Y^2$  then  $M(\cdot) = 0$ .
- When  $t_0 = 0$ , we denote  $\Phi^i(t, \lambda, y) = \Phi^i(t, 0, \lambda, y)$ ,  $L(t) = L([0, t])$  and  $M(t) = M([0, t])$ .
- Finally, it is interesting to observe that, in Lemma ??, actually only the first vector field  $Y^1$  is required to be locally Lipschitz. Concerning the second, it is only required that  $Y^2$  is regular enough so that (??) is well defined, and also that the flow  $\Phi^2$  is well defined.

*Proof.* Given any  $(\lambda_1, y_1) \in \text{supp}(\mu^1(t_0))$  and  $(\lambda_2, y_2) \in \text{supp}(\mu^2(t_0))$ , using (??) we have

$$\begin{aligned}& \partial_t \|\Phi^1(t, t_0, \lambda_1, y_1) - \Phi^1(t, t_0, \lambda_2, y_2)\|_E \\ & \leq \|Y^1(t, \lambda_1, \Phi^1(t, t_0, \lambda_1, y_1)) - Y^1(t, \lambda_2, \Phi^1(t, t_0, \lambda_2, y_2))\|_E \\ & \leq L([t_0, t]) (d_\Lambda(\lambda_1, \lambda_2) + \|\Phi^1(t, t_0, \lambda_1, y_1) - \Phi^1(t, t_0, \lambda_2, y_2)\|_E)\end{aligned}$$

<sup>8</sup>Note that  $S(t)$  is compact and that  $\Phi^i(t, t_0, \text{supp}(\mu^i(t_0))) = \text{supp}(\mu^i(t))$ .

because  $\Phi^1(t, t_0, \lambda_1, y_1) \in \Phi^1(t, t_0, \text{supp}(\mu^1(t_0)))$  and  $\Phi^1(t, t_0, \lambda_2, y_2) \in \Phi^1(t, t_0, \text{supp}(\mu^2(t_0)))$  (this motivates the definition of  $S(t)$ ), and by integration we get that

$$d_\Lambda(\lambda_1, \lambda_2) + \|\Phi^1(t, t_0, \lambda_1, y_1) - \Phi^1(t, t_0, \lambda_2, y_2)\|_E \leq e^{(t-t_0)L([t_0, t])} (d_\Lambda(\lambda_1, \lambda_2) + \|y_1 - y_2\|_E) \quad (112)$$

for every  $t \geq t_0$ , for  $i = 1, 2$  (we have used the fact that  $t \mapsto L([t_0, t])$  is nondecreasing).

Taking an optimal coupling  $\Pi_{t_0} \in \mathcal{P}((\Lambda \times E)^2)$  between  $\mu^1(t_0)$  and  $\mu^2(t_0)$ , the probability measure  $\Pi_t = (\Phi^1(t, t_0) \otimes \Phi^2(t, t_0))_* \Pi_{t_0}$  couples (maybe not optimally)  $\mu^1(t)$  with  $\mu^2(t)$ .<sup>9</sup> Therefore, using the definition (??) of  $W_p$ ,

$$\begin{aligned} & W_p(\mu^1(t), \mu^2(t))^p \\ & \leq \int_{(\Lambda \times E)^2} (d_\Lambda(\lambda_1, \lambda_2) + \|y_1 - y_2\|_E)^p d\Pi_t(\lambda_1, y_1, \lambda_2, y_2) \\ & = \int_{(\Lambda \times E)^2} (d_\Lambda(\lambda_1, \lambda_2) + \|\Phi^1(t, t_0, \lambda_1, y_1) - \Phi^2(t, t_0, \lambda_2, y_2)\|_E)^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \\ & \leq \int_{(\Lambda \times E)^2} \left( d_\Lambda(\lambda_1, \lambda_2) + \|\Phi^1(t, t_0, \lambda_1, y_1) - \Phi^1(t, t_0, \lambda_2, y_2)\|_E \right. \\ & \quad \left. + \|\Phi^1(t, t_0, \lambda_2, y_2) - \Phi^2(t, t_0, \lambda_2, y_2)\|_E \right)^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \end{aligned}$$

and thus, and using the triangular inequality in  $L^p$ , we get

$$\begin{aligned} & W_p(\mu^1(t), \mu^2(t)) \\ & \leq \left( \int_{(\Lambda \times E)^2} (d_\Lambda(\lambda_1, \lambda_2) + \|\Phi^1(t, t_0, \lambda_1, y_1) - \Phi^1(t, t_0, \lambda_2, y_2)\|_E)^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \right)^{1/p} \\ & \quad + \left( \int_{(\Lambda \times E)^2} \|\Phi^1(t, t_0, \lambda_2, y_2) - \Phi^2(t, t_0, \lambda_2, y_2)\|_E^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \right)^{1/p} \end{aligned} \quad (113)$$

Using (??), the first term of the sum at the right-hand side of (??) is less than or equal to

$$\begin{aligned} & e^{(t-t_0)L([t_0, t])} \left( \int_{(\Lambda \times E)^2} (d_\Lambda(\lambda_1, \lambda_2) + \|y_1 - y_2\|_E)^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \right)^{1/p} \\ & = e^{(t-t_0)L([t_0, t])} W_p(\mu^1(t_0), \mu^2(t_0)), \end{aligned}$$

the latter equality being because  $\Pi_{t_0}$  is an optimal coupling between  $\mu^1(t_0)$  and  $\mu^2(t_0)$ .

To treat the second term, we first observe that, for  $(\lambda, y) \in \text{supp}(\mu^2(t_0))$ ,

$$\begin{aligned} \partial_t \|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E & \leq \|Y^1(t, \lambda, \Phi^1(t, t_0, \lambda, y)) - Y^1(t, \lambda, \Phi^2(t, t_0, \lambda, y))\|_E \\ & \quad + \|Y^1(t, \lambda, \Phi^2(t, t_0, \lambda, y)) - Y^2(t, \lambda, \Phi^2(t, t_0, \lambda, y))\|_E \\ & \leq L([t_0, t]) \|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E \\ & \quad + \|Y^1(t, \lambda, \Phi^2(t, t_0, \lambda, y)) - Y^2(t, \lambda, \Phi^2(t, t_0, \lambda, y))\|_E \end{aligned}$$

where we have used (??), noting that  $(\lambda, \Phi^1(t, t_0, \lambda, y)) \in S(t)$  and  $(\lambda, \Phi^2(t, t_0, \lambda, y)) \in S(t)$ , and thus, using the Gronwall lemma and the fact that  $\tau \mapsto L([t_0, \tau])$  is nondecreasing,

$$\begin{aligned} & \|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E \\ & \leq \int_{t_0}^t e^{(t-\tau)L([t_0, t])} \|Y^1(\tau, \lambda, \Phi^2(\tau, t_0, \lambda, y)) - Y^2(\tau, \lambda, \Phi^2(\tau, t_0, \lambda, y))\|_E d\tau. \end{aligned} \quad (114)$$

<sup>9</sup>Indeed, denoting by  $\pi_i$  the projection of  $(\Lambda \times E)^2$  onto the  $i^{\text{th}}$ -copy of  $\Lambda \times E$ , we have  $\pi_i \circ (\Phi^1 \otimes \Phi^2) = \Phi^i \circ \pi_i$ .

Using the definition (??) of  $M([t_0, t])$  and the fact that  $\Phi^2(\tau, t_0, \text{supp}(\mu^2(t_0))) = \text{supp}(\mu^2(\tau))$ , we get

$$\|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E \leq M([t_0, t]) \frac{e^{(t-t_0)L([t_0, t])} - 1}{L([t_0, t])}.$$

Therefore, the second term of the sum at the right-hand side of (??) is estimated by

$$\begin{aligned} & \left( \int_{(\Lambda \times E)^2} \|\Phi^1(t, t_0, \lambda_2, y_2) - \Phi^2(t, t_0, \lambda_2, y_2)\|_E^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \right)^{1/p} \\ &= \left( \int_{\Lambda \times E} \|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E^p d\mu_{t_0}^2(\lambda, y) \right)^{1/p} \leq M([t_0, t]) \frac{e^{(t-t_0)L([t_0, t])} - 1}{L([t_0, t])} \end{aligned}$$

where we have used that the second marginal of  $\Pi_{t_0}$  is  $\mu_{t_0}^2 = \mu^2(t_0)$ . The estimate (??) follows.

To obtain the alternative estimate with the term (??), we apply the Hölder inequality to the right-hand side of (??), obtaining

$$\begin{aligned} & \|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E \\ & \leq \left( \frac{e^{p'(t-t_0)L([t_0, t])} - 1}{p'L([t_0, t])} \right)^{1/p'} \left( \int_{t_0}^t \|Y^1(\tau, \lambda, \Phi^2(\tau, t_0, \lambda, y)) - Y^2(\tau, \lambda, \Phi^2(\tau, t_0, \lambda, y))\|_E^p d\tau \right)^{1/p}. \end{aligned}$$

Therefore, the second term of the sum at the right-hand side of (??) is estimated by

$$\begin{aligned} & \left( \int_{(\Lambda \times E)^2} \|\Phi^1(t, t_0, \lambda_2, y_2) - \Phi^2(t, t_0, \lambda_2, y_2)\|_E^p d\Pi_{t_0}(\lambda_1, y_1, \lambda_2, y_2) \right)^{1/p} \\ &= \left( \int_{\Lambda \times E} \|\Phi^1(t, t_0, \lambda, y) - \Phi^2(t, t_0, \lambda, y)\|_E^p d\mu_{t_0}^2(\lambda, y) \right)^{1/p} \\ & \leq \left( \frac{e^{p'(t-t_0)L([t_0, t])} - 1}{p'L([t_0, t])} \right)^{1/p'} \left( \int_{t_0}^t \int_{\Lambda \times E} \|Y^1(\tau, \lambda, \Phi^2(\tau, t_0, \lambda, y)) \right. \\ & \qquad \qquad \qquad \left. - Y^2(\tau, \lambda, \Phi^2(\tau, t_0, \lambda, y))\|_E^p d\mu_{t_0}^2(\lambda, y) d\tau \right)^{1/p} \\ & \leq \left( \frac{e^{p'(t-t_0)L([t_0, t])} - 1}{p'L([t_0, t])} \right)^{1/p'} \left( \int_{t_0}^t \int_{\Lambda \times E} \|Y^1(\tau, \lambda, y) - Y^2(\tau, \lambda, y)\|_E^p d\mu_{\tau}^2(\lambda, y) d\tau \right)^{1/p} \\ & \leq \left( \frac{e^{p'(t-t_0)L([t_0, t])} - 1}{p'L([t_0, t])} \right)^{1/p'} (t - t_0)^{1/p} M_p([t_0, t]) \end{aligned}$$

The lemma is proved.  $\square$

**Lemma 17.** *Let  $Y(t, \lambda, \cdot)$  be a continuous time-varying vector field on  $E$ , depending on the parameter  $\lambda \in \Lambda$ , locally Lipschitz with respect to  $y \in E$  uniformly with respect to  $(t, \lambda)$  on any compact, generating a flow  $(\Phi(t, t_0, \lambda, \cdot))_{t \in \mathbb{R}}$  (assumed to be well defined for every  $t \in \mathbb{R}$ ) for any  $t_0 \in \mathbb{R}$  (as in Lemma ??). Given any  $t_0 \in \mathbb{R}$  and any  $\mu_{t_0} \in \mathcal{P}_c(\Lambda \times E)$ , we set  $\mu(t) = \Phi(t, t_0)_* \mu_{t_0}$  for every  $t \geq t_0$ . For every  $p \in [1, +\infty)$ , we have*

$$W_p(\mu(t), \mu(t_0)) \leq M([t_0, t]) |t - t_0| \quad \forall t \geq t_0$$

where  $M([t_0, t]) = \max \{ \|Y(\tau, \lambda, y)\| \mid t_0 \leq \tau \leq t, (\lambda, y) \in \text{supp}(\mu(\tau)) \}$ .

*Proof.* It would suffice to apply Lemma ?? with  $Y^1 = 0$  and  $Y^2 = Y$ , if  $Y$  were also Lipschitz with respect to  $\lambda$  (or, to adapt this lemma to vector fields that depend only continuously on  $\lambda$ ). Without this assumption, let us give a quick proof. We first establish the following general result.

**Lemma 18.** *Let  $F$  be a Polish space, endowed with a distance  $d_F$ , let  $\mu \in \mathcal{P}_c(F)$  and let  $\phi : F \rightarrow F$  be a measurable mapping. For every  $p \in [1, +\infty)$ , we have*

$$W_p(\phi_*\mu, \mu) \leq \left( \int_F d_F(y, \phi(y))^p d\mu(y) \right)^{1/p}$$

*Proof of Lemma ??.* With a slight abuse of notation, we define  $\Pi \in \mathcal{P}(F \times F)$  by  $\Pi(y, y') = \mu(y) \delta_{y'=\Phi(y)}$ . Since we also have  $\Pi = (\Phi_*\mu)(y') \delta_{y'=\Phi(y)}$ , it follows that  $\Pi$  couples  $\mu$  and  $\phi_*\mu$ . Therefore  $W_p(\phi_*\mu, \mu)^p \leq \int_{F^2} d_F(y, y')^p d\Pi(y, y') = \int_F d_F(y, \phi(y))^p d\mu(y)$ . Lemma ?? is proved.  $\square$

Applying Lemma ?? with  $F = \Lambda \times E$ ,  $\mu = \mu_{t_0}$  and  $\phi = \Phi(t, t_0)$ , we have

$$W_p(\mu(t), \mu(t_0))^p \leq \int_{\Lambda \times E} d_{\Lambda \times E}((\lambda, y), (\lambda, \Phi(t, t_0, \lambda, y)))^p d\mu_{t_0}(\lambda, y)$$

and we note that  $d_{\Lambda \times E}((\lambda, y), (\lambda, \Phi(t, t_0, \lambda, y))) = \|\Phi(t, t_0, \lambda, y) - y\|$ . Now, since  $\Phi(t, t_0, \lambda, y) = y + \int_{t_0}^t Y(\tau, \lambda, \Phi(\tau, t_0, \lambda, y)) d\tau$  and  $\text{supp}(\mu(t)) = \Phi(t, t_0, \text{supp}(\mu_{t_0}))$ , Lemma ?? easily follows.  $\square$

### A.1.7 Moment of order one

Let  $\Omega$  be a Polish space and let  $d \in \mathbb{N}^*$ .

**Lemma 19.** *For  $i = 1, 2$ , let  $\mu_i \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , disintegrated as  $\mu_i = \int_{\Omega} (\mu_i)_x d\nu_i(x)$  with respect to its marginal  $\nu_i$  on  $\Omega$ , and let  $y_i$  be the moment of order one of  $\mu_i$ , defined by  $y_i(x) = \int_{\mathbb{R}^d} \xi d(\mu_i)_x(\xi)$  for  $\nu_i$ -almost every  $x \in \Omega$ . Then*

$$W_1(y_1\nu_1, y_2\nu_2) \leq W_1(\mu_1, \mu_2).$$

*Proof.* By the definition (??) of  $W_1$ , we have  $W_1(y_1\nu_1, y_2\nu_2) = \int_{\Omega} y_1 g d\nu_1 - \int_{\Omega} y_2 g d\nu_2$  for some  $g \in \text{Lip}(\Omega)$  such that  $\text{Lip}(g) \leq 1$ . Hence  $W_1(y_1\nu_1, y_2\nu_2) = \int_{\Omega \times \mathbb{R}^d} f d(\mu_1 - \mu_2)$  where  $f(x, \xi) = g(x)\xi$ . Since  $f \in \text{Lip}(\Omega \times \mathbb{R}^d)$  and  $\text{Lip}(f) \leq 1$ , the result follows.  $\square$

## A.2 More precise facts on the marginals of a symmetrization

Let  $N \in \mathbb{N}^*$  be arbitrary. Recall that the symmetrization of a measure is defined by (??) (see Appendix ??).

### A.2.1 First marginal of the symmetrization

For every  $i \in \{1, \dots, N\}$ , we denote by  $p^i$  the projection of  $E^N$  onto the  $i^{\text{th}}$  copy of  $E$ , i.e., in coordinates,  $p^i(y) = y_i$ .

**Lemma 20.** *Let  $\mu \in \mathcal{P}(E^N)$  be arbitrary.*

- *The first marginal  $\mu_{N:1}^s = p_*^1 \mu^s$  of the symmetrization  $\mu^s$  of  $\mu$  is given by*

$$\mu_{N:1}^s = \frac{1}{N} \sum_{i=1}^N p_*^i \mu$$

*where  $p_*^i \mu$  is the image of  $\mu$  under the projection  $p^i$ . In other words,  $\mu_{N:1}^s$  is the average of the marginals of  $\mu$  on the copies of  $E$ .*

- We have  $p_*^i \mu^s = \frac{1}{N} \sum_{j=1}^N p_*^j \mu^s$  for every  $i \in \{1, \dots, N\}$  and thus  $p_*^i \mu^s$  does not depend on  $i$ .

In other words, the marginals of a symmetric measure on the copies of  $E$  are all equal; the same is true for the marginals of higher order.

*Proof.* Given any  $f \in \mathcal{C}_c^0(E)$ , we have

$$\begin{aligned} \langle \mu_{N:1}^s, f \rangle &= \langle p_*^1 \mu^s, f \rangle = \langle \mu^s, (p^1)^* f \rangle = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \langle \sigma_* \mu, (p^1)^* f \rangle = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \langle \mu, \sigma^* (p^1)^* f \rangle \\ &= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{E^N} f \circ p^1(\sigma \cdot y) d\mu(y) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{E^N} f(y_{\sigma(1)}) d\mu(y) \end{aligned}$$

When designing a permutation  $\sigma \in \mathfrak{S}_N$ , we have  $N$  choices for  $\sigma(1)$ , among  $\{1, \dots, N\}$ , and the rest is a permutation of  $N - 1$  elements. Since  $\text{card}(\mathfrak{S}_{N-1}) = (N - 1)!$ , we get that

$$\langle \mu_{N:1}^s, f \rangle = \frac{1}{N} \sum_{i=1}^N \int_{E^N} f(y_i) d\mu(y) = \frac{1}{N} \sum_{i=1}^N \int_{E^N} f \circ p^i(y) d\mu(y) = \frac{1}{N} \sum_{i=1}^N \langle p_*^i \mu, f \rangle$$

whence the first item.

The second item is proved in the same way, replacing  $\mu$  by  $\mu^s$ .  $\square$

## A.2.2 Marginals of symmetric measures

We have seen in Lemma ?? (Appendix ??) that  $W_p^{[q]}((\mu_1)_{N:k}, (\mu_2)_{N:k}) \leq W_p^{[q]}(\mu_1, \mu_2)$ , for every  $k \in \{1, \dots, N\}$ , for any  $\mu_1, \mu_2 \in \mathcal{P}(E^N)$ . We have a stronger estimate when  $\mu_1$  and  $\mu_2$  are symmetric, i.e., when  $\mu_1 = \mu_1^s$  and  $\mu_2 = \mu_2^s$ .

**Lemma 21.** *Let  $\mu_1, \mu_2 \in \mathcal{P}(E^N)$  be symmetric measures. Then, for any  $k \in \{1, \dots, N\}$ ,*

$$W_p^{[q]}((\mu_1)_{N:k}, (\mu_2)_{N:k}) \leq \left( \frac{k}{N} \right)^{1/q} W_p^{[q]}(\mu_1, \mu_2). \quad (115)$$

In (??), the  $W_p^{[q]}$  distances are computed with respect to the  $\ell^q$  distances  $d_{E^k}^{[q]}$  and  $d_{E^N}^{[q]}$  defined by (??).

*Proof.* Assume that  $q < +\infty$  (for  $q = +\infty$ , it suffices to take limits). Let  $\Pi \in \mathcal{P}((E^N)^2)$  be an optimal coupling between  $\mu_1$  and  $\mu_2$  for the  $W_p^{[q]}$  distance, i.e., using the definitions (??) and (??) of  $W_p^{[q]}$ ,

$$W_p^{[q]}(\mu_1, \mu_2)^p = \int_{(E^N)^2} \left( \sum_{i=1}^N d_E(y_i^1, y_i^2) \right)^{p/q} d\Pi(y^1, y^2) = \mathbb{E} \left( \sum_{i=1}^N d_E(Y_i^1, Y_i^2) \right)^{p/q}$$

where  $y^j = (y_1^j, \dots, y_N^j)$  for  $j = 1, 2$ , and where the  $Y_i^j$  are random variables of laws the respective marginals of  $\Pi$ . Using that the cost is symmetric, without loss of generality we assume that  $\Pi$  is symmetric, i.e.,  $\Pi = (\sigma \otimes \sigma)_* \Pi$  for every  $\sigma \in \mathfrak{S}_N$  (the symmetrization is performed in each copy  $E^N$  of  $(E^N)^2$ ). By an obvious adaptation of Lemma ?? in Appendix ??, the marginals of  $\Pi$  (which are, accordingly, probability measures on  $E^2$ , by considering the product of the  $i^{\text{th}}$  copy of  $E$  with



the  $i^{\text{th}}$  copy of  $E$ ) are all equal. The same is true for the marginals of higher order. It follows that  $Y_i^1$  and  $Y_i^2$  do not depend on  $i$  and thus

$$\begin{aligned} W_p^{[q]}(\mu_1, \mu_2)^p &= N^{p/q} \mathbb{E} d_E(Y_1^1, Y_1^2)^p = N^{p/q} \int_{E^2} d_E(y_1^1, y_1^2)^p d\Pi_{N:1}(y_1^1, y_1^2) \\ &= \left(\frac{N}{k}\right)^{p/q} \mathbb{E} \left( \sum_{i=1}^k d_E(Y_i^1, Y_i^2)^q \right)^{p/q} = \left(\frac{N}{k}\right)^{p/q} \int_{(E^k)^2} d_{E^k}^{[q]}(y^1, y^2)^p d\Pi_{N:k}(y^1, y^2) \end{aligned}$$

for every  $k \in \{1, \dots, N\}$ . The latter quantity is greater than or equal to  $W_p^{[q]}((\mu_1)_{N:k}, (\mu_2)_{N:k})^p$  because  $\Pi_{N:k}$  couples  $(\mu_1)_{N:k}$  and  $(\mu_2)_{N:k}$ . The lemma follows.  $\square$

### A.2.3 A technical lemma towards propagation of chaos

**Lemma 22.** *Let  $\mu_1, \dots, \mu_N \in \mathcal{P}(E)$ , and let  $\rho \in \mathcal{P}(E^N)$  be defined by*

$$\rho = \mu_1 \otimes \dots \otimes \mu_N.$$

*The symmetrization of  $\rho$  is given by*

$$\rho^s = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \mu_{\sigma(1)} \otimes \dots \otimes \mu_{\sigma(N)}. \quad (116)$$

*The first marginal  $\rho_{N:1}^s \in \mathcal{P}(E)$  of  $\rho^s$  is*

$$\rho_{N:1}^s = \frac{1}{N} \sum_{i=1}^N \mu_i \quad (117)$$

*and, for every  $k \in \{2, \dots, N\}$ , its  $k^{\text{th}}$ -order marginal  $\rho_{N:k}^s \in \mathcal{P}(E)$  is*

$$\rho_{N:k}^s = (1 + \varepsilon_k) (\rho_{N:1}^s)^{\otimes k} - \varepsilon_k \beta_k \quad (118)$$

*where*

$$\varepsilon_k = \frac{N^k (N-k)!}{N!} - 1 \in \left[0, e^{\frac{k^2}{2N}} - 1\right] \quad (119)$$

*and*

$$\beta_k = \frac{1}{\varepsilon_k} \frac{(N-k)!}{N!} \sum \mu_{i_1} \otimes \dots \otimes \mu_{i_k} \in \mathcal{P}(E^k) \quad (120)$$

*where the sum in (??) is taken over all  $k$ -tuples  $(i_1, \dots, i_k) \in \{1, \dots, N\}^k$  for which at least two elements are equal. For every  $p \in [1, +\infty)$ , for every  $k \in \mathbb{N}^*$  such that  $k^2 \leq 2N \ln(1 + \frac{1}{2p})$ , we have*

$$W_p^{[q]}(\rho_{N:k}^s, (\rho_{N:1}^s)^{\otimes k}) \leq 2 \left(\frac{k^2}{N}\right)^{1/p} W_p^{[q]}((\rho_{N:1}^s)^{\otimes k}, \beta_k) \quad (121)$$

*and therefore, assuming moreover that  $\mu_1, \dots, \mu_N \in \mathcal{P}_c(E)$ ,*

$$W_p^{[q]}(\rho_{N:k}^s, (\rho_{N:1}^s)^{\otimes k}) \leq 2k^{1/q} \left(\frac{k^2}{N}\right)^{1/p} \text{diam}_E \left( \bigcup_{i=1}^N \text{supp}(\mu_i) \right). \quad (122)$$

In (??) and (??), the Wasserstein distance  $W_p^{[q]}$  is computed with respect to the  $\ell^q$  distance  $d_{E^k}^{[q]}$ . The estimate (??) is used several times in the proofs of our main results, in an instrumental way to express that, for  $N$  large, the  $k^{\text{th}}$ -order marginal  $\rho_{N:k}^s$  of the symmetric measure  $\rho^s$  is close to the tensor power  $(\rho_{N:1}^s)^{\otimes k}$ , with an error that is precisely estimated.

The first part of the lemma, in particular the formulas (??) and (??), are certainly known by experts and can be found, e.g., in [?, Section 3], for Dirac measures.

*Proof.* The formula (??) straightforwardly follows from (??), and the formula (??) follows from Lemma ?? in Appendix ?? because  $p_*^i \rho = \mu_i$ .

Let us now compute the  $k^{\text{th}}$ -order marginal  $\rho_{N:k}^s$  of  $\rho^s$ , for every  $k \in \{2, \dots, N\}$ . Let  $I_k^N$  be the set of all  $k$ -tuples  $(i_1, \dots, i_k)$  consisting of distinct integers chosen in  $\{1, \dots, N\}$ . We have  $\text{card}(I_k^N) = \frac{N!}{(N-k)!}$ . Denoting by  $\mathfrak{S}_N^{i_1, \dots, i_k}$  the set of all  $\sigma \in \mathfrak{S}_N$  such that  $(\sigma(1), \dots, \sigma(k)) = (i_1, \dots, i_k)$ , we have  $\text{card}(\mathfrak{S}_N^{i_1, \dots, i_k}) = (N-k)!$ . Now, since

$$\sum_{\sigma \in \mathfrak{S}_N} \mu_{\sigma(1)} \otimes \dots \otimes \mu_{\sigma(N)} = \sum_{(i_1, \dots, i_N) \in I_k^N} \mu_{i_1} \otimes \dots \otimes \mu_{i_k} \otimes \sum_{\sigma \in \mathfrak{S}_N^{i_1, \dots, i_k}} \mu_{\sigma(n+1)} \otimes \dots \otimes \mu_{\sigma(N)}$$

we infer that

$$\rho_{N:k}^s = \frac{(N-k)!}{N!} \sum_{(i_1, \dots, i_k) \in I_k^N} \mu_{i_1} \otimes \dots \otimes \mu_{i_k}. \quad (123)$$

Now, writing  $I_k^N = \{1, \dots, N\}^k \setminus (\{1, \dots, N\}^k \setminus I_k^N)$ , we write the sum in (??) as a sum over  $\{1, \dots, N\}^k$  minus a sum over  $\{1, \dots, N\}^k \setminus I_k^N$  (where at least two of the indices are equal). For the first sum, we have

$$\sum_{(i_1, \dots, i_k) \in \{1, \dots, N\}^k} \mu_{i_1} \otimes \dots \otimes \mu_{i_k} = \left( \sum_{i=1}^N \mu_i \right)^{\otimes k} = N^k (\rho_{N:1}^s)^{\otimes k}. \quad (124)$$

We infer from (??) and (??) that

$$\rho_{N:k}^s = \frac{N^k (N-k)!}{N!} (\rho_{N:1}^s)^{\otimes k} - \frac{(N-k)!}{N!} \beta$$

where

$$\beta = \sum_{(i_1, \dots, i_k) \in \{1, \dots, N\}^k \setminus I_k^N} \mu_{i_1} \otimes \dots \otimes \mu_{i_k}$$

is a nonnegative Radon measure of total mass  $|\beta| = \text{card}(\{1, \dots, N\}^k \setminus I_k^N) = N^k - \frac{(N-k)!}{N!}$ . Besides, we have

$$1 \leq \frac{N^k (N-k)!}{N!} = \frac{N^k}{N(N-1) \dots (N-k+1)} = \frac{1}{\prod_{i=1}^{k-1} \left(1 - \frac{i}{N}\right)} \leq e^{\frac{k^2}{2N}}$$

where we have used the inequality

$$\ln \prod_{i=1}^{k-1} \left(1 - \frac{i}{N}\right) = - \sum_{i=1}^{k-1} \ln \left(1 - \frac{i}{N}\right) \geq - \frac{1}{N} \sum_{i=1}^{k-1} i = - \frac{(k-1)k}{2N} \geq - \frac{k^2}{2N}.$$

Therefore, defining  $\varepsilon_k$  by (??) and

$$\beta_k = \frac{1}{\varepsilon_k} \frac{(N-k)!}{N!} \beta \in \mathcal{P}(\mathbb{R}^{dk}),$$

we obtain  $\rho_{N:k}^s = (1 + \varepsilon_k)(\rho_{N:1}^s)^{\otimes k} - \varepsilon_k \beta_k$ , which is (??). Then, applying Lemma ?? in Appendix ??, using that  $\varepsilon_k < 1$  if  $e^{k^2/2N} - 1 < 1$ , or equivalently,  $k^2 < 2N \ln(2)$ , we obtain that

$$W_p^{[q]}((\rho_{N:1}^s)^{\otimes k}, \rho_{N:k}^s) \leq \frac{\varepsilon_k^{1/p}}{1 - \varepsilon_k^{1/p}} W_p^{[q]}((\rho_{N:1}^s)^{\otimes k}, \beta_k) \quad \text{if } k^2 < 2 \ln(2)N. \quad (125)$$

The estimate (??) is now inferred from (??) as follows: if  $k^2 \leq 2N \ln(1 + \frac{1}{2^p})$  then  $e^{k^2/2N} - 1 \leq \frac{1}{2^p}$ , hence  $\varepsilon_k^{1/p} \leq \frac{1}{2}$  (using (??)) and thus  $\frac{\varepsilon_k^{1/p}}{1 - \varepsilon_k^{1/p}} \leq 2\varepsilon_k^{1/p}$ , and it follows from (??) that

$$W_p^{[q]}((\rho_{N:1}^s)^{\otimes k}, \rho_{N:k}^s) \leq 2(e^{\frac{k^2}{2N}} - 1)^{1/p} W_p^{[q]}((\rho_{N:1}^s)^{\otimes k}, \beta_k).$$

Using the inequality  $\frac{e^x - 1}{x} \leq 1/2^p \ln(1 + \frac{1}{2^p}) \leq 2$  for every  $x \in (0, \ln(1 + \frac{1}{2^p})]$ , we obtain (??).

Let us finally establish (??). Using (??) and (??), which express  $(\rho_{N:1}^s)^{\otimes k}$  and  $\beta_k$  as linear combinations, applying two times Lemma ?? in Appendix ?? and then Lemma ?? in Appendix ??, we infer that

$$W_p^{[q]}((\rho_{N:1}^s)^{\otimes k}, \beta_k) \leq \max \left( \sum_{i=1}^k d_E(y_i, y'_i)^q \right)^{1/q} \leq k^{1/q} \text{diam}_E \left( \bigcup_{i=1}^N \text{supp}(\mu_i) \right)$$

where, above, the maximum has been taken over all possible  $y_i, y'_i \in \text{supp}(\mu_i)$ , for  $i \in \{1, \dots, k\}$ . Then, (??) follows from (??) combined with the above inequality.  $\square$

### A.3 Density of empirical measures in the set of probability measures

Let  $E$  be a Polish space, endowed with a distance  $d_E$ . For every  $N \in \mathbb{N}^*$ , let  $Y^N = (y_1^N, \dots, y_N^N) \in E^N$ , and define the *empirical measure*  $\mu_{Y^N}^e \in \mathcal{P}(E)$  by

$$\mu_{Y^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{y_i^N}.$$

The points  $y_i^N$  are not required to be distinct, so that the empirical measure  $\mu_{Y^N}^e$  can equivalently be defined as a *convex combination with rational coefficients of Dirac masses*. Note that

$$\int_E f d\mu_{Y^N}^e = \frac{1}{N} \sum_{i=1}^N f(y_i^N) \quad \forall f \in \mathcal{C}^0(E).$$

A sequence  $(\mu_j)_{j \in \mathbb{N}^*}$  of  $\mathcal{P}(E)$  converges weakly to  $\mu \in \mathcal{P}(E)$  if  $\int_E f d\mu_j \rightarrow \int_E f d\mu$  as  $j \rightarrow +\infty$  for any  $f \in \mathcal{C}_b(E)$  (narrow convergence), where  $\mathcal{C}_b(E)$  is the Banach space of bounded functions on  $E$ .

**Lemma 23.** *When  $E$  is compact, the set  $\{\mu_{Y^N}^e \mid N \in \mathbb{N}^*, Y^N \in E^N\}$  is weakly dense in  $\mathcal{P}(E)$ . In other words, any probability measure on  $E$  is the weak limit of a sequence of empirical measures.*

*Proof.* This is a well known consequence of the Krein-Milman theorem (see, e.g., [?, Lemma 7]). Let us anyway recall a proof. The set  $\mathcal{P}(E)$  is convex and weak star compact, and its extreme points are Dirac masses. The Krein-Milman theorem implies that any  $\mu \in \mathcal{P}(E)$  is the limit of a finite convex combination  $\sum_i \lambda_i \delta_{y_i}$  of Dirac masses. By density of rationals, without loss of generality we can moreover assume that  $\lambda_i \in \mathbb{Q}$ . The statement follows.  $\square$

Recall that the Wasserstein distance  $W_p$  metrizes the weak convergence in  $\mathcal{P}_p(E)$  (which also entails the convergence of first moments), for any  $p \in [1, +\infty)$ . We have then the following variant of the above lemma (see [?, Theorem 6.18]).

**Lemma 24.** *The set  $\{\mu_{Y^N}^e \mid N \in \mathbb{N}^*, Y^N \in E^N\}$  is dense in  $\mathcal{P}_p(E)$  for the Wasserstein distance  $W_p$ . In other words, any  $\mu \in \mathcal{P}_p(E)$  is the limit of a sequence of empirical measures for the Wasserstein distance  $W_p$ .*

*Proof.* It suffices to consider  $R > 0$  sufficiently large such that  $\int_{E \setminus B(y_0, R)} d_E(y_0, y)^p d\mu(y) < \varepsilon$ , for  $\varepsilon > 0$  small enough, so that the argument can be performed in the compact set  $\overline{B}(y_0, R)$ , and the statement readily follows (see also [?, Chap. 5]).  $\square$

There exist a number of results in the literature quantifying the convergence of empirical measures  $\mu_{Y^N}^e$  towards  $\mu \in \mathcal{P}(E)$  and providing rates of convergence, most in a probabilistic context, like [?] where  $Y$  consists of  $N$  random variables having the same distribution as  $\mu$ . In the result hereafter,  $Y$  is deterministic and the rate of convergence is the one obtained by Riemann integration.

**Lemma 25.** *Let  $\mu \in \mathcal{P}_c(E)$  and let  $N \in \mathbb{N}^*$ . We assume that there exists a family of tagged partitions of  $\text{supp}(\mu)$  associated with  $\mu$  (see (??)), i.e., for every  $N \in \mathbb{N}^*$  there exists a partition of  $\text{supp}(\mu) = \cup_{i=1}^N F_i^N$  such that all subsets  $F_i^N$  are  $\mu$ -measurable, pairwise disjoint,  $\mu(F_i^N) = \frac{1}{N}$ , and satisfy  $\text{diam}_E(F_i^N) \leq \frac{C_E}{N^r}$  for some  $C_E > 0$  not depending on  $N$ , and a  $N$ -tuple  $Y^N = (y_1^N, \dots, y_N^N) \in E^N$  such that  $y_i^N \in F_i^N$  for every  $i \in \{1, \dots, N\}$ . Then*

$$W_1(\mu_{Y^N}^e, \mu) \leq \frac{C_E}{N^r}$$

and thus also, using (??),  $W_p(\mu_{Y^N}^e, \mu) \leq \text{diam}_E(\text{supp}(\mu))^{1-1/p} \frac{C_E^{1/p}}{N^{r/p}}$ , for any  $p \in [1, +\infty)$ .

Note that, when  $E$  is a finite-dimensional manifold,  $r = 1/\dim(E)$ .

When one wants that the assumption on the tagged partition be satisfied for any  $N \in \mathbb{N}^*$ , this requires that the mass of  $\mu$  be quite well uniformly distributed; for instance it is satisfied if  $\mu$  is absolutely continuous with respect to a Lebesgue measure with a density that is bounded above and below on  $\text{supp}(\mu)$ . This result is quite obvious and has nothing to see with much deeper and general results like those of [?].

*Proof.* For every  $i \in \{1, \dots, N\}$ , we have  $\int_{F_i^N} f(y_i^N) d\mu(y) = f(y_i^N) \mu(F_i^N) = \frac{1}{N} f(y_i^N)$  because  $\mu(F_i^N) = \frac{1}{N}$  and thus, for every  $f \in \text{Lip}(E)$  such that  $\text{Lip}(f) \leq 1$ ,

$$\begin{aligned} \left| \int_E f d(\mu - \mu_{Y^N}^e) \right| &= \left| \sum_{i=1}^N \int_{F_i^N} f(y) d\mu(y) - \frac{1}{N} \sum_{i=1}^N f(y_i^N) \right| = \left| \sum_{i=1}^N \int_{F_i^N} (f(y) - f(y_i^N)) d\mu(y) \right| \\ &\leq \sum_{i=1}^N \int_{F_i^N} |f(y) - f(y_i^N)| d\mu(y) \leq \sum_{i=1}^N \int_{F_i^N} d_E(y, y_i^N) d\mu(y) \leq \sum_{i=1}^N \mu(F_i^N) \text{diam}_E(F_i^N) \leq \frac{C_E}{N^r} \end{aligned}$$

and the conclusion follows by taking the supremum over all  $f$ .  $\square$

## A.4 Convergence of empirical and semi-empirical measures

Let  $(\Omega, d_\Omega)$  be a complete metric space and let  $\nu \in \mathcal{P}_c(\Omega)$ . We assume that there exists a family of tagged partitions  $(\mathcal{A}^N, X^N)$  of  $\text{supp}(\nu)$  associated with  $\nu$  satisfying (??) (see Section ??), with

$\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$  and  $X^N = (x_1^N, \dots, x_N^N)$ . We define the *empirical measure*  $\nu_{X^N}^e \in \mathcal{P}(\Omega)$  by

$$\nu_{X^N}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}.$$

Note that, when  $\Omega$  is a  $n$ -dimensional manifold, one has  $r = 1/n$  in (??).

#### A.4.1 Convergence of empirical measures on $\Omega$

**Lemma 26.** • *Let  $f$  be a bounded and  $\nu$ -almost everywhere continuous (i.e.,  $\nu$ -Riemann integrable) function on  $\Omega$ , of compact support. Then*

$$\int_{\Omega} f d(\nu - \nu_{X^N}^e) = \int_{\Omega} f d\nu - \frac{1}{N} \sum_{i=1}^N f(x_i^N) = o(1) \quad (126)$$

as  $N \rightarrow +\infty$ . As a consequence,  $\nu_{X^N}^e$  converges weakly to  $\nu$  as  $N \rightarrow +\infty$ ; equivalently,  $W_p(\nu_{X^N}^e, \nu) = o(1)$  as  $N \rightarrow +\infty$ .

- Given any  $\alpha \in (0, 1]$  and any  $N \in \mathbb{N}^*$ , we have

$$\left| \int_{\Omega} f d\nu - \frac{1}{N} \sum_{i=1}^N f(x_i^N) \right| \leq \frac{C_{\Omega}^{\alpha}}{N^{r\alpha}} \text{Hol}_{\alpha}(f) \quad (127)$$

for every  $f \in \mathcal{C}_c^{0,\alpha}(\Omega)$ . As a consequence of (??) for  $\alpha = 1$ , we have

$$W_1(\nu_{X^N}^e, \nu) \leq \frac{C_{\Omega}}{N^r} \quad (128)$$

and thus also, using (??),  $W_p(\nu_{X^N}^e, \nu) \leq \text{diam}_{\Omega}(\text{supp}(\nu))^{1-1/p} \frac{C_{\Omega}^{1/p}}{N^{r/p}}$ , for any  $p \in [1, +\infty)$ .

*Proof.* In the first item, (??) follows from the theorem of convergence of Riemann sums, as already recalled in (??). Interpreted in terms of the empirical measure  $\nu_{X^N}^e$ , this means that  $\nu_{X^N}^e$  converges weakly to  $\nu$  as  $N \rightarrow +\infty$ . In accordance with the Portmanteau theorem (see, e.g., [?, Chapter 1, Section 2, Theorem 2.1]), since  $W_p$  metrizes the weak convergence, we have  $W_p(\nu_{X^N}^e, \nu) = o(1)$  as  $N \rightarrow +\infty$  since  $\text{supp}(\nu)$  is compact.

Writing  $\int_{\Omega} f d\nu = \sum_{i=1}^N \int_{\Omega_i^N} f d\nu$  and using that  $\nu(\Omega_i^N) = \frac{1}{N}$  (thus  $\frac{1}{N} f(x_i^N) = \int_{\Omega_i^N} f(x_i^N) d\nu(x)$ ) and that  $\text{diam}_{\Omega}(\Omega_i^N) \leq \frac{C_{\Omega}}{N^r}$  (see (??)), we have

$$\begin{aligned} \left| \int_{\Omega} f d\nu - \frac{1}{N} \sum_{i=1}^N f(x_i^N) \right| &= \left| \sum_{i=1}^N \int_{\Omega_i^N} (f(x) - f(x_i^N)) d\nu(x) \right| \leq \sum_{i=1}^N \int_{\Omega_i^N} |f(x) - f(x_i^N)| d\nu(x) \\ &\leq \text{Hol}_{\alpha}(f) \sum_{i=1}^N \int_{\Omega_i^N} d_{\Omega}(x, x_i^N)^{\alpha} d\nu(x) \leq \text{Hol}_{\alpha}(f) \sum_{i=1}^N \nu(\Omega_i^N) \text{diam}_{\Omega}(\Omega_i^N)^{\alpha} \leq \frac{C_{\Omega}^{\alpha}}{N^{r\alpha}} \text{Hol}_{\alpha}(f) \end{aligned}$$

which gives (??). Taking  $\alpha = 1$ , (??) follows by the definition (??) of  $W_1$ .  $\square$

### A.4.2 Convergence of semi-empirical measures

Let  $d \in \mathbb{N}^*$ . Let  $\mu \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , disintegrated as  $\mu = \int_{\Omega} \mu_x d\nu(x)$  with respect to its marginal  $\nu = \pi_* \mu$  on  $\Omega$ . We define the *semi-empirical measure*  $\mu_{X^N}^{se} \in \mathcal{P}(\Omega \times \mathbb{R}^d)$  by

$$\mu_{X^N}^{se} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \mu_{x_i^N} = \int_{\Omega} \mu_x d\nu_{X^N}^e(x).$$

Its marginal on  $\Omega$  is the empirical measure  $\nu_{X^N}^e$ . In other words, the disintegration of  $\mu_{X^N}^{se}$  with respect to  $\nu_{X^N}^e$  is the family of probability measures given by  $\mu_{x_i^N}$  when  $x = x_i^N$  for some  $i \in \{1, \dots, N\}$  and 0 otherwise.

**Lemma 27.**

- We assume that  $x \mapsto \mu_x$  is  $\nu$ -almost everywhere continuous for the Wasserstein distance  $W_1$  (equivalently,  $W_p$ ). Let  $f$  be a bounded and  $\mu$ -almost everywhere continuous (i.e.,  $\mu$ -Riemann integrable) function on  $\Omega \times \mathbb{R}^d$ , of compact support, Lipschitz with respect to  $\xi \in \mathbb{R}^d$  with a Lipschitz constant that is uniform with respect to  $x \in \Omega$ . Then

$$\int_{\Omega \times \mathbb{R}^d} f d(\mu - \mu_{X^N}^{se}) = o(1) \quad (129)$$

as  $N \rightarrow +\infty$ . As a consequence,  $\mu_{X^N}^{se}$  converges weakly to  $\mu$ ; equivalently,  $W_p(\mu_{X^N}^{se}, \mu) = o(1)$  as  $N \rightarrow +\infty$ .

- We assume that  $x \mapsto \mu_x$  is Lipschitz for the Wasserstein distance  $W_1$ , i.e., that there exists  $L > 0$  such that  $W_1(\mu_x, \mu_y) \leq L d_{\Omega}(x, y)$  for  $\nu$ -almost all  $x, y \in \Omega$ . Then, given any  $N \in \mathbb{N}^*$ ,

$$\left| \int_{\Omega \times \mathbb{R}^d} f d(\mu - \mu_{X^N}^{se}) \right| \leq \frac{(L+1)C_{\Omega}}{N^r} \text{Lip}(f) \quad (130)$$

for every  $f \in \mathcal{C}_0^0(\Omega \times \mathbb{R}^d) \cap \text{Lip}(\Omega \times \mathbb{R}^d)$ . As a consequence,

$$W_1(\mu_{X^N}^{se}, \mu) \leq \frac{(L+1)C_{\Omega}}{N^r}, \quad (131)$$

and thus also, using (??),  $W_p(\mu_{X^N}^{se}, \mu) \leq \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu))^{1-1/p} \frac{((L+1)C_{\Omega})^{1/p}}{N^{r/p}}$ , for any  $p \in [1, +\infty)$ .

*Proof.* Let  $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a bounded and  $\mu$ -almost everywhere continuous function, of compact support, Lipschitz with respect to  $\xi \in \mathbb{R}^d$ . The function  $F$  defined by  $F(x) = \int_{\mathbb{R}^d} f(x, \xi) d\mu_x(\xi)$  is bounded on  $\Omega$ , and

$$\begin{aligned} |F(x) - F(x')| &\leq \int_{\mathbb{R}^d} |f(x, \xi) - f(x', \xi)| d\mu_x(\xi) + \left| \int_{\mathbb{R}^d} f(x', \xi) d(\mu_x - \mu_{x'})(\xi) \right| \\ &\leq \int_{\mathbb{R}^d} |f(x, \xi) - f(x', \xi)| d\mu_x(\xi) + W_1(\mu_x, \mu_{x'}) \text{Lip}(f(x', \cdot)) \end{aligned} \quad (132)$$

for all  $x, x' \in \Omega$ . Now:

- First, if moreover  $x' \mapsto \text{Lip}(f(x', \cdot))$  is bounded on  $\Omega$  and if  $x \mapsto \mu_x$  is  $\nu$ -almost everywhere continuous for the Wasserstein distance  $W_1$ , then we infer from (??) that  $F$  is  $\nu$ -almost everywhere continuous. Therefore

$$\int_{\Omega \times \mathbb{R}^d} f d(\mu - \mu_{X^N}^{se}) = \int_{\Omega} F d(\nu - \nu_{X^N}^e) = \int_{\Omega} F d\nu - \frac{1}{N} \sum_{i=1}^N F(x_i^N) = o(1)$$

as  $N \rightarrow +\infty$  by convergence of Riemann sums ( $f$  and thus  $F$  being fixed), which gives (??).

- Second, if  $f \in \text{Lip}(\Omega \times \mathbb{R}^d)$  and if  $x \mapsto \mu_x$  is  $L$ -Lipschitz for the Wasserstein distance  $W_1$  then we infer from (??) that

$$|F(x) - F(x')| \leq \text{Lip}(f) d_\Omega(x, x') + W_1(\mu_x, \mu_{x'}) \text{Lip}(f) \leq \text{Lip}(f)(1 + L) d_\Omega(x, x')$$

and thus, using Lemma ??, that  $\int_\Omega F d(\nu - \nu_{X^N}^e) \leq \frac{C_\Omega}{N^r} \text{Lip}(F)$ , whence (??) and (??).  $\square$

**Remark 17.** In the first item of Lemma ??, the boundedness assumption on  $f$  can be slightly weakened to:  $x \mapsto f(x, 0)$  bounded and  $\mu \in \mathcal{P}_1(\Omega \times \mathbb{R}^d)$ . Indeed, writing  $|f(x, \xi)| \leq |f(x, 0)| + \text{Lip}(f(x, \cdot))|\xi|$ , we infer that  $F$  is bounded. The rest of the proof is the same.

## A.5 Discrepancy between empirical and $\nu$ -monokinetic measures

Recall that:

- given any  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$  and any  $\Xi^N = (\xi_1^N, \dots, \xi_N^N) \in \mathbb{R}^{dN}$ , the empirical measure  $\mu_{(X^N, \Xi^N)}^e$  on  $\Omega \times \mathbb{R}^d$  is defined by (??);
- given any  $\nu \in \mathcal{P}(\Omega)$  and any measurable function  $y : \Omega \rightarrow \mathbb{R}^d$ , the  $\nu$ -monokinetic measure  $\mu_y^\nu$  on  $\Omega \times \mathbb{R}^d$  is defined by (??).

**Lemma 28.** Let  $\nu \in \mathcal{P}(\Omega)$  and let  $(\mathcal{A}^N, X^N)_{N \in \mathbb{N}^*}$  be a family of tagged partitions associated with  $\nu$  (see (??)), with  $\mathcal{A}^N = (\Omega_1^N, \dots, \Omega_N^N)$  and  $X^N = (x_1^N, \dots, x_N^N)$ .

- (i) Let  $y \in \text{Lip}(\Omega, \mathbb{R}^d)$ . For every  $N \in \mathbb{N}^*$ , taking  $\Xi^N = (\xi_1^N, \dots, \xi_N^N)$  with  $\xi_i^N = y(x_i^N)$  for every  $i \in \{1, \dots, N\}$ , we have

$$\left| \left\langle \mu_y^\nu - \mu_{(X^N, \Xi^N)}^e, f \right\rangle \right| \leq \frac{C_\Omega}{N^r} \text{Lip}(x \mapsto f(x, y(x))) \quad \forall f \in \text{Lip}_c(\Omega \times \mathbb{R}^d).$$

- (ii) For every  $N \in \mathbb{N}^*$ , let  $\Xi^N = (\xi_1^N, \dots, \xi_N^N) \in \mathbb{R}^{dN}$ . Defining the piecewise continuous function

$$y^N(x) = \sum_{i=1}^N \xi_i^N \mathbf{1}_{\Omega_i^N}(x) \quad \forall x \in \Omega,$$

so that  $y^N(x_i^N) = \xi_i^N$  for every  $i \in \{1, \dots, N\}$ , we have

$$\left| \left\langle \mu_y^\nu - \mu_{(X^N, \Xi^N)}^e, f \right\rangle \right| \leq \frac{C_\Omega}{N^r} \max_{1 \leq i \leq N} \text{Lip}(f(\cdot, \xi_i^N)) \quad \forall f \in \text{Lip}_c(\Omega \times \mathbb{R}^d).$$

*Proof.* Let us prove (i). We have  $\langle \mu_y^\nu, f \rangle = \int_\Omega f(x, y(x)) d\nu(x) = \sum_{i=1}^N \int_{\Omega_i^N} f(x, y(x)) d\nu(x)$  and (using that  $\nu(\Omega_i^N) = \frac{1}{N}$ )

$$\left\langle \mu_{(X^N, \Xi^N)}^e, f \right\rangle = \frac{1}{N} \sum_{i=1}^N f(x_i^N, y(x_i^N)) = \sum_{i=1}^N \int_{\Omega_i^N} f(x_i^N, y(x_i^N)) d\nu(x)$$

hence

$$\begin{aligned} \left| \left\langle \mu_y^\nu - \mu_{(X^N, \Xi^N)}^e, f \right\rangle \right| &\leq \sum_{i=1}^N \int_{\Omega_i^N} |f(x, y(x)) - f(x_i^N, y(x_i^N))| d\nu(x) \\ &\leq \text{Lip}(x \mapsto f(x, y(x))) \sum_{i=1}^N \int_{\Omega_i^N} d_\Omega(x, x_i^N) d\nu(x) \end{aligned}$$

and ?? follows because  $\int_{\Omega_i^N} d_{\Omega}(x, x_i^N) d\nu(x) \leq \nu(\Omega_i^N) \text{diam}_{\Omega}(\Omega_i^N) \leq \frac{C_{\Omega}}{N^{1+r}}$  (using ??).

The estimate of ?? is proved similarly: we have  $\langle \mu_{y^N}^{\nu}, f \rangle = \sum_{i=1}^N \int_{\Omega_i^N} f(x, \xi_i^N) d\nu(x)$  and thus

$$\begin{aligned} \left| \langle \mu_{y^N}^{\nu} - \mu_{(X^N, \Xi^N)}^e, f \rangle \right| &\leq \sum_{i=1}^N \int_{\Omega_i^N} |f(x, \xi_i^N) - f(x_i^N, \xi_i^N)| d\nu(x) \\ &\leq \sum_{i=1}^N \text{Lip}(f(\cdot, \xi_i^N)) \int_{\Omega_i^N} d_{\Omega}(x, x_i^N) d\nu(x) \end{aligned}$$

and ?? follows.  $\square$

**Remark 18.** Actually, we see from the proof that, in the estimates stated in the above lemma, it suffices that all functions of which we consider the Lipschitz constant, be Lipschitz on each subset  $\Omega_i^N$ . In particular, they may be discontinuous at the boundary of  $\Omega_i^N$ .

With that remark, we recover ?? as a consequence of ??.

## A.6 Mean field and variance

Let  $\mu \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  be arbitrary. Recall that the mean field  $\mathcal{X}[\mu](t, x, \xi)$  is defined by (??), which is the expectation of  $G(t, x, \cdot, \xi, \cdot)$  for the measure  $\mu$ , performed with respect to  $(x', \xi') \in \Omega \times \mathbb{R}^d$ :

$$\mathcal{X}[\mu](t, x, \xi) = \int_{\Omega \times \mathbb{R}^d} G(t, x, x', \xi, \xi') d\mu(x', \xi') = \mathbb{E}^{\mu} G(t, x, \cdot, \xi, \cdot)$$

Given any  $t \in \mathbb{R}$ , any  $x, x' \in \Omega$  and any  $\xi, \xi' \in \mathbb{R}^d$ , we set

$$e_t[\mu](x, x', \xi, \xi') = G(t, x, x', \xi, \xi') - \mathcal{X}[\mu](t, x, \xi).$$

Of course, we have  $\mathbb{E}^{\mu} e_t[\mu](x, \cdot, \xi, \cdot) = 0$  and thus also

$$\mathbb{E}^{\mu \otimes \mu} e_t[\mu] = 0.$$

This naturally leads to consider the variance of  $e_t[\mu]$  with respect to  $\mu \otimes \mu$ :

$$\text{Var}(e_t[\mu]) = \mathbb{E}^{\mu \otimes \mu} \|e_t\|^2 = \int_{\Omega^2 \times \mathbb{R}^{2d}} \|G(t, x, x', \xi, \xi') - \mathcal{X}[\mu](t, x, \xi)\|^2 d\mu(x', \xi') d\mu(x, \xi).$$

Note that

$$\text{Var}(e_t[\mu]) \leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^0}^2. \quad (133)$$

Let  $N \in \mathbb{N}^*$  be fixed. Recall that the particle (time-dependent) vector field  $Y^N = (Y_1^N, \dots, Y_N^N)$  is defined by (??) with  $Y_i^N$  defined by (??), i.e.,  $Y_i^N(t, X, \Xi) = \frac{1}{N} \sum_{j=1}^N G(t, x_i, x_j, \xi_i, \xi_j)$ , where we use the notations  $X = (x_1, \dots, x_N) \in \Omega^N$  and  $\Xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^d)^N$ .

**Lemma 29.** *We assume that the norm  $\|\cdot\|$  on  $\mathbb{R}^d$  is induced by a scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^d$ . For every  $i \in \{1, \dots, N\}$  we have*

$$\int_{\Omega^N \times \mathbb{R}^{dN}} \|Y_i^N(t, X, \Xi) - \mathcal{X}[\mu](t, x_i, \xi_i)\|^2 d\mu^{\otimes N}(X, \Xi) = \frac{1}{N} \text{Var}(e_t[\mu]) \leq \frac{4}{N} \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^0}^2.$$



*Proof of Lemma ??.* By definition of  $e_t$ , we have, for every  $i \in \{1, \dots, N\}$ ,

$$Y_i^N(\tau, X, \Xi) - \mathcal{X}[\mu](t, x_i, \xi_i) = \frac{1}{N} \sum_{j=1}^N e_t[\mu](x_i, x_j, \xi_i, \xi_j).$$

Therefore,

$$\begin{aligned} & \int_{\Omega^N \times \mathbf{R}^{dN}} \|Y_i^N(\tau, X, \Xi) - \mathcal{X}[\mu](t, x_i, \xi_i)\|^2 d\mu^{\otimes N}(X, \Xi) \\ &= \frac{1}{N^2} \int_{\Omega^N \times \mathbf{R}^{dN}} \sum_{j=1}^N \|e_t[\mu](x_i, x_j, \xi_i, \xi_j)\|^2 d\mu^{\otimes N}(X, \Xi) \\ & \quad + \frac{1}{N^2} \int_{\Omega^N \times \mathbf{R}^{dN}} \sum_{\substack{j,k=1 \\ j \neq k}}^N \langle e_t[\mu](x_i, x_j, \xi_i, \xi_j), e_t[\mu](x_i, x_k, \xi_i, \xi_k) \rangle d\mu^{\otimes N}(X, \Xi). \end{aligned} \quad (134)$$

The first term at the right-hand side of (??) is equal to

$$\frac{1}{N} \int_{\Omega^2 \times \mathbf{R}^{2d}} \|e_t[\mu](x, x', \xi, \xi')\|^2 d\mu(x', \xi') d\mu(x, \xi) = \frac{1}{N} \text{Var}(e_t[\mu]). \quad (135)$$

The second term at the right-hand side of (??) is equal to

$$\begin{aligned} & \frac{N^2 - N}{N^2} \int_{\Omega^3 \times \mathbf{R}^{3d}} \langle e_t[\mu](x, x', \xi, \xi'), e_t[\mu](x, x'', \xi, \xi'') \rangle d\mu(x, \xi) d\mu(x', \xi') d\mu(x'', \xi'') \\ &= \frac{N^2 - N}{N^2} \int_{\Omega \times \mathbf{R}^d} \left\| \int_{\Omega \times \mathbf{R}^d} e_t[\mu](x, x', \xi, \xi') d\mu(x', \xi') \right\|^2 d\mu(x, \xi) = 0 \end{aligned} \quad (136)$$

because the expectation of  $e_t[\mu](x_i, \cdot, \xi_i, \cdot)$  is equal to 0. The lemma is proved, using (??).  $\square$

Although Lemma ?? is not used as such in this article, we believe that it has its own interest. Actually, in the proof of Theorem ?? (in Appendix ??), we will need the following result, in the spirit of Lemma ?? but more technical.

**Lemma 30.** *As in Lemma ??, we assume that the norm  $\|\cdot\|$  on  $\mathbf{R}^d$  is induced by a scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathbf{R}^d$ . Let  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_N) \in \Omega^N$  be arbitrary, and let*

$$\rho = \delta_{\bar{x}_1} \otimes \dots \otimes \delta_{\bar{x}_N} \otimes \mu_{\bar{x}_1} \otimes \dots \otimes \mu_{\bar{x}_N}.$$

Then

$$\begin{aligned} M(t) &= \left( \int_{\Omega^N \times \mathbf{R}^{dN}} \left( \sum_{i=1}^N \|Y_i^N(t, X, \Xi) - \mathcal{X}[\mu](t, x_i, \xi_i)\| \right)^2 d\rho(X, \Xi) \right)^{1/2} \\ &\leq 2 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^{0,1}} \left( \sqrt{N} \sqrt{1 + 70 \text{diam}_{\Omega \times \mathbf{R}^d}(\text{supp}(\mu))} + N \sqrt{5 W_1(\mu, \mu_{\bar{X}}^{se})} \right) \end{aligned} \quad (137)$$

where  $\mu_{\bar{X}}^{se} = \frac{1}{N} \sum_{i=1}^N \delta_{\bar{x}_i} \otimes \mu_{\bar{x}_i} = \rho_{N;1}^s$  (semi-empirical measure).

*Proof.* As a first remark, we note that, since the function inside the integral at the left-hand side of the inequality (??) is symmetric, we can replace  $\rho$  by the symmetrization  $\rho^s$  in the integral (indeed, when  $F : \Omega^N \times \mathbb{R}^{dN} \rightarrow \mathbb{R}$  is symmetric, we have  $\int F d\rho = \int F d\rho^s$ ). As a second remark, since  $M(t)$  is defined as the  $L^2$  norm of a sum, we infer from the triangular inequality that

$$\begin{aligned} M(t) &\leq \sum_{i=1}^N \left( \int_{\Omega^N \times \mathbb{R}^{dN}} \|Y_i^N(t, X, \Xi) - \mathcal{X}[\mu](t, x_i, \xi_i)\|^2 d\rho^s(X, \Xi) \right)^{1/2} \\ &\leq N \max_{1 \leq i \leq N} \sqrt{I_i(t)} \quad \text{with} \quad I_i(t) = \int_{\Omega^N \times \mathbb{R}^{dN}} \|Y_i^N(t, X, \Xi) - \mathcal{X}[\mu](t, x_i, \xi_i)\|^2 d\rho^s(X, \Xi) \end{aligned}$$

Note that it was important to symmetrize the measure  $\rho$  *before* applying the triangular inequality. Let us now estimate  $I_i(t)$ , for any fixed  $i \in \{1, \dots, N\}$ . We cannot apply directly Lemma ?? because in the integral  $I_i(t)$  the integration is performed with respect to  $\rho^s$ , and not with respect to  $\mu^{\otimes N}$ . However, following the proof of Lemma ??, we expand  $I_i(t)$  similarly as in (??); replacing  $\mu^{\otimes N}$  by  $\rho^s$  and thus the second-order marginal  $\rho_{N:2}^s$  and third-order marginal  $\rho_{N:3}^s$  appear. Note that, by Lemma ?? in Appendix ??, since  $\rho^s$  is symmetric all its second-order (resp., third-order) marginals on the various copies of  $(\Omega \times \mathbb{R}^d)^2$  (resp., of  $(\Omega \times \mathbb{R}^d)^2$ ) are equal. We obtain

$$I_i(t) = \frac{1}{N} \int_{\Omega^2 \times \mathbb{R}^{2d}} \|e_t[\mu]\|^2 d\rho_{N:2}^s + \frac{N^2 - N}{N^2} \int_{\Omega^3 \times \mathbb{R}^{3d}} F_t[\mu] d\rho_{N:3}^s \quad (138)$$

with

$$F_t[\mu](x, x', x'', \xi, \xi', \xi'') = \langle e_t[\mu](x, x', \xi, \xi'), e_t[\mu](x, x'', \xi, \xi'') \rangle$$

To estimate the first term at the right-hand side of (??), we observe that (using the definition (??) of the Wasserstein distance  $W_1$ )

$$\begin{aligned} \int_{\Omega^2 \times \mathbb{R}^{2d}} \|e_t[\mu]\|^2 d(\rho_{N:2}^s - \mu^{\otimes 2}) &\leq \text{Lip}(\|e_t[\mu]\|_{\text{supp}(\mu)^2}) W_1^{[1]}(\rho_{N:2}^s, \mu^{\otimes 2}) \\ &\leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^{0,1}}^2 W_1^{[1]}(\rho_{N:2}^s, \mu^{\otimes 2}) \end{aligned}$$

(the choice of  $q = 1$ , above, has little importance; other choices would change the constant 4, see Lemma ??) and that, using (??) and (??),

$$\int_{\Omega^2 \times \mathbb{R}^{2d}} \|e_t[\mu]\|^2 d\mu^{\otimes 2} = \text{Var}(e_t[\mu]) \leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^0}^2 \leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^{0,1}}^2.$$

To estimate the second term at the right-hand side of (??), similarly, we observe that

$$\begin{aligned} \int_{\Omega^3 \times \mathbb{R}^{3d}} F_t[\mu] d(\rho_{N:3}^s - \mu^{\otimes 3}) &\leq \text{Lip}(F_t[\mu]_{|\text{supp}(\mu)^2}) W_1^{[1]}(\rho_{N:3}^s, \mu^{\otimes 3}) \\ &\leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^{0,1}}^2 W_1^{[1]}(\rho_{N:3}^s, \mu^{\otimes 3}) \end{aligned}$$

and that, as in (??),

$$\int_{\Omega^3 \times \mathbb{R}^{3d}} F_t[\mu] d\mu^{\otimes 3} = 0.$$

It follows that

$$I_i(t) \leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^{0,1}}^2 \left( \frac{1}{N} + W_1^{[1]}(\rho_{N:2}^s, \mu^{\otimes 2}) + W_1^{[1]}(\rho_{N:3}^s, \mu^{\otimes 3}) \right).$$

Now, applying Lemma ?? (Appendix ??), we infer from (??) that  $\rho_{N:1}^s = \frac{1}{N} \sum_{i=1}^N \delta_{\bar{x}_i} \otimes \mu_{\bar{x}_i} = \mu_X^{se}$  (semi-empirical measure) and from (??) (taking  $k = 2, 3$ ) that

$$W_1^{[1]}(\rho_{N:2}^s, (\mu_X^{se})^{\otimes 2}) + W_1^{[1]}(\rho_{N:3}^s, (\mu_X^{se})^{\otimes 3}) \leq \frac{70}{N} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu))$$

and thus, using the triangular inequality,

$$\sum_{k=2,3} W_1^{[1]}(\rho_{N:k}^s, \mu^{\otimes k}) \leq \frac{70}{N} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu)) + \sum_{k=2,3} W_1^{[1]}(\mu^{\otimes k}, (\mu_X^{se})^{\otimes k}).$$

Applying Lemma ?? in Appendix ??, we have

$$W_1^{[1]}(\mu^{\otimes k}, (\mu_X^{se})^{\otimes k}) \leq k W_1(\mu, \mu_X^{se}).$$

Finally,

$$I_i(t) \leq 4 \|G(t, \cdot, \cdot, \cdot, \cdot)_{|\text{supp}(\mu)^2}\|_{\mathcal{C}^{0,1}}^2 \left( \frac{1}{N} + \frac{70}{N} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu)) + 5 W_1(\mu, \mu_X^{se}) \right)$$

and the estimate (??) follows.  $\square$

## B Proofs

### B.1 Proof of Theorem ??

Let  $T > 0$  be arbitrary. Let  $F$  be either equal to  $\Omega \times \mathbb{R}^d$ , or a compact subset of  $\Omega \times \mathbb{R}^d$  that is the closure of an open set. Let  $\mathcal{C}_0(F)$  be the Banach space of continuous functions on  $F$  vanishing at infinity (when  $F$  is compact we have  $\mathcal{C}_0(F) = \mathcal{C}(F)$ ), and let  $\mathcal{M}^1(F) = \mathcal{C}_0(F)'$  be the Banach space of Radon measures on  $F$ , endowed with the total variation norm  $\|\cdot\|_{TV}$  (which is the dual norm). We have  $\mathcal{P}_c(F) \subset \mathcal{M}^1(F)$  and  $\mathcal{C}^0([0, T], \mathcal{P}_c(F)) \subset L^\infty([0, T], \mathcal{M}^1(F))$ .

The Banach space  $L^\infty([0, T], \mathcal{M}^1(F))$  is endowed with its strong topology, induced by the  $L^\infty$  norm in time of the total variation in space, but can also be endowed with its weak star topology, as follows. Recall the general fact of Bochner integral theory that  $L^1([0, T], E)' = L^\infty([0, T], E')$  (isometric isomorphism) for any Banach space  $E$  such that  $E'$  is separable, where the prime is the topological dual. Applying this fact to the Banach space  $E = \mathcal{C}_0(F)$ , observing that  $E' = \mathcal{M}^1(F)$  is separable in weak star topology (because the set of rational convex combinations of Dirac measures over points with rational coordinates is dense in it), the Banach space  $L^\infty([0, T], \mathcal{M}^1(F))$  coincides with  $L^1([0, T], \mathcal{C}_0(F))'$ , i.e., with the dual of a Banach space, and thus is endowed with the corresponding weak star topology.

We have the following preliminary lemma.

**Lemma 31.** *Let  $K$  be a compact subset of  $\Omega \times \mathbb{R}^d$ , let  $\mu_0 \in \mathcal{P}_c(K)$  and let  $T > 0$  be arbitrary. Assume that there exists a sequence of measures  $\mu^k \in \mathcal{C}^0([0, T], \mathcal{P}_c(K))$  solutions of the Vlasov equation  $\partial_t \mu^k + L_{\mathcal{X}[\mu^k]} \mu^k = 0$  in the sense (??), such that:*

- $\mu^k(0)$  converges weakly to  $\mu_0$  in  $\mathcal{P}_c(K)$ ,
- $\mu^k$  converges to  $\mu \in L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$  for the weak star topology,

as  $k \rightarrow +\infty$ . Then  $\mu \in C^0([0, T], \mathcal{P}_c(K))$  and  $t \mapsto \mu(t)$  is Lipschitz continuous in  $W_p$  distance (for any  $p \in [1, +\infty)$ ) and is a solution of the Vlasov equation  $\partial_t \mu + L_{\mathcal{X}[\mu]} \mu = 0$  (in the sense (??)) such that  $\mu(0) = \mu_0$ . Moreover,  $\mu^k(t)$  converges weakly to  $\mu(t)$  as  $k \rightarrow +\infty$  (equivalently,  $W_p(\mu^k(t), \mu(t)) \rightarrow 0$ ), uniformly with respect to  $t \in [0, T]$ .

*Proof.* For every  $k \in \mathbb{N}^*$ , since  $\mu^k \in \mathcal{C}^0([0, T], \mathcal{P}_c(K))$  is solution of the Vlasov equation in the sense (??) and thus in the distributional sense, we have  $\langle \partial_t \mu^k + L_{\mathcal{X}[\mu^k]} \mu^k, f \rangle = 0$  for any  $f \in \mathcal{C}^\infty([0, T] \times K)$  and thus, integrating by parts,

$$\begin{aligned} \int_K f(0, x, \xi) d\mu_0^k(x, \xi) + \int_0^T \int_K \partial_t f(t, x, \xi) d\mu^k(t, x, \xi) \\ + \int_0^T \int_K \langle \nabla_\xi f(t, x, \xi), G(\tau, x, x', \xi, \xi') \rangle d\mu^k(t, x', \xi') d\mu^k(t, x, \xi) d\tau = 0. \end{aligned}$$

Passing to the limit, we obtain the same equation with  $\mu^k$  replaced by  $\mu$  and  $\mu_0^k$  replaced by  $\mu_0$ , for any  $f \in \mathcal{C}^\infty([0, T] \times K)$ . Hence  $\mu$  is solution of the Vlasov equation  $\partial_t \mu + L_{\mathcal{X}[\mu]} \mu = 0$  in the distributional sense, with  $\mu(0) = \mu_0$ .

Let us prove that  $\mu \in \mathcal{C}^0([0, T], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  and that  $\mu$  is a solution of the Vlasov equation in the sense (??) and that  $\mu^k(t)$  converges weakly to  $\mu(t)$  for every  $t \in [0, T]$ .

By assumption,  $\text{supp}(\mu^k) \subset [0, T] \times K$  for every  $k \in \mathbb{N}^*$ , hence, by the Prokhorov theorem, a subsequence of  $\mu^k$  converges weakly (i.e., in  $(\mathcal{C}^0([0, T] \times K))'$ ) to some measure, which must then be equal to  $\mu$ . Then  $\text{supp}(\mu)$  is contained in the Kuratowski liminf of  $\text{supp}(\mu^k)$  (see, e.g., [?, Proposition 5.1.8]), i.e., for every  $(t, x, \xi) \in \text{supp}(\mu)$  there exists a sequence of  $(t_k, x_k, \xi_k) \in \text{supp}(\mu^k)$  such that  $(t_k, x_k, \xi_k) \rightarrow (t, x, \xi)$  as  $k \rightarrow +\infty$ . We infer that  $\text{supp}(\mu) \subset [0, T] \times K$ . Since  $\mu \in L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$ , we must have  $\text{supp}(\mu(t)) \subset K$  for almost every  $t \in [0, T]$ .

For every  $k \in \mathbb{N}^*$ , we consider on  $[0, T] \times K$  the time-dependent vector field  $v^k(t, x, \xi) = \mathcal{X}[\mu_t^k](t, x, \xi)$ , which is continuous and Lipschitz with respect to  $\xi$  (thanks to ??). Since  $\text{supp}(\mu^k) \subset [0, T] \times K$ , we have  $\|v^k\|_{\mathcal{C}^0([0, T] \times K)} \leq C$  for some  $C > 0$  not depending on  $k$ . Let  $(\Phi_{v^k}(t))_{t \in [0, T]}$  be the flow on  $[0, T] \times K$  generated by  $v^k$ . Since  $\mu^k$  is a solution of the transport equation  $\partial_t \mu^k + L_{v^k} \mu^k = 0$ , by the usual existence and uniqueness theorem for linear transport equations (see, e.g., [?, Theorem 5.34]), we have  $\mu^k(t) = \Phi_{v^k}(t)_* \mu_0^k$  for every  $t \in [0, T]$ . This means, denoting by  $\nu^k = \pi_* \mu^k(t)$  the (constant in time) marginal of  $\mu^k(t)$  on  $\Omega$  and disintegrating  $\mu_t^k = \mu^k(t) = \int_\Omega \mu_{t,x}^k d\nu^k(x)$ , that  $\mu_{t,x}^k = \Phi_{v^k}(t, x, \cdot)_* \mu_{0,x}^k$  for  $\nu^k$ -almost every  $x \in \pi(K)$ , for every  $t \in [0, T]$ . It follows from Lemma ?? of Appendix ?? that  $W_p(\mu^k(t_1), \mu^k(t_2)) \leq C|t_1 - t_2|$  for all  $t_1, t_2 \in [0, T]$ , i.e.,  $t \mapsto \mu^k(t)$  is Lipschitz continuous on  $[0, T]$  in  $W_p$  distance, uniformly with respect to  $k$ . Since  $\mu^k$  converges to  $\mu \in L^\infty([0, T], \mathcal{M}^1(K))$  for the weak star topology, it follows from the Ascoli theorem that  $\mu^k(t)$  converges weakly to  $\mu(t)$  in  $\mathcal{M}^1(K) = \mathcal{C}^0(K)'$ , uniformly with respect to  $t \in [0, T]$ , and thus that  $\mu \in \mathcal{C}^0([0, T], \mathcal{P}_c^1(K))$  and that  $t \mapsto \mu(t)$  is Lipschitz continuous in  $W_p$  distance. Lemma ?? is proved.  $\square$

In view of establishing Item ??, let us first prove the existence of a solution of the Vlasov equation. Given  $\mu_0 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$ , we consider a sequence of empirical measures  $\mu_{(X^N, \Xi_0^N)}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_{0,i}^N}$  converging weakly to  $\mu_0$  as  $N \rightarrow +\infty$ . Setting  $X^N = (x_1^N, \dots, x_N^N) \in \Omega^N$  and  $\Xi_0^N = (\xi_1^N, \dots, \xi_N^N) \in (\mathbb{R}^d)^N$ , let  $t \mapsto \Xi^N(t) = (\xi_1^N(t), \dots, \xi_N^N(t))$  be the unique solution of the particle system (??) with parameter  $X^N$  such that  $\Xi^N(0) = \Xi_0^N$ . It is well defined on  $[0, T]$  for any  $T \in (0, T_{\max}(\text{supp}(\mu_0)))$  thanks to Assumption ?? and Lemma ?. Using the first part of Proposition ??, which does not use anything from Theorem ?? (see its proof),  $t \mapsto \mu_{(X^N, \Xi^N(t))}^e = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}$  is a solution of the Vlasov equation (??) in the sense (??).

Without loss of generality, we can assume that  $(X^N, \Xi_0^N) \in (\text{supp}(\mu_0))^N$  for every  $N \in \mathbb{N}^*$ , where we recall that  $\text{supp}(\mu_0)$  is compact. Since  $\mu_{(X^N, \Xi^N(t))}^e$  is supported on the corresponding solutions of the particle system, it follows from Lemma ?? that there exists a compact subset  $K \subset \Omega \times \mathbb{R}^d$  such that  $\text{supp}(\mu_{(X^N, \Xi^N(t))}^e) \subset K$  for every  $t \in [0, T]$  and for every  $N \in \mathbb{N}^*$ , i.e., the measures  $\mu_{(X^N, \Xi^N(\cdot))}^e$  are equi-compactly supported on  $[0, T]$ , uniformly with respect to  $N$ .

Besides, since  $\mu_{(X^N, \Xi^N(t))}^e$  is a probability measure, we have  $\|\mu_{(X^N, \Xi^N(t))}^e\|_{TV} = 1 < +\infty$  for every  $t \in [0, T]$ , and thus the sequence  $(\mu_{(X^N, \Xi^N(\cdot))}^e)_{N \in \mathbf{N}^*}$  is bounded in  $L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$  for the strong topology, i.e., in  $(L^1([0, T], \mathcal{C}_0(\Omega \times \mathbb{R}^d)))'$  for the strong (dual norm) topology. By the Banach-Alaoglu theorem, there exists a subsequence of  $(\mu_{(X^N, \Xi^N(\cdot))}^e)_{N \in \mathbf{N}^*}$  converging to some  $\mu \in L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$  for the weak star topology.

Therefore, a subsequence of the sequence of measures  $\mu_{(X^N, \Xi^N(\cdot))}^e \in \mathcal{C}_{\text{comp}}^0([0, T], \mathcal{P}_c(K))$  satisfies all assumptions of Lemma ???. It follows from that lemma that  $\mu \in \mathcal{C}^0([0, T], \mathcal{P}_c(K))$  and that  $\mu$  is a solution on  $[0, T]$  of the Vlasov equation  $\partial_t \mu + L_{\mathcal{X}[\mu]} \mu = 0$  (in the sense (??)) such that  $\mu(0) = \mu_0$ , and is Lipschitz continuous with respect to  $t$  in  $W_p$  distance.

At this step, we have obtained the existence of solutions in  $\mathcal{C}_{\text{comp}}^0([0, T_{\max}(\text{supp}(\mu_0))], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  (not yet uniqueness).

**Remark 19.** In [?, ?, ?, ?], the proof of the existence is done by constructing a sequence of piecewise constant measures converging to a solution, under the stronger assumption that  $G$  be globally Lipschitz continuous. The proof given above relies on approximation by empirical measures and propagation of them, in the spirit of [?] (see also [?] and [?, Part I, Theorem 5.1]), which is more appropriate to exploit Lemma ???.

**Remark 20.** Before going ahead, let us observe that, when  $G$  is locally Lipschitz with respect to all variables  $(x, x', \xi, \xi')$ , we have, for all  $\mu^1, \mu^2 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  and for every  $(t, x, \xi) \in [0, T] \times \Omega \times \mathbb{R}^d$ ,

$$\begin{aligned} \|\mathcal{X}[\mu^1](t, x, \xi) - \mathcal{X}[\mu^2](t, x, \xi)\| &= \left\| \int_{\Omega \times \mathbb{R}^d} G(t, x, x', \xi, \xi') d(\mu^1(x', \xi') - \mu^2(x', \xi')) \right\| \\ &\leq \text{Lip}(G(t, x, \cdot, \xi, \cdot)|_S) W_1(\mu^1, \mu^2) \leq \text{Lip}(G(t, x, \cdot, \xi, \cdot)|_S) W_p(\mu^1, \mu^2) \end{aligned} \quad (139)$$

where  $S = \text{supp}(\mu^1) \cup \text{supp}(\mu^2)$  (compact set). We have used that  $W_1 \leq W_p$ .

In the case ???, there is however a weaker assumption: under Assumption ???,  $G$  is locally Lipschitz only with respect to  $(\xi, \xi')$  and thus the classical Wasserstein distance  $W_1$  cannot be used as above. The main difference then comes from the following observation: given any  $\mu^1, \mu^2 \in \mathcal{P}_c(\Omega \times \mathbb{R}^d)$  having the same marginal  $\nu \in \mathcal{P}_c(\Omega)$  on  $\Omega$ , we have, by disintegration,

$$\mathcal{X}[\mu^1](t, x, \xi) - \mathcal{X}[\mu^2](t, x, \xi) = \int_{\Omega} \int_{\mathbb{R}^d} G(t, x, x', \xi, \xi') d(\mu_{x'}^1(\xi') - \mu_{x'}^2(\xi')) d\nu(x')$$

and thus

$$\begin{aligned} \|\mathcal{X}[\mu^1](t, x, \xi) - \mathcal{X}[\mu^2](t, x, \xi)\| &\leq \max_{x' \in \text{supp}(\nu)} \text{Lip}(G(t, x, x', \xi, \cdot)|_{S_{x'}}) L_\nu^1 W_1(\mu^1, \mu^2) \\ &\leq \max_{x' \in \text{supp}(\nu)} \text{Lip}(G(t, x, x', \xi, \cdot)|_{S_{x'}}) L_\nu^1 W_p(\mu^1, \mu^2) \end{aligned} \quad (140)$$

where  $S_{x'} = \text{supp}(\mu_{x'}^1) \cup \text{supp}(\mu_{x'}^2)$  (compact) and  $L_\nu^1 W_p(\mu^1, \mu^2) = \int_{\Omega} W_p(\mu_{x'}^1, \mu_{x'}^2) d\nu(x')$  is defined by (??).

Let us now establish (??) in the item ??? (which also entails uniqueness). Let  $\mu^1, \mu^2 \in \mathcal{C}_{\text{comp}}^0([0, T], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$  be two solutions of the Vlasov equation for some  $T > 0$ , having the same (constant in time) marginal  $\nu = \pi_* \mu^i \in \mathcal{P}_c(\Omega)$  on  $\Omega$ . Let  $K \subset \Omega \times \mathbb{R}^d$  be a compact subset containing  $\text{supp}(\mu^i(t))$  for  $i = 1, 2$  and for every  $t \in [0, T]$ .

For  $i = 1, 2$ , we consider on  $[0, T] \times K$  the continuous time-dependent vector field  $v^i(t, x, \xi) = \mathcal{X}[\mu^i](t, x, \xi)$  (which is  $C^1$  with respect to  $\xi$ ), so that  $\mu^i$  is a solution of the transport equation  $\partial_t \mu^i + L_{v^i} \mu^i = 0$ . Since we are going to apply Lemma ??? (in Appendix ??) with  $t_0 \neq 0$ , for every

$t_0 \in [0, T]$  we consider the flow  $(\Phi_{v^i}(t, t_0))_{t \in [0, T]}$  on  $[0, T] \times K$  generated by  $v^i$ , i.e., defined as the unique solution of  $\partial_t \Phi_{v^i}(t, t_0, x, \cdot) = v^i(t, x, \cdot) \circ \Phi_{v^i}(t, t_0, x, \cdot)$  such that  $\Phi_{v^i}(t_0, t_0, x, \cdot) = \text{id}_{\mathbb{R}^d}$  for  $\nu$ -almost every  $x \in \text{supp}(\nu)$ . Then, we have  $\mu^i(t) = \Phi_{v^i}(t, t_0)_* \mu^i(t_0)$  for every  $t \in [0, T]$ . This means, disintegrating  $\mu_t^i = \mu^i(t) = \int_{\Omega} \mu_{t,x}^i d\nu(x)$ , that  $\mu_{t,x}^i = \Phi_{v^i}(t, t_0, x, \cdot)_* \mu_{t_0,x}^i$  for  $\nu$ -almost every  $x \in \text{supp}(\nu)$ , for every  $t \in [0, T]$ .

It follows from Lemma ?? (in Appendix ??), applied with  $\Lambda = \emptyset$  and  $E = \mathbb{R}^d$  to the vector fields  $v^i(t, x, \cdot)$  for any fixed  $x \in \text{supp}(\nu)$ , that

$$W_p(\mu_{t,x}^1, \mu_{t,x}^2) \leq e^{(t-t_0)L([t_0,t])} W_p(\mu_{t_0,x}^1, \mu_{t_0,x}^2) + M([t_0, t]) \frac{e^{(t-t_0)L([t_0,t])} - 1}{L([t_0, t])} \quad \forall t \in [t_0, T]$$

where, setting  $S(t) = \text{supp}(\nu) \times \Phi_{v^1}(t, t_0, \text{supp}(\mu_{t_0}^1) \cup \text{supp}(\mu_{t_0}^2)) \cup \text{supp}(\mu^2(t))$  (compact),

$$L([t_0, t]) = \text{ess sup} \{ \|(\partial_\xi G, \partial_{\xi'} G)(\tau, x, x', \xi, \xi')\| \mid t_0 \leq \tau \leq t, (x, \xi), (x', \xi') \in S(\tau) \},$$

$$M([t_0, t]) = \max \{ \|\mathcal{X}[\mu_\tau^1](\tau, x, \xi) - \mathcal{X}[\mu_\tau^2](\tau, x, \xi)\| \mid t_0 \leq \tau \leq t, (x, \xi) \in \text{supp}(\mu_\tau^2) \}.$$

Since  $\mu_\tau^1$  and  $\mu_\tau^2$  have the same marginal  $\nu$  on  $\Omega$ , it follows from (??) and from the above definition of  $L([t_0, t])$  and of  $S(t)$  that

$$M([t_0, t]) \leq L([t_0, t]) \max_{t_0 \leq \tau \leq t} L_\nu^1 W_p(\mu_\tau^1, \mu_\tau^2).$$

Therefore

$$W_p(\mu_{t,x}^1, \mu_{t,x}^2) \leq e^{(t-t_0)L([t_0,t])} W_p(\mu_{t_0,x}^1, \mu_{t_0,x}^2) + (e^{(t-t_0)L([t_0,t])} - 1) \max_{t_0 \leq \tau \leq t} L_\nu^1 W_p(\mu_\tau^1, \mu_\tau^2).$$

Integrating with respect to  $x \in \Omega$  for the measure  $\nu$ , we obtain

$$L_\nu^1 W_p(\mu^1(t), \mu^2(t)) \leq e^{(t-t_0)L([t_0,t])} L_\nu^1 W_p(\mu^1(t_0), \mu^2(t_0)) + (e^{(t-t_0)L([t_0,t])} - 1) \max_{t_0 \leq \tau \leq t} L_\nu^1 W_p(\mu^1(\tau), \mu^2(\tau)). \quad (141)$$

We have the following general lemma.

**Lemma 32.** *For every  $t_0 \in \mathbb{R}$ , let  $a_{t_0} : [t_0, +\infty) \rightarrow [0, +\infty)$  be a nondecreasing function, continuous at  $t_0$ , depending continuously on  $t_0$ . Let  $h : \mathbb{R} \rightarrow [0, +\infty)$  be an absolutely continuous function such that*

$$h(t) \leq e^{(t-t_0)a_{t_0}(t)} h(t_0) + (e^{(t-t_0)a_{t_0}(t)} - 1) \max_{t_0 \leq \tau \leq t} h(\tau) \quad \forall t \geq t_0 \quad \forall t_0 \in \mathbb{R}.$$

Then

$$h(t) \leq h(0) \exp \left( 2 \int_0^t a_\tau(\tau) d\tau \right) \quad \forall t \in \mathbb{R}.$$

*Proof.* Taking  $t_0 < t < t_1$ , writing

$$\frac{h(t) - h(t_0)}{t - t_0} \leq \frac{e^{(t-t_0)a_{t_0}(t_1)} - 1}{t - t_0} h(t_0) + \frac{e^{(t-t_0)a_{t_0}(t_1)} - 1}{t - t_0} \max_{t_0 \leq \tau \leq t} h(\tau)$$

and taking the limit as  $t \rightarrow t_0$ , since  $t_1$  is arbitrary, we obtain  $h'(t_0) \leq 2h(t_0)a_{t_0}(t_0)$ , for almost every  $t_0 \in \mathbb{R}$ . The lemma follows by integration.  $\square$

Applying Lemma ?? to  $h(t) = L_\nu^1 W_p(\mu^1(t), \mu^2(t))$  and  $a_{t_0}(t) = L([t_0, t])$ , and using (??), we obtain (??). In particular, the uniqueness statement follows.

At this step, we have proved existence and uniqueness of solutions of the Vlasov equation in the space  $\mathcal{C}_{\text{comp}}^0([0, T], \mathcal{P}_c(\Omega \times \mathbb{R}^d))$ . We can thus now define the Vlasov flow by (??), and we obtain (??) by uniqueness.

Establishing (??) in the item ?? follows straightforwardly the same lines as above, by applying Lemma ?? (in Appendix ??) with  $\Lambda = \Omega$  and  $E = \mathbb{R}^d$ , and using (??) instead of (??). We do not give any further details.

It remains to establish the item ?. For  $K \subset \Omega \times \mathbb{R}^d$  compact and  $T \in (0, T_{\max}(K))$ , we consider a sequence of measures  $\mu^k \in \mathcal{C}^0([0, T], \mathcal{P}_c(K))$  solutions of the Vlasov equation such that  $\mu_0^k = \mu^k(0)$  converges weakly to  $\mu_0 = \mu(0)$  as  $k \rightarrow +\infty$ . Our objective is to prove that  $\mu^k(t)$  converges weakly to  $\mu(t)$ , uniformly with respect to  $t \in [0, T]$ .

Since  $\mu^k(t)$  is a probability measure, we have  $\|\mu^k(t)\|_{TV} = 1 < +\infty$  for every  $t \in [0, T]$ , and thus the sequence  $(\mu^k(\cdot))_{k \in \mathbb{N}^*}$  is bounded in  $L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$  (for the strong topology), i.e., in  $(L^1([0, T], \mathcal{C}_0(\Omega \times \mathbb{R}^d)))'$  for the strong (dual norm) topology. By the Banach-Alaoglu theorem, a subsequence of  $(\mu^k(\cdot))_{k \in \mathbb{N}^*}$  converges to some  $\tilde{\mu}(\cdot) \in L^\infty([0, T], \mathcal{M}^1(\Omega \times \mathbb{R}^d))$  for the weak star topology.

It follows from Lemma ?? that  $\tilde{\mu} \in \mathcal{C}^0([0, T], \mathcal{P}_c(K))$ , that  $\tilde{\mu}$  is a solution of the Vlasov equation such that  $\tilde{\mu}(0) = \mu_0$ , and that  $\mu^k(t)$  converges weakly to  $\tilde{\mu}$  uniformly with respect to  $t$ . By uniqueness, we must have  $\tilde{\mu} = \mu$ . This finishes the proof of the theorem.

## B.2 Proof of Theorem ??

We have  $\rho^N(t) = \delta_{X^N} \otimes \delta_{\Xi^N(t)}$ . By (??) in Lemma ?? of Appendix ?? (applied with  $\gamma_i = \delta_{\xi_i^N(t)}$ ), we have

$$\rho^N(t)_{N:1}^s = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)} = \mu_{(X^N, \Xi^N(t))}^e,$$

which gives the preliminary remark to Theorem ?. The statement ?? for  $k = 1$  then follows from the item ?? of Theorem ??, and the estimate (??) follows from the item ?? of Theorem ?.

For any  $k \in \mathbb{N}^*$ , the  $k^{\text{th}}$ -order marginal  $\rho^N(t)_{N:k}^s$  is given by (??) in Appendix ?? (applied with  $\mu_i = \delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}$ ). By the triangular inequality, we have

$$W_p^{[q]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) \leq W_p^{[q]}(\rho^N(t)_{N:k}^s, (\mu_{(X^N, \Xi^N(t))}^e)^{\otimes k}) + W_p^{[q]}((\mu_{(X^N, \Xi^N(t))}^e)^{\otimes k}, \mu(t)^{\otimes k}) \quad (142)$$

For the first term at the right-hand side of (??), noting that  $(X^N, \Xi_0^N) \in (\text{supp}(\mu_0))^N$  and thus

$$\text{diam}_{\Omega \times \mathbb{R}^d} \left( \bigcup_{i=1}^N \text{supp}(\delta_{x_i^N} \otimes \delta_{\xi_i^N(t)}) \right) \leq \text{diam}_{\Omega}(\text{supp}(\nu)) + \text{diam}_{\mathbb{R}^d}(\Xi^N(t)),$$

and noting that  $\mu_{(X^N, \Xi^N(t))}^e = \rho^N(t)_{N:1}^s$ , we infer from (??) in Lemma ?? of Appendix ?? that

$$W_p^{[q]}(\rho^N(t)_{N:k}^s, (\mu_{(X^N, \Xi^N(t))}^e)^{\otimes k}) \leq 2k^{1/q} \left( \frac{k^2}{N} \right)^{1/p} (\text{diam}_{\Omega}(\text{supp}(\nu)) + \text{diam}_{\mathbb{R}^d}(\Xi^N(t))). \quad (143)$$

For the second term at the right-hand side of (??), as a consequence, successively, of the estimate (??) of Lemma ?? in Appendix ?? and of Theorem ??, we have

$$W_p^{[q]}((\mu_{(X^N, \Xi^N(t))}^e)^{\otimes k}, \mu(t)^{\otimes k}) \leq k^{1/q} W_p(\mu_{(X^N, \Xi^N(t))}^e, \mu(t)) \leq k^{1/q} C_\mu^N(t) W_p(\mu_{(X^N, \Xi_0^N)}^e, \mu(0)) \quad (144)$$

where the constant  $C_\mu^N(t)$  is defined by (??) (or equivalently by  $C_{\mu, \mu_{(X^N, \Xi^N)}}^\varepsilon(t)$ , with the notation used in (??) in Theorem ??). Therefore, (??) follows from (??), (??) and (??). Note that, for  $k = 1$ , the first term of the right-hand side of (??) is equal to 0, which gives (??) again.

The statement ?? for any  $k \in \mathbb{N}^*$  is proved by replacing the right inequality in (??) with the application of the item ?? of Theorem ??.

### B.3 Proof of Theorem ??

First of all, since  $\rho_0^N = \delta_{X^N} \otimes \rho_{0, X^N}^N$ , with  $\delta_{X^N} = \delta_{x_1^N} \otimes \cdots \otimes \delta_{x_N^N}$  and  $\rho_{0, X^N}^N = \mu_{0, x_1^N} \otimes \cdots \otimes \mu_{0, x_N^N}$ , it follows from (??) in Lemma ?? of Appendix ?? (applied with  $\mu_i = \delta_{x_i^N} \otimes \mu_{0, x_i^N}$ ) that  $(\rho_0^N)_{N:1}^s = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \mu_{0, x_i^N} = (\mu_0)_{X^N}^{se}$  (semi-empirical measure), which gives (??), and the weak convergence to  $\mu_0$  stated in Item ?? for  $k = 1$  is obtained by Lemma ?? of Appendix ??, which gives the preliminary remark to the theorem.

Recall that  $\rho^N(t) = \Phi^N(t)_* \rho_0^N$  and  $\mu(t) = \varphi_{\mu_0}(t)_* \mu_0$ . Setting

$$\tilde{\rho}^N(t) = \varphi_{\mu_0}(t)_*^{\otimes N} \rho_0^N = \delta_{x_1^N} \otimes \cdots \otimes \delta_{x_N^N} \otimes \mu_{t, x_1^N} \otimes \cdots \otimes \mu_{t, x_N^N}$$

(the latter equality is because  $\varphi_{\mu_0}(t, x_i, \cdot)_* \mu_{0, x_i^N} = \mu_{t, x_i^N}$ ), we note that  $\tilde{\rho}^N(t)^s = \varphi_{\mu_0}(t)_*^{\otimes N} (\rho_0^N)^s$  and that

$$\tilde{\rho}^N(t)_{N:k}^s = \varphi_{\mu_0}(t)_*^{\otimes k} (\rho_0^N)_{N:k}^s \quad \forall k \in \{1, \dots, N\}.$$

Indeed, this follows from the following obvious lemma.

**Lemma 33.** *Let  $E$  be a measure space,  $\varphi : E \rightarrow E$  be a measurable mapping,  $N \in \mathbb{N}^*$  and  $\rho \in \mathcal{P}(E^N)$ . Then*

$$(\varphi_*^{\otimes N} \rho)_{N:k} = \varphi_*(\rho_{N:k}) \quad \forall k \in \{1, \dots, N\}.$$

*Proof of Lemma ??.* Denoting by  $\pi_k : E^N = E^k \times E^{N-k} \rightarrow E^k$  the canonical projection, the lemma straightforwardly follows from the fact that  $\pi_k \circ \varphi_*^{\otimes N} = \varphi_*^{\otimes k} \circ \pi_k$ .  $\square$

In particular, we have

$$\tilde{\rho}^N(t)_{N:1}^s = \varphi_{\mu_0}(t)_* (\rho_0^N)_{N:1}^s = \varphi_{\mu_0}(t)_* (\mu_0)_{X^N}^{se} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N} \otimes \mu_{t, x_i^N} = \mu(t)_{X^N}^{se}.$$

In order to establish (??), we start by applying the triangular inequality:

$$W_p^{[q]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) \leq W_p^{[q]}(\rho^N(t)_{N:k}^s, \tilde{\rho}^N(t)_{N:k}^s) + W_p^{[q]}(\tilde{\rho}^N(t)_{N:k}^s, \mu(t)^{\otimes k}), \quad (145)$$

and we next show how to estimate each of the two terms of the sum at the right-hand side of (??).

**First term.** Applying successively Lemma ?? in Appendix ?? and Lemma ?? in Appendix ??, and using that  $W_p^{[q]} \leq W_2^{[q]} \leq W_2^{[1]}$  (see (??) and (??)) because  $p \leq 2$ , we have

$$W_p^{[q]}(\rho^N(t)_{N:k}^s, \tilde{\rho}^N(t)_{N:k}^s) \leq \left(\frac{k}{N}\right)^{1/q} W_2^{[1]}(\rho^N(t), \tilde{\rho}^N(t)) \quad (146)$$

where we insist that the latter Wasserstein distance  $W_2$  is computed with respect to the distance  $d_{(\Omega \times \mathbb{R}^d)_N}^{[1]}$  defined by (??): the choice of  $q = 1$  is important. Given that  $\rho^N(t) = \Phi^N(t)_* \rho_0^N$  and  $\tilde{\rho}^N(t) = \varphi_{\mu_0}(t)_*^{\otimes N} \rho_0^N$ , to estimate this distance, we apply Lemma ?? (in Appendix ??) with  $\Lambda = \Omega^N$  and  $E = \mathbb{R}^{dN}$  to the flows  $\Phi^N(t)$  and  $\varphi_{\mu_0}(t)^{\otimes N}$  in the space  $\Omega^N \times \mathbb{R}^{dN}$  endowed with the



distance  $d_{(\Omega \times \mathbb{R}^d)^N}^{[1]}$ , respectively generated by the time-dependent vector fields  $Y^N(t, \cdot, \cdot)$  (defined by (??)) and  $\mathcal{X}[\mu_t](t, \cdot, \cdot)^{\otimes N}$  (with  $\mathcal{X}[\mu_t]$  defined by (??)), obtaining from the alternative estimate of that lemma, with  $p = 2$ , that

$$W_2^{[1]}(\rho^N(t), \tilde{\rho}^N(t)) \leq M_2(t) \sqrt{t} \left( \frac{e^{tL_2(t)} - 1}{L_2(t)} \right)^{1/2} \quad (147)$$

where

$$L_2(t) = \max_{0 \leq \tau \leq t} \text{Lip}(Y^N(\tau, \cdot, \cdot)|_{\text{supp}(\rho^N(\tau)) \cup \text{supp}(\tilde{\rho}^N(\tau))})$$

and, using (??),

$$\begin{aligned} M_2(t) &= \max_{0 \leq \tau \leq t} \left( \int_{\Omega^N \times \mathbb{R}^{dN}} \|Y^N(\tau, \cdot, \cdot) - \mathcal{X}[\mu_\tau](\tau, \cdot, \cdot)^{\otimes N}\|_{\ell^1}^2 d\tilde{\rho}_\tau^N \right)^{1/2} \\ &= \max_{0 \leq \tau \leq t} \left( \int_{\Omega^N \times \mathbb{R}^{dN}} \left( \sum_{i=1}^N \|Y_i^N(\tau, X, \Xi) - \mathcal{X}[\mu_\tau](\tau, x_i, \xi_i)\| \right)^2 d\tilde{\rho}_\tau^N(X, \Xi) \right)^{1/2} \end{aligned}$$

where we recall that  $\|\Xi\|_{\ell^1} = \sum_{i=1}^N \|\xi_i\|$  for any  $\Xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^d)^N$ . Let us estimate  $L_2(t)$  and  $M_2(t)$ .

Since the  $\ell^1$  distance  $d_{(\Omega \times \mathbb{R}^d)^N}^{[1]}$  has been used, according to Lemma ?? in Section ?? we have, using the definition (??) of  $Y_i$ ,

$$\begin{aligned} L_2(t) &= \max_{0 \leq \tau \leq t} \max_{1 \leq i \leq N} \text{Lip}(Y_i^N(\tau, \cdot, \cdot)|_{\text{supp}(\rho^N(t)) \cup \text{supp}(\tilde{\rho}^N(t))}) = \max_{0 \leq \tau \leq t} \text{Lip}(G(\tau, \cdot, \cdot, \cdot, \cdot)|_{S_\mu^N(\tau)^2}) \\ &\leq \max_{0 \leq \tau \leq t} \|G(\tau, \cdot, \cdot, \cdot, \cdot)|_{S_\mu^N(\tau)^2}\|_{\mathcal{C}^{0,1}} = L(t) \end{aligned}$$

where  $S_\mu^N(\tau)$  is defined by (??). The choice  $q = 1$  has been crucial here (for a choice  $q > 1$  we would get a positive power of  $N$  in the exponential term in (??), which is not desirable).

Besides, by Lemma ?? in Appendix ?? (the choice of  $p = 2$  has been done to be able to apply this lemma), we have

$$M_2(t) \leq 2L(t) \left( \sqrt{N} \sqrt{1 + 70 \max_{0 \leq \tau \leq t} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu(t)))} + N \sqrt{5 W_1(\mu(t)_{X^N}^{se}, \mu(t))} \right).$$

Since the map  $s \mapsto \frac{e^{ts} - 1}{s}$  is increasing for  $s > 0$ , and since  $\sqrt{y(e^y - 1)} \leq e^y$  for every  $y \geq 0$ , we infer from (??) and (??) that

$$W_2^{[q]}(\rho^N(t)_{N:k}^s, \tilde{\rho}^N(t)_{N:k}^s) \leq 2 \left( \frac{k}{N} \right)^{1/q} \left( \sqrt{N} C'_\mu(t) + N \sqrt{5 W_1(\mu(t)_{X^N}^{se}, \mu(t))} \right) e^{tL(t)}.$$

where  $C'_\mu(t) = (1 + 70 \max_{0 \leq \tau \leq t} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu(t))))^{1/2}$ , for every  $t \geq 0$ . Applying Lemma ?? (in Appendix ??) with  $\Lambda = \Omega$  and  $E = \mathbb{R}^d$  to the Vlasov flow  $\varphi_{\mu_0}(t)$  in  $\Omega \times \mathbb{R}^d$  generated by the vector field  $\mathcal{X}[\mu_t](t, \cdot, \cdot)$ , we obtain

$$W_1(\mu(t)_{X^N}^{se}, \mu(t)) \leq e^{tL(t)} W_1((\mu_0)_{X^N}^{se}, \mu_0).$$

Finally,

$$W_p^{[q]}(\rho^N(t)_{N:k}^s, \tilde{\rho}^N(t)_{N:k}^s) \leq 2k^{1/q} \left( \frac{C'_\mu(t)}{N^{\frac{1}{q} - \frac{1}{2}}} + N^{1 - \frac{1}{q}} \sqrt{5 W_1((\mu_0)_{X^N}^{se}, \mu_0)} \right) e^{2tL(t)} \quad (148)$$

**Second term.** Applying Lemma ?? (in Appendix ??) with  $\Lambda = \Omega^k$  and  $E = (\mathbb{R}^d)^k$  to the Vlasov flow  $\varphi_{\mu_0}(t)^{\otimes k}$  in the space  $\Omega^k \times (\mathbb{R}^d)^k$  endowed with the distance  $d_{(\Omega \times \mathbb{R}^d)^k}^{[q]}$  defined by (??), generated by the vector field  $\mathcal{X}[\mu_t](t, \cdot, \cdot)^{\otimes k}$ , we obtain

$$W_p^{[q]}(\tilde{\rho}^N(t)_{N:k}^s, \mu(t)^{\otimes k}) = W_p^{[q]}(\varphi_{\mu_0}(t)_{*}^{\otimes k}(\rho_0^N)_{N:k}^s, \varphi_{\mu_0}(t)_{*}^{\otimes k}\mu_0^{\otimes k}) \leq e^{tL(t)}W_p^{[q]}((\rho_0)_{N:k}^s, \mu_0^{\otimes k})$$

where  $L(t)$  is defined as before.

Similarly as in the proof of Theorem ?? (see Appendix ??), we note that, for any  $k \in \{1, \dots, N\}$ , the measure  $(\rho_0^N)_{N:k}^s$  is given by the formula (??) of Lemma ?? in Appendix ?? with  $\beta_k$  given by (??) and  $\gamma_i = \mu_{0,x_i^N}$ . Hence we infer from (??) in Lemma ?? (Appendix ??) that, if  $k^2 \leq 2N \ln(1 + \frac{1}{2^p})$ , since  $(\rho_0^N)_{N:1}^s = (\mu_0)_{X^N}^{se}$ ,

$$W_p^{[q]}((\rho_0^N)_{N:k}^s, ((\mu_0)_{X^N}^{se})^{\otimes k}) \leq 2k^{1/q} \left(\frac{k^2}{N}\right)^{1/p} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu_0))$$

(the above term is zero and thus does not appear in the final estimate when  $k = 1$ ). Therefore, by the triangular inequality and by (??) in Lemma ?? (Appendix ??),

$$\begin{aligned} W_p^{[q]}(\tilde{\rho}^N(t)_{N:k}^s, \mu(t)^{\otimes k}) &\leq \left(W_p^{[q]}((\rho_0)_{N:k}^s, ((\mu_0)_{X^N}^{se})^{\otimes k}) + W_p^{[q]}(((\mu_0)_{X^N}^{se})^{\otimes k}, \mu_0^{\otimes k})\right) e^{tL(t)} \\ &\leq k^{1/q} \left(2 \left(\frac{k^2}{N}\right)^{1/p} \text{diam}_{\Omega \times \mathbb{R}^d}(\text{supp}(\mu_0)) + W_p((\mu_0)_{X^N}^{se}, \mu_0)\right) e^{tL(t)}. \end{aligned} \quad (149)$$

**Conclusion.** From (??), (??) and (??), we conclude that, for every  $t \geq 0$ ,

$$\begin{aligned} W_p^{[q]}(\rho^N(t)_{N:k}^s, \mu(t)^{\otimes k}) &\leq 2k^{1/q} \left( \left(\frac{k^2}{N}\right)^{1/p} C'_\mu(0) + \frac{C'_\mu(t)}{N^{\frac{1}{q}-\frac{1}{2}}} \right. \\ &\quad \left. + N^{1-\frac{1}{q}} \sqrt{5 W_1((\mu_0)_{X^N}^{se}, \mu_0)} + W_p((\mu_0)_{X^N}^{se}, \mu_0) \right) e^{2tL(t)} \end{aligned}$$

and (??) finally follows.

To obtain the statement ?? for any  $k \in \mathbb{N}^*$ , we have to adapt all the above arguments and in particular Lemma ?? in Appendix ?? to the case where  $G$  is locally Lipschitz only with respect to  $(\xi, \xi')$ . This is lengthy but straightforward and we do not give any details.

## B.4 Proof of Theorem ??

We start by proving the second item of Theorem ?. Hence, we assume that  $G$  is locally  $\alpha$ -Hölder continuous with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact).

**Lemma 34.** *Let  $x, x' \in \Omega$  be arbitrary. We have*

$$\|y(t, x) - y(t, x')\| \leq e^{tL_y(t)} (\|y^0(x) - y^0(x')\| + d_\Omega(x, x')^\alpha) \quad \forall t \geq 0. \quad (150)$$

*Proof of Lemma ??.* By definition, we have  $\partial_t y(t, z) = \int_\Omega G(t, z, x'', y(t, z), y(t, x'')) d\nu(x'')$  for every  $z \in \Omega$ , hence

$$\begin{aligned} \partial_t y(t, x) - \partial_t y(t, x') &= \int_\Omega G(t, x, x'', y(t, x), y(t, x'')) d\nu(x'') - \int_\Omega G(t, x', x'', y(t, x), y(t, x'')) d\nu(x'') \\ &\quad + \int_\Omega G(t, x', x'', y(t, x), y(t, x'')) d\nu(x'') - \int_\Omega G(t, x', x'', y(t, x'), y(t, x'')) d\nu(x'') \end{aligned} \quad (151)$$

and using the definition of  $L_y(t)$  we obtain

$$\|\partial_t(y(t, x) - y(t, x'))\| \leq L_y(t) (d_\Omega(x, x')^\alpha + \|y(t, x) - y(t, x')\|)$$

and (??) follows by integration (noting that  $\tau \mapsto L_y(\tau)$  is nondecreasing).  $\square$

By assumption,  $\|y^0(x) - y^0(x')\| \leq \text{Hol}_\alpha(y^0)d_\Omega(x, x')^\alpha$  for all  $x, x' \in \Omega$ , hence, using (??) in Lemma ?? we infer that  $y(t, \cdot)$  is  $\alpha$ -Hölder continuous and (??) follows.

Let us establish (??). We set  $r_i^N(t) = y(t, x_i^N) - \xi_i^N(t)$ , for  $i = 1, \dots, N$ . By definition, we have

$$r_i^N(t) = \frac{1}{N} \sum_{j=1}^N (G(t, x_i^N, x_j^N, y(t, x_i^N), y(t, x_j^N))) - G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t)) + \epsilon_i^N(t) \quad (152)$$

where

$$\epsilon_i^N(t) = \int_\Omega G(t, x_i^N, x', y(t, x_i^N), y(t, x')) d\nu(x') - \frac{1}{N} \sum_{j=1}^N G(t, x_i^N, x_j^N, y(t, x_i^N), y(t, x_j^N)) \quad (153)$$

with  $r_i^N(0) = 0$ , for every  $i \in \{1, \dots, N\}$ . On the one part, we have

$$\|G(t, x_i^N, x_j^N, y(t, x_i^N), y(t, x_j^N)) - G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t))\| \leq L_y^N(t) (\|r_i^N(t)\| + \|r_j^N(t)\|) \quad (154)$$

where  $L_y^N(t)$  is defined by (??). On the other part, using (??) in Lemma ?? (see Appendix ??), we have

$$\|\epsilon_i^N(t)\| \leq \frac{C_\Omega^\alpha}{N r_\alpha} \text{Hol}_\alpha(x' \mapsto G(t, x_i^N, x', y(t, x_i^N), y(t, x'))) \quad (155)$$

and we claim that

$$\text{Hol}_\alpha(x' \mapsto G(t, x_i^N, x', y(t, x_i^N), y(t, x'))) \leq L_y(t) (1 + e^{tL_y(t)} (\text{Hol}_\alpha(y^0) + 1)). \quad (156)$$

Indeed, writing for short  $g(x', y(t, x')) = G(t, x_i^N, x', y(t, x_i^N), y(t, x'))$ , we have

$$\begin{aligned} & \|g(x'_1, y(t, x'_1)) - g(x'_2, y(t, x'_2))\| \\ & \leq \|g(x'_1, y(t, x'_1)) - g(x'_2, y(t, x'_1))\| + \|g(x'_2, y(t, x'_1)) - g(x'_2, y(t, x'_2))\| \\ & \leq L_y(t) d_\Omega(x'_1, x'_2)^\alpha + L_y(t) \|y(t, x'_1) - y(t, x'_2)\| \\ & \leq L_y(t) d_\Omega(x'_1, x'_2)^\alpha + L_y(t) \text{Hol}_\alpha(y(t, \cdot)) d_\Omega(x'_1, x'_2)^\alpha \end{aligned}$$

and (??) follows by using (??). Finally, setting  $R^N(t) = (r_1^N(t), \dots, r_N^N(t))$ , noting that  $L_y(t) \leq L_y^N(t)$ , we infer from (??), (??), (??) and (??) that

$$\frac{d}{dt} \|R^N(t)\|_\infty \leq \|\dot{R}^N(t)\|_\infty \leq L_y^N(t) \left( 2\|R^N(t)\|_\infty + \frac{C_\Omega^\alpha}{N r_\alpha} (1 + e^{tL_y^N(t)} (\text{Hol}_\alpha(y^0) + 1)) \right)$$

and, noting that  $\tau \mapsto L_y^N(\tau)$  (defined by (??)) is nondecreasing and by integration, we obtain (??).

Let us establish (??). For every  $x \in \Omega$  there exists  $i \in \{1, \dots, N\}$  such that  $x \in \Omega_i^N$ , and thus  $d_\Omega(x, x_i^N) \leq \text{diam}_\Omega(\Omega_i^N) \leq \frac{C_\Omega}{N^r}$  (by (??)). It follows from (??) that

$$\|y(t, x) - y(t, x_i^N)\| \leq \text{Hol}_\alpha(y(t, \cdot)) d_\Omega(x, x_i^N)^\alpha \leq \frac{C_\Omega^\alpha}{N r_\alpha} e^{tL_y^N(t)} (\text{Hol}_\alpha(y(0, \cdot)) + 1)$$

and, noting that  $y^N(t, x) = \xi_i^N(t)$ , (??) follows by the triangular inequality, using (??).

Let us now prove the first item of Theorem ???. Starting as in the proof of Lemma ??, by continuity of  $G$ , we infer from (??) that, for any  $\varepsilon > 0$ , if  $x$  and  $x'$  are sufficiently close then

$$\|\partial_t(y(t, x) - y(t, x'))\| \leq L_y(t) (\varepsilon + \|y(t, x) - y(t, x')\|)$$

and by integration we obtain

$$\|y(t, x) - y(t, x')\| \leq e^{tL_y(t)} (\|y^0(x) - y^0(x')\| + \varepsilon). \quad (157)$$

By assumption,  $y^0$  is continuous  $\nu$ -almost everywhere on  $\Omega$ . It follows from (??) that, for every  $t \geq 0$ ,  $y(t, \cdot)$  is continuous  $\nu$ -almost everywhere on  $\Omega$  with the same continuity set as  $y^0$  (thus, not depending on  $t$ ).

Let us finally establish (??). By the Riemann integration theorem (see (??)), we have  $\varepsilon_i^N(t) = o(1)$  (where  $\varepsilon_i^N(t)$  is defined by (??)) as  $N \rightarrow +\infty$ , uniformly with respect to  $t$  on every compact. Besides, we still have the inequality (??), but with  $L_y^N(t)$  now defined by

$$L_y^N(t) = \max_{\substack{x, x' \in \Omega \\ 0 \leq \tau \leq t}} \text{Lip}(G(\tau, x, x', \cdot, \cdot)|_{S_y^N(\tau)^2},$$

i.e., like in (??) but without the first term involving  $\text{Hol}_\alpha(G)$ . With this substitution, we obtain

$$\frac{d}{dt} \|R^N(t)\|_\infty \leq \|\dot{R}^N(t)\|_\infty \leq L_y^N(t) (2\|R^N(t)\|_\infty + o(1))$$

and integrating we get  $\|R^N(t)\|_\infty \leq e^{2tL_y^N(t)} o(1)$ , which yields (??), noting that  $L_y^N(t)$  is uniformly bounded with respect to  $t \in [0, T]$  and to  $N \in \mathbb{N}^*$  (as a consequence of Lemma ??). Then, (??) follows by the triangular inequality, using the  $\nu$ -almost everywhere continuity of  $y(t, \cdot)$ .

## B.5 Proof of Theorem ???

The proof is a slight adaptation of the proof of Theorem ???. We start by establishing (??). Hence, we assume that  $G$  is locally  $\alpha$ -Hölder continuous with respect to  $(x, x', \xi, \xi')$  (uniformly with respect to  $t$  on any compact).

**Lemma 35.** *Let  $i \in \{1, \dots, N\}$  and  $x, x' \in \Omega_i^N$  be arbitrary. We have*

$$\|y_N(t, x) - y_N(t, x')\| \leq e^{tL_{y_N}(t)} d_\Omega(x, x')^\alpha \quad (158)$$

where  $y^0(\cdot) = y(0, \cdot)$ , where  $L_{y_N}(t)$  is defined as  $L_y(t)$  in Theorem ??? with  $y$  replaced by  $y_N$ .

*Proof.* Following the proof of Lemma ??, we arrive at

$$\|\partial_t(y_N(t, x) - y_N(t, x'))\| \leq L_{y_N}(t) (d_\Omega(x, x')^\alpha + \|y_N(t, x) - y_N(t, x')\|)$$

and (??) follows by integration, noting that  $y_N(0, x) - y_N(0, x') = 0$  if  $x, x' \in \Omega_i^N$ .  $\square$

It follows from Lemma ?? that  $y_N(t, \cdot)$  is  $\alpha$ -Hölder continuous in each  $\Omega_i$ , with Hölder constant  $e^{tL_{y_N}(t)}$ .

We set  $r^N(t, x) = y_N(t, x) - y^N(t, x)$  for every  $x \in \Omega$ . By definition, if  $x \in \Omega_i^N$  then  $y^N(t, x) = \xi_i^N(t)$  and thus

$$\partial_t r^N(t, x) = \frac{1}{N} \sum_{j=1}^N (G(t, x, x_j^N, y_N(t, x), y_N(t, x_j^N)) - G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t))) + \epsilon^N(t, x)$$

where

$$\epsilon^N(t, x) = \int_{\Omega} G(t, x, x', y_N(t, x), y_N(t, x')) d\nu(x') - \frac{1}{N} \sum_{j=1}^N G(t, x, x_j^N, y_N(t, x), y_N(t, x_j^N))$$

with  $r^N(0, x) = 0$ . We have on the one part

$$\begin{aligned} & \|G(t, x, x_j^N, y_N(t, x), y_N(t, x_j^N)) - G(t, x_i^N, x_j^N, \xi_i^N(t), \xi_j^N(t))\| \\ & \leq L_{y_N}^N(t)(d_{\Omega}(x, x_i^N)^{\alpha} + \|r^N(t, x)\| + \|r^N(t, x_j^N)\|) \quad \forall x \in \Omega_i^N \end{aligned}$$

and on the other part, proceeding like in the proof of Theorem ?? (see Appendix ??),

$$\|\epsilon^N(t, x)\| \leq \frac{C_{\Omega}^{\alpha}}{Nr^{\alpha}} L_{y_N}^N(t)(1 + e^{tL_{y_N}^N(t)}) \quad \forall x \in \Omega_i^N.$$

Using that  $d_{\Omega}(x, x_i^N) \leq \text{diam}_{\Omega}(\Omega_i^N) \leq \frac{C_{\Omega}}{Nr}$  (see (??)), we finally obtain

$$\frac{d}{dt} \|r^N(t, \cdot)\|_{L^{\infty}(\Omega)} \leq \|\partial_t r^N(t, \cdot)\|_{L^{\infty}(\Omega)} \leq L_{y_N}^N(t) \left( 2\|r^N(t, \cdot)\|_{L^{\infty}(\Omega)} + \frac{C_{\Omega}^{\alpha}}{Nr^{\alpha}} (2 + e^{tL_{y_N}^N(t)}) \right)$$

and by integration, noting that  $\tau \mapsto L_{y_N}^N(\tau)$  is nondecreasing, (??) follows.

Finally, (??) is established as in the proof of Theorem ??.

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