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► **To cite this version:**

Titus Lupu. An equivalence between gauge-twisted and topologically conditioned scalar Gaussian free fields. 2022. hal-03779011v1

HAL Id: hal-03779011

<https://hal.science/hal-03779011v1>

Preprint submitted on 28 Sep 2022 (v1), last revised 11 Oct 2023 (v2)

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AN EQUIVALENCE BETWEEN GAUGE-TWISTED AND TOPOLOGICALLY CONDITIONED SCALAR GAUSSIAN FREE FIELDS

TITUS LUPU

ABSTRACT. We study on the metric graphs two types of scalar Gaussian free fields (GFF), the usual one and the one twisted by a $\{-1, 1\}$ -valued gauge field. We show that the latter can be obtained, up to an additional deterministic transformation, by conditioning the first on a topological event. This event is that all the sign clusters of the field should be trivial for the gauge field, that is to say should not contain loops with holonomy -1 . We also express the probability of this topological event as a ratio of two determinants of Laplacians to the power $1/2$, the usual Laplacian and the gauge-twisted Laplacian. As an example, this gives on annular planar domains the probability that no sign cluster of the metric graph GFF surrounds the inner hole of the domain.

1. INTRODUCTION

In this article we consider two types of scalar Gaussian free fields (GFF), the usual one and the one twisted by a $\{-1, 1\}$ -valued gauge field, and observe that the second is essentially obtained from the first by conditioning on a topological (more precisely homotopical) event.

We will work on an abstract finite electrical network $\mathcal{G} = (V, E)$ endowed with conductances $C(x, y) = C(y, x) > 0$ for $\{x, y\} \in E$. A the set of vertices V will be divided into two parts, V_{int} and V_{∂} , with V_{int} being considered as the interior vertices, and V_{∂} as the boundary. The discrete GFF ϕ with 0 boundary conditions on V_{∂} is given by the distribution

$$(1.1) \quad \frac{1}{Z} \exp\left(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x,y)(\phi(y) - \phi(x))^2\right) \prod_{z \in V_{\text{int}}} d\phi(z).$$

Further, in the language of the gauge theory, we consider $\{-1, 1\}$ as our gauge group, and take a gauge field $\sigma \in \{-1, 1\}^E$. The σ -twisted GFF ϕ_{σ} with 0 boundary conditions on V_{∂} has for distribution

$$(1.2) \quad \frac{1}{Z_{\sigma}} \exp\left(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x,y)(\sigma(x,y)\phi(y) - \phi(x))^2\right) \prod_{z \in V_{\text{int}}} d\phi(z).$$

The gauge field σ corresponds to disorder operators in the language of the Ising model [KC71].

To see the relation between ϕ and ϕ_{σ} one has to look at the level of metric graphs. The metric graph $\tilde{\mathcal{G}}$ associated to \mathcal{G} is obtained by replacing each discrete edge $\{x, y\}$ by a continuous line of length $C(x, y)^{-1}$ joining x and y . The discrete GFF ϕ has a natural extension $\tilde{\phi}$ to the metric graph $\tilde{\mathcal{G}}$, which is a continuous Gaussian random field satisfying a Markov property. This extension has been introduced in [Lup16]. The field $\tilde{\phi}$, unlike ϕ , is known to satisfy some exact identities, including for instance the probabilities of crossings. These relations have been explored in the articles [Lup16, LW18, DPR22, DPR21]; see Section 2.4 for details. The twisted discrete GFF ϕ_{σ} also has a natural extension $\tilde{\phi}_{\sigma}$ to the metric graph $\tilde{\mathcal{G}}$. It is introduced in this paper in Section 3.1. Unlike $\tilde{\phi}$, the field $\tilde{\phi}_{\sigma}$ is not continuous in general, and has one discontinuity point inside each $e \in E$ for which $\sigma(e) = -1$. However, the absolute value $|\tilde{\phi}_{\sigma}|$ is a continuous field on the whole $\tilde{\mathcal{G}}$, since the discontinuities of $\tilde{\phi}_{\sigma}$ consist in switching to the opposite sign by keeping the same absolute value. However, by taking a double cover of $\tilde{\mathcal{G}}$ induced by the gauge field σ , one can extend $\tilde{\phi}_{\sigma}$ to a continuous field on the double cover. This is explained in Section 3.3.

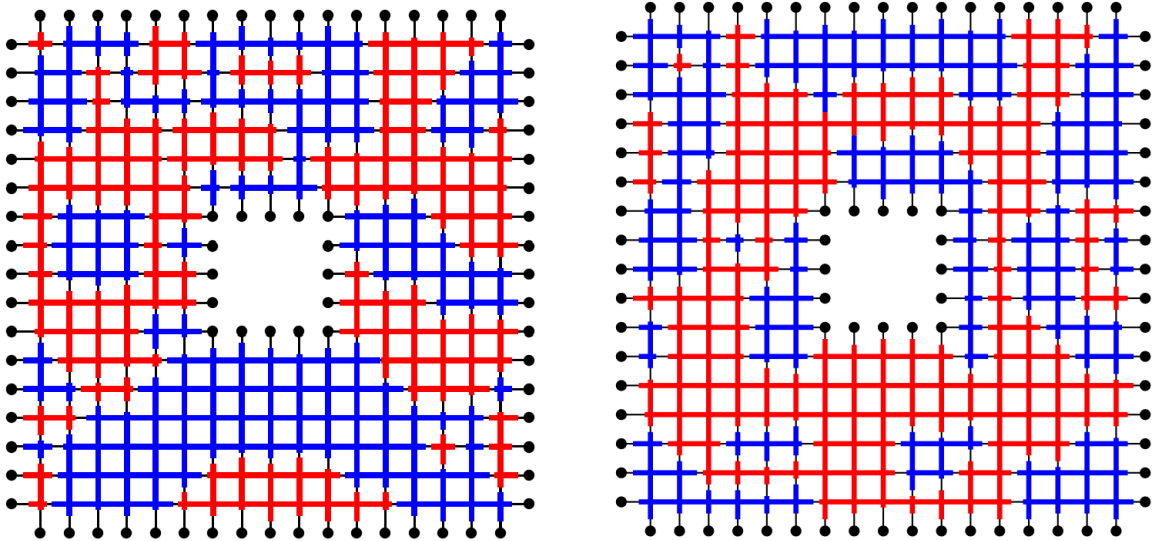


FIGURE 1. Conceptual depiction of the metric graph GFF $\tilde{\phi}$ on an annular domain. The black dots represent the boundary V_∂ . The positive, resp. negative values of the fields are in red, resp. blue. Left: no sign cluster of $\tilde{\phi}$ surrounds the inner hole of the annulus. Right: there is a sign cluster (in red) surrounding the inner hole.

Let \mathcal{T}_σ denote the subset of continuous functions on $\tilde{\mathcal{G}}$, made of functions f such that for every connected component U of the non-zero set $\{f \neq 0\}$, U does not contain loops of holonomy -1 for σ . For the notion of holonomy in this setting (product of the values of σ along the edges of the loop), we refer to Section 2.1. But let us give an example. Consider a planar annular domain (one hole), such as depicted on Figures 3 and 1. One can consider a gauge field σ on this annular domain that gives a holonomy -1 to loops that turn an odd number of times around the inner hole, and holonomy 1 to other loops; see Figure 3. Then $f \in \mathcal{T}_\sigma$ if and only if no connected component of $\{f \neq 0\}$ surrounds the inner hole; see Figure 1.

It is easy to see that $|\tilde{\phi}_\sigma| \in \mathcal{T}_\sigma$ a.s., and this is proved in Lemma 3.15 by relying on the extension of $\tilde{\phi}_\sigma$ to the double cover of $\tilde{\mathcal{G}}$. Our main result is the following.

Theorem 1. *Let be a gauge field $\sigma \in \{-1, 1\}^E$. Then*

$$(1.3) \quad \mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) = \frac{Z_\sigma}{Z},$$

where Z and Z_σ are the partition functions appearing in (1.1) and (1.2). Moreover, conditionally on the event $\{\tilde{\phi} \in \mathcal{T}_\sigma\}$, the field $|\tilde{\phi}|$ has the same distribution as $|\tilde{\phi}_\sigma|$.

To obtain the field $\tilde{\phi}_\sigma$, rather than just the absolute value $|\tilde{\phi}_\sigma|$, from the field $\tilde{\phi}$ conditioned on $\tilde{\phi} \in \mathcal{T}_\sigma$, one has to additionally apply a deterministic sign flipping procedure across the discontinuity points. This is explained in Corollary 3.24.

The identity (1.3) is thus a newcomer to the family of exact identities known to be satisfied by $\tilde{\phi}$. We would like to emphasize that Theorem 1 does not require at all the graph \mathcal{G} to be planar. However, for planar graphs the subset \mathcal{T}_σ is simpler to interpret.

This article is organized as follows. Section 2 consists of preliminaries where we recall some background that is maybe not common knowledge. In Section 2.1 we recall the notions of gauge field, gauge equivalence and holonomy in our particular setting where the gauge group is $\{-1, 1\}$. Section 2.2 deals with the discrete GFFs, the usual one and the gauge twisted. Section 2.3 deals with random walk representations of these fields. Section 2.4 recalls the method of the metric graph and some results for the metric graph GFF. In Section 3 we present the results of this article and provide the corresponding proofs. In Section 3.1 we introduce the natural

extrapolation $\tilde{\phi}_\sigma$ of the gauge-twisted GFF ϕ_σ to the metric graph. In Section 3.2 we consider the double cover of the discrete graph induced by the gauge field and observe that the usual discrete GFF and the gauge twisted one are projections of the discrete GFF on the double cover on two orthogonal subspaces. In Section 3.3 we do the same at the level of the metric graph, which in particular provides us a continuous extension of $\tilde{\phi}_\sigma$ to the double cover of the metric graph. In Section 3.4 we give a more detailed statement of Theorem 1 and then prove it. In Section 3.5 we provide an isomorphism between $\tilde{\phi}_\sigma$ and Brownian loop soups on the metric graph that strengthens the result of Kassel and Lévy [KL21].

2. PRELIMINARIES

2.1. On gauge fields, holonomy and gauge equivalence. Let $\mathcal{G} = (V, E)$ be a finite connected undirected graph. We assume that there are no self-loops or multi-edges. We also assume that the set of vertices consists of two disjoint parts, $V = V_{\text{int}} \cup V_\partial$, $V_{\text{int}} \cap V_\partial = \emptyset$, with both V_{int} and V_∂ non-empty. We see V_{int} as the interior vertices and V_∂ as boundary vertices. Each edge $\{x, y\} \in E$ is endowed with a conductance $C(x, y) = C(y, x) > 0$. Thus, \mathcal{G} is an electrical network.

In this paper, a *gauge field* will mean a family $(\sigma(e))_{e \in E} \in \{-1, 1\}^E$. This is the simplest case when the gauge group is $\{-1, 1\}$. We will also use the notation $\sigma(x, y) = \sigma(\{x, y\})$, when $\{x, y\} \in E$. Given an other collection of signs $(\hat{\sigma}(x))_{x \in V} \in \{-1, 1\}^V$, this time above the vertices, it induces a *gauge transformation* $\sigma \mapsto \hat{\sigma} \cdot \sigma$, where $\hat{\sigma} \cdot \sigma \in \{-1, 1\}^E$ is the gauge field defined by

$$(\hat{\sigma} \cdot \sigma)(x, y) = \hat{\sigma}(x)\sigma(x, y)\hat{\sigma}(y).$$

Two gauge fields $\sigma, \sigma' \in \{-1, 1\}^E$ are said to be *gauge-equivalent* if there is $\hat{\sigma} \in \{-1, 1\}^V$ such that $\sigma' = \hat{\sigma} \cdot \sigma$. A gauge field $\sigma \in \{-1, 1\}^E$ is said *trivial* if it is gauge-equivalent to the gauge field with value 1 on every edge.

Given $\wp = (x_1, x_2, \dots, x_n)$ a discrete nearest-neighbor path in \mathcal{G} , the *holonomy* of $\sigma \in \{-1, 1\}^E$ along \wp is the product

$$\text{hol}^\sigma(\wp) = \sigma(x_1, x_2)\sigma(x_2, x_3) \dots \sigma(x_{n-1}, x_n).$$

If the nearest-neighbor path \wp with finitely many jumps is parametrized by continuous time, then $\text{hol}^\sigma(\wp)$ is the holonomy along the discrete skeleton of \wp .

- Lemma 2.1.** (1) *Assume that two gauge fields $\sigma, \sigma' \in \{-1, 1\}^E$ are gauge-equivalent. Then for every loop (i.e. closed path) $\wp = (x_1, x_2, \dots, x_{n-1}, x_n, x_1)$ in \mathcal{G} , $\text{hol}^\sigma(\wp) = \text{hol}^{\sigma'}(\wp)$.*
(2) *Conversely, assume that there is $x_1 \in V$ such that for every loop $\wp = (x_1, x_2, \dots, x_{n-1}, x_n, x_1)$ rooted in x_1 , $\text{hol}^\sigma(\wp) = \text{hol}^{\sigma'}(\wp)$. Then σ and σ' are gauge equivalent.*
(3) *In particular, a gauge field $\sigma \in \{-1, 1\}^E$ is trivial if and only if for every loop \wp , $\text{hol}^\sigma(\wp) = 1$.*

Proof. (1) If $\sigma' = \hat{\sigma} \cdot \sigma$, then for every path $\wp = (x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1})$,

$$\text{hol}^{\sigma'}(\wp) = \hat{\sigma}(x_1)\text{hol}^\sigma(\wp)\hat{\sigma}(x_{n+1}).$$

In particular, if $x_{n+1} = x_1$, $\text{hol}^{\sigma'}(\wp) = \text{hol}^\sigma(\wp)$.

(2) For each $x \in V$, take a path $\wp^{x_1, x}$ from x_1 to x in \mathcal{G} and set

$$\hat{\sigma}(x) = \text{hol}^\sigma(\wp^{x_1, x})\text{hol}^{\sigma'}(\wp^{x_1, x}).$$

The value of $\hat{\sigma}(x)$ does not depend on the particular choice of the path $\wp^{x_1, x}$. Indeed, if $\bar{\wp}^{x_1, x}$ is an other path from x_1 to x , then one can concatenate $\bar{\wp}^{x_1, x}$ with the time reversal $\overleftarrow{\wp}^{x_1, x}$ of $\wp^{x_1, x}$, so as to get a loop $\bar{\wp}^{x_1, x} \wedge \overleftarrow{\wp}^{x_1, x}$ from x_1 to x_1 , and then

$$\text{hol}^\sigma(\wp^{x_1, x})\text{hol}^\sigma(\bar{\wp}^{x_1, x}) = \text{hol}^\sigma(\bar{\wp}^{x_1, x} \wedge \overleftarrow{\wp}^{x_1, x}) = \text{hol}^{\sigma'}(\bar{\wp}^{x_1, x} \wedge \overleftarrow{\wp}^{x_1, x}) = \text{hol}^{\sigma'}(\wp^{x_1, x})\text{hol}^{\sigma'}(\bar{\wp}^{x_1, x}).$$

With $\hat{\sigma}$ defined in this way, we have that $\sigma' = \hat{\sigma} \cdot \sigma$. Note that $\hat{\sigma}$ is not the only element of $\{-1, 1\}^V$ to relate σ and σ' through induced gauge transformation. With \mathcal{G} being connected, there are exactly two such elements of $\{-1, 1\}^V$, $\hat{\sigma}$ and $-\hat{\sigma}$. \square

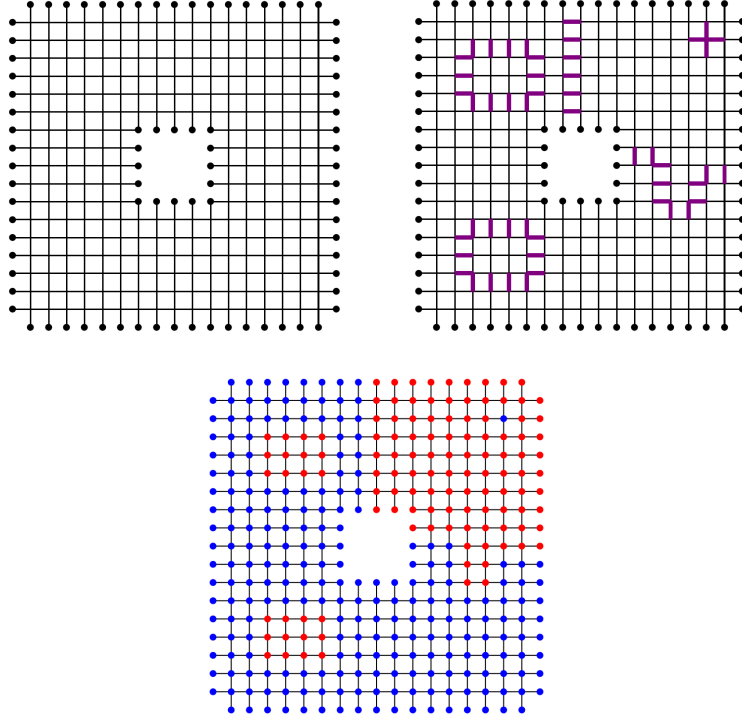


FIGURE 2. Top: two trivial gauge fields. The edges with sign -1 are in violet. Bottom: a gauge transformation relating the above gauge fields. In red are the $+1$ vertices and in blue the -1 vertices. The black dots represent the boundary V_∂ .

Figures 2, 3 and 4 provide examples of gauge-equivalent gauge fields together with the corresponding gauge transformation.

2.2. Discrete scalar GFF twisted by a gauge field. Let $\Delta_{\mathcal{G}}$ denote the discrete Laplacian on \mathcal{G} :

$$(\Delta_{\mathcal{G}}f)(x) = \sum_{y \sim x} C(x, y)(f(y) - f(x)), \quad x \in V,$$

where $y \sim x$ means that y is a neighbor of x , i.e. $\{x, y\} \in E$. Let $(G(x, y))_{x, y \in V}$ be the Green's function of $-\Delta_{\mathcal{G}}$ with 0 boundary conditions on V_∂ .

Further, if $\sigma \in \{-1, 1\}^E$ is a gauge field, the discrete Laplacian twisted by σ is

$$(\Delta_{\mathcal{G}, \sigma}f)(x) = \sum_{y \sim x} C(x, y)(\sigma(x, y)f(y) - f(x)), \quad x \in V.$$

The twisted operator $-\Delta_{\mathcal{G}, \sigma}$ is still non-negative in the sense that for every $f \in \mathbb{R}^V$,

$$\mathcal{E}_\sigma(f, f) := \sum_{x \in V} (-\Delta_{\mathcal{G}, \sigma}f)(x)f(x) = \sum_{\{x, y\} \in E} C(x, y)(\sigma(x, y)f(y) - f(x))^2 \geq 0.$$

Moreover, if f is a function such that $\mathcal{E}_\sigma(f, f) = 0$ and f is 0 on V_∂ , then f is uniformly 0 on V . Indeed, one can see this by induction on the graph distance of a vertex $x \in V$ from V_∂ . Therefore, one can consider the inverse of the restriction of the operator $-\Delta_{\mathcal{G}, \sigma}$ to the functions that are 0 on V_∂ . This is the gauge-twisted Green's function $(G_\sigma(x, y))_{x, y \in V}$ with 0 boundary conditions on V_∂ . It is defined by

$$\forall x \in V, \forall y \in V_\partial, \quad G_\sigma(x, y) = 0,$$

and

$$\forall x_0 \in V_{\text{int}}, \forall x \in V_{\text{int}}, \quad \sum_{y \sim x} C(x, y)(\sigma(x, y)G_\sigma(x_0, y) - G_\sigma(x_0, x)) = -\mathbf{1}_{x=x_0}.$$

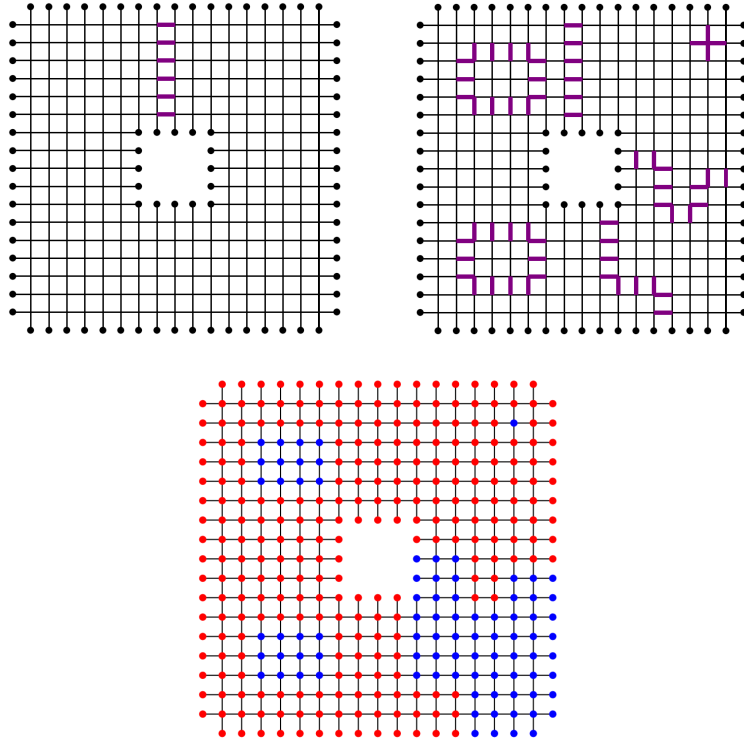


FIGURE 3. Top: two non-trivial and gauge-equivalent gauge fields. The edges with sign -1 are in violet. The holonomy of a loop is -1 to the power the number of turns it performs around the inner hole. Bottom: a gauge transformation relating the above gauge fields. In red are the $+1$ vertices and in blue the -1 vertices. The black dots represent the boundary V_{∂} .

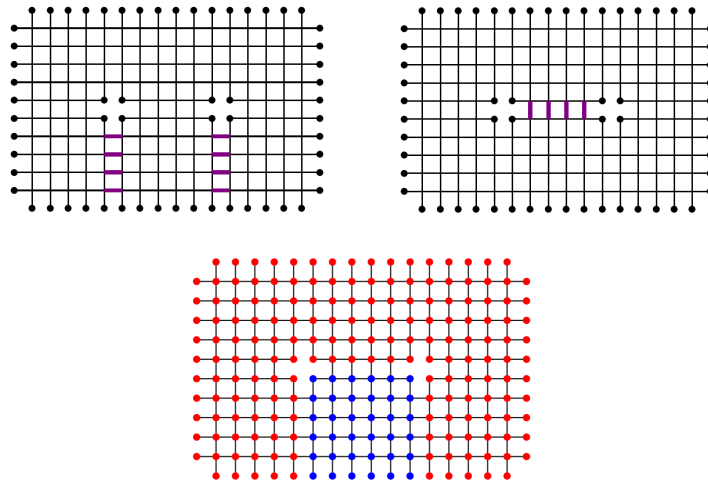


FIGURE 4. Top: two non-trivial and gauge-equivalent gauge fields. The edges with sign -1 are in violet. In the holonomy of a loop one multiplies by -1 each time one surrounds one of the two holes but not both. Bottom: a gauge transformation relating the above gauge fields. In red are the $+1$ vertices and in blue the -1 vertices. The black dots represent the boundary V_{∂} .

The function G_{σ} is symmetric: $G_{\sigma}(x, y) = G_{\sigma}(y, x)$. However, unlike for the usual Green's function $G(x, y)$, some of the values $G_{\sigma}(x, y)$ may be negative. Still, the operator induced by

G_σ is non-negative: for every $f \in \mathbb{R}^V$,

$$\sum_{x,y \in V} f(x)G_\sigma(x,y)f(y) \geq 0.$$

Further, if $\sigma, \sigma' \in \{-1, 1\}^E$ are two gauge-equivalent gauge fields, with a gauge transformation between σ and σ' induced by a $\hat{\sigma} \in \{-1, 1\}^V$, then for every $x, y \in V$,

$$G_{\sigma'}(x, y) = \hat{\sigma}(x)G_\sigma(x, y)\hat{\sigma}(y).$$

The discrete scalar *Gaussian free field* (GFF) on \mathcal{G} with boundary conditions 0 on V_∂ is the random Gaussian field $(\phi(x))_{x \in V}$, with $\phi(x) = 0$ for $x \in V_\partial$, and with the probability distribution

$$(2.1) \quad \frac{1}{Z} \exp\left(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x,y)(\varphi(y) - \varphi(x))^2\right) \prod_{z \in V_{\text{int}}} d\varphi(z).$$

The expectation of ϕ is 0. The covariance function of ϕ is the Green's function $(G(x, y))_{x,y \in V}$.

Further, given $\sigma \in \{-1, 1\}^E$ a gauge field, the GFF twisted by σ is the random Gaussian field $(\phi_\sigma(x))_{x \in V}$, with $\phi_\sigma(x) = 0$ for $x \in V_\partial$, and with the probability distribution

$$(2.2) \quad \frac{1}{Z_\sigma} \exp\left(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x,y)(\sigma(x,y)\varphi(y) - \varphi(x))^2\right) \prod_{z \in V_{\text{int}}} d\varphi(z).$$

The expectation of ϕ_σ is 0. Actually, $\phi_\sigma \stackrel{(\text{law})}{=} -\phi_\sigma$. The covariance function of ϕ_σ is the gauge-twisted Green's function $(G_\sigma(x, y))_{x,y \in V}$. If $\sigma, \sigma' \in \{-1, 1\}^E$ are two gauge-equivalent gauge fields, with a gauge transformation between σ and σ' induced by a $\hat{\sigma} \in \{-1, 1\}^V$, then

$$(\phi_{\sigma'}(x))_{x \in V} \stackrel{(\text{law})}{=} (\hat{\sigma}(x)\phi_\sigma(x))_{x \in V}.$$

In particular, if the gauge field σ is trivial, then the field ϕ_σ can be obtained via a deterministic transformation from the usual GFF ϕ .

2.3. Measures on paths. Let $(X_t)_{t \geq 0}$ be the nearest-neighbor Markov jump process on \mathcal{G} with the jump rates given by the conductances $C(x, y)$. Let T_{V_∂} denote the first hitting time of V_∂ . We will denote by $p(t, x, y)$ the transition probabilities of the killed process $(X_t)_{0 \leq t \leq T_{V_\partial}}$, so that

$$\sum_{y \in V_{\text{int}}} p(t, x, y) = \mathbb{P}_x(T_{V_\partial} > t).$$

Since the process is symmetric, $p(t, x, y) = p(t, y, x)$. Moreover,

$$\int_0^{+\infty} p(t, x, y) dt = G(x, y).$$

Given $x, y \in V_{\text{int}}$ and $t > 0$, let $\mathbb{P}_t^{x,y}$ denote the bridge probability measure from x to y of duration t , where one conditions on $T_{V_\partial} > t$. The *excursion measure* from x to y is

$$\mu^{x,y}(d\varphi) = \int_0^{+\infty} \mathbb{P}_t^{x,y}(d\varphi) p(t, x, y) dt.$$

So $\mu^{x,y}$ is a measure on nearest-neighbor paths from x to y , parametrized by continuous time, of finite duration. The total mass of $\mu^{x,y}$ is $G(x, y)$. The measure $\mu^{y,x}$ is the image of $\mu^{x,y}$ by time-reversal. The induced measure on discrete skeletons is

$$\mu^{x,y}(x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{n-1} \rightarrow y) = \frac{C(x, x_1)C(x_1, x_2) \dots C(x_{n-2}, x_{n-1})C(x_{n-1}, y)}{W(x)W(x_1)W(x_2) \dots W(x_{n-1})W(y)},$$

where

$$W(z) = \sum_{w \sim z} C(z, w).$$

Consider now $\sigma \in \{-1, 1\}^E$ a gauge field. Kassel and Lévy showed in [KL21, Theorem 5.1] the following.

Theorem 2.2 (Kassel-Lévy, [KL21]). *For every $x, y \in V_{\text{int}}$,*

$$G_\sigma(x, y) = \int \text{hol}^\sigma(\varphi) \mu^{x,y}(d\varphi).$$

In other words,

$$\frac{G_\sigma(x, y)}{G(x, y)} = \mathbb{E}[\text{hol}^\sigma(\varphi^{x,y})],$$

where $\varphi^{x,y}$ is a random excursion from x to y distributed according to the probability measure $\mu^{x,y}/G(x, y)$. In particular, for every $x \in V_{\text{int}}$,

$$\int \text{hol}^\sigma(\varphi) \mu^{x,x}(d\varphi) = G_\sigma(x, x) > 0.$$

The measure on continuous time random walk loops introduced by Le Jan [LJ10, LJ11] is

$$(2.3) \quad \mu^{\text{loop}}(d\varphi) = \sum_{x \in V_{\text{int}}} \int_0^{+\infty} \mathbb{P}_t^{x,x}(d\varphi) p(t, x, x) \frac{dt}{t} = \frac{1}{T(\varphi)} \sum_{x \in V_{\text{int}}} \mu^{x,x}(d\varphi),$$

where $T(\varphi)$ denotes the duration of a path φ . There are two types of loops, those that visit at least two different vertices, and those that stay in one vertex and do not perform jumps. For $n \geq 2$ jumps, the measure induced on discrete skeletons is

$$\mu^{\text{loop}}(x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \rightarrow x_1) = \frac{1}{n} \frac{C(x_1, x_2) \cdots C(x_{n-1}, x_n) C(x_n, x_1)}{W(x_1) W(x_2) \cdots W(x_{n-1}) W(x_n)}.$$

So this is the same measure on discrete-time loops as the one that appeared in [LTF07] and [LL10, Chapter 9]. The total mass of the loops that visit at least two vertices is

$$(2.4) \quad \mu^{\text{loop}}(\{\text{Loops that visit at least 2 vertices}\}) = \log \left(\det((G(x, y))_{x,y \in V_{\text{int}}}) \prod_{x \in V_{\text{int}}} W(x) \right);$$

see [LJ11, Equation (2.5)]. Besides the loops that visit at least two vertices, μ^{loop} also puts weight on loops that stay in one vertex and do not jump. For every $x \in V_{\text{int}}$, the induced measure on the duration of loops that only stay in x is

$$e^{-t/G(x,x)} \frac{dt}{t}.$$

Given $\sigma \in \{-1, 1\}^E$ a gauge field, the (signed) measure on loops twisted by σ is

$$\mu_\sigma^{\text{loop}}(d\varphi) = \text{hol}^\sigma(\varphi) \mu^{\text{loop}}(d\varphi).$$

The measure μ_σ^{loop} is signed and its total variation measure is μ^{loop} . The measure μ_σ^{loop} is the same for the whole gauge-equivalence class of σ . The signed measure μ_σ^{loop} can be decomposed as

$$\mu_\sigma^{\text{loop}} = \mu_{\sigma,+}^{\text{loop}} - \mu_{\sigma,-}^{\text{loop}},$$

where $\mu_{\sigma,+}^{\text{loop}}$ is the restriction of μ_σ^{loop} to loops φ with $\text{hol}^\sigma(\varphi) = 1$, and $\mu_{\sigma,-}^{\text{loop}}$ is the restriction of μ_σ^{loop} to loops φ with $\text{hol}^\sigma(\varphi) = -1$. According to [KL21, Corollary 3.7], a formula similar to (2.4) also holds in the presence of a gauge field:

$$(2.5) \quad \mu_\sigma^{\text{loop}}(\{\text{Loops that visit at least 2 vertices}\}) = \log \left(\det((G_\sigma(x, y))_{x,y \in V_{\text{int}}}) \prod_{x \in V_{\text{int}}} W(x) \right).$$

So in particular, one gets the following.

Corollary 2.3. *The following identity holds:*

$$\frac{\det((G_\sigma(x, y))_{x,y \in V_{\text{int}}})}{\det((G(x, y))_{x,y \in V_{\text{int}}})} = \exp \left(-2\mu^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^\sigma(\varphi) = -1\}) \right).$$

Proof. By combining the formulas (2.4) and (2.5), one obtains

$$\begin{aligned} \log \left(\frac{\det((G_\sigma(x, y))_{x, y \in V_{\text{int}}})}{\det((G(x, y))_{x, y \in V_{\text{int}}})} \right) &= (\mu_\sigma^{\text{loop}} - \mu^{\text{loop}})(\{\text{Loops that visit at least 2 vertices}\}) \\ &= -2\mu_{\sigma, -}^{\text{loop}}(\{\text{Loops that visit at least 2 vertices}\}). \end{aligned}$$

Further, a loop \wp with $\text{hol}^\sigma(\wp) = -1$ has to visit at least two vertices. Thus,

$$\mu_{\sigma, -}^{\text{loop}}(\{\text{Loops that visit at least 2 vertices}\}) = \mu^{\text{loop}}(\{\text{Loops } \wp \text{ with } \text{hol}^\sigma(\wp) = -1\}). \quad \square$$

Given a path \wp on \mathcal{G} parametrized by continuous time, $\ell^x(\wp)$ will denote its total time spent at vertex x , $x \in V$. Given $\alpha > 0$, \mathcal{L}^α will denote the Poisson point process of loops with intensity measure $\alpha\mu^{\text{loop}}$. This is the continuous time *random walk loop soup* [LJ10, LJ11]. For $x \in V$, $\ell^x(\mathcal{L}^\alpha)$ will denote its *occupation field* in x :

$$\ell^x(\mathcal{L}^\alpha) = \sum_{\wp \in \mathcal{L}^\alpha} \ell^x(\wp).$$

For the particular value $\alpha = 1/2$, this occupation field is Gaussian. More precisely one has the Le Jan's isomorphism.

Theorem 2.4 (Le Jan, [LJ10, LJ11]). *For $\alpha = 1/2$, the following identity in law holds:*

$$(\ell^x(\mathcal{L}^{1/2}))_{x \in V} \stackrel{\text{(law)}}{=} \left(\frac{1}{2} \phi(x)^2 \right)_{x \in V},$$

where ϕ is the GFF (2.1).

Now take $\sigma \in \{-1, 1\}^E$ a gauge field. In [KL21, Theorem 6.6], Kassel and Lévy proved the following extension of Le Jan's isomorphism.

Theorem 2.5 (Kassel-Lévy, [KL21]). *Denote by $\mathcal{L}_{\sigma, +}^{1/2}$, resp. $\mathcal{L}_{\sigma, -}^{1/2}$, a Poisson point process with intensity measure $\frac{1}{2}\mu_{\sigma, +}^{\text{loop}}$, resp. $\frac{1}{2}\mu_{\sigma, -}^{\text{loop}}$. Recall that ϕ_σ denotes the gauge-twisted GFF (2.2). Take ϕ_σ and $\mathcal{L}_{\sigma, -}^{1/2}$ to be independent. Then the following identity in law holds*

$$(2.6) \quad (\ell^x(\mathcal{L}_{\sigma, +}^{1/2}))_{x \in V} \stackrel{\text{(law)}}{=} \left(\frac{1}{2} \phi_\sigma(x)^2 + \ell^x(\mathcal{L}_{\sigma, -}^{1/2}) \right)_{x \in V}.$$

In particular, the field ϕ_σ^2 is stochastically dominated by the field ϕ^2 .

The identity (2.6) tells that in some sense the field $\frac{1}{2}\phi_\sigma^2$ is distributed as the occupation field of a Poisson point process with intensity measure $\frac{1}{2}\mu_\sigma^{\text{loop}}$. However, the measure $\frac{1}{2}\mu_\sigma^{\text{loop}}$ is signed, unless the gauge field σ is trivial, and thus cannot be an intensity for a Poisson. Actually, the loops \wp with $\text{hol}^\sigma(\wp) = -1$ have to be counted negatively.

2.4. GFF on metric graphs. Here we will briefly recall the notion of the GFF on metric graphs. For details we refer to [Lup16].

The *metric graph* associated to the electrical network \mathcal{G} , which we will denote by $\tilde{\mathcal{G}}$, is obtained by replacing each edge $e = \{x, y\} \in E$ by a continuous line segment $I_e = I_{\{x, y\}}$, with endpoint x and y . Moreover, $\tilde{\mathcal{G}}$ is endowed with a metric, by setting the length of I_e to be $C(x, y)^{-1}$ (i.e. the resistance, the inverse of the conductance), and with the corresponding length measure, which we will denote by \tilde{m} . So $\tilde{\mathcal{G}}$ is a continuous and connected metric space.

The discrete GFF ϕ on \mathcal{G} (2.1) can be interpolated to a continuous Gaussian field $\tilde{\phi}$. The restriction of $\tilde{\phi}$ to the vertices V is the discrete GFF ϕ . Conditionally on ϕ , the values of $\tilde{\phi}$ along an edge-line $I_{\{x, y\}}$ are distributed as a standard one-dimensional Brownian bridge between $\phi(x)$ and $\phi(y)$ of length $C(x, y)^{-1}$, with values on different edge-lines being independent. The field $\tilde{\phi}$ is the *metric graph GFF*. It satisfies a strong Markov property when cutting not only along the vertices, but also inside edge-lines; see [Lup16, Section 3]. The covariance of $\tilde{\phi}$ is given by the extension of the Green's function to $\tilde{\mathcal{G}} \times \tilde{\mathcal{G}}$, which we will still denote $G(x, y)$.

The Markov jump process $(X_t)_{t \geq 0}$ on \mathcal{G} can be extended to a continuous Markov diffusion process on \mathcal{G} , which we will denote by $(\tilde{X}_t)_{t \geq 0}$. Inside an edge-line I_e , \tilde{X}_t behaves as a one-dimensional Brownian motion, and once the process reaches a vertex $x \in V$, it performs Brownian excursions inside each adjacent edge-line. See [Lup16, Section 2] for details. In order to fit with the isomorphism identities, we normalize \tilde{X}_t so that inside every edge-line I_e it behaves like a Brownian motion with quadratic variation $2dt$. Just as a one-dimensional Brownian motion, the process $(\tilde{X}_t)_{t \geq 0}$ admits a family of local times $(\ell_t^x(\tilde{X}))_{x \in \tilde{\mathcal{G}}, t \geq 0}$, continuous in (x, t) , characterized by

$$\int_0^t f(\tilde{X}_s) ds = \int_{\tilde{\mathcal{G}}} f(x) \ell_t^x(\tilde{X}) \tilde{m}(dx).$$

Consider the continuous additive functional $A(t)$ and its inverse $A^{-1}(t)$:

$$(2.7) \quad A(t) = \sum_{x \in V} \ell_t^x(\tilde{X}), \quad A^{-1}(t) = \inf\{s \geq 0 | A(s) > t\}.$$

Then $(\tilde{X}_{A^{-1}(t)})_{t \geq 0}$ is distributed as the Markov jump process $(X_t)_{t \geq 0}$.

Let \tilde{T}_{V_∂} denote the first time \tilde{X}_t hits V_∂ . By construction, $A(\tilde{T}_{V_\partial}) = T_{V_\partial}$. Considered the process $(\tilde{X}_t)_{0 \leq t \leq \tilde{T}_{V_\partial}}$ killed upon hitting V_∂ . Let $\tilde{p}(t, x, y)$ be the transition densities of the killed process, so that

$$\int_{\tilde{\mathcal{G}}} \tilde{p}(t, x, y) \tilde{m}(dy) = \mathbb{P}_x(\tilde{T}_{V_\partial} > t).$$

For $x, y \in \tilde{\mathcal{G}} \setminus V_\partial$ and $t > 0$, let $\tilde{\mathbb{P}}_t^{x,y}$ denote the bridge probability measure from x to y in time t , where one conditions on $\tilde{T}_{V_\partial} > t$. The measure on loops on the metric graph is

$$(2.8) \quad \tilde{\mu}^{\text{loop}}(d\wp) = \int_{\tilde{\mathcal{G}}} \int_0^{+\infty} \tilde{\mathbb{P}}_t^{x,x}(d\wp) \tilde{p}(t, x, x) \frac{dt}{t} \tilde{m}(dx).$$

For $\alpha > 0$, $\tilde{\mathcal{L}}^\alpha$ will denote the Poisson point process of loops on $\tilde{\mathcal{G}}$ of intensity $\alpha \tilde{\mu}^{\text{loop}}$. This is the *metric graph loop soup*. The loops in $\tilde{\mathcal{L}}^\alpha$ can be divided into two: those that visit vertices in V and those that only stay in the interior of an edge-line. If one takes the trace on V of the loops in $\tilde{\mathcal{L}}^\alpha$ that intersect V , by applying the time change A^{-1} (2.7), one gets the continuous time random walk loop soup \mathcal{L}^α , actually up to a rerooting of the loops. See [Lup16, Section 2] for details. Given $\wp \in \tilde{\mathcal{L}}^\alpha$ and $x \in \tilde{\mathcal{G}}$, $\ell^x(\wp)$ will denote the cumulative local time of \wp in x . The *occupation field* of $\tilde{\mathcal{L}}^\alpha$ is

$$\ell^x(\tilde{\mathcal{L}}^\alpha) = \sum_{\wp \in \tilde{\mathcal{L}}^\alpha} \ell^x(\wp).$$

For $x \in V$, we have that $\ell^x(\tilde{\mathcal{L}}^\alpha) = \ell^x(\mathcal{L}^\alpha)$.

Next we will consider the clusters of loops in $\tilde{\mathcal{L}}^\alpha$. Two loops $\wp, \wp' \in \tilde{\mathcal{L}}^\alpha$ belong to the same *cluster* if they are connected by a finite chain of loops in $\tilde{\mathcal{L}}^\alpha$. We will also see a cluster \mathcal{C} of $\tilde{\mathcal{L}}^\alpha$ as a subset of $\tilde{\mathcal{G}}$ by taking the union of the ranges of loops forming the cluster. The clusters of $\tilde{\mathcal{L}}^\alpha$ are exactly the connected components of $\{x \in \tilde{\mathcal{G}} | \ell^x(\tilde{\mathcal{L}}^\alpha) > 0\}$. In [Lup16, Theorem 1] is shown that for $\alpha = 1/2$, there is a one to one correspondence between the clusters of $\tilde{\mathcal{L}}^{1/2}$ and the sign components of $\tilde{\phi}$.

Theorem 2.6 (Lupu, [Lup16]). *Take $\tilde{\mathcal{L}}^{1/2}$ a metric graph loop soup on $\tilde{\mathcal{G}}$ of parameter $\alpha = 1/2$. For each cluster \mathcal{C} additionally sample a conditionally independent sign $\sigma(\mathcal{C})$ with*

$$\mathbb{P}(\sigma(\mathcal{C}) = 1 | \tilde{\mathcal{L}}^{1/2}) = \mathbb{P}(\sigma(\mathcal{C}) = -1 | \tilde{\mathcal{L}}^{1/2}) = 1/2.$$

For $x \in \tilde{\mathcal{G}}$ such that $\ell^x(\tilde{\mathcal{L}}^{1/2}) > 0$, let $\mathcal{C}(x)$ denote the cluster of $\tilde{\mathcal{L}}^{1/2}$ containing x . Then the field

$$(\sigma(\mathcal{C}(x)) \sqrt{2\ell^x(\tilde{\mathcal{L}}^{1/2})})_{x \in \tilde{\mathcal{G}}}$$

is distributed as a metric graph GFF $\tilde{\phi}$.

The result above comes from an application of the Le Jan's isomorphism (Theorem 2.4) on the metric graph level and the use of the intermediate value property of the continuous field $\tilde{\phi}$. Beside the exact correspondence between the sign components of $\tilde{\phi}$ and the clusters of $\tilde{\mathcal{L}}^{1/2}$, the metric graph GFF $\tilde{\phi}$ satisfies many other exact solvability properties that one does not get for the discrete GFF ϕ . Next we give a brief list of these exact identities, and further in Section 3 we will prove a new one, related to a special kind of topological conditioning; see Theorem 1.

- As observed in [Lup16, Proposition 5.2], given $x, y \in \tilde{\mathcal{G}}$, the probability that x and y belong to the same sign component of $\tilde{\phi}$, or alternatively to the same cluster of $\tilde{\mathcal{L}}^{1/2}$, equals

$$\mathbb{E}[\text{sign}(\tilde{\phi}(x)) \text{sign}(\tilde{\phi}(y))] = \frac{2}{\pi} \arcsin \left(\frac{G(x, y)}{\sqrt{G(x, x)G(y, y)}} \right).$$

- If V_{∂} is divided into 3 parts $V_{\partial,0}$, $V_{\partial,1}$ and $V_{\partial,2}$, with $V_{\partial,0}$ allowed to be empty, and if the boundary condition of a metric graph GFF $\tilde{\phi}$ is strictly positive on $V_{\partial,1}$ and $V_{\partial,2}$, and zero on $V_{\partial,0}$ (rather than zero on the whole V_{∂}), then there is an explicit formula for the existence of a positive crossing from $V_{\partial,1}$ to $V_{\partial,2}$:

$$\mathbb{P}\left(V_{\partial,1} \xrightarrow{\tilde{\phi} > 0} V_{\partial,2}\right) = 1 - \exp\left(-2 \sum_{x \in V_{\partial,1}} \sum_{y \in V_{\partial,2}} \tilde{\phi}(x) H(x, y) \tilde{\phi}(y)\right),$$

where $H(x, y)$ is the boundary Poisson kernel on $V_{\partial} \times V_{\partial}$. This is [LW18, Formula (18)]. If the boundary condition mixes values of different sign, no analogous explicit formula is known.

- In the articles [DPR22, DPR21] the authors provide the exact law for the effective conductance (called capacity there) between the boundary V_{∂} and the connected component containing x_0 of the level set

$$\{x \in \tilde{\mathcal{G}} \mid \tilde{\phi}(x) \geq h\}$$

($x_0 \in \tilde{\mathcal{G}}$ and $h \geq 0$ fixed).

- The Lévy transformation for the one-dimensional Brownian motion can be extended to the general metric graph GFFs, as explained in [LW18].

3. GAUGE-TWISTED GFF ON METRIC GRAPH, DOUBLE COVER AND EQUIVALENCE TO TOPOLOGICAL CONDITIONING

3.1. Gauge-twisted GFF and subdivision of edges. Consider the electrical network $\mathcal{G} = (V, E)$ as in Section 2.1. For $N \in \mathbb{N} \setminus \{0\}$, we will denote by $\mathcal{G}^{(N)} = (V^{(N)}, E^{(N)})$ the electrical network obtained from $\mathcal{G} = (V, E)$ by subdividing each edge $e \in E$ into N . In this way, $\mathcal{G}^{(1)} = \mathcal{G}$. In general,

$$|E^{(N)}| = N|E|, \quad |V^{(N)}| = |V| + (N-1)|E|.$$

Moreover, if N divides N' , then $V^{(N)}$ is naturally a subset of $V^{(N')}$. In particular, we always have $V \subset V^{(N)}$. Given $e \in E$, we will denote $E^{(N)}(e)$ the subset of $E^{(N)}$ made of edges obtained by subdivision of e . We also denote by $V^{(N)}(e)$ the endpoints of edges in $E^{(N)}(e)$. So $|E^{(N)}(e)| = N$, $|V^{(N)}(e)| = N+1$ and $|V^{(N)}(e) \setminus V| = N-1$. We endow $\mathcal{G}^{(N)}$ with the following conductances: for every $e \in E$ and $e' \in E^{(N)}(e)$, $C^{(N)}(e') = NC(e)$. Let be the energy

$$\mathcal{E}^{(N)}(f, f) = \sum_{\{x, y\} \in E^{(N)}} C^{(N)}(x, y) (f(y) - f(x))^2.$$

We also see the electrical network $\mathcal{G}^{(N)}$, more precisely $V^{(N)}$ as a subset of the metric graph $\tilde{\mathcal{G}}$. In this way, $V^{(N)}(e)$ is a set of $N+1$ points on I_e with equal spacing $(NC(e))^{-1}$, which includes the two endpoints of e .

If $(\phi^{(N)}(x))_{x \in V^{(N)}}$ is a GFF on $\mathcal{G}^{(N)}$, then its restriction to $V = V^{(1)}$ is a GFF on \mathcal{G} . Indeed,

$$\mathcal{E}^{(N)}(f, f) = \mathcal{E}^{(1)}(f|_V, f|_V) + N(N-1) \sum_{e \in E} C(e) \sum_{i=1}^N (f(x_i) - f(x_{i-1}) - (f(x_N) - f(x_0))/N)^2,$$

where x_0, x_1, \dots, x_N is the ordered set of points in $V^{(N)}(e)$. Moreover, if $(\tilde{\phi}(x))_{x \in \tilde{\mathcal{G}}}$ is a metric graph GFF, then its restriction $\tilde{\phi}|_{V^{(N)}}$ is distributed as a GFF $\phi^{(N)}$ on $\mathcal{G}^{(N)}$. So, in a sense, $\tilde{\phi}$ is the limit of $\phi^{(N)}$ as $N \rightarrow +\infty$.

Now, given a gauge field $\sigma \in \{-1, 1\}^E$, we similarly want a natural extension of a σ -twisted GFF ϕ_σ to the subdivided graphs $\mathcal{G}^{(N)}$, and ultimately to the metric graph $\tilde{\mathcal{G}}$. So, given $\sigma \in \{-1, 1\}^E$, we will denote by $\sigma^{(N)}$ the following element of $\{-1, 1\}^{E^{(N)}}$. If $e \in E$ and $\sigma(e) = 1$, then for every $e' \in E^{(N)}(e)$, $\sigma^{(N)}(e') = 1$. However, if $e \in E$ and $\sigma(e) = -1$, then among the edges in $E^{(N)}(e)$ there is exactly one with sign -1 under $\sigma^{(N)}$, all the $N-1$ other having sign $+1$ (so there are actually choices to make for $\sigma^{(N)}$, N choices for each $e \in E$ with $\sigma(e) = -1$).

Proposition 3.1. *Let $N \in \mathbb{N} \setminus \{0\}$, $\sigma \in \{-1, 1\}^E$ and $\sigma^{(N)} \in \{-1, 1\}^{E^{(N)}}$ as above. Let $(\phi_{\sigma^{(N)}}^{(N)}(x))_{x \in V^{(N)}}$ be the $\sigma^{(N)}$ -twisted GFF on $\mathcal{G}^{(N)}$ with 0 boundary conditions on $V_\partial \subset V \subset V^{(N)}$. Then its restriction to V is distributed as $(\phi_\sigma(x))_{x \in V}$, the σ -twisted GFF on \mathcal{G} with 0 boundary conditions.*

Proof. Consider the energy

$$\mathcal{E}_\sigma^{(N)}(f, f) = \sum_{\{x, y\} \in E^{(N)}} C^{(N)}(x, y) (\sigma^{(N)}(x, y) f(y) - f(x))^2.$$

Let $e \in E$ and let x_0, x_1, \dots, x_N be the ordered set of points in $V^{(N)}(e)$. For $i \in \{1, \dots, N\}$, let be

$$\tilde{\sigma}_i = \sigma^{(N)}(x_0, x_1) \dots \sigma^{(N)}(x_{i-1}, x_i).$$

Then

$$(3.1) \quad \sum_{i=1}^N C^{(N)}(x_{i-1}, x_i) (\sigma^{(N)}(x_{i-1}, x_i) f(x_i) - f(x_{i-1}))^2 = C(x_0, x_N) (\sigma(x_0, x_N) f(x_N) - f(x_0))^2 \\ + N(N-1) C(x_0, x_N) \sum_{i=1}^N (\tilde{\sigma}_i f(x_i) - \tilde{\sigma}_{i-1} f(x_{i-1}) - (\sigma(x_0, x_N) f(x_N) - f(x_0))/N)^2.$$

This induces a decomposition of the energy $\mathcal{E}_\sigma^{(N)}$. This implies that $\phi_{\sigma^{(N)}}^{(N)}$ restricted to V is distributed as ϕ_σ . Moreover, the fields

$$(\tilde{\sigma}_i f(x_i) + i(\sigma(x_0, x_N) f(x_N) - f(x_0))/N)_{1 \leq i \leq N-1, e \in E},$$

are independent from $\phi_{\sigma^{(N)}}^{(N)}$ restricted to V , with also independence across $e \in E$. \square

In the sequel, $\sigma \in \{-1, 1\}^E$ and, for the sake of simplicity, N will be odd and $\sigma^{(N)}$ is chosen such that for $e \in E$ with $\sigma(e) = -1$, the middle edge in $E^{(N)}(e)$ has the sign -1 under $\sigma^{(N)}$. We will define a random Gaussian field $\tilde{\phi}_\sigma$ on the metric graph $\tilde{\mathcal{G}}$ as follows. We consider the following independent objects.

- The twisted discrete GFF $(\phi_\sigma(x))_{x \in V}$.
- For every edge $e \in E$, and independent Brownian bridge $(W_e(u))_{0 \leq u \leq C(e)-1}$ of length $C(e)-1$, from 0 to 0.

We will also chose for each edge $e \in E$ and arbitrary orientation and denote its endpoints e_- and e_+ . For $u \in [0, C(e)-1]$, $x_{e,u}$ will denote the point of I_e at distance u from e_+ . We define $(\tilde{\phi}_\sigma(x))_{x \in \tilde{\mathcal{G}}}$ as follows.

- For $x \in V$, $\tilde{\phi}_\sigma(x) = \phi_\sigma(x)$.
- If $e \in E$ and $\sigma(e) = 1$, then for every $u \in (0, C(e)^{-1})$,

$$(3.2) \quad \tilde{\phi}_\sigma(x_{e,u}) = W_e(u) + C(e)(u\phi_\sigma(e_-) + (C(e)^{-1} - u)\phi_\sigma(e_+)).$$

- If $e \in E$ and $\sigma(e) = -1$, then for $u \in (0, C(e)^{-1}/2)$,

$$(3.3) \quad \tilde{\phi}_\sigma(x_{e,u}) = W_e(u) + C(e)(-u\phi_\sigma(e_-) + (C(e)^{-1} - u)\phi_\sigma(e_+)),$$

and for $u \in (C(e)^{-1}/2, C(e)^{-1})$,

$$(3.4) \quad \tilde{\phi}_\sigma(x_{e,u}) = -W_e(u) + C(e)(u\phi_\sigma(e_-) - (C(e)^{-1} - u)\phi_\sigma(e_+)).$$

The value for $u = C(e)^{-1}/2$ is not specified and can be chosen arbitrary.

Defined in this way, the field $\tilde{\phi}_\sigma$ is discontinuous in the middle of each I_e for $e \in \{e \in E | \sigma(e) = -1\}$, with

$$\lim_{u \rightarrow (C(e)^{-1}/2)_-} \tilde{\phi}_\sigma(x_{e,u}) = - \lim_{u \rightarrow (C(e)^{-1}/2)_+} \tilde{\phi}_\sigma(x_{e,u}).$$

However, the absolute value $|\tilde{\phi}_\sigma|$ extends to a continuous field on $\tilde{\mathcal{G}}$. Note that the law of $\tilde{\phi}_\sigma$ does not depend on the arbitrary choice of orientations for the edges, since $\phi_\sigma \stackrel{(\text{law})}{=} -\phi_\sigma$ and $W_e \stackrel{(\text{law})}{=} -W_e$.

Proposition 3.2. *For every N odd, the restriction of $\tilde{\phi}_\sigma$ to $V^{(N)}$ is distributed as $\phi_{\sigma(N)}^{(N)}$.*

Proof. This is an immediate consequence of the decomposition (3.1). \square

So the metric graph field $\tilde{\phi}_\sigma$ appears as the limit of $\phi_{\sigma(N)}^{(N)}$ as $N \rightarrow +\infty$. However, it is not continuous. To recover continuous fields we will work on a double cover of $\tilde{\mathcal{G}}$; see Section 3.3.

3.2. Double covers induced by gauge fields. Let $\sigma \in \{-1, 1\}^E$. We will introduce a double cover $\mathcal{G}_\sigma^{\text{db}} = (V^{\text{db}}, E_\sigma^{\text{db}})$ of the graph \mathcal{G} , which is as follows. V^{db} consist of two copies of V , $V^{\text{db}} = V_1 \cup V_2$, and the projection $\pi_\sigma : V^{\text{db}} \rightarrow V$ induces a bijection between V_1 and V and between V_2 and V . The set of edges E_σ^{db} is as follows. Let $\{x, y\} \in E$ and let $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ such that $\pi_\sigma(x_1) = \pi_\sigma(x_2) = x$ and $\pi_\sigma(y_1) = \pi_\sigma(y_2) = y$.

- If $e \in E$ and $\sigma(e) = 1$, then $\{x_1, y_1\}, \{x_2, y_2\} \in E_\sigma^{\text{db}}$.
- If $e \in E$ and $\sigma(e) = -1$, then $\{x_1, y_2\}, \{x_2, y_1\} \in E_\sigma^{\text{db}}$.

See Figure 5 for an example. We also have a projection $\pi_\sigma : E_\sigma^{\text{db}} \rightarrow E$ such that

$$\pi_\sigma(\{\hat{x}, \hat{y}\}) = \{\pi_\sigma(\hat{x}), \pi_\sigma(\hat{y})\}.$$

In this way π_σ is a graph covering map from $\mathcal{G}_\sigma^{\text{db}}$ to \mathcal{G} . Given $x_1 \in V_1$ and $x_2 \in V_2$ such that $\pi_\sigma(x_1) = \pi_\sigma(x_2)$, we will define $\psi(x_1) = x_2$ and $\psi(x_2) = x_1$. In this way, ψ is a bijection from V^{db} to V^{db} . It is also a graph automorphism of $\mathcal{G}_\sigma^{\text{db}}$, that is to say if $\{\hat{x}, \hat{y}\} \in E_\sigma^{\text{db}}$ then $\{\psi(\hat{x}), \psi(\hat{y})\} \in E_\sigma^{\text{db}}$. The map ψ is also a covering automorphism, because $\pi_\sigma \circ \psi = \pi_\sigma$.

Lemma 3.3. *Let $\wp = (x_1, x_2, \dots, x_{n-1}, x_n, x_1)$ be a nearest neighbor loop on \mathcal{G} and let $\hat{\wp} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n-1}, \hat{x}_n, \hat{x}_{n+1})$ be a lift of \wp on $\mathcal{G}_\sigma^{\text{db}}$ with respect to the covering map π_σ : $\pi_\sigma(\hat{x}_i) = x_i$ for $i \in \{1, 2, \dots, n\}$, $\pi_\sigma(\hat{x}_{n+1}) = x_1$, and $\{\hat{x}_i, \hat{x}_{i+1}\} \in E_\sigma^{\text{db}}$ for $i \in \{1, 2, \dots, n\}$. Then $\text{hol}^\sigma(\wp) = 1$ if and only if $\hat{x}_{n+1} = \hat{x}_1$. Otherwise $\text{hol}^\sigma(\wp) = -1$ and $\hat{x}_{n+1} = \psi(\hat{x}_1)$.*

Proof. By construction, whenever $\sigma(x_i, x_{i+1}) = 1$, then \hat{x}_i and \hat{x}_{i+1} belong to the same copy V_1 or V_2 . Whenever $\sigma(x_i, x_{i+1}) = -1$ then either $\hat{x}_i \in V_1$ and $\hat{x}_{i+1} \in V_2$, or vice-versa. So $\hat{x}_{n+1} = \hat{x}_1$ if and only even there is an even number of transitions from V_1 to V_2 or from V_2 to V_1 , that is to say that the number of $i \in \{1, 2, \dots, n\}$ such that $\sigma(x_i, x_{i+1}) = -1$ is even, which exactly means that $\text{hol}^\sigma(\wp) = 1$. \square

Lemma 3.4. *Let $\sigma, \sigma' \in \{-1, 1\}^E$.*

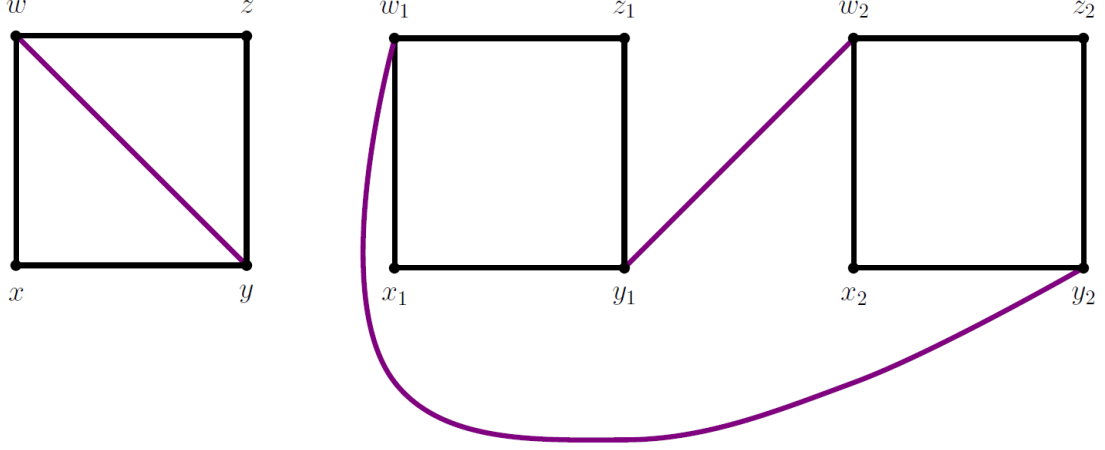


FIGURE 5. Left: a graph with 4 vertices and 5 edges. Right: its double cover. The edges in violet correspond to $\sigma(e) = -1$.

- (1) Assume that σ and σ' are gauge equivalent, and let $\hat{\sigma} \in \{-1, 1\}^V$ such that $\sigma' = \hat{\sigma} \cdot \sigma$. Define $\psi_{\hat{\sigma}} : V^{\text{db}} \rightarrow V^{\text{db}}$ as follows. Let $\hat{x} \in V^{\text{db}}$. If $\hat{\sigma}(\pi_{\sigma}(\hat{x})) = 1$, then $\psi_{\hat{\sigma}}(\hat{x}) = \hat{x}$. If $\hat{\sigma}(\pi_{\sigma}(\hat{x})) = -1$, then $\psi_{\hat{\sigma}}(\hat{x}) = \psi(\hat{x})$. Then $\psi_{\hat{\sigma}}$ induces a graph isomorphism between $\mathcal{G}_{\sigma}^{\text{db}}$ et $\mathcal{G}_{\sigma'}^{\text{db}}$, which is moreover a covering isomorphism, i.e. $\pi_{\sigma'} = \pi_{\sigma} \circ \psi_{\hat{\sigma}}$.
- (2) Conversely, if there exists a covering isomorphism between $\mathcal{G}_{\sigma}^{\text{db}}$ et $\mathcal{G}_{\sigma'}^{\text{db}}$, then σ and σ' are gauge equivalent.

Proof. (1) The map $\psi_{\hat{\sigma}}$ is clearly a bijection from V^{db} to itself, and it is clear that $\pi_{\sigma'} = \pi_{\sigma} \circ \psi_{\hat{\sigma}}$. One needs only to check that whenever $\{\hat{x}, \hat{y}\} \in E^{\text{db}}$, then also $\{\psi_{\hat{\sigma}}(\hat{x}), \psi_{\hat{\sigma}}(\hat{y})\} \in E^{\text{db}}$. But this is clear from the definition of $\psi_{\hat{\sigma}}$ and the fact that

$$\sigma'(\pi_{\sigma'}(\psi_{\hat{\sigma}}(\hat{x})), \pi_{\sigma'}(\psi_{\hat{\sigma}}(\hat{y}))) = \sigma'(\pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y})) = \hat{\sigma}(\pi_{\sigma}(\hat{x}))\sigma(\pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y}))\hat{\sigma}(\pi_{\sigma}(\hat{y})).$$

(2) Let $\hat{\psi}$ be a covering a covering isomorphism between $\mathcal{G}_{\sigma}^{\text{db}}$ et $\mathcal{G}_{\sigma'}^{\text{db}}$. Since $\pi_{\sigma'} = \pi_{\sigma} \circ \hat{\psi}$, for every $\hat{x} \in V^{\text{db}}$,

- either $(\hat{\psi}(\hat{x}), \hat{\psi}(\psi(\hat{x}))) = (\hat{\psi}(\hat{x}), \hat{\psi}(\psi(\hat{x})))$,
- or $(\hat{\psi}(\hat{x}), \hat{\psi}(\psi(\hat{x}))) = (\hat{\psi}(\psi(\hat{x})), \hat{\psi}(\hat{x}))$.

Note that here \hat{x} and $\psi(\hat{x})$ symmetric roles. In the first case we set $\hat{\sigma}(\pi_{\sigma}(\hat{x})) = 1$. In the second case we set $\hat{\sigma}(\pi_{\sigma}(\hat{x})) = -1$. Then it is easy to check that $\sigma' = \hat{\sigma} \cdot \sigma$. \square

Lemma 3.5. Let $\sigma \in \{-1, 1\}^E$ be a gauge field and let $\mathcal{G}_{\sigma}^{\text{db}}$ be the corresponding double cover. Then the graph $\mathcal{G}_{\sigma}^{\text{db}}$ is connected if and only if σ is non-trivial. Otherwise $\mathcal{G}_{\sigma}^{\text{db}}$ consists of two disconnected copies of \mathcal{G} .

Proof. If $\sigma(e) = 1$ for every $e \in E$, then by construction, V_1 and V_2 are not connected in $\mathcal{G}_{\sigma}^{\text{db}}$ and one has just two copies \mathcal{G} with vertex sets V_1 and V_2 respectively. In the more general case of σ trivial, one can apply Lemma 3.4 to get that $\mathcal{G}_{\sigma}^{\text{db}}$ is isomorphic as a covering to the previous case.

Assume now that σ is non-trivial. Let $x \in V$, and let $x_1 \in V_1, x_2 \in V_2$ such that $\pi_{\sigma}(x_1) = \pi_{\sigma}(x_2) = x$. According to Lemma 2.1, there is a loop \wp rooted in x such that $\text{hol}^{\sigma}(\wp) = -1$. Let $\hat{\wp}$ be a lift in $\mathcal{G}_{\sigma}^{\text{db}}$ of the loop \wp . By Lemma 3.3, $\hat{\wp}$ is a path joining x_1 and x_2 . Thus x_1 and x_2 belong to the same connected component. Further, by construction, if $\{x, y\} \in E$, then $\pi_{\sigma}^{-1}(\{x\})$ and $\pi_{\sigma}^{-1}(\{y\})$ are connected. Since \mathcal{G} is connected, then so is $\mathcal{G}_{\sigma}^{\text{db}}$. \square

Further, we will endow the double covers $\mathcal{G}_{\sigma}^{\text{db}}$ with conductances as follows. Given $\{\hat{x}, \hat{y}\} \in E_{\sigma}^{\text{db}}$,

$$(3.5) \quad C(\hat{x}, \hat{y}) = C(\pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y})).$$

Let be $V_{\partial}^{\text{db}} = \pi_{\sigma}^{-1}(V_{\partial})$ and $V_{\text{int}}^{\text{db}} = \pi_{\sigma}^{-1}(V_{\text{int}})$. We consider the nearest-neighbor Markov jump process on $\mathcal{G}_{\sigma}^{\text{db}}$ with the transition rates given by the conductances, and killed upon hitting V_{∂}^{db} . Let $p_{\sigma}(t, \hat{x}, \hat{y})$ be the transition probabilities of this killed process. The following is immediate.

Lemma 3.6. (1) For every $\hat{x}, \hat{y} \in V_{\text{int}}^{\text{db}}$,

$$p_{\sigma}(t, \hat{x}, \hat{y}) + p_{\sigma}(t, \hat{x}, \psi(\hat{y})) = p(t, \pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y})).$$

(2) For every $\hat{x}, \hat{y} \in V_{\text{int}}^{\text{db}}$, such that \hat{x} and \hat{y} are both in V_1 or both in V_2 ,

$$\mathbb{P}_t^{\pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y})}(\text{hol}^{\sigma}(\varphi) = -1) = \frac{p_{\sigma}(t, \hat{x}, \psi(\hat{y}))}{p(t, \pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y}))},$$

where $\mathbb{P}_t^{\pi_{\sigma}(\hat{x}), \pi_{\sigma}(\hat{y})}$ is the bridge probability measure for the Markov jump process on \mathcal{G} conditioned on staying in V_{int} .

A fundamental domain of π_{σ} is a subset $\mathcal{D} \subset V^{\text{db}}$ such that π_{σ} induces a bijection from \mathcal{D} to V . By construction, V_1 and V_2 are both fundamental domains. The following is an immediate consequence of (2.3) and Lemma 3.6.

Corollary 3.7. Let be a gauge field $\sigma \in \{-1, 1\}^E$. Then

$$\mu^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^{\sigma}(\varphi) = -1\}) = \sum_{\hat{x} \in \mathcal{D} \cap V_{\text{int}}^{\text{db}}} \int_0^{+\infty} p_{\sigma}(t, \hat{x}, \psi(\hat{x})) \frac{dt}{t},$$

where \mathcal{D} is any fundamental domain.

Let be the space $\mathcal{S}_0^{\text{db}}$:

$$\mathcal{S}_0^{\text{db}} = \{f \in \mathbb{R}^{V^{\text{db}}} \mid \forall \hat{x} \in V_{\text{int}}^{\text{db}}, f(\hat{x}) = 0\}.$$

We endow $\mathcal{S}_0^{\text{db}}$ with the positive definite inner product

$$\mathcal{E}_{\sigma}^{\text{db}}(f, g) = \sum_{\{\hat{x}, \hat{y}\} \in E_{\sigma}^{\text{db}}} C(\hat{x}, \hat{y})(f(\hat{y}) - f(\hat{x}))(g(\hat{y}) - g(\hat{x})).$$

Let $\mathcal{S}_{0,+}^{\text{db}}$ and $\mathcal{S}_{0,-}^{\text{db}}$ be the following subspaces of $\mathcal{S}_0^{\text{db}}$:

$$\mathcal{S}_{0,+}^{\text{db}} = \{f \in \mathcal{S}_0^{\text{db}} \mid f \circ \psi = f\}, \quad \mathcal{S}_{0,-}^{\text{db}} = \{f \in \mathcal{S}_0^{\text{db}} \mid f \circ \psi = -f\}.$$

Note that $\dim \mathcal{S}_{0,+}^{\text{db}} = \dim \mathcal{S}_{0,-}^{\text{db}} = |V_{\text{int}}|$, and $\dim \mathcal{S}_0^{\text{db}} = 2|V_{\text{int}}|$.

Lemma 3.8. The space $\mathcal{S}_0^{\text{db}}$ is a direct orthogonal (for $\mathcal{E}_{\sigma}^{\text{db}}$) sum of the subspaces $\mathcal{S}_{0,+}^{\text{db}}$ and $\mathcal{S}_{0,-}^{\text{db}}$.

Proof. It is clear that $\mathcal{S}_{0,+}^{\text{db}} \cap \mathcal{S}_{0,-}^{\text{db}} = \{0\}$, and every $f \in \mathcal{S}_0^{\text{db}}$ can be decomposed as

$$f = \frac{1}{2}(f + f \circ \psi) + \frac{1}{2}(f - f \circ \psi).$$

So $\mathcal{S}_0^{\text{db}}$ is a direct sum of $\mathcal{S}_{0,+}^{\text{db}}$ and $\mathcal{S}_{0,-}^{\text{db}}$. To check that the decomposition is orthogonal, consider $f \in \mathcal{S}_{0,+}^{\text{db}}$ and $g \in \mathcal{S}_{0,-}^{\text{db}}$. Given $\{\hat{x}, \hat{y}\} \in E_{\sigma}^{\text{db}}$, then $\{\psi(\hat{x}), \psi(\hat{y})\} \in E_{\sigma}^{\text{db}}$ and

$$(f(\hat{y}) - f(\hat{x}))(g(\hat{y}) - g(\hat{x})) = -(f(\psi(\hat{y})) - f(\psi(\hat{x})))(g(\psi(\hat{y})) - g(\psi(\hat{x}))).$$

So $\mathcal{E}_{\sigma}^{\text{db}}(f, g) = 0$. □

Let $\Delta_{\sigma}^{\text{db}}$ denote the discrete Laplacian on $\mathcal{G}_{\sigma}^{\text{db}}$:

$$(\Delta_{\sigma}^{\text{db}} f)(\hat{x}) = \sum_{\substack{\hat{y} \in V^{\text{db}} \\ \{\hat{x}, \hat{y}\} \in E_{\sigma}^{\text{db}}}} C(\hat{x}, \hat{y})(f(\hat{y}) - f(\hat{x})), \quad \hat{x} \in V^{\text{db}}.$$

One can see $\Delta_\sigma^{\text{db}}$ as an operator on $\mathcal{S}_0^{\text{db}}$ by taking $((\Delta_\sigma^{\text{db}} f)(\hat{x}))_{\hat{x} \in V_{\text{int}}^{\text{db}}}$ and extending to V_∂^{db} by 0. Then, clearly, the subspaces $\mathcal{S}_{0,+}^{\text{db}}$ and $\mathcal{S}_{0,-}^{\text{db}}$ are stable by $\Delta_\sigma^{\text{db}}$. Therefore,

$$\det_{\mathcal{S}_0^{\text{db}}}(-\Delta_\sigma^{\text{db}}) = \det_{\mathcal{S}_{0,+}^{\text{db}}}(-\Delta_\sigma^{\text{db}}) \det_{\mathcal{S}_{0,-}^{\text{db}}}(-\Delta_\sigma^{\text{db}}),$$

where the subscripts indicate the space considered.

Let $f : V^{\text{db}} \rightarrow \mathbb{R}$, and let f_1 denote the following function from V to \mathbb{R} . Given $x \in V$ and $x_1 \in V_1$ such that $\pi_\sigma(x_1) = x$,

$$f_1(x) = f(x_1).$$

If $f \in \mathcal{S}_{0,+}^{\text{db}}$, then

$$(\Delta_\sigma^{\text{db}} f)(x_1) = (\Delta_{\mathcal{G}} f_1)(x),$$

and if $f \in \mathcal{S}_{0,-}^{\text{db}}$, then

$$(\Delta_\sigma^{\text{db}} f)(x_1) = (\Delta_{\mathcal{G},\sigma} f_1)(x).$$

The following is an immediate consequence of the above.

Corollary 3.9. *We have that*

$$\det_{\mathcal{S}_{0,+}^{\text{db}}}(-\Delta_\sigma^{\text{db}}) = \det((G(x, y))_{x, y \in V_{\text{int}}})^{-1}, \quad \det_{\mathcal{S}_{0,-}^{\text{db}}}(-\Delta_\sigma^{\text{db}}) = \det((G_\sigma(x, y))_{x, y \in V_{\text{int}}})^{-1}.$$

In particular,

$$\frac{\det_{\mathcal{S}_{0,+}^{\text{db}}}(-\Delta_\sigma^{\text{db}})}{\det_{\mathcal{S}_{0,-}^{\text{db}}}(-\Delta_\sigma^{\text{db}})} = \exp(-2\mu^{\text{loop}}(\{\text{Loops } \wp \text{ with } \text{hol}^\sigma(\wp) = -1\})).$$

Let $(G_\sigma^{\text{db}}(\hat{x}, \hat{y}))_{\hat{x}, \hat{y} \in V^{\text{db}}}$ denote the Green's function on $\mathcal{G}_\sigma^{\text{db}}$ with 0 boundary conditions on V_∂^{db} . Note that $G_\sigma^{\text{db}}(\psi(\hat{x}), \psi(\hat{y})) = G_\sigma^{\text{db}}(\hat{x}, \hat{y})$

Lemma 3.10. *Let $x, y \in V$ and let $x_1, y_1 \in V_1$ and $y_2 \in V_2$ such that $\pi_\sigma(x_1) = x$ and $\pi_\sigma(y_1) = \pi_\sigma(y_2) = y$. Then*

$$G(x, y) = G_\sigma^{\text{db}}(x_1, y_1) + G_\sigma^{\text{db}}(x_1, y_2), \quad G_\sigma(x, y) = G_\sigma^{\text{db}}(x_1, y_1) - G_\sigma^{\text{db}}(x_1, y_2),$$

or equivalently

$$G_\sigma^{\text{db}}(x_1, y_1) = \frac{1}{2}(G(x, y) + G_\sigma(x, y)), \quad G_\sigma^{\text{db}}(x_1, y_2) = \frac{1}{2}(G(x, y) - G_\sigma(x, y)).$$

Proof. We have that

$$G_\sigma^{\text{db}}(x_1, y_i) = \int_0^{+\infty} p_\sigma(t, x_1, y_i) dt, \quad G(x, y) = \int_0^{+\infty} p(t, x, y) dt,$$

and according to Theorem 2.2,

$$G_\sigma(x, y) = \int_0^{+\infty} p(t, x, y) \mathbb{P}_t^{x, y}(\text{hol}^\sigma(\wp) = -1) dt.$$

We conclude by Lemma 3.6. □

Let $(\phi_\sigma^{\text{db}}(\hat{x}))_{\hat{x} \in V^{\text{db}}}$ be the discrete GFF on the double cover $\mathcal{G}_\sigma^{\text{db}}$. We have that

$$\mathbb{E}[\phi_\sigma^{\text{db}}(\hat{x})] = 0, \quad \mathbb{E}[\phi_\sigma^{\text{db}}(\hat{x})\phi_\sigma^{\text{db}}(\hat{y})] = G_\sigma^{\text{db}}(\hat{x}, \hat{y}).$$

Let $\phi_{\sigma,+}^{\text{db}}$ and $\phi_{\sigma,-}^{\text{db}}$ be the fields

$$\phi_{\sigma,+}^{\text{db}} = \frac{1}{\sqrt{2}}(\phi_\sigma^{\text{db}} + \phi_\sigma^{\text{db}} \circ \psi), \quad \phi_{\sigma,-}^{\text{db}} = \frac{1}{\sqrt{2}}(\phi_\sigma^{\text{db}} - \phi_\sigma^{\text{db}} \circ \psi).$$

The field $\phi_{\sigma,+}^{\text{db}}$, respectively $\phi_{\sigma,-}^{\text{db}}$, is a random element of $\mathcal{S}_{0,+}^{\text{db}}$, respectively $\mathcal{S}_{0,-}^{\text{db}}$. Since $\mathcal{S}_{0,+}^{\text{db}}$ and $\mathcal{S}_{0,-}^{\text{db}}$ are orthogonal for $\mathcal{E}_\sigma^{\text{db}}$, the fields $\phi_{\sigma,+}^{\text{db}}$ and $\phi_{\sigma,-}^{\text{db}}$ are independent. Lemma 3.10 immediately implies the following.

Corollary 3.11. *Let $(\phi(x))_{x \in V}$ and $(\phi_\sigma(x))_{x \in V}$ be the following fields:*

$$\phi(x) = \phi_{\sigma,+}^{\text{db}}(x_1), \quad \phi_\sigma(x) = \phi_{\sigma,-}^{\text{db}}(x_1),$$

where $x_1 \in V_1$ and $\pi_\sigma(x_1) = x$. Then ϕ is distributed as the GFF on \mathcal{G} with 0 boundary conditions (2.1), and ϕ_σ is distributed as the σ -twisted GFF on \mathcal{G} with 0 boundary conditions (2.2).

3.3. Double covers of metric graphs and related GFFs. Consider a gauge field $\sigma \in \{-1, 1\}^E$. Let be the associated double cover $\mathcal{G}_\sigma^{\text{db}}$, endowed with the conductances (3.5). We will consider the metric graph associated to $\mathcal{G}_\sigma^{\text{db}}$, denoted by $\tilde{\mathcal{G}}_\sigma^{\text{db}}$. The projection π_σ can be extended into a covering map $\pi_\sigma : \tilde{\mathcal{G}}_\sigma^{\text{db}} \rightarrow \tilde{\mathcal{G}}$ which is locally an isometry. In this way, $\tilde{\mathcal{G}}_\sigma^{\text{db}}$ is a double cover of the metric graph $\tilde{\mathcal{G}}$. Similarly, the map $\psi : V^{\text{db}} \rightarrow V^{\text{db}}$ extends into a covering automorphism $\psi_\sigma : \tilde{\mathcal{G}}_\sigma^{\text{db}} \rightarrow \tilde{\mathcal{G}}_\sigma^{\text{db}}$ ($\pi_\sigma \circ \psi_\sigma = \pi_\sigma$) which interchanges the two sheets of the covering. Lemma 3.5 extends to the metric graph setting: the metric space $\tilde{\mathcal{G}}_\sigma^{\text{db}}$ is connected if and only if the gauge field σ is non-trivial. Otherwise it consists of two disjoint copies of $\tilde{\mathcal{G}}$.

Let $\tilde{\phi}_\sigma^{\text{db}}$ be the metric graph GFF on $\tilde{\mathcal{G}}_\sigma^{\text{db}}$ with 0 boundary conditions on V_∂^{db} . The field $\tilde{\phi}_\sigma^{\text{db}}$ is continuous. Its restriction to V^{db} is the field ϕ_σ^{db} introduced previously. Let $\tilde{\phi}_{\sigma,+}^{\text{db}}$ and $\tilde{\phi}_{\sigma,-}^{\text{db}}$ be the fields

$$\tilde{\phi}_{\sigma,+}^{\text{db}} = \frac{1}{\sqrt{2}}(\tilde{\phi}_\sigma^{\text{db}} + \tilde{\phi}_\sigma^{\text{db}} \circ \psi_\sigma), \quad \tilde{\phi}_{\sigma,-}^{\text{db}} = \frac{1}{\sqrt{2}}(\tilde{\phi}_\sigma^{\text{db}} - \tilde{\phi}_\sigma^{\text{db}} \circ \psi_\sigma).$$

The fields $\tilde{\phi}_{\sigma,+}^{\text{db}}$ and $\tilde{\phi}_{\sigma,-}^{\text{db}}$ are again continuous and their restrictions to V^{db} are $\phi_{\sigma,+}^{\text{db}}$ and $\phi_{\sigma,-}^{\text{db}}$ respectively.

Lemma 3.12. *The fields $\tilde{\phi}_{\sigma,+}^{\text{db}}$ and $\tilde{\phi}_{\sigma,-}^{\text{db}}$ are independent.*

Proof. We already know that the fields $\phi_{\sigma,+}^{\text{db}}$ and $\phi_{\sigma,-}^{\text{db}}$ are independent. To conclude, we need to look at what happens inside the edge-lines. Given $(W_1(u))_{0 \leq u \leq C(e)-1}$ and $(W_2(u))_{0 \leq u \leq C(e)-1}$ two independent Brownian bridges from 0 to 0 of the same length, then indeed $(W_1 + W_2)/\sqrt{2}$ and $(W_1 - W_2)/\sqrt{2}$ are indeed independent and again distributed as Brownian bridges from 0 to 0. \square

Since $\tilde{\phi}_{\sigma,+}^{\text{db}} \circ \psi_\sigma = \tilde{\phi}_{\sigma,-}^{\text{db}}$, there is a continuous field $\tilde{\phi}$ on $\tilde{\mathcal{G}}$ (i.e. on the base of the covering) such that

$$\tilde{\phi}_{\sigma,+}^{\text{db}} = \tilde{\phi} \circ \pi_\sigma.$$

Lemma 3.13. *The field $\tilde{\phi}$ is distributed as the metric graph GFF on $\tilde{\mathcal{G}}$ with 0 boundary conditions on V_∂ (see Section 2.4), hence the same notation, and in particular its distribution is the same whatever the gauge field σ .*

Proof. By Corollary 3.11 we already know that the restriction of the field $\tilde{\phi}$ to V is distributed as the discrete GFF with 0 boundary conditions on V_∂ . We need to check that the identity in law also extends inside the edge-lines. Given $(W_1(u))_{0 \leq u \leq C(e)-1}$ and $(W_2(u))_{0 \leq u \leq C(e)-1}$ two independent Brownian bridges from 0 to 0 of the same length, then indeed $(W_1 + W_2)/\sqrt{2}$ is distributed as a Brownian bridge from 0 to 0. \square

Next we will introduce a section $\mathbf{s}_\sigma : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}_\sigma^{\text{db}}$, such that $\pi_\sigma \circ \mathbf{s}_\sigma = \text{Id}_{\tilde{\mathcal{G}}}$. In general, \mathbf{s}_σ will not be continuous and will actually have finitely many discontinuity points. The section \mathbf{s}_σ will be defined by the following conditions:

- (1) $\pi_\sigma \circ \mathbf{s}_\sigma = \text{Id}_{\tilde{\mathcal{G}}}$.
- (2) For every $x \in V$, $\mathbf{s}_\sigma(x) \in V_1$.
- (3) For every edge $e \in E$, the map $u \mapsto \mathbf{s}_\sigma(x_{e,u})$ (see Section 3.1 for the notations) is continuous on the intervals $[0, C(e)^{-1}/2)$ and $(C(e)^{-1}/2, C(e)^{-1}]$, i.e. a discontinuity is possible only in the middle of the edge-line at $u = C(e)^{-1}/2$.

The above entirely specifies \mathbf{s}_σ outside the middle points of edge-lines in $\tilde{\mathcal{G}}$. Further, if $\{x, y\} \in E$ and $\sigma(x, y) = 1$, then $\{\mathbf{s}_\sigma(x), \mathbf{s}_\sigma(y)\} \in E_\sigma^{\text{db}}$, and thus \mathbf{s}_σ extends continuously to the whole edge-line $I_{\{x, y\}}$. However, if $\sigma(x, y) = -1$, then $\{\mathbf{s}_\sigma(x), \mathbf{s}_\sigma(y)\} \notin E_\sigma^{\text{db}}$, and thus \mathbf{s}_σ has a discontinuity in the middle of the edge-line $I_{\{x, y\}}$. We will not specify which is the value assumed by \mathbf{s}_σ at such a discontinuity point, i.e. which of the left or right limit, as this will not be of any importance.

Now, consider on $\tilde{\mathcal{G}}$ the field

$$(3.6) \quad \tilde{\phi}_\sigma = \tilde{\phi}_{\sigma, -}^{\text{db}} \circ \mathbf{s}_\sigma.$$

Lemma 3.14. *The field $\tilde{\phi}_\sigma$ has the same distribution as the Gaussian field on $\tilde{\mathcal{G}}$ introduced in Section 3.1, hence the same notation.*

Proof. By Corollary 3.11 we already know that the restriction of the field $\tilde{\phi}_\sigma$ to V is distributed as the discrete σ -twisted GFF ϕ_σ with 0 boundary conditions on V_∂ . We need only to check what happens inside the edge lines. Given $(W_1(u))_{0 \leq u \leq C(e)-1}$ and $(W_2(u))_{0 \leq u \leq C(e)-1}$ two independent Brownian bridges from 0 to 0 of the same length, then $W_e = (W_1 - W_2)/\sqrt{2}$ is again distributed as a Brownian bridge from 0 to 0. If $\sigma(e) = 1$, then \mathbf{s}_σ is continuous on the edge-line I_e and the restriction of $\tilde{\phi}_\sigma$ to I_e is obtained by interpolating between $\tilde{\phi}_\sigma(e_-)$ and $\tilde{\phi}_\sigma(e_+)$ with W_e (plus the deterministic linear part). If $\sigma(e) = -1$, then \mathbf{s}_σ has a discontinuity in the middle of I_e , and the restriction of $\tilde{\phi}_\sigma$ to I_e is obtained by interpolating between $-\tilde{\phi}_\sigma(e_-)$ and $\tilde{\phi}_\sigma(e_+)$ with W_e (plus the deterministic linear part), then by reflecting the bridge on half of I_e . So this is indeed the same construction as the one given by the formulas (3.2), (3.3) and (3.4). \square

The main point of the Sections 3.2 and 3.3 was actually to introduce the field $\tilde{\phi}_{\sigma, -}^{\text{db}}$. The advantage of $\tilde{\phi}_{\sigma, -}^{\text{db}}$ over $\tilde{\phi}_\sigma$ is that the former is always continuous, and in particular satisfies the intermediate value property. This will be exploited in the forthcoming Section 3.4. See Lemma 3.15.

3.4. Topological events for the metric graph GFF. Here we consider a gauge field $\sigma \in \{-1, 1\}^E$. Given U an open non-empty connected subset of the metric graph $\tilde{\mathcal{G}}$, the inverse image $\pi_\sigma^{-1}(U) \subset \tilde{\mathcal{G}}^{\text{db}}$ is either connected or consists of two isometric connected components. It is easy to verify (see e.g. Lemmas 3.3 and 3.5) that $\pi_\sigma^{-1}(U)$ is connected if and only there is a continuous loop \wp contained in U such that $\text{hol}^\sigma(\wp) = -1$. The holonomy of a continuous loop on the metric graph is given by the parity of the number of crossings of edge-lines I_e with $\sigma(e) = -1$.

In the example of Figure 3, $\pi_\sigma^{-1}(U)$ is connected if and only if U surrounds the inner hole of the domain. In the example of Figure 4, $\pi_\sigma^{-1}(U)$ is connected if and only if U separates one hole from the other. By *separate* we mean separate as a subset of the continuum plane, since the planar metric graph on the figure can be embedded into the plane. However, if U surrounds both holes without separating one from the other, then $\pi_\sigma^{-1}(U)$ is not connected.

So whether $\pi_\sigma^{-1}(U)$ is connected or not is a topological, or rather homotopical property of U . It can be reformulated in a more abstract way as follows. Let $\pi_1(\tilde{\mathcal{G}})$ be the fundamental group of the metric graph $\tilde{\mathcal{G}}$ (the notation π_1 should not be confused with the notation π_σ for the covering map; π_1 is just the standard notation for the fundamental group). Let $\pi_1(U)$ be the fundamental group of U . Since U is a subset of $\tilde{\mathcal{G}}$, there is a natural group homomorphism $\theta_U : \pi_1(U) \rightarrow \pi_1(\tilde{\mathcal{G}})$, obtained by considering the loops in U as loops in $\tilde{\mathcal{G}}$. The gauge field σ induces a group homomorphism $h_\sigma : \pi_1(\tilde{\mathcal{G}}) \rightarrow \{-1, 1\}$ obtained by taking the holonomy of loops. The homomorphism h_σ depends only on the gauge equivalence class of σ . The gauge field σ is trivial if and only if $\ker h_\sigma = \pi_1(\tilde{\mathcal{G}})$. Further, $\pi_\sigma^{-1}(U)$ is **not** connected if and only if $\ker h_\sigma \circ \theta_U = \pi_1(U)$. This depends on σ only through its gauge equivalence class, and if σ is trivial, $\pi_\sigma^{-1}(U)$ cannot be connected.

Let $\mathcal{C}(\tilde{\mathcal{G}})$ denote the space of continuous functions $\tilde{\mathcal{G}} \rightarrow \mathbb{R}$. Given a function $f \in \mathcal{C}(\tilde{\mathcal{G}})$, $\{f \neq 0\}$ will be a short notation for the non-zero set of f :

$$\{f \neq 0\} = \{x \in \tilde{\mathcal{G}} \mid f(x) \neq 0\}.$$

Let \mathcal{T}_σ be the following subset of $\mathcal{C}(\tilde{\mathcal{G}})$:

$$\mathcal{T}_\sigma = \{f \in \mathcal{C}(\tilde{\mathcal{G}}) \mid \forall U \text{ connected component of } \{f \neq 0\}, \pi_\sigma^{-1}(U) \text{ is **not** connected}\}.$$

It is easy to see that \mathcal{T}_σ is a closed subset of $\mathcal{C}(\tilde{\mathcal{G}})$ for the uniform norm. In the example of Figure 3, $f \in \mathcal{T}_\sigma$ if and only if no connected component of $\{f \neq 0\}$ surrounds the inner hole of the domain; see Figure 1. In the example of Figure 4, $f \in \mathcal{T}_\sigma$ if and only if no connected component of $\{f \neq 0\}$ separates one hole from the other. Again, \mathcal{T}_σ depends only on the gauge-equivalence class of σ , and if σ is trivial, then $\mathcal{T}_\sigma = \mathcal{C}(\tilde{\mathcal{G}})$.

We recall that, although the field $\tilde{\phi}_\sigma$ is usually not continuous, its absolute value $|\tilde{\phi}_\sigma|$ can always be continuously extended to $\tilde{\mathcal{G}}$. Thus, we see $|\tilde{\phi}_\sigma|$ as a random element of $\mathcal{C}(\tilde{\mathcal{G}})$.

Lemma 3.15. *Almost surely, $|\tilde{\phi}_\sigma| \in \mathcal{T}_\sigma$.*

Proof. We will show that a.s., for every U connected component of $\{|\tilde{\phi}_\sigma| \neq 0\}$, and for every $\hat{x} \in \pi_\sigma^{-1}(U)$, the points \hat{x} and $\psi_\sigma(\hat{x})$ are not connected inside $\pi_\sigma^{-1}(U)$. If this were not the case, there would be a point $\hat{x} \in \tilde{\mathcal{G}}_\sigma^{\text{db}}$ and a continuous path $(\hat{\gamma}(t))_{0 \leq t \leq 1}$ joining \hat{x} and $\psi_\sigma(\hat{x})$ such that for every $t \in [0, 1]$, $|\tilde{\phi}_\sigma|(\pi_\sigma(\hat{\gamma}(t))) > 0$. Consider on $\tilde{\mathcal{G}}_\sigma^{\text{db}}$ the field $\tilde{\phi}_{\sigma,-}^{\text{db}}$, related to $\tilde{\phi}_\sigma$ through (3.6). Then, for every $t \in [0, 1]$, $\tilde{\phi}_{\sigma,-}^{\text{db}}(\hat{\gamma}(t)) \neq 0$. But the field $\tilde{\phi}_{\sigma,-}^{\text{db}}$ is continuous, and $\tilde{\phi}_{\sigma,-}^{\text{db}}(\hat{\gamma}(0)) = -\tilde{\phi}_{\sigma,-}^{\text{db}}(\hat{\gamma}(1))$. So this is a contradiction. \square

Now consider the usual metric graph GFF $\tilde{\phi}$, without the gauge field σ .

Lemma 3.16. *We have that $\mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) > 0$. Moreover, if σ is non-trivial, $\mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) < 1$.*

Proof. Let us first prove that $\mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) > 0$. Let A be the event that the field $\tilde{\phi}$ has zeroes inside every edge-line I_e for $e \in E$. With the construction with Brownian bridges, it is clear that $\mathbb{P}(A) > 0$. Let us argue that on the event A , we have that $\tilde{\phi} \in \mathcal{T}_\sigma$, whatever the gauge field σ . Indeed, on the event A , no connected component of $\{\tilde{\phi} \neq 0\}$ contains a full edge-line, and thus cannot contain a loop with holonomy -1 .

Assume now that the gauge field σ is non-trivial. Then there is $(\gamma(t))_{0 \leq t \leq 1}$ a continuous (deterministic) loop in $\tilde{\mathcal{G}}$ such that $\text{hol}^\sigma(\varphi) = -1$. With positive probability, $\tilde{\phi}$ has no zeroes on the range of φ . On this event, the range of φ belongs to the same connected component of $\{\tilde{\phi} \neq 0\}$, and because $\text{hol}^\sigma(\varphi) = -1$ we have that $\tilde{\phi} \notin \mathcal{T}_\sigma$. \square

We are ready to state the main result of this paper.

Theorem 1. *Let be a gauge field $\sigma \in \{-1, 1\}^E$. Then*

$$(3.7) \quad \mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) = \exp\left(-\mu^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^\sigma(\varphi) = -1\})\right).$$

So in particular (see Corollaries 2.3, 3.7 and 3.9),

$$\begin{aligned} \mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) &= \frac{\det((G_\sigma(x, y))_{x, y \in V_{\text{int}}})^{1/2}}{\det((G(x, y))_{x, y \in V_{\text{int}}})^{1/2}} \\ &= \frac{\det_{\mathcal{S}_{0,+}^{\text{db}}}(-\Delta_\sigma^{\text{db}})^{1/2}}{\det_{\mathcal{S}_{0,-}^{\text{db}}}(-\Delta_\sigma^{\text{db}})^{1/2}} \\ &= \exp\left(-\sum_{\hat{x} \in \mathcal{D} \cap V_{\text{int}}^{\text{db}}} \int_0^{+\infty} p_\sigma(t, \hat{x}, \psi(\hat{x})) \frac{dt}{t}\right), \end{aligned}$$

where \mathcal{D} is a fundamental domain of the covering map π_σ . Moreover, conditionally on the event $\{\tilde{\phi} \in \mathcal{T}_\sigma\}$, the field $|\tilde{\phi}|$ has the same distribution as $|\phi_\sigma|$.

Remark 3.17. Theorem 1 implies in particular that for every $x \in \tilde{\mathcal{G}}$,

$$\mathbb{E}[\tilde{\phi}(x)^2 | \tilde{\phi} \in \mathcal{T}_\sigma] = G_\sigma(x, x).$$

Further, for any $k \in \mathbb{N} \setminus \{0\}$ and $x_1, x_2, \dots, x_k \in \tilde{\mathcal{G}}$, for the conditional moments (products of squares)

$$\mathbb{E}[\tilde{\phi}(x_1)^2 \tilde{\phi}(x_2)^2 \dots \tilde{\phi}(x_k)^2 | \tilde{\phi} \in \mathcal{T}_\sigma]$$

the Wick's formula with kernel G_σ holds.

Remark 3.18. Theorem 1 immediately extends to interacting bosonic fields on the metric graph where the interaction potential depends only on the absolute value of the field. For instance, one can consider the φ^4 fields on $\tilde{\mathcal{G}}$. Let $g > 0$ be a coupling constant and consider the relative partition functions

$$Z_g^{\varphi^4} = \mathbb{E}\left[\exp\left(-g \int_{\tilde{\mathcal{G}}} \tilde{\phi}(x)^4 \tilde{m}(dx)\right)\right], \quad Z_{g,\sigma}^{\varphi^4} = \mathbb{E}\left[\exp\left(-g \int_{\tilde{\mathcal{G}}} \tilde{\phi}_\sigma(x)^4 \tilde{m}(dx)\right)\right].$$

Let $\tilde{\rho}$ be the field on $\tilde{\mathcal{G}}$ with density

$$\frac{1}{Z_g^{\varphi^4}} \exp\left(-g \int_{\tilde{\mathcal{G}}} \tilde{\phi}(x)^4 \tilde{m}(dx)\right)$$

with respect to the GFF $\tilde{\phi}$. This is the φ^4 field on the metric graph. Similarly, let $\tilde{\rho}_\sigma$ be the field on $\tilde{\mathcal{G}}$ with density

$$\frac{1}{Z_{g,\sigma}^{\varphi^4}} \exp\left(-g \int_{\tilde{\mathcal{G}}} \tilde{\phi}_\sigma(x)^4 \tilde{m}(dx)\right)$$

with respect to the GFF $\tilde{\phi}_\sigma$. This is the σ -twisted φ^4 field. Then Theorem 1 implies that

$$(3.8) \quad \mathbb{P}(\tilde{\rho} \in \mathcal{T}_\sigma) = \frac{\det((G_\sigma(x, y))_{x, y \in V_{\text{int}}})^{1/2} Z_{g,\sigma}^{\varphi^4}}{\det((G(x, y))_{x, y \in V_{\text{int}}})^{1/2} Z_g^{\varphi^4}},$$

and that conditionally on $\tilde{\rho} \in \mathcal{T}_\sigma$, the field $|\tilde{\rho}|$ is distributed as $|\tilde{\rho}_\sigma|$. In (3.8) one again gets a ratio of partition functions, this time for the φ^4 field and its σ -twisted version. One can replace the interaction potential φ^4 by any other potential $\mathcal{V}(|\varphi|)$ bounded from below and depending only on the absolute value of the field.

The rest of this section is dedicated to the proof of Theorem 1.

Let be

$$\underline{r} = \min_{e \in E} C(e)^{-1}.$$

Given $e \in E$, x_e^{m} will denote the middle point of the edge-line I_e :

$$x_e^{\text{m}} = x_{e, C(e)^{-1/2}}.$$

For $\varepsilon \leq C(e)^{-1}/2$, we will denote by $x_{e,\varepsilon}^-$ and $x_{e,\varepsilon}^+$ the two points of I_e at distance ε from x_e^{m} :

$$x_{e,\varepsilon}^- = x_{e, C(e)^{-1/2} + \varepsilon}, \quad x_{e,\varepsilon}^+ = x_{e, C(e)^{-1/2} - \varepsilon}.$$

The points $x_{e,\varepsilon}^-$ and $x_{e,\varepsilon}^+$ will play symmetric roles, the distinction is just for the notations. Let $J_{e,\varepsilon}$ denote the open subinterval of I_e delimited by $x_{e,\varepsilon}^-$ and $x_{e,\varepsilon}^+$:

$$J_{e,\varepsilon} = \{x_{e,u} | C(e)^{-1}/2 - \varepsilon < u < C(e)^{-1}/2 + \varepsilon\}.$$

See Figure 6.

Given a gauge field $\sigma \in \{-1, 1\}^E$, we will denote

$$E_{\sigma,-} = \{e \in E | \sigma(e) = -1\}.$$

For $\varepsilon \in (0, \underline{r}/2)$, let $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$ denote

$$\tilde{\mathcal{G}}_{\sigma,\varepsilon} = \tilde{\mathcal{G}} \setminus \bigcup_{e \in E_{\sigma,-}} J_{e,\varepsilon}.$$

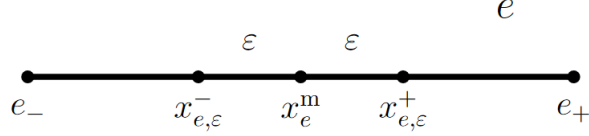


FIGURE 6. Inside an edge-line.

In this way, $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$ is a closed subset of the metric graph $\tilde{\mathcal{G}}$, and is itself a metric graph, but not necessarily connected. Consider the discrete graph $\mathcal{G}_{\sigma,\varepsilon} = (V_{\sigma,\varepsilon}, E_{\sigma,\varepsilon})$ constructed as follows. The set of vertices is given by

$$V_{\sigma,\varepsilon} = V \cup \{x_{e,\varepsilon}^- | e \in E_{\sigma,-}\} \cup \{x_{e,\varepsilon}^+ | e \in E_{\sigma,-}\},$$

and the set of edges given by

$$E_{\sigma,\varepsilon} = (E \setminus E_{\sigma,-}) \cup \{\{x_{e,\varepsilon}^-, e_-\} | e \in E_{\sigma,-}\} \cup \{\{x_{e,\varepsilon}^+, e_+\} | e \in E_{\sigma,-}\}.$$

The graph $\mathcal{G}_{\sigma,\varepsilon}$ is endowed with the following conductances: for $e \in E \setminus E_{\sigma,-}$, we keep the same conductance $C(e)$. Further, $e \in E_{\sigma,-}$, we set

$$C(x_{e,\varepsilon}^-, e_-) = C(x_{e,\varepsilon}^+, e_+) = (C(e)^{-1}/2 - \varepsilon)^{-1}.$$

Then $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$ is actually the metric graph associated to the electrical network $\mathcal{G}_{\sigma,\varepsilon}$. We will denote by $\tilde{\phi}_{\sigma,\varepsilon}$ the metric graph GFF on $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$ with 0 boundary conditions on V_∂ . The field $\tilde{\phi}_{\sigma,\varepsilon}$ is a usual metric graph GFF, i.e. **not** gauge-twisted. We will also consider the restrictions of the metric graph GFF $\tilde{\phi}$ and $\tilde{\phi}_\sigma$ to $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$.

For $x, y \in \tilde{\mathcal{G}}$, we will denote

$$G(x, y) = \mathbb{E}[\tilde{\phi}(x), \tilde{\phi}(y)], \quad G_\sigma(x, y) = \mathbb{E}[\tilde{\phi}_\sigma(x), \tilde{\phi}_\sigma(y)].$$

In this way we have a natural extension of Green's functions G and G_σ beyond $V \times V$. note that G defined in this way is continuous on $\tilde{\mathcal{G}} \times \tilde{\mathcal{G}}$, but G_σ has discontinuities at the middle points x_e^m for $e \in E_{\sigma,-}$. We will also denote by $G_{\sigma,\varepsilon}(x, y)$ the covariance kernel of $\tilde{\phi}_{\sigma,\varepsilon}$ on $\tilde{\mathcal{G}}_{\sigma,\varepsilon} \times \tilde{\mathcal{G}}_{\sigma,\varepsilon}$.

Lemma 3.19. *The restrictions $\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ and $\tilde{\phi}_\sigma|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ are both absolutely continuous with respect to $\tilde{\phi}_{\sigma,\varepsilon}$. The Radon-Nikodym derivatives are given by*

$$(3.9) \quad \frac{d\mathbb{P}_{\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}}(\tilde{\varphi})}{d\mathbb{P}_{\tilde{\phi}_{\sigma,\varepsilon}}}(\tilde{\varphi}) = \frac{(\det(G_{\sigma,\varepsilon}(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} \exp\left(-\frac{1}{4\varepsilon} \sum_{e \in E_{\sigma,-}} (\tilde{\varphi}(x_{e,\varepsilon}^+) - \tilde{\varphi}(x_{e,\varepsilon}^-))^2\right),$$

$$(3.10) \quad \frac{d\mathbb{P}_{\tilde{\phi}_\sigma|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}}(\tilde{\varphi})}{d\mathbb{P}_{\tilde{\phi}_{\sigma,\varepsilon}}}(\tilde{\varphi}) = \frac{(\det(G_{\sigma,\varepsilon}(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G_\sigma(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} \exp\left(-\frac{1}{4\varepsilon} \sum_{e \in E_{\sigma,-}} (\tilde{\varphi}(x_{e,\varepsilon}^+) + \tilde{\varphi}(x_{e,\varepsilon}^-))^2\right).$$

Proof. Let us introduce the auxiliary electrical network $\mathcal{G}_{\sigma,\varepsilon}^*$ as follows. Its set of vertices is $V_{\sigma,\varepsilon}$, the same as for $\mathcal{G}_{\sigma,\varepsilon}$. Its set of edges is

$$E_{\sigma,\varepsilon}^* = E_{\sigma,\varepsilon} \cup \{\{x_{e,\varepsilon}^-, x_{e,\varepsilon}^+\} | e \in E_{\sigma,-}\}.$$

The conductances on $E_{\sigma,\varepsilon}^*$ are as follows. If $e \in E_{\sigma,\varepsilon}$, then we keep the same conductance $C(e)$. Moreover, we set

$$C(x_{e,\varepsilon}^-, x_{e,\varepsilon}^+) = \frac{1}{2\varepsilon},$$

for $e \in E_{\sigma,-}$. Then the metric graph associated to the electrical network $\mathcal{G}_{\sigma,\varepsilon}^*$ is again $\tilde{\mathcal{G}}$, the same as for \mathcal{G} . This is because for every $e \in E_{\sigma,-}$,

$$\begin{aligned} C(e_-, x_{e,\varepsilon}^-)^{-1} + C(x_{e,\varepsilon}^-, x_{e,\varepsilon}^+)^{-1} + C(x_{e,\varepsilon}^+, e_+)^{-1} \\ = (C(e)^{-1}/2 - \varepsilon) + 2\varepsilon + (C(e)^{-1}/2 - \varepsilon) = C(e)^{-1}, \end{aligned}$$

as for resistors in series. Therefore, the restriction of $\tilde{\phi}$ to $V_{\sigma,\varepsilon}$ is the discrete GFF on the electrical network $\mathcal{G}_{\sigma,\varepsilon}^*$, with 0 boundary conditions on V_∂ . The right-hand side of (3.9) is then the density of $(\tilde{\phi}(x))_{x \in V_{\sigma,\varepsilon}}$ with respect to $(\tilde{\phi}_{\sigma,\varepsilon}(x))_{x \in V_{\sigma,\varepsilon}}$, both being discrete GFFs, on $\mathcal{G}_{\sigma,\varepsilon}^*$ and $\mathcal{G}_{\sigma,\varepsilon}$ respectively. Indeed, compared to $\mathcal{G}_{\sigma,\varepsilon}$, $\mathcal{G}_{\sigma,\varepsilon}^*$ contains only the extra edges $\{x_{e,\varepsilon}^-, x_{e,\varepsilon}^+\}$ for $e \in E_{\sigma,-}$, with conductance $1/(2\varepsilon)$ each, so the corresponding factors appear in the density. The ration of determinants of Green's functions to the power $1/2$ is the normalization factor. Finally, the expression of the density is the same when we consider the whole metric graph $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$, because both for $\tilde{\phi}$ and $\tilde{\phi}_{\sigma,\varepsilon}$ we interpolate with the same independent Brownian bridges.

Now let us deal with $\tilde{\phi}_\sigma$. We consider on $\mathcal{G}_{\sigma,\varepsilon}^*$ the gauge-field $\sigma^* \in \{-1, 1\}^{E_{\sigma,\varepsilon}^*}$ defined as follows. For $e \in E \setminus E_{\sigma,-}$, $\sigma^*(e) = \sigma(e) = 1$. Further,

$$\sigma^*(e_-, x_{e,\varepsilon}^-) = \sigma^*(e_+, x_{e,\varepsilon}^+) = 1,$$

and

$$\sigma^*(x_{e,\varepsilon}^-, x_{e,\varepsilon}^+) = -1.$$

In this way, for every \wp nearest-neighbor path in $\mathcal{G}_{\sigma,\varepsilon}^*$ joining two points, $x, y \in V$,

$$\text{hol}^{\sigma^*}(\wp) = \text{hol}^\sigma(\wp_{\mathcal{G}}),$$

where $\wp_{\mathcal{G}}$ is the nearest-neighbor path in \mathcal{G} from x to y obtained by taking the trace of \wp on V . The restriction of $\tilde{\phi}_\sigma$ to $V_{\sigma,\varepsilon}$ is the σ^* -twisted GFF on $\mathcal{G}_{\sigma,\varepsilon}^*$ with 0 boundary conditions on V_∂ . This is similar to Proposition 3.2. Then (3.10) is just the density of $(\tilde{\phi}_\sigma(x))_{x \in V_{\sigma,\varepsilon}}$ with respect to $(\tilde{\phi}_{\sigma,\varepsilon}(x))_{x \in V_{\sigma,\varepsilon}}$. Because of σ^* , there is a + instead of a - in (3.10) compared to (3.9). \square

Note that Lemma 3.19 implies in particular that $\tilde{\phi}_{\sigma|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ is absolutely continuous with respect to $\tilde{\phi}_{|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$. However, if one considers the whole metric graph $\tilde{\mathcal{G}}$, then $\tilde{\phi}_\sigma$ is singular with respect to $\tilde{\phi}$, unless $E_{\sigma,-} = \emptyset$. For instance, $\tilde{\phi}_\sigma$ is a.s. discontinuous at the middle points x_e^m for $e \in E_{\sigma,-}$. So that is the reason why we restrict to ε -far away from these middle points.

Let $\mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$ denote the space of continuous real-valued functions on $\tilde{\mathcal{G}}_{\sigma,\varepsilon}$. For $f \in \mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$, $\{f \neq 0\}$ will denote the non-zero set of f , with $\{f \neq 0\} \subset \tilde{\mathcal{G}}_{\sigma,\varepsilon}$. Let $\mathcal{T}_{\sigma,\varepsilon}$ be the following subset of $\mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$:

$$\mathcal{T}_{\sigma,\varepsilon} = \left\{ f \in \mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon}) \mid \forall U \text{ connected component of } \left(\{f \neq 0\} \bigcup_{e \in E_{\sigma,-}} J_{e,\varepsilon} \right), \pi_\sigma^{-1}(U) \text{ is } \mathbf{not} \text{ connected} \right\}.$$

In the above definition, we enlarge $\{f \neq 0\}$ with the intervals $J_{e,\varepsilon}$ for $e \in E_{\sigma,-}$, and then consider the connected components.

Given $f \in \mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$, we will define an equivalence relation \sim_f on the set

$$(3.11) \quad \{x_{e,\varepsilon}^- \mid e \in E_{\sigma,-}\} \cup \{x_{e,\varepsilon}^+ \mid e \in E_{\sigma,-}\} = V_{\sigma,\varepsilon} \setminus V.$$

Two points x, x' in the above set are in the same class if they are in the same connected component of $\{f \neq 0\}$. If $f(x) = 0$, then x is alone in its class. We will denote by \mathbf{V}_f the set (3.11) quotiented by the equivalence relation \sim_f , and by $[x]_f$ the equivalence class of a point x . Next we will introduce an undirected multigraph Γ_f induced by the function f . Note that in general Γ_f is not connected and may have self-loops and multiple edges. The set of vertices of Γ_f is the quotient set \mathbf{V}_f . Further, for each $e \in E_{\sigma,-}$, add to the multigraph Γ_f and edge with the ends $[x_{e,\varepsilon}^-]_f$ and $[x_{e,\varepsilon}^+]_f$. So the number of edges of Γ_f is $|E_{\sigma,-}|$.

Lemma 3.20. *Let $f \in \mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$. Then the multigraph Γ_f is bipartite, i.e. does not contain cycles with odd number of edges, if and only if $f \in \mathcal{T}_{\sigma,\varepsilon}$.*

Proof. Given a continuous loop \wp in the subset

$$(3.12) \quad \left(\{f \neq 0\} \bigcup_{e \in E_{\sigma,-}} J_{e,\varepsilon} \right),$$

It induces a nearest-neighbor loop in the multigraph Γ_f , which will denote by $[\wp]_f$ by a slight abuse of notation. If \wp does not visit any point in $V_{\sigma,\varepsilon} \setminus V$ (3.11), then $[\wp]_f$ is just the empty loop. Conversely, every nearest-neighbor loop in Γ_f can be lifted to a continuous loop in (3.12). This follows from the way Γ_f has been constructed. The edges visited by $[\wp]_f$ correspond to the intervals $J_{e,\varepsilon}$ (with $e \in E_{\sigma,-}$) crossed by \wp from one end to the opposite. The holonomy $\text{hol}^\sigma(\wp)$ is given by the parity of the number of edge-lines I_e , for $e \in E_{\sigma,-}$, crossed by \wp . This parity is the same as for the crossings of $J_{e,\varepsilon}$, for $e \in E_{\sigma,-}$, although the number of crossings itself might be different. This is because each time \wp crosses an edge-line I_e , it will cross the corresponding subinterval $J_{e,\varepsilon}$ an odd number of times. Thus, $\text{hol}^\sigma(\wp)$ is also given by the parity of the number of edges of multigraph loop $[\wp]_f$. Thus, Γ_f is bipartite if and only if the subset (3.12) does not contain loops \wp with $\text{hol}^\sigma(\wp) = -1$, which is the same as $f \in \mathcal{T}_{\sigma,\varepsilon}$. \square

Note that although $\mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$ is infinite, the number of different partitions of $V_{\sigma,\varepsilon} \setminus V$ (3.11) induced by the functions f can be only finite. Given $f \in \mathcal{T}_{\sigma,\varepsilon}$, one can choose a ‘‘coloring’’ $\omega_f : V_{\sigma,\varepsilon} \setminus V \rightarrow \{-1, 1\}$ satisfying the following two properties.

- (1) The value of ω_f is the same across every equivalence class for \sim_f .
- (2) For every $e \in E_{\sigma,-}$, $\omega_f(x_{e,\varepsilon}^-) = -\omega_f(x_{e,\varepsilon}^+)$.

The existence of such ω_f is ensured by the fact that Γ_f is bipartite. Moreover, one can choose ω_f to depend only on the partition \mathbf{V}_f induced by \sim_f .

Next we define a transformation $\mathbf{T}_{\sigma,\varepsilon}$ acting on $\mathcal{T}_{\sigma,\varepsilon}$. Given $f \in \mathcal{T}_{\sigma,\varepsilon}$, the function $\mathbf{T}_{\sigma,\varepsilon}f$ is defined as follows.

- (1) For every $x \in \tilde{\mathcal{G}}_{\sigma,\varepsilon}$ such that $f(x) = 0$, we also have $(\mathbf{T}_{\sigma,\varepsilon}f)(x) = 0$.
- (2) On every connected component U of $\{f \neq 0\}$ such that U does not intersect the set $V_{\sigma,\varepsilon} \setminus V$, we have $(\mathbf{T}_{\sigma,\varepsilon}f)|_U = f|_U$.
- (3) On every connected component U of $\{f \neq 0\}$ intersecting $V_{\sigma,\varepsilon} \setminus V$, we have $(\mathbf{T}_{\sigma,\varepsilon}f)|_U = \omega_f(U)f|_U$, where $\omega_f(U)$ is the common value of ω_f on U .

So, in essence, $\mathbf{T}_{\sigma,\varepsilon}f$ is obtained by flipping the signs on some connected components of $\{f \neq 0\}$. In this way, $\mathbf{T}_{\sigma,\varepsilon}f$ is continuous too, and $|\mathbf{T}_{\sigma,\varepsilon}f| \equiv f$. Therefore, $\mathbf{T}_{\sigma,\varepsilon}f \in \mathcal{T}_{\sigma,\varepsilon}$ too. Moreover, the equivalence relation $\sim_{\mathbf{T}_{\sigma,\varepsilon}f}$ is the same as \sim_f . Thus, $\mathbf{T}_{\sigma,\varepsilon}^2f = \mathbf{T}_{\sigma,\varepsilon}(\mathbf{T}_{\sigma,\varepsilon}f) = f$.

Further, we will extend $\mathbf{T}_{\sigma,\varepsilon}$ to the whole $\mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$. For $f \in \mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon}) \setminus \mathcal{T}_{\sigma,\varepsilon}$, we set $\mathbf{T}_{\sigma,\varepsilon}f = f$.

Let us denote by $\tilde{\xi}_\varepsilon$ and $\tilde{\eta}_\varepsilon$ the fields

$$\tilde{\xi}_\varepsilon = \mathbf{T}_{\sigma,\varepsilon}(\tilde{\phi}_{|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}), \quad \tilde{\eta}_\varepsilon = \mathbf{T}_{\sigma,\varepsilon}(\tilde{\phi}_{\sigma,\varepsilon}).$$

Lemma 3.21. *The field $\tilde{\xi}_\varepsilon$ is absolutely continuous with respect to $\tilde{\eta}_\varepsilon$, and the corresponding density is*

$$\frac{d\mathbb{P}_{\tilde{\xi}_\varepsilon}}{d\mathbb{P}_{\tilde{\eta}_\varepsilon}}(\tilde{\varphi}) = \frac{(\det(G_{\sigma,\varepsilon}(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} \exp\left(-\frac{1}{4\varepsilon} \sum_{e \in E_{\sigma,-}} (\tilde{\varphi}(x_{e,\varepsilon}^+) - (-1)^{\mathbf{1}_{\tilde{\varphi} \in \mathcal{T}_{\sigma,\varepsilon}}} \tilde{\varphi}(x_{e,\varepsilon}^-))^2\right).$$

Proof. Since $\tilde{\phi}_{|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ is absolutely continuous with respect to $\tilde{\phi}_{\sigma,\varepsilon}$ (Lemma 3.19) and $\mathbf{T}_{\sigma,\varepsilon}$ is a deterministic and measurable transformation, we get that $\tilde{\xi}_\varepsilon$ is absolutely continuous with respect to $\tilde{\eta}_\varepsilon$, with

$$\frac{d\mathbb{P}_{\tilde{\xi}_\varepsilon}}{d\mathbb{P}_{\tilde{\eta}_\varepsilon}}(\tilde{\varphi}) = \frac{d\mathbb{P}_{\tilde{\phi}_{|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}(\mathbf{T}_{\sigma,\varepsilon}\tilde{\varphi})}{d\mathbb{P}_{\tilde{\phi}_{\sigma,\varepsilon}}(\tilde{\varphi})}.$$

Further we use the expression (3.9). Take $e \in E_{\sigma,-}$. For $\tilde{\varphi} \notin \mathcal{T}_{\sigma,\varepsilon}$,

$$((\mathbf{T}_{\sigma,\varepsilon}\tilde{\varphi})(x_{e,\varepsilon}^+) - (\mathbf{T}_{\sigma,\varepsilon}\tilde{\varphi})(x_{e,\varepsilon}^-))^2 = (\tilde{\varphi}(x_{e,\varepsilon}^+) - \tilde{\varphi}(x_{e,\varepsilon}^-))^2.$$

For $\tilde{\varphi} \in \mathcal{T}_{\sigma,\varepsilon}$,

$$((\mathbf{T}_{\sigma,\varepsilon}\tilde{\varphi})(x_{e,\varepsilon}^+) - (\mathbf{T}_{\sigma,\varepsilon}\tilde{\varphi})(x_{e,\varepsilon}^-))^2 = (\tilde{\varphi}(x_{e,\varepsilon}^+) + \tilde{\varphi}(x_{e,\varepsilon}^-))^2.$$

This because $\omega_f(x_{e,\varepsilon}^-)\omega_f(x_{e,\varepsilon}^+) = -1$. \square

Lemma 3.22. *The field $\tilde{\eta}_\varepsilon = \mathbb{T}_{\sigma,\varepsilon}(\tilde{\phi}_{\sigma,\varepsilon})$ has the same distribution as $\tilde{\phi}_{\sigma,\varepsilon}$.*

Proof. First, by construction, $|\mathbb{T}_{\sigma,\varepsilon}(\tilde{\phi}_{\sigma,\varepsilon})| \equiv |\tilde{\phi}_{\sigma,\varepsilon}|$. Further, the signs $\omega_{\tilde{\phi}_{\sigma,\varepsilon}}$ (that is to say ω_f with $f = \tilde{\phi}_{\sigma,\varepsilon}$) are by construction measurable with respect to $|\tilde{\phi}_{\sigma,\varepsilon}|$. Then, we use the fact that the signs of $\tilde{\phi}_{\sigma,\varepsilon}$ (do not confuse them with $\omega_{\tilde{\phi}_{\sigma,\varepsilon}}$) are distributed, conventionally on $|\tilde{\phi}_{\sigma,\varepsilon}|$ as independent uniform $\{-1, 1\}$ -valued r.v.s, one for each connected component of $\{\tilde{\phi}_{\sigma,\varepsilon} \neq 0\}$; see [Lup16, Lemma 3.2]. This implies that the product of signs $\omega_{\tilde{\phi}_{\sigma,\varepsilon}} \text{sign}(\tilde{\phi}_{\sigma,\varepsilon})$ has the same conditional distribution given $|\tilde{\phi}_{\sigma,\varepsilon}|$ as $\text{sign}(\tilde{\phi}_{\sigma,\varepsilon})$. Therefore, $\mathbb{T}_{\sigma,\varepsilon}(\tilde{\phi}_{\sigma,\varepsilon})$ has the same distribution as $\tilde{\phi}_{\sigma,\varepsilon}$. \square

Lemma 3.23. *The function*

$$\varepsilon \mapsto \frac{(\det(G_\sigma(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}$$

is constant on $(0, \underline{r}/2)$. More precisely, for every $\varepsilon \in (0, \underline{r}/2)$,

$$\begin{aligned} \frac{(\det(G_\sigma(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} &= \frac{(\det(G_\sigma(x, y))_{x,y \in V_{\text{int}}})^{1/2}}{(\det(G(x, y))_{x,y \in V_{\text{int}}})^{1/2}} \\ &= \exp(-\mu^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^\sigma(\varphi) = -1\})). \end{aligned}$$

Proof. Consider the measure $\tilde{\mu}^{\text{loop}}$ (2.8) on Brownian loops on the metric graph $\tilde{\mathcal{G}}$. Similarly to Corollary (2.3), we have that

$$\frac{(\det(G_\sigma(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} = \exp(-\tilde{\mu}^{\text{loop}}(\{\text{Loops } \varphi \text{ visiting } V_{\sigma,\varepsilon} \setminus V \text{ and with } \text{hol}^\sigma(\varphi) = -1\})).$$

Moreover,

$$\begin{aligned} \frac{\det((G_\sigma(x, y))_{x,y \in V_{\text{int}}})^{1/2}}{\det((G(x, y))_{x,y \in V_{\text{int}}})^{1/2}} &= \exp(-\mu^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^\sigma(\varphi) = -1\})) \\ &= \exp(-\tilde{\mu}^{\text{loop}}(\{\text{Loops } \varphi \text{ visiting } V \text{ and with } \text{hol}^\sigma(\varphi) = -1\})), \end{aligned}$$

where the second equality is due to the fact that the measure on discrete loops μ^{loop} can be obtained by taking the trace on V of metric graph loops under $\tilde{\mu}^{\text{loop}}$. We conclude by using the fact that any continuous loop φ on $\tilde{\mathcal{G}}$ with $\text{hol}^\sigma(\varphi) = -1$ has to visit both V and $V_{\sigma,\varepsilon} \setminus V$. That is to say a loop not visiting either of two subsets has holonomy 1, because then it cannot cross any of the edge-lines I_e for $e \in E_{\sigma,-}$. Therefore,

$$\begin{aligned} \tilde{\mu}^{\text{loop}}(\{\text{Loops } \varphi \text{ visiting } V_{\sigma,\varepsilon} \setminus V \text{ and with } \text{hol}^\sigma(\varphi) = -1\}) &= \tilde{\mu}^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^\sigma(\varphi) = -1\}) \\ &= \mu^{\text{loop}}(\{\text{Loops } \varphi \text{ visiting } V \text{ and with } \text{hol}^\sigma(\varphi) = -1\}) = \mu^{\text{loop}}(\{\text{Loops } \varphi \text{ with } \text{hol}^\sigma(\varphi) = -1\}). \quad \square \end{aligned}$$

Proof of Theorem 1. By Lemma 3.19, the field $\tilde{\phi}_{\sigma|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ is absolutely continuous with respect to the field $\tilde{\phi}_{\sigma,\varepsilon}$. By Lemma 3.22, the field $\tilde{\eta}_\varepsilon$ has the same distribution as $\tilde{\phi}_{\sigma,\varepsilon}$. By Lemma 3.21, the fields $\tilde{\eta}_\varepsilon$ and $\tilde{\xi}_\varepsilon$ are mutually absolutely continuous, the corresponding Radon–Nikodym derivative being positive on $\mathcal{C}(\tilde{\mathcal{G}}_{\sigma,\varepsilon})$. Therefore, the field $\tilde{\phi}_{\sigma|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ is absolutely continuous with respect to the field $\tilde{\xi}_\varepsilon$ and the Radon–Nikodym derivative is given by

$$(3.13) \quad \frac{d\mathbb{P}_{\tilde{\phi}_{\sigma|\tilde{\mathcal{G}}_{\sigma,\varepsilon}}}}{d\mathbb{P}_{\tilde{\xi}_\varepsilon}}(\tilde{\varphi}) = \frac{(\det(G(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G_\sigma(x, y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} \left(\mathbf{1}_{\tilde{\varphi} \in \mathcal{T}_{\sigma,\varepsilon}} + \mathbf{1}_{\tilde{\varphi} \notin \mathcal{T}_{\sigma,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \sum_{e \in E_{\sigma,-}} \tilde{\varphi}(x_{e,\varepsilon}^+) \tilde{\varphi}(x_{e,\varepsilon}^-)\right) \right).$$

Note that $\tilde{\xi}_\varepsilon \in \mathcal{T}_{\sigma,\varepsilon}$ if and only if $\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \in \mathcal{T}_{\sigma,\varepsilon}$. So in particular,

$$\begin{aligned} & \mathbb{P}(\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \in \mathcal{T}_{\sigma,\varepsilon}) + \mathbb{E}\left[\mathbf{1}_{\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \notin \mathcal{T}_{\sigma,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \sum_{e \in E_{\sigma,-}} \tilde{\phi}(x_{e,\varepsilon}^+) \tilde{\phi}(x_{e,\varepsilon}^-)\right)\right] \\ &= \frac{(\det(G_\sigma(x,y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x,y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} = \exp(-\mu^{\text{loop}}(\{\text{Loops } \wp \text{ with } \text{hol}^\sigma(\wp) = -1\})), \end{aligned}$$

where the second equality is due to Lemma 3.23. Further,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \in \mathcal{T}_{\sigma,\varepsilon}) = \mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma)$$

and

$$\mathbb{E}\left[\mathbf{1}_{\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \notin \mathcal{T}_{\sigma,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \sum_{e \in E_{\sigma,-}} \tilde{\phi}(x_{e,\varepsilon}^+) \tilde{\phi}(x_{e,\varepsilon}^-)\right)\right] = \frac{(\det(G_\sigma(x,y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}}{(\det(G(x,y))_{x,y \in V_{\sigma,\varepsilon} \setminus V})^{1/2}} \mathbb{P}(|\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \notin \mathcal{T}_{\sigma,\varepsilon}),$$

with

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(|\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}} \notin \mathcal{T}_{\sigma,\varepsilon}) = \mathbb{P}(|\tilde{\phi}| \notin \mathcal{T}_\sigma) = 0;$$

see Lemma 3.15. So we get (3.7).

Since $|\tilde{\xi}_\varepsilon| = |\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$, the field $|\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$ is absolutely continuous with respect to $|\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}$, and

$$\lim_{\varepsilon \rightarrow 0} \frac{d\mathbb{P}_{|\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}}}{d\mathbb{P}_{|\tilde{\phi}|_{\tilde{\mathcal{G}}_{\sigma,\varepsilon}}}}(\tilde{\varphi}) = \frac{1}{\mathbb{P}(\tilde{\varphi} \in \mathcal{T}_\sigma)} \mathbf{1}_{\tilde{\varphi} \in \mathcal{T}_\sigma},$$

with convergence in L^1 . So $\mathbb{P}(\tilde{\varphi} \in \mathcal{T}_\sigma)^{-1} \mathbf{1}_{\tilde{\varphi} \in \mathcal{T}_\sigma}$ is the density of $|\tilde{\phi}|_\sigma$ with respect to $|\tilde{\phi}|$. \square

Next we explain how to get the field $\tilde{\phi}_\sigma$ given the field $\tilde{\phi}$ conditioned on $\tilde{\phi} \in \mathcal{T}_\sigma$, that is to say how to get the signs, not just the absolute value. In essence, one has to flip the signs on some of the connected components of $\{\tilde{\phi} \neq 0\} \setminus \{x_e^m | e \in E_{\sigma,-}\}$.

To avoid trivialities, we assume that the gauge field σ is not uniformly 1. Let us denote

$$\mathfrak{M}_\sigma = \{x_e^m | e \in E_{\sigma,-}\}.$$

We endow the subset $\tilde{\mathcal{G}} \setminus \mathfrak{M}_\sigma$ with the correspondent length metric d' , which is not the same as the distance inherited from $\tilde{\mathcal{G}}$ since now we are not allowed to cross the points in \mathfrak{M}_σ . The metric space $(\tilde{\mathcal{G}} \setminus \mathfrak{M}_\sigma, d')$ is not complete, but we can consider its completion for d' , which is not $\tilde{\mathcal{G}}$. This completion is

$$(\tilde{\mathcal{G}} \setminus \mathfrak{M}_\sigma) \cup \mathfrak{M}_\sigma^\pm,$$

with

$$\mathfrak{M}_\sigma^\pm = \{x_e^{m,+} | e \in E_{\sigma,-}\} \cup \{x_e^{m,-} | e \in E_{\sigma,-}\},$$

where $x_e^{m,-}$ and $x_e^{m,+}$ are to be understood as left and right infinitesimal neighborhoods of x_e^m :

$$x_e^{m,-} = \lim_{\varepsilon \rightarrow 0} x_{e,\varepsilon}^-, \quad x_e^{m,+} = \lim_{\varepsilon \rightarrow 0} x_{e,\varepsilon}^+.$$

The fields $\tilde{\phi}$ and $\tilde{\phi}_\sigma$ can be both seen as continuous fields on $(\tilde{\mathcal{G}} \setminus \mathfrak{M}_\sigma) \cup \mathfrak{M}_\sigma^\pm$, with

$$\tilde{\phi}(x_e^{m,-}) = \tilde{\phi}(x_e^{m,+}), \quad \tilde{\phi}_\sigma(x_e^{m,-}) = -\tilde{\phi}_\sigma(x_e^{m,+}), \quad e \in E_{\sigma,-}.$$

A sample of $\tilde{\phi}$ induces a partition $\mathbf{V}_{\tilde{\phi}}$ of \mathfrak{M}_σ^\pm , where two points $x, x' \in \mathfrak{M}_\sigma^\pm$ are in the same class if they are in the same connected component of $\{\tilde{\phi} \neq 0\}$ seen as a subset of $(\tilde{\mathcal{G}} \setminus \mathfrak{M}_\sigma) \cup \mathfrak{M}_\sigma^\pm$ and not as a subset of $\tilde{\mathcal{G}}$. We will denote by $[x]_{\tilde{\phi}}$ the equivalence class of x for $x \in \mathfrak{M}_\sigma^\pm$. The condition $\tilde{\phi} \in \mathcal{T}_\sigma$ is equivalent to $\mathbf{V}_{\tilde{\phi}}$ being bicolorable in the following sense: there is a map $\omega : \mathbf{V}_{\tilde{\phi}} \rightarrow \{-1, 1\}$, such that for every $e \in E_{\sigma,-}$, $\omega([x_e^{m,-}]_{\tilde{\phi}}) = -\omega([x_e^{m,+}]_{\tilde{\phi}})$. This is similar to Lemma 3.20. The number of different bicolourings is 2^k where k is the number of connected

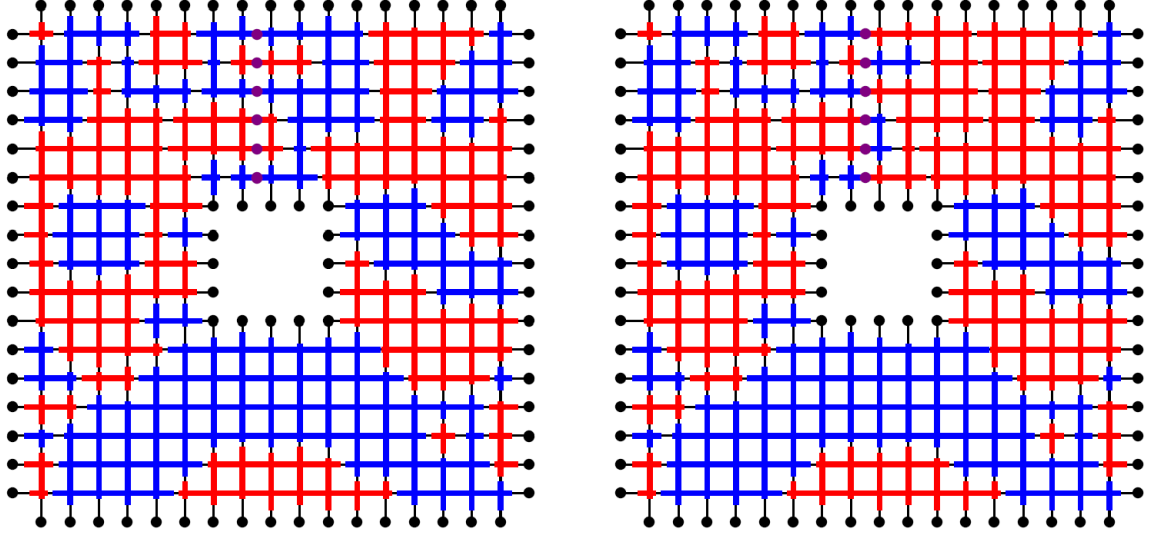


FIGURE 7. Left: the field $\tilde{\phi}$ on the event $\tilde{\phi} \in \mathcal{T}_\sigma$, which in this example means that there are no sign clusters surrounding the inner hole of the domain. Right: the associated $\tilde{\xi}$ field. The black dots represent the boundary V_∂ . The violet dots represent \mathfrak{M}_σ . The positive, resp. negative values of the fields are in red, resp. blue.

components of $\{\tilde{\phi} \neq 0\}$ in $\tilde{\mathcal{G}}$ (not in $(\tilde{\mathcal{G}} \setminus \mathfrak{M}_\sigma) \cup \mathfrak{M}_\sigma^\pm$!) that intersect \mathfrak{M}_σ . Indeed, there are two different colorings per such connected component, one being the opposite of the other.

Let $\text{Bic}(\mathfrak{M}_\sigma)$ denote the set of bicolored partitions of \mathfrak{M}_σ . To each such partition $\mathbf{p} \in \text{Bic}(\mathfrak{M}_\sigma)$ we will associate (in a deterministic way) a bicoloring $\omega_{\mathbf{p}}$. We define the random field $\tilde{\xi}$ on $\tilde{\mathcal{G}}$ as follows. On the event $\tilde{\phi} \notin \mathcal{T}_\sigma$, we set $\tilde{\xi} = \tilde{\phi}$. On the event $\tilde{\phi} \in \mathcal{T}_\sigma$, $\tilde{\xi}$ is defined by the following.

- (1) For every $x \in \tilde{\mathcal{G}}$, $|\tilde{\xi}(x)| = |\tilde{\phi}(x)|$.
- (2) On every connected component U of $\{\tilde{\phi} \neq 0\}$, such that $U \cap \mathfrak{M}_\sigma = \emptyset$, we have $\tilde{\xi}|_U = \tilde{\phi}|_U$.
- (3) On every connected component U of $\{\tilde{\phi} \neq 0\} \setminus \mathfrak{M}_\sigma$ such that $\bar{U} \cap \mathfrak{M}_\sigma \neq \emptyset$, $\tilde{\xi}|_U = \omega_{\mathbf{v}_{\tilde{\phi}}}(U) \tilde{\phi}|_U$, where $\omega_{\mathbf{v}_{\tilde{\phi}}}$ is $\omega_{\mathbf{p}}$ with \mathbf{p} being the random partition $\mathbf{V}_{\tilde{\phi}}$, and $\omega_{\mathbf{v}_{\tilde{\phi}}}(U)$ is the common value of $\omega_{\mathbf{v}_{\tilde{\phi}}}[x]_{\tilde{\phi}}$ for $x \in \bar{U} \cap \mathfrak{M}_\sigma$.

So $\tilde{\xi}$ is obtained from $\tilde{\phi}$ through a deterministic transformation. It corresponds to flipping the signs of some of the connected components of $\{\tilde{\phi} \neq 0\} \setminus \mathfrak{M}_\sigma$, on the event $\tilde{\phi} \in \mathcal{T}_\sigma$, so as to achieve a bicoloring. See Figure 7.

Corollary 3.24. *The conditional distribution of $\tilde{\xi}$ on the event $\tilde{\phi} \in \mathcal{T}_\sigma$, is that of $\tilde{\phi}_\sigma$.*

Proof. The couple $(\tilde{\phi}, \tilde{\xi})$ can be obtained as limit in law as $\varepsilon \rightarrow 0$ of $(\tilde{\phi}_{|\tilde{\mathcal{G}}_{\sigma, \varepsilon}}, \tilde{\xi}_\varepsilon)$. Then the result is obtained by passing the density (3.13) to the limit. \square

3.5. A remark on the isomorphism for $\tilde{\phi}_\sigma$. Recall the measure on metric graph loops $\tilde{\mu}^{\text{loop}}$ (2.8). Let $\sigma \in \{-1, 1\}^E$. Let the signed measure on metric graph loops be

$$\tilde{\mu}_\sigma^{\text{loop}}(d\varphi) = \text{hol}^\sigma(\varphi) \tilde{\mu}^{\text{loop}}(d\varphi),$$

which can be decomposed according to the sign into

$$\tilde{\mu}_\sigma^{\text{loop}} = \tilde{\mu}_{\sigma,+}^{\text{loop}} - \tilde{\mu}_{\sigma,-}^{\text{loop}}.$$

Let $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$, respectively $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$, be the Poisson point processes of metric graph loops on $\tilde{\mathcal{G}}$ with intensity measure $\frac{1}{2} \tilde{\mu}_{\sigma,+}^{\text{loop}}$, respectively $\frac{1}{2} \tilde{\mu}_{\sigma,-}^{\text{loop}}$. The version on Le Jan's isomorphism due to

Kassel and Lévy (2.6) extends in a straightforward way to the metric graphs:

$$(\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}))_{x \in \tilde{\mathcal{G}}} \stackrel{(\text{law})}{=} \left(\frac{1}{2} \tilde{\phi}_\sigma(x)^2 + \ell^x(\tilde{\mathcal{L}}_{\sigma,-}^{1/2}) \right)_{x \in \tilde{\mathcal{G}}},$$

where $\tilde{\phi}_\sigma$ and $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$ are taken independent, and ℓ^x above denotes the Brownian local times. One can use for instance an approximation from discrete through a subdivision of edges as in Section 3.1.

Now consider \mathcal{C} a connected component on

$$\left\{ x \in \tilde{\mathcal{G}} \mid \frac{1}{2} \tilde{\phi}_\sigma(x)^2 + \ell^x(\tilde{\mathcal{L}}_{\sigma,-}^{1/2}) \neq 0 \right\}.$$

Then $\pi_\sigma^{-1}(\mathcal{C})$ is either connected or not. If $\pi_\sigma^{-1}(\mathcal{C})$ is not connected, then \mathcal{C} cannot contain any loop with holonomy -1 , and in particular, \mathcal{C} cannot contain any loop in $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$. So in this case, \mathcal{C} is actually a connected component of $\{|\tilde{\phi}_\sigma| \neq 0\}$. We summarize this remark in the corollary below.

Corollary 3.25. *One can couple on the same probability space the metric graph loop soups $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ and $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$, and the field $\tilde{\phi}_\sigma$, such that all of the following conditions hold.*

- (1) *The field $\tilde{\phi}_\sigma$ and the loop soup $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$ are independent.*
- (2) *For every $x \in \tilde{\mathcal{G}}$,*

$$\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}) = \frac{1}{2} \tilde{\phi}_\sigma(x)^2 + \ell^x(\tilde{\mathcal{L}}_{\sigma,-}^{1/2}).$$

- (3) *For every cluster \mathcal{C} of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ such that $\pi_\sigma^{-1}(\mathcal{C})$ is not connected, \mathcal{C} is also a connected component of $\{|\tilde{\phi}_\sigma| \neq 0\}$ and $\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2})$ coincides with $\frac{1}{2} \tilde{\phi}_\sigma(x)^2$ on \mathcal{C} .*

In particular, one can couple on the same probability space the metric graph loop soups $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ and the field $\tilde{\phi}_\sigma$, such that the following conditions hold.

- (1) *For every $x \in \tilde{\mathcal{G}}$,*

$$\frac{1}{2} \tilde{\phi}_\sigma(x)^2 \leq \ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}).$$

- (2) *For every cluster \mathcal{C} of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ such that $\pi_\sigma^{-1}(\mathcal{C})$ is not connected, \mathcal{C} is also a connected component of $\{|\tilde{\phi}_\sigma| \neq 0\}$ and $\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2})$ coincides with $\frac{1}{2} \tilde{\phi}_\sigma(x)^2$ on \mathcal{C} . In other words, $\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2})$ and $\frac{1}{2} \tilde{\phi}_\sigma(x)^2$ can differ only on clusters \mathcal{C} of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ such that $\pi_\sigma^{-1}(\mathcal{C})$ is connected.*

In the example of Figure 3, $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ consists of loops that turn around the inner hole an even number of times, including those that do not surround it, and $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$ consists of loops that turn around the inner hole an odd number of times. Further, $\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2})$ and $\frac{1}{2} \tilde{\phi}_\sigma(x)^2$ coincide on clusters of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ that do not surround the inner hole. At the risk of being redundant, let us emphasize that the dichotomy for loops in $\tilde{\mathcal{L}}^{1/2}$ and the dichotomy for clusters of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ are different. For loops in $\tilde{\mathcal{L}}^{1/2}$ distinguishes between the loops that turn an even number of times around the hole, and the loops turn an odd number of time around the hole. In this way, the loops that turn twice around the hole are in the same class as the loops that do not surround the hole at all. For the clusters of loops however, the dichotomy is just surrounding or not surrounding the inner hole.

In view of the above corollary, perhaps it is worth pointing out the difference between the clusters of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ on the metric graph $\tilde{\mathcal{G}}$ and the clusters of $\mathcal{L}_{\sigma,+}^{1/2}$ on the discrete graph \mathcal{G} .

Proposition 3.26. *The discrete loop soup $\mathcal{L}_{\sigma,+}^{1/2}$ is obtained, up to a rerooting of the loops, from the metric graph loop soup $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ by taking the trace of loops on V with the time change A^{-1} (2.7). By doing this, one only takes into account the loops in $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ that visits at least one vertex.*

In particular, for every $x \in V$, $\ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}) = \ell^x(\mathcal{L}_{\sigma,+}^{1/2})$. Further, the crossings of edges-lines I_e by $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ correspond to the jumps through discrete edges e by $\mathcal{L}_{\sigma,+}^{1/2}$.

If an edge $e \in E$ is visited by $\mathcal{L}_{\sigma,+}^{1/2}$, then $\forall x \in I_e, \ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}) > 0$ a.s. Moreover, for every $e \in E$,

$$(3.14) \quad \mathbb{P}\left(\forall x \in I_e, \ell^x(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}) > 0 \mid \mathcal{L}_{\sigma,+}^{1/2}, e \text{ not visited by } \mathcal{L}_{\sigma,+}^{1/2}\right) \\ = 1 - \exp\left(-2C(e)(\ell^{e_-}(\tilde{\mathcal{L}}_{\sigma,+}^{1/2})\ell^{e_+}(\tilde{\mathcal{L}}_{\sigma,+}^{1/2}))^{1/2}\right),$$

with conditional independence (given $\mathcal{L}_{\sigma,+}^{1/2}$) across the edges $e \in E$.

Proof. These are the properties already satisfied by the loop soups $\tilde{\mathcal{L}}^{1/2}$ and $\mathcal{L}^{1/2}$, proven in [Lup16]. The fact that $\mathcal{L}^{1/2}$ is the trace of $\tilde{\mathcal{L}}^{1/2}$ on the vertices V immediately implies that $\mathcal{L}_{\sigma,+}^{1/2}$ is the trace of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ on V , as well as that $\mathcal{L}_{\sigma,-}^{1/2}$ is the trace of $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$ on V .

As for the formula (3.14), given $e \in E$ not visited by $\mathcal{L}_{\sigma,+}^{1/2}$, the connection between the two endpoints of e can be created by a superposition of three objects: the Brownian excursions from e_+ to e_+ inside I_e , the Brownian excursions from e_- to e_- inside I_e , and the Brownian loops of $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$ that stay inside I_e . The formula (3.14) is then the same as for $\tilde{\mathcal{L}}^{1/2}$ and $\mathcal{L}^{1/2}$ appearing in [Lup16] as the same three types of Brownian paths appear in both settings. This is in particular due to the fact that all the Brownian loops in $\tilde{\mathcal{L}}^{1/2}$ that stay inside I_e have a holonomy 1, and thus also appear in $\tilde{\mathcal{L}}_{\sigma,+}^{1/2}$. Note that for $\tilde{\mathcal{L}}_{\sigma,-}^{1/2}$ and $\mathcal{L}_{\sigma,-}^{1/2}$, a similar formula is no longer true, precisely because the Brownian loops staying inside I_e no longer participate to connecting the two ends of e , as they all have a wrong holonomy. \square

ACKNOWLEDGMENTS

The author thanks Adrien Kassel (ENS Lyon) and Thierry Lévy (Sorbonne Université) for inspiring discussions.

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