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# A UNIVERSAL FINITE TYPE INVARIANT OF KNOTS IN HOMOLOGY 3–SPHERES

BENJAMIN AUDOUX AND DELPHINE MOUSSARD

ABSTRACT. We construct a universal finite type invariant for knots in homology 3–spheres, refining Kricker’s lift of the Kontsevich integral. This provides a full diagrammatic description of the graded space of finite type invariants of knots in homology 3–spheres.

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## 1. INTRODUCTION

Given a set of topological objects, in general knots or manifolds, and an operation on them, like crossing change for knots or some surgery on manifolds, the finite type invariants for these objects are defined by their behaviour with respect to this operation. The notion of finite type invariants was introduced independently by Goussarov and Vassiliev in the early 90's for the study of invariants of knots in  $S^3$ . It was extended by Ohtsuki to invariants of integral homology 3-spheres [Oht96]. Then different theories were developed, including in particular a theory due to Goussarov and Habiro independently which applies to invariants of all 3-manifolds and their knots [Gou99, Hab00]. Garoufalidis, Kricker and Rozansky built upon the latter to study invariants of knots in homology 3-spheres [GK04, GR04].

To a theory of finite type invariants, *ie* a set of objects and an operation, one associates the graded space defined by a filtration, provided by the operation, on the vector space generated by the objects; its dual is the space of finite type invariants, graded by their degree. In the study of such a theory, the goal is to obtain a combinatorial description of this graded space, by identifying it with a graded space of Feynman diagrams; this requires the construction of a universal finite type invariant. This has been achieved by Bar-Natan and Kontsevich for knots in  $S^3$  using the Kontsevich integral [Kon93, Bar95]. For integral homology 3-spheres in the Goussarov–Habiro theory, the description of the graded space follows from works of Garoufalidis–Goussarov–Polyak [GGP01], Habiro [Hab00] and Le [Le97], with the LMO invariant of Le–Murakami–Ohtsuki as universal finite type invariant. This was generalized to rational homology 3-spheres by works of Massuyeau [Mas15] and the second author [Mou12a]. In their study of finite type invariants of knots in homology 3-spheres, Garoufalidis–Kricker–Rozansky described the graded space in the case of knots whose Alexander polynomial is trivial with, as universal invariant, the Kricker invariant defined in [GK04] following a first construction of [Kri00]. A graded space of diagrams suitable for knots with any Alexander polynomial was proposed in [Mou19], but the universal invariant that would make the description complete was still missing. Here, we construct a refinement of the Kricker invariant and we prove that it is a universal finite type invariant of knots in homology 3-spheres. Since this refinement is strict, it shows in particular that the Kricker invariant is not universal.

The universal invariants that appear in these theories all derive from the Kontsevich integral of knots and links in  $S^3$  in some sense. The constructions of the LMO invariant and the Kricker invariant apply the Kontsevich integral to surgery presentations of the manifold or the pair (manifold, knot). Another approach is to extend directly the idea of the Kontsevich integral: in this case, the invariants of 3-manifolds and their knots are obtained by computing integrals in associated configuration spaces. This approach has led to the KKT invariant of Kontsevich–Kuperberg–Thurston for homology 3-spheres [Kon94, KT99, Les04a] and to the Lescop invariant for knots in homology 3-spheres [Les11]. These two methods are expected to produce equivalent invariants, but no direct comparison is known for the LMO and KKT invariants, nor for the Lescop and Kricker invariants. However, these invariants may be compared through finite type invariants theories. The LMO and KKT invariants are both universal finite type invariants of integral homology 3-spheres, so that they are equivalent [Le97, Les04b]; this also holds for rational homology 3-spheres with homology groups of a given cardinality [Mas15, Mou12a]. Similarly, the equivalence of the Kricker and Lescop invariants for knots in homology 3-spheres with trivial Alexander polynomial stems from [GK04, Mou19]. In [Les13], Lescop conjectured

that this equivalence holds for all null-homologous knots in rational homology 3–spheres. The description of the graded space given here is a great step toward this conjecture.

*Notations.* For  $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$ , a  $\mathbb{K}$ –sphere is a closed 3–manifold which has the same homology with coefficients in  $\mathbb{K}$  as the standard 3–sphere. A  $\mathbb{K}$ SK–pair is a pair  $(M, K)$  where  $M$  is a  $\mathbb{K}$ –sphere and  $K$  is a knot in  $M$  with trivial homology class in  $H_1(M; \mathbb{Z})$ .

We consider finite type invariants of  $\mathbb{Q}$ SK–pairs with respect to a surgery move called null LP–surgery. It turns out that the classes of  $\mathbb{Q}$ SK–pairs up to null LP–surgeries are classified by their Blanchfield modules, namely their Alexander modules equipped with the Blanchfield form [Mou15]. Hence we fix a Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  and we work with  $\mathbb{Q}$ SK–pairs whose Blanchfield module is isomorphic to  $(\mathfrak{A}, \mathfrak{b})$ . We denote  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$  the associated graded space. In [Mou19], a graded space of diagrams  $\mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b})$  was constructed, together with a canonical onto map  $\varphi : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}, \mathfrak{b})$ . The superscript “aug” refers to diagrams that are “augmented” with isolated vertices, as detailed in the next section.

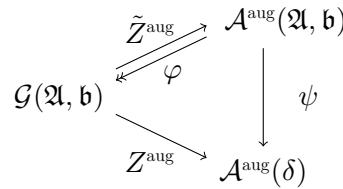


FIGURE 1. Invariants and maps on the graded space  
 Here,  $Z^{\text{aug}}$  is the augmented version of either  $Z^{\text{Kri}}$  or  $Z^{\text{Les}}$ .

The Kriker invariant  $Z^{\text{Kri}}$  and the Lescop invariant  $Z^{\text{Les}}$  are valued in a graded space of diagram  $\mathcal{A}(\delta)$  which depends on the annihilator  $\delta$  of the Alexander module  $\mathfrak{A}$ . They can be written as series of finite type invariants and they induce maps on  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$  [Les13, Mou20]. These invariants can be completed into an invariant with values in an “augmented” diagram space  $\mathcal{A}^{\text{aug}}(\delta)$ . The composition with the map  $\varphi$  is in both cases the same map  $\psi : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}^{\text{aug}}(\delta)$ . However, we proved in [AM19] that the map  $\psi$  is not injective in general. The main goal of this article is to refine the Kriker invariant into an invariant  $\tilde{Z}$  with values in  $\mathcal{A}(\mathfrak{A}, \mathfrak{b})$  whose augmented version provides the inverse of the map  $\varphi$ . This will show that the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$  is naturally isomorphic to  $\mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b})$  and that our invariant is a universal finite type invariant for  $\mathbb{Q}$ SK–pairs with respect to null LP–surgeries.

**Theorem 1.1** (Theorem 6.11). *There is an invariant  $\tilde{Z}^{\text{aug}}$  of  $\mathbb{Q}$ SK–pairs which induces the inverse of the map  $\varphi : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}, \mathfrak{b})$ .*

The invariant  $\tilde{Z}$  is a lift of the Kriker invariant. The fact that the map  $\psi$  is not always injective, while  $\varphi$  is an isomorphism, proves that  $\tilde{Z}$  strictly contains  $Z^{\text{Kri}}$ .

Moreover, the description of the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$  for all Blanchfield modules shows that the Kriker invariant and the Lescop invariant induce the same map on  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$ . A refinement of the Lescop invariant similar to the one we construct for the Kriker invariant would allow to prove the Lescop conjecture.

One can focus on knots in  $\mathbb{Z}$ -spheres, or  $\mathbb{Z}$ SK-pairs. There is a natural  $\mathbb{Z}$ -version of the null LP-surgery move, which provides a similar notion of finite type invariants. All the results of this paper can be transposed to this setting.

*Plan of the paper.* In Section 2, we give definitions and notations and we recall some background. In Section 3, we introduce the surgery presentations that we will use to define the invariant and we give a combinatorial computation of the associated equivariant linking matrices. Section 4 is devoted to diagram spaces and diagrammatic operations that underly the construction of the invariant. Section 5 gives the construction of the invariant  $\tilde{Z}$ . In Section 6, we describe the behaviour of our invariant with respect to null LP-surgeries and we establish universality. Finally, Section 7 adapts the results to the setting of knots in  $\mathbb{Z}$ -spheres.

*Conventions and notations.*

The boundary of an oriented manifold is oriented with the outward normal first convention. For submanifolds  $X$  and  $Y$  of a manifold  $M$  such that  $\dim(X) + \dim(Y) = \dim(M)$ ,  $\langle X, Y \rangle$  denotes the algebraic intersection number of  $X$  and  $Y$ .

For  $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$ , a *genus- $g$   $\mathbb{K}$ -handlebody* is a compact 3-manifold which has the same homology with coefficients in  $\mathbb{K}$  as the standard genus- $g$  handlebody.

Given a matrix  $W$ ,  ${}^tW$  denotes the transpose of  $W$ .

For a matrix  $A(t)$  with polynomial coefficients, we set  $\bar{A}(t) = A(t^{-1})$ .

## 2. FINITE TYPE INVARIANTS AND DIAGRAM SPACES

**2.1. Filtration defined by null LP-surgeries.** We first recall the definition of the Alexander module and the Blanchfield form.

Let  $(M, K)$  be a  $\mathbb{Q}$ SK-pair. The *exterior*  $X$  of  $K$  is the complement in  $M$  of an open tubular neighborhood of  $K$ . Let  $\tilde{X}$  be the *infinite cyclic covering* of  $X$ , ie the covering associated with the kernel of the map  $\pi_1(X) \rightarrow \mathbb{Z} = \langle t \rangle$  which sends the positive meridian of  $K$  to  $t$ . The automorphism group of the covering is isomorphic to  $\mathbb{Z}$ ; let  $\tau$  be the generator associated with the action of the positive meridian. Denoting the action of  $\tau$  as the multiplication by  $t$ , we get a structure of  $\mathbb{Q}[t^{\pm 1}]$ -module on  $\mathfrak{A}(M, K) = H_1(\tilde{X}; \mathbb{Q})$ . This  $\mathbb{Q}[t^{\pm 1}]$ -module is called the *Alexander module* of  $(M, K)$ . It is a finitely generated torsion  $\mathbb{Q}[t^{\pm 1}]$ -module.

Let  $\delta \in \mathbb{Q}[t^{\pm 1}]$  be the annihilator of  $\mathfrak{A}$  (it is defined up to an invertible element of  $\mathbb{Q}[t^{\pm 1}]$ , which has no importance here). Given two disjoint knots  $J_1$  and  $J_2$  in  $\tilde{X}$ , define the *equivariant linking number* of  $J_1$  and  $J_2$  as:

$$\text{lk}_e(J_1, J_2) = \frac{1}{\delta(t)} \sum_{k \in \mathbb{Z}} \langle S, \tau^k(J_2) \rangle t^k,$$

where  $S$  is a rational 2-chain  $S$  such that  $\partial S = \delta(\tau)J_1$ . It is a well-defined element of  $\frac{1}{\delta(t)}\mathbb{Q}[t^{\pm 1}]$  which satisfies  $\text{lk}_e(J_2, J_1)(t) = \text{lk}_e(J_1, J_2)(t^{-1})$  and  $\text{lk}_e(P(\tau)J_1, J_2)(t) = P(t)\text{lk}_e(J_1, J_2)(t)$ . Now define the *Blanchfield form*  $\mathfrak{b} : \mathfrak{A} \times \mathfrak{A} \rightarrow \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]}$  as follows: if  $\gamma$  (resp.  $\eta$ ) is the homology class of  $J_1$  (resp.  $J_2$ ) in  $\mathfrak{A}$ , set

$$\mathfrak{b}(\gamma, \eta) = \text{lk}_e(J_1, J_2) \text{ mod } \mathbb{Q}[t^{\pm 1}].$$

The Blanchfield form is *hermitian*:  $\mathfrak{b}(\gamma, \eta)(t) = \mathfrak{b}(\eta, \gamma)(t^{-1})$  and  $\mathfrak{b}(P(t)\gamma, \eta)(t) = P(t)\mathfrak{b}(\gamma, \eta)(t)$  for all  $\gamma, \eta \in \mathfrak{A}$  and all  $P, Q \in \mathbb{Q}[t^{\pm 1}]$ . Moreover, it is *non degenerate* (see Blanchfield in [Bla57]):  $\mathfrak{b}(\gamma, \eta) = 0$  for all  $\eta \in \mathfrak{A}$  implies  $\gamma = 0$ .

The Alexander module of a QSK–pair  $(M, K)$  endowed with its Blanchfield form is its *Blanchfield module* denoted by  $(\mathfrak{A}, \mathfrak{b})(M, K)$ . In the sequel, by a *Blanchfield module*  $(\mathfrak{A}, \mathfrak{b})$ , we mean a pair  $(\mathfrak{A}, \mathfrak{b})$  that can be realized as the Blanchfield module of a QSK–pair. An isomorphism between Blanchfield modules is an isomorphism between the underlying Alexander modules which preserves the Blanchfield form.

We now define LP–surgeries. Note that the boundary of a genus– $g$   $\mathbb{Q}$ –handlebody is homeomorphic to the standard genus– $g$  surface. The *Lagrangian*  $\mathcal{L}_A$  of a  $\mathbb{Q}$ –handlebody  $A$  is the kernel of the map  $i_* : H_1(\partial A; \mathbb{Q}) \rightarrow H_1(A; \mathbb{Q})$  induced by the inclusion; it is indeed a Lagrangian subspace of  $H_1(\partial A; \mathbb{Q})$  with respect to the intersection form. Two  $\mathbb{Q}$ –handlebodies  $A$  and  $B$  have *LP–identified* boundaries if  $(A, B)$  is equipped with a homeomorphism  $h : \partial A \rightarrow \partial B$  such that  $h_*(\mathcal{L}_A) = \mathcal{L}_B$ .

Let  $M$  be a  $\mathbb{Q}$ –sphere, let  $A \subset M$  be a  $\mathbb{Q}$ –handlebody and let  $B$  be a  $\mathbb{Q}$ –handlebody whose boundary is LP–identified with  $\partial A$ . Set  $M \left(\frac{B}{A}\right) = (M \setminus \text{Int}(A)) \cup_{\partial A =_h \partial B} B$ . We say that the  $\mathbb{Q}$ –sphere  $M \left(\frac{B}{A}\right)$  is obtained from  $M$  by *Lagrangian-preserving surgery*, or *LP–surgery*.

Given a QSK–pair  $(M, K)$ , a  $\mathbb{Q}$ –*handlebody null in*  $M \setminus K$  is a  $\mathbb{Q}$ –handlebody  $A \subset M \setminus K$  such that the map  $i_* : H_1(A; \mathbb{Q}) \rightarrow H_1(M \setminus K; \mathbb{Q})$  induced by the inclusion has a trivial image. A *null LP–surgery* on  $(M, K)$  is an LP–surgery  $\left(\frac{B}{A}\right)$  such that  $A$  is null in  $M \setminus K$ . The QSK–pair obtained by surgery is denoted by  $(M, K) \left(\frac{B}{A}\right)$ .

We can now define a filtration on the rational vector space  $\mathcal{F}_0$  generated by all QSK–pairs up to orientation-preserving homeomorphism. For that, we define  $\mathcal{F}_n$  as the subspace of  $\mathcal{F}_0$  generated by the *brackets*

$$\left[ (M, K); \left( \frac{B_i}{A_i} \right)_{1 \leq i \leq n} \right] = \sum_{I \subset \{1, \dots, n\}} (-1)^{|I|} (M, K) \left( \left( \frac{B_i}{A_i} \right)_{i \in I} \right)$$

for all QSK–pairs  $(M, K)$  and all families of  $\mathbb{Q}$ –handlebodies  $(A_i, B_i)_{1 \leq i \leq n}$ , where the  $A_i$  are null in  $M \setminus K$  and disjoint, and each pair  $(A_i, B_i)$  has LP–identified boundaries. Note that  $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ . An invariant  $\lambda$  of QSK–pairs valued in some rational vector space is a *finite type invariant of degree at most  $n$  with respect to null LP–surgeries* if its  $\mathbb{Q}$ –linear extension to  $\mathcal{F}_0$  satisfies  $\lambda(\mathcal{F}_{n+1}) = 0$ .

As proven in [Mou15, Theorem 1.14], two QSK–pairs are related by null LP–surgeries if and only if their Blanchfield modules are isomorphic. This allows to work with QSK–pairs with a given Blanchfield module. Hence, we fix an abstract Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  and we consider the subspace  $\mathcal{F}_0(\mathfrak{A}, \mathfrak{b})$  of  $\mathcal{F}_0$  generated by the QSK–pairs whose Blanchfield module is isomorphic to  $(\mathfrak{A}, \mathfrak{b})$ . Let  $(\mathcal{F}_n(\mathfrak{A}, \mathfrak{b}))_{n \in \mathbb{N}}$  be the filtration defined on  $\mathcal{F}_0(\mathfrak{A}, \mathfrak{b})$  by null LP–surgeries. Then, for  $n \in \mathbb{N}$ ,  $\mathcal{F}_n$  is the direct sum, over all isomorphism classes  $(\mathfrak{A}, \mathfrak{b})$  of Blanchfield modules, of the  $\mathcal{F}_n(\mathfrak{A}, \mathfrak{b})$ . Set  $\mathcal{G}_n(\mathfrak{A}, \mathfrak{b}) = \mathcal{F}_n(\mathfrak{A}, \mathfrak{b}) / \mathcal{F}_{n+1}(\mathfrak{A}, \mathfrak{b})$  and  $\mathcal{G}(\mathfrak{A}, \mathfrak{b}) = \bigoplus_{n \in \mathbb{N}} \mathcal{G}_n(\mathfrak{A}, \mathfrak{b})$ . Our goal is to describe the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$ ; note that  $\mathcal{G}_0(\mathfrak{A}, \mathfrak{b}) \cong \mathbb{Q}$ .

**2.2. Borromean surgeries.** We now introduce a specific type of LP–surgeries. The *standard  $Y$ –graph* is the graph  $\Gamma_0 \subset \mathbb{R}^2$  represented in Figure 2. The looped edges of  $\Gamma_0$  are called *leaves* and the vertex incident to three different edges is called the *internal vertex*. With  $\Gamma_0$  is associated a regular neighborhood  $\Sigma(\Gamma_0)$  of  $\Gamma_0$  in the plane. The surface  $\Sigma(\Gamma_0)$  is oriented with the usual convention. This induces an orientation of the leaves and an *orientation* of the internal vertex, *ie* a cyclic order of the three edges, shown in Figure 2. Consider a 3–manifold  $M$  and

an embedding  $h : \Sigma(\Gamma_0) \rightarrow M$ . The image  $\Gamma$  of  $\Gamma_0$  is a *Y-graph*, endowed with its *associated surface*  $\Sigma(\Gamma) = h(\Sigma(\Gamma_0))$ . The Y-graph  $\Gamma$  is equipped with the framing induced by  $\Sigma(\Gamma)$ . A *Y-link* in a 3-manifold is a collection of disjoint Y-graphs.

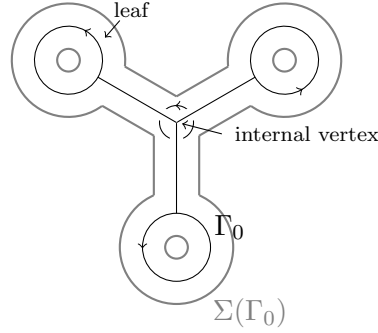


FIGURE 2. The standard Y-graph and its associated surface

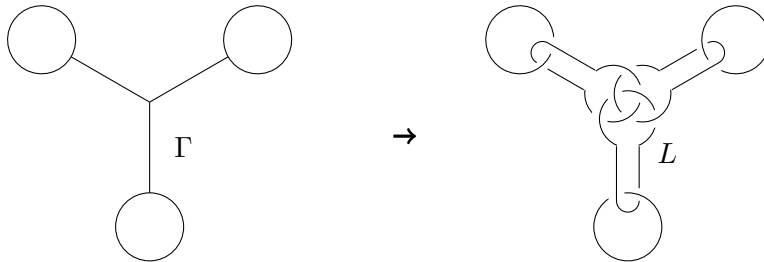



FIGURE 3. Y-graph and associated surgery link

Let  $\Gamma$  be a Y-graph in a 3-manifold  $M$  and let  $\Sigma$  be its associated surface. In  $\Sigma \times [-1, 1]$ , associate with  $\Gamma$  the six components link  $L$  represented in Figure 3. The *borromean surgery on  $\Gamma$*  is the surgery along the framed link  $L$ ; the surgered manifold is denoted  $M(\Gamma)$ . Borromean surgeries were introduced and studied by Matveev in [Mat87]. He proved in particular that two 3-manifolds are related by a sequence of borromean surgeries if and only if they have isomorphic first homology group and linking pairing. Moreover, he showed that a borromean surgery can be realized by cutting a genus-3 handlebody (a regular neighborhood of the Y-graph) and regluing it in another way, which preserves the Lagrangian. It follows that borromean surgeries are specific LP-surgeries.

**2.3. Colored Jacobi diagrams.** Fix a Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  and let  $\delta \in \mathbb{Q}[t^{\pm 1}]$  be the annihilator of  $\mathfrak{A}$ . An  $(\mathfrak{A}, \mathfrak{b})$ -colored diagram  $D$  is a unitrivalent graph without *strut* (isolated edge), with the following data:

- trivalent vertices are oriented (an *orientation of a trivalent vertex* is a cyclic order of the three half-edges that meet at this vertex; by convention, we fix it as  in the pictures),

- edges are oriented and labeled by elements of  $\mathbb{Q}[t^{\pm 1}]$ ;
- univalent vertices are labeled by elements of  $\mathfrak{A}$ ;
- for all  $v \neq v'$  in the set  $V$  of univalent vertices of  $D$ , a rational fraction  $f_{vv'}^D(t) \in \frac{1}{\delta(t)}\mathbb{Q}[t^{\pm 1}]$  is fixed such that  $f_{vv'}^D(t) \bmod \mathbb{Q}[t^{\pm 1}] = \mathfrak{b}(\gamma, \gamma')$ , where  $\gamma$  (resp.  $\gamma'$ ) is the coloring of  $v$  (resp.  $v'$ ); we require that  $f_{v'v}^D(t) = f_{vv'}^D(t^{-1})$ .

When it does not seem to cause confusion, we write  $f_{vv'}$  for  $f_{vv'}^D$ . The *degree* of a colored diagram is the number of trivalent vertices of its underlying graph. The unique degree 0 diagram is the empty diagram. For  $n \geq 0$ , set:

$$\tilde{\mathcal{A}}_n(\mathfrak{A}, \mathfrak{b}) = \frac{\mathbb{Q}\langle (\mathfrak{A}, \mathfrak{b})\text{-colored diagrams of degree } n \rangle}{\mathbb{Q}\langle \text{AS, IHX, LE, OR, Hol, LV, EV, LD} \rangle},$$

where the relations are described in Figures 4 and 5.

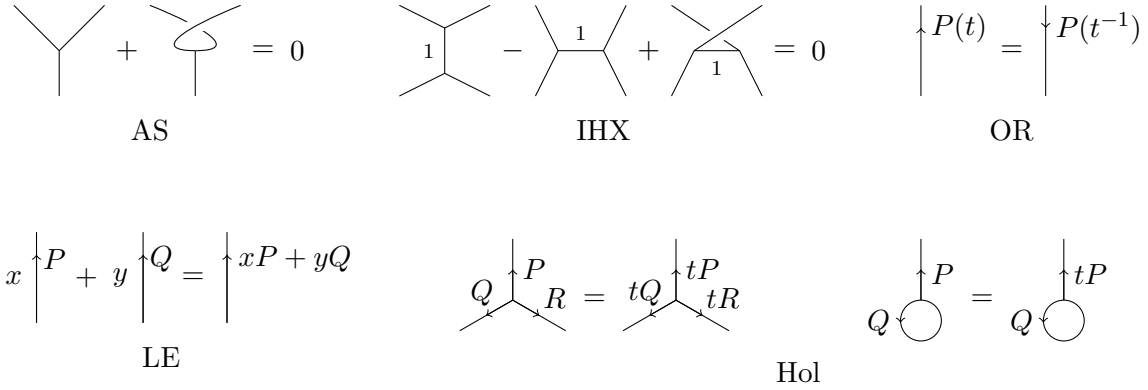


FIGURE 4. Relations, where  $x, y \in \mathbb{Q}$  and  $P, Q, R \in \mathbb{Q}[t^{\pm 1}]$

The automorphism group  $\text{Aut}(\mathfrak{A}, \mathfrak{b})$  of the Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  acts on  $\tilde{\mathcal{A}}_n(\mathfrak{A}, \mathfrak{b})$  by acting on the colorings of all the univalent vertices of a diagram simultaneously. Denote by  $\text{Aut}$  the relation which identifies two diagrams obtained from one another by the action of an element of  $\text{Aut}(\mathfrak{A}, \mathfrak{b})$ . Set:

$$\mathcal{A}_n(\mathfrak{A}, \mathfrak{b}) = \tilde{\mathcal{A}}_n(\mathfrak{A}, \mathfrak{b}) / \langle \text{Aut} \rangle \quad \text{and} \quad \mathcal{A}(\mathfrak{A}, \mathfrak{b}) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n(\mathfrak{A}, \mathfrak{b}).$$

An  $(\mathfrak{A}, \mathfrak{b})$ -*augmented diagram* is the union of an  $(\mathfrak{A}, \mathfrak{b})$ -colored diagram (its *Jacobi part*) and of finitely many isolated vertices colored by prime integers. The *degree* of an  $(\mathfrak{A}, \mathfrak{b})$ -augmented diagram is the number of its vertices of valence 0 or 3. Set:

$$\mathcal{A}_n^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) = \frac{\mathbb{Q}\langle (\mathfrak{A}, \mathfrak{b})\text{-augmented diagrams of degree } n \rangle}{\mathbb{Q}\langle \text{AS, IHX, LE, OR, Hol, LV, EV, LD, Aut} \rangle} \quad \text{for } n \geq 0,$$

and  $\mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n^{\text{aug}}(\mathfrak{A}, \mathfrak{b})$ .

**2.4. The map  $\varphi$ .** We shall now recall the construction of the map  $\varphi : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}, \mathfrak{b})$ . For this, we fix a QSK-pair  $(M, K)$  with Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  and we associate surgery data to  $(\mathfrak{A}, \mathfrak{b})$ -colored diagrams. For the labels of univalent vertices to make sense in the Blanchfield module of  $(M, K)$ , we need to choose an isomorphism  $\xi : (\mathfrak{A}, \mathfrak{b}) \rightarrow (\mathfrak{A}, \mathfrak{b})(M, K)$ . However,



$$\begin{array}{ccc}
x \begin{array}{c} \bullet \\ \uparrow \gamma_1 \\ | \\ D_1 \end{array} + y \begin{array}{c} \bullet \\ \uparrow \gamma_2 \\ | \\ D_2 \end{array} = \begin{array}{c} \bullet \\ \uparrow x\gamma_1 + y\gamma_2 \\ | \\ D \end{array} & \begin{array}{c} \bullet \\ \uparrow PQ \\ | \\ D \end{array} = \begin{array}{c} \bullet \\ \uparrow Q(t)\gamma \\ | \\ D' \end{array} & f_{vv'}^{D'}(t) = Q(t)f_{vv'}^D(t) \\
& & \forall v' \neq v \\
x f_{vv'}^{D_1}(t) + y f_{vv'}^{D_2}(t) = f_{vv'}^D(t) \quad \forall v' \neq v & & \text{EV} \\
\text{LV} & & 
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} \bullet \\ \uparrow \gamma_1 \\ | \\ 1 \end{array} \begin{array}{c} \bullet \\ \uparrow \gamma_2 \\ | \\ 1 \end{array} = \begin{array}{c} \bullet \\ \uparrow \gamma_1 \\ | \\ 1 \end{array} \begin{array}{c} \bullet \\ \uparrow \gamma_2 \\ | \\ 1 \end{array} + \begin{array}{c} \text{P} \\ \frown \\ | \\ D'' \end{array} \\
f_{v_1 v_2}^D = f_{v_1 v_2}^{D'} + P & & \text{LD}
\end{array}$$

FIGURE 5. Relations, where  $x, y \in \mathbb{Q}$ ,  $P, Q, R \in \mathbb{Q}[t^{\pm 1}]$  and  $\gamma, \gamma_1, \gamma_2 \in \mathfrak{A}$

different choices produce the same map  $\varphi$  (which amounts to saying that the map  $\varphi$  respects the relation Aut); hence we will keep implicit the isomorphism  $\xi$  in the sequel.



FIGURE 6. Replacement of an edge

An  $(\mathfrak{A}, \mathfrak{b})$ -colored diagram is *elementary* if its edges that connect two trivalent vertices are colored by powers of  $t$  and if its edges adjacent to univalent vertices are colored by 1. Let  $D$  be an elementary diagram. Embed  $D$  in  $M \setminus K$  in such a way that the vertices of  $D$  are embedded in some ball  $B \subset M \setminus K$  and, for each edge colored by  $t^k$ , the closed curve obtained by connecting the extremities of this edge by a path in  $B$  has linking number  $k$  with  $K$ . Equip  $D$  with the framing induced by an immersion in the plane which induces the fixed orientation of the trivalent vertices. If an edge connects two trivalent vertices, then insert a positive Hopf link in this edge, as shown in Figure 6. At each univalent vertex  $v$ , glue a leaf  $\ell_v$ , trivial in  $H_1(M \setminus K; \mathbb{Q})$ , in order to obtain a null Y-link  $\Gamma$ . Let  $V$  be the set of all univalent vertices of  $D$ . Let  $\tilde{B}$  be a lift of the ball  $B$  in the infinite cyclic covering  $\tilde{X}$  of the exterior of  $K$  in  $M$ . For  $v \in V$ , let  $\hat{\ell}_v$  be the extension of  $\ell_v$  in  $\Gamma$  (see Figure 7) and let  $\tilde{\ell}_v$  be the lift of  $\hat{\ell}_v$  in  $\tilde{X}$  defined by lifting the basepoint (the point  $p$  on Figure 7) in  $\tilde{B}$ . The null Y-link  $\Gamma$  is a *realization* of  $D$  if:

- for all  $v \in V$ ,  $\tilde{\ell}_v$  is homologous to the label  $\gamma_v$  of  $v$ ,
- for all  $(v, v') \in V^2$ ,  $\text{lk}_e(\tilde{\ell}_v, \tilde{\ell}_{v'}) = f_{vv'}$ .

If such a realization exists, the elementary diagram  $D$  is *realizable*.

Realizable elementary diagrams turn out to generate the graded space  $\mathcal{A}(\mathfrak{A}, \mathfrak{b})$ . Any realization of such a diagram, of degree  $n$ , provides a family of  $n$  disjoint null borromean surgeries in  $M \setminus K$ , defining a bracket in  $\mathcal{F}_n(\mathfrak{A}, \mathfrak{b})$ . By [Mou19, Section 4], this gives a well-defined graded

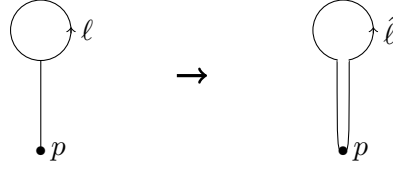


FIGURE 7. Extension of a leaf in a Y–graph

$\mathbb{Q}$ –linear map  $\varphi : \mathcal{A}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}, \mathfrak{b})$ . Further, this map can be extended to  $\mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b})$ : to an isolated vertex labeled by  $p$ , we associate a surgery  $\left(\frac{B_p}{B^3}\right)$ , where  $B_p$  is a  $\mathbb{Q}$ –ball satisfying  $|H_1(B_p; \mathbb{Z})| = p$ .

**Theorem 2.1** ([Mou19, Theorem 2.7]). *The graded  $\mathbb{Q}$ –linear map  $\varphi : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}, \mathfrak{b})$  is well-defined, canonical and surjective.*

That our map is canonical means that it does not depend of any of the choices we made, including the choice of the  $\mathbb{Q}$ SK–pair  $(M, K)$ .

**2.5. Kriker and Lescop invariants.** We first introduce the diagram space in which the Kriker invariant  $Z^{\text{Kri}}$  and the Lescop invariant  $Z^{\text{Les}}$  take values.

Let  $\delta \in \mathbb{Q}[t^{\pm 1}]$ . A  $\delta$ –colored diagram is a trivalent graph whose vertices are oriented and whose edges are oriented and colored by  $\frac{1}{\delta(t)}\mathbb{Q}[t^{\pm 1}]$ . The degree of a  $\delta$ –colored diagram is the number of its vertices. Set:

$$\mathcal{A}_n(\delta) = \frac{\mathbb{Q}\langle \delta\text{-colored diagrams of degree } n \rangle}{\mathbb{Q}\langle \text{AS, IHX, LE, OR, Hol} \rangle},$$

where AS, IHX, LE, OR, Hol are the relations represented in Figure 4 with  $P, Q, R \in \frac{1}{\delta(t)}\mathbb{Q}[t^{\pm 1}]$ . We have a graded algebra  $\mathcal{A}(\delta) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n(\delta)$ , where the product is defined by the disjoint union. Since any trivalent graph has an even number of vertices, we have  $\mathcal{A}_{2n+1}(\delta) = 0$  for all  $n \geq 0$ .

There is a natural “closing” map from  $\mathcal{A}(\mathfrak{A}, \mathfrak{b})$  to  $\mathcal{A}(\delta)$ . With an  $(\mathfrak{A}, \mathfrak{b})$ –colored diagram  $D$  of degree  $n$ , we associate a  $\delta$ –colored diagram defined as the sum of all ways of pairing all vertices as indicated in Figure 8. This provides a well-defined  $\mathbb{Q}$ –linear map:  $\psi_n : \mathcal{A}_n(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}_n(\delta)$ .

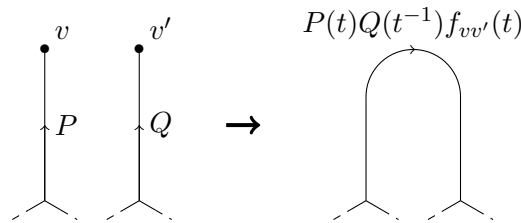


FIGURE 8. Pairing of vertices

The following result asserts the existence and the properties of an invariant  $Z$  which may be either the Lescop invariant or the Kriker invariant. Although it is not known whether they

are equal or not, they both satisfy the properties of the theorem. Until the end of this section, we shall refer to any of the two by “the invariant  $Z$ ”.

**Theorem 2.2** ([Les11, Les13, Kri00, GK04, Mou20]). *There is an invariant  $Z = (Z_n)_{n \in \mathbb{N}}$  of  $\mathbb{Q}\text{SK}$ -pairs with the following properties.*

- If  $(M, K)$  is a  $\mathbb{Q}\text{SK}$ -pair with Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$ , then  $Z_n(M, K) \in \mathcal{A}_n(\delta)$ , where  $\delta$  is the annihilator of  $\mathfrak{A}$ .
- Fix a Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  and let  $\delta$  be the annihilator of  $\mathfrak{A}$ . The  $\mathbb{Q}$ -linear extension  $Z_n : \mathcal{F}_0(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}_n(\delta)$  vanishes on  $\mathcal{F}_{n+1}(\mathfrak{A}, \mathfrak{b})$  and  $Z_n \circ \varphi_n = \psi_n$ .
- The invariant  $Z$  is multiplicative under connected sum.

In order to take into account the whole quotient  $\mathcal{G}_n(\mathfrak{A}, \mathfrak{b})$ , we extend the invariant  $Z$ . Define a  $\delta$ -augmented diagram as the disjoint union of a  $\delta$ -colored diagram with finitely many isolated vertices colored by prime integers. The *degree* of such a diagram is the number of its vertices. Set:

$$\mathcal{A}_n^{\text{aug}}(\delta) = \frac{\mathbb{Q}\langle \delta\text{-augmented diagrams of degree } n \rangle}{\mathbb{Q}\langle \text{AS, IHX, LE, OR, Hol} \rangle}.$$

Once again, the disjoint union makes  $\mathcal{A}^{\text{aug}}(\delta) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n^{\text{aug}}(\delta)$  a graded algebra. The map  $\psi_n$  naturally extends to a map  $\psi_n : \mathcal{A}_n^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}_n^{\text{aug}}(\delta)$  preserving the isolated vertices.

We shall complete the invariant  $Z$  with degree-1 invariants. For each prime integer  $p$ , we define as follows an invariant  $\rho_p$  of  $\mathbb{Q}$ -spheres: for a  $\mathbb{Q}$ -sphere  $M$ ,  $\rho_p(M) = -v_p(|H_1(M; \mathbb{Z})|) \bullet_p$  where  $v_p$  is the  $p$ -adic valuation. These invariants turn out to be degree-1 invariants of  $\mathbb{Q}$ -spheres with respect to LP-surgeries [Mou12a, Proposition 1.9]. In turn, they are also degree-1 invariants of  $\mathbb{Q}\text{SK}$ -pairs. Set:

$$Z^{\text{aug}} = Z \sqcup \exp_{\sqcup} \left( \sum_{p \text{ prime}} \rho_p \right)$$

and denote  $Z_n^{\text{aug}}$  the degree- $n$  part of  $Z^{\text{aug}}$ .

**Theorem 2.3** ([Mou19] Theorem 2.10). *Fix a Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$ . Let  $\delta$  be the annihilator of  $\mathfrak{A}$ . The  $\mathbb{Q}$ -linear extension  $Z_n^{\text{aug}} : \mathcal{F}_0(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}_n^{\text{aug}}(\delta)$  vanishes on  $\mathcal{F}_{n+1}(\mathfrak{A}, \mathfrak{b})$  and satisfies  $Z_n^{\text{aug}} \circ \varphi_n = \psi_n$ .*

To summarize, both the Kricker invariant and the Lescop invariant give rise to an invariant  $Z^{\text{aug}}$  that fits into the following commutative diagram, where all space and maps are graded.

$$\begin{array}{ccc} & & \mathcal{G}(\mathfrak{A}, \mathfrak{b}) \\ & \nearrow \varphi & \downarrow Z^{\text{aug}} \\ \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) & & \mathcal{A}^{\text{aug}}(\delta) \\ & \searrow \psi & \end{array}$$

To reach a full description of the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$ , we shall construct a lift of the augmented Kricker invariant with values in  $\mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b})$ .

3. SURGERY PRESENTATIONS AND WINDING MATRICES

**3.1. Surgery presentation and Blanchfield module.** Throughout the article, we fix a trivial knot  $\mathcal{O} \subset S^3$ . By a *surgery link*, we mean a link  $L = \sqcup_{i=1}^s L_i \subset (S^3 \setminus \mathcal{O})$  whose connected components  $L_i$  satisfy  $\text{lk}(L_i, \mathcal{O}) = 0$ . This condition ensures that the pair  $(M, K)$  obtained from  $(S^3, \mathcal{O})$  by surgery on  $L$  is a QSK–pair. Moreover, any QSK–pair admits such a surgery presentation (see [Mou12b, Section 2.1]). In this section, we define the equivariant linking matrix of a surgery link and we give a diagrammatic computation of it.

Let  $L \subset S^3 \setminus \mathcal{O}$  be a surgery link. Let  $\mathcal{D}$  be a disk bounded by  $\mathcal{O}$ . An *admissible diagram* of  $L$  is a projection of  $L \cup \mathcal{D}$  onto a square  $[-1, 1]^2$  where:

- the image of  $\mathcal{D}$  is the segment line  $[(0, 0), (1, 0)]$ ,
- the multiple points of the projection restricted to  $L$  are transverse double points disjoint from  $\mathcal{D}$ ,
- the points of  $L$  that project onto  $[(0, 0), (1, 0)]$  are the points of  $L \cap \mathcal{D}$ .

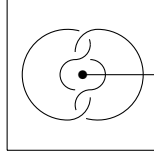


FIGURE 9. An admissible diagram of a surgery presentation

Let  $E$  be the exterior of  $\mathcal{O}$  in  $S^3$  and let  $\tilde{E}$  be the infinite cyclic covering of  $E$ . Note that  $\tilde{E}$  is homeomorphic to  $\mathcal{D} \times \mathbb{R}$ . In particular, the equivariant linking number of knots in  $\tilde{E}$  takes values in  $\mathbb{Z}[t^{\pm 1}]$ .

Fix an admissible diagram of  $L$  and base points  $\star_i$  of its components, away from the crossings and the disk  $\mathcal{D}$ . Let  $\tilde{E}_0 \subset \tilde{E}$  be a copy of  $E$  cut along  $\mathcal{D}$  and define the lift  $\tilde{L}_i$  of  $L_i$  in  $\tilde{E}$  by lifting  $\star_i$  in  $\tilde{E}_0$ . Consider the matrix of equivariant linkings  $W_L = \left( \text{lk}_e(\tilde{L}_i, \tilde{L}_j) \right)_{1 \leq i, j \leq n}$ . If the link  $L$  is a surgery presentation for a QSK–pair  $(M, K)$ , then the matrix  ${}^t W_L$  is a presentation matrix of the Alexander module of  $(M, K)$  with generators the classes of meridians  $m_i$  of the components  $\tilde{L}_i$  [Mou12b, Proposition 2.5]. Moreover, the Blanchfield form is given on these generators by the matrix  $-W_L^{-1}$  [Mou12b, Corollary 3.2].

**3.2. Winding matrix.** We now give a diagrammatic computation of the matrix  $W_L$ .

Given an admissible diagram of  $L$ , define the *winding number*  $w(L_i, L_j) \in \mathbb{Z}[t^{\pm 1}]$  of  $L_i$  and  $L_j$  in the following way. For a crossing  $c$  between  $L_i$  and  $L_j$ , denote  $\varepsilon_{ij}(c)$  the algebraic intersection number of the disk  $\mathcal{D}$  with the path that goes from  $\star_i$  to  $c$  along  $L_i$  and then from  $c$  to  $\star_j$  along  $L_j$ . If  $i = j$ , change component at the first occurrence of  $c$ . Set

$$w(L_i, L_j) = \begin{cases} \frac{1}{2} \sum_c \text{sg}(c) t^{\varepsilon_{ij}(c)} & \text{if } i \neq j \\ \frac{1}{2} \sum_c \text{sg}(c) (t^{\varepsilon_{ii}(c)} + t^{-\varepsilon_{ii}(c)}) & \text{if } i = j \end{cases}$$

where the sums are over all crossings between  $L_i$  and  $L_j$ . Note that  $w(L_j, L_i)(t) = w(L_i, L_j)(t^{-1})$ .

**Lemma 3.1.** *The winding numbers are invariant by isotopies that do not allow the base points to pass through the disk  $\mathcal{D}$ .*

*Proof.* First note that the winding numbers are preserved when a crossing passes through the disk  $\mathcal{D}$ . It is also preserved when the base point of a component passes through a crossing since the algebraic intersection number of this component with  $\mathcal{D}$  is trivial. Hence it only remains to check invariance with respect to framed Reidemeister moves performed far from the base points and the disks, which is direct.  $\square$

**Lemma 3.2.**  $W_L = (w(L_i, L_j))_{1 \leq i, j \leq n}$

*Proof.* First note that, since  $L_i$  and  $L_j$  are null-homologous in  $S^3 \setminus \mathcal{O}$ ,  $\text{lk}_e(\tilde{L}_i, \tilde{L}_j)$  is equal to  $\sum_{k \in \mathbb{Z}} \text{lk}(\tilde{L}_i, \tau^k(\tilde{L}_j)) t^k$ . From the diagram of  $L$ , we can get a diagram of  $\tilde{L}$  and its translates: cut the diagram along the image of  $\mathcal{D}$  and glue together  $\mathbb{Z}$  copies of it, see Figure 10. A crossing  $c$  between  $L_i$  and  $L_j$  such that  $\varepsilon_{ij}(c) = k$  lifts as a crossing between  $\tilde{L}_i$  and  $\tau^k(\tilde{L}_j)$ , so that it contributes equally to  $w(L_i, L_j)$  and  $\text{lk}_e(\tilde{L}_i, \tilde{L}_j)$ . When  $i = j$  and  $k \neq 0$ , the crossing  $c$  lifts as two crossings of  $\tilde{L}_i$ , one with  $\tau^k(\tilde{L}_i)$  and one with  $\tau^{-k}(\tilde{L}_i)$ .  $\square$

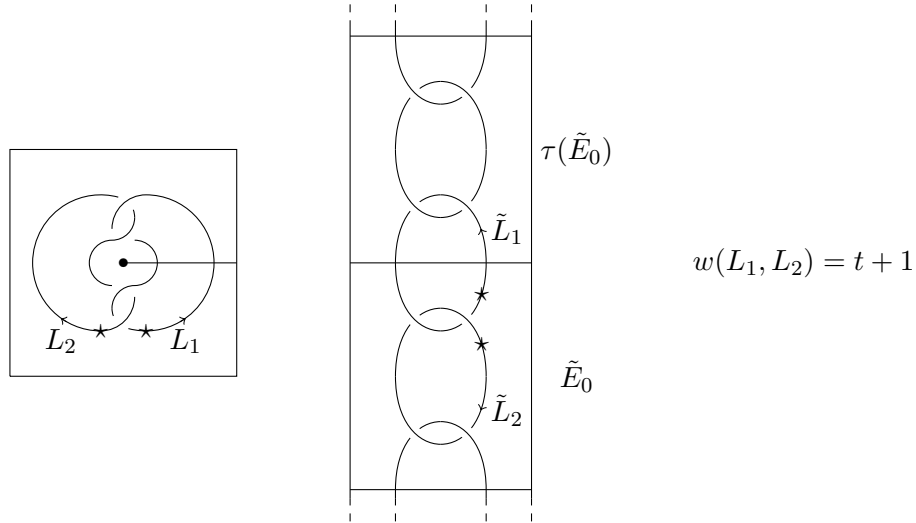


FIGURE 10. An admissible diagram and its lift

In the sequel, we call  $W_L$  the *winding matrix* of  $L$ . To fully understand the effect of an isotopy on this matrix, we shall describe its modification when a base point passes through the disk  $\mathcal{D}$ . Fix a component  $L_i$ . Fix an admissible diagram of  $L$  with the base point of  $L_i$  located “just before” the disk, as shown in the first part of Figure 11. Consider another admissible diagram of  $L$  which differs from the previous one only by the position of the base point  $\star_i$ , which is as shown on the second part of Figure 11. Let  $\varepsilon = \pm 1$  give the sign of the intersection point of  $\mathcal{D}$  and  $L_i$  that the base point passes through. It is easily seen that the winding matrix of the latter diagram is obtained from the winding matrix of the previous one by multiplying the coefficients of the  $i$ -th column (resp. row) by  $t^\varepsilon$  (resp.  $t^{-\varepsilon}$ ).



FIGURE 11. Base point passing through the disk

#### 4. DIAGRAM SPACES

In this section, we introduce the diagram spaces that are needed in the construction of the invariant  $\tilde{Z}$ , and we define an operation that will play, in our construction, the role of the formal Gaussian integral in the construction of the Kricker invariant.

**4.1. Beaded Jacobi diagrams.** For a compact oriented 1–manifold  $X$  (resp. a finite set  $C$ ), a *Jacobi diagram on  $X$*  (resp. a *Jacobi diagram on  $C$* ) is a unitrivalent graph whose trivalent vertices are oriented and whose univalent vertices are embedded in  $X$  (resp. labeled by  $C$ ). When relevant, the manifold  $X$  is called the *skeleton* of the diagram. A *beaded Jacobi diagram on  $X$  or  $C$*  is a Jacobi diagram on  $X$  or  $C$  whose graph edges are oriented and labeled by  $\mathbb{Q}[t^{\pm 1}]$ . A *w–beaded Jacobi diagram on  $X$*  is a beaded Jacobi diagram on  $X$  whose skeleton is viewed as a union of edges —defined by the embedded vertices— that are labeled by powers of  $t$ , with the condition that the product of the labels on each component of  $X$  is 1. The *degree* of a unitrivalent diagram is the number of its trivalent vertices (sometimes called i–degree). Set:

$$\begin{aligned} \tilde{\mathcal{A}}(X) &= \frac{\mathbb{Q}\langle \text{beaded Jacobi diagrams on } X \rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol} \rangle}, \\ \tilde{\mathcal{A}}_w(X) &= \frac{\mathbb{Q}\langle \text{w–beaded Jacobi diagrams on } X \rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol, Hol}_w \rangle}, \\ \tilde{\mathcal{A}}(*_C) &= \frac{\mathbb{Q}\langle \text{beaded Jacobi diagrams on } C \rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol} \rangle}, \end{aligned}$$

where the relations are defined in Figures 4 and 12. In the STU relation, corresponding edges have the same orientation and label. In the pictures, the skeleton is represented with bold lines and the graph with thin lines. We consider the degree completion of these vector spaces, keeping the same notation.

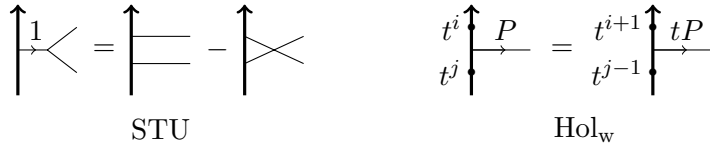


FIGURE 12. Relations STU and  $\text{Hol}_w$  on Jacobi diagrams.

*Remark.* For diagrams in  $\tilde{\mathcal{A}}(X)$ , the condition on the labels on the skeleton implies that all labels can be pushed off each component of the skeleton using the relation  $\text{Hol}_w$ . When the component is an interval, there is a unique way to do so.

For a finite set  $C$ , denote by  $\uparrow_C$  (resp.  $\circ_C$ ) the manifold made of  $|C|$  intervals (resp. circles) indexed by the elements of  $C$ . The above remark provides an isomorphism  $\tilde{\mathcal{A}}_w(\uparrow_C) \cong \tilde{\mathcal{A}}(\uparrow_C)$ . In the case of a squeeleton with closed components, we need to add a relation to get such an isomorphism.

Given a beaded Jacobi diagram  $D$  on  $\circ_C$ , a label  $c \in C$  and an integer  $k$ , the associated *winding relation* identifies  $D$  with the diagram obtained from  $D$  by *pushing  $t^k$  at each vertex glued on  $\circ_c$ , ie* by multiplying the label of each edge adjacent to a univalent vertex glued on  $\circ_c$  by  $t^k$  if the orientation of the edge goes backward the vertex and by  $t^{-k}$  otherwise, see Figure 13. Denote by  $\sim_w$  the induced equivalence relation. It provides an isomorphism  $\tilde{\mathcal{A}}_w(\circ_C) \cong \tilde{\mathcal{A}}(\circ_C)/\sim_w$ . We want to relate this quotient to the space  $\tilde{\mathcal{A}}(\uparrow_C)$ . The winding relation  $\sim_w$  is defined as above on  $\tilde{\mathcal{A}}(\uparrow_C)$ . We also define a *link relation* on  $\tilde{\mathcal{A}}(\uparrow_C)$  as follows. Given two beaded Jacobi diagrams  $D_1$  and  $D_2$  on  $\uparrow_C$ , we have  $D_1 \sim_\ell D_2$  if, for an index  $c \in C$  and two extra indices  $c_1$  and  $c_2$ , there is a beaded Jacobi diagram  $D$  on  $\uparrow_{(C \setminus \{c\}) \cup \{c_1, c_2\}}$  such that  $D_1$  and  $D_2$  are obtained from  $D$  by gluing together the squeeleton components  $\uparrow_{c_1}$  and  $\uparrow_{c_2}$  in the two possible orders. It is easily checked that  $\tilde{\mathcal{A}}(\circ_C)/\sim_w \cong \tilde{\mathcal{A}}(\uparrow_C)/\sim_w, \sim_\ell$ . The link relation  $\sim_\ell$  is similarly defined on  $\tilde{\mathcal{A}}_w(\uparrow_C)$ .

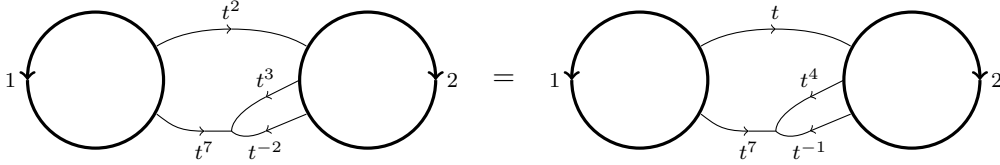


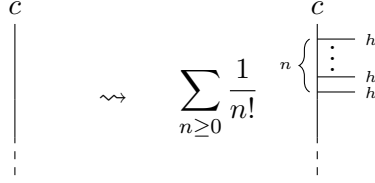
FIGURE 13. A winding relation (for  $c = 2$  and  $k = 1$ )

In [Bar95, Theorem 8], Bar-Natan defines a formal PBW isomorphism:

$$\chi_C : \tilde{\mathcal{A}}(*_C) \xrightarrow{\cong} \tilde{\mathcal{A}}(\uparrow_C).$$

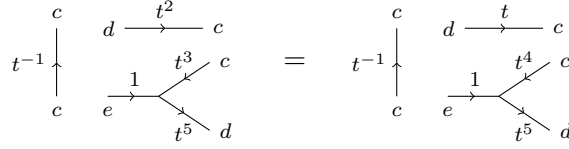
For a beaded Jacobi diagram  $D$ , the image  $\chi_C(D)$  is the average of all possible ways to attach the  $c$ -colored vertices of  $D$  on the interval  $\uparrow_c$  for each  $c \in C$ . The setting of [Bar95] is not exactly the same, but the argument adapts directly. To recover an isomorphism onto  $\tilde{\mathcal{A}}(\circ_C)$ , one needs a version of the link relations on  $\tilde{\mathcal{A}}(*_C)$ . These relations were first introduced in [BGRT02, Section 5.2]; the ones we use here mainly come from [GK04].

Given a beaded Jacobi diagram  $D$  on  $C$  and distinct elements  $c, \bar{c} \in C$ , we define  $\langle D \rangle_{c-\bar{c}}$  as the sum of all diagrams obtained by gluing all  $c$ -labeled vertices to all  $\bar{c}$ -labeled vertices when there are as many  $c$  and  $\bar{c}$ -labeled vertices, and as 0 otherwise. We also denote  $D|_{c \rightarrow ce^h}$  the diagram obtained from  $D$  by *pushing  $e^h$  on each  $c$ -labeled vertex of  $D$* , where pushing  $e^h$  is the operation pictured in Figure 14; note that the  $h$ -ended edges are added on the right when going toward the  $c$ -labeled vertex. We define the *link relation*  $\sim_\ell$  on  $\tilde{\mathcal{A}}(*_C)$  as generated by the following: given two beaded Jacobi diagrams  $D_1$  and  $D_2$  on  $C$ , we have  $D_1 \sim_\ell D_2$  if, for some  $c \in C$  and some extra vertices  $h, \check{h}$ , there is a beaded Jacobi diagram  $D$  on  $C \cup \{h, \check{h}\}$  such that  $D_1 = \langle D \rangle_{h-\check{h}}$  and  $D_2 = \langle D|_{c \rightarrow ce^h} \rangle_{h-\check{h}}$ .


 FIGURE 14. Pushing  $e^h$  on a  $c$ -labeled vertex

**Proposition 4.1** (Garoufalidis–Kriker [GK04, Lemma 3.6]). *For beaded Jacobi diagrams  $D_1$  and  $D_2$  on  $C$ , we have  $D_1 \sim_\ell D_2$  if and only if  $\chi_C(D_1) \sim_\ell \chi_C(D_2)$ .*

Pushing  $t^k$  at the  $c$ -labeled vertices defines as above a winding relation  $\sim_w$  on  $\tilde{\mathcal{A}}(*_C)$ , see Figure 15. Set  $\tilde{\mathcal{A}}(\otimes_C) = \tilde{\mathcal{A}}(*_C)/\sim_w, \sim_\ell$ . The following is a corollary of Proposition 4.1.


 FIGURE 15. A winding relation on  $\tilde{\mathcal{A}}(*_C)$  (on  $c$  for  $k = 1$ )

**Proposition 4.2.** *The isomorphism  $\chi_C : \tilde{\mathcal{A}}(*_C) \xrightarrow{\cong} \tilde{\mathcal{A}}(\uparrow_C)$  descends to an isomorphism  $\chi_C : \tilde{\mathcal{A}}(\otimes_C) \xrightarrow{\cong} \tilde{\mathcal{A}}(\circ_C)/\sim_w$ .*

We finally have the following commutative diagram of diagram spaces.

$$\begin{array}{ccccc}
 \tilde{\mathcal{A}}_w(\uparrow_C) & \xrightarrow{\cong} & \tilde{\mathcal{A}}(\uparrow_C) & \xrightarrow{\cong} & \tilde{\mathcal{A}}(*_C) \\
 \downarrow / \sim_\ell & & \downarrow / \sim_\ell, \sim_w & & \downarrow / \sim_\ell \\
 \tilde{\mathcal{A}}_w(\circ_C) & \xrightarrow{\cong} & \tilde{\mathcal{A}}(\circ_C)/\sim_w & \xrightarrow{\cong} & \tilde{\mathcal{A}}(\otimes_C)
 \end{array}$$

Given a subset  $S$  of a finite set  $C$ , one can also consider Jacobi diagrams with univalent vertices either labeled by  $S$  or embedded in  $\uparrow_{C \setminus S}$  or  $\circ_{C \setminus S}$ . This provides diagram spaces  $\tilde{\mathcal{A}}(*_S, \uparrow_{C \setminus S})$  and  $\tilde{\mathcal{A}}(*_S, \circ_{C \setminus S})$ , and their quotients. As above, we have an isomorphism

$$\chi_S : \tilde{\mathcal{A}}(*_S, \uparrow_{C \setminus S}) \xrightarrow{\cong} \tilde{\mathcal{A}}(\uparrow_C).$$

**4.2. Product and coproduct.** We first define a coproduct on the diagram spaces of the previous subsection. Given a (w-)beaded Jacobi diagram  $D$  on  $X$  or  $C$ , denote by  $\check{D}$  its graph part, and by  $\check{D}_i$ ,  $i \in I$ , the connected components of  $\check{D}$ . Define  $D_J = D \setminus (\sqcup_{i \in I} \check{D}_i)$  by multiplying the labels of the concatenated edges of the skeleton, when relevant. The coproduct of a diagram  $D$  is defined by

$$\Delta(D) = \sum_{J \subset I} D_J \otimes D_{I \setminus J}.$$



Note that the different relations on beaded Jacobi diagrams respect the coproduct. This provides a notion of *group-like* elements, *ie* elements  $G$  such that  $\Delta(G) = G \otimes G$ . Also, the isomorphisms  $\chi$  of the previous subsection preserve the coproduct.

We now define a Hopf algebra structure on  $\tilde{\mathcal{A}}(*_C)$ . Define the product of two diagrams as the disjoint union. The unit  $\epsilon : \mathbb{Q} \rightarrow \tilde{\mathcal{A}}(*_C)$  is defined by  $\epsilon(1) = \emptyset$  and the counit  $\varepsilon : \tilde{\mathcal{A}}(*_C) \rightarrow \mathbb{Q}$  is given by  $\varepsilon(D) = 0$  if  $D \neq \emptyset$  and  $\varepsilon(\emptyset) = 1$ . The antipode is given by  $D \mapsto (-1)^s D$ , where  $s$  is the number of connected components in  $D$ . We finally have a structure of graded Hopf algebra on  $\tilde{\mathcal{A}}(*_C)$ , where the grading is given by the degree. It is known that an element in a graded Hopf algebra is group-like if and only if it is the exponential of a *primitive* element, *ie* an element  $G$  such that  $\Delta(G) = 1 \otimes G + G \otimes 1$ . Here, the primitive elements are the series of connected diagrams. We denote by  $\exp_{\sqcup}$  the exponential of diagrams with respect to the disjoint union.

**4.3. Operation  $\omega$ .** This part is devoted to the definition of an operation on  $\tilde{\mathcal{A}}(*_{\{1, \dots, n\}})$  that will play the role of the formal Gaussian integration in our refinement of the Kricker invariant.

A hermitian matrix  $W(t)$  with coefficients in  $\mathbb{Q}[t^{\pm 1}]$  such that  $\det(W(1)) \neq 0$  defines a Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  by  $\mathfrak{A} = \frac{\mathbb{Q}[t^{\pm 1}]^n}{tW\mathbb{Q}[t^{\pm 1}]^n}$  and  $\mathfrak{b}(x_i, x_j) = -(W^{-1})_{ij}(t) \bmod \mathbb{Q}[t^{\pm 1}]$ , where the  $x_i$  are the generators associated with the presentation (see [Mou12b] for details). Given a beaded Jacobi diagram  $D$  on  $\{1, \dots, n\}$ , we define an  $(\mathfrak{A}, \mathfrak{b})$ -colored diagram  $\omega_W(D)$  by replacing the label  $i$  on univalent vertices by  $x_i$  and fixing  $f_{vv'}(t) = -(W^{-1})_{ij}(t)$  if the univalent vertices  $v$  and  $v'$  are labeled by  $i$  and  $j$  respectively.

A *strut* is an isolated edge in a graph. To a square matrix  $W$  of size  $n$ , we associate the sum of struts  $\sum_{1 \leq i, j \leq n} W_{ij} \left. \vphantom{\sum} \right\}^j_i \in \tilde{\mathcal{A}}(*_{\{1, \dots, n\}})$ . At the other end of the spectrum, we say that a beaded Jacobi diagram on some finite set is *substantial* if it has no strut.

**Definition 4.3.** An element  $G \in \tilde{\mathcal{A}}(*_{\{1, \dots, n\}})$  is *Gaussian* if  $G = \exp_{\sqcup}(\frac{1}{2}W(t)) \sqcup H$  where  $W(t)$  is a hermitian matrix of size  $n$  with coefficients in  $\mathbb{Q}[t^{\pm 1}]$  and  $H$  is substantial. If in addition  $\det(W(t)) \neq 0$ ,  $G$  is *non-degenerate* and we set  $\omega(G) = \omega_{\frac{1}{2}W}(H)$ .

*Notations.* For a Jacobi diagram  $D$  on some finite set, the subscript  $D|_{x \rightarrow y}$  means that the label  $x$  on univalent vertices is replaced by the label  $y$ . We also define the following exponential notation

on diagrams:  $\dots \left. \vphantom{\dots} \right\}^e \dots = \sum_{s \geq 0} \frac{1}{s!} \dots \left. \vphantom{\dots} \right\}^{\overbrace{h \dots h}^{s \text{ times}}} \dots$ . A term  $e^h - 1$  instead of  $e^h$  means that the sum is over  $s > 0$ , whereas a term  $e^{-h}$  means that the  $h$ -ended edges are added on the other side of the supporting edge. Thanks to the relation AS, this latter variant is equivalent to replacing the  $\frac{1}{s!}$  factor by  $\frac{(-1)^s}{s!}$ ; note in particular that an  $e^h$  notation followed immediately by an  $e^{-h}$  just amounts to no notation at all.

**Lemma 4.4.** Let  $G_1 = \exp_{\sqcup}(\frac{1}{2}W_1) \sqcup H_1$  and  $G_2 = \exp_{\sqcup}(\frac{1}{2}W_2) \sqcup H_2$  be non-degenerate Gaussians in  $\tilde{\mathcal{A}}(*_{\{1, \dots, n\}})$ . Assume  $\chi_{\{1, \dots, n\}}(G_1)$  and  $\chi_{\{1, \dots, n\}}(G_2)$  are related by one link relation on  $\uparrow_i$ . Then  $W_1 = W_2$  and  $G_1 \sim_{\ell} G_2$ , using the label  $i$  and a Gaussian beaded Jacobi diagram  $G =$

$\exp_{\sqcup}(\frac{1}{2}W) \sqcup H$  with  $W$  of the form  $\begin{pmatrix} W_1 & \zeta & 0 \\ {}^t\zeta & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

*Proof.* The proof follows very closely that of [GK04, Lemma 3.6]. Up to relabeling, we assume that  $i = n$ . There is a diagram  $D \in \tilde{\mathcal{A}}(\uparrow_{\{1, \dots, n-1, n_1, n_2\}})$  such that  $\chi_{\{1, \dots, n\}}(G_1)$  is obtained from  $D$  by gluing the head of  $\uparrow_{n_2}$  to the tail of  $\uparrow_{n_1}$ , and  $\chi_{\{1, \dots, n\}}(G_2)$  by gluing the head of  $\uparrow_{n_1}$  to the tail of  $\uparrow_{n_2}$ ; hence  $\chi_{\{n\}}(G_1)$  and  $\chi_{\{n\}}(G_2)$  are obtained similarly from  $\chi_{\{1, \dots, n-1\}}^{-1}(D)$ . Writing  $\chi^{-1}(D)|_{\substack{n_1 \rightarrow h \\ n_2 \rightarrow k}} = \sum \alpha_i D_i$  with the  $D_i$  in  $\tilde{\mathcal{A}}(*_{\{1, \dots, n-1, h, k\}})$ , we have:

$$\chi_{\{1, \dots, n-1\}}^{-1}(D) = \chi_{\{n_1, n_2\}} \circ \chi^{-1}(D) = \left\langle \sum \alpha_i \begin{array}{c} \boxed{D_i} \\ \begin{array}{cc} \cdots & \cdots \\ k & h \end{array} \end{array} \xrightarrow{\begin{array}{c} e^{\check{k}} \\ \cdots \\ n_2 \end{array}} \xrightarrow{\begin{array}{c} e^{\check{h}} \\ \cdots \\ n_1 \end{array}} \right\rangle_{\substack{h-\check{h} \\ k-\check{k}}},$$

and hence

$$\chi_{\{n\}}(G_1) = \left\langle \sum \alpha_i \begin{array}{c} \boxed{D_i} \\ \begin{array}{cc} \cdots & \cdots \\ k & h \end{array} \end{array} \xrightarrow{\begin{array}{c} e^{\check{k}} \\ \cdots \\ n \end{array}} \xrightarrow{\begin{array}{c} e^{\check{h}} \\ \cdots \\ n \end{array}} \right\rangle_{\substack{h-\check{h} \\ k-\check{k}}}.$$

In this bracket, the sum is over all the ways of gluing the  $h$ -labeled legs of diagrams in  $\sum D_i$  on the “head half” of  $\uparrow_n$  and the  $\check{k}$ -legs on the “tail half”, so basically, we have

$$\chi_{\{n\}}(G_1) = m_n^{kh} \circ \chi_{\{k, h\}} \left( \sum \alpha_i \begin{array}{c} \boxed{D_i} \\ \begin{array}{cc} \cdots & \cdots \\ k & h \end{array} \end{array} \right)$$

where, on each diagram,  $m_n^{kh}$  glues the head of  $\uparrow_k$  to the tail of  $\uparrow_h$  to form  $\uparrow_n$ . Then, we can use [BGR02, Prop. 5.4] to write

$$G_1 = \left\langle \exp_{\sqcup}(\Lambda_n^{\check{k}\check{h}}) \sqcup \sum \alpha_i \begin{array}{c} \boxed{D_i} \\ \begin{array}{cc} \cdots & \cdots \\ k & h \end{array} \end{array} \right\rangle_{\substack{h-\check{h} \\ k-\check{k}}}$$

where  $\Lambda_n^{\check{k}\check{h}}$  is the Baker–Campbell–Hausdorff sum

$$(*) \quad \Lambda_n^{\check{k}\check{h}} = \begin{array}{c} \check{k} \\ | \\ n \end{array} + \begin{array}{c} \check{h} \\ | \\ n \end{array} + \frac{1}{2} \begin{array}{c} \check{k} \quad \check{h} \\ \diagdown \quad \diagup \\ | \\ n \end{array} + \frac{1}{12} \begin{array}{c} \check{k} \quad \check{k} \quad \check{h} \\ \diagdown \quad | \quad \diagup \\ | \\ n \end{array} - \frac{1}{12} \begin{array}{c} \check{h} \quad \check{k} \quad \check{h} \\ \diagdown \quad | \quad \diagup \\ | \\ n \end{array} + \dots$$

Now, we want to perform a partial contraction of  $\check{h}$  on  $h$ . Note that

$$\begin{aligned} & \left\langle \exp_{\sqcup}(\Lambda_n^{\check{k}\check{h}}) \sqcup \sum \alpha_i \begin{array}{c} \boxed{D_i} \\ \begin{array}{cc} \cdots & \cdots \\ k & h \end{array} \end{array} \right\rangle_{\substack{h-\check{h} \\ k-\check{k}}} \\ &= \left\langle \exp_{\sqcup} \left( \Lambda_n^{\check{k}\check{h}} - \begin{array}{c} \check{h} \\ | \\ n \end{array} \right) \sqcup \exp_{\sqcup} \left( \begin{array}{c} \check{h}' \\ | \\ n \end{array} \right) \sqcup \sum \alpha_i \begin{array}{c} \boxed{D_i^p} \\ \begin{array}{cc} \cdots & \cdots \\ k & h, h' \end{array} \end{array} \right\rangle_{\substack{h'-\check{h}' \\ h-\check{h} \\ k-\check{k}}}, \end{aligned}$$

where the  $D_i^p$  run over all the ways to partition the  $h$ -labeled legs into  $h$ - and  $h'$ -labeled ones.

Now, setting

$$G = \left\langle \exp_{\sqcup} \left( \Lambda_n^{\check{k}\check{h}} - \left| \begin{array}{c} \check{h} \\ n \end{array} \right. \right) \sqcup \exp_{\sqcup} \left( \left| \begin{array}{c} \check{h}' \\ n \end{array} \right. \right) \sqcup \sum \alpha_i \left[ \begin{array}{c} D_i^p \\ \cdots \quad \cdots \\ k \quad h, h' \end{array} \right] \right\rangle_{\substack{h'=\check{h}' \\ k=\check{k}}},$$

we obtain  $G_1 = \langle G \rangle_{h=\check{h}}$ .

Note that  $G$  is group-like because  $G_1$  is and the map  $\chi$  preserves this property. Hence  $G$  is Gaussian. To obtain the form of the matrix  $W$ , observe that  $G$  has no strut with a label  $\check{h}$  and its struts with both labels in  $\{1, \dots, n\}$  are exactly that of  $G_1$ .

We now consider  $G_2$ . As for  $G_1$ , we have

$$\chi_{\{n\}}(G_2) = \left\langle \sum \alpha_i \left[ \begin{array}{c} D_i \\ \cdots \quad \cdots \\ k \quad h \end{array} \right] \xrightarrow{\begin{array}{c} e^{\check{h}} \quad e^{\check{k}} \\ \cdots \quad \cdots \\ n \end{array}} \right\rangle_{\substack{h=\check{h} \\ k=\check{k}}}.$$

Further, we have

$$\xrightarrow{\begin{array}{c} e^{\check{h}} \quad e^{\check{k}} \\ \cdots \quad \cdots \\ n \end{array}} = \xrightarrow{\begin{array}{c} e^{\check{h}} \quad e^{\check{k}} \quad e^{-\check{h}} \quad e^{\check{h}} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ n \end{array}} = \xrightarrow{\begin{array}{c} e^{\check{k}} \\ \overbrace{\begin{array}{c} e^{\check{h}} \quad \cdots \quad e^{\check{h}} \\ \vdots \quad \quad \quad \vdots \\ e^{\check{h}} \quad \cdots \quad e^{\check{h}} \end{array}} \\ \cdots \quad \cdots \\ n \end{array}},$$

so

$$\chi_{\{n\}}(G_2) = \left\langle \sum \alpha_i \left[ \begin{array}{c} D_i \\ \cdots \quad \cdots \\ k \quad h \end{array} \right] \xrightarrow{\begin{array}{c} e^{\check{k}} \\ \overbrace{\begin{array}{c} e^{\check{h}} \quad \cdots \quad e^{\check{h}} \\ \vdots \quad \quad \quad \vdots \\ e^{\check{h}} \quad \cdots \quad e^{\check{h}} \end{array}} \\ \cdots \quad \cdots \\ n \end{array}} \right\rangle_{\substack{h=\check{h} \\ k=\check{k}}} = \left\langle \sum \alpha_i \left[ \begin{array}{c} D_i^{\check{h}} \\ \cdots \quad \cdots \\ k \quad h \end{array} \right] \xrightarrow{\begin{array}{c} e^{\check{k}} \quad e^{\check{h}} \\ \cdots \quad \cdots \\ n \end{array}} \right\rangle_{\substack{h=\check{h} \\ k=\check{k}}}$$

where  $D_i^{\check{h}}$  is obtained from  $D_i$  by pushing  $e^{\check{h}}$  on each  $k$ -labeled leg. As above, we get then

$$G_2 = \left\langle \exp_{\sqcup} \left( \Lambda_n^{\check{k}\check{h}} \right) \sqcup \sum \alpha_i \left[ \begin{array}{c} D_i^{\check{h}} \\ \cdots \quad \cdots \\ k \quad h \end{array} \right] \right\rangle_{\substack{h=\check{h} \\ k=\check{k}}}.$$

Now, the  $e^{\check{h}}$  on the  $k$ -labeled legs can be pushed on the  $\check{k}$ -labeled legs of  $\exp_{\sqcup} \left( \Lambda_n^{\check{k}\check{h}} \right)$ , where they become  $e^{-\check{h}}$ . Some  $e^{-\check{h}}$  can also be freely added to the  $\check{h}$ -labeled legs of  $\exp_{\sqcup} \left( \Lambda_n^{\check{k}\check{h}} \right)$  as all the diagrams corresponding to the non trivial terms in  $e^{-\check{h}}$  will vanish. All these  $e^{\check{h}}$  can then be pushed down through  $\Lambda_n^{\check{k}\check{h}}$  (using some IHX relations), so that

$$G_2 = \left\langle \exp_{\sqcup} \left( \tilde{\Lambda}_n^{\check{k}\check{h}} \right) \sqcup \sum \alpha_i \left[ \begin{array}{c} D_i \\ \cdots \quad \cdots \\ k \quad h \end{array} \right] \right\rangle_{\substack{h=\check{h} \\ k=\check{k}}}$$

where  $\tilde{\Lambda}_n^{\check{k}\check{h}}$  is obtained from  $\Lambda_n^{\check{k}\check{h}}$  by pushing  $e^{\check{h}}$  on each  $n$ -labeled leg. This can be rewritten as

$$G_2 = \left\langle \left\langle \exp_{\sqcup} \left( \tilde{\Lambda}_n^{\check{k}\check{h}} - e^{\check{h}} \left| \begin{array}{c} \check{h} \\ \vdots \\ \check{k} \end{array} \right| \right) \sqcup \exp_{\sqcup} \left( e^{\check{h}} \left| \begin{array}{c} \check{h}' \\ \vdots \\ \check{k}' \end{array} \right| \right) \sqcup \sum \alpha_i \boxed{D_i^p} \right\rangle_{\substack{h'-\check{h}' \\ k-\check{k}}} \right\rangle_{h-\check{h}},$$

that is  $G_2 = \left\langle G_{|n \rightarrow ne^{\check{h}}}\right\rangle_{h-\check{h}}$ .  $\square$

**Proposition 4.5.** *Let  $G_1 = \exp_{\sqcup}(\frac{1}{2}W_1(t)) \sqcup H_1$  and  $G_2 = \exp_{\sqcup}(\frac{1}{2}W_2(t)) \sqcup H_2$  be non-degenerate Gaussians in  $\tilde{\mathcal{A}}(*_{\{1, \dots, n\}})$ .*

- If  $G_1 \sim_w G_2$ , then  $\omega(G_1) = \omega(G_2)$ .
- If  $G_1 \sim_{\ell} G_2$ , then  $W_1(t) = W_2(t)$  and  $\omega(G_1) = \omega(G_2)$ .

*Proof.* Assume  $G_2$  is obtained from  $G_1$  by pushing  $t^k$  on the  $i$ -labeled vertices. Denote by  $\text{Diag}_i(t)$  the diagonal matrix with a  $t$  at the  $i^{\text{th}}$  position and 1's elsewhere. We have  $W_2(t) = \text{Diag}_i(t^k) \cdot W_1(t) \cdot \text{Diag}_i(t^{-k})$ . Hence the map  $\frac{\mathbb{Q}[t^{\pm 1}]^n}{tW_1\mathbb{Q}[t^{\pm 1}]^n} \rightarrow \frac{\mathbb{Q}[t^{\pm 1}]^n}{tW_2\mathbb{Q}[t^{\pm 1}]^n}$  that maps  $x_i^{(1)}$  to  $t^{-k}x_i^{(2)}$  and  $x_j^{(1)}$  to  $x_j^{(2)}$  is an isomorphism of Blanchfield modules. Pushing a  $t^k$  on the  $i$ -labeled vertices of  $H$  precisely applies this isomorphism to the univalent vertices of  $\omega(G_1)$ , thanks to the relation EV. Hence  $\omega(G_1) = \omega(G_2)$ .

Now assume  $G_2$  is obtained from  $G_1$  by a single link relation on the  $n$ -labeled vertices. Lemma 4.4 gives the equality  $W_2 = W_1$ . Let  $G$ ,  $W$  and  $H$  be the diagrams and matrix given by the lemma. Since  $G$  is a degenerate Gaussian, we cannot directly apply the operation  $\omega$ .

Instead, we pertube the matrix  $W$ : set  $\widehat{W} = \begin{pmatrix} W_1 & \zeta & 0 \\ t\bar{\zeta} & \lambda & -1 \\ 0 & -1 & 0 \end{pmatrix}$ . Applying  $\omega_{\widehat{W}}$  to  $H$ , the added

coefficients  $-1$  and the relation LD will play the role of the contraction  $\check{h} - h$ . Note that  $\widehat{W}$  and  $W_1$  define the same Blanchfield module. Indeed, denoting  $x_i$ ,  $1 \leq i \leq n$ ,  $x_h$  and  $x_{\check{h}}$  the generators associated to the presentation  $\widehat{W}$ , the last two columns of  $\widehat{W}$  give  $x_h = 0$  and  $x_{\check{h}} = \sum_{i=1}^n \zeta_i x_i$ . The relations between the  $x_i$  are given by the matrix  $W_1$ . Moreover, since

$$\widehat{W}^{-1} = \begin{pmatrix} W_1^{-1} & 0 & W_1^{-1}\zeta \\ 0 & 0 & -1 \\ t\bar{\zeta}W_1^{-1} & -1 & t\bar{\zeta}W_1^{-1}\zeta - \lambda \end{pmatrix},$$

the Blanchfield forms also coincide.

We now conclude in three steps.

**First step:**  $\omega(G_1) = \omega_{\widehat{W}}(H)$ .

We have  $G_1 = \langle \exp_{\sqcup}(\frac{1}{2}W) \sqcup H \rangle_{\check{h}-h}$ , which gives:

$$\begin{aligned} G_1 &= \left\langle \exp_{\sqcup} \left( \frac{1}{2}W_1 \right) \sqcup \exp_{\sqcup} \left( \sum_{i=1}^n \zeta_i \left| \begin{array}{c} h \\ \vdots \\ i \end{array} \right| + \frac{1}{2} \lambda \left| \begin{array}{c} h \\ \vdots \\ h \end{array} \right| \right) \sqcup H \right\rangle_{\check{h}-h} \\ &= \exp_{\sqcup} \left( \frac{1}{2}W_1 \right) \sqcup \left\langle \exp_{\sqcup} \left( \sum_{i=1}^n \zeta_i \left| \begin{array}{c} h \\ \vdots \\ i \end{array} \right| + \frac{1}{2} \lambda \left| \begin{array}{c} h \\ \vdots \\ h \end{array} \right| \right) \sqcup H \right\rangle_{\check{h}-h}. \end{aligned}$$

Hence  $H_1 = \left\langle \exp_{\sqcup} \left( \sum_{i=1}^n \zeta_i \left\{ \begin{matrix} h \\ i \end{matrix} \right\} + \frac{1}{2} \lambda \left\{ \begin{matrix} h \\ h \end{matrix} \right\} \right) \sqcup H \right\rangle_{\check{h}-h}$ , and we need to prove the equality

$\omega_{W_1}(H_1) = \omega_{\widehat{W}}(H)$ . The operation  $\omega$ , as well as the contraction, are applied on each diagram of a series. Hence it suffices to prove the result for each summand, so we assume here that  $H$  is a single diagram.

In  $\omega_{\widehat{W}}(H)$ , apply successive relations LD to join the  $x_h$ -labeled legs to the  $x_{\check{h}}$ -labeled legs. If there are initially more  $h$ -labeled legs, then some  $x_h$  remains, with linking 0 to all other vertices, and the relation LV shows that  $\omega_{\widehat{W}}(H) = 0$ . Else, we are led to the sum of all diagrams obtained by joining all the  $x_h$ -labeled legs to some  $x_{\check{h}}$ -labeled legs. We can work on one of these diagrams, which amounts to assume that  $H$  initially had no  $h$ -labeled leg. Now, in  $\omega_{\widehat{W}}(H)$ , choose one  $x_{\check{h}}$ -labeled vertex and, using the relation LV and the equality  $x_{\check{h}} = \lambda x_h + \sum_{i=1}^n \zeta_i x_i$ , replace it by a sum of diagrams where the  $x_{\check{h}}$  is alternatively replaced by some  $\zeta_i x_i$  or  $\lambda x_h$ , with linking to other vertices as prescribed by  $\widehat{W}$ . Use then the relation LD to join the latter  $\lambda x_h$ -labeled vertex in all possible ways to other univalent vertices; by the relations LE and LV, the only surviving terms appear when this vertex is connected to a  $x_{\check{h}}$ -labeled one, acting hence like the insertion of a  $\lambda$ -labeled edge connecting two formerly  $x_{\check{h}}$ -labeled vertices. Perform recursively these last two steps on all the  $x_{\check{h}}$ -labeled vertices.

On the  $\omega_{W_1}(H_1)$  side, write  $H_1$  as above and apply the contractions  $\check{h} - h$  to  $H$ . Once again, if there are more  $h$ -labeled legs in  $H$ , we get  $H_1 = 0$ . Else, we get a similar sum of diagrams obtained by joining all the  $h$ -labeled legs to some  $\check{h}$ -labeled legs in  $H$ . Then contract the  $\check{h}$ -labeled legs of  $H$  with the  $h$ -labeled legs of  $\exp_{\sqcup} \left( \sum_{i=1}^n \zeta_i \left\{ \begin{matrix} h \\ i \end{matrix} \right\} + \frac{1}{2} \lambda \left\{ \begin{matrix} h \\ h \end{matrix} \right\} \right)$  and apply  $\omega_{W_1}$ .

Finally, noting that there is two different ways to contract a strut  $\frac{1}{2} \lambda \left\{ \begin{matrix} h \\ h \end{matrix} \right\}$  to two given  $h$ -labeled vertices, it is easily observed that, up to some relations EV, the above two processes, for  $\omega_{\widehat{W}}(H)$  and  $\omega_{W_1}(H_1)$ , give the same result.

**Second step:**  $G_{|n \rightarrow ne^{\check{h}}} = \exp_{\sqcup} \left( \frac{1}{2} W \right) \sqcup H'$  and  $\omega(G_2) = \omega_{\widehat{W}}(H')$ .

We have  $G_{|n \rightarrow ne^{\check{h}}} = \left( \exp_{\sqcup} \left( \frac{1}{2} W \right) \sqcup H \right)_{|n \rightarrow ne^{\check{h}}} = \exp_{\sqcup} \left( \frac{1}{2} W_{|n \rightarrow ne^{\check{h}}} \right) \sqcup H_{|n \rightarrow ne^{\check{h}}}$ . Now  $\frac{1}{2} W_{|n \rightarrow ne^{\check{h}}} = \frac{1}{2} W + J$  where  $J$  is substantial. It follows that  $G_{|n \rightarrow ne^{\check{h}}} = \exp_{\sqcup} \left( \frac{1}{2} W \right) \sqcup H'$ , where  $H' = \exp_{\sqcup} J \sqcup H_{|n \rightarrow ne^{\check{h}}}$ . Recall that  $G_2 = \left\langle G_{|n \rightarrow ne^{\check{h}}} \right\rangle_{\check{h}-h}$ , so that the second equality is given by the first step.

**Third step:**  $\omega_{\widehat{W}}(H) = \omega_{\widehat{W}}(H')$ .

As we have seen above,  $H' = \exp_{\sqcup} J \sqcup H_{|n \rightarrow ne^{\check{h}}}$  where  $J = \frac{1}{2} W_{|n \rightarrow ne^{\check{h}}} - \frac{1}{2} W$ . This gives

$$J = \sum_{1 \leq i \leq n} n \begin{array}{c} e^{\check{h}-1} \\ | \quad \cdots \quad | \\ \rightarrow W_{ni} \\ | \\ i \end{array} + \frac{1}{2} n \begin{array}{c} e^{\check{h}-1} \\ | \quad \cdots \quad | \\ \rightarrow W_{nn} \\ | \quad \cdots \quad | \\ e^{\check{h}-1} \end{array} n .$$

Writing  $H = \sum_{\kappa} \left[ \begin{array}{c} \boxed{D_{\kappa}} \\ \vdots \\ \end{array} \right]_{n, n}$ , we get

$$H' = \sum \frac{1}{s! \prod_{i=1}^n r_i!} \left[ \begin{array}{c} \boxed{D_{\kappa}} \\ \vdots \\ \end{array} \right]_{n, n} \left. \begin{array}{l} n \xrightarrow{e^{\check{h}-1} \dots} W_{ni_1} i_1 \\ \vdots \\ n \xrightarrow{e^{\check{h}-1} \dots} W_{ni_r} i_r \end{array} \right\} \begin{array}{l} n \xrightarrow{e^{\check{h}-1} \dots} \frac{1}{2} W_{nn} \\ \vdots \\ n \xrightarrow{e^{\check{h}-1} \dots} \frac{1}{2} W_{nn} \end{array} \left. \right\} s \text{ times}$$

where  $r_i = |\{1 \leq j \leq r \mid i_j = i\}|$ , and the sum is over all  $\kappa$  as above, all  $s \geq 0$  and all increasing sequences  $1 \leq i_1 \leq \dots \leq i_r \leq n$ .

Applying  $\omega_{\widehat{W}}$  and EV leads to the following.

$$\omega_{\widehat{W}}(H') = \sum \frac{1}{s! \prod_{i=1}^n r_i!} \left[ \begin{array}{c} \boxed{D_{\kappa}} \\ \vdots \\ \end{array} \right]_{n, n} \left. \begin{array}{l} x_n \xrightarrow{e^{x\check{h}-1} \dots} W_{ni_1} x_{i_1} \\ \vdots \\ x_n \xrightarrow{e^{x\check{h}-1} \dots} W_{ni_r} x_{i_r} \end{array} \right\} \begin{array}{l} x_n \xrightarrow{e^{x\check{h}-1} \dots} \frac{1}{2} W_{nn} \\ \vdots \\ x_n \xrightarrow{e^{x\check{h}-1} \dots} \frac{1}{2} W_{nn} \end{array} \left. \right\} s \text{ times}$$

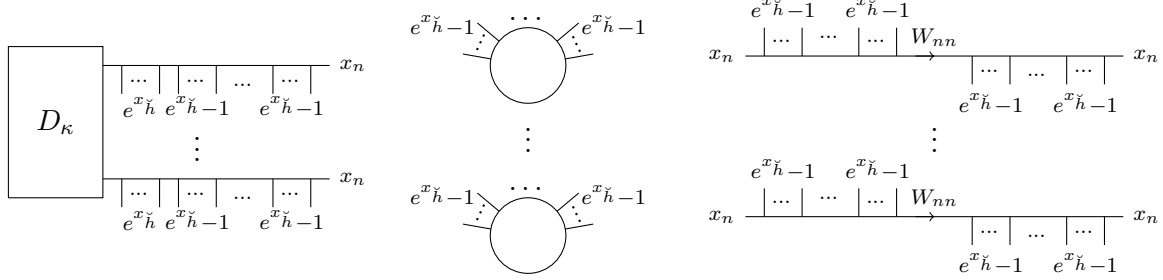
In this large sum, all combinations of values for the  $i_j$  indices arise and can be combined linearly using LV and the multinomial formula to make  $\sum_i \widehat{W}_{ni} x_i$  appear as vertex labels. Since this sum vanishes in the Blanchfield module  $\mathfrak{A}$ , we obtain

$$\omega_{\widehat{W}}(H') = \sum \frac{1}{r!s!} \left[ \begin{array}{c} \boxed{D_{\kappa}} \\ \vdots \\ \end{array} \right]_{n, n} \left. \begin{array}{l} x_n \xrightarrow{e^{x\check{h}-1} \dots} 0 \\ \vdots \\ x_n \xrightarrow{e^{x\check{h}-1} \dots} 0 \end{array} \right\} r \text{ times} \left. \begin{array}{l} x_n \xrightarrow{e^{x\check{h}-1} \dots} \frac{1}{2} W_{nn} \\ \vdots \\ x_n \xrightarrow{e^{x\check{h}-1} \dots} \frac{1}{2} W_{nn} \end{array} \right\} s \text{ times} .$$

At the level of the linkings  $f_{vv'}$ , these 0-labeled vertices are linked to other  $x_j$ -labeled vertices by  $-\sum_i \widehat{W}_{ni} (\widehat{W}^{-1})_{ij} = -\delta_{nj}$ , where  $\delta$  is the Kronecker symbol, and with other 0-labeled vertices by  $-\sum_{i,j} \widehat{W}_{ni} (\widehat{W}^{-1})_{ij} \widehat{W}_{nj} = -\widehat{W}_{nn}$ . Using iteratively the LD relation, all these 0-labeled vertices can be attached in all possible ways to the other vertices (the remaining diagrams have a 0-labeled vertex with trivial linkings with all other univalent vertices; such a diagram vanishes thanks to LV). Everytime a 0-labeled vertex is attached to an  $x_n$ -labeled vertex, it produces

a  $-1$  label on an edge; when attached to another  $0$ -labeled vertex, it produces a  $-\widehat{W}_{nn}$  label; otherwise it produces a  $0$  label which makes the whole diagram vanish thanks to LE.

These gluings allow to write  $\omega_{\widehat{W}}(H')$  as a sum of diagrams of the following form.



Let  $u$  be the number of terms in the right hand side of the picture. These terms comes from  $v$

terms  $x_n \xrightarrow{\frac{1}{2}W_{nn}} x_n$  in the original diagram, for  $0 \leq v \leq u$ , and from  $u - v$  gluings

of pairs of  $x_n \xrightarrow{e^{x_h-1}} 0$  (plus the gluing of many other  $x_n \xrightarrow{e^{x_h-1}} 0$ , but we will ignore these additive gluings which provide the same coefficient in all cases). For a given  $v$ , we have in the original diagram  $r = \rho - 2v$  and  $s = v$  for a fixed integer  $\rho \geq 2u$ . The coefficient that appears when performing the  $u - v$  gluings is

$$(-1)^{u-v} \frac{1}{(\rho - 2v)!v!} \frac{1}{2^v} \binom{\rho - 2v}{2(u-v)} \frac{(2(u-v))!}{2^{u-v}(u-v)!},$$

where the different terms, from left to right, come from the  $u - v$  gluings, the coefficient  $\frac{1}{r!s!}$ , the labels  $\frac{1}{2}W_{nn}$ , the choice of the  $2(u - v)$  terms to be glued by pairs, and the choice of the pairs. This gives  $\frac{(-1)^{u-v}}{2^u(\rho-2u)!u!} \binom{u}{v}$ , which, summed over  $v \in \{0, \dots, u\}$ , gives  $0$  when  $u > 0$ . Hence we can assume  $u = 0$ .

Finally,  $\omega_{\widehat{W}}(H')$  is a linear combination of diagrams

$$D(\kappa, k_1, \dots, k_s, \ell_1, \dots, \ell_t) = \left[ \begin{array}{c} \boxed{D_\kappa} \\ \dots \\ \underbrace{x_h \quad x_h}_{k_1} \\ \vdots \\ \underbrace{x_h \quad x_h}_{k_s} \end{array} \right] \cdot \left[ \begin{array}{c} \overbrace{x_h \quad x_h}^{\ell_1} \\ \vdots \\ \underbrace{\quad \quad}_{\ell_t} \end{array} \right].$$

We fix the parameters  $\kappa, k_1, \dots, k_s, \ell_1, \dots, \ell_t$  and we compute the coefficient  $\Lambda_D$  of the corresponding diagram. We have

$$\Lambda_D = \sum \frac{(-1)^r}{r!} \times \frac{r!}{\prod_{i=1}^t v_i} \times \frac{1}{\prod_{i=1}^s \prod_{j=0}^{u_i} k_i^j! \times \prod_{i=1}^t \prod_{j=1}^{v_i} \ell_i^j!}$$

where the sum is over the set of integers

$$\left\{ \begin{array}{l} k_1^0, \dots, k_s^0 \geq 0 \\ k_i^j, \ell_i^j \geq 1 \end{array} \middle| \begin{array}{l} k_1^0 + \dots + k_1^{u_1} = k_1 \quad \ell_1^1 + \dots + \ell_1^{v_1} = \ell_1 \\ \vdots \\ k_s^0 + \dots + k_s^{u_s} = k_s \quad \ell_t^1 + \dots + \ell_t^{v_t} = \ell_t \end{array} \right\}$$

and  $r = \sum_{i=1}^s u_i + \sum_{i=1}^t v_i$ . In the  $\frac{(-1)^r}{r!}$  factor, the  $(-1)^r$  comes from the LD relations, while the  $r!$  comes from the expansion of  $\exp_{\square} J$ ; the  $\frac{r!}{\prod_{i=1}^t v_i}$  factor comes from all the ways to arrange the elements of the expansion of  $\exp_{\square} J$  altogether using the LD relations, noting that on the circular components these elements are cyclically and not linearly arranged; and the last factor corresponds to the expansion of the  $e^{x_h} - 1$ . The renormalized coefficient  $\Lambda'_D = \prod_{i=1}^s k_s! \times \prod_{i=1}^t \ell_s! \times \Lambda_D$  factorizes as

$$\begin{aligned} \Lambda'_D &= \sum_{i=1}^s \prod_{i=1}^s (-1)^{u_i} \binom{k_i}{k_i^0, \dots, k_i^{u_i}} \prod_{i=1}^t \frac{(-1)^{v_i}}{v_i} \binom{\ell_i}{\ell_i^1, \dots, \ell_i^{v_i}} \\ &= \prod_{i=1}^s \left( \sum_{p=0}^{k_i} (-1)^p \sum_{\substack{k_i^0 + \dots + k_i^p = k_i \\ k_i^0 \geq 0, k_i^1, \dots, k_i^p \geq 1}} \binom{k_i}{k_i^0, \dots, k_i^p} \right) \times \prod_{i=1}^t \left( \sum_{p=1}^{\ell_i} \frac{(-1)^p}{p} \sum_{\substack{\ell_i^1 + \dots + \ell_i^p = \ell_i \\ \ell_i^1, \dots, \ell_i^p \geq 1}} \binom{\ell_i}{\ell_i^1, \dots, \ell_i^p} \right). \end{aligned}$$

Using the multinomial formula, it follows from an induction on  $p > 0$  and  $k \in \{0, \dots, p\}$  that

$$\sum_{\substack{a_1 + \dots + a_p = a \\ a_1, \dots, a_{p-k} \geq 0 \\ a_{p-k+1}, \dots, a_p \geq 1}} \binom{a}{a_1, \dots, a_p} = \sum_{j=0}^k (-1)^j \binom{k}{j} (p-j)^a.$$

This leads to

$$\begin{aligned} \Lambda'_D &= \prod_{i=1}^s \left( \sum_{p=0}^{k_i} \sum_{j=0}^p (-1)^{p+j} \binom{p}{j} (p+1-j)^{k_i} \right) \times \prod_{i=1}^t \left( \sum_{p=1}^{\ell_i} \sum_{j=0}^p \frac{(-1)^{p+j}}{p} \binom{p}{j} (p-j)^{\ell_i} \right) \\ &= \prod_{i=1}^s \left( \sum_{p=0}^{k_i} \sum_{j=0}^p (-1)^j \binom{p}{j} (j+1)^{k_i} \right) \times \prod_{i=1}^t \left( \sum_{p=1}^{\ell_i} \sum_{j=1}^p \frac{(-1)^j}{p} \binom{p}{j} j^{\ell_i} \right) \\ &= \prod_{i=1}^s \left( \sum_{p=0}^{k_i} \sum_{j=0}^p (-1)^j \binom{p}{j} (j+1)^{k_i} \right) \times \prod_{i=1}^t \left( \sum_{p=1}^{\ell_i} \sum_{j=1}^p (-1)^j \binom{p-1}{j-1} j^{\ell_i-1} \right) \\ &= \prod_{i=1}^s \left( \sum_{p=0}^{k_i} \sum_{j=0}^p (-1)^j \binom{p}{j} (j+1)^{k_i} \right) \times \prod_{i=1}^t \left( \sum_{p=0}^{\ell_i-1} \sum_{j=0}^p (-1)^{j+1} \binom{p}{j} (j+1)^{\ell_i-1} \right) \\ &= \prod_{i=1}^s \left( \sum_{j=0}^{k_i} (-1)^j (j+1)^{k_i} \sum_{p=j}^{k_i} \binom{p}{j} \right) \times \prod_{i=1}^t \left( \sum_{j=0}^{\ell_i-1} (-1)^{j+1} (j+1)^{\ell_i-1} \sum_{p=j}^{\ell_i-1} \binom{p}{j} \right) \end{aligned}$$



$$\begin{aligned}
&= \prod_{i=1}^s \left( \sum_{j=0}^{k_i} (-1)^j \binom{k_i+1}{j+1} (j+1)^{k_i} \right) \times \prod_{i=1}^t \left( \sum_{j=0}^{\ell_i-1} (-1)^{j+1} \binom{\ell_i}{j+1} (j+1)^{\ell_i-1} \right) \\
&= \prod_{i=1}^s \left( \sum_{j=1}^{k_i+1} (-1)^{j+1} \binom{k_i+1}{j} j^{k_i} \right) \times \prod_{i=1}^t \left( \sum_{j=1}^{\ell_i} (-1)^j \binom{\ell_i}{j} j^{\ell_i-1} \right).
\end{aligned}$$

But it is a well-known corollary of the binomial formula and its derivative that  $\sum_{j=1}^a (-1)^j \binom{a}{j} j^b$  is zero whenever  $a > b \geq 1$ . It follows that  $\Lambda_D$  is zero except if all  $k_i$  are zero, and all  $\ell_i$  are 1. But if some  $\ell_i$  is 1, then  $D$  vanishes because of the AS relation. It follows that the only non trivial terms in  $\omega_{\widehat{W}}(H')$  correspond to  $t = 0$  and  $k_1 = \dots = k_r = 0$ , hence that  $\omega_{\widehat{W}}(H') = \omega_{\widehat{W}}(H)$ .  $\square$

## 5. CONSTRUCTION OF THE INVARIANT $\tilde{Z}$

**5.1. Invariant of a surgery presentation.** We use the functor  $Z$  defined in [CHM08], which is a renormalization of the Le–Murakami functor [LM95, LM96].

The domain of this functor is the category  $\mathcal{T}_q$  with objects the non-associative words in the letters  $(+, -)$  and morphisms the  $q$ -tangles. Composition is given by vertical juxtaposition. We also define a tensor product by horizontal juxtaposition.

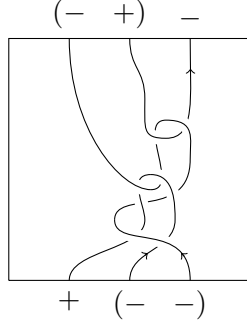


FIGURE 16. Diagram of a  $q$ -tangle

Define a category  $\tilde{\mathcal{A}}$  whose objects are associative words in the letters  $(+, -)$  and whose set of morphisms are  $\tilde{\mathcal{A}}(v, u) = \bigoplus_X \tilde{\mathcal{A}}(X)$ , where  $X$  runs over all compact oriented 1-manifolds with boundary identified with the set of letters of  $u$  and  $v$ , with the following sign convention: for  $u$ , a “+” when the orientation of  $X$  goes towards the boundary point and a “-” when it goes backward, and the converse for  $v$ . Composition is given by vertical juxtaposition, where the label of a created edge is the product of the labels on the initial two edges. The tensor product given by disjoint union defines a strict monoidal structure on  $\tilde{\mathcal{A}}$ .

We recall in Figure 17 the definition of  $Z$  on the elementary  $q$ -tangles, where  $\nu \in \mathcal{A}(\bigcirc) \cong \mathcal{A}(\uparrow)$  is the value of the Kontsevich integral on the zero framed unknot,  $\Phi \in \mathcal{A}(\downarrow\downarrow\downarrow)$  is a Drinfeld associator with rational coefficients and  $\Delta_{u_1, u_2, u_3}^{+++} : \mathcal{A}(\downarrow\downarrow\downarrow) \rightarrow \mathcal{A}(\downarrow_{u_1} u_2 u_3)$  is obtained by applying  $(|u_i| - 1)$  times the coproduct  $\Delta$  on the  $i$ -th factor.

Let  $L$  be a surgery presentation. Fixing an admissible diagram of  $L$ , one can view the surgery presentation as a  $q$ -tangle with empty top and bottom words and write it as the product of

$$\begin{aligned}
 Z \left( \begin{array}{c} (+ \ +) \\ \diagdown \ \diagup \\ (+ \ +) \end{array} \right) &= \exp \left( \frac{1}{2} \begin{array}{c} \text{---} \\ \diagdown \ \diagup \\ \text{---} \end{array} \right) \in \mathcal{A} \left( \begin{array}{c} \diagdown \ \diagup \\ \diagup \ \diagdown \end{array} \right) & Z \left( \begin{array}{c} (+ \ +) \\ \diagup \ \diagdown \\ (+ \ +) \end{array} \right) &= \exp \left( -\frac{1}{2} \begin{array}{c} \text{---} \\ \diagdown \ \diagup \\ \text{---} \end{array} \right) \in \mathcal{A} \left( \begin{array}{c} \diagdown \ \diagup \\ \diagup \ \diagdown \end{array} \right) \\
 Z \left( \begin{array}{c} \curvearrowright \\ (+ \ -) \end{array} \right) &= \begin{array}{c} \text{---} \\ \curvearrowright \\ \text{---} \end{array} \in \mathcal{A} \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) & Z \left( \begin{array}{c} (+ \ -) \\ \curvearrowleft \end{array} \right) &= \begin{array}{c} \curvearrowleft \\ \text{---} \end{array} \in \mathcal{A} \left( \begin{array}{c} \curvearrowleft \\ \curvearrowleft \end{array} \right) \\
 Z \left( \begin{array}{c} (u \ \ (vw)) \\ \downarrow \ \ \downarrow \\ ((uv) \ \ w) \end{array} \right) &= \Delta_{u,v,w}^{+++}(\Phi) \in \mathcal{A}(\downarrow uvw)
 \end{aligned}$$

FIGURE 17. The functor  $Z : \mathcal{T}_q \rightarrow \mathcal{A}$ .

two  $q$ -tangles  $\gamma_t$  and  $\gamma_b$ , see Figure 18. The word at the top of  $\gamma_b$  and at the bottom of  $\gamma_t$  is a product  $(v)(w)$ , where  $w$  corresponds to the part of the tangle which meets the disk  $\mathcal{D}$ . Set:

$$Z^\bullet(L) = Z(\gamma_b) \circ (I_v \otimes G_w) \circ Z(\gamma_t) \in \tilde{\mathcal{A}}(L),$$

where  $I_v$  is the identity on the word  $v$  and  $G_w$  is obtained from  $I_w$  by adding a label  $t$  (resp.  $t^{-1}$ ) on skeleton components associated with a  $-$  sign (resp. a  $+$  sign), see Figure 19. The invariance with respect to isotopy and to the cutting of  $\gamma$  is due to the invariance of the functor  $Z$  and the following observation of Kricker [Kri00, Lemma 3.2.4].

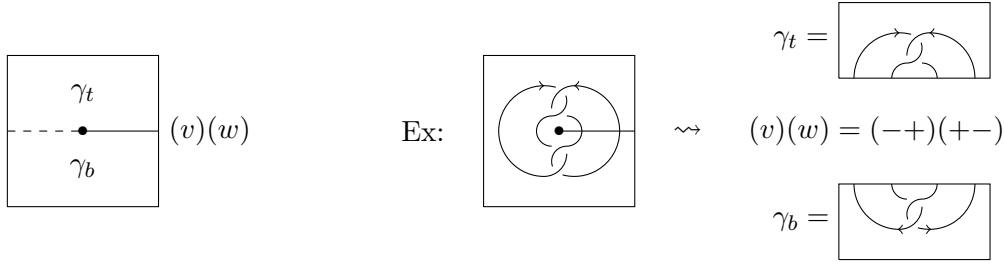


FIGURE 18. Cutting a surgery presentation into two  $q$ -tangles

$$I_{--+-} = \begin{array}{c} \uparrow \uparrow \downarrow \uparrow \end{array} \qquad G_{--+-} = \begin{array}{c} \uparrow_t \uparrow_t \downarrow_{t^{-1}} \uparrow_t \end{array}$$

FIGURE 19. The diagrams  $I_v$  and  $G_v$ .

**Lemma 5.1.** For a beaded Jacobi diagram  $D \in \tilde{\mathcal{A}}(w, v)$ , we have  $G_v \circ D = D \circ G_w$ .

*Proof.* Apply the relations Hol and Hol<sub>w</sub> at all vertices of the diagram. □

**Lemma 5.2.** For any surgery presentation  $L$ ,  $Z^\bullet(L)$  is group-like.

*Proof.* The fact that  $Z(\gamma)$  is group-like for a  $q$ -tangle  $\gamma$  follows from [LM97, Theorem 5.1]. This concludes since the  $G_v$  are obviously group-like and the coproduct commutes with the composition.  $\square$

**5.2. Invariant of  $\mathbb{Q}\text{SK}$ -pairs.** Set:

$$Z^\circ(L) = \chi^{-1} \left( \nu^{\otimes \pi_0(L)} \#_{\pi_0(L)} Z^\bullet(L) \right) \in \tilde{\mathcal{A}}(\otimes_{\pi_0(L)})$$

where the connected sum means that a copy of  $\nu$  is summed to each component of  $L$ . Note that  $Z^\circ(L)$  is group-like since  $Z^\bullet(L)$  and  $\nu$  are group-like and  $\chi$  preserves the coproduct.

Let  $\overline{Z^\circ(L)} \in \tilde{\mathcal{A}}(*_{\pi_0(L)})$  be a lift of  $Z^\circ(L)$ . Fix an admissible diagram of  $L$  and base points  $\star_i$  on each component  $L_i$  of  $L$ . Construct  $\overline{Z^\circ(L)}$  following the construction from the beginning of Section 5 for this diagram, with the skeleton components corresponding to the components of  $L$  defined as intervals by cutting each component  $L_i$  at the base point  $\star_i$ .

**Lemma 5.3.** *The lift  $\overline{Z^\circ(L)}$  is group-like and we have:*

$$\overline{Z^\circ(L)} = \exp_{\sqcup} \left( \frac{1}{2} W_L \right) \sqcup H,$$

where  $W_L$  is the winding matrix associated with our choice of diagram and base points and  $H$  is substantial.

*Proof.* Check as in Lemma 5.2 that  $\overline{Z^\circ(L)}$  is group-like. We have to compute the part of  $\overline{Z^\circ(L)}$  made of struts. The group-like property implies that  $\overline{Z^\circ(L)}$  is the exponential of a series of connected diagrams. Since  $\nu$  and the associator  $\Phi$  have no terms with exactly two vertices, the only contributions to the strut part come from the crossings between components of  $L$ . For  $i \neq j$ , the definition of  $Z$  and the  $\text{Hol}_w$  relation show that the contribution of a crossing  $c$

between  $L_i$  and  $L_j$  is  $\chi^{-1} \left( \frac{1}{2} \text{sg}(c) \begin{array}{c} \uparrow t^{\varepsilon_{ij}(c)} \uparrow \\ \left| \quad \quad \quad \right| \\ L_i \quad \quad \quad L_j \end{array} \right)$ . Hence the contribution of all crossings between

$L_i$  and  $L_j$  is  $\begin{array}{c} L_j \\ \left| (W_L)_{ij} \right. \\ L_i \end{array} = \begin{array}{c} L_i \\ \left| (W_L)_{ji} \right. \\ L_j \end{array}$ . For  $i = j$ , the contribution of a self-crossing of  $L_i$  is:

$$\chi^{-1} \left( \frac{1}{2} \text{sg}(c) \begin{array}{c} \uparrow \\ \left| t^{\varepsilon_{ii}(c)} \right. \\ L_i \end{array} \right) = \text{sg}(c) \left( \begin{array}{c} L_i \\ \left| t^{\varepsilon_{ii}(c)} \right. \\ L_i \end{array} + \frac{1}{2} \begin{array}{c} \circlearrowleft t^{\varepsilon_{ii}(c)} \\ \left| \\ L_i \end{array} \right) \right).$$

Summed over all self-crossings of  $L_i$ , we get as strut part:

$$\sum_c \frac{1}{2} \text{sg}(c) \begin{array}{c} L_i \\ \left| t^{\varepsilon_{ii}(c)} \right. \\ L_i \end{array} = \frac{1}{2} \begin{array}{c} L_i \\ \left| (W_L)_{ii} \right. \\ L_i \end{array}.$$

Hence  $\overline{Z^\circ(L)} = \exp_{\sqcup} \left( \frac{1}{2} W_L \right) \sqcup H$  where  $H \in \tilde{\mathcal{A}}(*_{\pi_0(L)})$  is substantial.  $\square$

The matrix  $W_L(1)$  is the linking matrix of the link  $L$ , hence it is a presentation matrix for the first homology group of a  $\mathbb{Q}$ -sphere. Thus  $\det(W_L(1)) \neq 0$  and  $\overline{Z^\circ(L)}$  is a non-degenerate Gaussian.

Proposition 4.5 implies the following lemma.

**Lemma 5.4.** *The diagram  $\omega(\overline{Z^\circ(L)}) \in \mathcal{A}(\mathfrak{A}, \mathfrak{b})$  does not depend on the lift  $\overline{Z^\circ(L)} \in \tilde{\mathcal{A}}(*_{\pi_0(L)})$  of  $Z^\circ(L) \in \tilde{\mathcal{A}}(\otimes_{\pi_0(L)})$ .*

This allows to set:

$$\omega(Z^\circ(L)) = \omega(\overline{Z^\circ(L)}) \in \mathcal{A}(\mathfrak{A}, \mathfrak{b}).$$

**Proposition 5.5.** *Let  $U_\pm$  be a trivial knot with framing  $\pm 1$  split from  $\mathcal{O} \subset S^3$ . For a QSK-pair  $(M, K)$  with an admissible surgery presentation  $L$  and Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$ , we denote  $\sigma_\pm(L)$  the positive/negative signature of  $L$ . Then*

$$\tilde{Z}(M, K) = Z^\circ(U_+)^{-\sigma_+(L)} \sqcup Z^\circ(U_-)^{-\sigma_-(L)} \sqcup \omega(Z^\circ(L)) \in \mathcal{A}(\mathfrak{A}, \mathfrak{b})$$

defines an invariant  $\tilde{Z}$  of QSK-pairs.

*Proof.* We have to check that  $\tilde{Z}(M, K)$  does not depend on the surgery presentation. We first consider the orientation of the components of  $L$ . Let  $L'$  be the surgery link that differs from  $L$  only by the orientation of the  $i$ -th component. By [LM96, Theorem 4],  $Z^\bullet(L')$  (and thus  $Z^\circ(L')$ ) is obtained from  $Z^\bullet(L)$  (resp.  $Z^\circ(L)$ ) by flipping the sign of all diagrams with an odd number of univalent vertices glued to the  $i$ -th skeleton component (resp. labelled by  $L_i$ ). Now, the winding matrix  $W_{L'}$  is obtained from  $W_L$  by multiplying the  $i$ -th row and column by  $-1$ , and the meridian  $m(L'_j)$  equals  $m(L_j)$  if  $j \neq i$  and  $-m(L_i)$  if  $j = i$ . Hence, using the relation LV, we see that the signs cancel and we get  $\omega(Z^\circ(L')) = \omega(Z^\circ(L))$ . It concludes since the signature is unmodified.

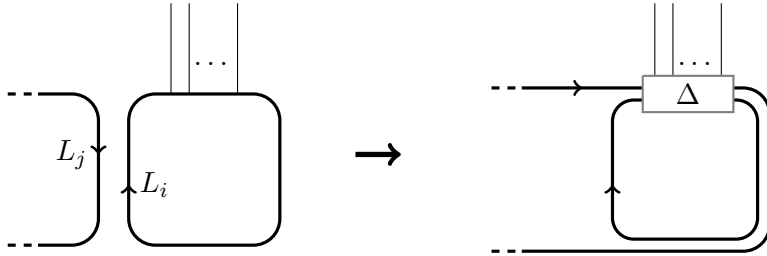


FIGURE 20. Effect of a KII move on  $\chi(Z^\circ(L))$

The  $\Delta$ -box stands for the sum over all possibilities to attach each thin line to one of the two skeleton components.

The normalization term  $Z^\circ(U_+)^{-\sigma_+(L)} \sqcup Z^\circ(U_-)^{-\sigma_-(L)}$  ensures independance with respect to the Kirby I move as usual. Independance with respect to the Kirby II move is based on a result of Le–Murakami–Murakami–Ohtsuki [LMMO99, Proposition 1] which expresses the effect of a Kirby II move on the Kontsevich integral. Set  $L' = (L \setminus L_j) \cup L_{j'}$ , where  $L_{j'}$  is obtained from  $L_j$



of values for the  $x_k$  and the  $x_\ell$  can occur, and since the sums  $\sum_k W_{kj}x_k$  and  $\sum_k W_{ki}x_k$  vanish in  $\mathfrak{A}$ , we get

$$\omega_{W'}(R) = \sum \begin{array}{c} \begin{array}{cccc} W_{ii} & W_{ii} & 0_i & 0_i \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \\ \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \\ \begin{array}{cccc} W_{jj} & W_{jj} & 0_j & 0_j \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \end{array} \cdot D_H .$$

In this picture,  $0_j$  and  $0_i$  are 0-labels for which we record whether they were coming from  $j\text{--}\check{j}$  or  $i'\text{--}\check{i}'$  gluings; it has in particular an effect on the linkings. The  $0_j$ -labeled vertices are indeed linked by  $-\sum_k W_{kj}(W^{-1})_{\ell k} = -\delta_{\ell j}$  to  $x_\ell$ -labeled vertices, by  $-\sum_{k,\ell} W_{kj}(W^{-1})_{\ell k} W_{\ell j} = -W_{jj}$  to  $0_j$ -labeled vertices and by  $-\sum_{k,\ell} W_{kj}(W^{-1})_{\ell k} W_{\ell i} = -W_{ji}$  to  $0_i$ -labeled vertices. Similarly,  $0_i$ -labeled vertices are linked by  $-1$  to  $x_i$ -labeled vertices,  $-W_{ii}$  to  $0_i$ -labeled vertices,  $-W_{ij}$  to  $0_j$ -labeled vertices and  $0$  to all other vertices. We can now use iteratively the LD relations on all  $0_j$  and  $0_i$ -labeled vertices, in all possible ways, and get

$$\omega_{W'}(R) = \sum (-1)^{k_j+k_i+k_{jj}+k_{ji}+k_{ii}} \begin{array}{c} \begin{array}{cccc} W_{ii} & W_{ii} & & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{array} \\ \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \\ \begin{array}{cccc} W_{jj} & W_{jj} & & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{array} \end{array} \cdot D_H .$$

where  $k_j$  (resp.  $k_i, k_{jj}, k_{ji}, k_{ii}$ ) is the number of gluing of  $0_j$  to  $x_j$ -vertices (resp.  $0_i$  to  $x_i, 0_j$  to  $0_j, 0_j$  to  $0_i, 0_i$  to  $0_i$ ) occurring during the LD relations step. Gathering all similar diagrams in the sum, which corresponds to all ways of separating  $D_\Lambda$ - $D_H$  arcs and  $W_{jj}, W_{ji}, W_{ii}$ -labeled arcs (we denote by  $n_j, n_i, n_{jj}, n_{ji}$  and  $n_{ii}$  the number, for each kind, of these arcs) into those which come from the  $\langle \rangle_{j\text{--}\check{j}, i'\text{--}\check{i}'}$  bracket or from the LD relations, we get a overall factor

$$\begin{aligned} \sum (-1)^{k_j} \binom{n_j}{k_j} \cdot (-1)^{k_i} \binom{n_i}{k_i} \cdot (-1)^{k_{jj}} \binom{n_{jj}}{k_{jj}} \cdot (-1)^{k_{ji}} \binom{n_{ji}}{k_{ji}} \cdot (-1)^{k_{ii}} \binom{n_{ii}}{k_{ii}} \\ = (1-1)^{n_j+n_i+n_{jj}+n_{ji}+n_{ii}} = 0, \end{aligned}$$

meaning that  $\omega_{W'}(R) = \emptyset$  and  $\omega(G') = \omega_{W'}\left(H \Big|_{j \rightarrow j'}^{i \rightarrow i+j'}\right) = \omega\left(G \Big|_{j \rightarrow j'}^{i \rightarrow i+j'}\right)$ .

From  $\omega(G)$  to  $\omega\left(G \Big|_{j \rightarrow j'}^{i \rightarrow i+j'}\right)$ , the labels  $m(L_j)$  and  $m(L_i)$  are respectively replaced by  $m(L_{j'})$  and  $m(L_i) + m(L_{j'})$  and the winding matrix is modified accordingly. Since  $m(L_j) = m(L_{j'})$  and

$m(L_i) = m(L_{i'}) + m(L_{j'})$  (see Figure 21), we get  $\omega(G) = \omega\left(G\Big|_{\substack{i \rightarrow i+j' \\ j \rightarrow j'}}\right)$ , so that  $\omega(Z^\circ(L')) = \omega(Z^\circ(L))$ . Once again, the signature is preserved.  $\square$

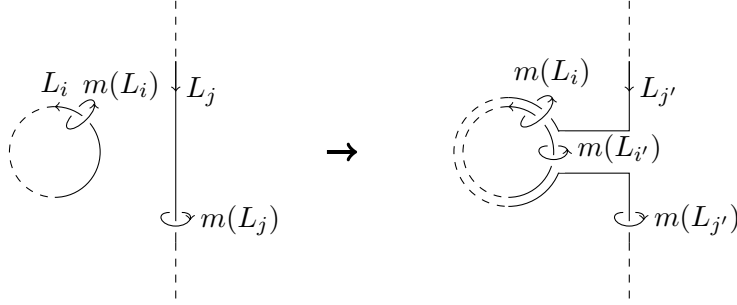


FIGURE 21. KII move and meridians

**5.3. Recovering the Kricker invariant.** We now explicit the fact that our invariant  $\tilde{Z}$  is a refinement of the Kricker invariant  $Z^{\text{Kri}}$ .

The construction of the invariant  $Z^{\text{Kri}}$  is the same as for  $\tilde{Z}$  until the last step. Instead of applying the operation  $\omega$ , Garoufalidis and Kricker apply a formal Gaussian integral which merges the strut part  $\exp_{\square}(\frac{1}{2}W_L)$  and the substantial part  $H$  by summing all possible ways to glue all vertices of  $\exp_{\square}(-\frac{1}{2}W_L^{-1})$  with all vertices of  $H$  that have the same label. This defines an invariant with values in the space  $\mathcal{A}(\delta)$ . Now, applying the operation  $\omega$  first and the map  $\psi : \mathcal{A}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}(\delta)$  then has the same effect as applying the formal Gaussian integral.

**Proposition 5.6.** *For any QSK-pair  $(M, K)$ , we have  $Z^{\text{Kri}}(M, K) = \psi \circ \tilde{Z}(M, K)$ .*

*Remark.* The formal Gaussian integration was initially introduced by Bar-Natan, Garoufalidis, Rozansky and Thurston to define the Aarhus integral, which recovers the LMO invariant. This version of the LMO invariant is constructed as the Kricker invariant, forgetting the knots in the 3-manifolds and the beads on the diagrams.

*Remark.* Since  $Z^{\text{Kri}}$  can be deduced from  $\tilde{Z}$ , the invariance of  $Z^{\text{Kri}}$  and its behaviour with respect to null LP-surgeries stem from that of  $\tilde{Z}$ .

**5.4. Behaviour of  $\tilde{Z}$  under connected sum.** By construction, the invariant  $\tilde{Z}$  behaves well under connected sum.

**Lemma 5.7.** *Let  $(M_1, K_1)$  and  $(M_2, K_2)$  be QSK-pairs. Let  $(\mathfrak{A}_1, \mathfrak{b}_1)$  and  $(\mathfrak{A}_2, \mathfrak{b}_2)$  denote their Blanchfield modules. The invariant  $\tilde{Z}$  is given on their connected sum by:*

$$\tilde{Z}((M_1, K_1)\sharp(M_2, K_2)) = \tilde{Z}(M_1, K_1) \sqcup \tilde{Z}(M_2, K_2) \in \mathcal{A}((\mathfrak{A}_1, \mathfrak{b}_1) \oplus (\mathfrak{A}_2, \mathfrak{b}_2)).$$

*Proof.* If  $L_1$  and  $L_2$  are surgery links for  $(M_1, K_1)$  and  $(M_2, K_2)$  respectively, a surgery link for  $(M_1, K_1)\sharp(M_2, K_2)$  is obtained by stacking  $L_1$  and  $L_2$ . Then  $L_1$  and  $L_2$  can be separated by an isotopy, see Figure 22.  $\square$

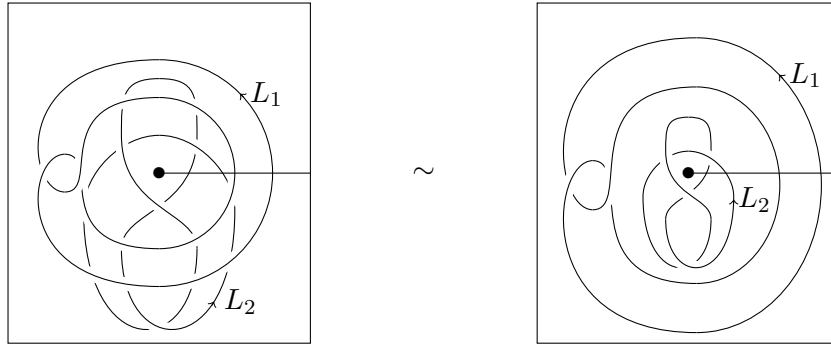


FIGURE 22. Stacking admissible diagrams.

## 6. UNIVERSALITY

We want to describe the behaviour of the invariant  $\tilde{Z}$  under null LP–surgeries. For this, we fix an abstract Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  and we restrict to QSK–pairs whose Blanchfield module is isomorphic to  $(\mathfrak{A}, \mathfrak{b})$ . For such a QSK–pair  $(M, K)$ , in order to see  $\tilde{Z}(M, K)$  in the diagram space  $\mathcal{A}(\mathfrak{A}, \mathfrak{b})$ , we need to fix an isomorphism from the Blanchfield module of  $(M, K)$  to  $(\mathfrak{A}, \mathfrak{b})$ . However, the relation  $\text{Aut}$  implies that the value of  $\tilde{Z}(M, K) \in \mathcal{A}(\mathfrak{A}, \mathfrak{b})$  does not depend on the chosen isomorphism, so that we will ignore it in the sequel.

**6.1. Preliminaries: the LMO invariant.** We recall here some properties of the LMO invariant we will need below. This invariant  $Z^{\text{LMO}}$  of  $\mathbb{Q}$ –spheres, constructed by Le–Murakami–Ohtsuki in [LMO98], is valued in the graded space  $\mathcal{A}(\emptyset)$  of trivalent diagrams with oriented trivalent vertices, quotiented out by the AS and IHX relations. The degree of a diagram is the number of its vertices; in particular,  $\mathcal{A}_n = 0$  when  $n$  is odd. Finiteness properties for this invariant were established by Le [Le97] with respect to borromean surgeries and generalized by Massuyeau [Mas15] to LP–surgeries. It follows that the LMO invariant induces a map on the graded space  $\mathcal{G}$  associated to finite type invariants of  $\mathbb{Q}$ –spheres with respect to LP–surgeries. Further, the map  $\varphi : \mathcal{A}(\emptyset) \rightarrow \mathcal{G}$  constructed in [GGP01] is defined as in Subsection 2.4 (without univalent vertices to deal with).

**Theorem 6.1** ([Le97, Mas15]). *The LMO invariant induces a map  $Z^{\text{LMO}} : \mathcal{G} \rightarrow \mathcal{A}(\emptyset)$  and the composition  $Z^{\text{LMO}} \circ \varphi$  is the identity on  $\mathcal{A}(\emptyset)$ .*

**6.2. Elementary surgeries.** To understand the behaviour of our invariant under null LP–surgeries, we will work on a restricted set of surgeries which generate all of them.

Given a positive integer  $d$ , we define a  $d$ –torus as a  $\mathbb{Q}$ –torus  $T_d$  satisfying, for some simple closed curves  $\alpha$  and  $\beta$  on  $\partial T$ :

- $H_1(\partial T_d; \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ , with  $\langle \alpha, \beta \rangle = 1$ ,
- $d\alpha = 0$  in  $H_1(T_d; \mathbb{Z})$ ,
- $\beta = d\gamma$  in  $H_1(T_d; \mathbb{Z})$ , where  $\gamma$  is a curve in  $T_d$ ,
- $H_1(T_d; \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}\alpha \oplus \mathbb{Z}\gamma$ .

We define a  $d$ –surgery as an LP–replacement of a solid torus by a  $d$ –torus. Finally, we define an *elementary surgery* as an LP–surgery among the following ones:



- connected sum (genus 0),
- $d$ -surgery (genus 1),
- borromean surgery (genus 3).

*Remark.* In terms of diagrams, the borromean surgeries will correspond to Jacobi diagrams, while connected sums will correspond to isolated vertices. Although it does not seem possible to remove the genus-1 elementary surgeries, they don't have a diagrammatic counterpart.

**Theorem 6.2** ([Mou12a] Theorem 1.15). *If  $A$  and  $B$  are two  $\mathbb{Q}$ -handlebodies with  $LP$ -identified boundaries, then  $B$  can be obtained from  $A$  by a finite sequence of elementary surgeries and their inverses in the interior of the  $\mathbb{Q}$ -handlebodies.*

In the proof of this theorem, arbitrary  $d$ -tori are used, so that we can reduce the genus-1 elementary surgeries to that defined by a fixed  $d$ -torus for each positive integer  $d$ . Here we will use the  $d$ -torus obtained from a standard solid torus by Dehn surgery on the link  $J_1 \cup J_2$  described in Figure 23; we denote it by  $T_d$  in the sequel. Note that  $T_1$  is the standard solid torus.

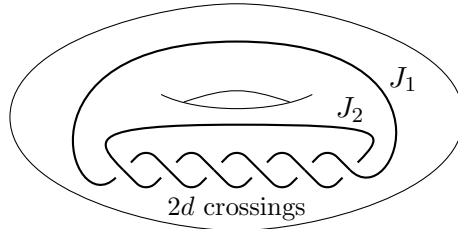


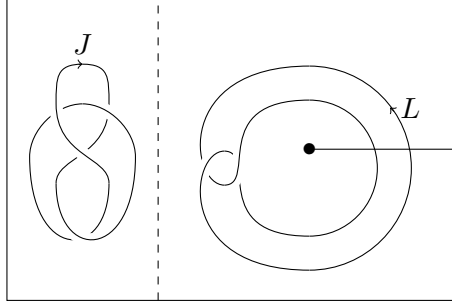
FIGURE 23. A  $d$ -torus constructed by Dehn surgery

**6.3. Behaviour of  $\tilde{Z}$  with respect to elementary surgeries.** A first step is to describe the behaviour of  $\tilde{Z}$  under connected sum. The idea is the same as in Lemma 5.7, but we now connect-sum with a  $\mathbb{Q}$ -sphere instead of a  $\mathbb{Q}SK$ -pair.

**Lemma 6.3.** *Let  $(M, K)$  be a  $\mathbb{Q}SK$ -pair with Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$ . Let  $N$  be a  $\mathbb{Q}$ -sphere. The invariant  $\tilde{Z}$  satisfies:*

$$\tilde{Z}(M \sharp N, K) = \tilde{Z}(M, K) \sqcup Z^{LMO}(N) \in \mathcal{A}(\mathfrak{A}, \mathfrak{b}).$$

*Proof.* If  $L$  and  $J$  are surgery links for  $(M, K)$  and  $N$  respectively, a surgery link for  $(M \sharp N, K)$  is obtained by stacking  $L$  and  $J$ , see Figure 24. At each step of the construction of the invariant  $\tilde{Z}$ , we have a disjoint union of two series of diagrams associated to  $L$  and  $J$  respectively. Moreover, the winding matrix of  $L \sqcup J$  is a bloc diagonal matrix with blocs  $W_L$  and  $W_J$ . Since the components of  $J$  do not meet the disk bounded by the unknot of the surgery presentation,  $W_J$  has all its coefficients in  $\mathbb{Z}$ . This implies that the Blanchfield module of  $(M \sharp N, K)$  is again  $(\mathfrak{A}, \mathfrak{b})$ . Further, the diagrams in  $\tilde{Z}(M \sharp N, K)$  coming from  $J$  have all their univalent vertices labeled by zero, so that these univalent vertices can be removed using the relation LV. By construction, this part of  $\tilde{Z}(M \sharp N, K)$  coming from  $J$  is the LMO invariant of  $N$  computed with the Aarhus method.  $\square$


 FIGURE 24. Stacking diagrams for a  $\mathbb{Q}$ -sphere and a  $\mathbb{Q}$ SK-pair.

Our second step is to describe the behaviour of  $\tilde{Z}$  under  $d$ -surgeries.

**Proposition 6.4.** *Let  $(M, K)$  be a  $\mathbb{Q}$ SK-pair. Fix a positive integer  $d$ . Consider a  $d$ -surgery  $\left(\frac{T_d}{T_1}\right)$  on  $(M, K)$ . Denote  $J_1 \sqcup J_2^d$  the surgery link defined on Figure 23. Let  $L$  be an admissible surgery presentation of  $(M, K)$ . Then  $L_1 = L \sqcup J_1 \sqcup J_2^1$  and  $L_d = L \sqcup J_1 \sqcup J_2^d$  are admissible surgery presentations of  $(M, K)$  and  $(M, K) \left(\frac{T_d}{T_1}\right)$  respectively, and  $\tilde{Z} \left( (M, K) \left(\frac{T_d}{T_1}\right) \right) - \tilde{Z}(M, K)$  is a series of diagrams of degree at least 1 containing a univalent vertex associated to  $J_2$ .*

*Proof.* The winding matrix of  $L_d$  is of the form  $W_d = \begin{pmatrix} W_L & \zeta & 0 \\ {}^t\bar{\zeta} & \lambda & d \\ 0 & d & 0 \end{pmatrix}$ , where  $W_L$  is the winding matrix of  $L$  and  $\zeta, \lambda$  do not depend on  $d$ . Hence the Blanchfield form of the pair  $(M, K)_d = (M, K) \left(\frac{T_d}{T_1}\right)$  is given by the matrix  $W_d^{-1} = \begin{pmatrix} W_L^{-1} & 0 & \eta_d \\ 0 & 0 & \frac{1}{d} \\ {}^t\bar{\eta}_d & \frac{1}{d} & \mu_d \end{pmatrix}$ , where  $\eta_d = -\frac{1}{d}W_L^{-1}\zeta$  and  $\mu_d = \frac{1}{d^2}({}^t\bar{\zeta}W_L^{-1}\zeta - \lambda)$ . If  $x_1^d, \dots, x_n^d, y_1^d, y_2^d$  is the associated basis of the Alexander module, then an automorphism  $f_d$  between the Blanchfield modules of  $(M, K)$  and  $(M, K)_d$  is given by  $x_i^1 \mapsto x_i^d$ ,  $y_1^1 \mapsto y_1^d$  and  $y_2^1 \mapsto dy_2^d$ ; we may note that  $y_1^d = 0$  and  $y_2^d$  is a  $\mathbb{Q}[t^{\pm 1}]$ -linear combination of the  $x_i^d$ .

In the computation of  $Z^\bullet(L_d)$ , the only part depending on  $d$  occurs at the level of the crossings between  $J_1$  and  $J_2$ , which can be presented as in the left hand side of Figure 25; the right hand side represents its contribution to  $Z^\bullet(L_d)$ . When applying  $\chi^{-1}$  to the right hand side of Figure 25, we get some struts which contribute to the winding matrix  $W_d$  and “correction terms” with at least one univalent vertex labeled by  $J_2$  and joined to a trivalent vertex. The latter contribute to  $H_d$  in  $\overline{Z^\circ(L_d)} = \exp_{\sqcup} \left(\frac{1}{2}W_d\right) \sqcup H_d$ . We get then  $\tilde{Z}((M, K)_d) = \omega_{W_d}(H_d)$ . To compare  $\tilde{Z}((M, K)_d)$  to  $\tilde{Z}(M, K)$ , we apply the above automorphism  $f_d$  to  $\omega_{W_1}(H_1)$ . The diagrams in the difference  $\tilde{Z}((M, K)_d) - \tilde{Z}(M, K)$  come from the above correction terms; they have degree at least one and one univalent vertex labeled by  $y_2^d$ .  $\square$

The last step is to describe the behaviour of  $\tilde{Z}$  under borromean surgeries. Once again, we have to deal with a local modification of the surgery presentation. To a  $Y$ -graph is associated a six-component surgery link, which can be turned to a trivial surgery link by separating the

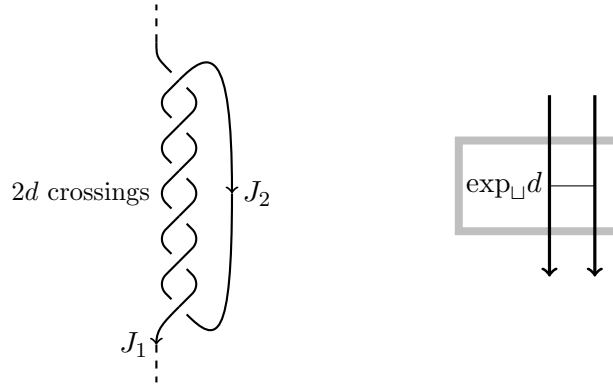


FIGURE 25. The local difference on the surgery presentation and on  $Z^\bullet$

central three components, see Figure 26. The effect of such a difference on the invariant  $Z$  has been computed by Le in [Le97].

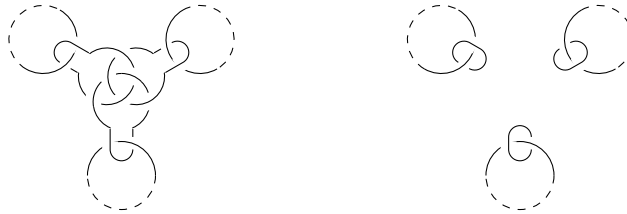


FIGURE 26. Surgery link associated to a Y-graph and corresponding trivial surgery

**Theorem 6.5** (Le).

$$Z \left( \text{diagram with crossings} \right) - Z \left( \text{diagram with arcs} \right) = \text{diagram with Y-junction} + \text{higher degree terms}$$

**6.4. Finiteness properties of  $\tilde{Z}$ .** The above computations provide us with the finiteness properties we seeked for the invariant  $\tilde{Z}$ .

**Theorem 6.6.** *The degree- $n$  part  $\tilde{Z}_n$  of the invariant  $\tilde{Z}$  is a degree- $n$  finite type invariant of QSK-pairs with respect to null LP-surgeries. Moreover,  $\tilde{Z}_n$  vanishes on any order- $n$  bracket containing a genus-0 elementary surgery.*

*Proof.* We proceed by induction on  $n$ . The image of  $\tilde{Z}_0$  on any QSK-pair is the empty diagram with coefficient 1, so that  $\tilde{Z}_0(\mathcal{F}_1) = 0$ . Fix  $n > 0$ . It is easily deduced from Theorem 6.2 that  $\mathcal{F}_{n+1}$  is generated by the brackets defined by elementary surgeries (see [Mou19, Corollary 5.5]).

Consider a bracket  $[(M, K); c_1, \dots, c_n]$  where the surgery  $c_n$  is the connected sum with some  $\mathbb{Q}$ -sphere  $N$ . We have

$$[(M, K); c_1, \dots, c_n] = [(M, K); c_1, \dots, c_{n-1}] - [(M \sharp N, K); c_1, \dots, c_{n-1}],$$

so that, by Lemma 6.3,

$$\tilde{Z}([(M, K); c_1, \dots, c_n]) = \tilde{Z}([(M, K); c_1, \dots, c_{n-1}]) \sqcup (\emptyset - Z^{\text{LMO}}(N)).$$

In the right hand side of the latter equality, the first term contains only diagrams of degree at least  $n - 1$ , by induction, and the second term contains only diagrams of degree at least 2 (see Theorem 6.1). Hence  $\tilde{Z}([(M, K); c_1, \dots, c_n])$  is made of diagrams of degree at least  $n + 1$ . Further, if an order- $n$  bracket is defined by elementary surgeries of genus 1 and 3, the local contributions explicited in Proposition 6.4 and Theorem 6.5 combine, so that the image of this bracket by  $\tilde{Z}$  contains only diagrams of degree at least  $n$ .  $\square$

This result implies that  $\tilde{Z}$  defines a map  $\tilde{Z} : \mathcal{G}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}(\mathfrak{A}, \mathfrak{b})$ .

We shall prove that  $\tilde{Z}_n$  also vanishes on order- $n$  brackets containing a genus-1 elementary surgery. Given a  $\mathbb{Q}$ -torus  $T$ , a *meridian* of  $T$  is a simple closed curve on  $\partial T$  that generates the Lagrangian of  $T$ ; it is well-defined up to isotopy. A *longitude* of  $T$  is a simple closed curve on  $\partial T$  that intersects the meridian exactly once. A *framed  $\mathbb{Q}$ -torus* is a  $\mathbb{Q}$ -torus with a fixed oriented longitude. Note that any two framed  $\mathbb{Q}$ -tori have a canonical LP-identification of their boundaries, which identifies the fixed longitudes. We define finite type invariant of framed  $\mathbb{Q}$ -tori with respect to LP-surgeries as we defined finite type invariants of QSK-pairs with respect to null LP-surgeries.

**Proposition 6.7** ([Mou12a, Corollary 5.10]). *For each prime integer  $p$ , let  $M_p$  be a  $\mathbb{Q}$ -sphere such that  $|H_1(M_p; \mathbb{Z})| = p$ . If  $\mu$  is a degree 1 invariant of framed  $\mathbb{Q}$ -tori, such that  $\mu(T_0) = 0$  and  $\mu(T_0 \sharp M_p) = 0$  for any prime  $p$ , then  $\mu = 0$ .*

**Corollary 6.8.** *For all  $n > 0$ , the invariant  $\tilde{Z}_n$  vanishes on any order- $n$  bracket containing a genus-1 elementary surgery.*

*Proof.* Consider a bracket  $[(M, K); c_1, \dots, c_n]$  where  $c_n = \left(\frac{T_d}{T_0}\right)$  is a genus-1 elementary surgery. Fix an oriented longitude of  $T_0$ . For any framed  $\mathbb{Q}$ -torus  $T$ , set

$$\lambda(T) = \tilde{Z}_n \left( \left[ (M, K); c_1, \dots, c_{n-1}, \frac{T}{T_0} \right] \right).$$

Then  $\lambda$  is a degree 1 invariant of framed  $\mathbb{Q}$ -tori: if  $s_1, s_2$  are two disjoint LP-surgeries on  $T$ ,

$$\lambda([T; s_1, s_2]) = \tilde{Z}_n \left( - \left[ (M, K) \left( \frac{T}{T_0} \right); c_1, \dots, c_{n-1}, s_1, s_2 \right] \right) = 0.$$

We have  $\lambda(T_0) = 0$ . Moreover, if  $M_p$  is a  $\mathbb{Q}$ -sphere such that  $|H_1(M_p; \mathbb{Z})| = p$  and  $B_p$  is the  $\mathbb{Q}$ -ball obtained from  $M_p$  by removing an open 3-ball, then

$$\lambda(T_0 \sharp M_p) = \tilde{Z}_n \left( \left[ (M, K); c_1, \dots, c_{n-1}, \frac{B_p}{B^3} \right] \right) = 0,$$

thanks to Theorem 6.6. Finally, by Proposition 6.7,  $\lambda = 0$ .  $\square$

We finally explicit the behaviour of  $\tilde{Z}_n$  on brackets defined by borromean surgeries. We start with a standard lemma.

**Lemma 6.9.** *Let  $J_1$  and  $J_2$  be disjoint knots in a 3-manifold  $M$  such that  $J_2$  is a meridian of  $J_1$  with 0 framing. Then the surgery on  $J_1 \sqcup J_2$  preserves the manifold  $M$  and a meridian of  $J_1$ , and changes a meridian of  $J_2$  into a longitude of  $J_1$ .*

*Proof.* Let  $\nu(J_1)$  and  $\nu(J_2)$  be tubular neighborhoods of  $J_1$  and  $J_2$  such that  $\nu(J_1) \cap \nu(J_2)$  is an annulus whose core is simultaneously a meridian of  $J_1$  and a longitude of  $J_2$ . The union  $T = \nu(J_1) \cup \nu(J_2)$  is a solid torus which shares a meridian with  $J_1$ . For  $i = 1, 2$ , the surgery replaces  $\nu(J_i)$  by a torus  $T_i$  whose meridian is a former longitude of  $J_i$ . Hence the core of  $\nu(J_1) \cap \nu(J_2)$  is a longitude of  $T_1$  and a meridian of  $T_2$ , so that  $T' = T_1 \cup T_2$  is again a solid torus with the same meridian as  $T$ . Further, a former meridian of  $J_2$  can be slid over a meridian disk of  $T_1$ , so that it is isotopic to a former longitude of  $J_1$ .  $\square$

**Proposition 6.10.** *For all  $n \geq 0$ , the composition  $\tilde{Z}_n \circ \varphi_n$  is the identity of  $\mathcal{A}_n(\mathfrak{A}, \mathfrak{b})$ .*

*Proof.* It is enough to prove it on the elementary diagrams introduced in Subsection 2.4 to define the map  $\varphi$ . Let  $D$  be such an elementary diagram, of degree  $n$ . The trivalent edges can be cut open using the relation LD to insert two univalent vertices labeled by zero (with the notations of Figure 5, one can set  $f_{v_1 v_2}^{D'} = 0$ , so that the diagram  $D'$  is trivial), thus we can assume that

$D$  is a union of graphs  $\begin{array}{c} \gamma_{3k} \\ \diagdown \quad \diagup \\ \gamma_{3k-2} \end{array}$  for  $k = 1, \dots, n$ , with fixed linkings  $f_{ij}$  between the

vertices labelled  $\gamma_i$  and  $\gamma_j$  respectively. Let  $\Gamma$  be an associated null Y-link in  $M \setminus K$  following the rules of Subsection 2.4. The  $6n$  components associated to  $\Gamma$  in a surgery link are the  $3n$  leaves  $\ell_i$  corresponding to the labels  $\gamma_i$  and  $3n$  components  $k_i$  corresponding to the edges of  $\Gamma$ , with coherent numbering. Thanks to Theorem 6.5, in  $\tilde{Z}_n \circ \varphi_n(D)$ , the only degree- $n$  term is

the union of the  $n$  graphs  $\begin{array}{c} x_{3k} \\ \diagdown \quad \diagup \\ x_{3k-2} \end{array}$ , where  $x_i$  is the generator of  $\mathfrak{A}$  associated with  $k_i$ . By

definition of the operation  $\omega$ ,  $x_i$  equals in  $\mathfrak{A}$  the homology class of a meridian  $m(\tilde{k}_i)$  of a lift  $\tilde{k}_i$  in the infinite cyclic covering of  $M \setminus K$ , and the fixed linking associated to the vertices labeled  $x_i$  and  $x_j$  is the equivariant linking  $\text{lk}_e(m(\tilde{k}_i), m(\tilde{k}_j))$ . Here, we work in  $(M, K)$ , so that the  $k_i$  are as represented on the right hand side of Figure 26. Hence we can apply Lemma 6.9 to conclude that  $m(\tilde{k}_i)$  is isotopic to  $\tilde{\ell}_i$ , so that  $x_i = \gamma_i$  and  $\text{lk}_e(m(\tilde{k}_i), m(\tilde{k}_j)) = f_{ij}$ . Eventually, we get  $\tilde{Z}_n \circ \varphi_n(D) = D$ .  $\square$

**6.5. Augmented invariant and universality.** The degree-1 invariants  $\rho_p$  can be merged with  $\tilde{Z}$  into a universal finite type invariant of QSK-pairs by setting, for a QSK-pair  $(M, K)$ :

$$\tilde{Z}^{\text{aug}}(M, K) = \tilde{Z}(M, K) \sqcup \exp_{\sqcup} \left( \sum_{p \text{ prime}} \rho_p(M) \right).$$

The following formula is classical and holds for any objects and any invariants with values in some ring (see for instance [Mou12a, Lemma 6.2]).

$$\left( \prod_{j=1}^n \lambda_j \right) \left( [(M, K); (c_i)_{i \in I}] \right) = \sum_{\emptyset = J_0 \subset \dots \subset J_n = I} \prod_{j=1}^n \lambda_j \left( [(M, K) \left( (c_i)_{i \in J_{j-1}} \right); (c_i)_{i \in J_j \setminus J_{j-1}}] \right)$$

It follows that the degree of finite type invariants is subadditive under multiplication. Moreover, the degree- $n$  part of  $\tilde{Z}^{\text{aug}}$  is given by

$$\tilde{Z}_n^{\text{aug}} = \sum_{k=0}^n \sum_{\substack{p_1 < \dots < p_s \\ \text{prime integers}}} \sum_{\substack{t_1 + \dots + t_s = n-k \\ t_i > 0}} \tilde{Z}_k \sqcup \left( \prod_{i=1}^s \frac{1}{t_i!} (\rho_{p_i})^{t_i} \right).$$

We deduce that  $\tilde{Z}_n^{\text{aug}}$  is a finite type invariant of degree  $n$ , so that the invariant  $\tilde{Z}^{\text{aug}}$  induces a map  $\tilde{Z}^{\text{aug}} : \mathcal{G}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b})$ .

**Theorem 6.11.** *The invariant  $\tilde{Z}^{\text{aug}}$  induces the inverse of the map  $\varphi : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}, \mathfrak{b})$ .*

*Proof.* We know from [Mou19, Theorem 2.7] that  $\varphi$  is surjective, so that it is enough to prove that  $\tilde{Z}^{\text{aug}} \circ \varphi$  is the identity. Let  $D$  be an  $(\mathfrak{A}, \mathfrak{b})$ -augmented diagram of degree  $n$ . Write  $D$  as the disjoint union of its Jacobi part  $D_J$  and its 0-valent part  $D_\bullet$ . Apply the above formula, noting that for a term in the right hand side of the obtained equality to be non trivial:

- the order of each bracket must be exactly the degree of the corresponding invariant,
- each invariant  $\rho_p$  must be evaluated on a bracket associated to the diagram  $\bullet_p$ ,
- the invariant  $\tilde{Z}_k$  must be evaluated on a bracket associated to a diagram without isolated vertices.

It follows that  $\tilde{Z}_n^{\text{aug}} \circ \varphi_n(D) = (\tilde{Z}_k \circ \varphi_k(D_J)) \sqcup D_\bullet = D_J \sqcup D_\bullet = D$ , where the third equality is due to Proposition 6.10.  $\square$

**6.6. Kriker invariant versus Lescop invariant.** The description of the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$  allows us to prove that the Kriker invariant and the Lescop invariant induce the same map on  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$ .

**Proposition 6.12.** *For a given Blanchfield module  $(\mathfrak{A}, \mathfrak{b})$  with annihilator  $\delta$ , the map induced by both the Kriker invariant and the Lescop invariant on the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$  is the map  $\psi \circ \tilde{Z} : \mathcal{G}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}(\delta)$ .*

*Proof.* If  $Z$  stands for  $Z^{\text{Kri}}$  or  $Z^{\text{Les}}$ , then  $Z \circ \varphi = \psi : \mathcal{A}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}(\delta)$  (see Theorem 2.2). Further,  $Z$  is multiplicative under connected sum, so that  $Z_n$  vanishes on order- $n$  brackets that contain a genus-0 elementary surgery (by the same argument as in Theorem 6.6). Thus  $Z \circ \varphi = \hat{\psi} : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}(\delta)$ , where we define  $\hat{\psi} : \mathcal{A}^{\text{aug}}(\mathfrak{A}, \mathfrak{b}) \rightarrow \mathcal{A}(\delta)$  as follows: for an augmented  $(\mathfrak{A}, \mathfrak{b})$ -colored diagram  $D$ ,  $\hat{\psi}(D) = 0$  if  $D$  contains an isolated vertex and  $\hat{\psi}(D) = \psi(D)$  otherwise. To conclude, we note that  $\hat{\psi} \circ \tilde{Z}^{\text{aug}} = \psi \circ \tilde{Z}$ .  $\square$

Here, we are able to explicit and compare  $Z^{\text{Kri}}$  and  $Z^{\text{Les}}$  on the graded space  $\mathcal{G}(\mathfrak{A}, \mathfrak{b})$ . In the case of  $Z^{\text{Kri}}$ , our refinement  $\tilde{Z}$  gives an expression of  $Z^{\text{Kri}}$  on the whole  $\mathcal{F}_0(\mathfrak{A}, \mathfrak{b})$  as the composition of the universal invariant  $\tilde{Z}$  and the explicit map  $\psi$  (Proposition 5.6). A similar refinement of  $Z^{\text{Les}}$  would give an alternative construction of a universal invariant and would enable us to compare more deeply the Kriker and Lescop invariants.

7. KNOTS IN  $\mathbb{Z}$ -SPHERES

In this section, we describe the differences that appear when we restrict our study to  $\mathbb{Z}\text{SK}$ -pairs. Let  $\mathcal{F}_0^{\mathbb{Z}}$  be the  $\mathbb{Q}$ -vector space generated by all  $\mathbb{Z}\text{SK}$ -pairs up to orientation-preserving homeomorphism. In the work of Garoufalidis–Kriker–Rozansky, the considered surgery move is the null borromean surgery (called null-move in [GR04]). One can also consider null  $\mathbb{Z}\text{LP}$ -surgeries, defined as the null  $\text{LP}$ -surgeries using  $\mathbb{Z}$ -handlebody and the  $\mathbb{Z}$ -homology. It turns out that these two moves define the same filtration of  $\mathcal{F}_0^{\mathbb{Z}}$ : Auclair and Lescop proved that any null  $\mathbb{Z}\text{LP}$ -surgery can be realized by a borromean surgery on a null  $Y$ -link [AL05, Lemma 4.11]. We denote  $\mathcal{G}^{\mathbb{Z}}$  the associated graded space.

To a  $\mathbb{Z}\text{SK}$ -pair  $(M, K)$ , we associate the *integral Alexander module*  $\mathfrak{A}_{\mathbb{Z}}(M, K)$  defined as the  $\mathbb{Z}[t^{\pm 1}]$  module  $H_1(\tilde{X}; \mathbb{Z})$ , with the notations of Section 2. The Blanchfield form is defined on  $\mathfrak{A}_{\mathbb{Z}}$  similarly. We may note that  $\mathfrak{A}(M, K) = \mathfrak{A}_{\mathbb{Z}}(M, K) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathfrak{A}_{\mathbb{Z}}(M, K)$  has no  $\mathbb{Z}$ -torsion (see [Mou15, Lemma 5.5]). Once again, the classes of  $\mathbb{Z}\text{SK}$ -pairs up to null  $\mathbb{Z}\text{LP}$ -surgeries are characterized by the isomorphism classes of integral Blanchfield modules. Hence our filtration splits along the isomorphism classes of integral Blanchfield modules. We shall fix an abstract integral Blanchfield module  $(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$  and consider the associated graded space  $\mathcal{G}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$ .

At the level of diagram spaces, we again use rational coefficients and univalent vertices labeled in the rational Alexander module; the only difference occurs in the definition of the relation  $\text{Aut}$ . On  $(\mathfrak{A}, \mathfrak{b})$ -colored diagrams, the relation  $\text{Aut}^{\mathbb{Z}}$  is reduced to automorphisms of the Blanchfield module that are induced by automorphisms of the integral Blanchfield module. This is restrictive in general (see [Mou19, remark after Proposition 7.9]), so that the associated diagram space  $\mathcal{A}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$  might be richer. At the level of invariants, we consider the same invariants  $Z^{\text{Kri}}$  and  $Z^{\text{Les}}$ , and the invariant  $\tilde{Z}^{\mathbb{Z}}$  is defined using the same construction, but with values in  $\mathcal{A}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$ . Note that there is no “augmented” diagrams or invariants in this setting.

In [Mou19, Theorem 2.17], a canonical surjective  $\mathbb{Q}$ -linear map  $\varphi^{\mathbb{Z}} : \mathcal{A}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b}) \rightarrow \mathcal{G}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$  is constructed.

**Theorem 7.1.** *The invariant  $\tilde{Z}^{\mathbb{Z}}$  of  $\mathbb{Z}\text{SK}$ -pairs induces the inverse of the map  $\varphi^{\mathbb{Z}} : \mathcal{A}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b}) \rightarrow \mathcal{G}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$ .*

Once again, the Kriker invariant can be recovered from  $\tilde{Z}^{\mathbb{Z}}$ :  $Z^{\text{Kri}} = \psi \circ \tilde{Z}^{\mathbb{Z}}$ . Further, both the Kriker and the Lescop invariants induce the same map on  $\mathcal{G}^{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}}, \mathfrak{b})$ , namely the map  $\psi \circ \tilde{Z}^{\mathbb{Z}}$ .

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