Multimaterial topology optimization of a heated channel

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Introduction

How to design the device to optimize the physical phenomenon?

- minimize temperature at output
- minimize pressure drop
- ...

minimize temperature at output
In a region $\Omega$, suppose a flow occupies a region $\Omega_f$ and the solid defines a region $\Omega_s$ such that $\Omega = \Omega_f \cup \Omega_s$. Suppose also that the channel is heated.

$\implies$ Navier-Stokes coupled with advection-diffusion equations.

(Focus only on steady-state model.)
We would like to control the distribution of solid. However, hard to put this in a control context.

\[ \nabla \cdot u = 0 \quad \text{in} \quad \Omega_f \]

\[ (u \cdot \nabla)u + \nabla p - Re^{-1} \Delta u - Ri \theta e_y = 0 \quad \text{in} \quad \Omega_f \]

\[ \nabla \cdot (u\theta) - \nabla \cdot (Re^{-1}Pr^{-1}k(x)\nabla \theta) = 0 \quad \text{in} \quad \Omega \]

\[ u = 0 \quad \text{in} \quad \Omega_s \]

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\[ u = 0 \quad \text{in} \quad \Omega_s \]
Model

\[ \nabla \cdot u = 0 \quad \text{in} \quad \Omega \]

\[ (u \cdot \nabla)u + \nabla p - Re^{-1} \Delta u - Ri \theta e_y + \eta 1_{\Omega_s} u = 0 \quad \text{in} \quad \Omega \]

\[ \nabla \cdot (u \theta) - \nabla \cdot (Re^{-1} Pr^{-1} k(x, 1_{\Omega_s}) \nabla \theta) = 0 \quad \text{in} \quad \Omega \]

Converges as \( \eta \to +\infty \) [1].

However, the indicator \( 1_{\Omega_s} \) is a binary function, so not suitable for optimization.

\( \implies \) Convexification and penalization of intermediate values.

How to penalize intermediate values?

Convexification. \( \{0, 1\} \rightarrow [0, 1] \).

Define \( \alpha : \Omega \rightarrow [0, 1] \) and \( h_\tau : [0, 1] \rightarrow [0, 1] \), s.t. \( h_\tau(\alpha(x)) \approx 1_{\Omega_s}(x) \).

How to avoid the values between 0 and 1?

Optimization approach: add a cost to push towards 0 or 1.

Dynamical approach: use a function \( h_\tau \) that will quickly evolve towards 0 or 1.
How to penalize intermediate values?

Cost penalization: usual concerning optimization with constraints (barrier methods...). In our case, we minimize the following cost:

$$\int_{\Omega} h_{\tau}(\alpha(x))(1 - h_{\tau}(\alpha(x)))dx$$

Since $h_{\tau} : [0, 1] \rightarrow [0, 1] \implies$ penalizes the values between 0 and 1, and $h_{\tau}(\alpha)$ approximates $1_{\Omega_5}$. 
How to penalize intermediate values?

$h_\tau$ can be used to penalize intermediate values, and obtain $h_\tau \xrightarrow{\tau \to +\infty} 0$ or 1.

- **SIMP**: $h_\tau(x) = x^\tau$
- **RAMP**: $h_\tau(x) = 1 - (1 - x)\frac{1+\tau}{1+\tau-x}$
- **Sigmoid**: $h_\tau(x) = \frac{1}{1+\exp(-\tau(x-x_0))} - \frac{1}{1+\exp(\tau x_0)}$
Model

\[ \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \]

\[(u \cdot \nabla)u + \nabla p - \text{Re}^{-1} \Delta u - \text{Ri} \theta e_y + \eta h_T(\alpha) \mathbf{u} = 0 \quad \text{in} \quad \Omega \]

\[ \nabla \cdot (u \theta) - \nabla \cdot (\text{Re}^{-1} \text{Pr}^{-1} k(x, 1_{\Omega_S}) \nabla \theta) = 0 \quad \text{in} \quad \Omega \]

\[ \Rightarrow \quad \alpha \text{ defines the distribution of solid.} \]

Another problem: how does this affect the conductivity function \( k(x, 1_{\Omega_S}) \)?
Usually, thermal conductivity = discrete set of constants. We would like to control the distribution of thermal conductivity also!
Interpolation of thermal conductivity constants

Advocates for the use of the sigmoid interpolation function. Introduces a new control function $\phi$. 
We would like to solve:

\[
\min_{\alpha, \phi} J(u, \theta, p)
\]

such that:

\[
\begin{align*}
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
(u \cdot \nabla) u + \nabla p - \text{Re}^{-1} \Delta u - \text{Ri} \theta e_y + \eta h_T(\alpha) u &= 0 \quad \text{in } \Omega \\
\nabla \cdot (u \theta) - \nabla \cdot (\text{Re}^{-1} \text{Pr}^{-1} k_T(\alpha, \phi) \nabla \theta) &= 0 \quad \text{in } \Omega
\end{align*}
\]

\(\alpha\) defines the distribution of solid, \(\phi\) the thermal conductivity.
Existence of solution

We first study the existence of solutions to these equations. Suppose:

$$\alpha, \phi \in U_{\text{ad}} = \{ \xi \in BV(\Omega) \mid 0 \leq \xi(x) \leq 1 \mid |D\xi|(\Omega) \leq \kappa \}.$$ 

and $h_\tau, k_\tau$ are bounded continuous functions with $k(x) \geq k_{\text{min}} > 0$.

**Theorem**

Given $\alpha, \phi \in U_{\text{ad}}$, and given $\text{Ri}$, input velocity and heat flux small enough, there exists a solution $(u, \theta, p) \in H^1(\Omega)^2 \times H^1(\Omega) \times L^2(\Omega)$. 
Suppose now we would like to minimize a functional $J(\alpha, \phi, u, \theta, p)$ Under some hypothesis, there exists an optimal solution. But more interesting:

**Theorem**

Define a sequence $(\alpha_h^*, \phi_h^*, u_h^*, \theta_h^*, p_h^*)$ of global optimal solutions to the discretized problem. Then it converges (weak-*, weak-*, strong, strong, strong) to a global optimal solution of the continuous problem.
Numerical example

\[
\max_{\alpha, \phi} \int_{\Gamma_{\text{out}}} \theta \\
\ \text{s.t.} \\
\begin{aligned}
(u, \theta, p) & \text{ solution of N.-S. + adv-diff} \\
\alpha_{\text{in}}(x, 0) & = 1.8x(1 - x), \ \varphi = 3 \\
Re & = 100, \ Ri = 2, \ Pr = 0.71, \ \eta = 10^8.
\end{aligned}
\]

Code done using Fenics, available online.
Numerical example

You may have noticed that the control $\phi$ depends on the control $\alpha$, since the distribution of the thermal diffusivity in the solid depends.... On the distribution of solid.

$\Rightarrow$ Advocates for alternating directions!

Optimize first on $\alpha$, then on $\phi$. In all numerical tests: it works way better.
Numerical example

For this presentation, I want to compare two approaches:

1. Gradually increase \( \tau \) defining \( h_\tau \) and \( k_\tau \).
2. Keep \( \tau \) fixed and penalize more and more in the cost \( \int_\Omega h_\tau(\alpha)(1 - h_\tau(\alpha)) \).
Numerical example: function

(a) Velocity  (b) Temperature  (c) $k_\tau$

Figure: Result with function penalization ($\tau \to +\infty$).
Numerical example: cost

Figure: Result with cost penalization.
How to solve a complex topology optimization problem applied to fluid dynamics?

- A nice and coherent way to approximate the problem.
- Mathematically sound, and now fully analyzed.
- Gives interesting numerical results.
- Gives hints concerning ways to enhance it!
Conclusion

What is left to enhance?

- Computations really long, may hardly converge.
- Works for some examples, and not for others: does there exist a uniform method for all problems?
- Optimal solution for the thermal conductivity still depends on the initial guess: how to enhance this?
- Solution needs to be "cleaned" (how do we remove the holes without altering the optimality?).
We have a preprint!

Topology Optimization for Steady-state anisothermal flow targeting solid with piecewise constant thermal diffusivity
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https://hal.archives-ouvertes.fr/hal-02569142
Question time!