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EXPLICIT DIRICHLET–NEUMANN OPERATOR FOR WATER WAVES

DIDIER CLAMOND

ABSTRACT. An explicit expression for the Dirichlet–Neumann operator for surface water waves is presented. For non-overturning waves, but without assuming small amplitudes, the formula is first derived in two dimensions, subsequently extrapolated in higher dimensions and with a moving bottom. Although described here for water waves, this elementary approach could be adapted to many other problems having similar mathematical formulations.

1. INTRODUCTION

In this note, we consider the classical problem of gravity waves propagating at the (non-overturning) free surface of a homogenous non-viscous fluid in irrotational motion over an impermeable (uneven but non-overturning) seabed. Mathematically, in two dimensions without obstacles (i.e., for a simply connected fluid domain extending to infinity in all horizontal directions), this leads to the system of equations (for $x \in \mathbb{R}$, $t \in \mathbb{R}$ or $t \geq t_0$) [24]

$$\partial_x^2 \phi + \partial_y^2 \phi = 0 \quad \text{for } -d(x) \leq y \leq \eta(x, t), \quad (1)$$

$$\partial_y \phi + (\partial_x d)(\partial_x \phi) = 0 \quad \text{at } y = -d(x), \quad (2)$$

$$\partial_y \phi - \partial_t \eta - (\partial_x \eta)(\partial_x \phi) = 0 \quad \text{at } y = \eta(x, t), \quad (3)$$

$$\partial_t \phi + g\eta + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}(\partial_y \phi)^2 = 0 \quad \text{at } y = \eta(x, t), \quad (4)$$

where $\phi(x, y, t)$ is a velocity potential such that $u \stackrel{\text{def}}{=} \partial_x \phi$ is the horizontal velocity and $v \stackrel{\text{def}}{=} \partial_y \phi$ is the vertical one, $g > 0$ is the acceleration due to gravity (directed downward), with (x, y) respectively the horizontal and upward-vertical Cartesian coordinates, and t is the time. $y = \eta(x, t)$, $y = 0$ and $y = -d(x)$ are, respectively, the equations of the free surface, of the still-water level and of the bottom; $h(x, t) \stackrel{\text{def}}{=} \eta(x, t) + d(x)$ is the total water depth. Physically, equation (1) means that the motion is irrotational and isochoric, equations (2) and (3) characterise the impermeability of the bottom and of the free surface, while (4) expresses that the pressure at the free surface equals the constant atmospheric pressure (set to zero without loss of generality). Capillarity and other surface effects can be considered but they do not affect the analysis below, so they are of no interest here. Also, extensions of equations (1)–(4) in higher dimensions and/or moving bottoms are straightforward; these generalisations are considered at the end of the present paper. However, further generalisations (e.g., overturning surface and/or bottom, submerged obstacles, floating bodies, lateral solid boundaries, rough bottom) are beyond the scope of the present study; they require *ad hoc* investigations.

A Dirichlet–Neumann (or Dirichlet-to-Neumann) operator (DNO) takes as input a function expressed at a point of the domain boundary and outputs its (outward) normal derivative at the same point. Here, the DNO producing the (non-unitary outgoing) normal derivative at the free surface is $G(\phi_s) \stackrel{\text{def}}{=} [\partial_y \phi - (\partial_x \eta)(\partial_x \phi)]_{y=\eta}$, where $\phi_s(x, t) \stackrel{\text{def}}{=} \phi(x, \eta, t)$ denotes the

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velocity potential at the free surface. Fulfilling the Laplace equation (1) and the bottom impermeability condition (2), the DNO is a homogeneous linear function of ϕ_s , i.e., $G(\phi_s) = \mathcal{G}\phi_s$ where \mathcal{G} is a self-adjoint positive-definite pseudo-differential operator depending nonlinearly of η and d [9, 11]. The operator \mathcal{G} is a fundamental mathematical object because it ‘encodes’ the domain geometry, the kinematic of the fluid motion and the bottom impermeability; moreover, it appears explicitly into the Hamiltonian formulation [27] of the equations (1)–(4). Understandably, \mathcal{G} has been the subject of many mathematical studies — see Lannes [20] and Nicholls and Reitich [22] for details — and it is at the heart of several rigorous investigations on water waves (e.g., Alazard and Baldi [2], Alazard et al. [3]). The knowledge of the DNO mathematical features is certainly important, but its explicit construction is at least as important, in particular for practical applications.

For flat horizontal free surface and bottom, the fluid domain is a strip and the DNO is easily obtained analytically, e.g., via Fourier transform. For wavy surface and bottom, the DNO can be constructed as a perturbation of the strip, assuming small amplitudes. This is the route followed in 2D by Craig and Sulem [9] and in 3D by Craig and Groves [8] for flat seabeds, then extended to varying bottoms [11], these authors providing recurrence relations for computing the DNO to an arbitrary order of their perturbative expansion. For small perturbations of the flat surface and seabed, other series representations of the DNO are available in the literature [12, 25]. Although all these series are formally equivalent, this is not necessarily the case with their truncations at the same order, as outlined by Schäffer [23]. Moreover, such expansions are badly conditioned, so prone to large numerical errors and instabilities [26]. An explicit formulation of the DNO is expected to facilitate various reformulations for more efficient computations, for example, but this is not the scope of the present paper.

The main purpose of this paper is to show how explicit Dirichlet–Neumann operators can be derived and, via few examples, to show their interest for analytic manipulations. Although some indications on potential issues and remedies with numerical computations are briefly discussed, it is not the purpose here to derive the most effective way to compute numerically a DNO.

The paper is organised as follow. In section 2, an explicit Dirichlet–Neumann operator is derived in two dimensions via rather elementary algebra. This DNO being in complex form, a real reformulation is introduced in section 3 in order to facilitate analytical approximations. Some approximations for small amplitudes in finite depth and for finite amplitudes in shallow water are then derived in section 4. The DNO is extended to higher dimensions in section 5, and its generalisation for moving bottoms is provided in section 6. Finally, summary and perspectives are briefly drawn in section 7.

2. TWO-DIMENSIONAL DIRICHLET–NEUMANN OPERATOR

Let be ψ the stream function harmonic conjugate of the velocity potential ϕ [21]. These two functions are related by the Cauchy–Riemann relations $\phi_x = \psi_y = u$ and $\phi_y = -\psi_x = v$. Thus, the complex potential $f \stackrel{\text{def}}{=} \phi + i\psi$ is a holomorphic function of $z \stackrel{\text{def}}{=} x + iy$, with $z = z_s \stackrel{\text{def}}{=} x + i\eta$ at the free surface and $z = z_b \stackrel{\text{def}}{=} x - id$ at the bottom. (As general notation, subscripts ‘s’ and ‘b’ denote quantities written, respectively, at the free surface and at the bottom.) The seabed being impermeable and static, it is a streamline where $\psi = \psi_b$ is constant. Without loss of generality, we then choose $\psi_b = 0$ for simplicity.

For any complex abscissa z_0 , the Taylor expansion around $z_0 = 0$ is (omitting temporal dependences for brevity)

$$f(z - z_0) = \exp[-z_0 \partial_z] f(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n z_0^n}{n!} \frac{\partial^n f(z)}{\partial z^n}. \quad (5)$$

For instance, taking $z_0 = ih = i(d + \eta)$, the relation (5) written at the free surface becomes

$$f(z_s - ih) = \exp[-ih \partial_{z_s}] f(z_s), \quad (6)$$

with the formal operator $\exp[-ih \partial_{z_s}] \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (n!)^{-1} (-ih)^n \partial_{z_s}^n$ together with $\partial_{z_s} \stackrel{\text{def}}{=} (1 + i\eta_x)^{-1} \partial_x$, $\partial_{z_s}^2 = (1 + i\eta_x)^{-1} \partial_x (1 + i\eta_x)^{-1} \partial_x$, etc. (Throughout this paper, we use the classical convention that any operator acts on everything it multiplies on its right, unless parenthesis enforce otherwise.)

It should be noticed that exponents denote differential compositions, so h^n is the n -th power of the function $h = d + \eta$, while $\partial_{z_s}^n$ is the n -th iteration of the differential operator ∂_{z_s} . Therefore, for example, if h is not constant then $h^2 = h(x)^2 \neq h(h(x))$, $h^2 \partial_{z_s}^2 \neq (h \partial_{z_s})^2 = h \partial_{z_s} h \partial_{z_s}$, $\exp[-ih \partial_{z_s}] \neq \sum_{n=0}^{\infty} (n!)^{-1} (-ih \partial_{z_s})^n$ and the operator inverse of $\exp[-ih \partial_{z_s}]$ is *not* $\exp[ih \partial_{z_s}]$ (but $\exp[ih \partial_{z_b}]$ as shown in section 3).

Since $z_s - ih = x - id = z_b$ then $f(z_s - ih) = f(z_b) = \phi_b$ is real (recall that $\psi_b = \text{Im} f_b = 0$ by definition), while $f(z_s) = \phi_s + i\psi_s$ is complex. Therefore, the imaginary part of (6), i.e.,

$$0 = \text{Re}\{\exp[-ih \partial_{z_s}]\} \psi_s + \text{Im}\{\exp[-ih \partial_{z_s}]\} \phi_s, \quad (7)$$

yields at once

$$\psi_s = -(\text{Re}\{\exp[-ih \partial_{z_s}]\})^{-1} \text{Im}\{\exp[-ih \partial_{z_s}]\} \phi_s. \quad (8)$$

The equation for the free surface impermeability being $\partial_t \eta = \mathcal{G} \phi_s = -\partial_x \psi_s = v_s - u_s \partial_x \eta$, an explicit definition of the Dirichlet–Neumann operator is obtained directly from (8) as

$$\mathcal{G} = \partial_x (\text{Re}\{\exp[-ih \partial_{z_s}]\})^{-1} \text{Im}\{\exp[-ih \partial_{z_s}]\}. \quad (9)$$

The formula (9) provides an explicit expression for the DNO, i.e., \mathcal{G} appears only on the left-hand side. It is the main result of this paper that can be generalised in higher dimensions and for moving bottoms (see below). It is also suitable to derive various approximations, in particular high-order shallow water approximations without assuming small amplitudes (see section 4.3 below; actually, this goal was the original motivation for deriving (9)).

For applications, it is convenient to introduce an operator \mathcal{I} such that $\mathcal{G} = -\partial_x \mathcal{I} \partial_x$, so

$$\mathcal{I} = -(\text{Re}\{\exp[-ih \partial_{z_s}]\})^{-1} \text{Im}\{\exp[-ih \partial_{z_s}]\} \partial_x^{-1}. \quad (10)$$

Since \mathcal{G} is a self-adjoint positive-definite operator [20], so is \mathcal{I} . Further, it is also convenient to introduce the operators \mathcal{R} and \mathcal{S} defined by

$$\mathcal{R} \stackrel{\text{def}}{=} \text{Re}\{\exp[-ih \partial_{z_s}]\}, \quad \mathcal{S} \stackrel{\text{def}}{=} -\text{Im}\{\exp[-ih \partial_{z_s}]\} \partial_x^{-1}, \quad (11)$$

so $\mathcal{I} = \mathcal{R}^{-1} \mathcal{S}$.

Although explicit, the formulae (9) and (10) are not quite in closed-form since they involve series (via the definition of the exponential operator) and operator inversion. Additional relations, suitable for practical applications, are then derived below.

3. AUXILIARY RELATIONS

With different choices of z and z_0 , the Taylor expansion (5) provides various relations of practical interest. Several variants of (9) can then be derived, their convenience depending on the problem at hand.

With the choice $z = z_b$ and $z_0 = -ih = -i(d + \eta)$, the relation (5) becomes

$$f(z_b + ih) = f(z_s) = \exp[ih \partial_{z_b}] f(z_b), \quad (12)$$

so a comparison with (6) yields at once

$$(\exp[-ih \partial_{z_s}])^{-1} = \exp[ih \partial_{z_b}] \iff (\exp[ih \partial_{z_b}])^{-1} = \exp[-ih \partial_{z_s}]. \quad (13)$$

With this relation, the operator involving ∂_{z_s} in the Dirichlet–Neumann operator (9) can be replaced by one involving ∂_{z_b} . This is somewhat convenient in constant depth because, then, $\partial_{z_b} = \partial_x$. However, two operators need then to be inverted instead of one with (9), so further simplifications are desirable.

Taking $z = x$ together with $z_0 = -i\eta$ and $z_0 = id$, (5) yields

$$f(x + i\eta) = f(z_s) = \exp[i\eta \partial_x] f(x), \quad f(x - id) = f(z_b) = \exp[-id \partial_x] f(x), \quad (14a, b)$$

and the elimination of $f(x)$ between these two relations, together with (6), yields

$$\exp[-ih \partial_{z_s}] = \exp[-id \partial_x] (\exp[i\eta \partial_x])^{-1} = \exp[-id \partial_x] (\exp[-i\eta \partial_x])^\dagger (1 + i\eta_x), \quad (15)$$

where a \dagger denotes the adjoint operator.¹ We have thus relations allowing to avoid the computation of the ∂_z operators, moreover without inversions. The operator \mathcal{R} remains to be inverted, however.

For a real or complex function γ depending on a single real variable x , let be the operators and their Hermitian adjoints

$$\mathcal{C}_\gamma \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \gamma^{2n} \partial_x^{2n}, \quad \mathcal{S}_\gamma \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \gamma^{2n+1} \partial_x^{2n+1}, \quad (16)$$

$$\mathcal{C}_\gamma^\dagger = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \partial_x^{2n} \gamma^{*2n}, \quad \mathcal{S}_\gamma^\dagger = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \partial_x^{2n+1} \gamma^{*2n+1}. \quad (17)$$

We then have

$$\exp[-id \partial_x] = \mathcal{C}_d - i\mathcal{S}_d, \quad (\exp[-i\eta \partial_x])^\dagger = \mathcal{C}_\eta^\dagger + i\mathcal{S}_\eta^\dagger, \quad (18)$$

and the relation (15) is split into real and imaginary parts as

$$\text{Re}\{\exp[-ih \partial_{z_s}]\} = \mathcal{C}_d \mathcal{C}_\eta^\dagger + \mathcal{S}_d \mathcal{S}_\eta^\dagger - \mathcal{C}_d \mathcal{S}_\eta^\dagger \eta_x + \mathcal{S}_d \mathcal{C}_\eta^\dagger \eta_x, \quad (19)$$

$$\text{Im}\{\exp[-ih \partial_{z_s}]\} = \mathcal{C}_d \mathcal{S}_\eta^\dagger - \mathcal{S}_d \mathcal{C}_\eta^\dagger + \mathcal{C}_d \mathcal{C}_\eta^\dagger \eta_x + \mathcal{S}_d \mathcal{S}_\eta^\dagger \eta_x. \quad (20)$$

With the operator relation $\eta_x = \partial_x \eta - \eta \partial_x$ (resulting from the Leibniz rule), we have

$$\mathcal{C}_\eta^\dagger \eta_x = \partial_x^{-1} \mathcal{S}_\eta^\dagger \partial_x - \mathcal{S}_\eta^\dagger, \quad \mathcal{S}_\eta^\dagger \eta_x = \mathcal{C}_\eta^\dagger - \partial_x^{-1} \mathcal{C}_\eta^\dagger \partial_x, \quad (21)$$

so the relations (19)–(20) yield

$$\mathcal{R} = \mathcal{C}_d \partial_x^{-1} \mathcal{C}_\eta^\dagger \partial_x + \mathcal{S}_d \partial_x^{-1} \mathcal{S}_\eta^\dagger \partial_x, \quad \mathcal{I} = \mathcal{S}_d \partial_x^{-1} \mathcal{C}_\eta^\dagger - \mathcal{C}_d \partial_x^{-1} \mathcal{S}_\eta^\dagger. \quad (22a, b)$$

¹For any complex function γ of a single real variable x , the operator $\exp[\gamma \partial_x] \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (n!)^{-1} \gamma^n \partial_x^n$ has for Hermitian adjoint $(\exp[\gamma \partial_x])^\dagger \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (n!)^{-1} \partial_x^n (-\gamma^*)^n$, a star denoting the complex conjugate. We then have $(\exp[\gamma \partial_x])^{-1} = (\exp[\gamma^* \partial_x])^\dagger (1 + \gamma_x)$.

The latter relations are particularly convenient to derive analytic approximations and to extrapolate the DNO in higher dimensions, as shown below.

4. APPROXIMATE DIRICHLET–NEUMANN OPERATORS

From the explicit DNO (9) and the relations derived in the previous section, several approximations of practical interest can be easily obtained. We consider here only two special cases.

4.1. Infinitesimal waves in arbitrary depth. Assuming that the free surface η remains close to zero, one can formally expand the DNO in increasing order of nonlinearities in η [9, 11]. Thus, writing $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2 + \dots$ and similarly for \mathcal{R} , \mathcal{I} and \mathcal{J} , one obtains at once from (22)

$$\mathcal{R}_0 = \mathcal{C}_d, \quad \mathcal{R}_1 = -\mathcal{S}_d \eta \partial_x, \quad \mathcal{R}_2 = -\frac{1}{2} \mathcal{C}_d \partial_x \eta^2 \partial_x, \quad \text{etc.}, \quad (23)$$

$$\mathcal{I}_0 = \mathcal{S}_d \partial_x^{-1}, \quad \mathcal{I}_1 = \mathcal{C}_d \eta, \quad \mathcal{I}_2 = -\frac{1}{2} \mathcal{S}_d \partial_x \eta^2, \quad \text{etc.} \quad (24)$$

The relation

$$\begin{aligned} \mathcal{R}^{-1} &= [\mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_2 + \dots]^{-1} = [1 + \mathcal{R}_0^{-1} \mathcal{R}_1 + \mathcal{R}_0^{-1} \mathcal{R}_2 + \dots]^{-1} \mathcal{R}_0^{-1} \\ &= [1 - \mathcal{R}_0^{-1} \mathcal{R}_1 - \mathcal{R}_0^{-1} \mathcal{R}_2 + \mathcal{R}_0^{-1} \mathcal{R}_1 \mathcal{R}_0^{-1} \mathcal{R}_1 + \dots] \mathcal{R}_0^{-1}, \end{aligned} \quad (25)$$

then yields after some algebra

$$\begin{aligned} \mathcal{I}_0 &= \mathcal{C}_d^{-1} \mathcal{S}_d \partial_x^{-1}, \quad \mathcal{I}_1 = \eta + \mathcal{I}_0 \partial_x \eta \partial_x \mathcal{I}_0, \\ \mathcal{I}_2 &= \frac{1}{2} \partial_x \eta^2 \partial_x \mathcal{I}_0 + \mathcal{I}_0 \partial_x \eta \partial_x \mathcal{I}_1 - \frac{1}{2} \mathcal{I}_0 \partial_x^2 \eta^2, \quad \text{etc.}, \end{aligned} \quad (26)$$

hence

$$\begin{aligned} \mathcal{G}_0 &= -\partial_x \mathcal{C}_d^{-1} \mathcal{S}_d, \quad \mathcal{G}_1 = -\partial_x \eta \partial_x - \mathcal{G}_0 \eta \mathcal{G}_0, \\ \mathcal{G}_2 &= \frac{1}{2} \partial_x^2 \eta^2 \mathcal{G}_0 - \mathcal{G}_0 \eta \mathcal{G}_1 - \frac{1}{2} \mathcal{G}_0 \partial_x \eta^2 \partial_x, \quad \text{etc.} \end{aligned} \quad (27)$$

In constant depth, the expansion of Craig and Sulem [9] is, as expected, recovered introducing the operator $\mathcal{D} \stackrel{\text{def}}{=} i\partial_x$, i.e., replacing ∂_x by $-i\mathcal{D}$. With a variable bottom, the expansion of Craig et al. [11] is also recovered, expect for the definition of \mathcal{G}_0 . Indeed, Craig et al. [11] define \mathcal{G}_0 with an expansion for small amplitudes of the bottom corrugation (i.e., $\max |d(x) - \bar{d}|$ is small, where \bar{d} is the mean depth), and they provide a recursion formula for computing this series. In (27), \mathcal{G}_0 is defined explicitly for arbitrary (non-overturning) bottom and no additional expansions are required.

4.2. Remarks. For higher-order approximations, the recursion formula of Craig and Sulem [9] can be used verbatim with \mathcal{G}_0 defined here in (27). This approach is convenient for the derivation of (rather low-order) analytical approximations. However, with numerical computations, this recursion is prone to cancelation errors leading to large numerical errors and instabilities [26]. This problem is more pronounced in higher dimensions.²

These difficulties come mostly from the expansion of the inverse operator \mathcal{R}^{-1} . For numerical computations, this expansion should be avoided to obtain $B = \mathcal{R}^{-1}A$ (for some functions A and B). It is generally more efficient to solve $\mathcal{R}B = A$ via an iterative procedure. This is

²W. Craig (2005), private communication.

a similar problem as the resolution of linear systems of equations, for which iterative methods are often more efficient [19]. For the DNO, the relation (22a)³ shows that \mathcal{R} behaves (roughly) like a cosh-function, so \mathcal{R}^{-1} behaves like a sech-function. The Maclaurin series of $\cosh(z)$ having an infinite radius of convergence, while the one of $\operatorname{sech}(z)$ convergences only for $|z| < \pi/2$, this provides an informal/heuristic argument showing why $B = \mathcal{R}^{-1}A$ should not be computed but $\mathcal{R}B = A$ should be solved instead. With other representations (than truncated Taylor series) of \mathcal{R}^{-1} , the computation of $B = \mathcal{R}^{-1}A$ may be efficient, however.

For linear waves in the context of a highly variable bathymetry, the improvements of the DNO expansion proposed by Andrade and Nachbin [4] could be exploited to reformulate the explicit DNO in a more effective form for numerical computations. However, when speed and high numerical accuracy are required, the DNO perturbation expansions are not competitive (specially for steep waves) and boundary integral formulations should be preferred [5, 15, 16].

4.3. Long waves in shallow water. For long waves in shallow water, the characteristic wavelength L_c is much larger than the characteristic depth d_c , so $\sigma \stackrel{\text{def}}{=} d_c/L_c \ll 1$ is a ‘shallowness’ dimensionless small parameter. The horizontal derivative ∂_x is then of first-order in shallowness and the DNO can be expanded in power series of σ , without assuming small amplitude for the waves and/or for the bottom corrugation. Thus, we do not need to explicitly introduce scalings to assess the order of terms, it is sufficient to count the number of derivatives. For instance, $\partial_x^3 \eta$, $(\partial_x^2 \eta)(\partial_x \eta)$ and $(\partial_x \eta)^3$ are all of third-order in shallowness, as well as $\partial_x^3 d$, $(\partial_x^2 d)(\partial_x d)$ and $(\partial_x d)^3$.

We then have the shallow water even terms expansions $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_2 + \mathcal{R}_4 + \dots$ (and similarly for \mathcal{I} and \mathcal{J}) so, from (22),

$$\mathcal{R}_0 = 1, \quad \mathcal{R}_2 = -\frac{1}{2} d^2 \partial_x^2 - d \partial_x \eta \partial_x - \frac{1}{2} \partial_x \eta^2 \partial_x, \quad \text{etc.}, \quad (28)$$

$$\mathcal{I}_0 = h, \quad \mathcal{I}_2 = -\frac{1}{6} d^3 \partial_x^2 - \frac{1}{2} d^2 \partial_x^2 \eta - \frac{1}{2} d \partial_x^2 \eta^2 - \frac{1}{6} \partial_x^2 \eta^3, \quad \text{etc.}, \quad (29)$$

hence, after some algebra,

$$\mathcal{J}_0 = h, \quad \mathcal{J}_2 = \frac{1}{2} h^2 d_{xx} + h h_x d_x - h d_x^2 + \frac{1}{3} \partial_x h^3 \partial_x, \quad \text{etc.} \quad (30)$$

Note that \mathcal{I}_0 and \mathcal{J}_2 are obviously self-adjoint, as it should be.

It should be emphasised that these approximations were obtained directly from the explicit DNO, considering weak variations in x (i.e., long waves in shallow water) but without assuming small amplitudes of the free surface and of the seabed (i.e., there are no restrictions on the magnitude of $|\eta|$ and $|d(x) - \bar{d}|$, \bar{d} being the mean depth).

5. DIRICHLET–NEUMANN OPERATOR IN HIGHER DIMENSIONS

It is rather straightforward to extrapolate the DNO given by (9) to three (and more) spacial dimensions. In higher dimensions, the holomorphic functions cannot be used but series representations remain. This feature is exploited here to obtain an explicit expression for the DNO in an arbitrary number of dimensions.

With $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ referring to the ‘horizontal’ coordinates, the mathematical problem is then posed in the $(N+1)$ -dimensional Cartesian (\mathbf{x}, y) -space, with y the ‘upward-vertical’ coordinate. Obviously, only the two-dimensional (i.e., $N = 1$) and three-dimensional (i.e., $N = 2$) cases are of physical interest for water waves. Let be $\nabla \stackrel{\text{def}}{=} (\partial_{x_1}, \dots, \partial_{x_N})$, $\Delta \stackrel{\text{def}}{=} \nabla \cdot \nabla$

³See also relation (41).

and $\mathcal{D} \stackrel{\text{def}}{=} (-\Delta)^{1/2}$ denote, respectively, the horizontal gradient, Laplacian and semi-Laplacian operators.

The Dirichlet–Neumann operator is naturally extended in higher dimensions extrapolating the relation $\mathcal{G} = -\partial_x \mathcal{R}^{-1} \mathcal{I} \partial_x$, the operators \mathcal{R} and \mathcal{I} having to be redefined. In the two-dimensional case, these operators are defined via complex expressions in section 2. In order to extend these operators in higher dimensions, one must consider their real form (22), so their extrapolation is natural.

One-dimensional operators involving only even-order derivatives have straightforward extensions in higher dimensions replacing the second-order horizontal derivative ∂_x^2 by the horizontal Laplacian Δ . For instance

$$\mathcal{C}_d \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} d^{2n} \Delta^n \stackrel{\text{def}}{=} \cosh(d\mathcal{D}), \quad (31)$$

$$\mathcal{C}_\eta^\dagger \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Delta^n \eta^{2n} \stackrel{\text{def}}{=} \cosh(\mathcal{D}\eta), \quad (32)$$

$$\mathcal{S}_d \partial_x^{-1} \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} d^{2n+1} \Delta^n \stackrel{\text{def}}{=} \sinh(d\mathcal{D}) \mathcal{D}^{-1}, \quad (33)$$

$$\partial_x^{-1} \mathcal{S}_\eta^\dagger \mapsto \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \Delta^n \eta^{2n+1} \stackrel{\text{def}}{=} -\mathcal{D}^{-1} \sinh(\mathcal{D}\eta). \quad (34)$$

It should be emphasised that, as in the one-dimensional case, the operators do not commute, so for example $\cosh(d\mathcal{D}) \neq \cosh(\mathcal{D}d)$ and $\cosh(d\mathcal{D})^{-1} \neq \text{sech}(d\mathcal{D})$, the equalities holding only in constant depth because then $d\mathcal{D} = \mathcal{D}d$.

A natural extension of $\mathcal{I} \partial_x$ is thus $\mathcal{I} \nabla$ with

$$\mathcal{I} \mapsto \sinh(d\mathcal{D}) \mathcal{D}^{-1} \cosh(\mathcal{D}\eta) + \cosh(d\mathcal{D}) \mathcal{D}^{-1} \sinh(\mathcal{D}\eta). \quad (35)$$

In order to find the extension of $\partial_x \mathcal{R}^{-1}$, the operator \mathcal{R} given by (22a) is rewritten as

$$\mathcal{R} = \mathcal{C}_d \partial_x^{-1} \left[\mathcal{C}_\eta^\dagger + \partial_x \mathcal{C}_d^{-1} (\mathcal{S}_d \partial_x^{-1}) \partial_x (\partial_x^{-1} \mathcal{S}_\eta^\dagger) \right] \partial_x. \quad (36)$$

Thus, we have the natural extension

$$\partial_x \mathcal{R}^{-1} \mapsto \left[\cosh(\mathcal{D}\eta) + \mathcal{G}_0 \mathcal{D}^{-1} \sinh(\mathcal{D}\eta) \right]^{-1} \nabla \cdot \cosh(d\mathcal{D})^{-1}, \quad (37)$$

where $\cosh(d\mathcal{D})^{-1}$ is the inverse operator of $\cosh(d\mathcal{D})$, and

$$\mathcal{G}_0 \stackrel{\text{def}}{=} -\nabla \cdot \cosh(d\mathcal{D})^{-1} \sinh(d\mathcal{D}) \mathcal{D}^{-1} \nabla. \quad (38)$$

Therefore, the DNO becomes at once

$$\mathcal{G} = - \left[\cosh(\mathcal{D}\eta) + \mathcal{G}_0 \mathcal{D}^{-1} \sinh(\mathcal{D}\eta) \right]^{-1} \nabla \cdot \cosh(d\mathcal{D})^{-1} \mathcal{I} \nabla. \quad (39)$$

In order to avoid misinterpretations of the formula (39), it is worthy to re-emphasise here that: (i) any operator acts on everything it multiplies on its right, so (39) should be applied successively leftward starting from the furthest right; (ii) exponents denote operator compositions, so an exponent -1 means an operator inversion.

Note that $\mathcal{G} \rightarrow \mathcal{G}_0$ as $\eta \rightarrow 0$. Moreover, processing as in section 4.1 for infinitesimal waves, one finds the expansion of Craig et al. [11], except for \mathcal{G}_0 that is defined implicitly by Craig et al. [11] but explicitly here.

5.1. Constant depth. In constant depth, d commuting then with both \mathcal{D} and ∇ , we have the simplified relation $\mathcal{G}_0 = \mathcal{D} \tanh(d\mathcal{D})$, while \mathcal{I} and \mathcal{G} become (see Appendix A for details)

$$\mathcal{I} = \mathcal{D}^{-1} [\sinh(d\mathcal{D}) \cosh(\mathcal{D}\eta) + \cosh(d\mathcal{D}) \sinh(\mathcal{D}\eta)] = \mathcal{D}^{-1} \sinh(\mathcal{D}h), \quad (40)$$

$$\begin{aligned} \mathcal{G} &= -[\cosh(d\mathcal{D}) \cosh(\mathcal{D}\eta) + \sinh(d\mathcal{D}) \sinh(\mathcal{D}\eta)]^{-1} \nabla \cdot \mathcal{I} \nabla \\ &= -\cosh(\mathcal{D}h)^{-1} \mathcal{D}^{-1} \nabla \cdot \sinh(\mathcal{D}h) \nabla. \end{aligned} \quad (41)$$

A better conditioned formulation, avoiding the computation of \mathcal{D} , is

$$\mathcal{G} = -[\operatorname{sech}(d\mathcal{D}) \cosh(\mathcal{D}h)]^{-1} \nabla \cdot [\operatorname{sech}(d\mathcal{D}) \operatorname{sinhc}(\mathcal{D}h) h] \nabla, \quad (42)$$

with

$$\operatorname{sech}(d\mathcal{D}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} d^{2n} \mathcal{D}^{2n}, \quad \operatorname{sinhc}(\mathcal{D}h) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \mathcal{D}^{2n} h^{2n}, \quad (43)$$

where E_n are the Euler numbers [1] (since d and \mathcal{D} commute, we have $\operatorname{sech}(d\mathcal{D}) = \cosh(d\mathcal{D})^{-1}$). The relation (42) involving only even powers of \mathcal{D} , only Laplacian and gradient operators need to be evaluated, i.e., the computation of the non-local operator \mathcal{D} can be avoided.

The DNO appearing in Hamiltonian formulations of water waves, its functional variations are crucial to derive the equations of motion and to investigate stability [13]. Thanks to the explicit DNO (41), these variations can be obtained quite effortlessly. Indeed, with the relations (c.f. Appendix A)

$$\cosh(\mathcal{D}(h + \delta h)) = \cosh(\mathcal{D}h) + \mathcal{D} \sinh(\mathcal{D}h) \delta h + O(\delta h^2), \quad (44)$$

$$\sinh(\mathcal{D}(h + \delta h)) = \sinh(\mathcal{D}h) + \mathcal{D} \cosh(\mathcal{D}h) \delta h + O(\delta h^2), \quad (45)$$

the first variation of the DNO is obtained at once as

$$\begin{aligned} \mathcal{G}(h + \delta h) &= \mathcal{G}(h) - \cosh(\mathcal{D}h)^{-1} \nabla \cdot \cosh(\mathcal{D}h) \delta h \nabla \\ &\quad + \cosh(\mathcal{D}h)^{-1} \mathcal{D} \sinh(\mathcal{D}h) \delta h \mathcal{G}(h) + O(\delta h^2), \end{aligned} \quad (46)$$

or

$$\begin{aligned} \mathcal{G}(h + \delta h) &= \mathcal{G}(h) - \cosh(\mathcal{D}h)^{-1} \nabla \cdot \cosh(\mathcal{D}h) \delta h \nabla + \mathcal{G}(h) \delta h \mathcal{G}(h) \\ &\quad - \cosh(\mathcal{D}h)^{-1} \nabla \cdot \cosh(\mathcal{D}h) (\nabla h) \delta h \mathcal{G}(h) + O(\delta h^2). \end{aligned} \quad (47)$$

Similarly, higher-order functional variations of \mathcal{G} can be easily obtained. This is one illustration of the advantage of dealing with an explicit DNO.

5.2. Remarks. Since the multidimensional DNO was derived extrapolating the bidimensional case, one can then naturally ask if (39) is a correct expression.

First, we note that the DNO explicit expression is not unique. For instance, as in 2D and as suggested by the Taylor expansion around $\eta = 0$, the DNO could also be written $\mathcal{G} = -\nabla \cdot \mathcal{J} \nabla$ for some operator $\mathcal{J}(d, \eta, \mathcal{D}, \nabla)$ to be specified.

In 2D (i.e., for $N = 1$), one can exploit the theory of holomorphic functions to directly check that the explicit DNO (39) is a correct one. This procedure is simply the reverse of the derivations made in §2 and §3. This is not possible in higher dimension (i.e., $N > 1$) because holomorphic functions cannot be used. The validity of (39) was then checked expanding it ala Craig & Sulem, checking that both expansions match. (This is detailed in Appendix A for constant depth.)

6. MOVING BOTTOM

We consider finally the generalisation of a moving bottom, i.e., $d = d(x, t)$. Of course, for simplicity, we begin with the two dimensional case, the generalisation in higher dimensions being straightforward.

When $\partial_t d \neq 0$ the bottom is no longer a streamline, so the stream function is not zero at the seabed, i.e., $\psi_b = \psi_b(x, t) \neq 0$. The lower boundary condition (2) becomes $\partial_t d = \partial_x \psi_b = -v_b - u_b \partial_x d$. With a moving bottom, the relations (5), (6) and (12) still hold, but (7) becomes

$$\psi_b = \operatorname{Re}\{\exp[-ih \partial_{z_s}]\} \psi_s + \operatorname{Im}\{\exp[-ih \partial_{z_s}]\} \phi_s. \quad (48)$$

The condition for the bottom impermeability yielding $\psi_b = \partial_x^{-1} \partial_t d$, the relation (48) gives

$$\psi_s = \operatorname{Re}\{\exp[-ih \partial_{z_s}]\}^{-1} \left(\partial_x^{-1} \partial_t d - \operatorname{Im}\{\exp[-ih \partial_{z_s}]\} \phi_s \right). \quad (49)$$

The relation (49) shows that the Dirichlet to Neumann transformation at the free surface is no longer a homogeneous linear function of ϕ_s . The impermeability of the free surface is then $\partial_t \eta = G(\phi_s)$, the generalised Dirichlet–Neumann operator G being

$$G(\phi_s) = \mathcal{G} \phi_s - \partial_x \operatorname{Re}\{\exp[-ih \partial_{z_s}]\}^{-1} \partial_x^{-1} \partial_t d, \quad (50)$$

where \mathcal{G} is given by (9). Note that $\partial_x^{-1} \partial_t d$ is not uniquely defined due to the antiderivative, unicity being enforced by the definitions of the mean water level and of the frame of reference.

In higher dimension, with $\partial_x^{-1} = \partial_x \partial_x^{-2} \mapsto \nabla \Delta^{-1} = -\mathcal{D}^{-2} \nabla$, the DNO obviously becomes

$$G(\phi_s) = \left[\cosh(\mathcal{D} \eta) + \mathcal{G}_0 \mathcal{D}^{-1} \sinh(\mathcal{D} \eta) \right]^{-1} \nabla \cdot \cosh(d \mathcal{D})^{-1} \left(\mathcal{D}^{-2} \nabla \partial_t d - \mathcal{I} \nabla \phi_s \right), \quad (51)$$

\mathcal{I} and \mathcal{G}_0 being defined, respectively, by (35) and (38).

7. DISCUSSION

Using elementary algebra, we obtained explicit formulae for the Dirichlet–Neumann operators involved in water wave problems. We first derived the DNO for two-dimensional waves over a static (uneven) bottom. We then extrapolated the formula to higher dimensions and generalised the formula for moving bottoms. The latter generalisation is interesting for its applications, such as tsunami generation [18], but also because it shows that extensions to fluids stratified in several homogeneous layers is possible [6, 10]. The DNO is also used in some water waves problems with vorticity [7, 17], and the derivation of an explicit DNO for rotational waves is conceivable.

In this note, the focus is on the DNO at the free surface assuming a given bottom shape and motion. Obviously, one can as easily obtain the DNO at the bottom from an assumed free surface that, in particular, should find applications in bottom detection from free surface measurements [14].

The explicit DNO derived here are expressed with pseudo-differential operators formally defined in terms of series. Such definition supposes sufficient regularity of the free surface and the bottom; regularity yet to be specified by rigorous mathematical analysis. When these regularity conditions are not met, other more general representations of the operators should be used instead, such as integral formulations. Once these operators properly defined, the explicit DNO should then be usable verbatim, allowing the investigation of rough bottoms and waves with angular crests, for example.

The main purpose of this paper is to show how explicit DNO can be derived and, via examples, to show their interest for analytic manipulations. Although some indications on potential issues and remedies with numerical computations are briefly discussed, it is not the

purpose here to derive the most effective way to compute numerically the DNO. For special functions, their definitions via power series are often not suitable for accurate fast computations, at least not in every cases and without extra knowledge (e.g., periodicity, symmetries, locations of singularities). The situation is similar with Dirichlet–Neumann operators defined via series, with the substantial extra difficulty that they involve non-commutative algebra.

Dirichlet–Neumann operators appear in many fields of research in Physics (acoustics, elasticity, electromagnetism, etc.) and, more generally, in the theory of partial differential equations. The use of a DNO is not restricted to problems involving the Laplace equation; it is also commonly employed in close relatives, such as the Helmholtz equation. The elementary formal approach presented here could then be adapted in these contexts.

APPENDIX A. SOME OPERATOR RELATIONS IN CONSTANT DEPTH

In constant depth, the algebra are significantly simplified because d commutes with both \mathcal{D} and ∇ . As mentioned at the end of section 5, we then have $\mathcal{G}_0 = \mathcal{D} \tanh(d\mathcal{D})$. We also have, from the definition of the operators,

$$\begin{aligned} \cosh(d\mathcal{D}) \cosh(\mathcal{D}\eta) &= \left(\sum_{i=0}^{\infty} \frac{d^{2i} \mathcal{D}^{2i}}{(2i)!} \right) \left(\sum_{j=0}^{\infty} \frac{\mathcal{D}^{2j} \eta^{2j}}{(2j)!} \right) = \sum_{i,j=0}^{\infty} \frac{d^{2i} \mathcal{D}^{2i+2j} \eta^{2j}}{(2i)!(2j)!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^i \frac{\mathcal{D}^{2i} d^{2i-2j} \eta^{2j}}{(2j)!(2i-2j)!} = \sum_{i=0}^{\infty} \sum_{j \text{ even}}^{2i} \frac{\mathcal{D}^{2i} d^{2i-j} \eta^j}{j!(2i-j)!}, \end{aligned} \quad (52)$$

$$\begin{aligned} \sinh(d\mathcal{D}) \sinh(\mathcal{D}\eta) &= \sum_{i,j=0}^{\infty} \frac{d^{2i+1} \mathcal{D}^{2i+2j+2} \eta^{2j+1}}{(2i+1)!(2j+1)!} = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \frac{\mathcal{D}^{2i} d^{2i-2j-1} \eta^{2j+1}}{(2j+1)!(2i-2j-1)!} \\ &= \sum_{i=1}^{\infty} \sum_{j \text{ odd}}^{2i-1} \frac{\mathcal{D}^{2i} d^{2i-j} \eta^j}{j!(2i-j)!}. \end{aligned} \quad (53)$$

Since $|n!| = \infty$ for all negative integers n , the summation $\sum_{i=1}^{\infty}$ in (53) can be replaced by $\sum_{i=0}^{\infty}$. Thus, we have

$$\begin{aligned} \cosh(d\mathcal{D}) \cosh(\mathcal{D}\eta) + \sinh(d\mathcal{D}) \sinh(\mathcal{D}\eta) &= \sum_{i=0}^{\infty} \frac{\mathcal{D}^{2i}}{(2i)!} \sum_{j=0}^{2i} \frac{(2i)! d^{2i-j} \eta^j}{j!(2i-j)!} \\ &= \sum_{i=0}^{\infty} \frac{\mathcal{D}^{2i} (d+\eta)^{2i}}{(2i)!} = \cosh(\mathcal{D}h). \end{aligned} \quad (54)$$

Similarly, one can easily derive the relations

$$\sinh(d\mathcal{D}) \cosh(\mathcal{D}\eta) + \cosh(d\mathcal{D}) \sinh(\mathcal{D}\eta) = \sinh(\mathcal{D}h), \quad (55)$$

$$\cosh(\eta\mathcal{D}) \sinh(\mathcal{D}d) + \sinh(\eta\mathcal{D}) \cosh(\mathcal{D}d) = \sinh(h\mathcal{D}), \quad (56)$$

$$\cosh(\eta\mathcal{D}) \cosh(\mathcal{D}d) + \sinh(\eta\mathcal{D}) \sinh(\mathcal{D}d) = \cosh(h\mathcal{D}), \quad (57)$$

and, obviously,

$$\cosh(\mathcal{D}h) \pm \sinh(\mathcal{D}h) = \exp(\pm \mathcal{D}h) \quad (58)$$

Note that \mathcal{D} commuting with d , but not with η and h , these relations are not valid for uneven bottoms and, in constant depth, $\sinh(\mathcal{D}h) \neq \sinh(h\mathcal{D})$ for example. However, for varying bottoms, similar relations can be easily obtained if η is constant.

Taylor expansions around $\eta = 0$ yield

$$\sinh(\mathcal{D}h) = \sinh(d\mathcal{D}) \left[1 + \frac{1}{2} \mathcal{D}^2 \eta^2 + \dots \right] + \cosh(d\mathcal{D}) \left[\mathcal{D}\eta + \frac{1}{6} \mathcal{D}^3 \eta^3 + \dots \right], \quad (59)$$

$$\cosh(\mathcal{D}h) = \cosh(d\mathcal{D}) \left[1 + \frac{1}{2} \mathcal{D}^2 \eta^2 + \dots \right] + \sinh(d\mathcal{D}) \left[\mathcal{D}\eta + \frac{1}{6} \mathcal{D}^3 \eta^3 + \dots \right], \quad (60)$$

hence, with $\mathcal{G}_0 \stackrel{\text{def}}{=} \mathcal{D} \tanh(d\mathcal{D})$,

$$\begin{aligned} (\cosh(\mathcal{D}h))^{-1} &= \left[1 + \mathcal{G}_0 \eta + \frac{1}{2} \mathcal{D}^2 \eta^2 + \frac{1}{6} \mathcal{G}_0 \mathcal{D}^2 \eta^3 + \dots \right]^{-1} \text{sech}(d\mathcal{D}) \\ &= \left[1 - \mathcal{G}_0 \eta - \frac{1}{2} \mathcal{D}^2 \eta^2 + \mathcal{G}_0 \eta \mathcal{G}_0 \eta + \dots \right] \text{sech}(d\mathcal{D}), \end{aligned} \quad (61)$$

$$\nabla \cdot \mathcal{D}^{-1} \sinh(\mathcal{D}h) \nabla = -\cosh(d\mathcal{D}) \left[\mathcal{G}_0 - \nabla \cdot \eta \nabla - \frac{1}{2} \mathcal{G}_0 \nabla \cdot \eta^2 \nabla + \dots \right], \quad (62)$$

Thus, with \mathcal{G} defined in (42), one gets

$$\mathcal{G} = \left[1 + \mathcal{G}_0 \eta + \frac{1}{2} \mathcal{D}^2 \eta^2 + \dots \right]^{-1} \left[\mathcal{G}_0 - \nabla \cdot \eta \nabla - \frac{1}{2} \mathcal{G}_0 \nabla \cdot \eta^2 \nabla + \dots \right]. \quad (63)$$

Expanding the DNO as $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2 + \dots$, with \mathcal{G} defined in (41), one obtains

$$\mathcal{G}_1 = -\mathcal{G}_0 \eta \mathcal{G}_0 - \nabla \cdot \eta \nabla, \quad \mathcal{G}_2 = -\frac{1}{2} \mathcal{D}^2 \eta^2 \mathcal{G}_0 - \mathcal{G}_0 \eta \mathcal{G}_1 - \frac{1}{2} \mathcal{G}_0 \nabla \cdot \eta^2 \nabla, \quad \text{etc.}, \quad (64)$$

so the expansion of Craig and Sulem [9] is recovered.

Substituting $h + \delta h$ for h , for some small δh , we have the first-order Taylor expansions

$$\begin{aligned} \cosh(\mathcal{D}(h + \delta h)) &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\mathcal{D}^{2n} (h + \delta h)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{\mathcal{D}^{2n} h^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{\mathcal{D}^{2n} h^{2n-1} \delta h}{(2n-1)!} + O(\delta h^2) \\ &= \cosh(\mathcal{D}h) + \mathcal{D} \sinh(\mathcal{D}h) \delta h + O(\delta h^2), \end{aligned} \quad (65)$$

$$\begin{aligned} \sinh(\mathcal{D}(h + \delta h)) &= \sum_{n=0}^{\infty} \frac{\mathcal{D}^{2n+1} h^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{\mathcal{D}^{2n+1} h^{2n} \delta h}{(2n)!} + O(\delta h^2) \\ &= \sinh(\mathcal{D}h) + \mathcal{D} \cosh(\mathcal{D}h) \delta h + O(\delta h^2), \end{aligned} \quad (66)$$

hence

$$\cosh(\mathcal{D}(h + \delta h))^{-1} = \left[1 - \cosh(\mathcal{D}h)^{-1} \mathcal{D} \sinh(\mathcal{D}h) \delta h \right] \cosh(\mathcal{D}h)^{-1} + O(\delta h^2). \quad (67)$$

We are then in position to compute explicitly the functional variations of the DNO.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, 1965.
- [2] T. Alazard and P. Baldi. Gravity capillary standing water waves. *Arch. Rational Mech. Anal.*, 217:741–830, 2015.
- [3] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198:71–163, 2012.
- [4] D. Andrade and A. Nachbin. A three-dimensional Dirichlet–Neumann operator for water waves over topography. *J. Fluid Mech.*, 845:321–345, 2018.
- [5] D. Clamond and J. Grue. A fast method for fully nonlinear water-wave computations. *J. Fluid Mech.*, 447:337–355, 2001.
- [6] A. Constantin and R. I. Ivanov. Equatorial wave-current interactions. *Comm. Math. Phys.*, 370:1–48, 2019.

- [7] A. Constantin, R. I. Ivanov, and C.-I. Martin. Hamiltonian formulation for wave-current interactions in stratified rotational flows. *Arch. Rational Mech. Anal.*, 221:1417–1447, 2016.
- [8] W. Craig and M. D. Groves. Hamiltonian long-wave approximations to the water-wave problem. *Wave Motion*, 19:367–389, 1994.
- [9] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comp. Phys.*, 108:73–83, 1993.
- [10] W. Craig, P. Guyenne, and H. Kalisch. Hamiltonian log-wave expansions for free surfaces and interfaces. *Comm. Pure App. Math.*, 58:1587–1641, 2005.
- [11] W. Craig, P. Guyenne, D. P. Nicholls, and C. Sulem. Hamiltonian long-wave expansions for water waves over a rough bottom. *Proc. R. Soc. A*, 461:839–873, 2005.
- [12] D. G. Dommermuth and D. K. P. Yue. A high-order spectral method for the study of nonlinear gravity waves. *J. Fluid. Mech.*, 184:267–288, 1987.
- [13] C. Fazioli and D. P. Nicholls. Stable computation of the functional variation of the Dirichlet–Neumann operator. *J. Comp. Phys.*, 229:906–920, 2010.
- [14] M. A. Fontelos, R. Lecaros, J. C. López-Ríos, and J. H. Ortega. Bottom detection through surface measurements on water waves. *SIAM J. Control Optim.*, 55(6):3890–3907, 2017.
- [15] D. Fructus, D. Clamond, J. Grue, and Ø. Kristiansen. Efficient numerical model for three-dimensional gravity waves simulations. Part I: Periodic domains. *J. Comp. Phys.*, 205:665–685, 2005.
- [16] D. Fructus and J. Grue. An explicit method for the nonlinear interaction between water waves and variable and moving bottom topography. *J. Comp. Phys.*, 222(2):720–739, 2007.
- [17] M. D. Groves and J. Horn. A variational formulation for steady surface water waves on a Beltrami flow. *Proc. R. Soc. A*, 476(2234):20190495, 2020.
- [18] T. Iguchi. A mathematical analysis of tsunami generation in shallow water due to seabed deformation. *Proc. Roy. Soc. Edinburgh Sect. A*, 141:551–608, 2011.
- [19] E. Isaacson and H. B. Keller. *Analysis of Numerical Methods*. Dover, 1994.
- [20] D. Lannes. *The water waves problem*, volume 188 of *Math. Surveys and Monographs*. Amer. Math. Soc., 2013.
- [21] L. M. Milne-Thomson. *Theoretical Hydrodynamics*. Dover Books on Physics. Dover, 5th edition, 2011.
- [22] D. P. Nicholls and F. Reitich. A new approach to analyticity of Dirichlet–Neumann operators. *Proc. Roy. Soc. Edinburgh A*, 131:1411–1433, 2001.
- [23] H. A. Schäffer. Comparison of Dirichlet–Neumann operator expansions for nonlinear surface gravity waves. *Coastal Engin.*, 55(4):288–294, 2008.
- [24] J. V. Wehausen and E. V. Laitone. Surface waves. In S. Flugge and C. Truesdell, editors, *Fluid Dynamics III*, volume IX of *Encyclopaedia of Physics*, pages 446–778. Springer-Verlag, 1960.
- [25] B. J. West, K. A. Brueckner, R. S. Janda, D. M. Milder, and R. L. Milton. A new numerical method for surface hydrodynamics. *J. Geophys. Res.*, 92:11803–11824, 1987.
- [26] J. Wilkening and V. Vasan. Comparison of five methods of computing the Dirichlet–Neumann operator for the water wave problem. In C. W. Curtis, A. Dzhamay, and W. A. Hereman, editors, *AMS Special Session on Nonlinear Wave and Integrable Systems*, volume 635 of *Contemporary Mathematics*, pages 175–210, 2015.
- [27] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.*, 9:1990–1994, 1968.

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