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Abstract. The aim of this paper is to study codimension one foliations on rational homogeneous spaces, with a focus on the moduli space of foliations of low degree on Grassmannians and cominuscule spaces. Using equivariant techniques, we show that codimension one degree zero foliations on (ordinary, orthogonal, symplectic) Grassmannians of lines, some spinor varieties, some Lagrangian Grassmannians, the Cayley plane (an E_6-variety) and the Freudenthal variety (an E_7-variety) are identified with restrictions of foliations on the ambient projective space. We also provide some evidence that such results can be extended beyond these cases.

1. Introduction

Let X be a compact connected complex manifold of dimension n. A a codimension-p holomorphic distribution on X is a rank $n-p$ saturated subsheaf of the tangent bundle $F \subset T_X$; here saturated means T_X/F is torsion free. If moreover F is stable under the Lie bracket, it is called a foliation. For a fixed line bundle $L \in \mathrm{Pic}(X)$ the space $\mathrm{Fol}_p(X,L)$ of codimension-p foliations with $\det(F) = L \otimes \omega_X^\vee$ is a locally closed subvariety of $\mathbb{P}(H^0(\Omega^p_X(L))^\vee)$. Being stable under the Lie bracket translates to a closed condition on the coefficients of the p-form, both are known as the Frobenius integrability condition.

For $p = 1$, as we shall explain in greater detail in §2, this condition is conveniently described by the quadratic map:

$$\psi_X \colon H^0(\Omega^1_X(L)) \to H^0(\Omega^3_X(2L)), \qquad \omega \mapsto \omega \wedge d\omega,$$

The zero-scheme of ψ_X inside $H^0(\Omega^1_X(L))$ is what we call the locus $\mathrm{IF}(X,L)$ of integrable forms, while we denote by $\mathrm{Dist}(X,L)$ the open set of 1-forms not vanishing in codimension one, as they correspond to distributions satisfying $\det(F) = L \otimes \omega_X^\vee$. Then

$$\mathrm{Fol}(X,L) := \mathrm{Fol}_1(X,L) = \mathrm{Dist}(X,L) \cap \mathrm{IF}(X,L) \subset \mathbb{P}(H^0(\Omega^1_X(L))^\vee).$$

The description of $\mathrm{Fol}_p(X,L)$ for given X and L is an interesting problem in the global theory of holomorphic foliations. The case $X = \mathbb{P}^n$ and $p = 1$ is already very challenging. Note that in this case $L \simeq \mathcal{O}_{\mathbb{P}^n}(d+2)$, with $d \geqslant 0$, so d is traditionally called the *degree* of the foliation. A full description of $\mathrm{Fol}(\mathbb{P}^n, d+2)$ is only known, at this moment, for $d \leqslant 2$. For degree $d = 0$ every foliation is given by a pencil of hyperplanes so $\mathrm{Fol}(\mathbb{P}^n, 2)$ is isomorphic to the Grassmannian $G(2, n+1)$, the inclusion in $\mathbb{P}(H^0(\Omega^p_{\mathbb{P}^n}(p+1))^\vee)$ being the Plücker embedding. It is unknown to us when this fact was first established but we refer to [DC05, Chapitre 3] and [ACM18, Theorem 4.3] for modern proofs. The case $d = 1$ was described in 1979 by Jouanolou [Jou79], $\mathrm{Fol}(\mathbb{P}^n, 3)$ has 2 irreducible components. The case $d = 2$ was established in 1996 by Cerveau and Lins Neto [CLN96]; $\mathrm{Fol}(\mathbb{P}^n, 4)$ has

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6 irreducible components. For $d = 3$ there exists, until this date, a partial classification due to da Costa, Lizarbe and Pereira [dCLP22]; they prove that $\mathrm{Fol}(\mathbb{P}^n, 5)$ has at least 24 components, some of them being not generically reduced – a phenomenon that was never observed before.

Many authors studied the geometry of foliations on other manifolds, especially when X is of low dimension (see [Bru97]) or when $-c_1(F)$ is positive or numerically trivial (see in particular [AD13, AD17] for $-c_1(F)$ ample and [LPT18] for $c_1(F) \equiv 0$). However, much less seems to be known about the behaviour of foliations under restriction, our main inspiration being [ACM18], where special attention is paid to the case of complete intersections.

The aim of our work is to describe the space $\mathrm{Fol}(X, d+2)$ of codimension one foliations on a manifold X which is G-homogeneous for the action of a simple complex Lie group G, bearing in mind that a prominent role should be played by the representation theory of G, or of the stabiliser P of a point of X. The spaces we consider are Grassmanians in their Plücker embedding, or more generally *cominuscule Grassmannians*, see §3.2, since for these varieties we only need the representation theory of the semisimple part of P, which affords a major simplification of our analysis. For a G-homogeneous variety X, there is an irreducible G-representation V such that $X \subset \mathbb{P}(V)$ is the minimal G-equivariant embedding. All line bundles on X are of the form $\mathcal{O}_X(t)$ for some $t \in \mathbb{Z}$, where $\mathcal{O}_X(1)$ is the G-linearized hyperplane section bundle of $X \subset \mathbb{P}(V)$. Then, considering the natural restriction maps i_p^* of p-forms from $\mathbb{P}$ to X, we get the following result.

Theorem A. *Let $X \subset \mathbb{P}(V)$ be a cominuscule variety. Then, for $d, p \geqslant 0$:*

i) The restriction map $i_p^*\colon H^0(\Omega^p_{\mathbb{P}(V)}(d+2)) \to H^0(\Omega^p_X(d+2))$ is surjective.
ii) The map $\pi\colon \mathrm{Fol}(\mathbb{P}(V), 2) \to \mathrm{Fol}(X, 2)$ induced by i_1^* is injective.
iii) The space of integrable forms $\mathrm{IF}(X, d+2) \subset \mathbb{P}(H^0(\Omega^1_X(d+2))^\vee)$ is defined by the quadratic equations given by the G*-equivariant inclusion:*

$$H^0(\Omega^3_X(d+2))^\vee \subset S^2 H^0(\Omega^1_X(d+2))^\vee,$$

induced by $\Psi^\vee_{\mathbb{P}(V)} \circ (i_3^*)^\vee = \pi^* \circ \Psi^\vee_X$.

The result above is a summary of statements spread along the paper – for instance i) is Corollary 3.6, while iii) is Theorem 3.7 – and some of these claims are actually proved in greater generality – for example ii) is shown in Proposition 2.8 with no need for homogeneity.

Then we look in more detail at the case of Grassmannians, in particular Grassmannians of lines, and a few other cominuscule varieties, where our results are particularly neat.

Theorem B. *Let $X \subset \mathbb{P}(V)$ be a cominuscule variety which is not a linear space nor a quadric. Then, restriction of 1-forms gives an isomorphism:*

$$\mathrm{Fol}(X, 2) \simeq \mathrm{Fol}(\mathbb{P}(V), 2) \simeq G(2, \bigwedge^2 V)$$

in the following cases:

i) Grassmannians of lines, $X = G(2, n)$;
ii) spinor varieties $OG(n, 2n)$, for $n = 4, 5$;
iii) the Cayley plane, $X = E_6/\mathrm{P}_1$;

iv) the following four varieties appearing in Freudenthal's magic square:

$$LG(3,6), \qquad G(3,6), \qquad OG(6,12), \qquad E_7/P_7.$$

A key point here is that in all cases above except iv) the isomorphism takes place because the maps i_1^* and i_3^* are isomorphisms (this is shown for items i), ii) and iii) respectively in Theorems 4.1, 6.1 and 6.4), so Theorem A is enough to conclude. This fact is true as well for quadrics of dimension at least 3, and these cases essentially cover all cominuscule Grassmannians for which this happens.

On the other hand, in case iv) we show in Theorem 6.6 that the map i_1^* induces a linear projection from a distinguished point lying away from $\mathrm{Fol}(\mathbb{P}(V),2)$ and that this map is actually an isomorphism onto $\mathrm{Fol}(X,2)$. We provide a uniform proof for these four cases, based on the observation that the point used to define the projection corresponds to G-invariant contact form on V. This is also the reason for the apparently awkward choice of listing the spinor variety $OG(6,12)$ among the cases of iv) rather than in ii).

To our knowledge, the only case known previously of Theorem B is $G(2,5)$, see [ACM18].

This leads to the expectation that i_1^* induces an isomorphism $\mathrm{Fol}(X,d+2) \simeq \mathrm{Fol}(\mathbb{P}(V),d+2)$ for all cominuscule Grassmannians and small d, though the evidence we provide is only for $d=0$ and mainly $G(3,n)$. Indeed, in this case Theorem 4.3 proves that i_1^* induces an embedding rather than a set-theoretic injection and moreover the ideal of its image agrees with that of $\mathrm{Fol}(G(3,n),2)$ up to degree 2. We prove more results in this direction, even slightly more generally than for homogeneous spaces, for instance in Theorem 5.2 we address the case of isotropic Grassiammians of lines for a skew-symmetric form of maximal rank. However, one should be warned that i_1^* sometimes induces a proper inclusion, for instance this happens for the (non-cominuscule) variety of isotropic lines for a non-degenerate quadratic form, see Proposition 5.4, and for products of projective spaces, see Proposition 6.11.

The paper is organized as follows. In Section 2 we introduce distributions, foliations and integrable forms taking values on a given line bundle on a manifold, with a focus on projective spaces, whereby defining the quadratic equations of integrability and the restriction maps mentioned above. In Section 3 we start the discussion of homogeneous spaces, recall the list of cominuscule varieties and provide some basic results about distributions and foliations over them. In Section 4 we look more closely to Grassmannians $G(k,n)$ by first treating the case $k=2$, then moving to $k=3$ and finally analyzing the special case $(k,n)=(3,6)$. In Section 5 we look at Grassmannians of isotropic lines, while Section 6 is devoted to other cominuscule spaces like spinor varieties or the Cayley plane. Finally, in Section 7 we discuss some further directions and open problems. The Appendix is devoted to some representation-theoretic lemmas needed for our treatment of $G(3,n)$.

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2. Distributions and Foliations

Let X be a smooth complex projective manifold of dimension n. A *codimension p distribution $\mathscr{F}$ on X* is a saturated subsheaf F of the tangent sheaf T_X of generic rank $n-p$. The inclusion $F \subset T_X$ induces an exact sequence of the form

$$\mathscr{F}\colon 0 \longrightarrow F \xrightarrow{\phi} T_X \xrightarrow{\eta} N \longrightarrow 0 \tag{1}$$

where N, called the *normal sheaf* of $\mathscr{F}$, is a torsion free sheaf of rank p. It follows that F, called the *tangent sheaf* of $\mathscr{F}$, must be a reflexive sheaf. Two codimension p distributions $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic if $\phi(F) = \phi'(F')$ as subsheaves of T_X.

Consider the induced morphism $\wedge^{n-p}\phi^\vee\colon \Omega_X^{n-p} \to \det(F)^\vee$; its image is the ideal sheaf $\mathcal{I}_Z$ of a subscheme Z of codimension at least 2 in X, twisted by $\det(F)^\vee$; Z is called the *singular scheme* of $\mathscr{F}$. The isomorphism $\bigwedge^{n-p} T_X \simeq \Omega_X^p(\omega_X^\vee)$ tells us that $\wedge^{n-p}\phi^\vee$ defines an element $\wedge^{n-p}\phi^\vee \in H^0(\Omega_X^p(\omega_X^\vee \otimes \det(F)^\vee))$. The induced contraction morphism

$$\wedge^{n-p}\phi^\vee : T_X \to \Omega_X^{p-1}(\omega_X^\vee \otimes \det(F)^\vee)$$

has $\phi(F)$ as its kernel, hence there exists an isomorphism $\beta\colon \operatorname{im}(\wedge^{n-p}\phi^\vee) \to N$ such that $\beta \circ \wedge^{n-p}\phi^\vee = \eta$ in (1).

Conversely let $\omega \in H^0(\Omega_X^p(\omega_X^\vee \otimes \det(F)^\vee))$ not vanishing in codimension one, it defines a codimension p distribution if and only if $F_\omega := \ker(\omega\colon T_X \to \Omega_X^{p-1}(\omega_X^\vee))$ has (generic) rank $n-p$. Due to [dM00, Proposition 1.2.1], $\operatorname{rk} F_\omega = n-p$ if and only if for every (closed) point $x \in X\backslash|Z|$ there exists an (affine) open neighborhood U of x and $\alpha_1, \dots, \alpha_p \in H^0(\Omega_U^1)$ such that ω decomposes as

$$\omega|_U = \alpha_1 \wedge \cdots \wedge \alpha_p. \tag{2}$$

Such p-forms are called *locally decomposable off the singular set* – LDS for short.

Therefore we get a set-theoretical bijection, for each $L \in \operatorname{Pic}(X)$,

$$\left\{\begin{matrix} F \hookrightarrow T_X \text{ saturated subsheaf of rank} \\ n-p,\ \det(F) = L^\vee \otimes \omega_X^\vee \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} [\omega] \in \mathbb{P}(H^0(\Omega_X^p(L))^\vee) \text{ LDS not} \\ \text{vanishing in codimension one} \end{matrix}\right\}$$

that to $\phi\colon F \to T_X$ associates $\omega = \wedge^{n-p}\phi^\vee$ and to ω associates F_ω as above. Then we define the algebraic set

$$\operatorname{Dist}_p(X, L) := \{[\omega] \in \mathbb{P}(H^0(\Omega_X^p(L))^\vee) \mid \omega \text{ is LDS},\ \operatorname{codim}\operatorname{Sing}(\omega) \geqslant 2\}$$

that parameterizes codimension p distributions with fixed determinant $\det(F) = L^\vee \otimes \omega_X^\vee$. We remark that $\operatorname{Dist}_p(X, L)$ is not a moduli space in the sense that it does not represent a functor parameterizing distributions. But it can be stratified into moduli spaces by fixing the Hilbert polynomial of F, see for instance [Qua15, CJM22].

A distribution is called *integrable* if it defines a *foliation*, which means that for each $x \in X\backslash|Z|$ there exists an unique analytic immersed subvariety $S \hookrightarrow X$ passing through x such that $T_{Sx} = F \otimes \mathcal{O}_{S,x}$. Due to a theorem of Frobenius, integrability of $\mathscr{F}$ is equivalent to $[F, F] \subset F$, where $[\cdot,\cdot]$ is the Lie bracket on T_X. In terms of differential forms this integrability condition reads locally as

$$d\alpha_j \wedge \alpha_1 \wedge \cdots \wedge \alpha_p = 0 \tag{3}$$

where α_j are the 1-forms from (2). If integrability holds we simply say that $\mathscr{F}$ is a foliation. Thus we define the (quasi-projective) algebraic set

$$\operatorname{Fol}_p(X, L) := \{[\omega] \in \operatorname{Dist}_p(X, L) \mid \omega \text{ is integrable}\}$$

whose points correspond to foliations on X with determinant $\det(F) = L^\vee \otimes \omega_X^\vee$.

Notation 2.1. In our cases of interest X will be embedded in some projective space and we will denote $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_X$. Hence we will write $\mathrm{Fol}_p(X,l) := \mathrm{Fol}_p(X, \mathcal{O}_X(l))$ and when $p = 1$ we may just write $\mathrm{Fol}(X,l) := \mathrm{Fol}_1(X, \mathcal{O}_X(l))$; the same for Dist and IF that will be defined below. Moreover, when $X = \mathbb{P}^n$ it is common to write $L = \mathcal{O}_{\mathbb{P}^n}(d+p+1)$, the integer d is called the *degree* of the foliation. Then $\mathrm{Fol}(\mathbb{P}^n, d+2)$ is the space of codimension one degree d foliations on $\mathbb{P}^n$.

For our purposes it will be useful to define also the set of general integrable forms

$$\mathrm{IF}_p(X,L) := \{[\omega] \in \mathbb{P}(H^0(\Omega^p_X(L))^\vee) \mid \omega \text{ is integrable}\}.$$

Notice that $\mathrm{Fol}_p(X,L) = \mathrm{IF}_p(X,L) \cap \mathrm{Dist}_p(X,L)$. Moreover, if $H^0(\Omega^p_X(L(-D))) = 0$ for every $D \neq 0$ effective divisor, then $\mathrm{Fol}_p(X,L) = \mathrm{IF}_p(X,L)$.

2.1. **Distributions and foliations on a projective space.** Let V be a (finite dimensional) complex vector space and let $\mathbb{P}(V)$ the associated projective space (of one dimensional quotients); in particular $V = H^0(\mathcal{O}_{\mathbb{P}(V)}(1))$. We will write $\mathbb{P}^n := \mathbb{P}(V)$, $n = \dim V$, unless we need to specify V. In the later sections we will be interested in describing distributions and foliations under the action of a semisimple linear algebraic group G, so that V will be a G-module. In order to do so we first establish a dictionary between the language of differential forms and representations of $\mathrm{SL}(V)$. For the representation theory of $\mathrm{SL}(V)$ and in particular Schur functors we refer to [Wey03].

First we note that distributions and foliations on $\mathbb{P}^n$ can be described via homogeneous polynomial differential forms. Recall the Euler sequence:

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes V \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

Taking exterior powers, twists and global sections we get

$$H^0\big(\Omega^p_{\mathbb{P}^n}(d+p+1)\big) \hookrightarrow S^{d+1}V \otimes \bigwedge^p V \xrightarrow{\iota_R} S^{d+2}V \otimes \bigwedge^{p-1} V \tag{4}$$

where S^kV is the k-th symmetric power of V. We have written V for the vector space generated by the homogeneous coordinates $\{x_j\}$ and for the space of their differentials $\{dx_j\}$. Similarly $H^0(T_{\mathbb{P}^n}(-1)) = V^\vee$ is generated by the (rational) vector fields $\frac{\partial}{\partial x_j}$. The map ι_R is the contraction with the radial vector field $R = \sum_j x_j \frac{\partial}{\partial x_j}$ and it can be written as the composition

$$\iota_R \colon S^{d+1}V \otimes \bigwedge^p V \xrightarrow{1\otimes\Delta} S^{d+1}V \otimes V \otimes \bigwedge^{p-1} V \xrightarrow{m\otimes 1} S^{d+2}V \otimes \bigwedge^{p-1} V, \tag{5}$$

where $m \colon S^{d+1}V \otimes V \to S^{d+2}V$ is the multiplication map $m(p(x) \otimes q(x)) = p(x)q(x)$ with p, q homogeneous polynomials of degrees $d+1$ and 1 respectively; and $\Delta \colon \bigwedge^p V \to V \otimes \bigwedge^{p-1} V$ is the diagonal map given by

$$\Delta(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} x_{i_j} \otimes dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_p}.$$

Therefore the contraction map with the radial vector field is

$$\iota_R(p(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = p(x) \sum_{j=1}^{p} (-1)^{j+1} x_{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_p}.$$

From this discussion we get that $H^0(\Omega^p_{\mathbb{P}^n}(d+p+1))$ are represented by homogeneous polynomial differential p-forms

$$\omega = \sum A_{i_1,\dots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \text{ such that } \iota_R \omega = 0.$$

On the other hand, computing the Lie derivative of ω with respect to R gives $(d+p+1)\omega$ because of homogeneity. Hence

$$\omega = \frac{1}{d+p+1}\mathcal{L}_R(\omega) = \frac{1}{d+p+1}(\iota_R d\omega + d\iota_R\omega) = \iota_R \frac{1}{d+p+1} d\omega$$

and $H^0(\Omega^p_{\mathbb{P}^n}(d+p+1))$ can also be seen as the image of

$$\iota_R \colon S^r V \otimes \bigwedge^{p+1} V \longrightarrow S^{d+1} V \otimes \bigwedge^{p} V. \tag{6}$$

Working with polynomial differential forms also simplifies the verification of the LDS and integrability conditions (2) and (3), since it can be done globally. Indeed, as a consequence of [dM00], ω is LDS if and only if

$$(\iota_u \omega) \wedge \omega = 0 \quad \forall u \in \bigwedge^{p-1} V^\vee \tag{7}$$

and it is integrable if in addition

$$(\iota_u \omega) \wedge d\omega = 0 \quad \forall u \in \bigwedge^{p-1} V^\vee. \tag{8}$$

Notation 2.2. We denote by $\Gamma^\lambda V$ the Schur functor of a decreasing sequence of integers $\lambda = (\lambda_1, \cdots, \lambda_k)$ applied to the vector space V. For instance $\Gamma^{(q)}V$ is just the symmetric power $S^q V$, while $\Gamma^{(1,\cdots,1)}V = \Gamma^{(1^k)}V$ is the exterior power $\bigwedge^k V$.

Remark 2.3. The sequence (4) is $\mathrm{SL}(V)$-equivariant, hence we can describe the space $H^0(\Omega^p_{\mathbb{P}^n}(d+p+1))$ in terms of irreducible representations, i.e. Schur functors applied to V. Indeed, from (5) we deduce that

$$H^0\big(\Omega^p_{\mathbb{P}^n}(d+p+1)\big) = \Gamma^{(d+1,1^p)}V;$$

in particular it is always an irreducible $\mathrm{SL}(V)$-module. This is established using [Wey03, example 2.1.17 (h)] (and noting that $\Gamma^{(d+1,1^p)}V = (K_{(d+1,1^p)}V^\vee)^\vee = L_{(p+1,1^r)}V$ in Weyman's notation). This result can also be obtained by using the Bott-Borel-Weil Theorem.

2.2. **The space of integrable 1-forms.** Hereafter we fix $p = 1$ and let X be a smooth complex projective variety. Let $L \in \mathrm{Pic}(X)$ and consider $\omega \in H^0(\Omega^1_X(L))$. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open (affine) covering that trivializes L and write $\omega = \{\omega_\alpha\}_{\alpha\in\Lambda}$, where $\omega_\alpha \in \Omega^1_X(U_\alpha)$ are such that, on $U_\alpha \cap U_\beta \neq \varnothing$, $\omega_\alpha = g_{\alpha\beta}\omega_\beta$ for $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ the cocycle of L. The integrability of ω is measured by the vanishing of $\omega_\alpha \wedge d(\omega_\alpha)$ for each α, see (3). Note that

$$\omega_\alpha \wedge d(\omega_\alpha) = g^2_{\alpha\beta}\omega_\beta \wedge d(\omega_\beta)$$

and $\{\omega_\alpha \wedge d(\omega_\alpha)\}_{\alpha\in\Lambda}$ defines a section in $H^0(\Omega^3_X(2L))$ that, by abuse of notation, we call $\omega \wedge d\omega$. Then we may say that

$$\omega \text{ is integrable} \iff \omega \wedge d\omega = 0.$$

Thus we get a quadratic map $\psi_X\colon H^0(\Omega^1_X(L)) \to H^0(\Omega^3_X(2L)), \omega \mapsto \omega \wedge d\omega$ whose vanishing locus is the cone over $\mathrm{IF}(X,L)$. Consider then the polarization

$$\begin{aligned}\Psi_X\colon S^2H^0(\Omega^1_X(L)) &\longrightarrow H^0(\Omega^3_X(2L)) \quad ,\\ \omega\cdot\eta &\longmapsto \frac{1}{2}(\omega\wedge d\eta + \eta\wedge d\omega)\end{aligned}$$

so that $\Psi_X(\omega\cdot\omega) = \psi_X(\omega)$, and notice that $S^2H^0(\Omega^1_X(L))^\vee = H^0(\mathcal{O}_{\mathbb{P}(H^0(\Omega^1_X(L))^\vee)}(2))$. Hence dualizing we get that

$$(\operatorname{im}\Psi_X)^\vee \hookrightarrow S^2H^0(\Omega^1_X(L))^\vee \tag{9}$$

is precisely the truncation in degree 2 of the homogeneous ideal of $\mathrm{IF}(X,L)$.

Lemma 2.4. *Let X be a smooth complex projective variety and let $L \in \mathrm{Pic}(X)$. Then the space of integrable 1-forms $\mathrm{IF}(X,L)$ is defined by the quadratic polynomials from (9).*

Proof. By construction $(\operatorname{im}\Psi_X)^\vee$ generates the whole ideal of integrable 1-forms. □

In the case of projective spaces we can say more, $\Psi_{\mathbb{P}^n}$ being always surjective.

Lemma 2.5. *Let $V \simeq \mathbb{C}^{n+1}$, $n \geqslant 3$, and let $l \geqslant 2$. Then*

$$\Psi_{\mathbb{P}^n}\colon S^2H^0(\Omega^1_{\mathbb{P}^n}(l)) \longrightarrow H^0(\Omega^3_{\mathbb{P}^n}(2l))$$

is surjective. As a consequence, the ideal of $\mathrm{IF}(\mathbb{P}^n,l)$ is generated by the quadratic polynomials given by

$$H^0(\Omega^3_{\mathbb{P}^n}(2l))^\vee \hookrightarrow S^2H^0(\Omega^1_{\mathbb{P}^n}(l))^\vee$$

Proof. Since pullbacks commute with exterior differential and exterior product of differential forms, we see that $\Psi_{\mathbb{P}^n}$ is $\mathrm{SL}(V)$-equivariant. From Remark 2.3, $H^0(\Omega^3_{\mathbb{P}^n}(2l))$ is an irreducible $\mathrm{SL}(V)$-module and this implies, by Schur's Lemma, that $\Psi_{\mathbb{P}^n}$ is either surjective or the zero map. In order to show that it is surjective, it is enough to provide an element $\omega \in H^0(\Omega^1_{\mathbb{P}^n}(l))$ such that $\omega\wedge d\omega \neq 0$.

Consider $\{x_0,\dots,x_n\}$ a basis of V and define

$$\omega = x_0^{l-2}(x_0dx_1 - x_1dx_0 + x_2dx_3 - x_3dx_2).$$

It is clear that

$$\begin{aligned}\omega\wedge d\omega = x_0^{2l-4}(&x_0dx_1\wedge dx_2\wedge dx_3 - x_1dx_0\wedge dx_2\wedge dx_3 +\\ &x_2dx_0\wedge dx_1\wedge dx_3 - x_3dx_0\wedge dx_1\wedge dx_2) \neq 0,\end{aligned}$$

concluding the proof. □

Remark 2.6. In the special case $l=2$ it follows from (6) that we have the diagram

$$\begin{array}{ccc} S^2\bigwedge^2 V & \xrightarrow{\Psi_{\mathbb{P}^n}} & \bigwedge^4 V \\ \downarrow\iota_R & & \downarrow\iota_R \\ S^2H^0(\Omega^1_{\mathbb{P}^n}(l)) & \xrightarrow{\Psi_{\mathbb{P}^n}} & H^0(\Omega^3_{\mathbb{P}^n}(2l)) \end{array}$$

whose vertical arrows are isomorphisms due to Remark 2.3. Then there exists a unique choice for the top horizontal arrow in order to make this diagram commutative, this is precisely the map $u\cdot v \mapsto u\wedge v$. By abuse of notation we also call this map $\Psi_{\mathbb{P}^n}$. We remark

that in this case $(\operatorname{im}\Psi_{\mathbb{P}^n})^\vee = \bigwedge^4 V^\vee$ is generated by the Plücker relations. From (2.4) we get that $\mathrm{Fol}(\mathbb{P}^n, 2) \simeq G(2, V^\vee)$. This is essentially the argument of [ACM18, Theorem 4.3]; in fact, they use the more general map $S^k \bigwedge^2 V \mapsto \bigwedge^{2k} V$ to get an isomorphism between the k-th secant variety of $G(2, V^\vee)$ and the space of so-called *class k distributions*, class 0 meaning integrable.

2.3. **Distributions and foliations on a projective variety.** Now consider a smooth subvariety $i\colon X \hookrightarrow \mathbb{P}^n$ and denote $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^n}(1)$. The embedding i induces the pullback (restriction) of twisted differential forms $i_p^*\colon \Omega^p_{\mathbb{P}^n}(k) \to \Omega^p_X(k)$. This morphism is surjective since we can express $i_p^* = \wedge^p i_1^* \otimes 1$ and i_1^* is the composition of the two surjections appearing in the following sequences:

$$0 \longrightarrow N_X^\vee \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0, \tag{10}$$

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes \mathcal{I}_X \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \longrightarrow 0. \tag{11}$$

Taking global sections gives us

$$i_p^*\colon H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(l)) \longrightarrow H^0(\Omega^p_X(l)) \tag{12}$$

for each p and l and it also induces a rational map between projective spaces

$$\pi_p\colon \mathbb{P}(H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(l))^\vee) \dashrightarrow \mathbb{P}(H^0(\Omega^p_X(l))^\vee).$$

Since pullbacks commute with exterior products and exterior differentials, the closure of the image of $\mathrm{IF}_p(\mathbb{P}^n, l)$ is contained in $\mathrm{IF}_p(X, l)$:

$$\overline{\pi_p(\mathrm{IF}_p(\mathbb{P}^n, l))} \subset \mathrm{IF}_p(X, l).$$

For $p \geqslant 2$ and general X the integrability condition is not easy to check. To study it for $p = 1$, we write the commutative diagram:

$$\begin{array}{ccc} S^2H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(l)) & \xrightarrow{S^2 i_1^*} & S^2H^0(\Omega^1_X(l)) \\ \downarrow \Psi_{\mathbb{P}^n} & & \downarrow \Psi_X \\ H^0(\mathbb{P}^n, \Omega^3_{\mathbb{P}^n}(2l)) & \xrightarrow{i_3^*} & H^0(\Omega^3_X(2l)). \end{array} \tag{13}$$

To simplify notation we write $W_X^l := H^0(\Omega^1_X(l))^\vee$ and $\pi = \pi_1$, so that $\mathrm{IF}(X, l) \subset \mathbb{P}(W^l_X)$ and the rational map we are concerned with is $\pi\colon \mathbb{P}(W^l_{\mathbb{P}^n}) \dashrightarrow \mathbb{P}(W^l_X)$. Hence the dual of (13) can be written as

$$\begin{array}{ccc} H^0(\mathbb{P}(W^l_{\mathbb{P}^n}), \mathcal{O}_{\mathbb{P}(W^l_{\mathbb{P}^n})}(2)) & \xleftarrow{S^2 i_1^*} & H^0(\mathbb{P}(W^l_X), \mathcal{O}_{\mathbb{P}(W^l_X)}(2)) \\ \Psi^\vee_{\mathbb{P}^n} \uparrow & & \Psi^\vee_X \uparrow \\ H^0(\mathbb{P}^n, \Omega^3_{\mathbb{P}^n}(2l))^\vee & \longleftarrow & H^0(\Omega^3_X(2l))^\vee. \end{array} \tag{14}$$

If i_1^* is surjective then every twisted 1-form in $H^0(\Omega^1_X(l))$ is the pullback of an element of $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(l))$. This raises the question of whether the integrable 1-forms on X as well come from integrable 1-forms on $\mathbb{P}^n$; in other words, when do we have the equality

$$\overline{\pi(\mathrm{IF}(\mathbb{P}^n, l))} = \mathrm{IF}(X, l)?$$

In this direction we have the following result.

Lemma 2.7. *Let X be a smooth complex projective variety and let $l \geqslant 2$ be an integer. Assume there exists an embedding $i\colon X \hookrightarrow \mathbb{P}^n$ such that $i_1^*\colon H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(l)) \to H^0(\Omega^1_X(l))$ and $i_3^*\colon H^0(\mathbb{P}^n, \Omega^3_{\mathbb{P}^n}(2l)) \to H^0(\Omega^3_X(2l))$ are surjective. Then*

(1) Ψ_X is surjective and the ideal of $\mathrm{IF}(X,d)$ is generated by the quadratic equations
$$\Psi_X^\vee(H^0(\Omega^3_X(2l))^\vee) \subset S^2H^0(\Omega^1_X(l))^\vee.$$
(2) If the square (14) is cartesian then we have equality on the degree 2 part of the ideals:
$$(I_{\overline{\pi(\mathrm{IF}(\mathbb{P}^n,l))}})_2 = (I_{\mathrm{IF}(X,l)})_2.$$

We remark that an analogous statement holds without assuming the surjectivity of i_3^* but instead replacing $H^0(\Omega^3_X(2l))$ by $\operatorname{im}\Psi_X$ in (13). However the surjectivity of both maps hold in our cases of interest.

Note also that if $\overline{\pi(\mathrm{IF}_p(\mathbb{P}^n,l))}$ is defined by quadrics and (2) holds then we get an isomorphism $\mathrm{IF}(\mathbb{P}^n,l) \simeq \mathrm{IF}(X,l)$. However the image $\overline{\pi(Z)}$ via a linear projection $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ of a variety $Z \subset \mathbb{P}^n$ defined by quadrics does not need to be defined by quadrics. For instance, if Z is a rational normal curve of degree $\geqslant 4$ and π is a projection from a point, then $\overline{\pi(Z)}$ is cut out by quadrics and cubics, see[AR02, Example 4.3].

Proof of Lemma 2.7 . Notice that since i_1^* is surjective, so is $S^2i_1^*$. Adding that $\Psi_{\mathbb{P}^n}$ is surjective, due to Lemma 2.5, we see that $\operatorname{im}\Psi_X = \operatorname{im} i_3^* = H^0(\Omega^3_X(2l))$.

From Lemma 2.4 we know that the vertical arrows of (14) are the inclusions of the ideals of $\mathrm{IF}(\mathbb{P}^n,l)$ and $\mathrm{IF}(X,l)$, proving our first assertion.

For the moment denote $Y := \mathrm{IF}(\mathbb{P}^n,l)$ and $Z := \mathrm{IF}(X,l)$. By definition,
$$(I_{\overline{\pi(Y)}})_2 = (I_Y)_2 \cap H^0(\mathcal{O}_{\mathbb{P}(W_X^l)}(2)),$$
i.e. $(I_{\overline{\pi(Y)}})_2$ is the pullback of $\Psi^\vee_{\mathbb{P}^n}$ by $S^2i_1^*$. By the universal property of pullback diagrams, (14) is cartesian if and only if the natural inclusion
$$H^0(\Omega^3_X(2l))^\vee = (I_Z)_2 \subset (I_{\overline{\pi(Y)}})_2$$
is an equality. □

Proposition 2.8. *Let X be a smooth projective variety such that $\mathrm{Pic}(X) = \mathbb{Z}$ with a very ample generator $\mathcal{O}_X(1)$. Consider the embedding $i\colon X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(1)))$. Then the restricted pullback map*
$$\pi\colon \mathrm{Fol}(\mathbb{P}(H^0(\mathcal{O}_X(1))),2) \to \mathrm{Fol}(X,2)$$
is injective.

Proof. Fix $\mathbb{P} := \mathbb{P}(H^0(\mathcal{O}_X(1)))$. Then we have $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(1)) = H^0(\mathcal{O}_X(1))$. It follows that π is well-defined on $\mathrm{Fol}(\mathbb{P}^n,2)$. Indeed, if π were not defined at $[\omega]$ with $\omega = fdg - gdf$ then X would be contained in some hyperplane of the pencil generated by f and g, but that is not possible.

Now let $\omega = fdg - gdf$ and $\omega' = f'dg' - g'df'$ such that their image define the same foliation, i.e. $i_1^*\omega = \lambda i_1^*\omega'$ for some $\lambda \in \mathbb{C}^*$. Passing to the pencils we see that there exists a Möbius transformation τ such that $\frac{f'}{g'}|_X = \tau(\frac{f}{g}|_X)$. Hence, up to composing with τ^{-1}, we may assume that $\frac{f'}{g'}|_X = \frac{f}{g}|_X$, which means
$$fg' - f'g \in H^0(\mathbb{P},\mathcal{I}_X(2)).$$

Observe that $fg' - f'g$ is a quadratic polynomial of rank 4; here we mean the rank of the associated symmetric (Hessian) matrix. To conclude we claim that under our hypothesis there exist no quadric of rank $\leqslant 4$ in the ideal of X. We will prove this claim in Lemma 2.9. □

Lemma 2.9. *Let X be a smooth projective variety such that $\mathrm{Pic}(X) = \mathbb{Z}$ with a very ample generator $\mathcal{O}_X(1)$. Consider the embedding $i\colon X \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(\mathcal{O}_X(1)))$. Then there is no element in $H^0(\mathbb{P}, \mathcal{I}_X(2))$ of rank $\leqslant 4$.*

Proof. Assume, aiming at a contradiction, that there exists $q \in H^0(\mathbb{P}, \mathcal{I}_X(2))\backslash\{0\}$ of rank $\leqslant 4$. Then, there exist linear polynomials $a, b, c, d \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ such that $q = ad - bc$. Then define the morphism

$$A\colon \mathcal{O}_X(-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} a|_X & b|_X \\ c|_X & d|_X \end{pmatrix}} \mathcal{O}_X^{\oplus 2}$$

whose generic rank is equal to one since $q \in H^0(\mathbb{P}, \mathcal{I}_X(2))$ and $H^0(\mathbb{P}, \mathcal{I}_X(1)) = 0$. Hence $\operatorname{im} A$ is a rank one torsion free sheaf. Since X is smooth (integral and locally factorial) $\operatorname{im} A = \mathcal{I}_Z(l)$ for some subscheme $Z \subset X$ of codimension $\geqslant 2$ and $l \in \mathbb{Z}$. Moreover, $\ker A = \mathcal{O}_X(-2-l)$. From $\mathcal{O}_X(1)$ being a generator of $\mathrm{Pic}(X)$ we deduce that $l \in \{-1, 0\}$ and we have two possibilities:

(1) $l = -1$: Then $\ker A = \mathcal{O}_X(-2-l) = \mathcal{O}_X(-1)$ and the inclusion $\ker A \hookrightarrow \mathcal{O}_X(-1)^{\oplus 2}$ is given by a constant vector $(u, v) \in H^0(\mathcal{O}_X)^{\oplus 2}$. Up to multiplying A on the left with some element of $\mathrm{GL}(2, \mathbb{C})$ we can assume that $(u, v) = (1, 0)$, thus

$$\begin{bmatrix} a|_X \\ c|_X \end{bmatrix} = \begin{pmatrix} a|_X & b|_X \\ c|_X & d|_X \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = H^0(\mathcal{O}_X(1))$ we have that $a = c = 0$ whence $q = 0$.

(2) $l = 0$: Then $\operatorname{im} A = \mathcal{I}_Z$ and the inclusion $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X^{\oplus 2}$ must factor as

$$\mathcal{I}_Z \hookrightarrow \mathcal{O}_X \hookrightarrow \mathcal{O}_X^{\oplus 2},$$

and the map on the right is given by a constant vector that we may assume to be $(1, 0)$. It follows that $c|_X = d|_X = 0$ and, as in the previous case, this implies $q = 0$.

□

We remark that we did not use that X is smooth but only that a rank one reflexive sheaf is locally free. This holds for X an integral and locally factorial variety, see [Har80, Proposition 1.9] for a proof.

3. Homogeneous spaces

We will focus now on homogeneous varieties, i.e., varieties which admit a transitive action of a Lie group. Using this action it will be possible, in some cases, to understand what are the codimension one minimal degree foliations on these varieties. Let us give a brief introduction to homogeneous varieties before studying 1-forms and foliations on them; we refer to [Ott95] for more details.

Let G be a Lie group over $\mathbb{C}$ and X a projective G-homogeneous variety. Then X is the product of an abelian variety by a rational variety. Let us suppose that X is

rational, so that we get rid of the abelian factor. One can then replace G by its image inside the automorphism group of X and thus suppose that G is a semisimple affine Lie group. By the transitivity assumption, the stabilizers of all the points in X are conjugated to some subgroup $\mathrm{P} \subset \mathrm{G}$. The fact that X is projective is equivalent to the fact that P is *parabolic*, i.e. that P contains a Borel (maximal connected solvable algebraic) subgroup of G. Sometimes we will write $X = \mathrm{G}/\mathrm{P}$ to make the group and the parabolic subgroup explicit. Recall finally that X has Picard number equal to one if and only if P is maximal (for inclusion); we will call such varieties *generalized Grassmannians*, following the literature.

Example 3.1. Let us fix $\mathrm{G} = \mathrm{SL}(n+1)$. Then a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ is given by upper triangular matrices. A parabolic subgroup P containing B can be written as a product $\mathrm{P} = \mathrm{DB}$, where

$$\mathrm{D} := \{(g_1, \cdots, g_{k+1}) \in \mathrm{GL}_{i_1} \times \cdots \times \mathrm{GL}_{i_k - i_{k-1}} \times \mathrm{GL}_{n+1-i_k} \mid \det(g_1) \cdots \det(g_{k+1}) = 1\}$$

is a subgroup of block diagonal matrices of fixed size given by an integer sequence $1 \leqslant i_1 < \cdots < i_k \leqslant n$. The quotient $X := \mathrm{G}/\mathrm{P}$ is then a Flag variety $\mathrm{Fl}(i_1, \cdots, i_k, n+1)$ parametrizing flags $[\mathbb{C}^{i_1} \subset \cdots \subset \mathbb{C}^{i_k}]$ contained in a $(n+1)$-dimensional vector space. The quotient G/B is thus the complete flag variety $\mathrm{Fl}(1, 2, \cdots, n+1)$, while Grassmannians and projective spaces are obtained by setting $k = 1$ and correspond to maximal parabolic subgroups. A similar description via flags of isotropic subspaces exists for rational homogeneous varieties for the classical groups $\mathrm{Sp}(2n)$ and $\mathrm{SO}(m)$.

We hereafter assume that G is a semisimple affine Lie group over $\mathbb{C}$. Let $\alpha_1 \ldots, \alpha_r$ be the set of fundamental roots of G. An irreducible representation of G is uniquely determined by its dominant weight, which has the form $\lambda = \sum_{i=1}^{r} a_i \lambda_i$, where $\{\lambda_1, \ldots, \lambda_r\}$ is the set of fundamental weights of G, and $a_i \in \mathbb{Z}_{\geqslant 0}$; such a representation will be denoted by V_λ. As it is made explicit by Example 3.1 in the case $\mathrm{G} = \mathrm{SL}(n+1)$, a parabolic subgroup P is defined by the choice of a subset of simple roots of G; moreover P is maximal if this subset consists of a single root. For instance, the parabolic P such that $\mathrm{SL}(n+1)/\mathrm{P} = \mathrm{Fl}(i_1, \cdots, i_k, n+1)$ is defined by the choice of the subset $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ of simple roots, while the Grassmannian $G(k, n+1)$ is defined by the choice of the root $\{\alpha_k\}$.

A vector bundle on X whose total space admits an action of G which extends the action on X is called *homogeneous*, or G-equivariant. There exists an equivalence of categories between homogeneous vector bundles on $X = \mathrm{G}/\mathrm{P}$ and representations of P. Given a representation V of P one can construct a vector bundle on X by $E := \mathrm{G} \times_{\mathrm{P}} V$, and one can check that the fiber of E over the point stabilized by P is isomorphic to V; see [Ott95, §9] for more details.

From now on, for any semisimple group G, let us suppose that P is maximal (i.e. X is a generalized Grassmannian) and defined by the k-th root α_k. Then the set of dominant weights for the irreducible representations of the semisimple part of P is precisely the set of weights of the form $\lambda = \sum_{i=1}^{r} a_i \lambda_i$, where $a_i \in \mathbb{Z}$ and $a_i \geqslant 0$ for $i \neq k$. An irreducible representation of P is uniquely defined by its restriction to the semisimple part of P, and thus ultimately to such a dominant weight λ. Vector bundles on X arising from an irreducible representation of P are called irreducible, and direct sum of those are called completely reducible. The irreducible bundle associated with the weight λ is denoted by E_λ. If $a_k \geqslant 0$, we have $H^0(E_\lambda) \simeq V_\lambda$, owing to the Borel-Weil Theorem [Ott95, Theorem 10.11].

We use the following convention. We set $\lambda_0 = \lambda_{m+1} = 0$ and, given a formal expression $\lambda = \sum_{i\in\mathbb{Z}} a_i\lambda_i$, we put $E_\lambda = 0$ and $V_\lambda = 0$ if there is $j \neq k$ such that $a_j < 0$, or if $a_j \neq 0$ for some $j < 0$ or $j > m + 1$. For instance we have:

$$S^2 V_{\lambda_k} \simeq \bigoplus_{j\in\mathbb{Z}_{\geqslant 0}} V_{\lambda_{k-2j}+\lambda_{k+2j}}.$$

Any G-equivariant vector bundle E over X has a filtration by completely reducible bundles.

3.1. **Grassmannians.** Let $\{e_1, \dots, e_n\}$ be the canonical basis of V and denote by l_i the weight of e_i with respect to the natural action of $\mathfrak{sl}(V)$. Then $\alpha_{ij} = l_i - l_j$ are the roots and $\lambda_i = l_1 + \cdots + l_i$ are the fundamental weights.

Consider $G(k, V)$ the Grassmaniann of k-dimensional quotients of V. It comes with the tautological sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow V \otimes \mathcal{O}_{G(k,V)} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{Q}$ is the rank k tautological quotient bundle and $\mathcal{U}$ is the rank $n - k$ tautological subbundle. In particular, the Plücker embedding is given by $\bigwedge^k \mathcal{Q} = \mathcal{O}_{G(k,V)}(1)$.

The Grassmannian $G(k, V)$ can be seen as a homogeneous space $\mathrm{SL}(n)/\mathrm{P}_k$ hence homogeneous bundles correspond to representations of P_k. Moreover the irreducible ones correspond to weights which are dominant for the Levi factor of P_k, which is the maximal reductive subgroup of P_k.

We also have that, according to [Wey03, P. 60],

$$\Omega^m_{G(k,V)} = \bigwedge^m (\mathcal{Q}^\vee \otimes \mathcal{U}) = \bigoplus_{|\mu|=m} \Gamma^\mu \mathcal{Q}^\vee \otimes \Gamma^{\mu'} \mathcal{U}.$$

Therefore one can prove the following result, see [Wey03, (2.3.3) Corollary].

Lemma 3.2. *We have the following equalities:*

$$\begin{aligned} H^0(\Omega^1_{G(k,V)}(d+2)) =& V_{\lambda_{k-1}+d\lambda_k+\lambda_{k+1}}, \\ H^0(\Omega^3_{G(k,V)}(2d+4)) =& V_{3\lambda_{k-1}+2d\lambda_k+\lambda_{k+3}} \oplus V_{\lambda_{k-2}+\lambda_{k-1}+2d\lambda_k+\lambda_{k+1}+\lambda_{k+2}} \\ & \oplus V_{\lambda_{k-3}+2d\lambda_k+3\lambda_{k+1}}. \end{aligned}$$

3.2. **Cominuscule spaces.** We want to study foliations on a few more homogeneous spaces. In order to restrict to a simpler situation, we will consider X a *cominuscule* homogeneous variety for a group G. These are varieties embedded in $\mathbb{P}(V_\lambda)$ via the positive generator of the Picard group, where V_λ is a cominuscule representation. The definition of a cominuscule representation is somewhat technical and not very useful in our context. However, there is a very explicit equivalent definition of a cominuscule variety which is more geometric friendly, see [BS19, Corolary 36].

Definition 3.3. A rational homogeneous variety $X = \mathrm{G}/\mathrm{P}$ is called a *cominuscule variety* (or *space*) if T_X is a completely reducible vector bundle. X is called a *cominuscule Grassmannian* if T_X is irreducible.

Remark 3.4. The distinction between cominuscule varieties and cominuscule Grassmannians is motivated by the following fact. Since G is semisimple, it is the product of simple groups $\mathrm{G} = \mathrm{G}_1 \times \cdots \times \mathrm{G}_k$ and the same holds for $X = \mathrm{G}_1/\mathrm{P}_1 \times \cdots \times \mathrm{G}_k/\mathrm{P}_k$. We can suppose that all P_i's are strictly contained in each G_i. Since the tangent bundle of a product

is the direct sum of the tangent bundles of the single factors, in order to have that T_X is irreducible the only possibility is that $k = 1$. Otherwise stated, if T_X is irreducible then one can suppose that G is simple. Moreover cominuscule varieties which are a quotient of a simple group are classified [BS19, Table 2] and consist of:

(1) Grassmannians $G(k, n) = \mathrm{SL}(n)/\mathrm{P}_k$;
(2) Quadrics $Q^n = \mathrm{SO}(n+2)/\mathrm{P}_1 \subset \mathbb{P}^{n+1}$;
(3) spinor varieties $OG(n, 2n)$ for the group D_n;
(4) Lagrangian Grassmannians $IG(n, 2n)$;
(5) The Cayley plane E_6/P_1 ;
(6) The Freudenthal variety E_7/P_7.

These are all generalized Grassmannians (i.e. P is a maximal parabolic subgroup). As a result of this discussion, we deduce that T_X is irreducible if and only if X is a cominuscule variety which is a generalized Grassmannian, hence our definition of cominuscule Grassmannian.

In what follows, we will start by studying foliations on (ordinary) Grassmannians and then we will study foliations on the remaining cominuscule Grassmannians. We will see that the hypothesis that T_X is irreducible will be very useful in order to understand the integrability condition on these spaces.

Lemma 3.5. *Let $X \subset \mathbb{P}^n$ be a homogeneous* G*-variety. Consider M a homogeneous* G*-bundle on $\mathbb{P}^n$ and N a homogeneous* G*-bundle on X such that*

(1) N is completely reducible;
(2) M is globally generated;
*(3) there exists a surjective (*G*-equivariant) morphism $M \twoheadrightarrow N$.*

Then the induced morphism on global sections $H^0(\mathbb{P}^n, M) \to H^0(X, N)$ is also surjective.

Proof. From the hypotheses we have

$$H^0(\mathbb{P}^n, M) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow\!\!\!\!\!\rightarrow M \longrightarrow\!\!\!\!\!\rightarrow N = \bigoplus_{\mu} E_\mu.$$

Up to composing further with a canonical projection $N \twoheadrightarrow E_\mu$ we may assume $N = E_\mu$ irreducible and we get an equivariant map

$$H^0(\mathbb{P}^n, M) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow\!\!\!\!\!\rightarrow E_\mu$$

that must factor through $H^0(X, E_\mu) = V_\mu$. In particular, the induced map on global sections is not zero and the surjectivity follows from Schur's Lemma. □

Corollary 3.6. *Let $i\colon X = \mathrm{G}/\mathrm{P} \to \mathbb{P}(V_\lambda)$ be a* G*-equivariant embedding of a (rational) homogeneous variety. Suppose that Ω^1_X is completely reducible. Then the induced pullback maps*

$$i_p^*\colon H^0(\Omega^p_{\mathbb{P}(V_\lambda)}(d+p+1)) \longrightarrow H^0(\Omega^p_X(d+p+1))$$

are surjective for every $p \geqslant 1$ and $d \geqslant 0$. In particular, this is true for cominuscule varieties.

Proof. Notice that for any $d \geqslant 0$ the vector bundle $\Omega^p_{\mathbb{P}^n}(d+p+1)$ is globally generated and homogeneous with respect to the action of $\mathrm{G} \subset \mathrm{SL}(V)$; moreover $i_p^*\colon \Omega^p_{\mathbb{P}^n}(d+p+1) \to$

$\Omega^p_X(d+p+1)$ is surjective and G-equivariant, since i is an equivariant embedding. By hypothesis Ω^1_X is completely reducible, the result then follows from Lemma 3.5. □

Later on we will use this result, together with Lemma 2.7, in order to obtain the equations of the set of integrable forms/foliations on cominuscule spaces. To make the computations and the results more explicit, we will study each case independently; however, let us start with a general result.

Theorem 3.7. *Let $X = \mathrm{G/P} \subset \mathbb{P}(V_\lambda)$ be a cominuscule variety. The space of codimension one degree d integrable forms $\mathrm{IF}(X, d+2) \subset \mathbb{P}(H^0(\Omega^1_X(d+2))^\vee)$ is defined by the quadratic equations given by the* G*-equivariant inclusion*

$$H^0(\Omega^3_X(d+2))^\vee \subset S^2 H^0(\Omega^1_X(d+2))^\vee \tag{15}$$

induced by $\Psi^\vee_{\mathbb{P}(V_\lambda)} \circ (i_3^*)^\vee = \pi^* \circ \Psi^\vee_X$.

Proof. As we have already recalled, the ideal of the space of integrable forms is generated by the quadratic equations in the image of $\Psi^\vee_X$. By the surjectivities in Corollary 3.6 and Lemma 2.7, $\Psi^\vee_X$ is an isomorphism onto its image. Moreover $\Psi^\vee_X$ is clearly G-equivariant since X is a G-homogeneous space. □

Notice that Proposition 2.8 applies to generalized Grassmannians (since their Picard number is equal to one). Let us rewrite this result for cominuscule Grassmannians for the sake of convenience.

Proposition 3.8. *Let $i\colon X = \mathrm{G/P} \hookrightarrow \mathbb{P}(V_\lambda)$ be a primitive embedding of a cominuscule Grassmannian. Then the projection map $\pi\colon \mathrm{Fol}(\mathbb{P}(V_\lambda), 2) \longrightarrow \mathrm{Fol}(X, 2)$ is injective.*

Remark 3.9. Notice that $H^0(\Omega^1_{\mathbb{P}(V_\lambda)}(2)) = H^0(\Omega^1_X(2)) \oplus W$ for $W := \ker i_1^*$. To show that π is an embedding it would be enough to show that each fibre of π intersects $\mathrm{Fol}(\mathbb{P}(V_\lambda), 2)$ at precisely one (reduced) point.

The restriction of π to $\mathrm{Fol}(\mathbb{P}(V_\lambda), 2)$ is an embedding if and only if $\mathbb{P}(W^\vee) \cap \mathrm{Sec}_1(\mathrm{Fol}(\mathbb{P}(V_\lambda)) = \varnothing$. Now recall that $\mathrm{Fol}(\mathbb{P}(V_\lambda), 2) \simeq G(2, V_\lambda^\vee)$ in its Plücker embedding and $\mathrm{Sec}_1(G(2, V_\lambda^\vee))$ correspond to skew-symmetric matrices of rank $\leqslant 4$. Then π is an embedding if and only if all the elements in $W \subset \bigwedge^2 V_\lambda$ have rank $\geqslant 6$. We will show that this fact holds for $X = G(3, n)$ and some other cases, but we do not know a uniform proof of this fact for any cominuscule Grassmannian.

4. Distributions and Foliations on $G(k, n)$

In this section we study the spaces $\mathrm{IF}(G(k,n), l)$ of integrable 1-forms on a Grassmannian $G(k, n)$ of k-dimensional quotients of the vector space $V \simeq \mathbb{C}^n$. The Plücker embedding i realizes $G(k, V)$ $\mathrm{SL}(V)$-equivariantly as a subvariety of $\mathbb{P}(V_{\lambda_k})$. We are going to consider the induced maps

$$i_p^*\colon H^0(\Omega^p_{\mathbb{P}(V_{\lambda_k})}(d+p+1)) \longrightarrow H^0(\Omega^p_{G(k,V)}(d+p+1)),$$

which are surjective due to Corollary 3.6. We will prove, in particular, that they induce isomorphisms between the spaces of degree zero foliations for $G(2, n)$, $n \geqslant 4$.

4.1. **The case of Grassmannians of lines $G(2,n)$.** Let us denote throughout this section by X the Grassmannian $G(2,V)$. The following result shows that the situation concerning foliations on Grassmannians of lines is completely analogous to the case of projective spaces.

Theorem 4.1. *Let V be a finite dimensional complex vector space. Then*

$$i_p^*\colon H^0(\Omega^p_{\mathbb{P}(V_{\lambda_2})}(p+1)) \longrightarrow H^0(\Omega^p_X(p+1))$$

is an isomorphism for $p=1,3$. In particular we have the isomorphism

$$\pi\colon G(2,V_{\lambda_2}^\vee) = \mathrm{Fol}(\mathbb{P}(V_{\lambda_2}),2) \xrightarrow{\sim} \mathrm{Fol}(X,2).$$

Proof. We already observed that the maps i_1^* and i_3^* are surjective. We only need to prove that both source and target have the same decomposition as irreducible $\mathrm{SL}(V)$-modules. On the one hand, Lemma 3.2 says that

$$H^0(\Omega^1_X(2)) = V_{\lambda_1+\lambda_3} \quad \text{and} \quad H^0(\Omega^3_X(4)) = V_{3\lambda_1+\lambda_5} \oplus V_{\lambda_1+\lambda_3+\lambda_4}.$$

On the other hand, it follows from [Wey03, Proposition 2.3.9], which is a consequence of the Littlewood-Richardson rule, that

$$H^0(\Omega^1_{\mathbb{P}(V_{\lambda_2})}(2)) = \bigwedge^2\left(\bigwedge^2 V\right) = V_{\lambda_1+\lambda_3}, \text{ and}$$

$$H^0(\Omega^3_{\mathbb{P}(V_{\lambda_2})}(4)) = \bigwedge^4\left(\bigwedge^2 V\right) = V_{3\lambda_1+\lambda_5} \oplus V_{\lambda_1+\lambda_3+\lambda_4}.$$

Therefore i_1^* and i_3^* are isomorphisms. It then follows that

$$\pi\colon \mathbb{P}(H^0(\Omega^1_{\mathbb{P}(V_{\lambda_2})}(2))^\vee) \longrightarrow \mathbb{P}(H^0(\Omega^1_X(2))^\vee)$$

is an isomorphism that takes $\mathrm{Fol}(\mathbb{P}(V_{\lambda_2}),2)$ to $\mathrm{Fol}(X,2)$. □

Remark 4.2. The case $n = \dim V = 5$ was already treated in [ACM18, Theorem 1.5, item (5.a)] of Araujo, Corrêa and Massarenti. Their proof relies on a previous result of Araujo and Druel about del Pezzo foliations and could not be generalized. One could also try to use our strategy to investigate distributions of class $k \geqslant 1$, see [ACM18] for a definition.

4.2. **The case of $G(3,n)$.** Let us denote throughout this section by X the variety $G(3,V)$. In this section we aim to prove the following result.

Theorem 4.3. *Let V be a finite dimensional complex vector space. Then the map*

$$\pi\colon \mathrm{Fol}(\mathbb{P}(V_{\lambda_3}),2) \longrightarrow \mathrm{Fol}(X,2)$$

is an embedding. Moreover, the homogeneous ideals of $\pi(\mathrm{Fol}(\mathbb{P}(V_{\lambda_3}),2))$ and $\mathrm{Fol}(X,2)$ agree in degree two, i.e.

$$(I_{\pi(\mathrm{Fol}(\mathbb{P}(V_{\lambda_3}),2))})_2 = (I_{\mathrm{Fol}(X,2)})_2.$$

First let us give a rough idea of the proof. Due to Proposition 3.8 π is well-defined and injective. The idea to prove that it is an embedding comes from Remark 3.9; we will prove that the intersection between $\ker i_1^*$ and the space of rank $\leqslant 4$ matrices is $\{0\}$ by studying $\mathrm{SL}(V)$-orbits. The second part boils down, owing to Lemma 2.7, to proving that the square (14) is cartesian in this case; this will be done explicitly by ad hoc computations.

From Lemma 3.2 we know that

$$H^0(\Omega^1_X(2)) = V_{\lambda_2+\lambda_4} \text{ and } H^0(\Omega^3_X(4)) = V_{3\lambda_2+\lambda_6} \oplus V_{\lambda_1+\lambda_2+\lambda_4+\lambda_5} \oplus V_{3\lambda_4}.$$

Moreover one can check (for instance with [vLCL92]) that

$$H^0(\Omega^1_{\mathbb{P}(V_{\lambda_3})}(2)) = \bigwedge^2 V_{\lambda_3} = V_{\lambda_6} \oplus V_{\lambda_2+\lambda_4} = V_{\lambda_6} \oplus H^0(\Omega^1_X(2)), \tag{16}$$

$$\begin{aligned} H^0(\Omega^3_{\mathbb{P}(V_{\lambda_3})}(4)) = \bigwedge^4 V_{\lambda_3} =& V_{\lambda_{12}} \oplus V_{\lambda_3+\lambda_9} \oplus V_{\lambda_2+\lambda_{10}} \oplus V_{2\lambda_6} \oplus V_{\lambda_4+\lambda_8} \oplus \\ &\oplus V_{2\lambda_2+\lambda_8} \oplus V_{\lambda_2+\lambda_3+\lambda_7} \oplus V_{\lambda_1+\lambda_4+\lambda_7} \oplus V_{\lambda_2+\lambda_4+\lambda_6} \oplus \\ &\oplus H^0(\Omega^3_{G(k,V)}(4)). \end{aligned} \tag{17}$$

In particular, neither i_1^* nor i_3^* are injective. Furthermore (by [vLCL92] again) we can compute

$$\begin{aligned} S^2 V_{\lambda_2+\lambda_4} = & V_{2\lambda_6} \oplus V_{\lambda_4+\lambda_8} \oplus V_{3\lambda_4} \oplus V_{\lambda_3+\lambda_4+\lambda_5} \oplus V^{\oplus 2}_{\lambda_2+\lambda_4+\lambda_6} \oplus V_{\lambda_2+\lambda_3+\lambda_7} \oplus \\ &\oplus V_{2\lambda_2+\lambda_8} \oplus V_{2\lambda_2+2\lambda_4} \oplus V_{3\lambda_2+\lambda_6} \oplus V_{\lambda_1+\lambda_5+\lambda_6} \oplus V_{\lambda_1+\lambda_4+\lambda_7} \oplus V_{\lambda_1+2\lambda_3+\lambda_5} \oplus \\ &\oplus V_{\lambda_1+\lambda_2+\lambda_4+\lambda_5} \oplus V_{\lambda_1+\lambda_2+\lambda_3+\lambda_6} \oplus V_{2\lambda_2+2\lambda_5} \oplus V_{2\lambda_1+\lambda_3+\lambda_7} \end{aligned} \tag{18}$$

and also

$$\begin{aligned} S^2 \bigwedge^2 V_{\lambda_3} = & V_{\lambda_{12}} \oplus V^{\oplus 2}_{2\lambda_6} \oplus V^{\oplus 3}_{\lambda_4+\lambda_8} \oplus V_{3\lambda_4} \oplus V_{\lambda_3+\lambda_9} \oplus V_{\lambda_3+\lambda_4+\lambda_5} \oplus V^{\oplus 2}_{\lambda_2+\lambda_{10}} \oplus \\ &\oplus V^{\oplus 3}_{\lambda_2+\lambda_4+\lambda_6} \oplus V^{\oplus 2}_{\lambda_2+\lambda_3+\lambda_7} \oplus V^{\oplus 2}_{2\lambda_2+\lambda_8} \oplus V_{2\lambda_2+2\lambda_4} \oplus V_{3\lambda_2+\lambda_6} \oplus V_{\lambda_1+\lambda_5+\lambda_6} \oplus \\ &\oplus V^{\oplus 2}_{\lambda_1+\lambda_4+\lambda_7} \oplus V_{\lambda_1+\lambda_3+\lambda_8} \oplus V_{\lambda_1+2\lambda_3+\lambda_5} \oplus V_{\lambda_1+\lambda_2+\lambda_9} \oplus V_{\lambda_1+\lambda_2+\lambda_4+\lambda_5} \oplus \\ &\oplus V_{\lambda_1+\lambda_2+\lambda_3+\lambda_6} \oplus V_{2\lambda_1+2\lambda_5} \oplus V_{2\lambda_1+\lambda_3+\lambda_7}. \end{aligned} \tag{19}$$

Let us write diagram (14) in this case:

$$\begin{array}{ccc} S^2 \bigwedge^2 V_{\lambda_3}^\vee & \xleftarrow{\pi^*} & S^2 V_{\lambda_2+\lambda_4}^\vee \\ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})} \uparrow & & \Psi^\vee_X \uparrow \\ \bigwedge^4 V_{\lambda_3}^\vee & \longleftarrow & H^0(\Omega^3_X(4))^\vee \end{array} \tag{20}$$

Note that π^* comes from the decomposition $\bigwedge^2 V_{\lambda_3} = V_{\lambda_6} \oplus V_{\lambda_2+\lambda_4}$ hence it is just the inclusion of a direct summand. On the other hand, recall that from Remark 2.6 the map $\Psi_{\mathbb{P}(V_{\lambda_3})}$ is the multiplication map $(a \wedge b) \cdot (c \wedge d) \mapsto a \wedge b \wedge c \wedge d$, hence its dual is the diagonal (or comultiplication) map

$$\Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(a \wedge b \wedge c \wedge d) = (a \wedge b) \cdot (c \wedge d) - (a \wedge c) \cdot (b \wedge d) + (a \wedge d) \cdot (b \wedge c), \tag{21}$$

see for instance [Wey03, Proposition 1.1.2].

Proposition 4.4. *The square* (20) *is cartesian.*

Proof. We want to show that the square (20) is cartesian, i.e.

$$\pi^*(S^2 V_{\lambda_2+\lambda_4}^\vee) \cap \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(\bigwedge^4 V_{\lambda_3}^\vee) = \pi^* \circ \Psi^\vee_X(H^0(\Omega^3_X(4))^\vee).$$

Since the maps are $\mathrm{SL}(V)$-equivariant, Schur's Lemma implies that we can show the equality above for each weight individually, i.e. we only need to prove that

$$\pi^*(V_\mu^\vee) \cap \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(V_\mu^\vee) = \pi^* \circ \Psi^\vee_X(V_\mu^\vee)$$

for each weight μ.

First we deal with the weights $3\lambda_2+\lambda_6, \lambda_1+\lambda_2+\lambda_4+\lambda_5$ and $3\lambda_4$ appearing in $H^0(\Omega^3_X(4))^\vee$. We remark that they appear only once (there is only one irreducible direct summand of each weight) in each term of the square (20). Indeed, check the decompositions (16), (17), (18) and (19). Hence $\pi^*(V_\mu^\vee) \cap \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(V_\mu^\vee)$ and $\pi^* \circ \Psi^\vee_X(V_\mu^\vee)$ can only be the unique $V_\mu^\vee$ appearing in $S^2 \bigwedge^2 V^\vee_{\lambda_3}$.

For a weight μ not appearing in $H^0(\Omega^3_X(4))^\vee$ we need to show that

$$\pi^*(V_\mu^\vee) \cap \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(V_\mu^\vee) = \{0\}. \tag{22}$$

Notice moreover that we only need to verify the weights appearing in both $S^2V^\vee_{\lambda_2+\lambda_4}$ and $\bigwedge^4 V^\vee_{\lambda_3}$. From (17) and (18) we see that common weights (not in $\bigwedge^4 V^\vee_{\lambda_3}$) are:

$$2\lambda_6,\ \lambda_4+\lambda_8,\ 2\lambda_2+\lambda_8,\ \lambda_2+\lambda_3+\lambda_7,\ \lambda_1+\lambda_4+\lambda_7 \text{ and } \lambda_2+\lambda_4+\lambda_6.$$

We claim that (22) holds for each one of these weights, concluding the proof of the proposition. The proof of the claim will be given in Lemmas A.4, A.5, A.6, A.7, A.8 and A.9. □

Proof of Theorem 4.3. The second part follows from Lemma 2.7 and Proposition 4.4, we only need to prove the first.

As already recalled, it follows from Proposition 3.8 that π is injective and from Remark 3.9 we only need to show that the center of projection $\mathbb{P}(V^\vee_{\lambda_6}) \subset \mathbb{P}(H^0(\Omega^1_{\mathbb{P}(V_{\lambda_3})}(2))^\vee)$ does not intersect the first secant variety to $\mathrm{Fol}(\mathbb{P}(V_{\lambda_3}),2) = G(2,V^\vee_{\lambda_3})$. Note that inside $H^0(\Omega^1_{\mathbb{P}(V_{\lambda_3})}(2)) = \bigwedge^2 V_{\lambda_3}$ the affine cone over $G(2,V^\vee_{\lambda_3})$ and its secant varieties are precisely

$$\mathrm{Sec}_k(G(2,V^\vee_{\lambda_3})) = \left\{ v \in \bigwedge^2 V_{\lambda_3} \mid v^{k+2} = \underbrace{v \wedge \cdots \wedge v}_{k+2 \text{ times}} = 0 \right\}$$

where $\mathrm{Sec}_0(G(2,V^\vee_{\lambda_3})) := G(2,V^\vee_{\lambda_3})$. Then let

$$Z := V_{\lambda_6} \cap \mathrm{Sec}_1(G(2,V^\vee_{\lambda_3})) = \{v \in V_{\lambda_6} \mid v^3 = 0\}.$$

We want to show that $Z = \{0\}$. Notice that Z is a closed algebraic subset of V_{λ_6} invariant under the action of $\mathrm{SL}(V)$. On the other hand, the unique closed (hence minimal) $\mathrm{PGL}(V)$-orbit in $\mathbb{P}(V_{\lambda_6})$ is $\mathrm{PGL}(V)\cdot[w_6]$, where w_6 is the highest weight vector, see [FH91, Claim 23.52]. Due to Remark A.3, $w_6^{10} \neq 0$ therefore $w_6 \notin Z$ and $Z = \{0\}$. □

Remark 4.5. Theorem 4.3 raises the question whether it is true that $\mathrm{Fol}(\mathbb{P}(V_{\lambda_3}),2) \to \mathrm{Fol}(X,2)$ is an isomorphism for any n. For $n=6$ we show in Subsection 6.5 that $\mathrm{Fol}(\mathbb{P}(V_{\lambda_3}),2) \simeq \mathrm{Fol}(G(3,6),2)_{red}$, where X_{red} denotes the underlying reduced structure of X.

5. Foliations for Grassmannians of isotropic lines

In the previous section we have shown that foliations on Grassmannians of lines $G(2,n)$ coincide with foliations on the ambient projective space $\mathbb{P}(\bigwedge^2 \mathbb{C}^n)$. In this section we study Grassmannians of lines for the symplectic and the orthogonal groups. These are not cominuscule varieties, but their inclusion inside $G(2,n)$ will allow us to understand their codimension one foliations. Let us give a brief introduction to these varieties. As rational homogeneous varieties, they can be seen as quotients $X = G/\mathrm{P}_2$, where G is one of the classical groups B, C, D and P_2 is the parabolic subgroup of G defined by the 2nd simple root α_2 (in Bourbaki's notation).

If G is of type C then $\mathrm{G} = \mathrm{Sp}(V)$, where V is an even dimensional vector space endowed with a maximal-rank skew-symmetric two-form $w \in \bigwedge^2 V$. Then $\mathrm{Sp}(V)/\mathrm{P}_2$ is the variety parametrizing isotropic lines inside $V^\vee$ which are isotropic with respect to w. This is clearly a subvariety of $G(2,V)$ and it is a homogeneous variety by classical linear algebra. Beware that if V is odd dimensional then w has corank equal to one (since its rank must be even) and the set of lines inside V which are isotropic with respect to w is not a homogeneous space. Indeed this set is composed of two orbits: the dense orbit of lines not containing the one-dimensional kernel of w and the closed orbit of lines containing the kernel of w. However, in both the even and the odd dimensional case, the isotropy condition is a codimension one condition and the set of lines isotropic with respect to w is a hyperplane section of $G(2,V)$ inside its Plücker embedding (notice that $\bigwedge^2 V \cong H^0(G(2,V),\mathcal{O}(1))$).

If G is of type B or D, then $\mathrm{G} = \mathrm{SO}(V)$, where V is a vector space endowed with a non-degenerate symmetric two-form $q \in S^2 V$: if G is of type B then V is odd dimensional, while if G is of type D then V is even dimensional. $\mathrm{SO}(V)/\mathrm{P}_2$ is the variety parametrizing isotropic lines inside $V^\vee$ which are isotropic with respect to q. This is clearly a subvariety of $G(2,V)$ and it is a homogeneous variety by classical linear algebra.

5.1. **Symplectic Grassmannians of lines.** In this section we will show that the analogue of Theorem 4.1 holds for hyperplane sections inside $G(2,V)$. As already recalled, these varieties are the so-called symplectic Grassmannians of lines: they are stabilized by the symplectic group $\mathrm{Sp}(V)$. When $\dim(V)$ is even, they are homogeneous spaces for the simple group $\mathrm{Sp}(V)$. Due to the homogeneity condition, the situation when $\dim(V)$ is even is easier to study, and we start with this situation. Let us denote throughout this section by X the variety $IG(2,V)$.

Let therefore $n = \dim(V)$ be an even integer. Let $w \in \bigwedge^2 V$ be the symplectic form defining $X := IG(2,V)$ and the group $\mathrm{Sp}(V)$. More explicitly w can be seen as a section of $\bigwedge^2 \mathcal{Q} \cong \mathcal{O}(1)$, where $\mathcal{Q}$ is the quotient tautological bundle, whose zero locus inside $G(2,V)$ is X. The dual tautological bundle $\mathcal{U}^\vee$ of X admits the following decomposition:

$$0 \to (\mathcal{Q}^\vee)^\perp/\mathcal{Q}^\vee \to \mathcal{U}^\vee \to \mathcal{Q} \to 0,$$

where $(\mathcal{Q}^\vee)^\perp$ is the subbundle of $V^\vee \otimes \mathcal{O}_X$ corresponding to the orthogonal with respect to the two-form w of $\mathcal{Q}^\vee$. Similarly, the cotangent bundle of X admits the following decomposition

$$0 \to S^2\mathcal{Q}^\vee \to \Omega^1_X \to (\mathcal{Q}^\vee)^\perp/\mathcal{Q}^\vee \otimes \mathcal{Q}^\vee \to 0. \tag{23}$$

The two-form w defines a trivial $\mathrm{Sp}(V)$-subrepresentation of $\bigwedge^2 V$, and the corresponding quotient will be denoted by $\bigwedge^{\langle 2\rangle} V$. Thus X is contained in the projective space $\mathbb{P}(\bigwedge^{\langle 2\rangle} V) \subset \mathbb{P}(\bigwedge^2 V)$.

Lemma 5.1. *We have the following isomorphisms:*

$$H^0(\Omega^1_X(2)) \cong \bigwedge^2 \bigwedge^{\langle 2\rangle} V;$$

$$H^0(\Omega^3_X(4)) \cong \bigwedge^4 \bigwedge^{\langle 2\rangle} V.$$

Proof. Apply the Borel-Weil-Bott Theorem to the cohomology of the exact sequence (23). □

The previous lemma allows, together with diagram (14), to directly deduce that $\mathrm{Fol}(X,2) \cong \mathrm{Fol}(\mathbb{P}(\bigwedge^{\langle 2\rangle} V),2) \cong G(2, \bigwedge^{\langle 2\rangle} V^\vee)$. However, since the same statement holds when n is odd (and in this case we cannot use the Borel-Weil-Bott Theorem), we will give a more general proof of the following theorem (we use the same notations as before, even though in this case n can either be even or odd):

Theorem 5.2. *For $X := IG(2,V)$, we have:*

$$\mathrm{Fol}(X,2) \simeq \mathrm{Fol}(\mathbb{P}(\bigwedge^{\langle 2\rangle} V),2) \cong G(N-2, \bigwedge^{\langle 2\rangle} V).$$

Proof. Since $\mathrm{Sp}(V)$ is not always (semi)simple, we will prove the result by using the inclusion $X \subset G(2,V)$. This inclusion gives the (twisted) conormal exact sequence

$$0 \to \mathcal{O}(1)|_X \to \Omega^1_{G(2,V)}(2)|_X \to \Omega^1_X(2) \to 0.$$

By the Borel-Weil-Bott Theorem and the exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{G(2,V)} \to \mathcal{O}_X \to 0,$$

one deduces that

$$H^0(\Omega^1_X(2)) \cong V_{\lambda_1+\lambda_3}/\bigwedge^{\langle 2\rangle} V = \bigwedge^2 \bigwedge^2 V/\bigwedge^{\langle 2\rangle} V = \bigwedge^2 \bigwedge^{\langle 2\rangle} V.$$

In order to compute the cohomology of $\Omega^3_X(4)$ one can take the exterior powers of the (twisted) conormal sequence, and use the fact that, for an exact sequence

$$0 \to M \to N \to L \to 0,$$

there exists a filtration of $\bigwedge^k N$ by terms of the form $\bigwedge^p M \otimes \bigwedge^q L$, for $p+q=k$. As a result, one obtains that

$$H^0(\Omega^3_X(4)) \cong V_{\lambda_1+\lambda_3+\lambda_4} \oplus V_{3\lambda_1+\lambda_5}/W \cong \bigwedge^4 \bigwedge^2 V/W,$$

where:

$$W = V_{2\lambda_3} \oplus V_{2\lambda_1+\lambda_4}/\bigwedge^2 \bigwedge^{\langle 2\rangle} V \cong \bigwedge^3 \bigwedge^2 V/\bigwedge^2 \bigwedge^{\langle 2\rangle} V.$$

These results are obtained by applying the Borel-Weil-Bott Theorem; a more careful use of this theorem shows that these quotients are the natural ones induced by

$$0 \to \mathbb{C}w \to \bigwedge^2 V \to \bigwedge^{\langle 2\rangle} V \to 0;$$

thus $H^0(\Omega^3_X(4)) \cong \bigwedge^4 \bigwedge^{\langle 2\rangle} V$. Therefore, as in the proof of Theorem 3.7 we obtain that the equations defining $\mathrm{Fol}(X,2)$ are the same as those defining $G(N-2, \bigwedge^{\langle 2\rangle} V)$; the result follows. □

5.2. **Orthogonal Grassmannians of lines.** Let us denote throughout this section by X the variety $OG(2,V)$. Let $n = \dim(V)$ and $q \in S^2V$ be the quadratic form defining $X := OG(2,V)$ and the group $\mathrm{SO}(V)$ inside $G(2,V)$ and $\mathrm{SL}(V)$ respectively. More explicitly, q can be seen as a section of the bundle $S^2\mathcal{Q}$, where $\mathcal{Q}$ is the quotient tautological bundle, whose zero locus inside $G(2,V)$ is X itself. The dual tautological bundle $\mathcal{U}^\vee$ of X admits the following decomposition:

$$0 \to (\mathcal{Q}^\vee)^\perp/\mathcal{Q}^\vee \to \mathcal{U}^\vee \to \mathcal{Q} \to 0,$$

where $(\mathcal{Q}^\vee)^\perp$ is the subbundle of $V^\vee \otimes \mathcal{O}_X$ corresponding to the orthogonal with respect to the two-form q of $\mathcal{Q}^\vee$. Similarly, the cotangent bundle of X admits the following decomposition

$$0 \to \bigwedge^2 \mathcal{Q}^\vee \to \Omega^1_X \to (\mathcal{Q}^\vee)^\perp/\mathcal{Q}^\vee \otimes \mathcal{Q}^\vee \to 0. \tag{24}$$

The variety X is contained in the projective space $\mathbb{P}(V_{\lambda_2}) \cong \mathbb{P}(\bigwedge^2 V)$.

Lemma 5.3. *We have the following isomorphisms:*

$$H^0(\Omega^1_X(2)) \cong \bigwedge^2 \bigwedge^2 V \cong V_{\lambda_1+\lambda_3} \oplus V_{\lambda_2};$$

$$H^0(\Omega^3_X(4)) \cong V_{\lambda_1+\lambda_3+\lambda_4} \oplus V_{\lambda_1+\lambda_2+\lambda_3} \oplus V_{3\lambda_1+\lambda_5} \oplus V_{2\lambda_2} \oplus V_{2\lambda_3} \oplus V_{2\lambda_1+\lambda_4}.$$

Proof. Apply the Borel-Weil-Bott Theorem to the cohomology of the exact sequence (24). □

Let us set $N = n(n-1)/2$.

Proposition 5.4. *The space of codimension one minimal degree foliations* $\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee) \cong \mathbb{P}(\bigwedge^2 V^\vee_{\lambda_2})$ *is defined by the quadratic equations given by the inclusion* $H^0(\Omega^3_X(4))^\vee \subset S^2 \bigwedge^2 V^\vee_{\lambda_2}$*, with:*

$$H^0(\Omega^3_X(4)) = V_{\lambda_1+\lambda_3+\lambda_4} \oplus V_{\lambda_1+\lambda_2+\lambda_3} \oplus V_{3\lambda_1+\lambda_5} \oplus V_{2\lambda_2} \oplus V_{2\lambda_3} \oplus V_{2\lambda_1+\lambda_4}.$$

Moreover we have a strict inclusion:

$$\mathrm{Fol}(\mathbb{P}(V_{\lambda_2}),2) \cong \mathrm{Fol}(G(2,V),2) \cong G(N-2, V_{\lambda_2}) \subsetneq \mathrm{Fol}(X,2).$$

Proof. By Lemma 2.7 we just need to prove that the natural morphisms

$$i_1^*\colon H^0(\Omega^1_{\mathbb{P}(\bigwedge^2 V)}(2)) \cong \bigwedge^2 \bigwedge^2 V \to H^0(\Omega^1_X(2)),$$

$$i_3^*\colon H^0(\Omega^3_{\mathbb{P}(\bigwedge^2 V)}(4)) \cong \bigwedge^4 \bigwedge^2 V \to H^0(\Omega^3_X(4))$$

are surjective. For this one can use the conormal sequence associated to $X \subset G(2,V)$ to see that $H^0(\Omega^1_{G(2,V)}(2)|_X) \to H^0(\Omega^1_X(2))$ since the twisted conormal bundle $S^2\mathcal{Q}^\vee(2)$ has trivial H^1. Then the Koszul complex for $X \subset G(2,V)$ and Borel-Weil-Bott allows to show that $H^0(G(2,V),\Omega^1_{G(2,V)}(2)) \to H^0(X,\Omega^1_G(2)|_X)$ is surjective. All together this shows that i_1^* is surjective. A similar argument holds for i_3^*. □

6. Foliations on other cominuscule spaces

As we recalled in Remark 3.4, cominuscule Grassmannians consist of: Grassmannians, quadrics, spinor varieties $OG(n,2n)$ for the group D_n, Lagrangian Grassmannians $IG(n,2n)$, E_6/P_1 and E_7/P_7. In Section 4 foliations on Grassmannians were studied. For a quadric hypersurface $Q \subset \mathbb{P}^n$ we also have the isomorphism $\mathrm{Fol}(Q,2) \cong \mathrm{Fol}(\mathbb{P}^n,2)$, this is a particular case of [ACM18, Theorem 5.6] which holds for general (weighted) complete intersections. Let us see what happens to the other cominuscule Grassmannians.

6.1. **spinor varieties.** Let us denote throughout this section by X the variety $OG(n,2n)$. Thus $X := OG(n,2n) = D_n/\mathrm{P}_n$ is a spinor variety of type D. This variety parametrizes (one of the two isomorphic connected components of) n-dimensional quotient spaces of a $2n$-dimensional space which are isotropic with respect to a symmetric non-degenerate two-form. If q denotes the symmetric form, X can be seen as the zero locus of q, seen as a section of $S^2\mathcal{Q}$, inside $G(n,2n)$. From the normal sequence one can check that the tangent bundle T_X is equal to $\bigwedge^2 \mathcal{Q}$, where we denoted by $\mathcal{Q}$ as well the restriction to X of the tautological quotient bundle on $G(n,2n)$. One gets:

$$\Omega^1_X(2) = E_{\lambda_{n-2}}, \qquad H^0(\Omega^1_X(2)) = V_{\lambda_{n-2}};$$
$$\Omega^3_X(4) = E_{2\lambda_{n-3}} \oplus E_{\lambda_{n-4}+2\lambda_{n-1}}, \qquad H^0(\Omega^3_X(4)) = V_{2\lambda_{n-3}} \oplus V_{\lambda_{n-4}+2\lambda_{n-1}}.$$

The spinor variety is primitively embedded in $\mathbb{P}(V_{\lambda_n})$. We denote by $N := \dim(V_{\lambda_n})$.

Theorem 6.1. *The space of codimension one minimal degree foliations* $\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee) \cong \mathbb{P}(V^\vee_{\lambda_{n-2}})$ *is defined by the quadratic equations given by the inclusion*

$$H^0(\Omega^3_X(4))^\vee = (E_{2\lambda_{n-3}} \oplus E_{\lambda_{n-4}+2\lambda_{n-1}})^\vee \subset S^2 V^\vee_{\lambda_{n-2}}.$$

Moreover, when $n = 4,5$*, these equations are exactly the equations of* $\mathrm{Fol}(\mathbb{P}(V_{\lambda_n}),2) \cong G(N-2,V_{\lambda_n})$*, thus identifying the two spaces of foliations.*

Remark 6.2. The last assertion is a new result only for $n = 5$ because for $n = 4$ the spinor variety X is just a six-dimensional quadric.

Proof. By Corollary 3.6, the morphisms

$$i_k^* \colon H^0(\Omega^k_{\mathbb{P}(V_{\lambda_n})}(k+1)) \to H^0(\Omega^k_X(k+1))$$

are surjective for $k = 1,3$ and any $n \geqslant 6$, and they are isomorphisms for $k = 1,3$ and $n = 4,5$ by a dimension argument. By applying Lemma 2.7 the result follows. □

6.2. **Lagrangian Grassmannians.** Let us denote throughout this section by X the variety $IG(n,2n)$. Thus $X := IG(n,2n) = C_n/\mathrm{P}_n$ is the symplectic Grassmannian of maximal isotropic quotient spaces. This variety parametrizes n-dimensional quotient spaces of a $2n$-dimensional space which are isotropic with respect to a skew-symmetric non-degenerate two-form. If w denotes the skew-symmetric form, X can be seen as the zero locus of w, seen as a section of $\bigwedge^2 \mathcal{Q}$, inside $G(n,2n)$. From the normal sequence one can check that the tangent bundle T_X is isomorphic to $S^2\mathcal{Q}$, where we denoted by $\mathcal{Q}$ as well the restriction to X of the tautological quotient bundle on $G(n,2n)$. One gets:

$$\begin{aligned} &\Omega^1_X(2) = E_{2\lambda_{n-1}}, && H^0(\Omega^1_X(2)) = V_{2\lambda_{n-1}};\\ &\Omega^3_X(4) = E_{3\lambda_{n-2}+\lambda_n} \oplus E_{\lambda_{n-3}+3\lambda_{n-1}}, && H^0(\Omega^3_X(4)) = V_{3\lambda_{n-2}+\lambda_n} \oplus V_{\lambda_{n-3}+3\lambda_{n-1}}. \end{aligned}$$

The variety X is primitively embedded in $\mathbb{P}(V_{\lambda_n})$.

Proposition 6.3. *The space of codimension one minimal degree foliations* $\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee) \cong \mathbb{P}(V^\vee_{2\lambda_{n-1}})$ *is defined by the quadratic equations given by the inclusion*

$$H^0(\Omega^3_X(4))^\vee = (V_{3\lambda_{n-2}+\lambda_n} \oplus V_{\lambda_{n-3}+3\lambda_{n-1}})^\vee \subset S^2 V^\vee_{2\lambda_{n-1}}.$$

Proof. By Corollary 3.6, the morphisms $i_k^* \colon H^0(\mathbb{P}(V_{\lambda_n}), \Omega^k_{\mathbb{P}(V_{\lambda_n})}(k+1)) \to H^0(\Omega^k_X(k+1))$ are surjective for $k = 1,3$ and any $n \geqslant 3$, but never isomorphisms. By applying Lemma 2.7 we deduce the result. □

6.3. **The Cayley plane.** The Cayley plane is the E_6-homogeneous variety E_6/P_1. Set $X = E_6/\mathrm{P}_1$ for the current subsection. The tangent bundle T_X is isomorphic to E_{λ_2}. One gets:

$$\begin{aligned} &\Omega^1_X(2) = E_{\lambda_3}, && H^0(\Omega^1_X(2)) = V_{\lambda_3};\\ &\Omega^3_X(4) = E_{\lambda_2+\lambda_5}, && H^0(\Omega^3_X(4)) = V_{\lambda_2+\lambda_5}. \end{aligned}$$

The variety $X = E_6/\mathrm{P}_1$ is primitively embedded in $\mathbb{P}(V_{\lambda_1}) = \mathbb{P}^{26}$, for $\dim(V_{\lambda_1}) = 27$. One checks with [vLCL92] that:

$$\begin{aligned} H^0(\Omega^1_{\mathbb{P}(V_{\lambda_1})}(2)) &\cong \bigwedge^2 V_{\lambda_1} \cong V_{\lambda_3} \cong H^0(\Omega^1_X(2)),\\ H^0(\Omega^3_{\mathbb{P}(V_{\lambda_1})}(4)) &\cong \bigwedge^4 V_{\lambda_1} \cong V_{\lambda_2+\lambda_5} \cong H^0(\Omega^3_X(4)). \end{aligned}$$

For $X = E_6/\mathrm{P}_1$, we deduce the following theorem, affording a complete description of the space of codimension one minimal degree foliations

$$\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee) \cong \mathbb{P}(V^\vee_{\lambda_3}) \cong \mathbb{P}(\bigwedge^2 V^\vee_{\lambda_1}).$$

Theorem 6.4. *For $X = E_6/\mathrm{P}_1$, we get:*

$$\mathrm{Fol}(X,2) \simeq \mathrm{Fol}(\mathbb{P}(V_{\lambda_1}),2) \cong G(25, V_{\lambda_1}).$$

Proof. The maps i_k^* for $k = 1,3$ are isomorphisms; the result follows from Lemma 2.7. □

6.4. **The Freudenthal variety.** The Freudenthal variety is the E_7-homogeneous variety $X = E_7/\mathrm{P}_7$. The tangent bundle T_X is E_{λ_1}. One gets:

$$\Omega^1_X(2) = E_{\lambda_6}, \qquad H^0(\Omega^1_X(2)) = V_{\lambda_6};$$
$$\Omega^3_X(4) = E_{\lambda_4}, \qquad H^0(\Omega^3_X(4)) = V_{\lambda_4}.$$

The variety E_7/P_7 is primitively embedded in $\mathbb{P}(V_{\lambda_7})$.

Proposition 6.5. *Set $X = E_7/P_7$. Then the space of codimension one minimal degree foliations $\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee) \cong \mathbb{P}(V^\vee_{\lambda_6})$ is defined by the quadratic equations given by the inclusion*

$$H^0(\Omega^3_X(4))^\vee = (V_{\lambda_4})^\vee \subset S^2 V^\vee_{\lambda_6}.$$

Proof. By Corollary 3.6 the morphisms

$$i_1^*\colon H^0(\Omega^1_{\mathbb{P}(V_{\lambda_7})}(2)) \cong \bigwedge^2 V_{\lambda_7} \to V_{\lambda_6} \cong H^0(\Omega^1_X(2))$$

and

$$i_3^*\colon H^0(\Omega^3_{\mathbb{P}(V_{\lambda_7})}(4)) \cong \bigwedge^4 V_{\lambda_7} \to V_{\lambda_4} \cong H^0(\Omega^3_X(4))$$

are both surjective, but not injective for a dimension argument. By applying Lemma 2.7 we deduce the statement of the proposition. □

6.5. **Some varieties from the Freudenthal magic square.** Let X be one of the following manifolds appearing in the third row of the Freudenthal magic square, see [LM01]: $IG(3,6)$, $G(3,6)$, $OG(6,12)$ or E_7/P_7. These are all cominuscule Grassmannians sharing the property that $\bigwedge^2 V_\lambda = \mathbb{C} \oplus H^0(\Omega^1_X(2))$ and $\bigwedge^4 V_\lambda = \bigwedge^2 V_\lambda \oplus H^0(\Omega^1_X(2))$. We will denote by $\mathrm{Fol}(X,2)_{\mathrm{red}}$ the classical variety $\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee)$ endowed with the reduced scheme structure. We can prove the following result.

Theorem 6.6. *Let $X \subset \mathbb{P}(V_\lambda)$ be the natural embedding. The projection*

$$\pi\colon \mathrm{Fol}(\mathbb{P}(V_\lambda),2) \longrightarrow \mathrm{Fol}(X,2)_{\mathrm{red}}$$

is an isomorphism. In particular $\mathrm{Fol}(X,2)_{\mathrm{red}} \simeq G(2, V_\lambda^\vee)$.

Let us break the proof into several steps. Let us begin with a better understanding of the trivial factor in $\bigwedge^2 V_\lambda$. Let us denote by w a of such trivial factor.

Let us recall the common construction of varieties from the third row of the Freudenthal magic square (we refer again to [LM01]). Let Y be one among the following adjoint varieties G'/P' for an exceptional group G': F_4/P_1, E_6/P_2, E_7/P_1, E_8/P_8. These are also the varieties from the fourth row of the Freudenthal magic square. From the general theory of homogeneous vector bundles, it follows that the fiber of the tangent bundle of Y at a point stabilized by P' is a P'-representation W'. Notice that the semisimple factor of P' is the group G for $X = G/P$ respectively equal to $IG(3,6)$, $G(3,6)$, $OG(6,12)$, E_7/P_7. Thus G acts on W' and it turns out that, as a G-representation, $W' = \mathbb{C} \oplus W$ for a certain G-representation W. The variety X is constructed as the minimal G-orbit in $\mathbb{P}(W)$, hence W is the representation V_λ appearing in the embedding $X \subset \mathbb{P}(V_\lambda)$.

Lemma 6.7. *A form $w \in \bigwedge^2 V_\lambda$ generating the trivial G-sub-representation is non-degenerate.*

Proof. Adjoint varieties are contact manifolds (see for instance [BM19]) with contact structure given by $\theta \in H^0(\Omega^1_Y(1))$. Being a contact structure means that the induced distribution is regular:

$$0 \longrightarrow F \longrightarrow T_Y \xrightarrow{\theta} \mathcal{O}_Y(1) \longrightarrow 0, \tag{25}$$

and the $\mathcal{O}_Y$-bilinear map $\bigwedge^2 F \to \mathcal{O}_Y(1)$ defined by $u \wedge v \mapsto d\theta(u,v) = \theta([u,v])$ is nondegenerate.

From our previous discussion, taking fibers gives $F_x \otimes k(x) = V_\lambda$. On the other hand θ_x is P'-invariant, hence G-invariant. Since G is semisimple, $d\theta_x$ defines a trivial one dimensional G-subrepresentation of $\bigwedge^2 V_\lambda^\vee$. One can check (for instance with LiE [vLCL92]) that this trivial factor is unique, thus w is also nondegenerate.

□

Proposition 6.8. *The map* $\pi\colon \mathrm{Fol}(\mathbb{P}(V_\lambda),2) \longrightarrow \mathrm{Fol}(X,2)$ *is an embedding.*

Proof. As in the proof of Theorem 4.3, we only need to show that the center of projection $\mathbb{P}(\mathbb{C}w) \subset \mathbb{P}(H^0(\Omega^1_{\mathbb{P}(V_\lambda)}(2))^\vee)$ does not intersect the first secant variety to $\mathrm{Fol}(\mathbb{P}(V_\lambda),2) = G(2,V_\lambda^\vee)$. However, the secant variety of $G(2,V_\lambda^\vee)$ is the set of forms of rank at most four (again, refer to the proof of Theorem 4.3). Then the result follows from Lemma 6.7. □

In view of Proposition 6.8 we only need to show that π is surjective. Recall the commutative diagram

$$\begin{array}{ccc} H^0(\Omega^1_{\mathbb{P}(V_\lambda)}(2)) = \bigwedge^2 V_\lambda & \xrightarrow{i_1^*} & H^0(\Omega^1_X(2)) \\ \downarrow \psi_{\mathbb{P}(V_\lambda)} & & \downarrow \psi_X \\ H^0(\Omega^3_{\mathbb{P}(V\lambda)}(4)) = \bigwedge^4 V_\lambda & \xrightarrow{i_3^*} & H^0(\Omega^3_X(4)). \end{array} \tag{26}$$

where $\psi_Y(v) = \Psi_Y(v \cdot v)$ is the quadratic map associated to Ψ_Y, Y being here $\mathbb{P}(V_\lambda)$ or X. Also recall that the cone over $\mathrm{Fol}(X,2) \subset \mathbb{P}(H^0(\Omega^1_X(2))^\vee)$ is precisely the vanishing locus of ψ_X.

Next, as already noticed, one can check with [vLCL92] that for all varieties in the third row of the Freudenthal magic square we have:

$$H^0(\Omega^1_{\mathbb{P}(V_\lambda)}(2)) = \bigwedge^2 V_\lambda = \mathbb{C} \oplus H^0(\Omega^1_X(2)), \text{ and}$$
$$H^0(\Omega^3_{\mathbb{P}(V_\lambda)}(4)) = \bigwedge^4 V_\lambda = \bigwedge^2 V_\lambda \oplus H^0(\Omega^3_X(4)),$$

with $H^0(\Omega^1_X(2))$ being an irreducible representation. Let us denote by w and w' the highest weight vectors in $\bigwedge^2 V_\lambda$, with $\mathbb{C}w$ being the trivial representation. Now we are ready to proceed to the proof of Theorem 6.6.

Proof of Theorem 6.6. As discussed above we only need to prove that π is surjective. We start with $v \in \bigwedge^2 V_\lambda$ such that $\psi_X \circ i_1^*(v) = 0$. Since w generates $\ker i_1^*$, it is enough to show that there exists $a \in \mathbb{C}$ and $u \in \bigwedge^2 V_\lambda$ such that

$$v = a\,w + u, \text{ and } u \wedge u = 0.$$

Recall that, owing to Remark 2.6, $\psi_{\mathbb{P}^n}(v) = v \wedge v$. Using the commutativity of (26) we may assume that

$$\psi_{\mathbb{P}^n}(v) = v \wedge v \in \ker i_3^* = \mathbb{C} \oplus H^0(\Omega^1_X(2)) \subset \bigwedge^4 V_\lambda$$

And we claim that $v \wedge v \in \ker i_3^*$ implies that w divides $v \wedge v$, i.e., there exists $x \in \bigwedge^2 V_\lambda$ such that $v \wedge v = w \wedge x$.

Let $\mathfrak{g}$ be the Lie algebra of G. To prove this claim we recall the universal enveloping algebra $U(\mathfrak{g}) = \bigoplus_{n \geqslant 0} \mathfrak{g}^{\otimes n}/I$, where I is the ideal generated by $x \otimes y - y \otimes x = [x, y]$ for every $x, y \in \mathfrak{g}$, and $\mathfrak{g}^{\otimes 0} = \mathbb{C}$. We can see the elements of $U(\mathfrak{g})$ as (non-commutative) polynomials on the elements of $\mathfrak{g}$. For more details we refer to [Ser06, Chapter III]. Any $\mathfrak{g}$-module affords an induced $U(\mathfrak{g})$-action and the property that we will use is the following. If $v_\mu \in V_\mu$ is the highest weight vector then $V_\mu = U(\mathfrak{g}) \cdot v_\mu$, see [Ser06, Chapter VII].

Now note that $w \wedge w, w \wedge w' \in \bigwedge^4 V_\lambda$ are highest weight vectors of weights 0 and the highest weight of $H^0(\Omega^1_X(2))$, respectively. Then for $v \wedge v \in \mathbb{C} \oplus H^0(\Omega^1_X(2)) \subset \bigwedge^4 V_\lambda$ there exist $P(X), Q(X) \in U(\mathfrak{g})$ such that

$$v \wedge v = P(X) \cdot (w \wedge w) + Q(X) \cdot (w \wedge w').$$

Note that $\mathfrak{g} \cdot w = \mathbb{C}\, w$, hence $P(X) \cdot w = p\, w$ and $Q(X) \cdot w = q\, w$, for some $p, q \in \mathbb{C}$. Developing the expression above we get

$$\begin{aligned} v \wedge v &= 2p\,(w \wedge w) + q\,(w \wedge w') + w \wedge (Q(X) \cdot w') \\ &= w \wedge \left(2p\, w + (q + Q(X)) \cdot w'\right). \end{aligned}$$

Then define $x := 2p\, w + (q + Q(X)) \cdot w' \in \bigwedge^2 V_\lambda$.

Next we claim that if w divides $v \wedge v$ then there exists $a \in \mathbb{C}$ and $u \in \bigwedge^2 V_\lambda$ such that $v = a\, w + u$ and $u \wedge u = 0$, concluding the proof of the theorem. The proof of this claim will be given in Lemma 6.9. □

Lemma 6.9. *Let $v, w \in \bigwedge^2 \mathbb{C}^{2n}$ such that $w^n \neq 0$. Suppose that w divides $v \wedge v$, i.e., there exists $u \in \bigwedge^2 \mathbb{C}^{2n}$ such that $v \wedge v = w \wedge u$. Then there exist $a \in \mathbb{C}$ and $y \in \bigwedge^2 \mathbb{C}^{2n}$ such that*

$$v = aw + y \text{ and } y \wedge y = 0.$$

Proof. Given $z \in \bigwedge^k \mathbb{C}^{2n}$ define the set $K_z^r := \{x \in \bigwedge^r \mathbb{C}^{2n} \mid x \wedge z = 0\}$. It follows from [DK11, Theorem 1] that w divides $v \wedge v$ if and only if $K_w^{2n-4} \subset K_{v \wedge v}^{2n-4}$.

Consider the pencil of 2-forms $v + tw$. Since each bilinear alternating form corresponds to a skew-symmetric matrix, we get a pencil $A + tB$ of $2n \times 2n$ matrices. Recall that linear change of coordinates $P \in \mathrm{GL}(2n, \mathbb{C})$ on $\mathbb{C}^{2n}$ correspond to a congruence $P^T(A + tB)P$.

Owing to [Tho91, Theorem 1], one has that $A + tB$ is, up to congruence, a block-diagonal matrix whose blocks depend on invariants of the pencil. Since $w^n \neq 0$ we have that $\det B \neq 0$, hence the only blocks that can appear have the form

$$B(a, m) := \begin{bmatrix} 0 & (a+t)\Delta_m + \Lambda_m \\ -(a+t)\Delta_m - \Lambda_m & 0 \end{bmatrix}_{2m \times 2m}$$

where

$$\Delta_m = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{m\times m}, \quad \Lambda_m = \begin{bmatrix} 0 & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & 0 \end{bmatrix}_{m\times m}.$$

The only invariants are the elementary divisors $(a+t)^{2m}$ of $\det(A+tB)$. Fix an integer partition $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$, $\lambda_1+\cdots+\lambda_r = n$, and complex numbers $a_1,\dots,a_r \in \mathbb{C}$ and define

$$M(t) = (p_{i,j} + tq_{i,j}) := B(a_1,\lambda_1)\oplus\cdots\oplus B(a_r,\lambda_r).$$

Moreover, $v = \sum_{i<j} p_{i,j} e_i \wedge e_j$ and $w = \sum_{i<j} q_{i,j} e_i \wedge e_j$; note that v depends on $a_1,\dots,a_r$ but w does not.

We note that if either $\lambda_1 \geqslant 3$ or $\lambda_2 \geqslant 2$ then $K_w^{2n-4} \not\subset K_{v\wedge v}^{2n-4}$. Indeed, if $\lambda_1 \geqslant 3$ then

$$w = \sum_{j=1}^{\lambda_1} e_j \wedge e_{2\lambda_1+1-j} + w' \text{ and } v = a_1 \sum_{j=1}^{\lambda_1} e_j \wedge e_{2\lambda_1+1-j} + \sum_{j=1}^{\lambda_1} e_{j+1} \wedge e_{2\lambda_1+1-j} + v'$$

where w' and v' only involve e_j for $j \geqslant 2\lambda_1+1$. Then take $\phi = \bigwedge_{j\notin\{2,3,2\lambda_1-1,2\lambda_1\}} e_j$. It follows that $\phi \in K_w^{2n-4}$ but $\phi\wedge v\wedge v = \pm 1$. Similarly, if $\lambda_1=\lambda_2=2$ we have

$$\begin{aligned} w &= e_1\wedge e_4 + e_2\wedge e_3 + e_5\wedge e_8 + e_6\wedge e_7 + w' \text{ and} \\ v &= a_1(e_1\wedge e_4 + e_2\wedge e_3) + a_2(e_5\wedge e_8 + e_6\wedge e_7) + e_2\wedge e_4 + e_6\wedge e_8 + v', \end{aligned}$$

where w' and v' only involve e_j for $j \geqslant 9$. Then $\phi = \bigwedge_{j\notin\{2,4,6,8\}} e_j \in K_w^{2n-4}\backslash K_{v\wedge v}^{2n-4}$. Therefore we only need to deal with the partitions $(2,1^{n-2})$ and (1^n).

For the partition $(2,1^{n-2})$ we get

$$\begin{aligned} w &= e_1\wedge e_4 + e_2\wedge e_3 + \sum_{j=3}^{n} e_{2j-1}\wedge e_{2j} \text{ and} \\ v &= a_1(e_1\wedge e_4 + e_2\wedge e_3) + e_2\wedge e_4 + \sum_{j=3}^{n} a_{j-1}(e_{2j-1}\wedge e_{2j}). \end{aligned}$$

Imposing $K_w^{2n-4} \subset K_{v\wedge v}^{2n-4}$ implies $a_1 = a_2 = \cdots = a_{n-1}$, hence $v = a_1 w + e_2\wedge e_4$ and we are done.

For the partition (1^n) we get

$$w = \sum_{j=1}^{n} e_{2j-1}\wedge e_{2j} \text{ and } v = \sum_{j=1}^{n} a_j(e_{2j-1}\wedge e_{2j})$$

and imposing $K_w^{2n-4} \subset K_{v\wedge v}^{2n-4}$ we get that at least $n-1$ of the n coefficients a_i must be equal. Hence, up to reordering, we have $v = a_2 w + (a_2-a_1)(e_1\wedge e_2)$ and we are done. □

Remark 6.10. It is still not clear to us whether $\mathrm{Fol}(X,2)$, seen as a scheme defined by the quadratic equations $H^0(\Omega^3_X(2))^\vee \subset S^2H^0(\Omega^1_X(2))^\vee$ (see Theorem 3.7), is already a reduced scheme. We also remark that experimental computations with Macaulay2 [GS] were crucial to discover the lemma above.

6.6. **Some products of projective spaces.** We give now a description of what happens for some examples of cominuscule spaces which are not Grassmannians. Clearly the easiest case to treat is that of product of projective spaces $X := \mathbb{P}(U) \times \mathbb{P}(V) \cong \mathbb{P}^m \times \mathbb{P}^n$. This variety embeds in $\mathbb{P}(U \otimes V) \cong \mathbb{P}^{N-1}$ via the Segre embedding, where $N = (m+1)(n+1)$. Let $\bigwedge^4(U \otimes V)^\vee$ be the space of quadrics defining $\mathrm{Fol}(\mathbb{P}(U \otimes V), 2) \cong G(N-2, U \otimes V)$. These quadrics can be decomposed in direct sum of $\mathrm{SL}(U) \times \mathrm{SL}(V)$-irreducible representations, among which one factor is $\Gamma^{2,2}U^\vee \otimes \Gamma^{2,2}V^\vee$; we denote by $\mathcal{J}$ the direct sum of irreducible representations of $\bigwedge^4(U \otimes V)^\vee$ not showing up in $\Gamma^{2,2}U^\vee \otimes \Gamma^{2,2}V^\vee$. Moreover let us denote by $\mathcal{O}_X(1,1)$ the ample bundle defining the Segre embedding. We have the following result.

Proposition 6.11. *The space of codimension one minimal degree foliations* $\mathrm{Fol}(X, 2) \subset \mathbb{P}(H^0(\Omega^1_X(2,2))^\vee) \cong \mathbb{P}(\bigwedge^2(U \otimes V)^\vee)$ *is defined by the quadratic equations given by the inclusion*

$$H^0(\Omega^3_X(4,4))^\vee = \mathcal{J} \subsetneq \bigwedge^4(U \otimes V)^\vee \subset S^2 \bigwedge^2(U \otimes V)^\vee.$$

Thus we have a scheme theoretical strict inclusion of $\mathrm{Fol}(\mathbb{P}(U \otimes V), 2) \cong G(N-2, U \otimes V)$ *inside* $\mathrm{Fol}(X, 2)$.

Proof. Since the cotangent bundle of a product $\mathbb{P}(U) \times \mathbb{P}(V)$ is the direct sum of the cotangent bundles of the factors $\Omega^1_{\mathbb{P}(U)} \oplus \Omega^1_{\mathbb{P}(V)}$, we also deduce that

$$\Omega^3_X = \Omega^3_{\mathbb{P}(U)} \oplus (\Omega^2_{\mathbb{P}(U)} \otimes \Omega^1_{\mathbb{P}(V)}) \oplus (\Omega^1_{\mathbb{P}(U)} \otimes \Omega^2_{\mathbb{P}(V)}) \oplus \Omega^3_{\mathbb{P}(V)}.$$

From the Borel-Weil-Bott Theorem we deduce that

$$H^0(\Omega^1_X(2,2)) = (\bigwedge^2 U \otimes S^2V) \oplus (S^2U \otimes \bigwedge^2 V)$$

and

$$H^0(\Omega^3_X(4,4)) = (\bigwedge^4 U \otimes S^4V) \oplus (\Gamma^{3,1}U \otimes \Gamma^{2,1,1}V) \oplus (\Gamma^{2,1,1}U \otimes \Gamma^{3,1}V) \oplus (S^4U \otimes \bigwedge^4 V).$$

We claim that $\bigwedge^2(U \otimes V) \cong H^0(\Omega^1_X(2,2))$ and $\bigwedge^4(U \otimes V) \cong H^0(\Omega^3_X(4,4)) \oplus (\Gamma^{2,2}U \otimes \Gamma^{2,2}V)$, thus showing that $H^0(\Omega^3_X(4,4))^\vee = \mathcal{J}$ and proving the statement of the proposition.

The computation of $\bigwedge^i(U \otimes V)$ is an application of the Littlewood-Richardson rule to the case of exterior powers. In terms of Young diagrams

$$\bigwedge^i(U \otimes V) = \bigoplus_\lambda \Gamma^\lambda U \otimes \Gamma^{\lambda'}V,$$

where the direct sum runs over all Young diagrams of size i with at most $m+1$ rows and $n+1$ columns, and λ' is the Young diagram obtained from λ by exchanging rows and columns. The claim follows. □

We believe that the situation is not easier for other cominuscule spaces. For instance if $X := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ then the morphism $i_1^*\colon H^0(\mathbb{P}^7, \Omega^1_{\mathbb{P}^7}(2)) \to H^0(\Omega^1_X(2,2,2))$ is already not injective.

7. Further directions

Let us list some possible open questions and research directions that arise from this work. We will denote by X a cominuscule Grassmannian. We could explicitly recover the space of foliations $\mathrm{Fol}(X,2)$ of some X by showing that it is equal to the space of foliations of the ambient projective space (see Theorems 4.1, 6.1, 6.4). A natural question is:

Question 7.1. *Is* $\overline{\pi(\mathrm{Fol}(\mathbb{P}(V_\lambda),2))}$ *isomorphic to* $\mathrm{Fol}(X,2)$ *for any cominuscule Grassmannian* X*?*

For Grassmannians $G(3,n)$ we could prove some results going in the direction of a positive answer to this question. These results concerned the ideal defining foliations and the properties of π and give rise to the following questions, which are weaker than 7.1:

Question 7.2. *Is the ideal of* $\overline{\pi(\mathrm{Fol}(\mathbb{P}(V_\lambda),2))}$ *equal to the ideal of* $\mathrm{Fol}(X,2)$*?*

For $X = G(3,n)$ we have already proved the equality in degree 2, i.e. for quadrics (Theorem 4.3).

Question 7.3. *Is the restriction of* π *to* $\mathrm{Fol}(\mathbb{P}(V_\lambda),2)$ *injective? Is it dominant?*

We have already shown that this restriction is an embedding for $X = G(3,n)$ (Theorem 4.3) and that it is a dominant embedding for the varieties in the third row of the Freudenthal magic square (Theorem 6.6).

Question 7.4. *Is* $\mathrm{Fol}(X,2)$ *smooth or at least reduced?*

If we knew this was true, we would get from Theorem 6.6 that $\mathrm{Fol}(\mathbb{P}(V_\lambda),2) \cong \mathrm{Fol}(X,2)$ for X one of the varieties appearing in the third row of the Freudenthal magic square.

In some cases (see Theorem 3.7, 6.1, Proposition 5.4, 6.3, 6.5) we were able to provide the equations of the space of foliations $\mathrm{Fol}(X,2)$. These equations are often equivariantly unique, meaning that the irreducible representations they involve are unique in their respective ambient representation spaces. This naturally brings out the following problem:

Question 7.5. *Is it possible to recover* $\mathrm{Fol}(X,2)$ *as a* G*-variety solely from a description of its* G*-equivariant ideal?*

For instance, take the case of Grassmannians $G(3,n)$; the question (which is now a priori independent of the understanding of foliations on $G(3,n)$) is whether one can find a geometric description of the variety defined by the quadrics

$$(V_{3\lambda_4} \oplus V_{3\lambda_2+\lambda_6} \oplus V_{\lambda_1+\lambda_2+\lambda_4+\lambda_5})^\vee \subset S^2 V^\vee_{\lambda_2+\lambda_4} \cong H^0(\mathbb{P}(V^\vee_{\lambda_2+\lambda_4}), \mathcal{O}_{\mathbb{P}(V^\vee_{\lambda_2+\lambda_4})}(2))$$

inside the projective space $\mathbb{P}(V^\vee_{\lambda_2+\lambda_4})$. Another remarkable example is that of $\mathbb{P}^m \times \mathbb{P}^n$: what is the variety inside $\mathbb{P}(\bigwedge^2(U\otimes V)^\vee)$ defined by $\mathcal{J} \subset S^2 \bigwedge^2(U\otimes V)^\vee$ (in the notation of Section 6.6) and containing $G(N-2, U\otimes V)$? This kind of questions shows that there exists a very interesting bidirectional interplay between the study of foliations and the equivariant geometric theory.

Appendix A. Technical lemmas in the proof of Proposition 4.4

We collected in this appendix some lemmas which are necessary to prove Proposition 4.4. From the square (20) we can derive the following diagram:

$$\begin{array}{ccccc} V_{\lambda_6}^\vee \cdot \bigwedge^2 V_{\lambda_3}^\vee & \xleftarrow{\xi} & S^2 \bigwedge^2 V_{\lambda_3}^\vee & \xleftarrow{\pi^*} & S^2 V_{\lambda_2+\lambda_4}^\vee \\ & & \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}\uparrow & & \\ & & \bigwedge^4 V_{\lambda_3}^\vee & & \end{array} \tag{27}$$

The top row comes from the second symmetric power of the decomposition $\bigwedge^2 V_{\lambda_3}^\vee = V_{\lambda_6}^\vee \oplus V_{\lambda_2+\lambda_4}^\vee$, in particular it is exact. Then to show that (22) holds for some μ it is enough to show, owing to Schur's Lemma, that $\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(V_\mu^\vee) \neq \{0\}$.

In order to properly describe the map ξ and do subsequent computations, let us fix some notation. Fix a basis $\{e_1, \dots, e_n\}$ of $V^\vee$ such that each e_i is a weight vector for the action of $\mathfrak{sl}(V^\vee)$. We write $e_{i_1,\dots,i_k}$ for the element $e_{i_1} \wedge \cdots \wedge e_{i_k} \in V_{\lambda_k}^\vee = \bigwedge^k V^\vee$. In particular, if $\sigma \in \mathcal{S}_k$ is a permutation,

$$e_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = (-1)^\sigma e_{i_1,\dots,i_k}$$

and $e_{i_1,\dots,i_k} = 0$ if $i_r = i_s$ for some pair of indices r and s. Even though we only need $e_{i_1,\dots,i_k}$ with $i_1 < i_2 < \cdots < i_k$ in the definition of $V_{\lambda_k}^\vee$, it will be useful to consider all possible indices in view of the induced $\mathfrak{sl}(V^\vee)$ action.

Let l_i denote the weight of e_i, then each $e_{i_1,\dots,i_k}$ has weight $l_{i_1} + \cdots + l_{i_k}$ for the induced $\mathfrak{sl}(V^\vee)$ action. If $\alpha_{r,s} = l_r - l_s$ is a root, then let $X_{r,s} \in \mathfrak{sl}(V^\vee)$ be the corresponding element: $X_{r,s}(e_t) = 0$ if $t \neq s$ and $X_{r,s}(e_s) = e_r$. Then the induced action gives

$$\begin{aligned} X_{r,s}(e_{i_1,\dots,i_k}) &= 0 \text{ if } s \notin I \text{ or } \{r,s\} \subset I, \\ X_{r,s}(e_{\dots,s,\dots}) &= e_{\dots,r,\dots} \text{ if } s \in I \text{ and } r \notin I \end{aligned}$$

where $I = \{i_1, \dots, i_k\}$.

Remark A.1. Let us denote by $\prec$ the lexicographic order. A basis of $V_{\lambda_6}^\vee$ is given by $\{e_{i_1,\cdots,i_6}\}$ with $1 \leqslant i_1 < \cdots < i_6 \leqslant n$, while a basis of $\bigwedge^2 V_{\lambda_3}^\vee$ is given by $\{e_{i_1,i_2,i_3} \wedge e_{j_1,j_2,j_3}\}$ with $1 \leqslant i_1 < i_2 < i_3 \leqslant n$, $1 \leqslant j_1 < j_2 < j_3 \leqslant n$ and $(i_1,i_2,i_3) \prec (j_1,j_2,j_3)$. Notice that the elements of the latter basis for which $\{i_1,i_2,i_3\} \cap \{j_1,j_2,j_3\} \neq 0$ belong to $V_{\lambda_2+\lambda_4}^\vee$. As a consequence of this, the elements $\{e_{i_1,i_2,i_3} \wedge e_{j_1,j_2,j_3} \cdot e_{k_1,\cdots,k_6}\}$ with the previous conditions are a generating set of $\bigwedge^2 V_{\lambda_3}^\vee \cdot V_{\lambda_6}^\vee$, and those such that $\{i_1,i_2,i_3\} \cap \{j_1,j_2,j_3\} \neq 0$ are linearly independent in $V_{\lambda_2+\lambda_4}^\vee \otimes V_{\lambda_6}^\vee \subset \bigwedge^2 V_{\lambda_3}^\vee \cdot V_{\lambda_6}^\vee$.

Lemma A.2. *The map $\xi\colon S^2 \bigwedge^2 V_{\lambda_3}^\vee \to V_{\lambda_6}^\vee \cdot \bigwedge^2 V_{\lambda_3}^\vee$ from (27) is defined by*

$$\xi((e_{i,j,k} \wedge e_{l,m,n}) \cdot (e_{o,p,q} \wedge e_{r,s,t})) = e_{i,j,k,l,m,n} \cdot (e_{o,p,q} \wedge e_{r,s,t}) + e_{o,p,q,r,s,t} \cdot (e_{i,j,k} \wedge e_{l,m,n})$$

Proof. First we note that the unique (up to scalar multiple) $\mathrm{SL}(V^\vee)$-equivariant map $\bigwedge^2 V_{\lambda_3}^\vee \to V_{\lambda_6}^\vee$ is the multiplication $m(e_{i_1,i_2,i_3} \wedge e_{j_1,j_2,j_3}) = e_{i_1,i_2,i_3,j_1,j_2,j_3}$. Then ξ can be described as the composition

$$\xi\colon S^2 \bigwedge^2 V_{\lambda_3}^\vee \xrightarrow{\delta} \bigwedge^2 V_{\lambda_3}^\vee \otimes \bigwedge^2 V_{\lambda_3}^\vee \xrightarrow{m\otimes 1} V_{\lambda_6}^\vee \cdot \bigwedge^2 V_{\lambda_3}^\vee$$

where $\delta(u \cdot v) = (u \otimes v + v \otimes u)$. Therefore

$$\begin{aligned}\xi((a \wedge b) \cdot (c \wedge d)) &= (m \otimes 1)((a \wedge b) \otimes (c \wedge d) + (c \wedge d) \otimes (a \wedge b)) \\ &= m(a \wedge b) \cdot (c \wedge d) + m(c \wedge d) \cdot (a \wedge b)\end{aligned}$$

and

$$\xi((e_{i,j,k} \wedge e_{l,m,n}) \cdot (e_{o,p,q} \wedge e_{r,s,t})) = e_{i,j,k,l,m,n} \cdot (e_{o,p,q} \wedge e_{r,s,t}) + e_{o,p,q,r,s,t} \cdot (e_{i,j,k} \wedge e_{l,m,n}).$$

□

Remark A.3. For further computations it will be useful to have the highest weight vectors of $\bigwedge^2 V_{\lambda_3}^\vee = V_{\lambda_6}^\vee \oplus V_{\lambda_2+\lambda_4}^\vee$. Firstly notice that

$$w_{2,4} := e_{1,2,3} \wedge e_{1,2,4}$$

is the unique vector in $\bigwedge^2 V_{\lambda_3}^\vee$ of weight $\lambda_2 + \lambda_4$, hence it must be cyclic. For λ_6 we may use the diagonal map $\delta\colon V_{\lambda_6}^\vee \hookrightarrow \bigwedge^2 V_{\lambda_3}^\vee$ from which we define

$$w_6 := \delta(e_{1,2,3,4,5,6}) = \sum_{\sigma \in \mathcal{S}_6} (-1)^\sigma e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}$$

where $(-1)^\sigma$ denotes the sign of the permutation σ. Also note that

$$w_6^{10} = \underbrace{w_6 \wedge \cdots \wedge w_6}_{10 \text{ times}} \neq 0$$

but $w_6^{11} = 0$.

Now we are ready to prove the technical bulk of Proposition 4.4.

Lemma A.4. *The equality* (22) *holds for* $\mu = 2\lambda_6$.

Proof. From Remark A.3 we have that $w_6 \in \bigwedge^2 V_{\lambda_3}^\vee$ is a highest weight vector of weight λ_6. Then

$$w_6 \wedge w_6 = \sum_{\sigma,\tau \in \mathcal{S}_6} (-1)^\sigma (-1)^\tau e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\tau_1,\tau_2,\tau_3} \wedge e_{\tau_4,\tau_5,\tau_6} \in \bigwedge^4 V_{\lambda_3}^\vee$$

is also a highest weight vector of weight $2\lambda_6$. We then compute

$$\Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_6 \wedge w_6) = \sum_{\sigma,\tau \in \mathcal{S}_6} (-1)^\sigma (-1)^\tau [(e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) \cdot (e_{\tau_1,\tau_2,\tau_3} \wedge e_{\tau_4,\tau_5,\tau_6})$$

$$-(e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\tau_1,\tau_2,\tau_3}) \cdot (e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\tau_4,\tau_5,\tau_6}) + (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\tau_4,\tau_5,\tau_6}) \cdot (e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\tau_1,\tau_2,\tau_3})].$$

Observe that

$$\begin{aligned}\xi((e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) \cdot (e_{\tau_1,\tau_2,\tau_3} \wedge e_{\tau_4,\tau_5,\tau_6})) &= (-1)^\sigma e_{1,2,3,4,5,6} \cdot (e_{\tau_1,\tau_2,\tau_3} \wedge e_{\tau_4,\tau_5,\tau_6}) \\ &\quad + (-1)^\tau e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6})\end{aligned}$$

and also notice that $\xi((e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\tau_1,\tau_2,\tau_3}) \cdot (e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\tau_4,\tau_5,\tau_6})) \neq 0$ if and only if $\{\tau_1,\tau_2,\tau_3\} = \{\sigma_4,\sigma_5,\sigma_6\}$. In this case there exist (unique) $\rho_1, \rho_2 \in \mathcal{S}_3$ such that $\rho_1(\tau_1,\tau_2,\tau_3) = (\sigma_4,\sigma_5,\sigma_6)$ and $\rho_2(\tau_4,\tau_5,\tau_6) = (\sigma_1,\sigma_2,\sigma_3)$ hence

$$\begin{aligned}&\xi((e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\tau_1,\tau_2,\tau_3}) \cdot (e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\tau_4,\tau_5,\tau_6})) = \\ &\quad = (-1)^{\rho_1}(-1)^{\rho_2}\xi((e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) \cdot (e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\sigma_1,\sigma_2,\sigma_3})) = \\ &\quad = -2(-1)^{\rho_1}(-1)^{\rho_2}(-1)^\sigma e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) = \\ &\quad = 2(-1)^\tau e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6})\end{aligned}$$

The last equality comes from the fact that the joint (or concatenated) permutation $(\rho_1 \mid \rho_2) \in \mathcal{S}_6$ satisfies $(\rho_1 \mid \rho_2)\tau = (\sigma_4, \sigma_5, \sigma_6, \sigma_1, \sigma_2, \sigma_3)$ hence $(-1)^{(\rho_1|\rho_2)} = (-1)^{\rho_1}(-1)^{\rho_2} = -(-1)^{\tau}(-1)^{\sigma}$. Analogously, we have

$$\begin{aligned}\xi((e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\tau_4,\tau_5,\tau_6}) \cdot (e_{\sigma_4,\sigma_5,\sigma_6} \wedge e_{\tau_1,\tau_2,\tau_3})) = \\ = -2(-1)^{\rho'_1}(-1)^{\rho'_2}(-1)^{\sigma} e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) = \\ = -2(-1)^{\tau} e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6})\end{aligned}$$

for (unique) ρ'_1, ρ'_2 such that $\rho'_1(\tau_4, \tau_5, \tau_6) = (\sigma_4, \sigma_5, \sigma_6)$ and $\rho'_2(\tau_1, \tau_2, \tau_3) = (\sigma_1, \sigma_2, \sigma_3)$, hence $(-1)^{\rho'_1}(-1)^{\rho'_2} = (-1)^{\tau}(-1)^{\sigma}$. Therefore

$$\xi \circ \Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_6 \wedge w_6) = A - B + C$$

where

$$\begin{aligned}A = &\sum_{\sigma,\tau \in \mathcal{S}_6} (-1)^{\sigma}(-1)^{\tau}[(-1)^{\sigma} e_{1,2,3,4,5,6} \cdot (e_{\tau_1,\tau_2,\tau_3} \wedge e_{\tau_4,\tau_5,\tau_6}) + \\ &+ (-1)^{\tau} e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6})] = \\ &= 6!\, 2\, e_{1,2,3,4,5,6} \cdot w_6 = 1440\, e_{1,2,3,4,5,6} \cdot w_6, \\ B = &\sum_{\sigma \in \mathcal{S}_6} \sum_{\substack{\tau \in \mathcal{S}_6 \\ \{\tau_1,\tau_2,\tau_3\} = \{\sigma_4,\sigma_5,\sigma_6\}}} (-1)^{\sigma}(-1)^{\tau} 2 (-1)^{\tau} e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) = \\ &= 72\, e_{1,2,3,4,5,6} \cdot w_6, \\ C = &\sum_{\sigma \in \mathcal{S}_6} \sum_{\substack{\tau \in \mathcal{S}_6 \\ \{\tau_1,\tau_2,\tau_3\} = \{\sigma_1,\sigma_2,\sigma_3\}}} -(-1)^{\sigma}(-1)^{\tau} 2 (-1)^{\tau} e_{1,2,3,4,5,6} \cdot (e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) = \\ &= -72\, e_{1,2,3,4,5,6} \cdot w_6.\end{aligned}$$

Therefore $\xi \circ \Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_6 \wedge w_6) = 1296\, e_{1,2,3,4,5,6} \cdot w_6 \neq 0$. □

Lemma A.5. *The equality* (22) *holds for* $\mu = \lambda_2 + \lambda_4 + \lambda_6$.

Proof. As in the previous lemma, we get from Remark A.3 that the highest weight vector in $\bigwedge^6 V^{\vee}_{\lambda_3}$ of weight $\lambda_2 + \lambda_4 + \lambda_6$ is $w_{2,4} \wedge w_6$. Then

$$\begin{aligned}\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,4} \wedge w_6) = \Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(e_{1,2,3} \wedge e_{1,2,4} \wedge \sum_{\sigma \in \mathcal{S}_6} (-1)^{\sigma} e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) = \\ = (e_{1,2,3} \wedge e_{1,2,4}) \cdot \sum_{\sigma \in \mathcal{S}_6} (-1)^{\sigma} e_{\sigma_1,\sigma_2,\sigma_3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) + \\ - \sum_{\sigma \in \mathcal{S}_6} (-1)^{\sigma} (e_{1,2,3} \wedge e_{\sigma_1,\sigma_2,\sigma_3}) \cdot (e_{1,2,4} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) + \\ + \sum_{\sigma \in \mathcal{S}_6} (-1)^{\sigma} (e_{1,2,3} \wedge e_{\sigma_4,\sigma_5,\sigma_6}) \cdot (e_{1,2,4} \wedge e_{\sigma_1,\sigma_2,\sigma_3}).\end{aligned}$$

Notice that $m(e_{1,2,3} \wedge e_{1,2,4}) = e_{1,2,3,1,2,4} = 0$. By the same reason we also have that $\xi((e_{1,2,3} \wedge e_{i,j,k}) \cdot (e_{1,2,4} \wedge e_{l,m,n})) \neq 0$ if and only if either $\{i,j,k\} = \{4,5,6\}$ or $\{l,m,n\} = \{3,5,6\}$. Following the same strategy of the proof of Lemma A.4 we arrive at

$$\xi \circ \Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,4} \wedge w_6) = 576\, e_{1,2,3,4,5,6} \cdot w_{2,4} \neq 0.$$

□

Lemma A.6. *The equality (22) holds for $\mu = \lambda_4 + \lambda_8$.*

Proof. Consider $V^\vee_{\lambda_4+\lambda_8} \subset V^\vee_{\lambda_4} \otimes V^\vee_{\lambda_8}$ generated by the highest weight vector $e_{1,2,3,4} \otimes e_{1,\dots,8}$. Now consider the diagonal maps:

$$V^\vee_{\lambda_4} \hookrightarrow (V^\vee)^{\otimes 4}: \quad e_{1,2,3,4} \longmapsto \sum_{\sigma\in\mathcal{S}_4} (-1)^\sigma e_{\sigma_1} \otimes e_{\sigma_2} \otimes e_{\sigma_3} \otimes e_{\sigma_4};$$
$$V^\vee_{\lambda_8} \hookrightarrow (V^\vee_{\lambda_2})^{\otimes 4}: \quad e_{1,\dots,8} \longmapsto \sum_{\tau\in\mathcal{S}_8} (-1)^\tau e_{\tau_1,\tau_2} \otimes e_{\tau_3,\tau_4} \otimes e_{\tau_5,\tau_6} \otimes e_{\tau_7,\tau_8}.$$

After applying the multiplication maps $V^\vee \otimes V^\vee_{\lambda_2} \to V^\vee_{\lambda_3}$ and $(V^\vee_{\lambda_3})^{\otimes 4} \to \bigwedge^4 V^\vee_{\lambda_3}$ we get a copy of $V^\vee_{\lambda_4+\lambda_8}$ inside $\bigwedge^4 V^\vee_{\lambda_3}$ determined by

$$w_{4,8} = \sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_8} (-1)^\sigma (-1)^\tau e_{\sigma_1,\tau_1,\tau_2} \wedge e_{\sigma_2,\tau_3,\tau_4} \wedge e_{\sigma_3,\tau_5,\tau_6} \wedge e_{\sigma_4,\tau_7,\tau_8}.$$

Let us show that $\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{4,8}) \neq 0$. First notice that, by symmetry,

$$\Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{4,8}) = \sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_8} (-1)^\sigma (-1)^\tau e_{\sigma_1,\tau_1,\tau_2} \wedge e_{\sigma_2,\tau_3,\tau_4} \cdot e_{\sigma_3,\tau_5,\tau_6} \wedge e_{\sigma_4,\tau_7,\tau_8}$$

and

$$\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{4,8}) = 6 \sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_8} (-1)^\sigma (-1)^\tau e_{\sigma_1,\tau_1,\tau_2} \wedge e_{\sigma_2,\tau_3,\tau_4} \cdot e_{\sigma_3,\sigma_4,\tau_5,\tau_6,\tau_7,\tau_8}.$$

In order to show that this element is different from zero, let us show that one of its coefficients in the basis described in Remark A.1 is nonzero. More precisely, let us compute its coefficient with respect to the element $e_{1,2,3} \wedge e_{2,1,4} \cdot e_{3,4,5,6,7,8}$. To compute this coefficient, we need to isolate the permutations σ and τ such that $e_{\sigma_1,\tau_1,\tau_2} \wedge e_{\sigma_2,\tau_3,\tau_4} \cdot e_{\sigma_3,\sigma_4,\tau_5,\tau_6,\tau_7,\tau_8} = \pm e_{1,2,3} \wedge e_{2,1,4} \cdot e_{3,4,5,6,7,8}$. In order for this to happen, σ must send $\{1,2\}$ to $\{1,2\}$ (there are two possibilities since we need to take into account the order) and $\{3,4\}$ to $\{3,4\}$ (2 possibilities). If $\sigma_1 = 1$ then either τ must send $\{1,2\}$ to $\{2,3\}$ (2 possibilities), $\{3,4\}$ to $\{1,4\}$ (2 possibilities) and $\{5,6,7,8\}$ to $\{5,6,7,8\}$ (4! possibilities) or it must send $\{1,2\}$ to $\{2,4\}$ (2 possibilities), $\{3,4\}$ to $\{1,3\}$ (2 possibilities) and $\{5,6,7,8\}$ to $\{5,6,7,8\}$ (4! possibilities); similarly if $\sigma_1 = 2$. Since all these terms come with a + sign, we obtain that the coefficient in question is equal to $2^8 3 \neq 0$. □

Lemma A.7. *The equality (22) holds for $\mu = 2\lambda_2 + \lambda_8$.*

Proof. Consider $V^\vee_{2\lambda_2+\lambda_8} \subset V^\vee_{\lambda_2} \otimes V^\vee_{\lambda_2} \otimes V^\vee_{\lambda_8}$ determined by $e_{1,2} \otimes e_{1,2} \otimes e_{1,\dots,8}$. Then consider the diagonal map

$$V^\vee_{\lambda_8} \hookrightarrow V^\vee \otimes V^\vee \otimes V^\vee_{\lambda_3} \otimes V^\vee_{\lambda_3}: \quad e_{1,\dots,8} \longmapsto \sum_{\sigma\in\mathcal{S}_8} (-1)^\sigma e_{\sigma_1} \otimes e_{\sigma_2} \otimes e_{\sigma_3,\sigma_4,\sigma_5} \otimes e_{\sigma_6,\sigma_7,\sigma_8}.$$

Aplying multiplication maps $V^\vee \otimes V^\vee_{\lambda_2} \to V^\vee_{\lambda_3}$ and $(V^\vee_{\lambda_3})^{\otimes 4} \to \bigwedge^4 V^\vee_{\lambda_3}$ we get $V^\vee_{2\lambda_2+\lambda_8} \subset \bigwedge^4 V^\vee_{\lambda_3}$ determined by

$$w_{2,2,8} = \sum_{\sigma\in\mathcal{S}_8} (-1)^\sigma e_{1,2,\sigma_1} \wedge e_{1,2,\sigma_2} \wedge e_{\sigma_3,\sigma_4,\sigma_5} \wedge e_{\sigma_6,\sigma_7,\sigma_8}.$$

Let us show that $\xi \circ \Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,2,8}) \neq 0$. Firstly by Lemma A.2 one computes $\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,2,8}) =$

$$\sum_{\sigma\in\mathcal{S}_8}(-1)^{\sigma}[e_{1,2,\sigma_1}\wedge e_{1,2,\sigma_2}\cdot e_{\sigma_3,\sigma_4,\sigma_5}\wedge e_{\sigma_6,\sigma_7,\sigma_8}-2e_{1,2,\sigma_1}\wedge e_{\sigma_3,\sigma_4,\sigma_5}\cdot e_{1,2,\sigma_2}\wedge e_{\sigma_6,\sigma_7,\sigma_8}]$$

and

$$\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,2,8})=\sum_{\sigma\in\mathcal{S}_8}(-1)^{\sigma}[e_A+e_B+e_C],$$

where $e_A = e_{1,2,\sigma_1}\wedge e_{1,2,\sigma_2}\cdot e_{\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7,\sigma_8}$, $e_B = -2e_{1,2,\sigma_1}\wedge e_{\sigma_3,\sigma_4,\sigma_5}\cdot e_{1,2,\sigma_2,\sigma_6,\sigma_7,\sigma_8}$, $e_C = -2e_{1,2,\sigma_2}\wedge e_{\sigma_6,\sigma_7,\sigma_8}\cdot e_{1,2,\sigma_1,\sigma_3,\sigma_4,\sigma_5}$. In order to show that $\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,2,8})\neq 0$, we want to show that the coefficient of $\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,2,8})$ corresponding to the element $e_{1,2,3}\wedge e_{1,2,4}\cdot e_{1,2,5,6,7,8}$ with respect to the basis of Remark A.1 is nonzero. Proceeding similarly to the previous proof, one can show that the coefficients of $\sum_{\sigma\in\mathcal{S}_8}(-1)^{\sigma}e_A$, $\sum_{\sigma\in\mathcal{S}_8}(-1)^{\sigma}e_B$, $\sum_{\sigma\in\mathcal{S}_8}(-1)^{\sigma}e_C$ are respectively $2\cdot 6!$, $4\cdot 3!4!$, $4\cdot 3!4!$, and the final coefficient is equal to $2^5 3^4\neq 0$. □

Lemma A.8. *The equality* (22) *holds for* $\mu=\lambda_2+\lambda_3+\lambda_7$.

Proof. Consider $V^{\vee}_{\lambda_2+\lambda_3+\lambda_7}\subset V^{\vee}_{\lambda_2}\otimes V^{\vee}_{\lambda_3}\otimes V^{\vee}_{\lambda_7}$ determined by $e_{1,2}\otimes e_{1,2,3}\otimes e_{1,\dots,7}$. Then consider the diagonal map

$$V^{\vee}_{\lambda_7}\hookrightarrow V^{\vee}\otimes V^{\vee}_{\lambda_3}\otimes V^{\vee}_{\lambda_3}:\quad e_{1,\dots,7}\longmapsto\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}e_{\sigma_1}\otimes e_{\sigma_2,\sigma_3,\sigma_4}\otimes e_{\sigma_5,\sigma_6,\sigma_7}.$$

Aplying multiplication maps $V^{\vee}\otimes V^{\vee}_{\lambda_2}\to V^{\vee}_{\lambda_3}$ and $(V^{\vee}_{\lambda_3})^{\otimes 4}\to\bigwedge^4 V^{\vee}_{\lambda_3}$ we get $V^{\vee}_{\lambda_2+\lambda_3+\lambda_7}\subset\bigwedge^4 V^{\vee}_{\lambda_3}$ determined by

$$w_{2,3,7}=e_{1,2,3}\wedge\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}e_{1,2,\sigma_1}\wedge e_{\sigma_2,\sigma_3,\sigma_4}\wedge e_{\sigma_5,\sigma_6,\sigma_7}.$$

Let us show that $\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,3,7})\neq 0$. A direct computation shows that $\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,3,7}) =$

$$\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}[e_{1,2,3}\wedge e_{1,2,\sigma_1}\cdot e_{\sigma_2,\sigma_3,\sigma_4}\wedge e_{\sigma_5,\sigma_6,\sigma_7}-2e_{1,2,3}\wedge e_{\sigma_2,\sigma_3,\sigma_4}\cdot e_{1,2,\sigma_1}\wedge e_{\sigma_5,\sigma_6,\sigma_7}]$$

and

$$\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,3,7})=\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}[e_A+e_B+e_C],$$

where $e_A = e_{1,2,3}\wedge e_{1,2,\sigma_1}\cdot e_{\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7}$, $e_B = -2e_{1,2,3}\wedge e_{\sigma_2,\sigma_3,\sigma_4}\cdot e_{1,2,\sigma_1,\sigma_5,\sigma_6,\sigma_7}$, $e_C = -2e_{1,2,\sigma_1}\wedge e_{\sigma_5,\sigma_6,\sigma_7}\cdot e_{1,2,3,\sigma_2,\sigma_3,\sigma_4}$. In order to show that $\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,3,7})\neq 0$, we want to show that the coefficient of $\xi\circ\Psi^{\vee}_{\mathbb{P}(V_{\lambda_3})}(w_{2,3,7})$ corresponding to the element $e_{1,2,3}\wedge e_{1,2,7}\cdot e_{1,2,3,4,5,6}$ with respect to the basis of Remark A.1 is nonzero. As in the previous proofs, one can show that the coefficients of $\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}e_A$, $\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}e_B$, $\sum_{\sigma\in\mathcal{S}_7}(-1)^{\sigma}e_C$ are respectively $6!$, $2\cdot 3!4!$, $-3!4!$, and the final coefficient is equal to $2^5 3^3\neq 0$. □

Lemma A.9. *The equality* (22) *holds for* $\mu=\lambda_1+\lambda_4+\lambda_7$.

Proof. Consider $V^\vee_{\lambda_1+\lambda_4+\lambda_7} \subset V^\vee_{\lambda_1} \otimes V^\vee_{\lambda_4} \otimes V^\vee_{\lambda_7}$ determined by $e_1 \otimes e_{1,2,3,4} \otimes e_{1,\dots,7}$. Then consider the diagonal maps

$$V^\vee_{\lambda_7} \hookrightarrow V^\vee \otimes V^\vee_{\lambda_3} \otimes V^\vee_{\lambda_3} : \quad e_{1,\dots,7} \longmapsto \sum_{\sigma\in\mathcal{S}_7} (-1)^\sigma e_{\sigma_1} \otimes e_{\sigma_2,\sigma_3,\sigma_4} \otimes e_{\sigma_5,\sigma_6,\sigma_7};$$

$$V^\vee_{\lambda_4} \hookrightarrow V^\vee \otimes V^\vee_{\lambda_3} : \quad e_{1,2,3,4} \longmapsto \sum_{\sigma\in\mathcal{S}_4} (-1)^\sigma e_{\sigma_1} \otimes e_{\sigma_2,\sigma_3,\sigma_4}.$$

Aplying multiplication maps $V^\vee \otimes V^\vee_{\lambda_2} \to V^\vee_{\lambda_3}$ and $(V^\vee_{\lambda_3})^{\otimes 4} \to \bigwedge^4 V^\vee_{\lambda_3}$ we get $V^\vee_{\lambda_1+\lambda_4+\lambda_7} \subset \bigwedge^4 V^\vee_{\lambda_3}$ determined by

$$w_{1,4,7} = \sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_7} (-1)^\sigma (-1)^\tau e_{1,\sigma_1,\tau_1} \wedge e_{\sigma_2,\sigma_3,\sigma_4} \wedge e_{\tau_2,\tau_3,\tau_4} \wedge e_{\tau_5,\tau_6,\tau_7}.$$

Let us show that $\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{1,4,7}) \neq 0$. Firstly notice that $\Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{1,4,7}) =$

$$= \sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_7} (-1)^\sigma (-1)^\tau [e_{1,\sigma_1,\tau_1} \wedge e_{\sigma_2,\sigma_3,\sigma_4} \cdot e_{\tau_2,\tau_3,\tau_4} \wedge e_{\tau_5,\tau_6,\tau_7} - 2e_{1,\sigma_1,\tau_1} \wedge e_{\tau_2,\tau_3,\tau_4} \cdot e_{\sigma_2,\sigma_3,\sigma_4} \wedge e_{\tau_5,\tau_6,\tau_7}]$$

and

$$\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{1,4,7}) = \sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_7} (-1)^\sigma (-1)^\tau [e_A + e_B + e_C],$$

where $e_A = e_{1,\sigma_1,\tau_1} \wedge e_{\sigma_2,\sigma_3,\sigma_4} \cdot e_{\tau_2,\tau_3,\tau_4,\tau_5,\tau_6,\tau_7}$, $e_B = -2e_{1,\sigma_1,\tau_1} \wedge e_{\tau_2,\tau_3,\tau_4} \cdot e_{\sigma_2,\sigma_3,\sigma_4,\tau_5,\tau_6,\tau_7}$, $e_C = -2e_{\sigma_2,\sigma_3,\sigma_4} \wedge e_{\tau_5,\tau_6,\tau_7} \cdot e_{1,\sigma_1,\tau_1,\tau_2,\tau_3,\tau_4}$. In order to show that $\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{1,4,7}) \neq 0$, we want to show that the coefficient of $\xi \circ \Psi^\vee_{\mathbb{P}(V_{\lambda_3})}(w_{1,4,7})$ corresponding to the element $e_{1,2,3} \wedge e_{1,2,4} \cdot e_{1,3,4,5,6,7}$ with respect to the basis of Remark A.1 is nonzero. Similarly to the previous proofs, one can compute that the coefficient of $\sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_7} (-1)^\sigma (-1)^\tau e_A$ is equal to $12 \cdot 6!$, the coefficient of $\sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_7} (-1)^\sigma (-1)^\tau e_B$ is equal to $4 \cdot (3!)^3$ and the coefficient of $\sum_{\sigma\in\mathcal{S}_4} \sum_{\tau\in\mathcal{S}_7} (-1)^\sigma (-1)^\tau e_C$ is equal to $4 \cdot (3!)^2 \cdot 4!$, thus giving a total coefficient of $2^5 3^4 5 \neq 0$. □

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