Robustification of stabilizing controllers for ODE-PDE-ODE systems: a filtering approach

Jean Auriol, Federico Bribiesca Argomedo, Florent Di Meglio

Robustification of stabilizing controllers for ODE-PDE-ODE systems: a filtering approach

Jean Auriol [a] , Federico Bribiesca Argomedo [b] Florent Di Meglio [c]

[a] *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France.*

[b] *Université de Lyon, INSA Lyon, CNRS Ampère, F-69621, Villeurbanne, France*

[c] *Centre Automatique et Systèmes, Mines ParisTech, PSL Research University, Paris, France*

Abstract

In this paper, we present a filtering technique that robustifies stabilizing controllers for systems composed of heterodirectional linear first-order hyperbolic Partial Differential Equations (PDEs) interconnected with Ordinary Differential Equations (ODEs) through their boundaries. The actuation is either available through one of the ODE or at the boundary of the PDE. The proposed framework covers a broad general class of interconnected systems. Assuming that a stabilizing controller is available, we derive simple sufficient conditions under which appropriate low-pass filters can be combined with the control law to robustify the closed-loop system. Our approach is based on a rewriting of the distributed dynamics as a delay-differential algebraic equation and an analysis in the Laplace domain. The proposed technique will simplify the design of stabilizing controllers for the class of systems under consideration (which can now be done only on a case-by-case basis due to the complexity and generality of the underlying interconnections) since it dissociates the stabilization problem from the robustness aspects. Indeed, it becomes possible to use convenient (but non-robust) techniques for the stabilization of such systems (as the cancellation of the boundary coupling terms or the inversion of the ODE dynamics), knowing that the resulting control law can be made robust (to delays and uncertainties) using the proposed filtering methodology.

Key words: Hyperbolic Partial Differential Equations; difference systems; backstepping; interconnection; filtering; robustness.

1 Introduction

This paper deals with the robustification of stabilizing controllers for systems of linear first-order hyperbolic Partial Differential Equations (PDEs) coupled with Ordinary Differential Equations (ODEs) at both boundaries of a one-dimensional spatial domain. Such systems model, e.g., the propagation of torsional waves in drilling systems [2,39].

The control of interconnections involving hyperbolic PDEs has been the focus of a large number of recent publications. In the seminal paper [29], a re-interpretation of the classical Finite Spectrum Assignment [34,31] is proposed, by modeling ODEs with input delays as PDE-ODE interconnections. Subsequently, this result has enabled the design of observers, controllers, parameter estimation methods, etc. for a plethora of interconnected systems: systems with varying delays [12,15], parabolic PDEs coupled with ODEs [40], cascades of PDEs [9], etc.

Here, we focus on interconnections involving ODEs at both boundaries of the spatial domain. We assume that actuation is located either at one boundary of the PDE domain or affects only one of the ODEs, referred to as *proximal*. When the ODE is actuated, a stabilizing observer-controller robust to delays is proposed in the case of a scalar proximal ODE in [24]. In [22], a controller is designed based on assumptions that guarantee the existence of a Byrnes–Isidori normal form for the proximal ODE, as well as a relative degree one condition. These restrictions are partially avoided in [13] where the proximal ODE is simply assumed to be minimum phase for the output that affects the PDE. Numerous contributions deal with the case where the PDE is directly actuated [23,16]. However, a large number of the proposed controllers feature vanishing robustness margins. Indeed, as detailed in [33,20,25], controls laws designed to stabilize hyperbolic systems can result in unstable closed-loop systems in the presence of arbitrarily small delays in the feedback loop or uncertainty in some pa-

Email address: jean.auriol@l2s.centralesupelec.fr (Jean Auriol).

rameters. Conversely, a controller ensuring stability even in the presence of small delays and parameter uncertainties is said to *strongly stabilize* the system [25]. This justifies the necessity for studying delay robustness while designing controllers for hyperbolic systems. In [4,6], modifications to standard backstepping controllers are proposed that guarantee the strong stability of the closed-loop dynamics for a variety of systems. The main idea behind these modifications is to avoid the complete cancellation of the reflections at the boundaries of the spatial domain of the PDE. The gained robustness comes at the price of degraded performance.

Here, rather than proposing a new stabilizing controller, we assume that one has already been designed for the systems of interest. Our main contribution is the simplification of the control design for the class of systems under analysis by decoupling the design of a stabilizing control law from its robustification. This robustification is achieved by the design of appropriate filters, generalizing the approaches in [14,7], that robustify the controller with respect to delays and parameter uncertainty. More precisely, when applied conjointly with the existing controller, they ensure strong stability of the closed-loop dynamics. Our approach is as follows. Using a triangular integral transformation, we rewrite the dynamics as a system of Delay-Differential Algebraic Equations, in the time and Laplace domains. Then, the filters are designed by leveraging the fact that robustness issues appear at high frequencies. We give sufficient conditions that ensure strong stabilization provided the original control laws ensured simple stabilization. This strategy can be considered a generalization of the PI controllers calibrated for drilling applications such as SoftSpeed [32] or Z-torque, which aim at canceling the boundary reflection only on a portion of the frequency domain, as detailed in [1].

2 Notations and preliminary results

In this section we detail the notations used through this paper and recall some important results. We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on $[0,1]$ with the standard L^2 norm, i.e., for any $f \in L^2([0,1],\mathbb{R})$, we have $||f||_{L^2} = \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}}$. Similarly, we denote the Hilbert space $L^2(0,\infty)$ as $L^2(0,\infty) = \{U : [0,\infty) \to \mathbb{C} \mid \int_0^\infty |U(t)|^2 dt < \infty\}$. The L^2-norm, $||U||_2 = \left(\int_0^\infty |U(t)|^2 dt\right)^{1/2}$ often has interpretation as the energy of the signal, and so it is reasonable to consider only inputs U in $L^2(0,\infty)$ and to ask that outputs are in $L^2(0,\infty)$. We denote the state space $\chi = \mathbb{R}^p \times L^2([0,1];\mathbb{R})^{n+m} \times \mathbb{R}^q$, where p,n,m,q are positive integers. For $(X,u,v,Y) \in \chi$, we introduce the corresponding $\chi-$norm $||(X,u,v,Y)||_\chi^2 = ||X_0||_{\mathbb{R}^p}^2 + ||u||_{L^2}^2 + ||v||_{L^2}^2 + ||Y||_{\mathbb{R}^q}^2$, i.e., the norm of each component is the usual Euclidean norm or the L^2-norm. The set $L^\infty([0,1],\mathbb{R})$ denotes the space of bounded real-valued functions defined on $[0,1]$ with the standard L^∞ norm, *i.e.*, for any $f \in L^\infty([0,1],\mathbb{R})$, $||f||_{L^\infty} = \underset{x\in[0,1]}{\text{ess sup}}|f(x)|$. For a function $\phi : [-\tau,\infty) \mapsto \mathbb{R}^m$, we define its partial trajectory $\phi_{[t]}$ by $\phi_{[t]}(\theta) = \phi(t+\theta)$, $-\tau \le \theta \le 0$. This maximum delay $\tau > 0$ will be related to the transport time of the PDE system we consider in this paper. The associated norm is given by $||\phi_{[t]}|| = \left(\int_{-\tau}^0 \phi^T(t+s)\phi(t+s)ds\right)^{\frac{1}{2}}$. The variable s denotes the Laplace variable. The space $\mathbb{C}^+$ corresponds to the complex right half plane: $\mathbb{C}^+ = \{s \in \mathbb{C},\ \text{Re}(s) \ge 0\}$, where Re denotes the real part of a complex number. For any $\eta \in \mathbb{R}$, the space $\mathbb{C}_\eta$ corresponds to $\{s \in \mathbb{C},\ \text{Re}(s) \ge -\eta\}$. Provided it is defined, the Laplace transform of a function $f(t)$ will be denoted $f(s)$ (or, when there is an ambiguity $\hat{f}(s)$). The Laplace transform of a function in $L^2(0,\infty)$ is analytic in the open right half-plane. For all $r \in \mathbb{N}$, we denote Id_r the identity matrix of dimension r (or Id if no confusion arises). We denote $\mathbb{C}^{n_1\times n_2}$ the set of complex matrices with n_1 rows and n_2 columns. For any proper and stable transfer matrix $G(s)$, we denote $\bar{\sigma}(G(s))$ the largest singular value of $G(s)$ ($s \in \mathbb{C}^+$) and $\underline{\sigma}(G(s))$ its lowest singular value. We recall the following results on singular values [27]. For any $(A,B) \in \mathbb{C}^{n\times m}$, we have $\bar{\sigma}(A+B) \le \bar{\sigma}(A) + \bar{\sigma}(B)$. For any $(A,B) \in \mathbb{C}^{n\times n}$, we have $\bar{\sigma}(AB) \le \bar{\sigma}(A)\bar{\sigma}(B)$.

The set $\mathcal{A}(0)$ stands for the convolution Banach algebra of BIBO-stable generalized functions in the sense of [42] (sometimes referred as Wiener algebra). A function $g(\cdot)$ belongs to $\mathcal{A}(0)$ if it can be expressed as $g(t) = g_r(t) + \sum_{i=0}^\infty g_i\delta(t-t_i)$, where $g_r \in L^1(\mathbb{R}^+,\mathbb{R})$, $\sum_{i\ge0}|g_i| < \infty$, $0 = t_0 < t_1 < \dots$ and $\delta(\cdot)$ is the Dirac distribution. If we only have $\mathrm{e}^{-\beta t}g_r \in L^1(\mathbb{R}^+,\mathbb{R})$ (β being a real), then we say that $g \in \mathcal{A}(\beta)$. The associated norm is $||g||_{\mathcal{A}} = ||g_r||_{L^1} + \sum_{i\ge0}|g_i|$. The set $\hat{\mathcal{A}}(0)$ of Laplace transforms of elements in $\mathcal{A}(0)$ is also a Banach algebra with associated norm $||\hat{g}||_{\hat{\mathcal{A}}(0)} = ||g||_{\mathcal{A}(0)}$. An input-output linear system given in the form of a convolution, $y = h \star u$ is BIBO-stable if its kernel h belongs to the class $\mathcal{A}$. The class $\mathcal{A}$ includes finite-dimensional linear systems with rational transfer, time-delay systems, or systems with distributed delays. We define the following subalgebras of $\hat{\mathcal{A}}(0)$: $\hat{\mathcal{A}}_-(0) = \{\hat{f} \mid \hat{f} \in \hat{\mathcal{A}}(\beta),\ \beta < 0\}$ and $\hat{\mathcal{A}}_\infty(0) = \{\hat{f} \mid \hat{f} \in \hat{\mathcal{A}}_-(0),\ \inf_{s\in\bar{\mathbb{C}}^+,|s|\ge\rho}|\hat{f}(s)| > 0 \text{ for some } \rho > 0\}$. Finally, we define the Callier Desoer class of transfer functions $\hat{\mathcal{B}}(0)$ as the quotient algebra $\hat{\mathcal{A}}_-[\hat{\mathcal{A}}_\infty(0)]^{-1}$. The multivariable extensions will be denoted with a $\mathcal{M}$ (e.g. $\mathcal{M}(\hat{\mathcal{B}}(0))$). We now recall several results adjusted from [19].

Definition 1 *If a system maps every input U in $L^2(0,\infty)$ to an output in $L^2(0,\infty)$, and if $\sup_{U\neq0}\frac{||y||_2}{||U||_2} <$*

∞, *the system is* ***stable****. A transfer function* $G(s)$ *is* ***proper*** *if for sufficiently large* ρ $\sup_{Re(s)\geq 0, |s|>\rho} |G(s)| < \infty$. *If the limit of* $G(s)$ *at infinity exists and is 0,* G *is said to be* ***strictly proper****.*

Theorem 2 *A linear system is stable if and only if its transfer function* G *belongs to* $H_\infty = \{G : \mathbb{C}^+ \to \mathbb{C} \mid G \text{ analytic and } \sup_{Re(s)>0} |G(s)| < \infty\}$, *with the norm* $||G||_\infty = \sup_{Re(s)>0} |G(s)|$. *In this case, the function* G *is called a stable transfer function.*

Finally, we recall different robustness concepts

Definition 3 (w-stability [20]) *Consider a plant transfer function* $G \in \mathcal{M}\hat{\mathcal{B}}(0)$ *and a feedback controller* $K \in \mathcal{M}\hat{\mathcal{B}}(0)$. *The closed-loop system is* w-stable *if and only if for any approximate identity* I_δ *(where* $0 \leq \delta < \mu$*), the closed-loop transfer function* $GK\left(I + I_\delta GK\right)^{-1}$ *is stable. An approximate identity is a family of transfer functions* $I_\delta \in \mathcal{M}\hat{\mathcal{A}}_-(0)$ *such that*

(1) $\|I_\delta\|_\infty < 1$, $I_0 = I$;
(2) On every compact set of $\bar{\mathbb{C}}_0^+$, I_δ *converges to* I *when* δ *goes to zero.*

Suppose that (G, K) *is input-output stable. Then (G,K) is* w-stable *if there exists a* $\rho > 0$ *such that*

$$\sup_{\{s\in\ \bar{\mathbb{C}}^+|\ |s|>\rho\}} ||G(s)K(s)|| < 1. \tag{1}$$

Approximate identities may include more general transfer functions than the ones stemming from uncertainties on the delays. Thus, w-stability implies delay-robust stability in the sense of [33]. It also includes robustness w.r.t some uncertainties (but not all, since it has been shown in [5] that uncertainties in the transport velocities cannot be modeled by approximate identities).

Definition 4 *Let* G *and* G_Δ *be transfer matrices in* $\mathcal{M}\hat{\mathcal{B}}(0)$. *A perturbation* $\Delta_a \in \mathcal{M}\hat{\mathcal{B}}(0)$ *is an* admissible multiplicative uncertainty *if* $G_\Delta = (I + \Delta_a)G$, *if* Δ_a *has no poles on the imaginary axis, and if* G *and* G_Δ *have the same number of unstable poles in* $\bar{\mathbb{C}}^+$. *It is an* admissible additive uncertainty *if* $G_\Delta = G + \Delta_a$ *and if either* Δ_a *is a transfer matrix satisfying the previous requirements, which is strictly proper on* $\bar{\mathbb{C}}^+$, *or it is a stable perturbation. If* $G = \tilde{M}^{-1}\tilde{N}$ *is a left-coprime factorization of* G *over* $\mathcal{M}\hat{\mathcal{A}}_-(0)$ *[20], a perturbation* $(\Delta_N, -\Delta_M) \in \mathcal{M}\hat{\mathcal{A}}_-(0)$ *is an admissible left-coprime-factor uncertainty if* $\det(\tilde{M} + \Delta_M) \in \hat{\mathcal{A}}_\infty(0)$ *and* $G_\Delta = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)$.

3 General context

3.1 Systems under consideration

We consider in this paper a $n \times m$ linear heterodirectional hyperbolic system coupled through its boundaries with linear ODEs. This structure can represent, for instance, systems with wave-like propagation between an actuator (with non-negligible dynamics) and a load to be stabilized. It is the case for vertical wells subject to coupled axial-torsional oscillations for which the top-drive and Bottom-Hole Assembly inertia cannot be neglected. More precisely, we consider systems with the following structure

$$\dot{X}_0(t) = A_0X_0(t) + E_0v(t,0) + B_XU(t), \tag{2}$$
$$u(t,0) = C_0X_0(t) + Qv(t,0) + B_uU(t), \tag{3}$$
$$u_t + \Lambda^+u_x = \Sigma^{++}(x)u + \Sigma^{+-}(x)v, \tag{4}$$
$$v_t - \Lambda^-v_x = \Sigma^{-+}(x)u + \Sigma^{--}(x)v, \tag{5}$$
$$v(t,1) = Ru(t,1) + C_1X_1(t), \tag{6}$$
$$\dot{X}_1(t) = A_1X_1(t) + E_1u(t,1), \tag{7}$$

for a.e. $(t,x) \in [0,+\infty) \times [0,1]$. The state of the system is $(X_0(t), u(t,\cdot), v(t,\cdot), X_1(t)) \in \chi$. The initial condition is taken as $((X_0)_0, u_0, v_0, (X_1)_0) \in \chi$ and we consider weak solutions to (2)-(7) [11]. The system is well-posed in the sense of [11, Theorem A.6, page 254]. The control input is denoted $U(t) \in \mathbb{R}^r$. In what follows, we do not consider the case where the PDE and the ODE can be simultaneously actuated, which means that either $B_X \equiv 0$ or $B_u \equiv 0$. The matrices Λ^+ and Λ^- are diagonal and represent the transport velocities. We have $\Lambda^+ = \text{diag}\ (\lambda_i)$ and $\Lambda_p^- = \text{diag}\ (\mu_i)$ and we assume that their coefficients satisfy $-\mu_m \leq \cdots \leq -\mu_1 < 0 < \lambda_1 \leq \cdots \leq \lambda_n$. The spatially-varying coupling matrices $\Sigma^{\cdot\cdot}$ are regular matrices (we assume here that each coefficient of the matrix is a continuous function). The different coupling matrices satisfy $A_0 \in \mathbb{R}^{p\times p}$, $E_0 \in \mathbb{R}^{p\times m}$, $B_X \in \mathbb{R}^{p\times r}$, $B_u \in \mathbb{R}^{n\times r}$, $C_0 \in \mathbb{R}^{n\times p}$, $A_1 \in \mathbb{R}^{q\times q}$, $E_1 \in \mathbb{R}^{q\times n}$, $C_1 \in \mathbb{R}^{m\times q}$, $R \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{n\times m}$. Finally, we denote τ the maximum transport delay for the PDE system $\tau = \max_{i,j}(\frac{1}{\lambda_i} + \frac{1}{\mu_j})$. The class of systems represented by equations (2)-(7) is extremely broad as the proposed representation can be used to model a large variety of interconnected ODE-PDE systems. The system can be actuated through the ODE or the PDE. Multiple particular cases have already been considered in the literature for the design of stabilizing control laws. We give below several examples.

- Example 1: *heterodirectional linear coupled hyperbolic PDEs.* Choosing $A_0 \equiv E_0 \equiv C_0 \equiv B_X \equiv A_1 \equiv E_1 \equiv C_1 \equiv 0$, $B_u = I_n$, $X_0(0) = 0$ and $X_1(0) = 0$, system (2)-(7) can be rewritten as a system of heterodirectional linear coupled hyperbolic PDEs:

$$u_t(t,x) + \Lambda^+u_x(t,x) = \Sigma^{++}(x)u + \Sigma^{+-}(x)v,$$

$$v_t(t,x) - \Lambda^- v_x(t,x) = \Sigma^{-+}(x)u + \Sigma^{--}(x)v,$$
$$v(t,1) = Ru(t,1),\ u(t,0) = Qv(t,0) + U(t).$$

For this class of systems, a stabilizing control law has been proposed in [18]. The proposed controller guarantees finite-time stabilization. However, it requires the cancellation of the reflection term $Qv(t,0)$, which results in a non-strictly-proper controller and raises some robustness issues with respect to small delays in the loop [8]. Modifications have been proposed in [10] (by canceling only a part of the reflection terms) to guarantee the existence of robustness margins.

- Example 2: *ODE-PDE-ODE system with scalar hyperbolic states.* In the case where $n = m = 1$ and $B_u \equiv 0$, stabilizing control laws have been designed in [44,22,14]. However, most of the proposed designs, even though mathematically correct, possess a zero robustness margin. In [14], the proposed control law combines the backstepping approach with a Laplace analysis. Several assumptions were required to guarantee the exponential stability. A low-pass filter has been added to the control law to make it strictly proper, thus guaranteeing the existence of robustness margins.

- Example 3: *underactuated network of two interconnected $n + m$ hyperbolic PDE systems.* In [7], the authors considered a system composed of two independent hyperbolic subsystems coupled through their boundaries. The system can be recast under the form (2)-(7) using a technique referred to as folding (see [7] for details). A robust stabilizing controller has been designed in [7] combining the backstepping approach with a flatness-based feedforward tracking control design. The robustness properties of the closed-loop system are guaranteed by combining the stabilizing control law with a well-tuned low-pass filter.

3.2 Objectives

As it can be seen through the various examples we presented above, multiple contributions in the literature have already considered the problem of stabilizing particular systems whose structure belongs to the class described by equations (2)-(7). However, the controllability analysis of (2)-(7) and the design of the corresponding generic stabilizing controllers is a challenging open question. Contributions in the literature mainly focus on specific configurations of networks for now. Again, we would like to emphasize that the representation (2)-(7) is a general representation that may not always be the most suitable for all systems, since it could hide specific network/cascade structures of the system that could be leveraged in the control design. For instance, recent contributions have introduced a recursive dynamics framework that explicitly uses the cascade structure of the network to recursively design a stabilizing controller [38].

To design stabilizing control laws for the class of system (2)-(7), it may be convenient either to cancel the reflection terms in the PDE or to inverse the ODE dynamics, using high order derivatives. Although such approaches considerably simplify the design, the resulting control laws are not *strictly proper*, which may result in vanishing robustness margins [8] (which is a major difference with finite-dimensional linear time invariant systems). Several alternative solutions have been proposed in the literature to overcome this issue. A first solution was to cancel only a part of the reflection terms in the PDE, using a convolutional procedure as performed, e.g., in [6,45,35]. Although somehow standard, this approach presents the drawback of not distinguishing the effects of high and low frequencies in terms of stability and robustness. In addition, such an approach can be challenging to implement when considering chains with a large number of subsystems. Recently, an alternative approach has been proposed in [14,7]. It combines the proposed non-proper control law with a well-tuned low-pass filter. The resulting control law then becomes strictly proper, which guarantees the existence of robustness margins. However, such a filtering approach has only been performed on a case-by-case basis for now.

In this paper, we give standard conditions under which it becomes possible to low-pass filter a control law that already stabilizes (2)-(7). This will simplify the design of future control laws for the considered class of *underactuated* systems. The proposed techniques will be applied in Section 6 on two different test cases.

4 Time-delay formulation

To analyse the properties of the controller combined with the low-pass filter in the frequency domain, we rewrite our interconnected ODE-PDE-ODE as a time-delay system, adjusting the backstepping method given in [10]. In what follows, we do not consider diagonal coupling terms in the PDEs (i.e. $\Sigma_{ii}^{++} = \Sigma_{ii}^{--} = 0$). This is not restrictive since these coefficients can be transferred to the anti-diagonal terms via a change of variables as in [41]. Using several backstepping transformations, we will move the local in-domain coupling terms $\Sigma^{\cdot\cdot}$ at the actuated boundary (where they will take the form of integral couplings). This will allow us to adjust the approach presented in [10] and to simplify the stability analysis.

4.1 Integral transformations

Inspired by [18], we first combine two integral transformations to move the local coupling terms $\Sigma^{\cdot\cdot}$ to the boundary (in the form of integral terms). Due to these transformations, non-local coupling terms and ODE terms may appear in the system. Consider the following Volterra transformation, similar to the one introduced

in [18,28,10]

$$u(t,x) = \alpha(t,x) - \int_x^1 L^{\alpha\alpha}(x,y)\alpha(t,y)dy - \int_x^1 L^{\alpha\beta}(x,y)\beta(t,y)dy + \gamma_\alpha(x)X_1(t), \quad (8)$$

$$v(t,x) = \beta(t,x) - \int_x^1 L^{\beta\alpha}(x,y)\alpha(t,y)dy, - \int_x^1 L^{\beta\beta}(x,y)\beta(t,y)dy + \gamma_\beta(x)X_1(t), \quad (9)$$

where the kernels $L^{\cdot\cdot}$ are bounded functions defined on $\mathfrak{T}_u = \{(x,y) \in [0,1]^2,\ x \leq y\}$, while the kernels γ_α and γ_β are bounded functions defined on $[0,1]$. They satisfy the following set of equations on their domains of definition

$$\Lambda L_x + L_y\Lambda = \Sigma(x)L, \quad \Lambda\gamma_x(x) = \Sigma(x)\gamma - \gamma^\top \bar{A}_1 \quad (10)$$

with the boundary conditions

$$\Lambda L(x,x) - L(x,x)\Lambda = \Sigma(x) \quad (11)$$

$$\gamma_\alpha(1) = D, \quad \gamma_\beta(1) = RD + C_1, \quad (12)$$

where $\Lambda = \text{diag}(\Lambda^+, -\Lambda-)$, $\Sigma = \begin{pmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{pmatrix}$ and $L = \begin{pmatrix} L^{\alpha\alpha} & L^{\alpha\beta} \\ L^{\beta\alpha} & L^{\beta\beta} \end{pmatrix}$, $\gamma = (\gamma_\alpha, \gamma_\beta)$, D is an arbitrary matrix and where the matrix $\bar{A}_1$ is defined by $\bar{A}_1 = A_1 + E_1 D$ (if the pair (A_1, E_1) is stabilizable, the matrix D is such that $\bar{A}_1$ is Hurwitz). To this set of equations, we add arbitrary continuous values for $L^{\alpha\alpha}_{ij}(x,1)$ and $L^{\beta\beta}_{ij}(x,1)$, for $i > j$ (or on the $(0,y)$-boundary for $j \geq i$). The set of PDEs and ODEs (10)-(12) has a unique solution which is piecewise continuous. This follows with minor adaptions, from the results presented in [23]. The transformation (8)-(9) maps the original system (2)-(7) to a target system that we do not detail here for lack of space. We now consider a second transformation whose objective is to obtain the framework of [10]. Let us consider the transformations defined by

$$\alpha(t,x) = \check{\alpha}(t,x) - \int_x^1 \check{L}(x,y)\check{\alpha}(t,y)dy \quad (13)$$

$$\check{\alpha}(t,x) = \bar{\alpha}(t,x) - \int_0^1 \bar{L}(x,y)\bar{\alpha}(t,y)dy, \quad (14)$$

The kernel $\check{L}$ is a lower triangular matrix (i.e. $(\check{L})_{ij} = 0$ if $i < j$) whose components are bounded functions. The kernel $\bar{L}$ is a strictly upper-triangular matrix (i.e. $\bar{L}_{ij} = 0$ if $i \geq j$) whose components are bounded. The invertibility of the transformation is a consequence of its triangular structure. The kernel $\check{L}$ satisfies the following set of equations if $i \geq j$

$$\Lambda^+\check{L}_x(x,y) + \check{L}_y(x,y)\Lambda^+ = 0, \quad (15)$$

$$\Lambda^+\check{L}(x,x) - \check{L}(x,x)\Lambda^+ = 0, \quad (16)$$

$$(\check{L}(x,1))_{ij} = (G_1(x)(\Lambda^+)^{-1})_{ij} + \int_x^1 \sum_{k=1}^n \check{L}_{ik}(x,y)\check{G}_{kj}(y)\frac{1}{\lambda_j}dy, \quad (17)$$

where the matrix $\check{G}(x)$ is strictly upper-triangular (i.e. $\check{G}_{i,j}(x) = 0$ if $i \geq j$) and satisfies for all $x \in [0,1]$

$$(\check{G}(x))_{ij} = (G_1(x))_{ij} + \int_x^1 \sum_{k=1}^n \check{L}_{ik}(x,y)\check{G}_{kj}(y)dy \quad \text{if } i < j. \quad (18)$$

and where the functions G_1 and G_2 satisfy

$$G_1(x) = \int_x^1 L^{\alpha\alpha}(x,y)G_1(y) + L^{\alpha\beta}(x,y)G_2(y)dy - L^{\alpha\alpha}(x,1)\Lambda^+ + L^{\alpha\beta}(x,1)\Lambda^- R - \gamma_\alpha(x)E_1, \quad (19)$$

$$G_2(x) = \int_x^1 L^{\beta\alpha}(x,y)G_1(y) + L^{\beta\beta}(x,y)G_2(y)dy - L^{\beta\alpha}(x,1)\Lambda^+ + L^{\beta\beta}\Lambda^- R - \gamma_\beta(x)E_1. \quad (20)$$

One can verify that the kernel equations (15)-(18) are well-defined due to the triangular structure of the different matrices. Indeed, for $j = 1$ equation (17) can be rewritten as $(\check{L}(x,1))_{i1} = (G_1(x)(\Lambda^+)^{-1})_{i1}$. Combining this boundary condition with equation (16), we can solve equation (15) to compute $\check{L}_{i1}$ on its domain of definition. For $j = 2$, equation (18) can be rewritten as $(\check{G}(x))_{12} = (G_1(x))_{12} + \int_x^1 \check{L}_{11}(x,y)\check{G}_{12}(y)dy$, which is a Volterra equation that can be solved to obtain $\check{G}_{12}(y)$ [46]. This in turns gives the kernels $\check{L}_{i2}$ using (17). Iterating the process allows us to compute the kernel matrix $\check{L}$ and the function $\check{G}$. The kernel $\bar{L}$ satisfies the following set of equations

$$\Lambda^+\bar{L}_x(x,y) + \bar{L}_y(x,y)\Lambda^+ = 0,\ \bar{L}(1,y) = 0, \quad (21)$$

$$(\bar{L}(x,1))_{ij} = (\check{G}(x)(\Lambda^+)^{-1})_{ij} \text{ if } i < j, \quad (22)$$

Due to its triangular structure, we can obtain a direct expression of $\bar{L}$ (and consequently show its existence) using the method of characteristics. We define the function $G_3(x)$ as $G_3(x) = \bar{L}(x,0)\Lambda^+ + \int_0^1 \bar{L}(x,y)G_3(y)dy$. This matrix is upper-triangular. The transformations (13)-(14) map the system to the target system

$$\dot{X}_0 = A_0X_0 + E_0\beta(t,0) + E_0\gamma_\beta(0)X_1(t) + B_XU - E_0(\int_0^1 L(y)\bar{\alpha}(t,y) + L^{\beta\beta}(0,y)\beta(t,y)dy), \quad (23)$$

$$\bar{\alpha}(t,0) = C_0X_0(t) + Q\beta(t,0) + Q\gamma_\beta(0)X_1(t) + B_uU, - \gamma_\alpha(0)X_1 + \int_0^1 L_1(y)\bar{\alpha}(t,y) + L^{\alpha\beta}(0,y)\beta(t,y)dy$$

$$- Q(\int_0^1 L(y)\bar{\alpha}(t,y) + L^{\beta\beta}(0,y)\beta(t,y)dy), \quad (24)$$

$$\bar{\alpha}_t(t,x) + \Lambda^+\bar{\alpha}_x(t,x) = G_3(x)\bar{\alpha}(t,0), \quad (25)$$

$$\beta_t(t,x) - \Lambda^-\beta_x(t,x) = G_2(x)\bar{\alpha}(t,1), \quad (26)$$

$$\beta(t,1) = R\bar{\alpha}(t,1), \quad \dot{X}_1 = \bar{A}_1 X_1 + E_1\bar{\alpha}(t,1), \quad (27)$$

where

$$L(y) = L^{\beta\alpha}(0,y) - \int_0^1 L^{\beta\alpha}(0,\nu)\bar{L}(\nu,y)d\nu - \int_0^y L^{\beta\alpha}(0,\nu)\check{L}(\nu,y)d\nu + \int_0^1\int_0^\eta L^{\beta\alpha}(0,\nu)\check{L}(\nu,\eta)\bar{L}(\eta,y)d\nu d\eta, \quad (28)$$

$$L_1(y) = L^{\alpha\alpha}(0,y) - \int_0^1 L^{\alpha\alpha}(0,\nu)\bar{L}(\nu,y)d\nu - \int_0^y L^{\alpha\alpha}(0,\nu)\check{L}(\nu,y)d\nu + \int_0^1\int_0^\eta L^{\alpha\alpha}(0,\nu)\check{L}(\nu,\eta)\bar{L}(\eta,y)d\nu d\eta. \quad (29)$$

The initial condition of the system (23)-(27) is denoted $(X_0, \bar{\alpha}_0, \beta_0, X_1) \in \chi$. It is obtained by applying the different inverse backstepping transformations on the initial condition (X_0, u_0, v_0, X_1) of the system (2)-(7). The original system (2)-(7) and the target system (23)-(27) have equivalent stability properties. In particular, we have the following lemma , whose proof is a simple consequence of the invertibility of the backstepping transformations and of the boundedness of the kernels [30].

Lemma 5 *Consider the system* (23)-(27) *with the initial condition* $((X_0)_0, \bar{\alpha}_0, \beta_0, (X_1)_0) \in \chi$ *and any arbitrary admissible feedback control law. Then, if the state* $(X_0, \bar{\alpha}, \beta, X_1)$ *exponentially converges to zero in the sense of the* χ*-norm, so does the state* (X_0, u, v, X_1)*, solution of* (2)-(7) *with corresponding initial condition* $((X_0)_0, u_0, v_0, (X_1)_0)$*.*

This lemma implies that the system (23)-(27) can be used for the stability analysis and the control design. Although the target system (23)-(27) may appear more complex than the original system (2)-(7) it presents the advantage that the right part of the PDE equations (25)-(26) does not contain any local in-domain coupling terms but only terms that depend on $\bar{\alpha}(t,1)$ and $\bar{\alpha}(t,0)$. This is essential to express the system in a time-delay form.

4.2 Target system in delay form

Using the method of characteristics, it is possible to express $\bar{\alpha}(t,0)$ as the solution of a neutral equation with distributed delays [10]. Following the steps proposed in [10], and denoting $z(t) = \bar{\alpha}(t,0)$, we can express $\bar{\alpha}(t,x)$ and $\beta(t,x)$ as functions of (delayed values of) $z(t)$. More precisely, we have the following lemma

Lemma 6 *There exist an integer* $N > 0$*, positive delays* $\tau_i \leq \tau$*, constant matrices* M_i, M_i^0 *and* M_i^1 *and* $L^\infty([0,\tau])$ *functions* H_1, H_2, H_3 *such that for all* $t > \tau$*, we have*

$$z(t) = \sum_{i=1}^N M_i z(t-\tau_i) + \int_0^\tau H_1(\nu)z(t-\nu)d\nu + C_0X_0(t) + (Q\gamma_\beta(0) - \gamma_\alpha(0))X_1(t) + B_uU(t), \quad (30)$$

$$\dot{X}_1(t) = \bar{A}_1X_1(t) + \sum_{i=1}^n M_i^1 z(t-\frac{1}{\lambda_i}) + \int_0^\tau H_2(\nu)z(t-\nu)d\nu, \quad (31)$$

$$\dot{X}_0(t) = A_0X_0 + \sum_{i=1}^N M_i^0 z(t-\tau_i) + \int_0^\tau H_3(\nu)z(t-\nu)d\nu + E_0\gamma_\beta(0)X_1(t) + B_XU(t). \quad (32)$$

There exists a constant κ_1 *such that for any linear bounded state-feedback law* $U(t)$ *(function of* $(X_0, \bar{\alpha}, \beta, X_1)$*), for any* $t > \tau$

$$||(X_0, \bar{\alpha}, \beta, X_1)||_\chi \leq \kappa_1||(z, X_1, X_0)|| \quad (33)$$

PROOF. The proof, although technical is similar to the one given in [10] and relies on the method of characteristics. Note that any feedback law function of $(X_0, \bar{\alpha}, \beta, X_1)$ (or equivalently of (X_0, u, v, X_1)) can be expressed as a function of (z, X_1, X_0). □

The different matrices M_i and H_i can be explicitly computed following the methodology of [10]. They only depend on the kernels of the backstepping transformations and on the parameters on the system. In what follows, we will assume that the delays τ_i are **rationally independent**. Indeed, as shown in [26], extending the variable z, it is always possible to rewrite the system in a situation where the delays are rationally independent. System (30)-(32) can be seen as a **comparison system** for the PDE system (23)-(27) (see, e.g., [37] and the references therein for some discussions for delay systems). The exponential stability of the state (z, X_1, X_0) implies the one of the state $(X_0, \bar{\alpha}, \beta, X_1)$ (since the initial condition of z can be expressed as a function of the initial condition of $(\bar{\alpha}, \beta)$). Note that any feedback law expressed in terms of (X_0, z, X_1) can, in the end, be expressed as a function of (X_0, u, v, X_1) using the different backstepping transformations.

4.3 Equation in the Laplace domain

The stability analysis will be done in the frequency domain using the Laplace transform. Without any loss of generality for the asymptotic stability analysis of the

plant, we assume all-zero initial conditions. The Laplace transform applied to (30)-(32) leads to

$$z(s) = F(s)z(s) + P_{11}(s)z(s) + P_{12}(s)X_0(s) + P_{13}(s)X_1(s) + B_uU(s), \quad (34)$$

$$(s\mathrm{Id} - A_0)X_0(s) = P_{21}(s)z(s) + P_{23}(s)X_1(s) + B_XU(s), \quad (35)$$

$$(s\mathrm{Id} - \bar{A}_1)X_1(s) = P_{31}(s)z(s), \quad (36)$$

where

$$F(s) = \sum_{i=1}^{N} M_i \mathrm{e}^{-\tau_i s},\ P_{11}(s) = \int_0^\tau H_1(\nu)\mathrm{e}^{-\nu s}d\nu, \quad (37)$$

$$P_{21}(s) = \sum_{i=1}^{N} M_i^0 \mathrm{e}^{-\tau_i s} + \int_0^\tau H_3(\nu)\mathrm{e}^{-\nu s}d\nu. \quad (38)$$

$$P_{31}(s) = \sum_{i=1}^{n} M_i^1 \mathrm{e}^{-\frac{s}{\chi_i}} + \int_0^\tau H_2(\nu)\mathrm{e}^{-\nu s}d\nu, \quad (39)$$

$$P_{12}(s) = C_0,\ P_{13}(s) = Q\gamma_\beta(0) - \gamma_\alpha(0), \quad (40)$$

$$P_{23}(s) = E_0\gamma_\beta(0). \quad (41)$$

We use general functions F, P_{ij} in the expression of (34)-(35) to state our results in the most general framework.

4.4 A remark on robustness

Equation (34) corresponds to a difference equation whose principal part $\mathrm{Id} - F(s)$ may have an infinite number of zeros in the right-half plane [26]. It has been shown in [33] that having an open-loop transfer function with a non finite number of poles in the right-half plane (RHP) makes delay-robust stabilization impossible. Having a principal part with an infinite number of poles in the RHP would imply having an infinite number of poles in the RHP for the whole open-loop system as the integral term is strictly proper. We can avoid such a case with the following assumption.

Assumption 7 *We have*

$$\sup_{\theta_p \in [0,2\pi]^N} Sp\,(\sum_{k=1}^{N} M_k \exp(j\theta_k)) < 1, \quad (42)$$

where Sp denotes the spectral radius and j is the imaginary unit. Then, there exists $\eta_0 > 0$ such that $Id - F(s)$ does not vanish on $\mathbb{C}_{\eta_0}$.

Assumption 7 is slightly stronger than a necessary condition for delay-robust stabilization and guarantees exponential stability of the characteristic equation $\mathrm{Id} - F(s) = 0$ (and prevents having an infinite number of poles on $\mathbb{C}^+$). Condition (42) is simplified if the delays are rationally dependent. Furthermore, since the spectral radius of a matrix is upper-bounded by any norm of the matrix, easy-to-compute sufficient conditions for this spectral radius condition to hold can be derived using different norms of the matrices involved at the cost of increased conservatism [17].

5 Filtering of the control law

Consider a dynamical system expressed in the Laplace domain by the set of equations (34)-(36). Let us denote $\bar{\ell}_0$ (resp. $\bar{\ell}_1$) the largest eigenvalue (in modulus) of $(sI - A_0)$ (resp. $(sI - A_0)$) Let us define $\mathbb{C}_u = \{s \in \mathbb{C}^+,\ |s| > \max(\bar{\ell}_0, \bar{\ell}_1)\}$, so that the functions $(sI - A_0)^{-1}$ and $(sI - A_1)^{-1}$ are properly defined on $\mathbb{C}_u$. Consider a feedback law $U(s)$ of the form

$$U(s) = K_z(s)z + K_{X_0}(s)X_0 + K_{X_1}(s)X_1, \quad (43)$$

where the operators K_z, K_{X_0} and K_{X_1} are holomorphic functions that will satisfy some conditions given later. Most (if not all) the feedback laws that have been designed in the literature [18,23,8,14] to stabilize systems of the form (2)-(7) have this form in the Laplace domain. However, a non strictly-proper control may not guarantee the existence of robustness margins [8]. This is why we will give in this section general conditions under which it becomes possible to low-pass filter the control law to obtain a strictly proper control law, thus allowing the existence of robustness margins.

The closed-loop characteristic equation associated to (34)-(36) with the feedback law (43) reads as

$$\det(Q(s)) = \det(I_1(s) - P_0(s) - BK(s)) = 0$$

where the matrix $I_1(s)$ is defined by $I_1(s) = \mathrm{diag}\ (\mathrm{Id} - F(s), s\mathrm{Id} - A_0, s\mathrm{Id} - \bar{A}_1)$, while the matrix $P_0(s)$ is defined by $P_0(s) = P_{ij}(s)$ (with $P_{22}(s) = P_{32}(s) = P_{33}(s) = 0$). The matrices B and $K(s)$ are defined by $B = \begin{pmatrix} B_u & B_X & 0 \end{pmatrix}^T$, $K(s) = \begin{pmatrix} K_z(s) & K_{X_0}(s) & K_{X_1}(s) \end{pmatrix}$. We consider that the proposed feedback law has been designed to exponentially stabilize the system, i.e., we make the following assumption

Assumption 8 *The system* (34)-(35) *with the feedback law* (43) *is exponentially stable. Then, there exist $\eta_1 > 0$ and $\epsilon > 0$ such that $|\det(Q(s))| > \epsilon$ for all $s \in \mathbb{C}_{\eta_1}$. Moreover, we assume that the explosion rate of the function $K(s)$ is at worst polynomial.*

The condition for exponentially stability should actually read $|\det(Q(s))| > 0$ for all $s \in \mathbb{C}_{\eta_1}$ [20]. However, simple computations show that the function $|\det(Q(s))|$ is lower-bounded $|s|$ tends to infinity. Thus, this implies the proposed characterization.

In what follows, we denote $\eta = \min\{\eta_0, \eta_1\}$ (where η_0 is defined in Assumption 7). The problem of designing

a stabilizing control for the system (2)-(7) has not been solved in the general case (in particular when $B_u \neq 0$). In most of the cases considered in the literature (see the examples given above in Section 3), it appears convenient for the control design to cancel some of the PDE boundary reflection terms or to invert a part of the ODE dynamics. However, this may result in a non-strictly proper control law, which implies robustness issues [8]. Here, we give general conditions under which it is possible to low-pass filter the (potentially non-strictly proper) control law to make it strictly proper, thus preserving suitable robustness properties while keeping the controller design simple. Indeed, the proposed approach allows separating the stabilization problem from the robustness problem.

5.1 Low-pass filter design

We now give general results that guarantee the possibility to low-pass filter the control $U(s)$ and make it strictly proper while stabilizing the system (34)-(35). We distinguish the two cases $B_X \equiv 0$ (only the PDE is actuated) and $B_u \equiv 0$ (only the ODE is actuated) as the requirements slightly differ depending on the considered case. We first rewrite Assumption 7 in a more amenable form.

Assumption 9 *There exists $0 < \epsilon_0 < 1$ such that $\bar{\sigma}(F(s)) < \epsilon_0 < 1$ on $\mathbb{C}_{\eta_1}$.*

We start with the case where only the PDE is actuated (i.e. $B_X \equiv 0$).

Theorem 10 *Consider the system (34)-(35) with the stabilizing feedback control law $U(s)$ given by (43). Consider that Assumptions 8, and 9 are satisfied. Assume that $B_X \equiv 0$ and that*

(1) The controller gains are such that the functions $B_u K_{X_1}(s)(sId - \bar{A}_1)^{-1}P_{31}(s)$, $B_u K_{X_0}(s)\ (sId - A_0)^{-1}P_{21}(s)$, and $B_u K_{X_0}(s)(sId - A_0)^{-1}P_{23}(s)(sId - \bar{A}_1)^{-1}$ are strictly proper
(2) The function $K_z(s)$ is defined by $K_z(s) = K_z^p(s) + K_z^u(s)$, where $K_z^p(s)$ is strictly proper and K_z^u satisfies $K_z^u(s) = -H_0(s)F(s)$, such that $B_u H_0(s)$ is similar to a diagonal matrix whose components belong to [0,1].

For any $M > 0$ and any integer $N > 0$, define $w(s)$ a low-pass filter (i.e. $w(s) \to 1$ as $|s| \to 0$ and $|w(s)| \to 0$ as $|s| \to +\infty$) with relative degree N, such that we have on $\mathbb{C}_{\eta_1}$, $|1 - w(s)| < 1, |w(s)| < 1$ and the following additional condition if $|s| \leq M$:

$$|1 - w(s)| < \underline{\sigma}(Q(s))(\bar{\sigma}(\begin{pmatrix} B_u & 0 & 0 \end{pmatrix}^T K(s)) + 1)^{-1}. \tag{44}$$

Then, there exists $M > 0$ and $N > 0$ such that the filtered control law $w(s)U(s)$ stabilizes the system (34)-(35) with $w(s)K(s)$ being strictly proper.

The proof of this theorem is given in Appendix. It provides a suitable value for the constant M (that depends on N). The constant N is chosen such that $w(s)K(s)$ is strictly proper (which is always possible as the explosion rate of the function $K(s)$ is at worst polynomial).

Although the different conditions given in the statement of Theorem 10 may appear complex at first sight, they are simple to verify. The first condition is always satisfied for bounded feedback gains. In particular, this condition is satisfied for all the cases that have been currently considered in the literature (see [7] for instance, where $X_0 \equiv 0$). The second condition (on the feedback gain K_z^u) is always satisfied when we cancel all the reflection terms at the actuated boundary of the PDE (as it is done in [18]). The requirements on the function K_z^u could be lowered by considering a matrix filter instead of a scalar one. This would allow canceling the reflection terms arbitrarily. However, in terms of stabilization objective, it appears relevant to cancel as many reflection terms as possible for each line where a control input is available. In pathological cases (for instance, a case for which the same control input acts on two boundary conditions), it may be necessary to perform a change of variables to write the system in a more amenable form. All in all, we believe that condition (2) is not restrictive in its present form. The following lemma gives a constructive way to design the filter w.

Lemma 11 *There exists $\nu_0 > 0$ such that the low-pass filter defined for all $s \in \mathbb{C}^+$ by $w_{\nu_0}(s) = \frac{1}{(1+\nu_0 s)^N}$ satisfies the requirements of Theorem 10.*

PROOF. We immediately have $|w_{\nu_0}| < 1$ and $|1 - w_{\nu_0}| < 1$. The set $\mathcal{S} = \{s \in \mathbb{C}^+,\ |s| \leq M\}$ is compact. Thus, we can define $\tilde{M} = \inf_{s\in\mathcal{S}} \underline{\sigma}(Q(s))(\bar{\sigma}(\begin{pmatrix} B_u & 0 & 0 \end{pmatrix}^T K(s)) + 1)^{-1} > 0$. Choosing $\nu_0 = \tilde{M}^{\frac{1}{N}}(2M)^{-1}$, we directly obtain $|1 - w_{\nu_0}(s)| < \tilde{M}$, which implies that condition (44) is always satisfied. The order of the filter can be chosen arbitrarily high to make the filtered control law strictly proper. □

Note that such a filter only has an illustrative purpose and may not be the most relevant choice. Butterworth filters may have more amenable properties while satisfying the required conditions. In the second theorem, we consider the case where only the ODE is actuated (i.e. $B_z \equiv 0$). To simplify the analysis, we assume that $K(s) = K^u(s) + K^b(s)$, where K^b somehow corresponds to the bounded parts of K while the functions K^u may not be proper (as it is the case in [14], when using a dynamical inversion of the ODE). We have the following result.

Theorem 12 *Consider the system (34)-(35) with the stabilizing feedback control law $U(s)$ given by (43). Consider that Assumptions 8 and 9 are verified. Assume that*

$B_z \equiv 0$. *Assume that there exists a matrix* $\bar{A}_0$ *such that* $(sId - \bar{A}_0)^{-1}$ *is properly defined on* $\mathbb{C}_u$. *Consider that*

(1) The function $K^u_{X_0}(s)$ *satisfies* $K^u_{X_0}(s) = \bar{K}^u(s)C_0$. *Moreover, the functions* $C_0(sId - \bar{A}_0)^{-1}B_X\bar{K}^u(s)$, $C_0(sId - \bar{A}_0)^{-1}B_XK^u_z(s)$, *and* $C_0(sId - \bar{A}_0)^{-1}B_XK^u_{X_1}(s)P_{31}(s)$ *are strictly proper.*

(2) The function $K^b_{X_0}(s)$ *satisfies* $B_XK^b_{X_0}(s) = \bar{K}^b(s)C_0 + \bar{A}_0 - A_0$. *Moreover, the functions* $(sId - \bar{A}_0)^{-1}\ B_XK^b_z(s)$, $(sId - \bar{A}_0)^{-1}B_XK^b_{X_1}(s)(sId - \bar{A}_1)^{-1}\ P_{31}(s)$, *and* $(sId - \bar{A}_0)^{-1}B_X\bar{K}^b_{X_0}$ *are strictly proper.*

For any $M_0 > 0$ *and any* $N_0 > 0$ *define* $w_0(s)$ *a low-pass filter with sufficiently high relative degree* N_0, *that satisfies for all* $s \in \mathbb{C}_{\eta_1}$ $|w_0(s)| < 1$, $|1 - w_0(s)| < 1$ *and the additional condition*

$$|1 - w_0| < \underline{\sigma}(Q(s))(\bar{\sigma}(\begin{pmatrix}0 & B_X & 0\end{pmatrix}^T K^u(s)) + 1)^{-1} \quad (45)$$

if $|s| \leq M_0$. *For any* $M_1 > 0$ *define* $w_1(s)$ *a low-pass filter with sufficiently high relative degree* N_1, *that satisfies for all* $s \in \mathbb{C}_{\eta_1}$ $|w_1(s)| < 1$, $|1 - w_1(s)| < 1$ *and the additional condition when* $|s| \leq M_1$

$$|1 - w_1| < \underline{\sigma}(\bar{Q}(s))(\bar{\sigma}(\begin{pmatrix}0 & B_X & 0\end{pmatrix}^T K^b(s)) + 1)^{-1} \quad (46)$$

where $\bar{Q}(s) = I_1 - P_0 - \begin{pmatrix}0 & B_X & 0\end{pmatrix}^T (w_0K^u + K^b)$. *Then, there exists* M_0, M_1, N_0, N_1 *such that the filtered control law defined by*

$$\begin{aligned} U_f(s) &= w_1(s)K^b_z(s)z(s) + w_1(s)K^b_{X_1}(s)X_1(s) \\ &+ w_1(s)K^b_{X_0}(s)X_0(s) + w_0(s)K^u_z(s)z(s) \\ &+ w_0(s)K^u_{X_0}(s)X_0(s) + w_0(s)K^u_{X_1}(s)X_1(s), \end{aligned} \quad (47)$$

is strictly proper and stabilizes the system (34)-(35).

The proof of this theorem is given in Appendix. Condition (1) is inspired by [14] and naturally appears when dealing with dynamical inversions of ODE dynamics. The functions K^u correspond to the unbounded parts of the control input and must grow slower at high frequencies than the ODE part of the dynamics $(s\mathrm{Id} - \bar{A}_0)$. Condition (2) is always satisfied in that case. Note that two filters are required (one for the unbounded part and one for the bounded part of the control input). This two-steps procedure is needed since the filter w_1 corresponding to the (bounded) ODE state-feedback needs to be *fast enough* with respect to the resulting closed-loop PDE-distal ODE subsystem (the dynamic of which depends already on the filter w_0).

5.2 Robustness properties

We now show that having a strictly proper control operator (which can be obtained using adequate low-pass filters) leads to the existence of robustness margins. In what follows, we denote $G(s)$ the input-output (the output being here the state) transfer functions associated to (34)-(36) and $K(s)$ the controller transfer matrix, we have that (G, K) is input-output stable, and that GK is strictly proper (since G is bounded).

Theorem 13 *Consider system* (34)-(35) *with the stabilizing feedback control law* (43) *(i.e. Assumption 8 is verified). Assume that the functions* $K_z(s)$, $K_{X_0}(s)$ *and* $K_{X_1}(s)$ *are* strictly proper. *Then, the closed-loop system is* $w-$*stable. Moreover,* K *stabilizes* $G + \Delta$ *for any admissible additive perturbation that verifies on* $\bar{\mathbb{C}}^+$

$$||\Delta(s)|| < (||K(s)(Id - G(s)K(s))^{-1}||)^{-1}. \quad (48)$$

Then, K *stabilizes* $(Id + \Delta)G$ *for any admissible multiplicative perturbation that verifies on* $\bar{\mathbb{C}}^+$

$$||\Delta(s)|| < (||G(s)K(s)(Id - G(s)K(s))^{-1}||)^{-1}. \quad (49)$$

Finally, if $G = \tilde{M}^{-1}\tilde{N}$ *is a left-coprime factorization of* G *over* $\mathcal{M}\hat{\mathcal{A}}_-(0)$, *then* K *stabilizes* $(\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)$ *for any left-coprime-factor perturbation* $\Delta = (\Delta_N, \ -\Delta_M)$ *that verifies on* $\bar{\mathbb{C}}^+$, $||\Delta_M(s)|| < ||\tilde{M}(s)||$ *and*

$$||\Delta|| < ||\begin{pmatrix} K(s)(Id - G(s)K(s))^{-1}\tilde{M}^{-1} \\ (Id - G(s)K(s))^{-1}\tilde{M}^{-1} \end{pmatrix}||^{-1} \quad (50)$$

Note that the right hand sides of equations (48) (49) and (50) are well defined since GK is continuous and strictly proper, since (G, K) is exponentially stable, and since $\det(\tilde{M}) \in \hat{\mathcal{A}}_\infty(0)$ (left-coprime factorization).

PROOF. The proof is adjusted from [20, Theorem 9.2.6]. The w-stability is a consequence of Definition 3 as equation (1) is verified since GK is strictly proper). Consider now an admissible additive perturbation that verifies (48). Denote $G_\Delta = G + \Delta$. We have $\det(\mathrm{Id} - G_\Delta K) = \det(\mathrm{Id} - GK)\det(\mathrm{Id} - \Delta(K(\mathrm{Id} - GK)^{-1}))$. Denote p_K (resp. p_G) the number of poles of K (resp. G) counted according to their McMillan degree. Since K stabilizes G, $\det(\mathrm{Id} - GK)$ has a well defined Nyquist index (ind) equal to $-p_K - p_G$ [20, Th. 9.1.8]. Consider the function $f(s) = \det(\mathrm{Id} - G_\Delta K)(s)$. We have $\mathrm{ind}(f) = \mathrm{ind}(\mathrm{Id} - GK) + \mathrm{ind}(\mathrm{Id} - \Delta(K(\mathrm{Id} - GK)^{-1}))$. The function $g_1(s) = \det(\mathrm{Id} - \Delta(K(\mathrm{Id} - GK)^{-1})$ has a well defined nonzero limit at infinity in $\bar{\mathbb{C}}^+$. Since K is a stabilizing controller for G, the function g_1 is meromorphic on some open set containing $\bar{\mathbb{C}}^+$. So g_1 has a well defined Nyquist index. Since Δ is an admissible perturbation that verifies (48), we have that $\sup_{\omega\in\mathbb{R}} ||\Delta(j\omega)(K(j\omega)(\mathrm{Id} - G(j\omega)K(j\omega))^{-1})|| < 1$. Define $h(s,t) : (-j\infty, j\infty) \times [0,1] \to \mathbb{C}$ by $h(j\omega, t) =$

$\det(\mathrm{Id} - t\Delta(j\omega)K(j\omega)(\mathrm{Id} - G(j\omega)K(j\omega))^{-1})$. The function h is continuous and $h(j\omega, t)$ and $h(\infty, t)$ are nonzero for every $t \in [0, 1]$. This implies that the Nyquist index of g_1 is equal to zero [20, Lemma A.1.18]. Consequently, K stabilizes G_Δ. The proof can be easily adjusted to deal with the case of multiplicative perturbations. For the case of left-coprime-factor perturbations, the proof can be adjusted noticing that $\det(\mathrm{Id} - G_\Delta) = \det(\mathrm{Id} - GK)\det(\mathrm{Id} + \Delta_M M^{-1}))\det(\mathrm{Id} - (K(\mathrm{Id} - GK)^{-1}\tilde{M}^{-1}, (\mathrm{Id} - GK)^{-1}\tilde{M}^{-1})^T)$. See [20, Th 9.2.6] for additional details. □

Theorem 13 guarantees the existence of robustness margins for a broad class of perturbations: input delays, uncertainties on the ODE parameters, uncertainties on the transport velocities. Provided we have a stabilizing control law that fulfills the different requirements of Theorem 10 or Theorem 12, it is possible to low-pass filter it to make it strictly proper. This in turns implies the robustness of the closed-loop system (Theorem 13). The proposed approach dissociates the stabilization problem from the robustness analysis which considerably simplifies the design. In particular this allows the cancellation of the PDE reflection terms or the inversion of the ODE dynamics, thus overcoming the limitations highlighted in [8]. The robustness results given in Theorem 13 do not depend on the model of the disturbances but on their bounds (that must be small enough). One significant advantage of such an approach is that, even for complex systems, it guarantees the existence of non-zero robustness margins by means of a simple low-pass filter that can be characterized by a single degree of freedom (its bandwidth). This bandwidth must verify some constraints that can be easily computed using the norms of the different functions. Qualitatively, increasing the bandwidth would imply reducing the robustness (at least w.r.t delays). We believe that a *quantitative* analysis is out of the scope of this paper. However, it is worth mentioning that such an analysis would be necessary to understand how the available degrees of freedom can be exploited to design a robust controller with an optimal behavior for a given uncertainty model for some industry-inspired constraints (similarly to what is done with H_∞-based approaches).

Remark 14 *Having a strictly proper control law is sufficient to guarantee the existence of robustness margins. However, it is not a necessary condition. As it can be seen in [8], it is possible to cancel a part of the reflection terms while guaranteeing robustness margins. However, the proof in this case can become technical.*

Finally, it is important to mention that such a strictly proper controller can be combined with a state-observer (a crucial step for practical implementation) and that the resulting output-feedback law can be made strictly proper by increasing the order of the filter.

6 Application: state feedback stabilization of an ODE-PDE-ODE system

In this section, we apply our filtering methodology to design strictly proper stabilizing state-feedback control laws for an ODE-PDE-ODE system (described by equations (2)-(7)) in different configurations. In the first configuration, only the ODE is actuated (i.e., $B_u \equiv 0$). Under some general conditions (adjusted from [14]), we design a stabilizing control law using an inversion of the ODE dynamics. We then apply our filtering procedure (Theorem 12) to obtain a strictly proper controller, thus guaranteeing the existence of robustness margins. In the second configuration, only the PDE is actuated (i.e., $B_X \equiv 0$). However, the PDE subsystem is considered scalar. Such a case has not been well studied in the literature, and the control design we propose is based on an Artstein-like transformation [3]. As it requires a complete cancellation of the reflection term $qv(t, 0)$, it is not strictly proper but can be robustified using our methodology (Theorem 10).

6.1 ODE-PDE-ODE system with actuation on the ODE ($B_u \equiv 0$)

We consider in this part that only the ODE is actuated (i.e., $B_u \equiv 0$). Such a system has been considered in [14] for the case of a scalar PDE subsystem and a state-observer was also proposed in [43]. Here, we extend the approach of [14] to deal with the difficulties resulting from the non-scalar PDE subsystem. We will still consider that Assumption 7 is verified.

6.1.1 General assumptions

Similarly to what has been done in [14], we make the following assumptions to design the control law

Assumption 15 *The pairs (A_1, E_1), (A_0, B_X) are stabilizable (i.e. there exist $F_0 \in \mathbb{R}^{r\times p}$ and $F_1 \in \mathbb{R}^{n\times q}$ such that $\bar{A}_0 = A_0 + B_X F_0$ and $\bar{A}_1 = A_1 + E_1 F_1$ are Hurwitz).*

This assumption is not very conservative since it is impossible to stabilize the ODE subsystems independently of the PDE or interconnection structure without the stabilizability of (A_1, E_1) and (A_0, B_X) at the very least. In this case, we ask for the slightly stronger condition (A_0, B_X) stabilizable in order to stabilize the actuator dynamics without requiring to go through the PDE dynamics. This condition, in exchange, allows us to obtain constructive control formulations with assumptions that can be easily checked based only on the ODE coefficients and basic finite-dimensional control tools known to any control engineer.

Assumption 16 *For all $s \in \mathbb{C}^+$, the matrices (A_0, B_X, C_0) satisfy rank* $\begin{pmatrix} sI - A_0 & B_X \\ C_0 & 0 \end{pmatrix} = p + n.$

This last assumption serves multiples purposes. It implies that the matrices C_0 and B_X are not identically zero. This is crucial for the stabilization of the PDE and of the X_1 subsystems through X_0. Under Assumption 16, we have that the function $P_0(s) = C_0(sI - \bar{A}_0)^{-1}B_X$ does not have any zeros in $\mathbb{C}^+$ that is common to all its components. Thus, the function $P_0(s)$ admits a right inverse whose entries have no unstable poles (such a right inverse is not proper) [36]. We denote P_0^+ any such right inverse. A possible starting point for the search of such an inverse is given by the Moore-Penrose right inverse $\bar{P}_0^+(s) = P_0^T(s)(P_0(s)P_0^T(s))^{-1}$ (which should be verified to be stable *a posteriori*). If this is not stable, then a more involved stable inversion procedure is needed, which is outside of the scope of this paper (such an inverse can always be found). This assumption is directly used in the constructive design of a control law and can be tested in a simple way. It is less restrictive than other conditions that can be found in the literature in terms of relative degree of the actuator (such as invertibility of B_X [13] or of C_0B_X [22]), and does not require the system to be written in any particular form [44]. The methodology we present here for the design of the stabilizing state-feedback control law is an extension of the design given in [14].

6.1.2 Stabilizing state-feedback law

In this section, we will use a frequency-domain formulation of the system as given by (34)-(36). We will choose the degree of freedom D introduced in (12) as $D = F_1$ such that the matrix $\bar{A}_1$ in equation (36) corresponds to the one introduced in Assumption 16. We will also choose the control input as $U(t) = \bar{U}(t)+F_0X_0(t)$. Thus, the system (30)-(32) can be rewritten as

$$\begin{aligned} z(s) &= F(s)z(s) + P_{11}(s)z(s) + C_0X_0(s) \\ &\quad + (Q\gamma_\beta(0) - \gamma_\alpha(0))X_1(s) \qquad (51) \\ (s\mathrm{Id} - \bar{A}_0)X_0(s) &= P_{21}(s)z(s) + E_0\gamma_\beta(0)X_1(s) \\ &\quad + B_X\bar{U}(s), \qquad (52) \\ (s\mathrm{Id} - \bar{A}_1)X_1(s) &= P_{31}(s)z(s). \qquad (53) \end{aligned}$$

Note that the stabilization of z actually implies the stabilization of the whole system provided that $\bar{U}(s)$ can be rewritten as a stable, dynamic state feedback of z. This is a consequence of the cascade structure of (51)-(53). Since the matrix $\bar{A}_0$ and $\bar{A}_1$ are Hurwitz, there exists $\eta > 0$ such that we can invert $(s\mathrm{Id} - \bar{A}_0)$ and $(s\mathrm{Id} - \bar{A}_1)$ on $\mathbb{C}_\eta$. Injecting the corresponding terms into (51), we obtain

$$\begin{aligned} z(s) &= (F(s) + G(s))z(s) + C_0(s\mathrm{Id} - \bar{A}_0)^{-1}B_X\bar{U}(s) \\ &= F(s)z(s) + G(s)z(s) + P_0(s)\bar{U}(s), \qquad (54) \end{aligned}$$

where

$$\begin{aligned} G(s) &= P_{11}(s) + (Q\gamma_\beta(0) - \gamma_\alpha(0))(s\mathrm{Id} - \bar{A}_1)^{-1}P_{31}(s) \\ &\quad + C_0(s\mathrm{Id} - \bar{A}_0)^{-1}P_{21}(s) + C_0(s\mathrm{Id} - \bar{A}_0)^{-1} \\ &\quad \cdot E_0\gamma_\beta(0)(s\mathrm{Id} - \bar{A}_1)^{-1}P_{31}(s). \qquad (55) \end{aligned}$$

Thus, we can define the control law $\bar{U}(s)$ as $\bar{U}(s) = -P_0^+(s)G(s)z(s)$, which gives us $z(s) = F(s)z(s)$. Due to Assumption 7, it implies the stabilization of z (and consequently of X_0 and X_1). Since the function $G(s)$ is strictly proper, we can apply Theorem 12 to filter the control law and obtain a strictly proper control law.

Theorem 17 *Consider system* (2)-(7) *under Assumptions 7, 15 and 16. Consider the stabilizing control* $U_1(s)$ *defined in the frequency domain by* $U_1(s) = F_0X_0(s) - P_0^+(s)G(s)z(s)$, *where* $z(s)$ *is defined by* $z(s) = \bar{\alpha}(s, 0)$, *the state* $\bar{\alpha}$ *being defined through the backstepping transformations* (8), (9), (13) *and* (14) *and where* $G(s)$ *is defined by equation* (55). *Then, there exist two low pass filters* $w_0(s)$ *and* $w_1(s)$ *that satisfy equations* (45) *and* (46) *such that the control law*

$$U(s) = w_1(s)F_0X_0(s) - w_0(s)P_0^+(s)G(s)z(s) \qquad (56)$$

is strictly proper and exponentially stabilizes the system (2)-(7).

PROOF. The different conditions of Theorem 12 are verified with $\bar{C} = C_0$, $K_{X_0}^b = F_0$, $K_{X_0}^u = K_{X_1}^u = 0$, $K_{X_1}^b = 0$, $K_z^b = 0$ and $K_z^u(s) = -P_0^+(s)G(s)$. Indeed, we have $B_XK_{X_0}^b = \bar{A}_0 - A_0$, and $C_0(s\mathrm{Id} - \bar{A}_0)^{-1}B_XK_z^u(s) = -G(s)$, that is strictly proper. It implies that the filtered control law exponentially stabilizes the system (34)-(36). The exponential stability of (X_1, z, X_2) implies the exponential of $(X_0, \bar{\alpha}, \beta, X_1)$ due to inequality (33). This in turns implies the exponential stability of (X_0, u, v, X_1) due to the invertibility of the backstepping transformations. □

Remark 18 *So far, we have not discussed the time-domain realization of such a feedback law. Nevertheless, the components of our feedback law are of two types: (i) distributed or pointwise delays of the states of the system and (ii) transfer matrices. The time-domain realization of the distributed delays corresponds to an integral operator with delayed values of the states, whereas a suitable state-space realization of the transfer matrices can be easily found. Thus the control law* (56) *is causal (it does not require future values of the state) and can be suitably approximated for implementation.*

6.2 ODE-PDE-ODE system with actuation on the PDE ($B_X \equiv 0$)

We now consider the case where the control input acts at the connection point between the PDE and ODE

($B_X \equiv 0$). This case has not been well studied in the literature and most of the contributions do not consider the ODE X_0 [8,5,21]. However, in all these works, canceling the PDE reflection term $Qu(t,0)$ simplified the analysis, and the resulting control laws verify the conditions of Theorem 10. In [22] a state observer was designed for an ODE-PDE-ODE system where the measurement corresponds to the PDE state. The dual controllability problem would correspond to $B_u = \mathrm{Id}$. As the observability problem is the dual of the controllability problem, the properties of the observer operator easily adjust to the controller operator. Consider $n = m = 1$ and $B_u = 1$. We assume $R \neq 0$ (otherwise the computations are slighlty more complicated). From system (23)-(27), we can define the intermediate control input $\bar{U}(t)$ as

$$\begin{aligned}\bar{U}(t) &= U(t) + C_0 X_0(t) + Q\beta(t,0) + Q\gamma_\beta(0)X_1 \\ &- \gamma_\alpha(0)X_1 + \int_0^1 L_1(y)\bar{\alpha}(t,y) + L^{\alpha\beta}(0,y)\beta(t,y)dy \\ &- Q(\int_0^1 L(y)\bar{\alpha}(t,y) + L^{\beta\beta}(0,y)\beta(t,y)dy), \end{aligned} \tag{57}$$

so that $\bar{\alpha}(t,0) = \bar{U}(t)$. Note that, due to the cancellation of the reflection term $Qv(t,0)$, the control law $U(t)$ is not strictly proper. Consequently, it will be necessary to low-pass filter it to guarantee the robustness of the closed-loop system. Imposing $L^{\alpha\alpha}(x,1) = (\Lambda^+)^{-1}(L^{\alpha\beta}(x,1)\Lambda^- R - \gamma_\alpha(x)E_1)$ and $L^{\beta\beta}(x,1) = (\Lambda^- R)^{-1}(L^{\beta\alpha}(x,1)\Lambda^- R + \gamma_\beta(x)E_1)$ we obtain $G_1 = 0$ and $G_2 = 0$. Using the method of characteristics, we get

$$\begin{aligned}\dot{X}_0 =& A_0 X_0 + E_0 R\bar{U}(t-\tau) + E_0\gamma_\beta(0)X_1 \\ &+ \int_0^\tau \bar{N}(\nu)\bar{U}(t-\nu)d\nu, \end{aligned} \tag{58}$$

$$\dot{X}_1 = \bar{A}_1 X_1 + E_1\bar{U}(t-\frac{1}{\lambda}), \tag{59}$$

where we recall that $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$, and where we have $\bar{N}(\nu) = -\mathbb{1}_{[0,\frac{1}{\lambda}]}(\nu)\lambda E_0 L^{\beta\alpha}(0,\lambda\nu) - \mathbb{1}_{]\frac{1}{\lambda},\tau]}(\nu)\mu E_0 L^{\beta\beta}(0, 1+\frac{\mu}{\lambda} - \nu\mu)$. Define the function Y_0 as

$$\begin{aligned}Y_0 = X_0 + \tau^2 \int_0^1 \int_0^x \mathrm{e}^{-A_0\tau(x-y)}\bar{N}(\tau(1-y))dy\cdot \\ \bar{U}(t-(1-x)\tau)dx. \end{aligned} \tag{60}$$

We obtain $\dot{Y}_0 = A_0 Y_0 + E_0\gamma_\beta(0)X_1 + \bar{E}_0 U(t) + E_0 R\bar{U}(t-\tau)$ where $\bar{E}_0 = \tau\int_0^1 \mathrm{e}^{-A_0\tau(1-y)}\bar{N}(\tau(1-y))dy$. Denote

$$Y(t) = \begin{pmatrix} Y_0 \\ X_1 \end{pmatrix},\ \bar{A} = \begin{pmatrix} A_0 & E_0\gamma_\beta(0) \\ 0 & \bar{A}_1 \end{pmatrix}.$$

Inspired by [3], we finally define the state Z as

$$\begin{aligned}Z =& Y + \int_{t-\tau}^t \mathrm{e}^{\bar{A}(t-s-\tau)} \begin{pmatrix} E_0 R & 0 \end{pmatrix}^T \bar{U}(s)ds \\ &+ \int_{t-\frac{1}{\lambda}}^t \mathrm{e}^{\bar{A}(t-s-\frac{1}{\lambda})} \begin{pmatrix} 0 & E_1 \end{pmatrix}^T \bar{U}(s)ds. \end{aligned} \tag{61}$$

We obtain $\dot{Z}(t) = \bar{A}Z(t) + \bar{B}\bar{U}(t)$, with

$$\bar{B} = \begin{pmatrix} \bar{E}_0 \\ 0 \end{pmatrix} + \mathrm{e}^{-\bar{A}\tau}\begin{pmatrix} E_0 R \\ 0 \end{pmatrix} + \mathrm{e}^{-\bar{A}\frac{1}{\lambda}}\begin{pmatrix} 0 \\ E_1 \end{pmatrix}.$$

We then have the following theorem

Theorem 19 *Consider system* (2)-(7) *(with $n = m = B_u = 1$) under Assumptions 7. Assume that there exists a matrix D in equation* (12) *such that the pair $(\bar{A}, \bar{B})$ is stabilizable (i.e., there exists K such that $\bar{A} + \bar{B}K$ is Hurwitz). Consider the control law $U(t)$ defined by equation* (57) *where $\bar{U}(t) = KZ(t)$, where Z is defined from X_0 and X_1 using transformations* (60) *and* (61)*. Then, there exists a low pass filter $w_0(s)$ that satisfies equation* (44) *such that the control law $U_f(s) = w_0(s)U(s)$ is strictly proper and exponentially stabilizes the system* (2)-(7)*.*

PROOF. Using the control law $U(t)$, the closed-loop stability is straightforward. Indeed, the exp. stability of Z implies that $\bar{U}$ converges to zero. This implies the exp. stability of the state Y (using (61)) and X_0 (using (60)). This in turns implies the exp. convergence of the state $\alpha(t,0)$ and consequently of (X_0, u, v, X_1). Finally, since the controller gains K_{X_1}, K_{X_0} are bounded and since $K_z = -F(s)$, the conditions of Theorem 10 are verified and the strictly proper controller U_f exponentially stabilizes the system (2)-(7). This concludes the proof. □

6.3 Simulation results

In this section, we illustrate the performances of our two output-feedback controllers. We implemented the proposed approach using Matlab and Simulink. The PDE system is simulated using an explicit in time, first-order, upwind finite difference method with 101 spatial discretization points. The ODE state was simulated using the Matlab medium order method ode45.

6.3.1 ODE-PDE-ODE system with actuation on the ODE

We consider the framework of Section 6.1. The transfer functions and the control law were transformed to a time-domain representation as discussed in Remark

18 for implementation and the integral appearing in the distributed-delay terms were approximated by a trapezoidal rule using discrete values for the delay. The numerical values used are $\Lambda^+ = \text{diag}(1, 2)$, $\Lambda^- = \text{diag}(1.2, 1.8)$

$$\Sigma^{++} = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}, \ \Sigma^{+-} = \begin{pmatrix} 0.7x & 0 \\ 0.1 & 0.2\mathrm{e}^{-0.1x} \end{pmatrix}, \ \Sigma^{-+} = \begin{pmatrix} -0.3 & 0 \\ -0.3 & 0 \end{pmatrix},$$

$$\Sigma^{--} = \begin{pmatrix} 0 & 0 \\ 0.4x^2 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} 0.3 & 0.3 \\ 0 & 0 \end{pmatrix}, \ R = \begin{pmatrix} 0.4 & 0 \\ 0.9 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -0.15 & 0.1 \\ 0 & 0.1 \end{pmatrix}, \ E_1 = \begin{pmatrix} 0 & 0.1 \\ 0 & 0.1 \end{pmatrix}, \ C_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$E_0 = \begin{pmatrix} 0.1 & 0.2 & 0 & 0.1 \\ -0.2 & 0 & 0.1 & 0 \end{pmatrix}^T, \ C_0 = \begin{pmatrix} -0.40 & -0.75 & -0.30 & 0.66 \\ 0.80 & -1.26 & -0.09 & 0.16 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} -0.15 & 0.85 & 0.65 & 0.85 \\ 0.5 & -2 & 0.5 & 0.5 \\ 1.35 & -0.15 & -0.85 & -0.15 \\ 0.5 & 0.5 & -0.5 & -2 \end{pmatrix}, \ B_X = \begin{pmatrix} 0.5 & 0 & -1 \\ 0.5 & -0.7 & -1 \\ -0.5 & 0 & 1 \\ 0.5 & 0.7 & -1 \end{pmatrix}.$$

Notice that the system satisfies Assumptions 15 and 16. Furthermore, the ODE systems are only stabilizable in that case and do not need to be transformed in any particular standard form. Each subsystem (ODEs and PDEs) are independently unstable (and remain so when plugged together). We added an input delay of $0.05s$ to show the robustness of the design to small delays in the loop. Some parameters are subject to a constant multiplicative uncertainty. We used simple low-pass filters of 4th order with different bandwidths, as illustrated by Lemma 11, one does not require extremely specific shapes for the low-pass filters. In this case, the stable right-inverse was obtained by diagonalizing the actuator system and computing the transfer functions and inversions based on a diagonal transfer matrix (thus simplifying the computations). We have plotted in Figure 1 the time evolution of the $\chi-$norm of the system in open-loop, with the unfiltered control law and with filtered controllers (for two different bandwiths: 125 rad.s^{-1} and 6.3 rad.s^{-1}). The control effort for one of the filter is shown in Figure 2. The input is subject to a 0.05s delay and the coupling matrices R, Q and A_0 are subject to 5% uncertainties. As expected, the states of the system converge to zero despite the presence of the input delay.

6.3.2 ODE-PDE-ODE system with actuation on the PDE

We now consider the framework of Section 6.2. The parameters are now $\Lambda^+ = 1$, $\Lambda^- = 2$, $\Sigma^{-+} = -0.4\mathrm{e}^{-x}$, $\Sigma^{+-} = 0.8x$, $Q = 0.7$, $R = 1$, $E_1 = (-0.1, 0.2)^T$, $C_1 = (0.1, 0.2)$, $C_0 = (-0.4 - 0.75)$, $E_0 = (-0.1, 0.3)^T$, $B_u = 1$. The matrix A_1 is identical to the ones of Section 6.3.1, while $A_0 = \begin{pmatrix} -0.15 & 0.1 \\ 0 & 0.1 \end{pmatrix}$. The matrix K in

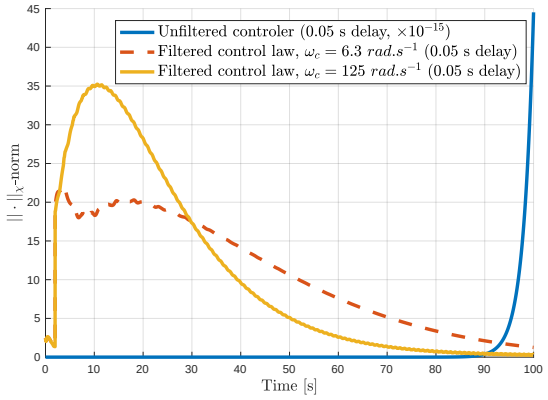

Fig. 1. Evolution of the $||\cdot||_\chi$-norm of the state with an input delay of 0.05s. and in presence of uncertainties for three situations: a) Without filter, b) With a fourth order filter ($\omega_c = 6.3$ rad.s^{-1}), c) With a fourth order filter ($\omega_c = 125$ rad.s^{-1})

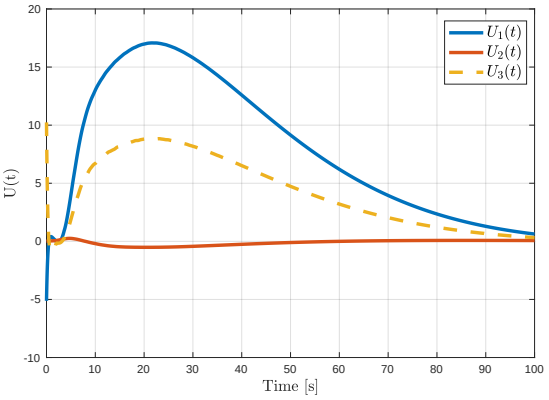

Fig. 2. Evolution of the filtered control input with an input delay of 0.05s ($\omega_c = 125$ rad.s^{-1}).

Theorem 19 is chosen such that the poles for the eigenvalues of $\bar{A} + \bar{B}K$ are -0.2, -0.3, -0.4 and -0.5. The coefficients Σ^{-+} and Σ^{+-} are subject to an additive sinusoidal uncertainty (amplitude 0.05). We have pictured in Figure 3 the time evolution of the χ-norm in four situations. The first case corresponds to the closed-loop behavior in the absence of low-pass filter and delay. In the second case, we consider a 0.05 input delay. Finally, in the third case and fourth case, we consider a filtered control law (with simple low-pass filter of 2nd order with a bandwidth of 40 rad.s^{-1} and 120 rad.s^{-1} as illustrated by Lemma 11) in presence of a $0.05s$ input delay. The nominal controller exponentially stabilizes the system. Note that the proposed design may be the cause of high transient values. It can be seen that, due to the cancellation of the reflection term $-Qv(t, 0)$ in the control design, the unfiltered control law is not robust to this (small) input delay. Conversely, the filtered control law still stabilizes the system in presence of this delay. Nevertheless, the filter slightly deteriorates the transient behavior. Increasing the bandwith improves the performance but can

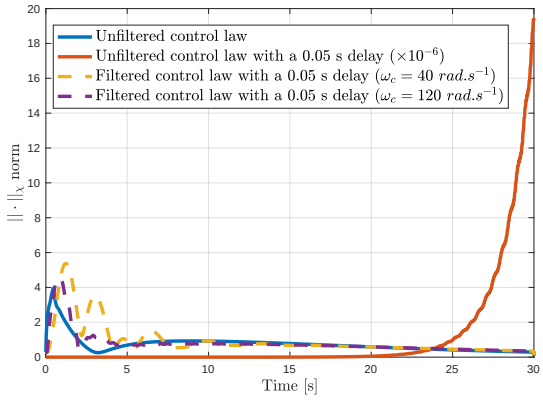

Fig. 3. Evolution of the $||\cdot||_\chi$-norm of the state for three situations: a) Without filter and without input delay (blue), b) without filter and with an input delay of 0.05s (red), c) with low-pass filter and input delay of 0.05s (yellow).

deteriorate the robustness margins. Knowing the structure of the uncertainty, it is theoretically possible to increase the bandwith (thus improving the performance) while guaranteeing that the conditions of Theorem 13 are still verified. However, due to the presence of the delay, a deeper analysis would be require to verify that the closed-loop system is still robust to delays.

7 Concluding remarks

This paper introduces a filtering methodology that robustifies stabilizing control laws for systems composed of interconnections of hyperbolic PDEs and ODEs. Assuming that a stabilizing controller is available, we derive simple sufficient conditions under which appropriate low-pass filters can be combined with the control law to obtain a strictly proper controller, thus enabling the existence of robustness margins. The proposed approach relies on a frequential analysis performed using the equivalence between hyperbolic systems and neutral delay-differential equations. This filtering technique simplifies the design of stabilizing controllers for the proposed class of systems as it dissociates the stabilization problem from the robustness problem. For future problems that belong to the broad class we consider in this paper, the stabilization problem can be solved using any arbitrary control technique, while the robustness aspects can be solved using the presented low-pass filter design. However, the proposed approach is **qualitative** for now as only a sufficient robustness condition has been given on the design of the low-pass filters. The impact of the tuning of the filter on performance and robustness margins, and a quantitative analysis should be the purpose of further investigations. Future discussions should also include the cases where the actuator can simultaneously act on the ODEs and the PDE sub-systems.

References

[1] U. J. F. Aarsnes, F. Di Meglio, and R. J. Shor. Benchmarking of industrial stick-slip mitigation controllers. *IFAC-PapersOnLine*, 51(8):233–238, 2018.

[2] U. J. F. Aarsnes and R. J. Shor. Torsional vibrations with bit off bottom: Modeling, characterization and field data validation. *Journal of Petroleum Science and Engineering*, 163:712–721, 2018.

[3] Z. Artstein. Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, 27(4):869–879, 1982.

[4] J. Auriol. *Robust design of backstepping controllers for systems of linear hyperbolic PDEs*. PhD thesis, PSL Research University, 2018.

[5] J. Auriol, U. J. F. Aarsnes, F. Di Meglio, and R. Shor. Robust control design of underactuated 2× 2 PDE-ODE-PDE systems. *IEEE Control Systems Letters*, 5(2):469–474, 2020.

[6] J. Auriol and F. Bribiesca Argomedo. Delay-robust stabilization of an $n + m$ hyperbolic PDE-ODE system. In *Conference on Decision and Control*, pages 4964–4970. IEEE, 2019.

[7] J. Auriol and D. Bresch Pietri. Robust state-feedback stabilization of an underactuated network of interconnected n + m hyperbolic PDE systems. *Automatica*, 2021.

[8] J. Auriol, F. Bribiesca Argomedo, D. Bou Saba, M. Di Loreto, and F. Di Meglio. Delay-robust stabilization of a hyperbolic PDE–ODE system. *Automatica*, 95:494–502, 2018.

[9] J. Auriol, F. Bribiesca-Argomedo, and F. Di Meglio. Delay robust state feedback stabilization of an underactuated network of two interconnected PDE systems. In *American and Control Conference*, 2019.

[10] J. Auriol and F. Di Meglio. An explicit mapping from linear first order hyperbolic PDEs to difference systems. *Systems & Control Letters*, 123:144–150, 2019.

[11] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*. Springer, 2016.

[12] N. Bekiaris-Liberis and M. Krstic. *Nonlinear control under nonconstant delays*. SIAM, 2013.

[13] D. Bou Saba, F. Bribiesca-Argomedo, M. Di Loreto, and D. Eberard. Backstepping stabilization of 2×2 linear hyperbolic PDEs coupled with potentially unstable actuator and load dynamics. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 2498–2503. IEEE, 2017.

[14] D. Bou Saba, F. Bribiesca-Argomedo, M. Di Loreto, and D. Eberard. Strictly proper control design for the stabilization of 2 × 2 linear hyperbolic ODE-PDE-ODE systems. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 4996–5001. IEEE, 2019.

[15] D. Bresch-Pietri. *Commande robuste de systèmes à retard variable: Contributions théoriques et applications au contrôle moteur*. PhD thesis, Ecole Nationale Supérieure des Mines de Paris, 2012.

[16] D. Bresch-Pietri and M. Krstic. Adaptive output feedback for oil drilling stick-slip instability modeled by wave PDE with anti-damped dynamic boundary. In *American Control Conference (ACC), 2014*, pages 386–391. IEEE, 2014.

[17] J-M. Coron, G. Bastin, and B. d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.

[18] J.-M. Coron, L. Hu, and G. Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. *Automatica*, 84:95–100, 2017.

[19] R. Curtain and K. Morris. Transfer functions of distributed parameter systems: A tutorial. *Automatica*, 45(5):1101–1116, 2009.

[20] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 2012.

[21] G. A. de Andrade, R. Vazquez, and D. J. Pagano. Backstepping stabilization of a linearized ODE–PDE rijke tube model. *Automatica*, 96:98–109, 2018.

[22] J. Deutscher, N. Gehring, and R. Kern. Output feedback control of general linear heterodirectional hyperbolic ODE–PDE–ODE systems. *Automatica*, 95:472–480, 2018.

[23] F. Di Meglio, F. Bribiesca Argomedo, L. Hu, and M. Krstic. Stabilization of coupled linear heterodirectional hyperbolic PDE–ODE systems. *Automatica*, 87:281–289, 2018.

[24] F. Di Meglio, P.-O. Lamare, and U. J. F. Aarsnes. Robust output feedback stabilization of an ODE–PDE–ODE interconnection. *Automatica*, 119:109059, 2020.

[25] J. Hale and S.M. Verduyn Lunel. Strong stabilization of neutral functional differential equations. *IMA Journal of Mathematical Control and Information*, 19(1 and 2):5–23, 2002.

[26] J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*. Springer-Verlag, 1993.

[27] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge university press, 2012.

[28] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic. Boundary exponential stabilization of 1-dimensional inhomogeneous quasi-linear hyperbolic systems. *SIAM Journal on Control and Optimization*, 57(2):963–998, 2019.

[29] M. Krstic and A. Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9):750–758, 2008.

[30] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*, volume 16. Siam, 2008.

[31] W. Kwon and A Pearson. Feedback stabilization of linear systems with delayed control. *IEEE Transactions on Automatic control*, 25(2):266–269, 1980.

[32] A. Kyllingstad and P.J. Nessjøen. A new stick-slip prevention system. In *SPE/IADC Drilling Conference and Exhibition*. Society of Petroleum Engineers, 2009.

[33] H. Logemann, R. Rebarber, and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. *SIAM Journal on Control and Optimization*, 34(2):572–600, 1996.

[34] A. Manitius and A. Olbrot. Finite spectrum assignment problem for systems with delays. *IEEE transactions on Automatic Control*, 24(4):541–552, 1979.

[35] T. Meurer and M. Zeitz. Flatness-based feedback control of diffusion-convection-reaction systems via k-summable power series. *IFAC Proceedings Volumes*, 37(13):177–182, 2004.

[36] P. Moylan. Stable inversion of linear systems. *IEEE Transactions on Automatic Control*, 22(1):74–78, 1977.

[37] S.-I. Niculescu. *Delay effects on stability: a robust control approach*, volume 269. Springer Science & Business Media, 2001.

[38] J. Redaud, J. Auriol, and S.-I. Niculescu. Output-feedback control of an underactuated network of interconnected hyperbolic PDE-ODE systems. *Systems and Control letters*, 2021.

[39] B. Saldivar, S. Mondié, J.-J. Loiseau, and V. Rasvan. Stick-slip oscillations in oillwell drilstrings: distributed parameter and neutral type retarded model approaches. *IFAC Proceedings Volumes*, 44(1):284–289, 2011.

[40] S. Tang and C. Xie. State and output feedback boundary control for a coupled pde–ode system. *Systems & Control Letters*, 60(8):540 – 545, 2011.

[41] R. Vazquez, M. Krstic, and J.-M. Coron. Backstepping boundary stabilization and state estimation of a 2× 2 linear hyperbolic system. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 4937–4942. IEEE, 2011.

[42] M. Vidyasagar. Input-output stability of a broad class of linear time-invariant multivariable systems. *SIAM J. Control*, 10(1):203–209, 1972.

[43] J. Wang and M. Krstic. Delay-compensated control of sandwiched ode–pde–ode hyperbolic systems for oil drilling and disaster relief. *Automatica*, 120:109131, 2020.

[44] J. Wang, M. Krstic, and Y. Pi. Control of a 2× 2 coupled linear hyperbolic system sandwiched between 2 odes. *International Journal of Robust and Nonlinear Control*, 28(13):3987–4016, 2018.

[45] F. Woittennek. Flatness based feedback design for hyperbolic distributed parameter systems with spatially varying coefficients. *IFAC Proceedings Volumes*, 46(26):37–42, 2013.

[46] K. Yoshida. *Lectures on differential and integral equations*, volume 10. Interscience Publishers, 1960.

Appendix

Proof of Theorem 10

We give here the proof of Theorem 10. Let us consider a constant $M > 0$ and define a low pass-filter $w(s)$ that satisfies $|1 - w(s)| < 1$, $|w(s)| < 1$ and equation (44). The relative order of this filter is denoted N and is chosen such that $w(s)K(s)$ is strictly proper (which is possible since the growth rate of $K(s)$ is at worst polynomial). Our objective is to prove that we can choose the constant M such that the new filtered control law $w_M(s)U(s)$ still guarantees the stabilization of (34)-(35). Plugging this filtered control law into the system (34)-(35), we obtain the characteristic equation:

$$\det(I_1(s) - P_0(s) - w(s)BK(s)) = 0. \qquad (62)$$

In what follows, we denote $\bar{Q}(s) = I_1(s) - P_0(s) - w(s)BK(s)$. For the sake of contradiction, assume that equation (62) admits a solution $s \in \mathbb{C}^+$. Consider in a first time that $s \in \mathbb{C}_u$ so that $\det(s\mathrm{Id} - A_0)$ and $\det(s\mathrm{Id} - \bar{A}_1)$ do not vanish. Since s is a solution of equation (62), there exists $\zeta \neq 0$ (where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$) such that $\bar{Q}(s)\zeta = 0$. We obtain

$$F_0(s)\zeta_1 = (P_{12}(s) + w(s)B_u K_{X_0}(s))\zeta_2$$

$$+(P_{13}(s)+w(s)B_uK_{X_1}(s))\zeta_3, \quad (63)$$
$$(s\mathrm{Id}-A_0)\zeta_2 = P_{21}(s)\zeta_1+P_{23}(s)\zeta_3, \quad (64)$$
$$(s\mathrm{Id}-\bar{A}_1)\zeta_3 = P_{31}\zeta_1, \quad (65)$$

where $F_0(s)=\mathrm{Id}-F(s)-P_{11}(s)-w(s)B_uK_z(s)$. Multiplying equation (65) by $(s\mathrm{Id}-\bar{A}_1)^{-1}$ and equation (64) by $(s\mathrm{Id}-A_0)^{-1}$ and injecting them into equation (63), we obtain

$$\bar{F}(s)\zeta_1 = 0, \quad (66)$$

where $\bar{F}(s) = F_0(s) - P_{13}(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s) - w(s)B_uK_{X_1}(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s) - P_{12}(s)(s\mathrm{Id}-A_0)^{-1}P_{21}(s) - w(s)B_uK_{X_0}(s)(s\mathrm{Id}-A_0)^{-1}P_{21}(s) - (P_{12}(s)+w(s)B_uK_{X_0}(s))(s\mathrm{Id}-A_0)^{-1}\cdot P_{23}(s)(s\mathrm{Id}-\bar{A}_1)^{-1}$. Due to condition (2) in the statement of the Theorem, and Assumption 9, we have

$$\bar{\sigma}(F+w(s)B_uK_z^u) \leq \bar{\sigma}(\mathrm{Id}-w(s)B_uH_0(s))\epsilon_0 \leq \epsilon_0,$$

since $B_uH_0(s)$ is similar to a diagonal matrix whose component belong to $[0,1]$ and since for all $d\in[0,1], |1-w(s)d|<1$ (due to the requirements on the filter). Moreover, due to condition (1) and the definitions of the different matrices, the remaining functions that appear in the definition of $\bar{F}$ (except the identity in $F_0(s)$) are strictly proper. Thus, equation (66) can be rewritten as $(Id+R_1(s)+R_2(s)+w(s)R_3(s))\zeta_1=0$, where R_2 and R_3 are strictly proper and where $R_1(s)$ satisfies $\bar{\sigma}(R_1(s))\leq\epsilon_0<1$. Consequently, using the fact that $|w(s)|<1$, there exist $\epsilon_1>0$ and $M_1>0$ that do not depend on the choice of w such that if $|s|>M_1$, we have $\bar{\sigma}(R_1(s)+R_2(s)+w(s)R_3(s))<1-\epsilon_1$, which implies $\underline{\sigma}(\bar{F}(s))>\epsilon_1$. Due to equation (66), we must have $\zeta_1=0$ which in turns implies $\zeta_2=\zeta_3=0$ due to equations (64)-(65) (since $s\in\mathbb{C}_u$). This is a contradiction. Thus, the characteristic equation does not admit a solution on $\mathbb{C}_u$ if $|s|>M_1$. Let us now consider the case $|s|<M_1$. Set $M=M_1$ in the definition of the filter given by equation (44). The closed-loop system (34)-(35) can be rewritten as $Q(s) = -(1-w(s))\begin{pmatrix}B_u & 0 & 0\end{pmatrix}^T K(s)$. We have $(1-w(s))\bar{\sigma}(\begin{pmatrix}B_u & 0 & 0\end{pmatrix}^T K(s)) < \underline{\sigma}(Q(s))$, due to equation (44). This results in a contradiction. Consequently, the characteristic equation has no zeros on $\mathbb{C}^+$. The proof could be adjusted to show that the asymptotic vertical chain of zeros of $\bar{Q}(s)$ cannot be the imaginary axis and that $\bar{Q}(s)$ does not have any zeros on $\mathbb{C}_{\eta_1}$. This proves the exponential stability of the closed-loop system [26]. □

Proof of Theorem 12

We give here the proof of Theorem 12. We will consider in a first time that $w_1\equiv 1$ and that w_0 is a low-pass filter that satisfies $|w_0(s)|<1$, $|1-w_0(s)|<1$, and equation (45). Consider that the control law is now $U_f(s) = w_0(s)K_z^u z(s) + w_0(s)K_{X_0}^u X_0(s) + w_0(s)K_{X_1}^u X_1(s) + K_z^b(s)z(s) + K_{X_1}^b(s)X_1(s) + K_{X_0}^b(s)X_0(s)$. For the sake of contradiction, assume that the characteristic equation of the system admits a solution $s\in\mathbb{C}_{\eta_1}$. Consider in a first time that this solution belongs to $s\in\mathbb{C}_u$ (where $\mathbb{C}_u$ is defined in Section 5). Thus, there exists $\zeta\neq 0$ (where $\zeta=(\zeta_1,\zeta_2,\zeta_3)$) such that

$$(\mathrm{Id}-F(s)-P_{11}(s))\zeta_1 = P_{12}(s)\zeta_2+P_{13}(s)\zeta_3, \quad (67)$$
$$\begin{aligned}(s\mathrm{Id}-A_0)\zeta_2 &= (P_{21}(s)+B_XK_z^b(s))\zeta_1\\ &+ w_0(s)B_XK_{X_0}^u(s)\zeta_2 + B_XK_{X_0}^b(s)\zeta_2\\ &+ w_0(s)B_XK_z^u(s)\zeta_1 + w_0(s)B_XK_{X_1}^u(s)\zeta_3\\ &+ (P_{23}(s)+B_XK_{X_1}^b(s))\zeta_3, \end{aligned} \quad (68)$$
$$(s\mathrm{Id}-\bar{A}_1)\zeta_3 = P_{31}(s)\zeta_1. \quad (69)$$

Multiplying equation (69) by $(s\mathrm{Id}-\bar{A}_1)^{-1}$ and injecting it into equation (67), we obtain

$$\bar{F}(s)\zeta_1 = P_{12}(s)\zeta_2 = C_0\zeta_2, \quad (70)$$

where $\bar{F}(s) = \mathrm{Id}-F(s)-P_{11}(s)-P_{13}(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)$. Due to Assumption 9, there exist $\epsilon_F>0$ and $M_F>0$ such that if $|s|>M_F$, we have $\underline{\sigma}(\bar{F}(s))>\epsilon_F$. Consequently, if $|s|>M_F$, we can multiply equation (70) by $(\bar{F}(s))^{-1}$ and inject it into (68). Using conditions (1) and (2), equation (68) can be rewritten as

$$\begin{aligned}(s\mathrm{Id}-\bar{A}_0)\zeta_2 &= w_0(s)B_X\bar{K}^u(s)C_0\zeta_2 + w_0(s)B_XK_z^u(s)\\ &\bar{F}^{-1}(s)C_0\zeta_2 + w_0(s)B_XK_{X_1}^u(s)(s\mathrm{Id}-\bar{A}_1)^{-1}\\ &P_{31}(s)\bar{F}^{-1}(s)C_0\zeta_2 + (R(s)+B_X\bar{K}_{X_0}^b)C_0\zeta_2, \end{aligned} \quad (71)$$

where the function $R(s)$ is defined by $R(s) = [P_{23}(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)+P_{21}(s)+B_XK_{X_1}^b(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)+B_XK_z^b(s)]\bar{F}^{-1}(s)$. Note that the function $(s\mathrm{Id}-\bar{A}_0(s))^{-1}(R(s)+B_X\bar{K}_{X_0}^b)$ is strictly proper due to the different requirements given in the statement of the theorem. Multiplying equation (71) by $C_0(s\mathrm{Id}-\bar{A}_0)^{-1}$, and denoting $\bar{\zeta}_2=C_0\zeta_2$ we obtain

$$\begin{aligned}\bar{\zeta}_2 &= w_0(s)C_0(s\mathrm{Id}-\bar{A}_0)^{-1}B_X(\bar{K}^u(s)+K_z^u(s)\bar{F}^{-1}(s)\\ &+ K_{X_1}^u(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)\bar{F}^{-1}(s))\bar{\zeta}_2 + C_0(s\mathrm{Id}\\ &- \bar{A}_0)^{-1}(R(s)+B_X\bar{K}_{X_0}^b(s))\bar{\zeta}_2 = G(s)\bar{\zeta}_2. \end{aligned} \quad (72)$$

Due to the second requirement of the theorem, $G(s)$ is strictly proper. There exists $\bar{M}_0>M_F$ (that does not depend on the choice of w_0) such that if $|s|>\bar{M}_0$, $\bar{\sigma}(G(s))<1$. This implies $\bar{\zeta}_2=0$. Injecting into equation (71), we obtain $(s\mathrm{Id}-\bar{A}_0)\zeta_2=0$. There exists $M_0>0$ such that if $s>M_0$ $\underline{\sigma}((s\mathrm{Id}-\bar{A}_0))>1$. It implies $\zeta_2=0$ which in turns results in $\zeta_1=\zeta_3=0$. This is a contradiction. Let us now consider the case $|s|<M_0$. Let us choose this M_0 for the definition of equation (45). The closed-loop system (34)-(35) can be rewritten as $Q(s)=-(1-$

$w_0(s))B\left(K_z^u(s)\ K_{X_0}^u(s)\ K_{X_1}^u(s)\right)$. In the meantime, we have $(1-w_0(s))\bar{\sigma}(B\left(K_z^u(s)K_{X_0}^u(s)K_{X_1}^u(s)\right)) < \underline{\sigma}(Q(s))$, due to equation (45). This results in a contradiction. Consequently, the characteristic equation cannot be verified on $\mathbb{C}_\eta$. Thus, the system is exponentially stable. Moreover, the order of the filter w_0 can always be chosen to make the function $w_0(s)B_{X_0}K^u(s)$ strictly proper (see Remark 11). Thus, we have that the function $(s\mathrm{Id}-\bar{A}_0)^{-1}\left(0\ B_X\ 0\right)^T K^u(s)C_0w_0(s))$ is strictly proper.

Consider now that $w_1(s)$ is a low-pass filter that satisfies $|w_1(s)|<1$, $|1-w_1(s)|<1$, and equation (46). Consider that the control law is now $U_f(s)=w_0(s)K_z^uz(s)+w_0(s)K_{X_0}^uX_0(s)+w_0(s)K_{X_1}^uX_1(s)+w_1(s)[K_z^b(s)z(s)+K_{X_1}^b(s)X_1(s)+K_{X_0}^b(s)X_0(s)]$. For the sake of contradiction, assume that the characteristic equation of the system admits a solution $s\in\mathbb{C}_{\eta_1}$. Consider in a first time that this solution belongs to $s\in\mathbb{C}_u$. Thus, there exists $\zeta=(\zeta_1,\zeta_2,\zeta_3)\neq 0$ such that

$$(\mathrm{Id}-F(s)-P_{11}(s))\zeta_1=P_{12}(s)\zeta_2+P_{13}(s)\zeta_3, \quad (73)$$

$$\begin{aligned}(s\mathrm{Id}-A_0)\zeta_2 &= (P_{21}(s)+w_1(s)B_XK_z^b(s))\zeta_1\\ &+ w_0(s)B_XK_{X_0}^u(s)\zeta_2+w_0(s)B_XK_z^u(s)\zeta_1\\ &+ w_0(s)B_XK_{X_1}^u(s)\zeta_3+w_1(s)B_XK_{X_0}^b(s)\zeta_2\\ &+ (P_{23}(s)+w_1(s)B_XK_{X_1}^b(s))\zeta_3, \quad (74)\end{aligned}$$

$$(s\mathrm{Id}-\bar{A}_1)\zeta_3=P_{31}(s)\zeta_1. \quad (75)$$

In a similar way to what has been done above, we obtain (when $|s|>M_F$)

$$\begin{aligned}&(s\mathrm{Id}-\bar{A}_0)\zeta_2=w_0(s)B_X(K_{X_0}^u(s)+K_z^u(s)\bar{F}^{-1}(s)\\ &+K_{X_1}^u(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)\bar{F}^{-1}(s))\zeta_2\\ &+(R_w(s)+w_1(s)B_XK_{X_0}^b(s))C_0\zeta_2\\ &-(1-w_1(s))(\bar{A}_0-A_0), \quad (76)\end{aligned}$$

where $R_w(s)=[P_{23}(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)+P_{21}(s)+w_1(s)B_XK_{X_1}^b(s)(s\mathrm{Id}-\bar{A}_1)^{-1}P_{31}(s)+B_Xw_1(s)K_z^b(s)]\bar{F}^{-1}(s)$. Multiplying the right-hand side of equation (76) by $(s\mathrm{Id}-\bar{A}_0)^{-1}$, we obtain strictly proper functions (since $w_0(s)B_XK^u(s)$ is strictly proper). Thus, there exists $M_1>M_F$ (that does not depend on the choice of w_1 since $|w_1(s)|<1$ and $|1-w_1(s)|<1$) such that if $|s|>M_1$, $\zeta_2=0$. This in turns implies $\zeta=0$ which is a contradiction. The rest of the proof (when $|s|<M_1$) is a consequence of equation (46). □