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# A class of nonlinear state observers for an SIS system counting primo-infections\*

Marcel Fang<sup>1</sup> and Pierre-Alexandre Bliman <sup>1</sup> and Denis Efimov<sup>2</sup> and Rosane Ushirobira<sup>2</sup>

**Abstract**— **Observation and identification are important issues for the practical use of compartmental models of epidemic dynamics. They are usually evaluated based on the number of infected individuals (the prevalence) or the newly infected cases (the incidence). We are interested in a general question: may the measure of the number of primo-infected individuals and the prevalence improve state estimation? To study this question, we analyze in this paper a simple model of infection with waning immunity and, consequently, the possibility of reinfections. A class of nonlinear observers is built for this model, and tractable sufficient conditions on the gain matrices are established, ensuring asymptotic convergence of the state estimate towards its actual value. Numerical simulations illustrate the method.**

## I. INTRODUCTION

Since their introduction by Kermack & McKendrick [11], compartmental models have enjoyed long and well-established success in the field of mathematical modeling in epidemiology. Applied researches in mathematical epidemiology are usually interested in the direct analysis and simulation of possible epidemic dynamics using a given model. Aside from theoretical and simplifying hypotheses, such a utilization implies reasonable assumptions on the existence of reliable estimates of parameter values for the disease and the initial condition or current state of the population. Thus, the inverse problem of estimating states and parameters is also essential for modeling epidemics, like any other reproducible phenomenon. Borrowing concepts from the control theory, these two estimation issues are equivalent to the identification and state observation problems of the epidemiological model. Although identification and observation are well studied in many fields, there is little literature available in mathematical epidemiology on identification and even less for observation. One refers to [6], [7] on the issue of observation of

epidemiological models and to [8] as an example of a survey on identification in mathematical epidemiology.

At the same time, the phenomena of reinfection have been little studied to date; see references related to the subject in [4]. In these papers, counting the number of reinfections was especially considered in connection with the information that such data may provide. In particular, algebraic observability and identifiability were established for a susceptible-infected-susceptible (SIS) model counting primo-infections, based on the measurement of the number of infected and the number of primo-infected.

The present paper takes a step further and is devoted to the observation of this model. We focus more precisely on providing a class of non-linear observers for the SIS state-space system previously mentioned. Section II recalls the model under study (system (1) below), transformed by a change of coordinates into a Persidskii system form [16], [10], [9], for which a class of observers is proposed. The asymptotic convergence of the latter is established in Section III under adequate assumptions on the gain coefficients and using techniques inspired from [13], [14]. Numerical simulations are provided in Section IV before some concluding remarks in Section V. For the sake of space, all proofs are omitted and will be published in an upcoming research report.

*Notations:* The sets of real, nonnegative, and nonpositive real numbers are denoted by  $\mathbb{R}$ ,  $\mathbb{R}_+$ , and  $\mathbb{R}_-$  respectively. The notions of classes of functions  $\mathcal{K}$ ,  $\mathcal{K}_\infty$ , and  $\mathcal{KL}$  used here are standard [17].

## II. MODELING

Section II-A provides the model studied in this paper, namely system (1), borrowed from [4]. A useful state estimation representation is obtained in Section II-B (system (2), written in matrix form in (3)) by a suitable change of variables. This allows us to propose a class of observers for this system in Section II-C.

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### A. A SIS model with primo-infections

The SIS model of the present study has the following state-space representation:

$$\dot{S} = \mu - \beta SI - \mu S + \gamma I, \quad \dot{I} = \beta SI - (\mu + \gamma)I, \quad (1a)$$

$$\dot{S}_1 = \mu - \beta S_1 I - \mu S_1, \quad \dot{I}_1 = \beta S_1 I - (\mu + \gamma)I_1, \quad (1b)$$

$$y = (\alpha I \quad \alpha I_1)^\top. \quad (1c)$$

The four states  $S(t)$ ,  $I(t)$ ,  $S_1(t)$ ,  $I_1(t)$ ,  $t \in \mathbb{R}_+$  represent the number of susceptible, infected, never infected susceptible and primo-infected individuals, respectively. As testified by the first term in the first formula of (1a), the constant total population  $S + I$  has been normalized to 1. The coefficients  $\mu, \beta, \gamma$  describe the natural mortality rate, the contact rate, and the recovery rate. Last, the output  $y$  describes the measurement of a given portion  $\alpha$  of the infected and primo-infected individuals. We assume that this portion, which corresponds to the infected individuals detected by the Public health system (e.g., the symptomatic cases), is identical for the primo-infected and the global infected populations. We assume in the remaining of the paper that the basic reproduction number  $\mathcal{R}_0 := \frac{\beta}{\mu + \gamma}$  is greater than 1 so that convergence to the (unique, positive) endemic equilibrium occurs.

*Theorem 1:* The set  $\mathcal{S} := \{(S, I, S_1, I_1) \in [0, 1]^4 : S_1 \leq S \leq 1, I_1 \leq I \leq 1, S + I = 1\}$  is positively invariant, as well as its topological interior<sup>1</sup>  $\text{int } \mathcal{S}$ .

### B. Alternative state-space representation

To simplify the observer design, a change of variables is proposed. Setting  $s_1 = \ln(S_1)$ ,  $\iota = \ln(I)$ ,  $z_1 = S_1 + I_1$  and  $z = S + I$  yields the following equations:

$$\dot{z} = \mu - \mu z, \quad (2a)$$

$$\dot{\iota} = \beta z - \beta(e^\iota - 1) - (\beta + \mu + \gamma), \quad (2b)$$

$$\dot{z}_1 = \mu - (\mu + \gamma)z_1 + \gamma(e^{s_1} - 1) + \gamma, \quad (2c)$$

$$\dot{s}_1 = \mu(e^{-s_1} - 1) - \beta(e^\iota - 1) - \beta, \quad (2d)$$

$$y_1 = \alpha(e^\iota - 1) + \alpha, \quad y_2 = \alpha z_1 - \alpha(e^{s_1} - 1) - \alpha, \quad (2e)$$

$$y_3 = \iota, \quad y_4 = z. \quad (2f)$$

Notice that for convenience, we added the two components  $y_3 = \iota = \ln(I)$  and  $y_4 = z = S + I = 1$  to the output vector  $y(t) := (y_1(t) \quad y_2(t) \quad y_3(t) \quad y_4(t))^\top \in \mathbb{R}^4$ . The expression of  $y$  could be simplified by eliminating its dependence on  $\alpha$ , the latter was kept here for future work on adaptive estimation (with unknown values of parameters  $\alpha$ ,  $\beta$ , and  $\gamma$ ).

<sup>1</sup>in the sense of the usual topology of  $\mathbb{R}^4$ .

Introduce the following notations:

$$A_0 := \begin{pmatrix} -\mu & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & -(\mu + \gamma) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_1 := \begin{pmatrix} 0 & 0 & 0 \\ -\beta & 0 & 0 \\ 0 & \gamma & 0 \\ -\beta & 0 & -\mu \end{pmatrix}, \quad B := \begin{pmatrix} \mu \\ -(\beta + \mu + \gamma) \\ \mu + \gamma \\ -\beta \end{pmatrix},$$

$$C_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D := \begin{pmatrix} \alpha \\ -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\phi(\zeta) := \begin{pmatrix} f_+(\zeta_1) \\ f_+(\zeta_2) \\ f_-(\zeta_2) \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

and  $f_\epsilon(\zeta) := \epsilon(e^{\epsilon\zeta} - 1)$ ,  $\epsilon = \pm$ , for all  $\zeta \in \mathbb{R}$ .

Putting  $x(t) := (z(t) \quad \iota(t) \quad z_1(t) \quad s_1(t))^\top \in \mathbb{R}^4$ , we express system (2) under the following form:

$$\begin{cases} \dot{x} = A_0 x + A_1 \phi(Kx) + B, \\ y = C_0 x + C_1 \phi(Kx) + D. \end{cases} \quad (3)$$

### C. An observer class for system (1)

Defining now

$$C_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the class of observers for system (3) we propose to study in the present paper is given in formula (4), comprising two gain matrices  $L^1 \in \mathbb{R}^{4 \times 4}$ ,  $L^2 \in \mathbb{R}^{4 \times 3}$ . Notice that we added in (4) term  $C_2 \phi(K(x - \hat{x}))$  that does not appear in (3), but which will be useful for the subsequent search of Lyapunov functions. As  $\iota$  is observed, the term  $C_2 \phi(K(x - \hat{x}))$  is accessible to computation.

One shows directly that with the previous definition, the observation error  $e := x - \hat{x}$  behaves according to equation (5).

Before proceeding to the study of the properties of the class of observers (4), we proceed to write the observer equations in the natural state coordinates  $(\hat{S} \quad \hat{I} \quad \hat{S}_1 \quad \hat{I}_1)$ . For this, let  $\Psi : [0, 1] \times (0, 1] \times [0, 1] \times (0, 1] \rightarrow [0, 1] \times \mathbb{R}_- \times [0, 1] \times \mathbb{R}_-$  be such that  $\Psi((S \quad I \quad S_1 \quad I_1)^\top) = x = (z \quad \iota \quad z_1 \quad s_1)^\top$ . Then  $(S \quad I \quad S_1 \quad I_1)^\top =$

$$\dot{\hat{x}} = A_0\hat{x} + A_1\phi(K\hat{x}) + B + L^1(y - C_0\hat{x} - C_1\phi(K\hat{x}) - D) + L^2C_2\phi(K(x - \hat{x})), \quad (4)$$

$$\dot{e} = (A_0 - L^1C_0)e + (A_1 - L^1C_1)(\phi(Kx) - \phi(K(x - e))) - L^2C_2\phi(Ke), \quad (5)$$

$$\begin{pmatrix} \dot{\hat{S}} \\ \dot{\hat{I}} \\ \dot{\hat{S}}_1 \\ \dot{\hat{I}}_1 \end{pmatrix} = \begin{pmatrix} \mu - \beta\hat{S}\hat{I} - \mu\hat{S} + \gamma\hat{I} \\ \beta\hat{S}\hat{I} - (\mu + \gamma)\hat{I} \\ \mu - \beta\hat{S}_1\hat{I} - \mu\hat{S}_1 \\ \beta\hat{S}_1\hat{I} - (\mu + \gamma)\hat{I}_1 \end{pmatrix} + \begin{pmatrix} 1 & -\hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_1 \\ 0 & 0 & 1 & -\hat{S}_1 \end{pmatrix} \left( L^1(y - (\alpha\hat{I} \quad \alpha\hat{I}_1 \quad \ln\hat{I} \quad \hat{S} + \hat{I})^\top) + L^2 \left( \frac{I}{\hat{I}} - 1 \quad 0 \quad 0 \right)^\top \right). \quad (6)$$

$\Psi^{-1}(x) = (z - e^t \quad e^t \quad e^{s_1} \quad z_1 - e^{s_1})^\top$ , and the corresponding Jacobian matrix is

$$J_{\Psi^{-1}}(x) = \begin{pmatrix} 1 & -I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & S_1 \\ 0 & 0 & 1 & -S_1 \end{pmatrix}.$$

One then has

$$\begin{pmatrix} \dot{\hat{S}} \\ \dot{\hat{I}} \\ \dot{\hat{S}}_1 \\ \dot{\hat{I}}_1 \end{pmatrix} = \begin{pmatrix} 1 & -\hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_1 \\ 0 & 0 & 1 & -\hat{S}_1 \end{pmatrix} \dot{\hat{x}},$$

which after computations yields (6). Under this form, the nonlinear character of the observer is evident, due to the presence of terms  $\ln\hat{I}$  and  $\frac{I-\hat{I}}{\hat{I}}$ .

We now study the behavior of the coupled system (3), (5).

### III. OBSERVER CONVERGENCE

We first recall in Section III-A the notion of stability that will be considered here, namely of *state-independent uniform output stability (SIuOS)* [17]. Taking advantage of the positivity of the original system, a natural attempt is to look for copositive Lyapunov functions, see *e.g.* [16], [12]. This is done in Section III-B, yielding a first sufficient condition for asymptotic convergence of the observer (Theorem 3). In preparation for the forthcoming result, some sector estimates for the nonlinearities appearing in the system are provided in Section III-C. This allows us to give in Section III-D a second sufficient condition for asymptotic convergence of the observer (Theorem 4), based on the search for Lyapunov function equal to a quadratic form plus some Lur'e integral terms.

#### A. Stability notions

To analyze the stability of the equilibrium  $e = 0$ , we consider the dynamics of the coupled model (3), (5) with state variable  $(x(t) \quad e(t))^\top$ , taking also  $e(t)$  as output of the system. Inspired by [17, Definition 3.1], we set the following definitions.

*Definition 1: A system*

$$\dot{x}(t) = f(x(t)), \quad y(t) = h(x(t)) \quad (7)$$

with state  $x(t) \in \mathbb{R}^n$  and output  $y(t) \in \mathbb{R}^p$  ( $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $h: \mathbb{R}^n \rightarrow \mathbb{R}^p$  are continuously differentiable functions) is called *state-independent uniformly output stable (SIuOS)* if there exists a  $\mathcal{KL}$ -function  $\eta$  such that, denoting  $y(t, x_0)$  the output at time  $t$  of the system with initial state  $x_0 \in \mathbb{R}^n$ , one has  $|y(t, x_0)| \leq \eta(|h(x_0)|, t)$ , for any  $t \geq 0$ , any  $x_0 \in \mathbb{R}^n$ .

*Definition 2: For system (7), a smooth function  $V_1: \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called a SIuOS-Lyapunov function if there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $\alpha_3 \in \mathcal{K}$  such that*

$$\alpha_1(\|y\|) \leq V_1(x) \leq \alpha_2(\|y\|), \quad (8)$$

$$\nabla V_1(x)f(x) \leq -\alpha_3(\|y\|) \quad (9)$$

for all  $x \in \mathbb{R}^n$  and  $y = h(x)$ .

The following result, based on [17, Theorem 3.2], links the SIuOS property to SIuOS-Lyapunov functions.

*Theorem 2: A forward complete system (7) is SIuOS iff it admits a SIuOS-Lyapunov function.*

Consequently, if the system is SIuOS, the origin  $e = 0$  of the system will be globally attracting all the trajectories, and therefore  $\hat{x}$  satisfies the requirement for being an observer of  $x$ . The SIuOS property and its conditions are formulated globally for all  $x \in \mathbb{R}^n$ , and their local counterparts (or ones defined on conic invariant sets) can be easily deduced under suitable restrictions on the amplitudes of  $x$  and  $y$ .

#### B. Stability through copositive Lyapunov functions

The result stated below provides a first set of gain matrices ensuring SIuOS.

*Theorem 3: Assume  $L_{4,2}^1 < 0 < L_{2,1}^1$  and all other coefficients of the matrices  $L^1, L^2$  zero. Then system (3), (5) is SIuOS with respect to the output  $e$ .*

#### C. Sector estimates

As preparation for the search of Lyapunov functions with Lur'e terms, investigated below in Section III-D, we here provide sector estimates on the nonlinearities. Notice first that the values of  $f_\varepsilon$  always have

the sign of their argument. Furthermore, the following result holds, which provides sector estimates for the nonlinearities in the dynamics of the observer.

*Lemma 1: For any trajectory of system (3), (5), if there is  $\varepsilon' > 0$  such that  $\|e(t)\| < \varepsilon'$  for all  $t \geq 0$ , then there exist  $0 \leq a_i \leq b_i$ ,  $i = 1, \dots, 9$  such that*

$$\begin{aligned} & (b_j e_2 - (\phi_j(K(x+e)) - \phi_j(Kx))) \\ & \quad \times (\phi_j(K(x+e)) - \phi_j(Kx) - a_j e_2) \geq 0, \\ & (b_{j+3} \phi_j(Ke) - (\phi_j(K(x+e)) - \phi_j(Kx))) \\ & \quad \times (\phi_j(K(x+e)) - \phi_j(Kx) - a_{j+3} \phi_j(Ke)) \geq 0, \\ & (b_{j+6} e_2 - \phi_j(Ke)) \times (\phi_j(Ke) - a_{j+6} e_2) \geq 0, \end{aligned}$$

for any  $j = 1, 2, 3$ .

One checks with no difficulty that the above sector inequalities can be rewritten in matrix form as

$$v_1^\top \begin{pmatrix} -E^\top \Lambda_1 E & F \Lambda_1 \\ \Lambda_1^\top F^\top & -\Lambda_1 \end{pmatrix} v_1 \geq 0, \quad (10a)$$

$$v_2^\top \begin{pmatrix} -\Lambda_2 & \Lambda_2 H \\ \Lambda_2 H & -\Lambda_2 G \end{pmatrix} v_2 \geq 0, \quad (10b)$$

$$v_3^\top \begin{pmatrix} -N^\top \Lambda_3 N & M \Lambda_3 \\ \Lambda_3^\top M^\top & -\Lambda_3 \end{pmatrix} v_3 \geq 0, \quad (10c)$$

for

$$v_1 = \begin{pmatrix} Ke \\ \phi(K(x+e)) - \phi(Kx) \end{pmatrix}, \quad (11a)$$

$$v_2 = \begin{pmatrix} \phi(K(x+e)) - \phi(Kx) \\ \phi(Ke) \end{pmatrix}, \quad (11b)$$

$$v_3 = \begin{pmatrix} Ke \\ \phi(Ke) \end{pmatrix}, \quad (11c)$$

where  $\Lambda_j := \text{diag}\{\lambda_{3j-2}, \lambda_{3j-1}, \lambda_{3j}\}$  for  $j = 1, 2, 3$ ,  $G := \text{diag}\{a_4 b_4, a_5 b_5, a_6 b_6\}$ ,  $H := \frac{1}{2} \text{diag}\{a_4 + b_4, a_5 + b_5, a_6 + b_6\}$ , and

$$\begin{aligned} E &= \begin{pmatrix} (a_1 b_1)^{1/2} & 0 \\ 0 & (a_2 b_2)^{1/2} \\ 0 & (a_3 b_3)^{1/2} \end{pmatrix}, \\ F &= \frac{1}{2} \begin{pmatrix} a_1 + b_1 & 0 & 0 \\ 0 & a_2 + b_2 & a_3 + b_3 \end{pmatrix}, \\ N &= \begin{pmatrix} (a_7 b_7)^{1/2} & 0 \\ 0 & (a_8 b_8)^{1/2} \\ 0 & (a_9 b_9)^{1/2} \end{pmatrix}, \\ M &= \frac{1}{2} \begin{pmatrix} a_7 + b_7 & 0 & 0 \\ 0 & a_8 + b_8 & a_9 + b_9 \end{pmatrix}. \end{aligned}$$

#### D. Stability through quadratic Lyapunov functions with Lur'e terms

Let us now study SIUOS of (3), (5). For a positive linear system, the existence of copositive Lyapunov function is equivalent to the existence of quadratic Lyapunov function represented by a diagonal matrix [5]. For the nonlinear system (5), searching for a

Lyapunov candidate function with Lur'e components is a priori a more powerful method [1], [18], so consider the Lyapunov candidate function

$$\begin{aligned} V_2(e) &= e^\top P e + 2 \left( \gamma_1 \int_0^{e_2} f_+(z) dz \right. \\ & \quad \left. + \gamma_2 \int_0^{e_4} f_+(z) dz + \gamma_3 \int_0^{e_4} f_-(z) dz \right), \end{aligned}$$

parameterized by  $P \in \mathbb{R}^{4 \times 4}$ ,  $P = P^\top$ , and the scalars  $\gamma_1, \gamma_2, \gamma_3 \geq 0$ . Consider

$$K_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and let  $\Gamma := \text{diag}(\gamma_1, \gamma_2, \gamma_3)$ . If one assumes that  $P + K_1^\top \Gamma K_1 > 0$  and  $P \geq 0$ , then  $V_2$  is positive definite by Finsler's lemma [3]. With  $v(t) := (e(t)^\top \quad \phi(Kx(t))^\top - \phi(K(x(t) - e(t)))^\top \quad \phi(Ke(t))^\top)^\top$ , we obtain  $\frac{d}{dt} V_2(e) = v^\top Q v$  along the trajectories, where  $Q$  is the  $10 \times 10$  symmetric matrix, defined by blocks in (12).

Defining now

$$w = \begin{pmatrix} Ke \\ \phi(K(x+e)) - \phi(Kx) \\ \phi(Ke) \end{pmatrix}$$

the matrix inequalities (10)-(11) imply the inequality  $w^\top R w \geq 0$ , with  $R = R^\top \in \mathbb{R}^{8 \times 8}$  defined below :

$$R := \begin{pmatrix} -(E^\top \Lambda_1 E + N^\top \Lambda_3 N) & F \Lambda_1 & M \Lambda_3 \\ \Lambda_1 F^\top & -(\Lambda_1 + \Lambda_2) & \Lambda_2 H \\ \Lambda_3 M^\top & \Lambda_2 H & -(\Lambda_2 G + \Lambda_3) \end{pmatrix}$$

One then has  $\dot{V}_2(e) \leq v^\top Q v + w^\top R w$  along the trajectories, where the last term is equal to

$$v^\top \begin{pmatrix} K & 0_{2 \times 3} & 0_{2 \times 3} \\ 0_{3 \times 4} & I_3 & 0_{3 \times 3} \\ 0_{3 \times 4} & 0_{3 \times 3} & I_3 \end{pmatrix}^\top R \begin{pmatrix} K & 0_{2 \times 3} & 0_{2 \times 3} \\ 0_{3 \times 4} & I_3 & 0_{3 \times 3} \\ 0_{3 \times 4} & 0_{3 \times 3} & I_3 \end{pmatrix} v.$$

We have demonstrated the following result.

*Theorem 4: Assume the gain matrices  $L^1, L^2$  are such that there exist a symmetric matrix  $P \in \mathbb{R}^{4 \times 4}$  and diagonal matrices  $\Gamma, \Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{R}^{3 \times 3}$  such that*

$$P \geq 0, P + K_1^\top \Gamma K_1 > 0, \quad \Lambda_1, \Lambda_2, \Lambda_3 \geq 0,$$

and for  $a_i, b_i$ ,  $i = 1, \dots, 9$  verifying the inequalities (10)-(11), the matrix inequality (13) is satisfied, then system (3), (5) is locally SIUOS-stable.

By locality, we mean that the initial conditions are chosen such that  $I(t) > \varepsilon_I$ ,  $S_1(t) > \varepsilon_{S_1}$  and  $\|e(t)\| < \varepsilon'$  for all  $t \geq 0$ . For the latter restriction on  $e$ , due to the substantiated properties of  $V$ , it is enough to take  $\delta' > 0$  such that  $\|e(0)\| < \delta'$  implies  $\|e(t)\| < \varepsilon'$  for all  $t \geq 0$  in such a case.



$$Q := \begin{pmatrix} (A_0 - L^1 C_0)^\top P + P(A_0 - L^1 C_0) & P(A_1 - LC_1) & (A_0 - L^1 C_0)^\top K_1^\top \Gamma - PL^2 C_2 \\ (A_1 - L^1 C_1)^\top P & 0 & (A_1 - L^1 C_1)^\top K_1^\top \Gamma \\ \Gamma K_1 (A_0 - L^1 C_0) - (L^2 C_2)^\top P & \Gamma K_1 (A_1 - L^1 C_1) & -\Gamma K_1 L^2 C_2 - (\Gamma K_1 L^2 C_2)^\top \end{pmatrix}, \quad (12)$$

$$Q + \begin{pmatrix} -K^\top (E^\top \Lambda_1 E + N^\top \Lambda_3 N) K & K^\top F \Lambda_1 & K^\top M \Lambda_3 \\ \Lambda_1 F^\top K & -(\Lambda_1 + \Lambda_2) & \Lambda_2 H \\ \Lambda_3 M^\top K & \Lambda_2 H & -(\Lambda_2 G + \Lambda_3) \end{pmatrix} < 0. \quad (13)$$

#### IV. NUMERICAL SIMULATIONS

As an application of the previous results, we now present numerical simulations of system (1) and state observer (6). For predefined gain matrices  $L^1$ ,  $L^2$  and values of  $\varepsilon_{S_1}, \varepsilon_I, \varepsilon'$  depending on the initial conditions, the LMIs of Theorem 4 are first verified using SDPT3 solvers and YALMIP in Matlab environment. Once existence of  $P, \Gamma, \Lambda_1, \Lambda_2, \Lambda_3$  is guaranteed numerically, the observer is admissible for the chosen value of the gain and we proceed to the numerical simulations of state observer dynamics. The figures correspond to parameter values set to  $\gamma^{-1} = 24$  days,  $\mu^{-1} = 60$  days,  $\alpha = 0.5$ ,  $\beta = 0.0972$  days $^{-1}$ . The gain matrices are set to  $L_1 = \frac{1}{2} C_0^\top, L_2 = \frac{1}{2} (0_{3 \times 1} \quad C_2^\top)^\top$ .

The estimates of  $S, I, S_1, I_1$  are shown respectively in Fig. 1-4 and compared with the estimates of a zero-gain observer likewise initialized. The convergence of the estimates  $\hat{S}, \hat{I}$  to the true states  $S, I$  of the system is almost instantaneous, the convergence of  $\hat{S}_1, \hat{I}_1$  is fast, and the error becomes negligible 50 days after the beginning of the measurements. The corresponding error norms are represented in Fig. 5 in semi-log scale, showing much faster convergence for the proposed observer.

#### V. CONCLUSION

A class of non-linear observers for an SIS system counting primo-infections has been proposed. first shown for a restricted set of gain matrices through copositive Lyapunov functions, and secondly, a tractable sufficient LMI condition was established without restriction on the choice of gain matrices. Simulations were done with the fulfillment of the LMI condition, which has been verified successfully beforehand using YALMIP and LMI solvers in Matlab. As illustrated, the convergence of the observer is exponentially faster than the convergence of the system to the global equilibrium. The Persidskii form of the system and the correlative use of Lur'e terms, is an original feature of the approach. Future work will make comparisons with other methods based on the search for a quadratic Lyapunov functions [15], possibly using triangular form [2] of (1).

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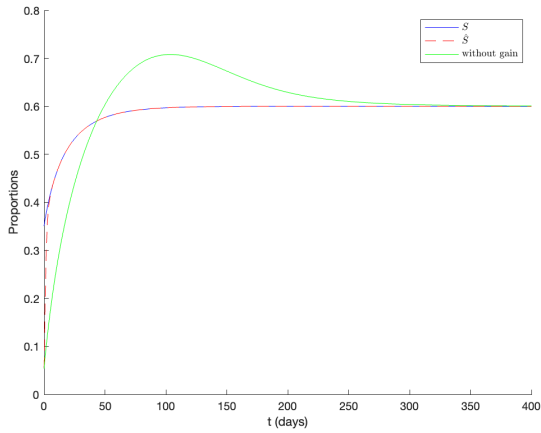


Fig. 1. Estimation  $\hat{S}$  of the susceptible population  $S$

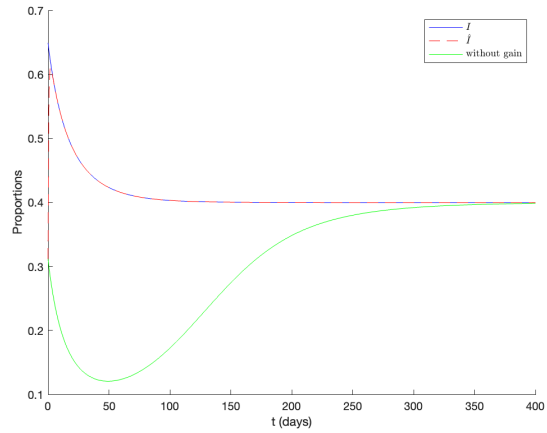


Fig. 2. Estimation  $\hat{I}$  of the infected population  $I$

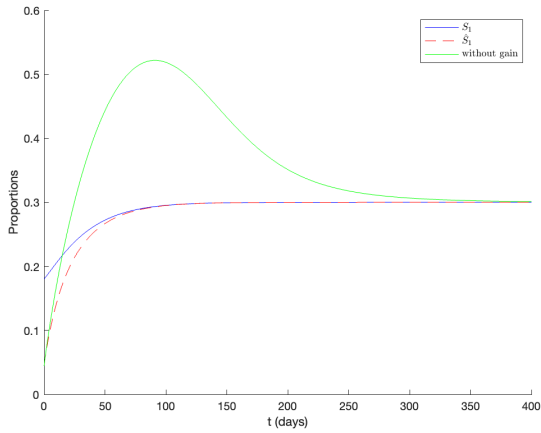


Fig. 3. Estimation  $\hat{S}_1$  of the never infected populations  $S_1$

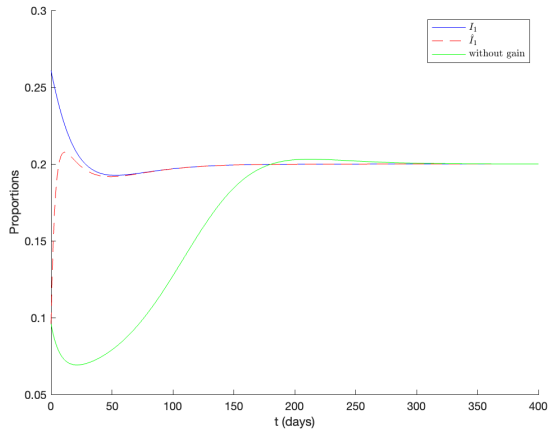


Fig. 4. Estimation  $\hat{I}_1$  of the primo-infected populations  $I_1$

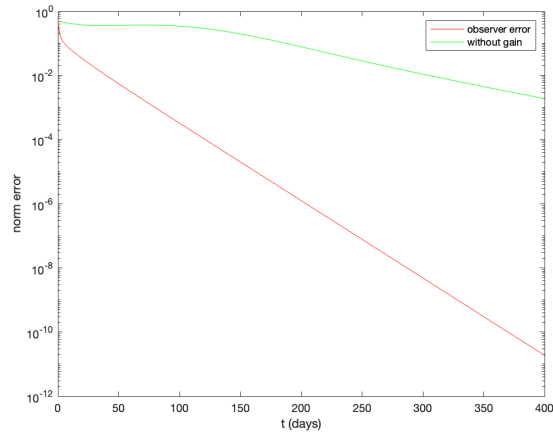


Fig. 5. Comparison of the estimation errors with (in red), and without (in green) observer.

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