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# Ergodicity for the Randomly Forced Navier–Stokes System in a Two-Dimensional Unbounded Domain

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# Ergodicity for the randomly forced Navier–Stokes system in a two-dimensional unbounded domain

Vahagn Nersesyan\*

March 4, 2019

## Abstract

The ergodic properties of the randomly forced Navier–Stokes system have been extensively studied in the literature during the last two decades. The problem has always been considered in bounded domains, in order to have, for example, suitable spectral properties for the Stokes operator, to ensure some compactness properties for the resolving operator of the system and the associated functional spaces, etc. In the present paper, we consider the Navier–Stokes system in an unbounded domain satisfying the Poincaré inequality. Assuming that the system is perturbed by a bounded non-degenerate noise, we establish uniqueness of stationary measure and exponential mixing in the dual-Lipschitz metric. The proof is carried out by developing the controllability approach of the papers [Shi19, KNS18] and using the asymptotic compactness of the dynamics.

**AMS subject classifications:** 35Q30, 37A25, 93B05.

**Keywords:** Navier–Stokes system, unbounded domain, exponential mixing, controllability, asymptotic compactness.

## Contents

<b>0</b>	<b>Introduction</b>	<b>2</b>
<b>1</b>	<b>Abstract criterion</b>	<b>6</b>
<b>2</b>	<b>Proof of the Main Theorem</b>	<b>10</b>
2.1	Preliminaries . . . . .	10
2.2	Regularity condition . . . . .	11
2.3	Asymptotic compactness . . . . .	12
2.4	Approximate controllability . . . . .	14

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<b>3 Appendix</b>	<b>15</b>
3.1 Proof of Proposition 2.1	15
3.2 Proof of Proposition 2.2	15
3.3 Proof of Lemma 2.3	16

## 0 Introduction

The ergodicity of randomly forced 2D Navier–Stokes (NS) system has been widely studied in the literature in the case of bounded domains (see the papers [FM95, KS00, EMS01, BKL02] for the first results and the book [KS12] and the reviews [Fla08, Deb13] for a detailed discussion of different methods and for further references). This paper is concerned with the ergodic behaviour of the NS system in an *unbounded* domain  $D$  in  $\mathbb{R}^2$  with smooth boundary  $\partial D$ :

$$\partial_t u - \nu \Delta u + \langle u, \nabla \rangle u + \nabla p = \eta(t, x), \quad x \in D, \quad (0.1)$$

$$\operatorname{div} u = 0, \quad u|_{\partial D} = 0, \quad (0.2)$$

$$u(0) = u_0. \quad (0.3)$$

Here  $\nu > 0$  is the kinematic viscosity of the fluid,  $u = (u_1(t, x), u_2(t, x))$  is the velocity field,  $p = p(t, x)$  is the pressure, and  $\eta$  is an external random force. To have a suitable dissipativity property for solutions, we assume that  $D$  is a *Poincaré domain*,<sup>1</sup> i.e., there is a number  $\lambda_1 > 0$  such that

$$\int_D |v|^2 dx \leq \lambda_1^{-1} \int_D |\nabla v|^2 dx, \quad v \in C_0^\infty(D, \mathbb{R}^2). \quad (0.4)$$

The random force  $\eta$  is a process of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1, k)}(t) \eta_k(t - k + 1, x), \quad t \geq 0, \quad x \in D, \quad (0.5)$$

where  $\mathbb{I}_{[k-1, k)}$  is the indicator function of the interval  $[k-1, k)$  and  $\{\eta_k\}$  is a sequence of independent and identically distributed (i.i.d.) random variables in the space<sup>2</sup>  $E := L^2([0, 1], H)$ . Moreover, the law of  $\eta_k$  is assumed to be *decomposable* in the following sense.

**Decomposability.** *There is an orthonormal basis  $\{e_j\}$  in  $E$  such that*

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j \quad (0.6)$$

for some real-valued independent random variables  $\xi_{jk}$  verifying  $|\xi_{jk}| \leq 1$  and some positive numbers  $b_j$  such that  $\sum_{j=1}^{\infty} b_j^2 < \infty$ . The law of the random variable  $\xi_{jk}$  is absolutely continuous with respect to the Lebesgue measure and the corresponding density  $\rho_j$  is  $C^1$ -smooth and  $\rho_j(0) > 0$  for all  $j \geq 1$ .

<sup>1</sup>E.g., we can assume that  $D$  is bounded in some direction  $a \in \mathbb{R}_*^2$ , i.e.,  $\sup_{x \in D} |\langle a, x \rangle| < \infty$ .

<sup>2</sup>We denote by  $H$  the usual functional space for the NS system defined by (0.9).

The restriction to integer times of the velocity field  $u_t$  defines a family of Markov processes  $(u_k, \mathbb{P}_u)$  parametrised by the initial condition  $u_0 = u \in H$ . The associated Markov operators are denoted by  $\mathfrak{P}_k$  and  $\mathfrak{P}_k^*$ . Recall that a measure  $\mu \in \mathcal{P}(H)$  is stationary for the family  $(u_k, \mathbb{P}_u)$  if  $\mathfrak{P}_1^* \mu = \mu$ . In this paper, we prove the following result.

**Main Theorem.** *Under the above assumptions, the family  $(u_k, \mathbb{P}_u)$  has a unique stationary measure  $\mu \in \mathcal{P}(H)$ . Moreover, it is exponentially mixing in the following sense: for any compact set  $\mathcal{H} \subset H$ , there are numbers  $C > 0$  and  $c > 0$  such that*

$$\|\mathfrak{P}_k^* \lambda - \mu\|_{L(H)}^* \leq C e^{-ck}, \quad k \geq 0 \quad (0.7)$$

for any initial measure  $\lambda \in \mathcal{P}(H)$  with  $\text{supp } \lambda \subset \mathcal{H}$ . Here  $\|\cdot\|_{L(H)}^*$  is the dual-Lipschitz metric defined by (0.11).

To the best of our knowledge, this is the first result that establishes uniqueness of stationary measure and exponential mixing for the NS system in an unbounded domain. In this unbounded setting there are at least two additional difficulties compared to the case of a bounded domain. First, the resolving operator of the equation is not compact and does not have a compact attracting set. The second and more important difficulty is related to the presence of a continuous component in the spectrum of the Stokes operator. Let us note that the existence of a stationary measure for the NS system perturbed by the random force (0.5) can be established by combining the *Bogolyubov–Krylov argument* and the *asymptotic compactness* of the dynamics. When the driving force is a white noise, existence results are obtained in [EH01], for a real Ginzburg–Landau equation, in [Rou02], for a complex Ginzburg–Landau equation, in [BL06], for the NS system, and in [EKZ17], for a damped Schrödinger equation.

The uniqueness of stationary measure and mixing properties for PDEs in unbounded domains have been studied previously only for the Burgers equation. The case of inviscid equation on the real line has been considered in the paper [BCK14], under the assumption that the driving force is a space-time homogeneous Poisson point process. The proof is based on a combination of Lagrangian methods and first/last passage percolation theory. This result is generalised to the viscous case in the recent paper [BL19], where the perturbation is a space-time homogeneous random kick force. In both papers, the stationary measure is space translation invariant. In the case of the NS system perturbed by a *homogeneous* noise, the uniqueness of a space-time translation invariant measure (and in some cases also the existence) remains an *open problem*.

The proof of the Main Theorem uses a controllability approach of the papers [Shi15, Shi19], where exponential mixing is established for the NS system with a space-time and boundary *localised* forcing. In [KNS18, KNS19], methods of control theory are used to study a family of parabolic PDEs with a random perturbation that is *highly degenerate* in the Fourier space; see also the paper [KZ18], where a non-degenerate version of these results is presented. In [JNPS19], controllability is used to derive large deviations principle for the Lagrangian trajectories of the NS system.

The main novelty of the controllability argument we use here is that the deterministic operator is not supposed to be regularising to a space that is compactly embedded into the main phase space (see Theorem 1.1). That regularisation condition is replaced by two weaker properties: we assume that the nonlinear dynamics is asymptotically compact and the linearised operator can be decomposed into a sum of two operators one of which is *dissipative* and the other is *compact* (see (1.3)). It is proved that these weaker conditions are still sufficient for the exponential mixing. We make an essential use of the form of the nonlinearity to verify these properties for the NS system. The reader is referred to Section 1 for further discussion of the abstract controllability criterion and to Section 2 for the verification of the conditions of this criterion for the NS system.

Let us close this section with two remarks. Note that the convergence rate  $c$  in (0.7) depends on the initial compact  $\mathcal{H}$ . Indeed, this is related to the fact that the NS system in the unbounded case does not have a compact invariant attracting set. Without going into the details, let us mention that the existence of such attracting set (and a uniform convergence rate) can be recovered if we take initial condition and forcing in weighted Sobolev spaces as in [Bab92].

The second remark is about the NS system with the *Ekman damping*. By literally repeating the arguments of the proof of the Main Theorem, one can establish exponential mixing in the case of the whole space  $D = \mathbb{R}^2$  or arbitrary unbounded<sup>3</sup> domain  $D \subset \mathbb{R}^2$  with smooth boundary for the following system:

$$\begin{aligned} \partial_t u - \nu \Delta u + au + \langle u, \nabla \rangle u + \nabla p &= \eta(t, x), \\ \operatorname{div} u &= 0, \quad u|_{\partial D} = 0, \end{aligned}$$

where  $a > 0$  is the damping parameter. The damping ensures a dissipativity property for the solutions without any assumption on the domain. We shall not discuss the details of this generalisation in this paper.

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## Notation

Let  $D \subset \mathbb{R}^2$  be an unbounded Poincaré domain with smooth boundary. In this paper, we use the following functional spaces.

$C_0^\infty(D, \mathbb{R}^2)$  is the space of compactly supported smooth functions  $u : D \rightarrow \mathbb{R}^2$ , and

$$\mathcal{V} = \{u \in C_0^\infty(D, \mathbb{R}^2) : \operatorname{div} u = 0\}. \quad (0.8)$$

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<sup>3</sup>I.e., domain  $D$  which does not necessarily satisfy the Poincaré inequality.

$H^s(D, \mathbb{R}^2)$  and  $L^p(D, \mathbb{R}^2)$  are the Sobolev and Lebesgue spaces on  $D$ . We consider the NS system in the usual functional spaces:

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(D, \mathbb{R}^2), \quad (0.9)$$

$$V = \text{closure of } \mathcal{V} \text{ in } H^1(D, \mathbb{R}^2) \quad (0.10)$$

endowed with the scalar products

$$\langle u, v \rangle = \int_D u \cdot v \, dx, \quad \langle u, v \rangle_1 = \int_D (\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2) \, dx$$

and the corresponding norms  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$  and  $\|\cdot\|_1 = \sqrt{\langle \cdot, \cdot \rangle_1}$ . The Poincaré inequality (0.4) implies that  $\|\cdot\|_1$  is equivalent to the norm inherited from  $H^1(D, \mathbb{R}^2)$ . The dual of  $V$  with respect to the scalar product in  $H$  is denoted by  $V'$ .

Let  $X$  be a Polish space with metric  $d$ .

$\mathcal{B}(X)$  denotes the Borel  $\sigma$ -algebra on  $X$ .

$C_b(X)$  is the space of continuous functions  $g : X \rightarrow \mathbb{R}$  endowed with the sup-norm  $\|g\|_\infty$ . When  $X$  is compact, we write  $C(X)$ .

$L_b(X)$  is the space of functions  $g \in C_b(X)$  for which the following norm is finite:

$$\|g\|_{L(X)} = \|g\|_\infty + \sup_{u \neq v} \frac{|g(u) - g(v)|}{d(u, v)}.$$

$\mathcal{P}(X)$  is the set of Borel probability measures on  $X$  endowed with the metric

$$\|\mu_1 - \mu_2\|_{L(X)}^* = \sup_{\|g\|_{L(X)} \leq 1} |\langle g, \mu_1 \rangle - \langle g, \mu_2 \rangle|, \quad \mu_1, \mu_2 \in \mathcal{P}(X), \quad (0.11)$$

where  $\langle g, \mu \rangle = \int_X g(u) \mu(du)$ .

Let  $E$  be a Banach space endowed with a norm  $\|\cdot\|_E$ , and let  $J_T = [0, T]$ .

$L^p(J_T, E)$ ,  $1 \leq p < \infty$  is the space of measurable functions  $u : J_T \rightarrow E$  such that

$$\|u\|_{L^p(J_T, E)} = \left( \int_0^T \|u(s)\|_E^p \, ds \right)^{\frac{1}{p}} < \infty.$$

$C(J_T, E)$  is the space of continuous functions  $u : J_T \rightarrow E$  with the norm

$$\|u\|_{C(J_T, E)} = \sup_{t \in J_T} \|u(t)\|_E.$$

$\mathcal{L}(E, Y)$  is the space of bounded linear operators from  $E$  to another Banach space  $Y$ . We write  $\mathcal{L}(E)$  when  $E = Y$ .

$B_E(a, R)$  is the closed ball in  $E$  of radius  $R$  centered at  $a$ . We write  $B_E(R)$  when  $a = 0$ .

$\mathcal{D}(\eta)$  is the law of  $E$ -valued random variable  $\eta$ .

# 1 Abstract criterion

Let  $H$  be a separable Hilbert space, and  $E$  be a separable Banach space. In this section, we consider a *random dynamical system* of the form

$$u_k = S(u_{k-1}, \eta_k), \quad k \geq 1, \quad (1.1)$$

where  $S : H \times E \rightarrow H$  is a continuous mapping and  $\{\eta_k\}$  is a sequence of i.i.d. random variables in  $E$ . Let  $\mathcal{K} \subset E$  be the support of the law  $\ell := \mathcal{D}(\eta_k)$ . For any sequence  $\{\zeta_k\}$  in  $\mathcal{K}$ , let us denote by  $S_k(u; \zeta_1, \dots, \zeta_k)$  the trajectory of (1.1) corresponding to the initial condition

$$u_0 = u \in H \quad (1.2)$$

and the vectors  $\eta_i = \zeta_i$ ,  $i = 1, \dots, k$ . For any set  $\mathcal{H} \subset H$ , we define the *set of attainability in time  $k \geq 1$* :

$$\mathcal{A}_k(\mathcal{H}) := \{S_k(u; \zeta_1, \dots, \zeta_k) : u \in \mathcal{H}, \zeta_1, \dots, \zeta_k \in \mathcal{K}\}$$

and the *set of attainability in infinite time*:

$$\mathcal{A}(\mathcal{H}) := \overline{\bigcup_{k=1}^{\infty} \mathcal{A}_k(\mathcal{H})}^H.$$

The following five conditions are assumed to be satisfied for the mapping  $S$  and the measure  $\ell$ .

**(i) Regularity.** *The mapping  $S : H \times E \rightarrow H$  is twice continuously differentiable, and its derivatives are bounded on bounded subsets. Moreover, for any  $(u, \eta) \in H \times \mathcal{K}$ , the derivative  $(D_u S)(u, \eta)$  can be represented as*

$$(D_u S)(u, \eta) = \Psi_1 + \Psi_2(u, \eta), \quad (1.3)$$

where the operators<sup>4</sup>  $\Psi_1, \Psi_2(u, \eta) \in L(H)$  are such that

$$\|\Psi_1\|_{\mathcal{L}(H)} =: \varkappa < 1 \quad (1.4)$$

and  $\Psi_2(u, \eta)$  is compact.

**(ii) Asymptotic compactness.** *For any bounded sequence  $\{u_0^n\}$  in  $H$ , any integers  $l_n \geq 1$  such that  $l_n \rightarrow \infty$ , and any family  $\{\zeta_m^n : m, n \geq 1\} \subset \mathcal{K}$ , the sequence  $\{S_{l_n}(u_0^n; \zeta_1^n, \dots, \zeta_{l_n}^n)\}$  is precompact in  $H$ .*

**(iii) Approximate controllability to a point.** *There is a point  $\hat{u} \in H$  with the following property: for any  $\varepsilon > 0$  and any compact  $\mathcal{H}$  in  $H$ , there is an integer  $n \geq 1$  such that, for any initial point  $u \in \mathcal{H}$ , there are vectors  $\zeta_1, \dots, \zeta_n \in \mathcal{K}$  satisfying*

$$\|S_n(u; \zeta_1, \dots, \zeta_n) - \hat{u}\| \leq \varepsilon. \quad (1.5)$$

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<sup>4</sup> $\Psi_1$  does not depend on  $(u, \eta)$ .

(iv) **Approximate controllability of the linearisation.** For any  $u \in H$  and  $\eta \in \mathcal{K}$ , the image of the linear mapping  $(D_\eta S)(u, \eta) : E \rightarrow H$  is dense in  $H$ .

(v) **Decomposability.** The set  $\mathcal{K}$  is compact in  $E$ . Moreover, there are sequences of closed subspaces  $\{F_n\}$  and  $\{G_n\}$  in  $E$  satisfying the following properties:

- $F_n$  are finite-dimensional,  $F_n \subset F_{n+1}$  for any  $n \geq 1$ , and  $E = \overline{\cup_n F_n}$ .
- $E$  is the direct sum of the spaces  $F_n$  and  $G_n$ , and the norms of the corresponding projections  $P_n$  and  $Q_n$  are bounded in  $n \geq 1$ .
- $\ell$  is the product of projections  $P_{n*}\ell$  and  $Q_{n*}\ell$  for any  $n \geq 1$ . Moreover, the measures  $P_{n*}\ell$  have  $C^1$ -smooth densities with respect to the Lebesgue measure on  $F_n$ .

System (1.1) defines a family of Markov processes  $(u_k, \mathbb{P}_u)$  in  $H$  parametrised by the initial condition (1.2). Let  $\mathfrak{P}_k : C_b(H) \rightarrow C_b(H)$  and  $\mathfrak{P}_k^* : \mathcal{P}(H) \rightarrow \mathcal{P}(H)$  be the associated Markov operators.

**Theorem 1.1.** *Under Conditions (i)-(v), the family  $(u_k, \mathbb{P}_u)$  has a unique stationary measure  $\mu \in \mathcal{P}(H)$ . Moreover, for any compact set  $\mathcal{H}$  in  $H$ , there are numbers  $C > 0$  and  $c > 0$  such that*

$$\|\mathfrak{P}_k^* \lambda - \mu\|_{L(H)}^* \leq C e^{-ck}, \quad k \geq 1 \quad (1.6)$$

for any initial measure  $\lambda \in \mathcal{P}(H)$  with  $\text{supp } \lambda \subset \mathcal{H}$ .

See the papers [Shi15, Shi19, KNS18, KNS19, KZ18, JNPS19] for related abstract criteria for uniqueness of stationary measure and mixing. The formulation of Theorem 1.1 is close to the results in [KNS19, JNPS19], but the proof is based on a theorem obtained in [Shi19]. There are two main differences in our formulation. First, in Condition (i), we do not suppose that the mapping  $S$  (or its linearisation) takes values in a space that is compactly embedded into  $H$ . Instead, we assume that  $(D_u S)(u, \eta)$  is a sum of *dissipative* and *compact* operators. The second difference is Condition (ii), which allows to recover some compactness properties<sup>5</sup> for the dynamics. These two new conditions make Theorem 1.1 applicable to the NS system in unbounded domains. Moreover, this theorem can be applied to dissipative PDEs without parabolic regularisation (this will be addressed in a subsequent publication).

As in this paper the random perturbation is *non-degenerate*, the image of the mapping  $(D_\eta S)(u, \eta)$  is assumed to be dense for any  $\eta$  in  $\mathcal{K}$ ; degenerate versions of this criterion can also be envisaged (cf. [KNS18, KNS19]).

*Proof of Theorem 1.1. Step 1: Existence of stationary measure.* The set  $X := \mathcal{A}(\mathcal{H})$  is compact in  $H$ . Indeed, it suffices to show that any sequence  $\{v_n\}$  of the form

$$v_n = S_{l_n}(u_0^n; \zeta_1^n, \dots, \zeta_{l_n}^n)$$

<sup>5</sup>The asymptotic compactness is a well-known property in the study of the attractors for deterministic PDEs (e.g., see [Lad91, Ros98]).



with some  $u_0^n \in \mathcal{H}$ ,  $\zeta_1^n, \dots, \zeta_{l_n}^n \in \mathcal{K}$ , and  $l_n \geq 1$ , is precompact in  $H$ . If the sequence  $\{l_n\}$  is bounded, then  $v_n \in \mathcal{A}_m(\mathcal{H})$  for all  $n \geq 1$ , where  $m = \max\{l_n\}$ . The fact that  $\mathcal{A}_m(\mathcal{H})$  is compact (as image of a compact set by a continuous mapping) implies that  $\{v_n\}$  is precompact. In the case  $l_n \rightarrow \infty$ , the conclusion follows from Condition (ii).

The compactness of  $X$ , combined with the invariance property  $S(X \times \mathcal{K}) \subset X$  and the usual Bogolyubov–Krylov argument (e.g., see Section 2.5 in [KS12]), implies the existence of a stationary measure  $\mu \in \mathcal{P}(X)$ .

*Step 2: Limit (1.6).* According to Theorem 1.1 in [Shi19], limit (1.6) will be established if we verify the following property.

**Local stabilisation.** Let  $\mathbf{D}_\delta := \{(u, u') \in X \times X : \|u - u'\| \leq \delta\}$ . For any  $R > 0$  and any compact  $\mathcal{K} \subset E$ , there is a finite-dimensional subspace  $\mathcal{E} \subset E$ , and a continuous mapping

$$\Phi : \mathbf{D}_\delta \times B_E(R) \rightarrow \mathcal{E}, \quad (u, u', \eta) \mapsto \eta',$$

which is continuously differentiable in  $\eta$  and satisfies the inequalities

$$\sup_{\eta \in B_E(R)} (\|\Phi(u, u', \eta)\|_E + \|D_\eta \Phi(u, u', \eta)\|_{\mathcal{L}(E)}) \leq C \|u - u'\|, \quad (1.7)$$

$$\sup_{\eta \in \mathcal{K}} \|S(u, \eta) - S(u', \eta + \Phi(u, u', \eta))\| \leq q \|u - u'\|, \quad (u, u') \in \mathbf{D}_\delta \quad (1.8)$$

for some positive constants  $C$ ,  $\delta$ , and  $q < 1$ .

Let us show that Conditions (i), (iv), and (v) imply this local stabilisation property. We use a construction of approximate right inverse for linear operators from Section 2.2 in [KNS18]. For any  $u \in X$  and  $\eta \in E$ , let  $A(u, \eta) : E \rightarrow H$  be given by  $A(u, \eta) := (D_\eta S)(u, \eta)$ . Then  $G(u, \eta) := A(u, \eta) A(u, \eta)^* : H \rightarrow H$  is non-negative self-adjoint operator and  $\text{Im}(G(u, \eta))$  is dense in  $H$  by Condition (iv). Thus  $(G(u, \eta) + \gamma I)^{-1}$  is well defined for any  $\gamma > 0$ , and we have the limit

$$G(u, \eta)(G(u, \eta) + \gamma I)^{-1} f \rightarrow f \quad \text{as } \gamma \rightarrow 0^+, \quad f \in H, \quad (1.9)$$

by Lemma 2.4 in [KNS18]. This shows that  $A(u, \eta)^*(G(u, \eta) + \gamma I)^{-1}$  is an approximate right inverse for  $A(u, \eta)$ . We truncate it to obtain an operator with finite-dimensional image:

$$\mathcal{R}_{M, \gamma}(u, \eta) := P_M A(u, \eta)^*(G(u, \eta) + \gamma I)^{-1},$$

where  $P_M$  is a projection as in Condition (v) and  $M \geq 1$  and  $\gamma > 0$  are parameters that will be chosen later. It is straightforward to see that

$$\|\mathcal{R}_{M, \gamma}(u, \eta)\|_{\mathcal{L}(H, E)} + \|(D_\eta \mathcal{R}_{M, \gamma})(u, \eta)\|_{\mathcal{L}(H \times E, E)} \leq C_1(R, M, \gamma) \quad (1.10)$$

for any  $u \in X$  and  $\eta \in B_E(R)$ , where the constant  $C_1(R, M, \gamma) > 0$  does not depend on  $(u, \eta)$ . By the Taylor formula, for any  $u, u' \in X$  and  $\eta, \eta' \in B_E(R)$ , we have

$$S(u', \eta') - S(u, \eta) = (D_u S)(u, \eta)(u' - u) + (D_\eta S)(u, \eta)(\eta' - \eta) + r(u, u', \eta, \eta') \quad (1.11)$$

where

$$\|r(u, u', \eta, \eta')\| \leq C_2(R) (\|u - u'\|^2 + \|\eta - \eta'\|_E^2). \quad (1.12)$$

The mapping  $\Phi$  is defined by

$$\Phi(u, u', \eta) := -\mathcal{R}_{M,\gamma}(u, \eta)\Psi_2(u, \eta)(u' - u).$$

Then (1.7) is verified due to (1.10) and

$$C_3 := \sup_{(u,\eta) \in X \times B_E(R)} (\|\Psi_2(u, \eta)\|_{\mathcal{L}(H)} + \|D_\eta \Psi_2(u, \eta)\|_{\mathcal{L}(H \times E, H)}) < \infty. \quad (1.13)$$

The fact that  $C_3$  is finite follows from the boundedness on  $X \times B_E(R)$  of the norms of the derivatives of  $S(u, \eta)$  and the representation (1.3). Assume that, for any  $\varepsilon > 0$ , we are able to find numbers  $M \geq 1$  and  $\gamma > 0$  such that

$$\|(D_\eta S)(u, \eta)\mathcal{R}_{M,\gamma}(u, \eta)\Psi_2(u, \eta)g - \Psi_2(u, \eta)g\| \leq \varepsilon \|g\| \quad (1.14)$$

for any  $u \in X$ ,  $\eta \in \mathcal{K}$ , and  $g \in H$ . Then combining (1.3), (1.4), and (1.11)-(1.14), we obtain

$$\begin{aligned} \|S(u, \eta) - S(u', \eta + \Phi(u, u', \eta))\| \\ \leq \varkappa \|u' - u\| + \varepsilon \|u' - u\| + \|r(u, u', \eta, \eta + \Phi(u, u', \eta))\| \\ \leq (\varkappa + \varepsilon + C_4(\varepsilon, R)) \|u' - u\| \leq q \|u' - u\| \end{aligned}$$

for any  $(u, u') \in \mathbf{D}_\delta$  and  $\eta \in \mathcal{K}$ , where  $\varepsilon > 0$  and  $\delta > 0$  are sufficiently small and  $q < 1$ . This completes the proof of the local stabilisation and limit (1.6).

Let us prove inequality (1.14). By (1.9), the continuous dependence of the operators  $(D_\eta S)(u, \eta)$  and  $\Psi_2(u, \eta)$  on  $(u, \eta)$ , and a simple compactness argument, we find a large integer  $M \geq 1$  and a small number  $\gamma > 0$  such that<sup>6</sup>

$$\|(D_\eta S)(u, \eta)\mathcal{R}_{M,\gamma}(u, \eta)f - f\| \leq \varepsilon$$

for any  $u \in X$ ,  $\eta \in \mathcal{K}$ , and  $f \in \Psi_2(u, \eta)(B_H(1))$ . Then inequality (1.14) follows by homogeneity.

*Step 3: Uniqueness of stationary measure.* To complete the proof, it remains to show the uniqueness of stationary measure in  $\mathcal{P}(H)$ . Assume that  $\mu$  is the stationary measure supported<sup>7</sup> in  $\mathcal{A}(\{\hat{u}\})$ , and let  $\lambda_1$  be any stationary measure for  $(u_k, \mathbb{P}_u)$  in  $\mathcal{P}(H)$ . As  $\lambda_1(\mathcal{A}(H)) = 1$ , there is a sequence  $\{\mathcal{H}_n\}$  of compacts in  $H$  such that

$$\lambda_1(\mathcal{A}(\mathcal{H}_n)) > 1 - 1/n \quad \text{for any } n \geq 1.$$

By Condition (iii), we have  $\mathcal{A}(\{\hat{u}\}) \subset \mathcal{A}(\mathcal{H}_n)$ , and  $\mu$  and  $\lambda_1/\lambda_1(\mathcal{A}(\mathcal{H}_n))$  are stationary measures for  $(u_k, \mathbb{P}_u)$  in  $\mathcal{P}(\mathcal{A}(\mathcal{H}_n))$ . So  $\mu = \lambda_1/\lambda_1(\mathcal{A}(\mathcal{H}_n))$  by the uniqueness of stationary measure on  $X = \mathcal{A}(\mathcal{H}_n)$ . Therefore,

$$\lambda_1(\Gamma) = \lim_{n \rightarrow \infty} \lambda_1(\Gamma \cap \mathcal{A}(\mathcal{H}_n)) = \lim_{n \rightarrow \infty} \lambda_1(\Gamma \cap \mathcal{A}(\mathcal{H}_n))/\lambda_1(\mathcal{A}(\mathcal{H}_n)) = \mu(\Gamma)$$

for any  $\Gamma \in \mathcal{B}(H)$ . Thus  $\lambda_1 = \mu$ .  $\square$

<sup>6</sup>Here we use also the boundedness of the projections  $\mathbf{P}_M$  (see Condition (v)), which implies the limit  $\mathbf{P}_M \rightarrow I$  as  $M \rightarrow \infty$  in the operator topology.

<sup>7</sup>Using Condition (iii), it is easy to see that  $\text{supp } \mu = \mathcal{A}(\{\hat{u}\})$ .

## 2 Proof of the Main Theorem

In this section, we prove the Main Theorem by applying Theorem 1.1. We begin with a short discussion of the deterministic NS system, then turn to the verification of Conditions (i)-(v) in an appropriate functional setting.

### 2.1 Preliminaries

Applying the Leray projection<sup>8</sup>  $\Pi$  to Eq. (0.1), we eliminate the pressure term and consider the evolution system

$$\dot{u} + \nu Lu + B(u) = \eta, \quad (2.1)$$

where  $L = -\Pi\Delta$  is the Stokes operator and  $B(u) = \Pi(\langle u, \nabla \rangle u)$ .

Let us define a bilinear symmetric form  $[\cdot, \cdot] : V \times V \rightarrow \mathbb{R}$  by

$$[u, v] := \langle u, v \rangle_1 - \frac{\lambda_1}{2} \langle u, v \rangle, \quad u, v \in V.$$

The Poincaré inequality (see (0.4))

$$\|u\|^2 \leq \lambda_1^{-1} \|u\|_1^2, \quad u \in V \quad (2.2)$$

implies that

$$\frac{1}{2} \|u\|_1^2 \leq [u]^2 := [u, u] \leq \|u\|_1^2. \quad (2.3)$$

Thus  $[\cdot, \cdot]$  defines a scalar product on  $V$  with norm  $[\cdot]$  equivalent to  $\|\cdot\|_1$ .

**Proposition 2.1.** *For any  $T > 0$ ,  $u_0 \in H$ , and  $\eta \in L^2(J_T, H)$ , there is a unique solution  $u \in C(J_T, H) \cap L^2(J_T, V)$  of problem (2.1), (0.3). It satisfies the following inequalities*

$$\|u(t)\|^2 \leq e^{-\nu\lambda_1 t} \|u_0\|^2 + \nu^{-2} \lambda_1^{-1} \|\eta\|_{L^2(J_T, V')}^2, \quad t \in J_T, \quad (2.4)$$

$$\|u\|_{L^2(J_T, V)}^2 \leq \nu^{-1} \|u_0\|^2 + \nu^{-2} \|\eta\|_{L^2(J_T, V')}^2, \quad (2.5)$$

and the equality

$$\|u(t)\|^2 = e^{-\nu\lambda_1 t} \|u_0\|^2 + 2 \int_0^t e^{-\nu\lambda_1(t-s)} (\langle \eta(s), u(s) \rangle - \nu [u(s)]^2) ds, \quad t \in J_T. \quad (2.6)$$

Let us define the mapping

$$S_t : H \times L^2(J_T, H) \rightarrow H, \quad (u_0, \eta) \mapsto u(t), \quad t \in J_T.$$

The following stability result holds.

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<sup>8</sup>That is the orthogonal projection in  $L^2(D, \mathbb{R}^2)$  onto  $H$ .

**Proposition 2.2.** *Assume that the sequence  $\{u_0^n\}$  converges weakly to  $u_0$  in  $H$  and the sequence  $\{\eta^n\}$  converges strongly to  $\eta$  in  $L^2(J_T, H)$ . Then*

$$S_t(u_0^n, \eta^n) \rightharpoonup S_t(u_0, \eta) \quad \text{weakly in } H \text{ for } t \in J_T, \quad (2.7)$$

$$S.(u_0^n, \eta^n) \rightharpoonup S.(u_0, \eta) \quad \text{weakly in } L^2(J_T, V). \quad (2.8)$$

The proofs of these two propositions are carried out by standard methods and are given in the Appendix.

Let us now describe the functional setting in which Theorem 1.1 is applied. The space  $H$  is defined by (0.9),  $E := L^2([0, 1], H)$ , and the mapping

$$S := S_1 : H \times E \rightarrow H, \quad (u_0, \eta) \mapsto u(1)$$

is the time-one shift along trajectories of Eq. (2.1). Then the restriction to integer times of the solution of (2.1), (0.3), (0.5) satisfies

$$u_k = S(u_{k-1}, \eta_k), \quad k \geq 1, \quad (2.9)$$

where  $\{\eta_k\}$  is a sequence of i.i.d. random variables as in the decomposability condition in the Introduction. It is straightforward to see that Condition (v) is satisfied.<sup>9</sup> In the next three subsections, we check Conditions (i)-(iv).

## 2.2 Regularity condition

The smoothness of the mapping  $S : H \times E \rightarrow H$  and the boundedness of its derivatives on bounded subsets of  $H \times E$  are proved using well-known methods (e.g., see Chapters I and VII in [BV92] for the case of bounded domain  $D$ ).

Let us consider the linearisation of Eq. (2.1) around the trajectory  $\tilde{u}(t) = S_t(u, \eta)$  corresponding to an initial condition  $\tilde{u}(0) = u \in H$  and control  $\eta \in \mathcal{K}$ :

$$\dot{w} + \nu Lw + Q(\tilde{u}, w) = 0, \quad (2.10)$$

$$w(0) = w_0, \quad (2.11)$$

where

$$Q(a, b) = \Pi(\langle a, \nabla \rangle b) + \Pi(\langle b, \nabla \rangle a). \quad (2.12)$$

Then  $(D_u S)(u, \eta)w_0 = w(1)$  for any  $w_0 \in H$ , and we can write  $w = v_1 + v_2$ , where  $v_1$  and  $v_2$  are the solutions of the problems

$$\dot{v}_1 + \nu L v_1 = 0, \quad v_1(0) = w_0, \quad (2.13)$$

$$\dot{v}_2 + \nu L v_2 + Q(\tilde{u}, w) = 0, \quad v_2(0) = 0. \quad (2.14)$$

The representation (1.3) holds with the linear operators

$$\begin{aligned} \Psi_1 : H &\rightarrow H, & w_0 &\mapsto v_1(1), \\ \Psi_2(u, \eta) : H &\rightarrow H, & w_0 &\mapsto v_2(1). \end{aligned}$$

---

<sup>9</sup>Note that  $\mathcal{K} := \text{supp } \mathcal{D}(\eta_k)$  is compact in  $E$ , since it is contained in a Hilbert cube.

Inequality

$$\|v_1(t)\| \leq e^{-\nu\lambda_1 t/2} \|w_0\|$$

implies (1.4) with  $\varkappa := e^{-\nu\lambda_1/2}$ . The following lemma (whose proof is given in the Appendix) completes the verification of Condition (i).

**Lemma 2.3.** *For any  $(u, \eta) \in H \times \mathcal{K}$ , the operator  $\Psi_2(u, \eta) : H \rightarrow H$  is compact.*

### 2.3 Asymptotic compactness

For any integer  $k \geq 1$  and any function  $\eta \in L^2(J_k, H)$  of the form (0.5) with some sequence  $\{\eta_n\} \subset E$ , we write

$$u(t) = S_t(u_0, \eta) =: S_t(u_0; \eta_1, \dots, \eta_k), \quad t \in J_k. \quad (2.15)$$

**Proposition 2.4.** *For any bounded sequence  $\{u_0^n\}$  in  $H$ , any sequence of integers  $l_n \geq 1$  such that  $l_n \rightarrow \infty$ , and any family  $\{\zeta_m^n : m, n \geq 1\} \subset \mathcal{K}$ , the sequence*

$$v_n = S_{l_n}(u_0^n; \zeta_1^n, \dots, \zeta_{l_n}^n)$$

*is precompact in  $H$ .*

*Proof.* Let us first explain the scheme of the proof. Using the dissipativity of the system, we show that  $\{v_n\}$  is bounded in  $H$ . As  $H$  is a Hilbert space, there is a subsequence  $\{v_{k_n}\}$  such that

$$v_{k_n} \rightharpoonup w \quad \text{weakly in } H. \quad (2.16)$$

Hence, we have

$$\|w\| \leq \liminf_{n \rightarrow \infty} \|v_{k_n}\|. \quad (2.17)$$

On the other hand, using energy equality (2.6) and Proposition 2.2, we show that

$$\limsup_{n \rightarrow \infty} \|v_{k_n}\| \leq \|w\|. \quad (2.18)$$

Inequalities (2.17) and (2.18) together imply that

$$v_{k_n} \rightarrow w \quad \text{strongly in } H,$$

which gives the required result.

*Step 1: Boundedness.* Let us show that, for any bounded set  $\mathcal{H} \subset H$ , the attainability set  $\mathcal{A}(\mathcal{H}) \subset H$  is bounded. Indeed, let

$$M_1 := \sup_{u \in \mathcal{H}} \|u\|^2 < \infty.$$

For any  $u_0 \in \mathcal{H}$  and  $\zeta_1, \dots, \zeta_k \in \mathcal{K}$ , let  $u_k := S_k(u_0; \zeta_1, \dots, \zeta_k)$ . Then by inequality (2.4), we have

$$\|u_k\|^2 \leq \varkappa \|u_{k-1}\|^2 + M_2,$$

where  $\varkappa := e^{-\nu\lambda_1} < 1$  and  $M_2 := \nu^{-2}\lambda_1^{-1} \sup_{\eta \in \mathcal{K}} \|\eta\|_{L^2([0,1],V')}^2$ . Iterating this, we get

$$\|u_k\|^2 \leq \varkappa^k \|u_0\|^2 + M_2(1 - \varkappa)^{-1} \leq M_1 + M_2(1 - \varkappa)^{-1} =: M.$$

This shows that

$$\sup_{u \in \mathcal{A}(\mathcal{H})} \|u\|^2 \leq M.$$

*Step 2: Proof of (2.18).* From the previous step it follows that  $\{v_n\}$  is bounded in  $H$ . Let  $\{v_{k_n}\}$  be a subsequence verifying (2.16). It is of the form

$$v_{k_n} = S_{\ell_n}(u_0^n; \eta_1^n, \dots, \eta_{\ell_n}^n)$$

for some vectors  $\eta_1^n, \dots, \eta_{\ell_n}^n \in \mathcal{K}$  and integers  $\ell_n \geq 1$  such that  $\ell_n \rightarrow \infty$ . Using the boundedness of the set  $\mathcal{A}(\{u_0^n, n \geq 1\})$  and passing to a subsequence if necessary (applying the diagonal process), we can assume that

$$w_{n,m} := S_{\ell_n - m}(u_0^n; \eta_1^n, \dots, \eta_{\ell_n - m}^n) \rightharpoonup w_m \quad \text{weakly in } H \quad (2.19)$$

for any  $m \geq 1$  and some  $w_m \in H$ . Using the compactness of  $\mathcal{K}$  and again passing to a subsequence if necessary, we can assume that

$$\eta_{\ell_n - i}^n \rightarrow \xi_{m-i}^m \quad \text{strongly in } E, \quad i = 0, \dots, m-1.$$

Combining this with (2.19) and Proposition 2.2, we get

$$v_{k_n} = S_m(w_{n,m}; \eta_{\ell_n - m + 1}^n, \dots, \eta_{\ell_n}^n) \rightharpoonup S_m(w_m; \xi_1^m, \dots, \xi_m^m) \quad \text{weakly in } H.$$

From (2.16) we derive the equality

$$w = S_m(w_m; \xi_1^m, \dots, \xi_m^m) \quad \text{for any } m \geq 1.$$

This and (2.6) imply that

$$\begin{aligned} \|w\|^2 &= \|u^m(m)\|^2 \\ &= e^{-\nu\lambda_1 m} \|w_m\|^2 + 2 \int_0^m e^{-\nu\lambda_1(m-s)} (\langle \xi^m(s), u^m(s) \rangle - \nu[u^m(s)]^2) ds, \end{aligned} \quad (2.20)$$

where

$$u^m(t) = S_t(w_m, \xi^m), \quad \xi^m(t) = \sum_{k=1}^m \mathbb{I}_{[k-1, k)}(t) \xi_k^m(t-k+1), \quad t \in J_m.$$

On the other hand, again by (2.6), we have

$$\begin{aligned} \|v_{k_n}\|^2 &= \|u^{n,m}(m)\|^2 \\ &= e^{-\nu\lambda_1 m} \|w_{n,m}\|^2 + 2 \int_0^m e^{-\nu\lambda_1(m-s)} (\langle \eta^{n,m}(s), u^{n,m}(s) \rangle - \nu[u^{n,m}(s)]^2) ds, \end{aligned} \quad (2.21)$$

where

$$u^{n,m}(t) = S_t(w_{n,m}, \eta^{n,m}), \quad \eta^{n,m}(t) = \sum_{k=1}^m \mathbb{I}_{[k-1,k)}(t) \eta_{k_n - m + k}^n(t - k + 1), \quad t \in J_m.$$

Note that

$$\int_0^m e^{-\nu\lambda_1(m-s)} \langle \eta^{n,m}(s), u^{n,m}(s) \rangle ds \rightarrow \int_0^m e^{-\nu\lambda_1(m-s)} \langle \xi^m(s), u^m(s) \rangle ds, \quad (2.22)$$

as  $\eta^{n,m} \rightarrow \xi^m$  strongly in  $L^2(J_m, H)$  and  $u^{n,m} \rightharpoonup u^m$  weakly in  $L^2(J_m, V)$ . Since

$$\left( \int_0^m e^{-\nu\lambda_1(m-s)} [\cdot]^2 ds \right)^{1/2}$$

is a norm in  $L^2(J_m, V)$  which is equivalent to the original one, the following inequality holds

$$\int_0^m e^{-\nu\lambda_1(m-s)} [u^m(s)]^2 ds \leq \liminf_{n \rightarrow \infty} \int_0^m e^{-\nu\lambda_1(m-s)} [u^{n,m}(s)]^2 ds.$$

Combining this with (2.20)-(2.22), we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|v_{k_n}\|^2 &\leq \limsup_{n \rightarrow \infty} (e^{-\nu\lambda_1 m} \|w_{n,m}\|^2) + \|w\|^2 - e^{-\nu\lambda_1 m} \|w_m\|^2 \\ &\leq e^{-\nu\lambda_1 m} M + \|w\|^2. \end{aligned}$$

As  $m \geq 1$  is arbitrary, we arrive at (2.18).  $\square$

## 2.4 Approximate controllability

By the decomposability assumption, we have  $0 \in \mathcal{K}$ . In view of (2.4), Condition (iii) is verified with  $\hat{u} = 0$ ,  $\zeta_1 = \dots = \zeta_n = 0$ , and sufficiently large  $n \geq 1$ .

To check Condition (iv), we consider the following linearisation of Eq. (2.1) around the same trajectory  $\tilde{u}$  as in Section 2.2:

$$\begin{aligned} \dot{w} + \nu Lw + Q(\tilde{u}, w) &= \zeta(t), \\ w(0) &= 0, \end{aligned} \quad (2.23)$$

where  $Q$  is given by (2.12). Then  $(D_\eta S)(u, \eta)\zeta = w(1)$  for any  $\zeta \in E$ . For any smooth function  $w_1 \in H$ , we can find a smooth function  $w : [0, 1] \times D \rightarrow \mathbb{R}^2$  such that  $w(0) = 0$ ,  $w(1) = w_1$ , and  $w(t) \in H$  for all  $t \in [0, 1]$ . Replacing  $w$  into Eq. (2.23), we find explicitly a control  $\zeta \in E$  such that  $(D_\eta S)(u, \eta)\zeta = w_1$ . This shows that the image of the mapping  $(D_\eta S)(u, \eta) : E \rightarrow H$  is dense in  $H$  for any  $(u, \eta) \in H \times \mathcal{K}$ . Thus Conditions (i)-(v) are verified. Applying Theorem 1.1, we complete the proof of the Main Theorem.

### 3 Appendix

#### 3.1 Proof of Proposition 2.1

The existence and uniqueness of solution is proved, e.g., in Chapter III of [Tem77]. Here we give a formal derivation of inequalities (2.4) and (2.5) and equality (2.6).

Taking the scalar product in  $H$  of Eq. (2.1) with  $2u$  and using the identity

$$\langle B(u), u \rangle = 0, \quad u \in V,$$

we get

$$\frac{d}{dt} \|u\|^2 + 2\nu \|u\|_1^2 = 2\langle \eta, u \rangle \leq 2\|\eta\|_{V'} \|u\|_1 \leq \nu \|u\|_1^2 + \nu^{-1} \|\eta\|_{V'}^2. \quad (3.1)$$

Combining this with the Poincaré inequality, we obtain (2.4). From (3.1) we also derive the inequality

$$\int_0^t \|u\|_1^2 ds \leq \nu^{-1} \|u_0\|^2 + \nu^{-2} \int_0^t \|\eta\|_{V'}^2 ds \quad t \in J_T \quad (3.2)$$

which implies (2.5). To prove (2.6), we rewrite the equality in (3.1) in the form

$$\frac{d}{dt} \|u\|^2 + \nu \lambda_1 \|u\|^2 = 2(\langle \eta, u \rangle - \nu[u]^2)$$

and apply the variation of constants formula.

#### 3.2 Proof of Proposition 2.2

By Proposition 2.1, the sequence  $u^n = S.(u_0^n, \eta^n)$  is bounded in the space  $C(J_T, H) \cap L^2(J_T, V)$ . From the inequality

$$\|B(u)\|_{V'} \leq C \|u\| \|u\|_1, \quad u \in V$$

and Eq. (2.1) we derive that

$$\|\dot{u}^n\|_{L^2(J_T, V')} \leq M_1, \quad n \geq 1. \quad (3.3)$$

For any  $R > 0$ , let us set  $D_R := D \cap B_{\mathbb{R}^2}(R)$ . Then the space  $H^1(D_R)$  is compactly embedded into  $L^2(D_R)$ . Applying Theorem 2.1 of Chapter III in [Tem77] with spaces  $X_0 = H^1(D_R)$ ,  $X = L^2(D_R)$ ,  $X_1 = H^{-1}(D_R)$  and numbers  $\alpha_0 = \alpha_1 = 2$ , and using the diagonal process, we find a subsequence  $\{u^{k_n}\}$  such that

$$u^{k_n} \rightharpoonup u \quad \text{weak-star in } L^\infty(J_T, H), \quad (3.4)$$

$$u^{k_n} \rightharpoonup u \quad \text{weakly in } L^2(J_T, V), \quad (3.5)$$

$$u^{k_n} \rightarrow u \quad \text{strongly in } L^2(J_T, L^2(D_R)) \quad (3.6)$$



for any  $R > 0$ . Passing to the limit in the equation for  $u^{k_n}(t)$ , we conclude that  $u(t)$  is the solution  $S_t(u_0, \eta)$ ,  $t \in J_T$ . Moreover, by the uniqueness of the limit, it is easy to see that limits (3.4)-(3.6) hold for the full sequence  $\{u^n\}$ . This proves (2.8).

Let us take any  $\varphi \in \mathcal{V}$  (see (0.8)). By inequality (2.4), we have

$$|\langle u^n(t), \varphi \rangle| \leq M_2, \quad n \geq 1, t \in J_T,$$

and by inequality (3.3),

$$\begin{aligned} |\langle u^n(t + \tau) - u^n(t), \varphi \rangle| &\leq \int_t^{t+\tau} |\langle \dot{u}^n(s), \varphi \rangle| ds \\ &\leq \tau^{\frac{1}{2}} \|\dot{u}^n\|_{L^2(J_T, V')} \|\varphi\|_1 \leq \tau^{\frac{1}{2}} M_1 \|\varphi\|_1 \end{aligned}$$

for any  $t \in J_T$  and  $\tau \in (0, T - t)$ . Thus the Arzelà–Ascoli theorem implies that

$$\langle u^n(t), \varphi \rangle \rightarrow \langle u(t), \varphi \rangle$$

uniformly in  $t \in J_T$ . Using the fact that  $\mathcal{V}$  is dense in  $H$ , we get (2.7).

### 3.3 Proof of Lemma 2.3

Let us set  $J := [0, 1]$ . By Proposition 2.1, we have  $\tilde{u} = S(u, \eta) \in L^2(J, V)$  for any  $u \in H$  and  $\eta \in \mathcal{K}$ . Using standard methods, one can show that the mapping  $\Psi : w_0 \mapsto w$  (i.e., the resolving operator of problem (2.10), (2.11)) is continuous from  $H$  to  $\mathcal{X} := L^2(J, V) \cap W^{1,2}(J, H^{-1})$ . Let us show that the linear mapping

$$\Phi^{\tilde{u}} : H \rightarrow L^2(J, H^{-1}), \quad w_0 \mapsto \langle \tilde{u}, \nabla \rangle w + \langle w, \nabla \rangle \tilde{u}$$

is compact. Indeed, let  $\{w_0^n\}$  be a bounded sequence in  $H$ . Then the sequence  $\{\Psi(w_0^n)\}$  is bounded in  $\mathcal{X}$ . As in the previous subsection, we apply Theorem 2.1 of Chapter III in [Tem77] with spaces  $X_0 = H^1(D_R)$ ,  $X = L^2(D_R)$ ,  $X_1 = H^{-1}(D_R)$  and numbers  $\alpha_0 = \alpha_1 = 2$ , and use the diagonal process to find a subsequence  $\{\Psi(w_0^{k_n})\}$  converging strongly in  $L^2(J, L^2(D_R))$  for any  $R > 0$ .

On the other hand, for any  $\varepsilon > 0$ , there is a number  $R > 0$  and a smooth function  $\varphi : J \times D \rightarrow \mathbb{R}^2$  with  $\text{supp } \varphi(t, \cdot) \subset D_R$  for any  $t \in J$  and

$$\|\tilde{u} - \varphi\|_{L^2(J, V)} < \varepsilon. \quad (3.7)$$

Using the boundedness of the sequence  $\{\Phi^{\tilde{u}}(w_0^{k_n})\}$  in  $L^2(J, V)$  and the estimate

$$\|\langle a, \nabla \rangle b\|_{H^{-1}} \leq C_1 \|a\|_{H^{1/2}} \|b\|_{H^{1/2}} \leq C_2 \|a\|_1 (\|b\| \|b\|_1)^{\frac{1}{2}}, \quad a, b \in V,$$

we obtain

$$\begin{aligned} \|\Phi^{\tilde{u}}(w_0^{k_n}) - \Phi^{\tilde{u}}(w_0^{k_m})\|_{L^2(J, H^{-1})} &\leq \|\Phi^\varphi(w_0^{k_n}) - \Phi^\varphi(w_0^{k_m})\|_{L^2(J, H^{-1})} \\ &\quad + \|\Phi^{\tilde{u}-\varphi}(w_0^{k_n}) - \Phi^{\tilde{u}-\varphi}(w_0^{k_m})\|_{L^2(J, H^{-1})} \\ &\leq C_3 \|\Psi(w_0^{k_n}) - \Psi(w_0^{k_m})\|_{L^2(J, L^2(D_R))}^{1/2} \\ &\quad + C_3 \|\tilde{u} - \varphi\|_{L^2(J, V)}, \end{aligned}$$

where the constant  $C_3 > 0$  does not depend on the numbers  $n, m, \varepsilon$ . Combining this with (3.7) and choosing  $n$  and  $m$  sufficiently large, we see that

$$\|\Phi^{\tilde{u}}(w_0^{k_n}) - \Phi^{\tilde{u}}(w_0^{k_m})\|_{L^2(J, H^{-1})} < 2C_3\varepsilon.$$

This shows that  $\Phi^{\tilde{u}}$  is compact. Thus the map  $\Psi_2 : H \rightarrow H$  is compact as a composition of  $\Phi^{\tilde{u}}$  with some linear continuous map.

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