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1 Local certification of graph decompositions and 2 applications to minor-free classes

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12 — Abstract —

13 Local certification consists in assigning labels to the vertices of a network to certify that some given
14 property is satisfied, in such a way that the labels can be checked locally. In the last few years,
15 certification of graph classes received considerable attention. The goal is to certify that a graph G
16 belongs to a given graph class \mathcal{G} . Such certifications with labels of size $O(\log n)$ (where n is the size
17 of the network) exist for trees, planar graphs and graphs embedded on surfaces. Feuilloley et al. ask
18 if this can be extended to any class of graphs defined by a finite set of forbidden minors.

19 In this work, we develop new decomposition tools for graph certification, and apply them to
20 show that for every small enough minor H , H -minor-free graphs can indeed be certified with labels
21 of size $O(\log n)$. We also show matching lower bounds using a new proof technique.

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27 **1** Introduction

28 Local certification is an active field of research in the theory of distributed computing. On a
29 high level it consists in certifying global properties in such a way that the verification can be
30 done locally. More precisely, for a given property, a local certification consists of a labeling
31 (called a *certificate assignment*), and of a local verification algorithm. If the configuration of
32 the network is correct, then there should exist a labeling of the vertices that is accepted by
33 the verification algorithm, whereas if the configuration is incorrect no labeling should make
34 the verification algorithm accept.

35 Local certification originates from self-stabilization, and was first concerned with certifying
36 that a solution to an algorithmic problem is correct. However, it is also important to
37 understand how to certify properties of the network itself, that is, to find locally checkable
38 proofs that the network belongs to some graph class. There are several reasons for that. First,
39 because certifying some solutions can be hard in general graphs, while they become simpler
40 on more restricted classes. To make use of this fact, it is important to be able to certify that
41 the network does belong to the restricted class. Second, because some distributed algorithms
42 work only on some specific graph classes, and we need a way to ensure that the network does
43 belong to the class, before running the algorithm. Third, the distinction between certifying
44 solutions and network properties is rather weak, in the sense that the techniques are basically

45 the same. So we should take advantage of the fact that a lot is known about graph classes to
 46 learn more about certification.

47 In the domain of graph classes certification, there have been several results on various
 48 classes such as trees [42], bipartite graphs [38] or graphs of bounded diameter [9], but until
 49 two years ago little was known about essential classes, such as planar graphs. Recently,
 50 it has been shown that planar graphs and graphs of bounded genus can be certified with
 51 $O(\log n)$ -bit labels [21, 27, 28]. This size, $O(\log n)$, is the gold standard of certification, in
 52 the sense that little can be achieved with $o(\log n)$ bits, thus $O(\log n)$ is often the best we
 53 can hope for.

54 Planar and bounded-genus graphs are classic examples of graphs classes defined by
 55 forbidden minors, a type of characterization that has become essential in graph theory since
 56 the Graph minor series of Robertson and Seymour [49]. Remember that a graph H is a
 57 minor of a graph G , if it is possible to obtain H from G by deleting vertices, deleting edges,
 58 contracting edges. At this point, the natural research direction is to try to get the big picture
 59 of graph classes certification, by understanding all classes defined by forbidden minors. In
 60 particular, we want to answer the following concrete question. By the graph minor theorem,
 61 every minor-closed class of graphs can be defined by a finite list of forbidden minors, which
 62 underlines how a positive answer for this question would apply to wide range of applications.

63 ► **Question 1** ([24, 27]). *Can any graph class defined by a finite set of forbidden minors be*
 64 *certified with $O(\log n)$ -bit certificates?*

65 This open question is quite challenging: there are as many good reasons to believe that
 66 the answer is positive as negative.

67 First, the literature provides some reasons to believe that the conjecture is true. Properties
 68 that are known to be hard to certify, that is, that are known to require large certificates, are
 69 very different from minor-freeness. Specifically, all these properties (*e.g.* small diameter [9],
 70 non-3-colorability [38], having a non-trivial automorphism [38]) are non-hereditary. That
 71 is, removing a vertex or an edge may yield a graph that is not in the class. Intuitively,
 72 hereditary properties might be easier to certify in the sense that one does not need to encode
 73 information about every single edge or vertex, as the class is stable by removal of edges
 74 and vertices. Minor-freeness is a typical example of hereditary property. Moreover, this
 75 property, that has been intensively studied in the last decades, is known to carry a lot of
 76 structure, which is an argument in favor of the existence of a compact certification (that is a
 77 certification with $O(\log n)$ -bit labels).

78 On the other hand, from a graph theory perspective, it might be surprising that a
 79 general compact certification existed for minor-free graphs. Indeed, for the known results,
 80 obtaining a compact certification is tightly linked to the existence of a precise constructive
 81 characterization of the class (*e.g.* a planar embedding for planar graphs [21, 28], or a canonical
 82 path to the root for trees [42]). Intuitively, this is because forbidden minor characterizations
 83 are about structures that are absent from the graphs, and local certification is often about
 84 certifying the existence of some structures. While such a characterization is known for some
 85 restricted minor-closed classes, we are far from having such a characterization for every
 86 minor-closed class. Note that there are a lot of combinatorial and algorithmic results on
 87 H -minor-free graphs, but they actually follow from properties satisfied by H -minor-free
 88 graphs, not from exact characterizations of such graphs. For certification, we need to rule out
 89 the graphs that do not belong to the class, hence a characterization is somehow necessary.

1.1 Our results

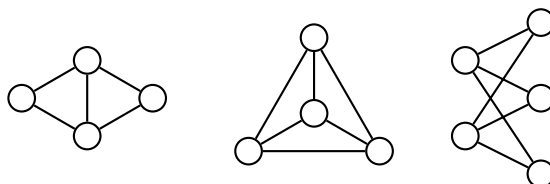
Answering Question 1 seems unfortunately out of reach, at the current state of our knowledge. We have explained above about why designing compact certification is hard for classes that do not have a constructive characterization. We will later give some intuition about why lower bounds seem equally difficult to get. In this paper, we intend to build the foundations needed to tackle Question 1. More precisely, we have four types of contributions.

First, we show how to certify some graph decompositions. Such decompositions state how to build a class based on a few elementary graphs and a few simple operations. They are essential in structural graph theory, and more specifically in the study of minor-closed classes. Amongst the most famous examples of these theorems is the proof of the 4-Color Theorem [3] or the Strong Perfect Graph Theorem [14].

Second, we show that by directly applying these tools, we can design compact certification for several H -minor-free classes, for which a precise characterization is known. See Fig. 1 and 2. That is, we answer positively Question 1, for several small minors, and show that our decomposition tools can easily be used.

Class	Optimal size	Result
K_3 -minor-free	$\Theta(\log n)$	Equivalent to acyclicity [38, 42].
Diamond-minor-free	$\Theta(\log n)$	Corollary 29
K_4 -minor-free	$\Theta(\log n)$	Corollary 29
$K_{2,3}$ -minor-free	$\Theta(\log n)$	Corollary 29
$(K_{2,3}, K_4)$ -minor-free (<i>i.e.</i> outerplanar)	$\Theta(\log n)$	Corollary 29
$K_{2,4}$ -minor-free	$\Theta(\log n)$	Lemma 36

■ **Figure 1** Our main results for the certification of minor-closed classes.



■ **Figure 2** From left to right: the diamond, the clique on 4 vertices K_4 , and the complete bipartite graph $K_{2,3}$.

Third, we do a systematic study of small minors to identify which is the first one that we cannot tackle. We first prove the following theorem.

► **Theorem 2.** H -minor-free classes can be certified in $O(\log n)$ bits when H has at most 4 vertices.

Then, we extend this theorem to minors on five vertices with a specific shape, proving along the way new purely graph-theoretic characterizations for the associated classes. After this study, we can conclude that the next challenge is to understand K_5 -minor-free graphs.

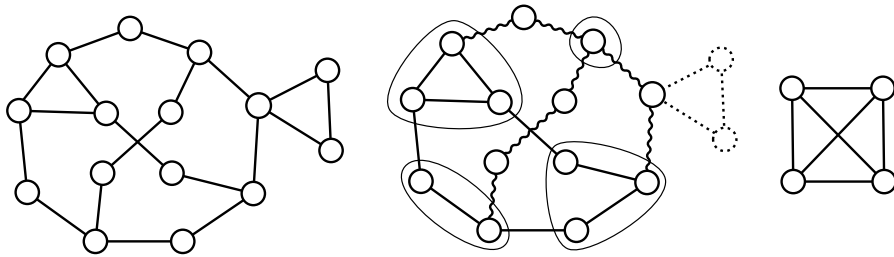
Finally, we prove a general $\Omega(\log n)$ lower bounds for H -minor-freeness for all 2-connected graphs H of size at least 3. This generalizes and simplifies the lower bounds of [28] which apply only to K_k and $K_{p,q}$ -minor-free graphs, and use ad-hoc and more complicated techniques.

115 At the end of the paper, we discuss why the current tools we have, both in terms of upper
 116 and lower bounds, do not allow settling Question 1. We list a few key questions that we need
 117 to answer before we can fully understand the certification of minor-closed classes, from the
 118 certification of classes with no tree minors to the certification of k -connectivity, for arbitrary
 119 k .

120 1.2 Our techniques

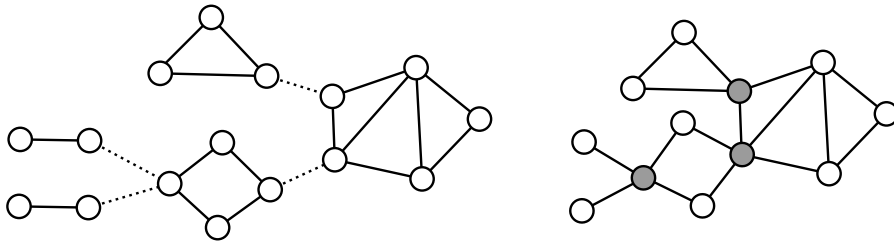
121 General approach and challenges

122 To give some intuition about our techniques, let us focus on a concrete example: K_4 -minor-
 123 free graphs. Remember that a graph has K_4 -minor if we can get a K_4 by deleting vertices
 124 and edges, and contracting edges. An alternative definition is that a graph has a K_4 -minor, if
 125 it is possible to find four disjoint sets of vertices, called *bags*, such that: each bag is connected,
 126 there is a path between each pair of bags, these paths and bags are all vertex-disjoint (except
 127 for the endpoints of the paths that coincide with vertices of the bags). See Figure 3.



128 **Figure 3** The graph on the left has a K_4 minor. Indeed, the bags of the second definition are
 129 depicted in the picture in the middle, and it is easy to find the six disjoint paths that link them.
 130 Alternatively, one can get a K_4 like the one of the right-most picture by contracting all the edges
 131 inside the bags, contracting the wavy paths between bags into edges, and deleting the dotted vertices
 132 and edges.

128 An important observation is that, if we take a collection F_1, \dots, F_k of K_4 -minor-free
 129 graphs, and organize them into a tree, by identifying pairs of vertices like in Figure 4, we get
 130 a K_4 -minor-free graph.



131 **Figure 4** The five graphs with plain edges on the left picture are K_4 -minor-free. Organizing them
 132 into a tree by identifying the vertices linked by dotted edges makes a larger K_4 -minor-free graph.

131 To see that, suppose that the graph we created has a K_4 -minor. Then there exist bags
 132 and paths as described above. If the bags and paths are all contained in the same former F_i ,
 133 then this F_i would not be K_4 -minor-free, which is a contradiction. If it is not the case, then
 134 the bags and paths use vertices that belong to different subgraphs F_i and F_j . And because

135 of connectivity, they should use a vertex v that connects two such subgraphs (grey vertices in
 136 Figure 4). Then the bags and paths cannot be vertex-disjoint as required, because v would
 137 them lie into at least two bags or interior of paths.

138 As a consequence of the observation above, a classic way to study K_4 -minor-free graphs
 139 (as well as other classes) is to decompose the graph into maximal 2-connected components
 140 organized into a tree. This is called the *block-cut tree* of the graph, where every maximal
 141 2-connected component is called a *block*. (Figure 4 actually shows the block-cut structure of
 142 the right-most graph.) This is relevant here because 2-connected K_4 -minor-free graphs have
 143 a specific structure; we will come back to this later.

144 Now, from the certification point of view, there is a natural strategy: first certify the
 145 structure of the block-cut tree, and then certify the special structure of each block. There
 146 are several challenges to face with this approach. First, to certify the block-cut tree, it is
 147 essential to be able to certify the connectivity of the blocks. Second, we need to avoid what
 148 we call certificate congestion, which is the issue of having too large certificates because we use
 149 too many layers of certification on some vertices. We now detail these two aspects, starting
 150 with the latter.

151 **Avoiding certificate congestion**

152 In the block-cut tree of a graph, the blocks are attached to each other by shared vertices,
 153 the *cut vertices*. There is no bound on the number of blocks that are attached to a given
 154 cut vertex, and this is problematic for certification. Indeed, we cannot give to every vertex
 155 the list of the blocks it belongs to, as we aim for $O(\log n)$ certificates, and such a list could
 156 contain $\Omega(n)$ blocks. And even if we could fix the certification of the block-cut tree, the same
 157 problem would appear with the certification of the specific structure of each block: the cut
 158 vertices would have to hold a piece of certification for each block.

159 We basically have two tools to deal with this problem. The first one is not new, it is a
 160 degeneracy argument that already appeared in [27, 28]. A graph is k -degenerate if in every
 161 subgraph there exists a vertex that has degree at most k . Intuitively (and a bit incorrectly),
 162 this means that when we need to put a large certificate on a vertex, we can spread it on
 163 its some of its neighbors that have lower degree. A more precise statement is that, for
 164 k -degenerate graphs, we can transform a certification with $O(f(n))$ labels *on the edges of*
 165 *the graphs*, into a classic certification with $O(k \cdot f(n))$ labels on the vertices. This is relevant
 166 for our problem, as a priori there is less congestion on the edges, and minor-free classes
 167 have bounded degeneracy. Unfortunately, this is not enough for our purpose. We then
 168 build a second, more versatile tool. It consists in proving that it is possible to transform
 169 in mechanical way any certification of a graph or subgraph, into a certification that would
 170 put an empty certificate on some given vertex. Once we have this tool, we can adapt the
 171 certification of the blocks to work well in the block-cut tree: build the block-cut tree by
 172 adding blocks iteratively, making sure that the connecting vertex has an empty label in the
 173 certification of the newly added block.

174 See Section 3 for the details on this topic.

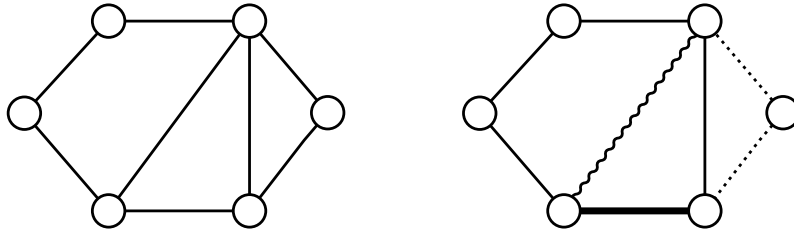
175 **Certifying connectivity properties**

176 Connectivity properties have been studied before in distributed certification. Specifically,
 177 certifying that for two given vertices s and t , the st -connectivity is at least k has been studied
 178 in [42] and [38]. But here we are interested in the connectivity of the graph itself, or in other
 179 words, in the st -connectivity between any pair of vertices. Clearly, proving st -connectivity for

6 Local certification of graph decompositions and applications to minor-free classes

180 any pair using the schemes of the literature would lead to huge certificates. Instead, we use
 181 the characterizations of k -connected graphs that are known for small values of k . There are
 182 various such characterizations, but they are all based on the same idea of *ear decomposition*.

183 To explain ear decompositions, consider a graph that we can build the following way (see
 184 Figure 5). Start from an edge, and iteratively apply the following process: take two vertices
 185 u, v of the current graph and link them by a path whose internal vertices are new vertices of
 186 the graph. This is called an *ear decomposition* (where the ears are the paths added at each
 187 step). When each time u and v are distinct vertices, it is not hard to see that such a graph
 188 is always 2-connected. Remarkably, the converse is also true: any 2-connected graph can
 189 be built (or decomposed this way). This is called an *open ear decomposition*, and similar
 190 constructions characterize 2-edge connected graphs and 3-connected graphs.



■ **Figure 5** Illustration of an open ear decomposition. The graph on the left can be built with the ear decomposition described on the right. First, put the bold edge. Then add the path of plain edges. Finally, add the dotted path, and the wavy path, which is just one edge.

191 The good thing about these constructions is that we can certify them, by describing and
 192 certifying every step. This requires some care, as when certifying a new path, we could
 193 increase the size of the certificates of the endpoints, that are already in the graph. Fortunately,
 194 the tools developed to avoid certificate congestions allow us to control the certificate size.

195 The details about the connectivity certification can be found in Section 5.

196 Putting things together

197 Combining these techniques, we can prove the following theorem.

198 ► **Theorem 3.** *For any 2-connected graph H , if the 2-connected H -minor-free graphs can be*
 199 *certified with $f(n)$ bits, then the H -minor-free graphs can be certified with $O(f(n) + \log n)$*
 200 *bits.*

201 Going back to our example, K_4 -minor-free graphs, given Theorem 3, we are left with
 202 certifying the 2-connected K_4 -minor-free graphs. As said above, these have a specific shape.
 203 More precisely, 2-connected K_4 -minor-free graphs have a nested ear decomposition, which
 204 is yet another type of ear decomposition, this time with additional constraints related
 205 to outerplanarity. We can certify this structure by adapting a construction from [28] for
 206 outerplanar graphs.

207 More generally the 2-connected graphs corresponding to most of the classes of Figure 1
 208 have specific shapes that we can certify quite easily, which imply our compact certification
 209 schemes. We do this in Section 6. A special case is $K_{2,4}$, that has a more complicated structure,
 210 requiring to consider 3-connected components, and some more complicated substructures.
 211 We study this case in Section 7.

212 Finally, in Section 8, we study all the minors on at most 4 vertices, and in Section 9 all
 213 the minors on 5 vertices of some simple form. For these, we do not need new techniques

214 on the certification side, but we need to work on the graph theory side to establish new
215 characterizations, as for these minors the literature does not help. The work we do in
216 Section 9 might be of independent interest as we study the natural notion of H -minimal
217 graph, which are the graph that have H as a minor, but for which any vertex deletion would
218 remove this property.

219 Lower bounds

220 Towards the end of the paper, we show that $\Omega(\log n)$ -bit labels are necessary to certify graph
221 classes excluding a 2-connected minor. When it comes to $\Omega(\log n)$ lower bounds in our
222 model, there are basically two complementary techniques (called *cut-and-plug techniques* in
223 [24]). Both techniques basically show that paths cannot be differentiated from cycles, if the
224 certificates use $o(\log n)$ bits. First, in [38], the idea is to use many correct path instances, and
225 to prove that we can plug them into an incorrect cycle instance, thanks to a combinatorial
226 result from extremal graph theory. Second, in [26], the idea is to consider a path, to cut it
227 into small pieces, and to show via Stirling formula, that there exists a shuffle of these pieces
228 that can be closed into a cycle.

229 Previous lower bounds for minor-free graphs in [28] followed the same kind of strategies
230 as [38] and [26], with the same type of counting arguments, more complicated constructions,
231 and tackled only minors that were cliques or bicliques.

232 In this paper, we are able to do a black-box reduction between the path/cycle problem and
233 the H -minor-freeness for any 2-connected H . This way we avoid explicit counting arguments,
234 and get a more general result with a simpler proof.

235 1.3 Related work

236 Local certification and graph classes

237 Local certification first appeared under the name of *proof-labeling schemes* in [42], inspired
238 by works on self-stabilizing algorithms (see [16] for a book on self-stabilization). It has then
239 been generalized under the name of *locally checkable proofs* in [38], and the field has been
240 very active since these seminal papers. In the following, we will focus on the papers about
241 local certification of graph classes, but we refer to [24] and [25] for an introduction and a
242 survey of local certification in general.

243 As said earlier, certification was first mostly about checking that the solution to an
244 algorithmic problem was correct, a typical example being the verification of a spanning
245 tree [42]. Some graph properties have also been studied, for example symmetry in [38], or
246 bounded diameter in [9]. Very recently, classes that are more central in graph theory have
247 attracted attention. It was first proved in [47], as an application of a more general method,
248 that planar graphs can be certified with $O(\log n)$ bits in the more general model of distributed
249 interactive proofs. Then it was proved in [28] that these graphs can actually be certified with
250 $O(\log n)$ bits in the classic model, that is, without interaction. This result was extended
251 to bounded-genus graphs in [27]. Later, [21] provided a simpler proof of both results via
252 different techniques. It was also proved in [41, 46] that cographs, distance-hereditary graphs,
253 and some intersection graphs have compact distributed interactive proofs. On the negative
254 side, it was established very recently that natural geometric classes, such as unit-disk graphs,
255 require quasi-linear certificates [15].

256 **Forbidden minors, meta-theorems and other labelings**

257 After the publication of the first version of this paper, some progress has been done on
 258 Question 1. On the one hand, a positive answer has been given for an approximate version
 259 of the question. More precisely, by allowing mistakes for graphs that are close to being
 260 H -minor-free (in the spirit of property testing) one can define a compact certification [22]
 261 (follows from Theorem 6). (An approximate certification for bounded degree planar graphs
 262 with constant size labels had been established before, in [18].) On the other hand, two papers
 263 have established meta-theorems that answer the question for specific minor shapes. More
 264 precisely, by proving that monadic second order properties can be certified with $O(\log n)$
 265 when the graph has bounded treedepth [29], and $O(\log^2 n)$ bits when the graph has bounded
 266 treewidth [32], these papers prove as corollaries that the same sizes suffice for path minors
 267 and planar minors, respectively. Indeed, graphs excluding a path (resp. planar) minor
 268 have bounded treedepth (resp. treewidth [50]). Let us also mention a generalization of the
 269 treewidth result to cliquewidth [33] (which has no further implications in terms of forbidden
 270 minors).

271 Graphs with forbidden minors have also attracted interest for other types of labelings.
 272 First, [34] established that one can design adjacency labelings of size $2 \log n + O(\log \log n)$
 273 for graphs excluding a specific minor. For approximate distance labelings, a key paper is [1],
 274 which introduced a specific decomposition of minor-free graphs.

275 **Forbidden subgraphs, testing and detection**

276 There are other structures that can be forbidden and give rise to interesting classes in graph
 277 theory, in particular subgraphs and induced subgraphs. These have been studied very recently
 278 for local certification, see [5].

279 Still in distributed computing, but outside local certification, a popular topic is the
 280 distributed detection of some subgraph H , which consists, in the CONGEST (or CONGEST-
 281 CLIQUE) model to decide whether the graph contains H as a subgraph or not (see [8]
 282 and the references therein). A related task is H -freeness testing, which is the similar but
 283 easier task consisting in deciding whether the graph is H -free or far from being H -free (in
 284 terms of the number of edges to modify to get a H -free graph). This line of work was
 285 formalized by [7] after the seminal work of [6] (see [31] and the references therein). To
 286 our knowledge, no detection or lower bounds have been designed for H -minor-freeness in
 287 general: examples like planarity testing [36] may however provide algorithms tailored to
 288 specific classes. Some results exists for H -minor-testing, see [10, 11], as well as for property
 289 testing on some minor-free classes [44].

290 **Distributed algorithms for specific graph classes**

291 Finally, we have mentioned in the introduction that certifying that the graph belongs to
 292 some given class is important because some algorithms are specially designed to work on
 293 some specific classes. For example, there is a large and growing literature on approximation
 294 algorithms for *e.g.* planar, bounded-genus, minor-free graphs. We refer to [23] for a
 295 bibliography of this area. There are also interesting works for exact problems in the
 296 CONGEST model, *e.g.* in planar graphs [35], graphs of bounded treewidth or genus [39]
 297 and minor-free graphs [40]. In particular the authors of [40] justify the focus on minor-free
 298 graphs by the fact that this class allows for significantly better results than general graphs,
 299 while being large enough to capture many interesting networks. Very recently, [37] proved

300 general tight results on low-congestion short-cuts (an essential tool for algorithms in the
301 CONGEST model) for graphs excluding a dense minor.

302 2 Preliminaries

303 In this section, we define formally the notions we use and describe some useful known
304 certification building blocks.

305 2.1 Graphs and minors

306 Let $G = (V, E)$ be a graph. Let $X \subseteq V$. The *subgraph of G induced by X* is the graph with
307 vertex set X and edge set $E \cap X^2$. The graph $G \setminus X$ is the subgraph of G induced by $V \setminus X$.
308 A graph G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. For every $v \in V$, $N(v)$ denotes the
309 *neighborhood of v* that is the set of vertices adjacent to v . The graph G is *d -degenerate* if there
310 exists an ordering v_1, \dots, v_n of the vertices such that, for every i , $N(v_i) \cap \{v_{i+1}, \dots, v_n\}$ has
311 size at most d . It refines the notion of maximum degree since any graph of maximum degree
312 Δ are indeed Δ -degenerate (but the gap between Δ and the degeneracy can be arbitrarily
313 large). Let $u, v \in V$, a *path* from u to v is a sequence of vertices $v_0 = u, v_1, \dots, v_\ell = v$ such
314 that for every $i \leq \ell - 1$, $v_i v_{i+1}$ is an edge. It is a *cycle* if $v_\ell v_0$ also exists.

315 A graph G is *connected* if there exists a path from u to v for every pair $u, v \in V$. All
316 along the paper, we only consider connected graphs. Indeed, in certification, the vertices
317 can only communicate with their neighbors, so no vertex can communicate with vertices of
318 another connected component.

319 A vertex v is a *cut-vertex* if $G \setminus \{v\}$ is not connected. If G does not contain any cut-vertex,
320 G is *2-(vertex)-connected*. If the removal of any edge does not disconnect the graph, we say
321 that G is *2-edge-connected*. A graph is *k -(vertex)-connected* if there does not exist any set
322 X of size $k - 1$ such that $G \setminus X$ is not connected. To avoid cumbersome notations, we will
323 simply write *k -connected* for *k -vertex-connected*.

324 A graph H is a *minor of G* if H can be obtained from G by deleting vertices, deleting
325 edges and contracting edges. Equivalently, it means that, if G is connected, there exists a
326 partition of V into connected sets $V_1, \dots, V_{|H|}$ such that there is (at least) an edge between
327 V_i and V_j if $h_i h_j$ is an edge of H . We say that $V_1, \dots, V_{|H|}$ is a *model of H* . The graph G is
328 *H -minor-free* if it does not contain H as a minor.

329 2.2 Local computation and certification

330 We assume that the graph is equipped with unique identifiers in polynomial range $[1, n^k]$,
331 thus these identifiers can be encoded on $O(\log n)$ bits.

332 Local certification is a mechanism for verifying properties of labeled or unlabeled graphs.
333 In this paper we will use a local certification at distance 1, which is basically the model
334 called *proof-labeling schemes* [42]. A convenient way to describe a local certification is with
335 a prover and a verifier. The *prover* is an external entity that assigns to every vertex v a
336 certificate $c(v)$. The *verifier* is a distributed algorithm, in which every vertex v acts as
337 follows: v collects the identifiers and the certificates of its neighbor and itself, and outputs a
338 decision *accept* or *reject*. A local certification certifies a graph class \mathcal{C} if the following two
339 conditions are verified:

- 340 1. For every graph of \mathcal{C} , the prover can find a certificate assignment such that the verifier
341 accepts, that is, all vertices output *accept*.

342 2. For every graph not in \mathcal{C} , there is no certificate assignment that makes the verifier accept,
 343 that is for every assignment, there is at least one vertex that rejects.

344 The size of the certificate of \mathcal{C} is the largest size of a certificate assigned to a vertex of a
 345 graph of \mathcal{C} .

346 Note that to describe a local certification, the only essential part is the verifier algorithm,
 347 the prover is just a way to facilitate the description of a scheme.

348 In this paper, we are going to use a variant of the model above, called *edge certification*,
 349 where the certificates can be assigned on both the vertices and the edges. See Subsection 3.1.

350 2.3 Known building blocks for graph certification

351 There are few known certification schemes that we are going to use extensively as building
 352 blocks in the paper.

353 ► **Lemma 4** ([2, 42]). *Acyclicity can be certified in $O(\log n)$ bits.*

354 The classic way to certify that the graph is acyclic, is for the prover to choose a root
 355 vertex, and then to give to every vertex as its certificate its distance to the root. The vertices
 356 can simply check that the distances are consistent.

357 The same idea can be used to certify a *spanning tree* of the graph, encoded locally at
 358 each vertex by the pointer to its parent, which is simply the ID of this parent. The scheme is
 359 the same, except that the prover, in addition to the distances, gives the ID of the root, and
 360 the verification algorithm checks that all vertices have been given the same root-ID, and only
 361 takes into account the edges that correspond to pointers (also the root checks that its ID
 362 is the root-ID). A spanning tree is a very useful tool to broadcast the *existence of a vertex*
 363 *satisfying a locally checkable property*: simply choose a spanning tree rooted at the special
 364 vertex, encode it locally with pointers and certify it. Then the root can check that indeed it
 365 has the right property, and all the other vertices know that such a vertex exists.

366 Finally, with the same ideas, one can easily deduce $O(\log n)$ certification for paths. We
 367 just add to the acyclicity scheme the verification that the degree of every vertex is at most 2.
 368 Note that cycles do not need certificates to be verified: every vertex just checks that it has
 369 degree exactly 2.

370 Let us now define a graph class that will appear in several decompositions.

371 ► **Definition 5.** *A path-outerplanar graph is a graph that admits a path P that can be drawn*
 372 *on a horizontal line, such that all the edges that do not belong to P can be drawn above that*
 373 *line without crossings. The edges outside of P are said to be nested.*

374 We are going to use the following result as a black box.

375 ► **Lemma 6** ([28]). *Path-outerplanar graphs can be certified with $O(\log n)$ -bit certificates.*

376 The following classic result will also be useful at some point of the paper.

377 ► **Lemma 7** ([42]). *Every graph class can be certified with $O(n^2)$ bits.*

378 The idea of the scheme is that the prover gives to every vertex v the map of the graph,
 379 e.g. as an adjacency matrix, along with the position of v in this map. Then every vertex can
 380 check that it has been given the same map as its neighbors, and that the map is consistent
 381 with its neighborhood in the network.

3 Avoiding certificate congestion

One can obtain many structured graph classes like minor-free graphs with "gluing" operations, for instance, by identifying vertices of two graphs of the class. If we have a certification for both graphs, we would like to simply take both certificate assignments to certify the new graph. However, for the vertex on which the two graphs are glued, the size of the certificate might have doubled. While it is not a problem for bounded degree graphs, it can become problematic if many gluing operations occur around the same vertex, since this vertex would get an additional certificate from each operation. In this section, we present two ways to tackle these issues, that will be used in the forthcoming sections.

The first one consists in shifting the certification on edges instead of vertices, which helps in the sense that when gluing on vertices the edge certificate can remain unchanged. As we will see, the edge setting is equivalent to the usual vertex certification for nice enough classes. The second option uses that one can (almost) freely assume that a given vertex has an empty label in a correct certification.

3.1 Edge certification and degeneracy

Note that any vertex certification is a special case of an edge certification (taking empty edge certificates). We can even actually transfer the certificates from vertices to edges to get an equivalent edge certification with empty vertex certificates, without additional asymptotic costs: just copy on every edge the certificate of the two endpoints, and adapt the verification algorithm accordingly. Transforming an edge certification into a vertex certification is also always possible, by giving a copy of the edge label to each of its endpoints. But this transformation can drastically increase the certificate size: if an edge certification uses $\Omega(f(n))$ -bit labels, the associated vertex certification might use $\Omega(n \cdot f(n))$ -bit if the maximum degree of the graph is linear. The following theorem ensures that in degenerate graph classes there is a more efficient transformation that permits to drastically reduce the size of the certificate.

► **Theorem 8** ([27]). *Consider an edge certification of a graph class \mathcal{C} where only the edges are labeled, and get $f(n)$ -bit certificates. If \mathcal{C} is d -degenerate, then there exists a (vertex) certification with $d \cdot f(n)$ -bit certificates.*

Note that H -minor-free graphs have degeneracy $O(h\sqrt{\log h})$ where $h = |V(H)|$ [43, 52]. Therefore, we can freely put labels on edges when certifying classes defined by forbidden minors.

3.2 Certification with one empty label

In this part, our goal is to erase the certificate of a vertex. To this end, we first consider certification of spanning trees and strengthen both Lemma 4 and the discussion that followed in Subsection 2.3. We then extend this intermediate step to every graph class in Lemma 10.

► **Lemma 9.** *Let T be a spanning tree of G . There exists a certification of T that does not assign a label to the root, and uses the same certificate as the classic tree certification (cf. Subsection 2.3) on the other vertices.*

Proof. On *yes*-instances, the prover assigns the labels as in the classic scheme, and removes the label of the root. Then the verification proceeds like in the classic scheme except for a vertex that has no label or a vertex that has a neighbor with no label. If two adjacent

424 vertices have been given an empty label, then they reject. If a vertex with no label sees that
 425 two of its neighbors have been given different root-ID, then it rejects. Otherwise, every vertex
 426 simulates the computation where the vertex with empty label has been given distance 0, and
 427 the same root-ID as its neighbors. Because of the previous checks, the labels used in the
 428 simulation are consistent, and on correct instance are the same as the one used in the classic
 429 certification. Thus, the correctness follows from the correctness of the classic scheme. ◀

430 A *pointed graph* is a graph with one selected vertex. Given a class, one can build its
 431 pointed version by taking for each graph all the pointed versions of it.

432 ▶ **Lemma 10.** *Consider a class \mathcal{C} that can be certified with certificates of size $f(n)$. One can*
 433 *certify the pointed class of \mathcal{C} with $O(f(n) + \log n)$ bits, without having to put certificates on*
 434 *the selected vertex.*

435 **Proof.** First, to certify that exactly one vertex is pointed, we can simply find a spanning
 436 tree rooted on the pointed vertex and assign to each vertex the spanning tree certification of
 437 Lemma 9 which uses $O(\log n)$ bits. For the rest of the certification, on a *yes*-instance, the
 438 prover first assigns the certificates following the original certification. Then it removes the
 439 certificate of the selected vertex and appends copies of it to the certificates of its neighbors.

440 Every vertex v runs the following verification. If v is not the selected vertex, nor one of
 441 its neighbors, then it does the same verification as before. If v is the selected vertex, it checks
 442 that its neighbors have been given the same label as "label of the selected vertex", and then
 443 takes this label as its own, and runs the previous verification algorithm. If v is a neighbor of
 444 the selected vertex, it runs the same verification algorithm as before, but simulating that the
 445 selected vertex has been given the certificate that was appended to its own certificate.

446 All vertices are simulating the computation in the graph where the selected vertex would
 447 have been given its certificates, thus the correctness of this new certification follows from the
 448 correctness of the original certification. ◀

449 Observe that the previous results can be easily iterated: one can always remove the labels
 450 of k vertices (as long as they are pairwise non-adjacent) to the cost of a factor k in the size
 451 of the certificates. Therefore, the result extends to the case of k -independent pointed classes
 452 (i.e. where an independent set of size at most k is selected instead of only one vertex).

453 ▶ **Corollary 11.** *Consider a class that can be certified with certificates of size $f(n)$. One can*
 454 *certify the k -independent pointed class with $O(kf(n) + k \log n)$ bits, without having to put*
 455 *certificates on the selected vertices.*

456 Moreover, with more constraints on the structure of the set of pointed vertices (for
 457 instance if they are all at distance at least 3), one could even obtain certificate of size
 458 $O(f(n) + k \log n)$ (since every vertex receives the certificate of at most one selected vertex).

459 **4 Compositions of certifications**

460 In this section, we show how to combine certification algorithms for several classes to certify
 461 larger ones, and we illustrate this idea on two constructions. The first one considers classes
 462 defined by the existence of some subgraph: we settle the intuition stating that it is often
 463 easier to test the existence of a structure rather than its absence, since we can pinpoint which
 464 vertices/edges lie in the structure.

465 The second construction mimics a natural operation on graphs, consisting in replacing
 466 some vertex/edge by another graph. This operation occurs quite often in the literature:

467 many classes, especially the ones defined by forbidden minors, get a characterization using
468 this operation.

469 Some results of this section will not be used to certify minor-free classes later in the
470 paper. We nonetheless include them since they correspond to very natural operations which
471 may be useful for future work.

472 4.1 Subgraphs

473 ► **Proposition 12.** *Let \mathcal{C} be a graph class that can be certified with $f(n)$ -bit labels. Let \mathcal{C}' be
474 the class of the graphs that contain a graph of \mathcal{C} as subgraph. Then \mathcal{C}' can be certified with
475 certificates of size $O(f(n) + \log n)$ on the vertices and $O(1)$ on the edges.*

476 **Proof.** On a *yes*-instance G , the prover assigns the certificates on vertices and edges in the
477 following way. First, it chooses a subgraph H that belongs to \mathcal{C} and assigns the certificates
478 that certify that H is in \mathcal{C} , as if the rest of the graph did not exist. This takes at most $f(n)$
479 bits. Second to every vertex and edge that belongs to H , the prover assigns a special label.
480 Third, the prover describes and certifies a spanning tree pointing to a vertex that has the
481 special label.

482 The verification algorithm is the following. The vertices that have the special label, run
483 the verification algorithm for \mathcal{C} , taking into account only the vertices and edges that have
484 the special label. The vertices also check the spanning tree structure, and the root of the
485 tree checks that it does have the special label.

486 Because of the spanning tree, there must exist a vertex with the special label, thus there
487 are vertices that run the verification algorithm for \mathcal{C} , and if they succeed it means that a
488 graph of \mathcal{C} appears as a subgraph in the graph G . ◀

489 The edge certificates in Proposition 12 can be inconvenient if we want a classic certification
490 (without edge certificates) and if the graph is not assumed to be degenerate, which prevents
491 us from using Theorem 8. However, observe that we give non-empty certificates only to
492 the edges of the subgraph, hence we can obtain a vertex-certification when the class \mathcal{C} is
493 degenerate.

494 ► **Corollary 13.** *Let \mathcal{C} be a d -degenerate graph class that can be certified with $f(n)$ -bit labels.
495 Let \mathcal{C}' be the class of the graphs that contain a graph of \mathcal{C} as a subgraph. Then \mathcal{C}' can be
496 certified with certificates of size $O(f(n) + d \log n)$ on the vertices.*

497 Observe also that when considering induced subgraphs, we only have to specify which
498 vertices are special, hence we do not need edge certificates either. Note that, since we do not
499 need to label edges, we do not need the class \mathcal{C} to be degenerate.

500 ► **Corollary 14.** *Let \mathcal{C} be a graph class that can be certified with $f(n)$ -bit labels. Let \mathcal{C}' be the
501 class of the graphs that contain a graph of \mathcal{C} as an induced subgraph. Then \mathcal{C}' can be certified
502 with certificates of size $O(f(n) + \log n)$ on the vertices.*

503 4.2 Expansions

504 Two common operations in characterizations of graph classes are what we call vertex and
505 edge expansions.

506 ► **Definition 15.** *Consider two graph classes \mathcal{C}_1 and \mathcal{C}_2 .*

507 ■ The vertex expansion of \mathcal{C}_1 by \mathcal{C}_2 is the class of graphs obtained by the following operation.
 508 Take a graph G in \mathcal{C}_1 and replace every vertex v by a graph $H(v)$ in \mathcal{C}_2 , in such a way
 509 that for every edge $uv \in E(G)$, there is (at least) one edge between $H(u)$ and $H(v)$ in G
 510 (and no such edge if $uv \notin E(G)$).

511 ■ The edge expansion of \mathcal{C}_1 by \mathcal{C}_2 is the class of graphs obtained by the following operation.
 512 Take a graph G in \mathcal{C}_1 and replace every edge uv by a graph $H(u, v)$ from \mathcal{C}_2 , in such a
 513 way that the vertices of the original graph that are contained in $H(u, v)$ are exactly u and
 514 v .

515 We would like to have results of the form: if \mathcal{C}_1 and \mathcal{C}_2 can be certified with $f(n)$ and
 516 $g(n)$ -bit labels respectively, then the expansion can be certified with $O(f(n) + g(n))$ -bit
 517 labels. While the natural approach (almost) works for edge-expansion, it does not give such
 518 a result for vertex-expansion. However, we can actually make it work with a bound that
 519 takes into account the maximum degree of the expanded graph.

520 ► **Proposition 16.** Consider two graph classes \mathcal{C}_1 and \mathcal{C}_2 that can be certified with $f(n)$ -bit
 521 and $g(n)$ -bit labels respectively, where all the graphs of \mathcal{C}_1 have maximum degree Δ . Then the
 522 vertex-expansion of \mathcal{C}_1 by \mathcal{C}_2 can be certified with $O(\Delta \cdot f(n) + g(n) + \Delta \log n)$ -bit certificates.

523 **Proof.** Consider a graph $G \in \mathcal{C}_1$ on ℓ vertices v_1, \dots, v_ℓ of maximum degree Δ , and let
 524 H_1, \dots, H_ℓ be graphs of \mathcal{C}_2 . We consider the vertex expansion of G where every v_i is replaced
 525 by H_i .

526 On a *yes*-instance, the prover assigns the certificates the following way. First it assigns to
 527 every vertex the index i corresponding to the graph H_i it belongs to and the certification of
 528 the fact that H_i belongs to \mathcal{C}_2 (without taking into accounts the other vertices and edges).
 529 This takes at most $g(n) + \log n$ bits per vertex. Second, the prover gives to each vertex of
 530 H_i the original certificate of v_i that G belongs to \mathcal{C}_1 as well as the original certificate of all
 531 the vertices in $N(v_i)$ in G together with their names, which takes $O(\Delta f(n))$ bits. Finally,
 532 for every $v_j \in N(v_i)$, the prover chooses a vertex w_j in H_i adjacent to a vertex in H_j , and
 533 certifies a spanning tree of H_i rooted at w_j . This takes $O(\Delta \log n)$ bits.

534 The verification algorithm is the following. Every vertex (labeled as) in H_i checks that
 535 the number of trees corresponds to the degree of v_i in G . Every vertex checks the correctness
 536 of the different trees. Moreover, every root v of a spanning tree in H_i checks that it has a
 537 neighbor in the corresponding H_j . All the vertices of H_i check that their neighbors are in
 538 H_i or in some H_j with v_j incident to v_i in G . Every vertex of each H_i runs the verification
 539 algorithm to check that H_i does belong to \mathcal{C}_2 . Finally, every vertex of H_i simulates the
 540 verification of the original vertex v_i , which is possible since every vertex of H_i receives the
 541 certificate of v_i and all its neighbors in G . And every vertex $w_i \in H_i$ incident to $w_j \in H_j$
 542 checks that $v_j \in N_G(v_i)$ and that the certificate of w_j indeed contains the certificates of v_i
 543 and v_j given for G . ◀

544 ► **Proposition 17.** Consider two graph classes \mathcal{C}_1 and \mathcal{C}_2 that can be certified with $f(n)$ -bit
 545 and $g(n)$ -bit labels respectively. Then the edge-expansion of \mathcal{C}_1 by \mathcal{C}_2 can be certified with
 546 $O(f(n) + g(n) + \log n)$ -bit certificates on the edges.

547 **Proof.** We use a similar reasoning as for the proof of Proposition 16, except that we first
 548 transform the vertex certifications of \mathcal{C}_1 and \mathcal{C}_2 into edge certifications (by putting the label
 549 of a vertex on all the edges incident with it).

550 Consider a graph $G \in \mathcal{C}_1$. We consider the edge expansion of G where every uv is replaced
 551 by $H(u, v)$. Each edge e from $H(u, v)$ receives the labels of u and v , the certificate of uv

552 in G for \mathcal{C}_1 , and the certificate of e in $H(u, v)$ for \mathcal{C}_2 . Therefore, the certificates have size
 553 $O(f(n) + g(n) + \log n)$.

554 Now each vertex can check that all the edges labeled in some $H(u, v)$ share the same
 555 certificate for uv . There are two kinds of vertices: some where all incident edges are labeled
 556 as in the same $H(u, v)$, and the others (the original vertices of G). All of them run the
 557 verification algorithm for \mathcal{C}_2 by considering each group of incident edges labeled as in the
 558 same $H(u, v)$. The latter also recover the certificates of their neighbors in G from the edge
 559 labeling, and run the verification algorithm for \mathcal{C}_1 . ◀

560 Before giving deeper applications of these results in future sections, let us prove that the
 561 *existence* of a minor in the graph is easy to certify. This was already mentioned in previous
 562 papers without formal proofs [27, 28]. We prove it here to show a simple application of our
 563 techniques, and we think it is a meaningful illustration of the fact that certifying that a
 564 structure is present or absent are two very different tasks in our model.

565 ▶ **Corollary 18.** *Given a graph H , one can certify that an n -vertex graph has H as a minor
 566 in $O(\log n)$ bits.*

567 **Proof.** As we already observed, a graph G has H as minor if and only if $V(G)$ can be
 568 partitioned into $|H|$ connected sets such that there is an edge between V_i and V_j when the
 569 corresponding vertices in H are connected. Free to delete edges, we can assume that each
 570 V_i is actually a spanning tree and there is a unique edge from V_i to V_j if and only if the
 571 corresponding vertices are connected in H . In other words, G has a subgraph that is a vertex
 572 expansion of H by trees. Moreover, we can choose such a subgraph with degree at most
 573 $|H| - 1 = O(1)$ since H is fixed.

574 Let us start from a certification of H and build a certification of G . The structure of H
 575 can be certified in a brute-force way, by providing to every vertex the complete map of the
 576 graph which takes constant space (since H is fixed). Then, since trees can be certified in
 577 $O(\log n)$ bits, thanks to Proposition 16, any vertex-expansion of H by trees can be certified
 578 with $O(\log n)$ certificates.

579 We finally get a vertex certification with certificates of size $O(\log n)$ using Corollary 13. ◀

580 **5 Connectivity and connectivity decompositions**

581 In this section, we explain how to certify connectivity properties and connectivity decompo-
 582 sitions, in particular the block-cut tree mentioned in the introduction.

583 An *ear decomposition* is a way to build a graph by iteratively adding paths, the so-called
 584 *ears*. Ear decompositions are central tools for decades in structural graph theory and are used
 585 in many decomposition or algorithmic results. There exist various variants of this process,
 586 that characterize different classes and properties. For certification, these decompositions
 587 happen to be easier to manipulate than some other types of characterizations since they
 588 are based on iterative construction of the graph, and use paths, which are easy to certify.
 589 These paths are convenient since we can "propagate" some quantity of information on them
 590 as long as every vertex belongs to a bounded number of paths. In this section, we remind
 591 several such decompositions, and use them to certify various connectivity properties and
 592 decompositions.

593 **5.1 Connectivity properties**

594 Let us start with 2-connectivity. A graph G has an *open ear decomposition* if G can be
 595 built, by starting from a single edge, and iteratively applying the following process: take

596 two different vertices of the current graph and link them by a path whose internal vertices
 597 are new vertices of the graph (such a path is called *an ear*). Note that this path can be a
 598 single edge, and then there is no new vertex. Let an *inner vertex* of an ear be a vertex that
 599 is created with this ear, and let a *long ear* be an ear with at least one inner vertex.

600 ► **Theorem 19** ([54] (reformulated)). *A graph is 2-connected if and only if it has an open*
 601 *ear decomposition.*

602 We use this characterization to certify 2-connectivity.

603 ► **Lemma 20.** *2-connected graphs can be certified with $O(\log n)$ bits.*

604 **Proof.** First observe that one can obtain a long-ear decomposition from an ear decomposition
 605 (and vice versa) by removing/adding short ears, i.e. edges. Therefore, having an open ear
 606 decomposition is equivalent to having a subgraph with an open long-ear decomposition. Note
 607 that if a graph G has an open long-ear decomposition, then it is 2-degenerate. Indeed, the
 608 vertices of the last added long-ear have degree two and their removal is still a graph with an
 609 open long-ear decomposition. So in order to get the conclusion, Corollary 13 ensures that we
 610 only have to certify open long-ear decomposition with $O(\log n)$ bits per vertex.

611 The certification works as follows. First the prover gives to every vertex the identifiers
 612 of the very first edge, and describes and certifies a spanning tree pointing to one of the
 613 endpoints of this edge. The vertices of this edge are given an *index* 0. Second, the prover
 614 gives to every vertex the information related to the step when it has been added, and only
 615 about this step. That is, the prover gives the *index* of the addition (that is the number of
 616 the ear in which the vertex is created), along with two oriented paths spanning the path
 617 and pointing to the two extremities of the ear. By Corollary 11, these paths can be certified
 618 without certificates on the extremities of the paths.

619 Every vertex checks the correctness of the spanning tree pointing to the first edge, and the
 620 fact that only these vertices have index 0. Then, every vertex also checks that the spanning
 621 paths it has been given are correct, that is: (1) the distances and root-ID are consistent (2)
 622 all vertices have the same index, and that (3) the declared endpoints are different. Also, the
 623 two vertices that are adjacent to the endpoints of the paths check that the endpoints have a
 624 smaller index than their own.

625 Let us now prove the correctness of the certification. Because of the spanning tree, the
 626 original edge exists, is unique, and is the only set of vertices with index 0. Because of the
 627 certified paths spanning the ears, one can also recover the path structure and the fact that a
 628 path is added after its endpoints. Note that in an instance where all vertices accept, there
 629 might be two different paths with the same index i , but this is not a problem: the only
 630 important feature is the precedence order. ◀

631 With similar construction we can certify the edge connectivity instead of the vertex
 632 connectivity.

633 ► **Corollary 21.** *2-edge-connected graphs can be certified with $O(\log n)$ bits.*

634 **Proof.** Robbins proved in [48] that a graph G is 2-edge connected if and only if G has an
 635 ear decomposition. An ear decomposition is the same as an open ear decomposition, except
 636 that it starts from a cycle and that the two endpoints of an ear do not need to be different.
 637 The proof above can thus be adapted to this class.

638 The only difference is that vertices with index 0 form a cycle (which can be certified).
 639 Then during the verification procedure we simply do not have to check that the extremities
 640 of the path of the ear decomposition are distinct, in other words we do not have to check
 641 (3). ◀

642 A more refined type of ear decomposition characterizes the 3-connected graphs.

643 ► **Definition 22** ([12, 45, 51]). *Let ru and rt be two edges of a graph G . A Mondshein*
 644 *sequence through rt , avoiding u is an open ear decomposition of G such that:*

- 645 1. *rt is in the first ear.*
- 646 2. *the ear that creates vertex u is the last long ear, u is its only inner vertex, and it does*
 647 *not contain ru .*
- 648 3. *the ear decomposition is non-separating, that is, for every long ear except the last one,*
 649 *every inner vertex has a neighbor that is created in a later ear.*

650 ► **Theorem 23** ([12, 51]). *Let ru and rt be two edges of a graph G . The graph G is 3-*
 651 *connected if and only if it has a Mondshein sequence through rt avoiding u , and there are*
 652 *three internally vertex-disjoint path between t and u .*

653 ► **Corollary 24.** *3-connected graphs can be certified with $O(\log n)$ bits on vertices.*

654 **Proof.** On *yes*-instances the prover chooses an arbitrary edge ru and certifies the ear
 655 decomposition as in Lemma 20. The prover also adds a spanning tree pointing to the edges
 656 ru and rt , and gives to every vertex the index of the last long ear created. These new pieces
 657 of information allow the vertices to check that the ear decomposition is a Mondshein sequence.
 658 The prover also encode the three vertex disjoint paths, by pointer on the vertices of these
 659 paths, and number them 1, 2 and 3, to allow the vertices to check disjointness. ◀

660 5.2 Block-cut tree

661 Now that we can certify connectivity properties, we introduce a way to certify decomposition
 662 of graphs into parts of higher connectivity. Let us start with a few definitions.

663 A *2-connected component* of a graph G is a connected subgraph H maximal by inclusion
 664 such that the removal of one vertex does not disconnect H . Observe that a 2-connected
 665 component can consists of just one edge in the case of a bridge, i.e. an edge whose removal
 666 disconnects the graph.

667 The intersection of any pair C, C' of 2-connected components has size at most one. Indeed,
 668 if it had size at least two, then we could merge these into a larger 2-connected component,
 669 which would contradict the maximality. So we can define an auxiliary graph from G where
 670 every vertex corresponds to a 2-connected component and there is an edge between two
 671 components if and only if they intersect on exactly one vertex. This graph is a tree, because
 672 a cycle would again create a larger 2-connected component, contradicting maximality. This
 673 tree is called the *block-cut tree*.

674 Let T be a block-cut tree of G , and D a maximal 2-connected component chosen to
 675 be the root of this tree. (Note that if G is 2-connected then the graph is reduced to this
 676 component). Let C be a component that is not the root of the tree. The *connecting vertex*
 677 of a component C is the vertex lying both in C and in its parent component. The *interior*
 678 of C is the set of vertices of C minus the connecting vertex of C . Note that the interior of a
 679 component is always non-empty.

680 This section is devoted to proving the following result and apply it for certification:

681 ► **Theorem 3.** *For any 2-connected graph H , if the 2-connected H -minor-free graphs can be*
 682 *certified with $f(n)$ bits, then the H -minor-free graphs can be certified with $O(f(n) + \log n)$*
 683 *bits.*

684 **Proof of Theorem 3.** Since H is 2-connected, a graph G is H -minor-free if and only if each
 685 of its 2-connected components is. (This is basically the observation we made at the beginning

686 of Subsection 1.2.) This is the property we certify. On a *yes*-instance, the prover will assign
 687 the certificates the following way. It first computes the block-cut tree and root it on some
 688 vertex C . It then does the following:

- 689 1. For each 2-connected component, the prover chooses a vertex from the interior of the
 690 component to be the *leader* of this component. Every vertex of the interior of a component
 691 C is given the identifier of the leader of C as well as a spanning tree of C pointing towards
 692 it. Since the component is 2-connected, the component minus the leader of the component
 693 is connected and such a tree exists.
- 694 2. Every vertex is given a label stating whether it is a connecting vertex or not.
- 695 3. Every vertex is given the identifier of the connecting vertex of its component closest from
 696 the root in the block-cut tree (called the *component* of the vertex), as well as a spanning
 697 tree pointing to it, using the certification of Lemma 9 that uses an empty certificate on
 698 the root.
- 699 4. In order to check acyclicity of the block-cut tree, every vertex is given the distance of its
 700 component to the root-component (in terms of number of components).
- 701 5. The prover certifies the 2-connectivity of each component using the certification of
 702 Lemma 20 and the fact that it is H -minor-free using the certification with $f(n)$ bits of
 703 the theorem. By Lemma 10 this can be done by only assigning labels to the interior
 704 vertices of the component.

705 Before we move on to the verification and the correctness of the scheme, note that every
 706 vertex is given a certificate of size $O(f(n) + \log n)$. Indeed, each piece of information we
 707 have given to the vertex is of size $O(\log n)$ or $f(n)$, and we have given a constant number of
 708 those to every vertex. In particular, a connecting vertex in the interior of a component C ,
 709 received only labels that are related C , and not labels related to other components it belongs
 710 to (since we consider pointed components).

711 Now, every vertex does the following verification. Every vertex checks that the spanning
 712 tree pointing to the leader is correct. If this step succeeds, we have a partition of the vertices
 713 in components. Every vertex also checks the correctness of the spanning tree pointing to the
 714 connecting vertex.

715 Every vertex v checks that, if it has an edge to a connecting vertex w with a different
 716 leader, then w is the connecting vertex of its own component. Every connecting vertex v
 717 checks that it is connected to a single vertex in its parent component and that it is the
 718 claimed neighbor in that component. If this step succeeds, we have a decomposition into
 719 components linked by connecting vertices. The consistency of the component distances are
 720 also checked by the vertices: this distance should be decremented at each connecting vertex,
 721 and only there. This ensures the acyclicity of the component structure. Finally, every vertex
 722 checks that the 2-connectivity and the H -minor-freeness of its component. Globally this
 723 verification ensures that the graph is H -minor-free. ◀

724 **6 Application to C_4 , C_5 , Diamond, K_4 and $K_{2,3}$ minor-free graphs**

725 This section is devoted to the certification of C_4 -minor-free, diamond-minor-free graphs, K_4 -
 726 minor-free graphs and $K_{2,3}$ -minor-free graphs. All the proofs will follow the same structure:
 727 prove that the 2-connected components, which are more structured, can be certified with
 728 small labels, and then use Theorem 3 to conclude for the general case.

729 Before proving this, let us describe how to certify *series-parallel graphs*, which in addition
 730 to being interesting network topologies [30], are closely related to K_4 -minor-free graphs [17].

731 ► **Definition 25.** A (2-terminal) series-parallel graph is a graph with two labeled vertices
 732 called the source and the sink that can be built recursively as follows. A single edge is a
 733 series-parallel graph where one endpoint is the source and the other is the sink. Let G_1, G_2 be
 734 two series-parallel graphs. The series of G_1 and G_2 which consists in merging the sink of G_1
 735 and the source of G_2 is a series-parallel graph. The parallel of G_1 and G_2 , which consists in
 736 merging the sources of G_1 and G_2 together and merging the sinks of G_1 and G_2 together, is
 737 a series-parallel graph.¹

738 A nested ear decomposition is an open ear decomposition that starts from a path, with
 739 two properties: (1) both ends of an ear e have to lie on a same ear added before e , and (2)
 740 for every ear e , the subpaths of e linking endpoints of ears plugged on e are pairwise either
 741 edge-disjoint or included one in the other. We say that the ears plugged on e are nested.
 742 Eppstein proved the following in [20] about series-parallel graphs.

743 ► **Theorem 26** ([20]). A 2-connected graph is series-parallel if and only if it has a nested
 744 ear decomposition.

745 We will use this decomposition theorem for our certification.

746 ► **Theorem 27.** 2-connected series-parallel graphs can be certified with $O(\log n)$ -bit labels.

747 **Proof.** The prover certifies the decomposition of Theorem 26. We have already described
 748 how to certify an open ear decomposition in the proof of Lemma 20. We can easily adapt it
 749 so that it starts from a path instead of an edge: there is a spanning tree pointing to one of
 750 the endpoints of the paths, and the path itself is certified with distances, the usual way.

751 It is also easy to certify that each ear e has both of its endpoints on the same older ear e' :
 752 just give to each vertex of e the identifiers of the endpoints of e and e' . The endpoints of an
 753 ear can check the consistency of these announced identifiers with the identifiers of their paths.
 754 A more tricky part is to certify that the ears are nested. Remember that Lemma 6 states
 755 that a path with nested edges (a path-outerplanar graph) can be certified with $O(\log n)$ -bit
 756 labels. This is exactly what we need except that we would like to have nested paths instead
 757 of nested edges. But then we can transfer the information from one endpoint of the paths to
 758 the other endpoint. ◀

759 ► **Lemma 28.** 2-connected C_5 -minor-free graphs are either graphs of size at most 4 or $K_{2,p}$
 760 or $K'_{2,p}$ which is the complete bipartite graph $K_{2,p}$ plus an edge between the two vertices on
 761 the set of size 2.

762 **Proof.** Since G is 2-connected, by Menger's theorem, for every pair x, y of non-adjacent
 763 vertices, there exist at least two vertex disjoint xy -paths. Since G is C_5 -minor-free, these
 764 paths have size at most 2, in particular x, y are at distance at most 2.

765 Let u, v be two non-adjacent vertices. Then the removal of $N(u) \cap N(v)$ disconnects u
 766 from v since otherwise we can find two vertex disjoint uv -paths, one being of size at least 3,
 767 which provides a C_5 . In particular, it implies that $|N(u) \cap N(v)| \geq 2$ since G is 2-connected.

768 Let $x \in N(u) \setminus (N(v) \cup \{v\})$. Since x, v are non-adjacent, they are at distance 2 hence
 769 there must be an edge between x and $N(v)$. But this creates a C_5 since $|N(u) \cap N(v)| \geq 2$.
 770 Therefore non-adjacent vertices have the same neighborhood.

771 Let I be a maximum independent set in G . Note that all the vertices of I have the same
 772 neighborhood. Therefore, by maximality, if $u \notin I$ then u is adjacent to every vertex of I .

¹ We only keep a single edge for each multi-edge created by this process.

773 Now either vertices of I have degree at least 3, and G contains $K_{3,3}$ hence a C_5 -minor, or
 774 vertices of I have degree 2 and G is $K_{2,p}$ or $K'_{2,p}$. ◀

775 We can now easily prove the claimed certifications.

776 ▶ **Corollary 29.** *The following classes of graphs can be certified with $O(\log n)$ bit certificates:*
 777 *C_4 -minor-free graphs, C_5 -minor-free graphs, diamond-minor-free graphs, house-minor-free*
 778 *graphs², outerplanar graphs (that is $(K_{2,3}, K_4)$ -minor-free graphs), $K_{2,3}$ -minor-free and*
 779 *K_4 -minor-free graphs.*

780 **Proof.** By Theorem 3, if we can certify the 2-connected graphs of these classes we obtain
 781 the conclusion. So we simply have to prove that for each class we can certify the 2-connected
 782 graph of the class.

783 ■ 2-connected C_4 -minor-free graphs are K_2 and K_3 [13], which can be certified with $O(1)$
 784 bits.

785 ■ 2-connected C_5 -minor-free graphs are either graphs of size at most 4 or a complete
 786 bipartite graph $K_{2,p}$ (with a potential edge between the two vertices in the set of size
 787 2 by Lemma 28. Since such graphs can be certified with $O(\log n)$ bits, the conclusion
 788 follows.

789 ■ 2-connected diamond-minor-free graphs are induced cycles. Cycles can be certified with
 790 $O(1)$ bits (see the discussion after Lemma 4).

791 ■ 2-connected house-minor-free graphs are either induced cycles or graphs of size at most
 792 four. Indeed, assume that there is a cycle of length at least 5. Then it should be induced,
 793 since otherwise it contains a house as a minor. Moreover, it should contain all the vertices
 794 of the graph otherwise there is an ear starting from this cycle and the cycle plus the
 795 ear provides a house. Since induced cycles can be easily certified with $O(\log n)$ bits, the
 796 conclusion follows.

797 ■ 2-connected outerplanar graphs are exactly path-outerplanar graphs with an edge between
 798 the first and the last vertex. Indeed, by 2-connectivity, the outer face must be a cycle,
 799 and removing any edge from it yields a path-outerplanar graph. One can then certify the
 800 existence and uniqueness of this edge using a spanning tree, and then certify that the rest
 801 of the graph is path-outerplanar. This yields a $O(\log n)$ -bit certification by Theorem 6.

802 ■ Let G be a 2-connected $K_{2,3}$ -minor-free graph. If G does not contain K_4 as a minor,
 803 then it is a 2-connected outerplanar graph and the result follows from the previous item.
 804 Otherwise, G contains K_4 as a minor. Observe that in particular G contains a subdivision
 805 of K_4 as a subgraph (by keeping only the edges of a spanning tree in each bag). But all
 806 such subdivisions contain $K_{2,3}$ as a minor except for K_4 itself. Therefore, G is K_4 , which
 807 can be certified easily.

808 ■ The 2-connected K_4 -minor-free graphs are exactly the 2-connected series-parallel graphs [17].
 809 Then the results follow directly from Theorem 27. ◀

810 **7 Application to $K_{2,4}$ -minor-free graphs**

811 When the size of the minors are increasing (and for most of the decomposition theorems
 812 known in structural graph theory), 2-connectivity is not enough. In this example we will
 813 illustrate how to use the certificate of 3-connectivity.

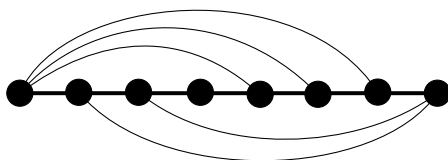
² The *house* being a C_4 plus a vertex connected to two consecutive vertices of the C_4 .

814 Let us illustrate it for this section on the characterization of $K_{2,4}$ -minor-free graphs
 815 from [19]. It is more involved than the other characterizations we have seen so far. We
 816 will follow the structure of [19], restricting first to 3-connected graphs, then to 2-connected
 817 graphs, and finally all $K_{2,4}$ -minor-free graphs.

818 7.1 3-connected case

819 Let us start with the definition of a graph class. We use notations similar to [19] (Section 2.1).
 820 See Figure 6.

821 ► **Definition 30.** *Let n, r, s be three integers, and p a Boolean. The graph $G_{n,r,s,p}$ consists
 822 of a path v_1, \dots, v_n , the edges v_1v_{n-i} for $1 \leq i \leq r$, the edges v_nv_{1+j} for $1 \leq j \leq s$, and
 823 the edge v_1v_n if $p = 1$. For a function $f(n, r, s, p)$ that associates a boolean value with each
 824 combination of parameters, let $\mathcal{G}[f]$ be the set of graphs $G_{n,r,s,p}$ such that $f(n, r, s, p) = 1$.*



825 ■ **Figure 6** Example for Definition 30. This graph is $G_{8,3,2,0}$: it has 8 vertices, edges v_1v_{n-i} for
 826 $i \in \{1, 2, 3\}$, edges v_nv_{1+j} for $j \in \{1, 2\}$, and it does not have the edge v_1v_n .

827 ► **Theorem 31** (Theorem 2.12 in [19] (adapted)). *There exists an f such that the set of
 828 3-connected $K_{2,4}$ -minor-free graphs is $\mathcal{G}[f]$, plus nine graphs on at most 8 vertices.*

827 In [19], the authors give an explicit description of f but we can avoid going into details
 828 here because of the following general lemma.

829 ► **Lemma 32.** *For all f , $\mathcal{G}[f]$ can be certified with $O(\log n)$ -bit labels.*

830 **Proof.** On a *yes*-instance, the prover certifies the spanning paths with root v_1 , with last
 831 vertex v_n , and writes in each certificate the values n, r, s and p . The vertices check the
 832 structure of the path and the fact that n, r, s and p are the same on all vertices. Second v_1
 833 checks the structure of its neighborhood, and in particular the values r and p . Similarly, v_n
 834 checks the structure of its neighborhood, and in particular the values s , and the fact that
 835 its distance to the root is indeed n . Finally, all vertices check that $f(n, r, s, p) = 1$. The
 836 correctness of the scheme is straightforward. ◀

837 This directly yields the following lemma.

838 ► **Lemma 33.** *3-connected $K_{2,4}$ -minor-free graphs can be certified with $O(\log n)$ -bit labels.*

839 **Proof.** Consider a *yes*-instance. By Theorem 31, either it is one of the nine small graphs of
 840 Theorem 31, and then we can use a constant size certification, or it is a graph of $\mathcal{G}[f]$ for the
 841 specific f of Theorem 31, and then we can use Lemma 32. ◀

842 For the 2-connected case, one of the types of graphs that we want to certify is of the
 843 following form: a 3-connected graph, where a set of edges with a special property is expanded

844 with another graph class. To be able to certify this, we will need the vertices to check that
 845 the set of edges that has been expanded has the special property. To capture the notion of
 846 special property, without going into the intricate details of what this property is exactly,
 847 let us define an *edge-set decider*. A function h is an edge-set decider if it takes as input a
 848 3-connected $K_{2,4}$ -minor-free graph whose edges are either unlabeled, or labeled with a special
 849 label, and outputs a Boolean.

850 ► **Lemma 34.** *Let h be an edge-set decider, such that for every graph G , there is at most*
 851 *$O(n)$ different sets of edges S such that $h(G, S) = 1$. The set of 3-connected $K_{2,4}$ -minor-free*
 852 *graphs G with labelled edges, where $h(G)$ is true, can be certified with $O(\log n)$ bits.*

853 **Proof.** First, for every graph G , we fix an indexing of edge sets S such that $h(G, S) = 1$.
 854 The prover first uses the same certificates as in Lemma 33 for the certification of unlabeled
 855 3-connected $K_{2,4}$ -minor-free graphs. Then it gives to all vertices the index of the set of
 856 labeled edges. Following the certification of the proof of Theorem 31, every vertex knows in
 857 which graph it lives and what is its position in that graph. Then, every vertex just checks
 858 that the labeled edges in its neighborhood correspond to the index announced by the prover.
 859 The labels have size $O(\log n)$ because of Lemma 33 and because there are at most $O(n)$
 860 different sets of edges S such that $h(G, S) = 1$. ◀

861 7.2 2-connected case

862 We now state the characterization theorem of the 2-connected case.

863 ► **Theorem 35** (Theorem 3.5 in [19]). *There exists a function h as in Lemma 34 such that*
 864 *the following holds. A graph G is 2-connected $K_{2,4}$ -minor-free graph if and only if one of the*
 865 *following holds:*

- 866 1. G is outerplanar.
- 867 2. G is the union of three path-outerplanar graphs H_1, H_2, H_3 with the same path endpoints x
 868 and y , and possibly the edge (x, y) , where $|V(H_i)| \geq 3$, for each i and $V(H_i) \cap V(H_j) = x, y$
 869 for $i \neq j$.
- 870 3. G is obtained from a 3-connected $K_{2,4}$ -minor-free graph G_0 by choosing a subset S
 871 such that $h(G_0, S) = 1$, and replacing each edges (x_i, y_i) of S by a path-outerplanar
 872 graphs H_i with endpoints (x_i, y_i) , where $V(H_i) \cap V(G_0) = \{x_i, y_i\}$ for each i , and
 873 $V(H_i) \cap V(H_j) \subset V(G_0)$ for $i \neq j$.

874 In [19], h is called the set of subdividable edges, and is fully characterized. Our proof
 875 works for any h , as long as it satisfies the properties of Lemma 33, and it is the case for the
 876 h of [19].

877 ► **Lemma 36.** *2-connected $K_{2,4}$ -minor-free graphs can be certified with $O(\log n)$ -bit labels.*

878 **Proof.** We show that each of the three cases can be certified with $O(\log n)$ bits.

- 879 1. Outerplanar graphs can be certified with $O(\log n)$ bits (Corollary 29).
- 880 2. This case basically consists in an edge expansion of a multigraph with three edges between
 881 two vertices by path outerplanar graphs. Note that the proof of Proposition 17 works
 882 here even if the original graph is a multigraph. Proposition 17 gives us an $O(\log n)$ edge
 883 certification because path-outerplanar graphs can be certified with $O(\log n)$ certificates,
 884 and the condition on the number of vertices can also be certified with $O(\log n)$ bits with
 885 a spanning tree counting the number of vertices (see *e.g.* in [24]). This edge certification
 886 can be transferred to a vertex certification with the same certificate size asymptotically
 887 because of Theorem 8, and because H -minor-free graphs have bounded degeneracy.

888 3. Again, this item basically corresponds to an edge-expansion: the edge expansion of a
 889 3-connected $K_{2,4}$ -minor-free graph by path-outerplanar graphs. We know by Lemma 33
 890 and Lemma 6 that both these classes can be certified on $O(\log n)$ bits, so the vanilla
 891 edge-expansion can also be certified with $O(\log n)$ bits (using the degeneracy like in the
 892 previous item). The only issue left is the fact that the only edges of G_0 that are allowed
 893 to be expanded by something different from an edge need to belong to an S such that
 894 $h(G_0, S) = 1$. But this is easy with Lemma 34: the edges that have a path-outerplanar
 895 expansion are the one that are considered to have a special label. ◀

886 **8** Certifying H -minor-free graphs with $|H| \leq 4$

887 In previous sections, we have proven that certifying H -minor-free graphs can be done with
 888 $O(\log n)$ bits for some graphs H . The graphs we have treated in previous sections are
 889 somehow amongst the hardest graphs of small size. When the connectivity of the graph
 890 H increases, the class of H -minor-free graph contains more and more graphs, and then is
 891 (morally speaking) harder to certify. Let us prove that the other graphs on 4 vertices (which
 892 have fewer edges, and then are less connected) can also be certified, with arguments either
 893 simpler than or similar to what has been done in previous sections, to establish the following
 894 theorem.

895 ▶ **Theorem 2.** *H -minor-free classes can be certified in $O(\log n)$ bits when H has at most 4
 896 vertices.*

897 We consider two cases depending on whether H contains a cycle.

898 ▶ **Lemma 37.** *If $|H| \leq 4$, and H contains a cycle, then H -minor-free graphs can be certified
 899 with $O(\log n)$ bits.*

900 **Proof.** Since H contains a cycle, either it is C_4 , and the result follows from Corollary 29, or
 901 it contains a triangle. Let us distinguish the cases depending on how the fourth vertex is
 902 connected to the triangle. If it is connected to two or three vertices, then H is either K_4 , or a
 903 diamond, and then H -minor-free graphs can be certified with $O(\log n)$ bits by Corollary 29.

904 So we can assume that H is a triangle plus one vertex attached to at most one vertex
 905 of the triangle. If G contains a cycle, let C be a shortest cycle in G , that is a cycle that
 906 contains the minimum number of vertices. Then C must contain all the vertices of the graph.
 907 Indeed, otherwise, since G is connected, there exists a vertex v attached to C , and $v \cup C$
 908 contains H as a minor. Therefore, G is either a cycle or a tree, which can be both certified
 909 in $O(\log n)$ bits, see Subsection 2.3. ◀

910 ▶ **Lemma 38.** *If $|H| \leq 4$ and H is acyclic, then we can certify H -minor-free graphs with
 911 $O(\log n)$ bits.*

912 **Proof.** If H has an isolated vertex then any graph G contain H as a minor as long as G
 913 contains a (non necessarily induced) path on three vertices and an isolated vertex. Since this
 914 property holds for every connected graph on 4 vertices, the conclusion follows. If H consists
 915 of two independent edges, then each H -minor-free graph is either a triangle or a star, which
 916 can be certified with $O(1)$ bit certificates.

917 So we can assume that H is connected. There are only two acyclic connected graphs on 4
 918 vertices: the star $S_{1,3}$ with 3 leaves, and the path P_4 on four vertices. If G does not contain
 919 a star with 3 leaves as a minor, it means that G is either a path or a cycle which can be
 920 easily certified. If G does not contain a path on four vertices as a minor, it means that G is a

931 star which, can be certified the following way. Give the identifier of the center to all vertices,
 932 and let the vertices check that they have been given the same ID, and that the non-center
 933 vertices have exactly one neighbor, and that this neighbor has this ID. ◀

934 This completes the picture for graphs H on at most 4 vertices.

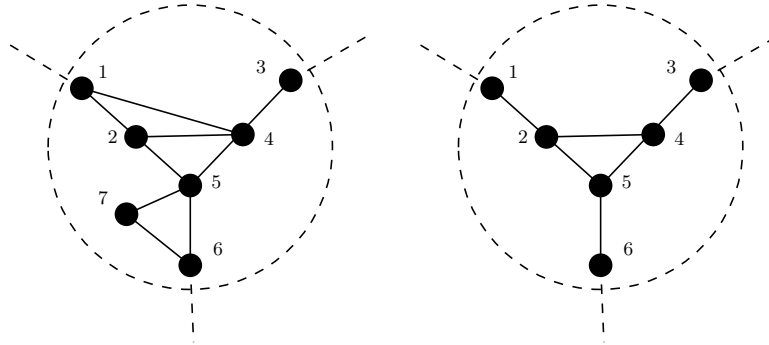
935 **9** Graphs on at most 5 vertices

936 Let us now focus on graphs H with at most 5 vertices. We were not able to deal with all of
 937 them, the most problematic one being K_5 , as we will discuss later on. However, we proved
 938 that H -minor-freeness can be certified for some dense graphs like $K_{2,3}$. The goal of this
 939 section is to provide evidence that again, the hardest case will be the case where H is dense.
 940 Before entering into the details of the proof, let us study some necessary conditions on the
 941 graph to be minimally not H -minor-free.

942 **9.1** H -minimal graphs

943 A graph G is H -minimal if G admits a H -minor but, for any vertex v , $G \setminus v$ does not admit
 944 any H -minor. Note that if H is not connected, H -minimal graphs may not be connected.
 945 Our goal is to study the structure of H -minimal graphs when H has at most 4 vertices, and
 946 show that they fall into a few easily-certifiable types.

947 Consider a model $V_1, \dots, V_{|H|}$ of H in a H -minimal graph G . Intuitively, for all i , the
 948 important part of the subgraph induced by V_i is a spanning tree that makes it connected,
 949 and connected to the neighboring V_j 's. For example, if a V_i contains a vertex that is only
 950 connected to other vertices of V_i , and whose removal does not disconnect the subgraph of
 951 V_i , then this vertex is unessential. In other words, such a vertex would not appear in a
 952 H -minimal graph, because we could remove it, and still have a model of H . Nevertheless, it
 953 is not true that the subgraph induced by every V_i is a tree (see Figure 7).



954 **Figure 7** The two pictures represent some set V_i in a H -model. The dashed edges represent
 955 connections to other vertices of the model. In the first picture, the graph cannot be H -minimal,
 956 indeed we can remove the vertices 7 and 2, and still have a proper model. In the second picture, no
 957 vertex can be removed without disconnecting the subgraph induced by V_i .

954 We now describe what the V_i 's subgraphs precisely look like in a H -minimal graph. Let
 955 T be a graph, and S_1, \dots, S_r be some prescribed subsets of vertices of T . A *Steiner tree* of T
 956 with respect to the S_i 's is a tree in T containing at least one element of each S_i . We say that
 957 T is an *almost tree* for the S_i 's if any Steiner tree with respect to the S_i 's contains all the
 958 vertices of T . Now, given a model $V_1, \dots, V_{|H|}$ of H , and $v_i \in H$, the prescribed sets we are

959 going to consider for V_i are the subsets $S_j \subseteq V_i$ containing all the vertices connected to V_j ,
 960 for every j such that $v_i v_j$ is an edge of H . A Steiner tree of V_i for the model $V_1, \dots, V_{|H|}$ of
 961 H is a Steiner tree containing at least one vertex of each prescribed set. When the model is
 962 clear from context, we simply say a Steiner tree of V_i .

963 With these notions, let us describe some properties of H -minimal graphs:

964 ► **Lemma 39.** *Let H be a graph and let G a H -minimal graph. For every H -model of G ,*
 965 *each V_i is an almost tree.*

966 **Proof.** The proof is straightforward. If some V_i is not an almost tree, then we can select a
 967 subset V'_i of V_i which is an almost tree. When we consider the subsets where all the V_j 's are
 968 the same but V_i which is replaced V'_i , we still have a model of H , and it does not contain all
 969 the vertices, a contradiction with the fact that G is H -minimal. ◀

970 It follows that we can characterize the form of the V_i 's such that h_i has small degree in
 971 H .

972 ► **Corollary 40.** *Let H be a graph, and G be a H -minimal graph. There exists a H -model of*
 973 *G such that:*

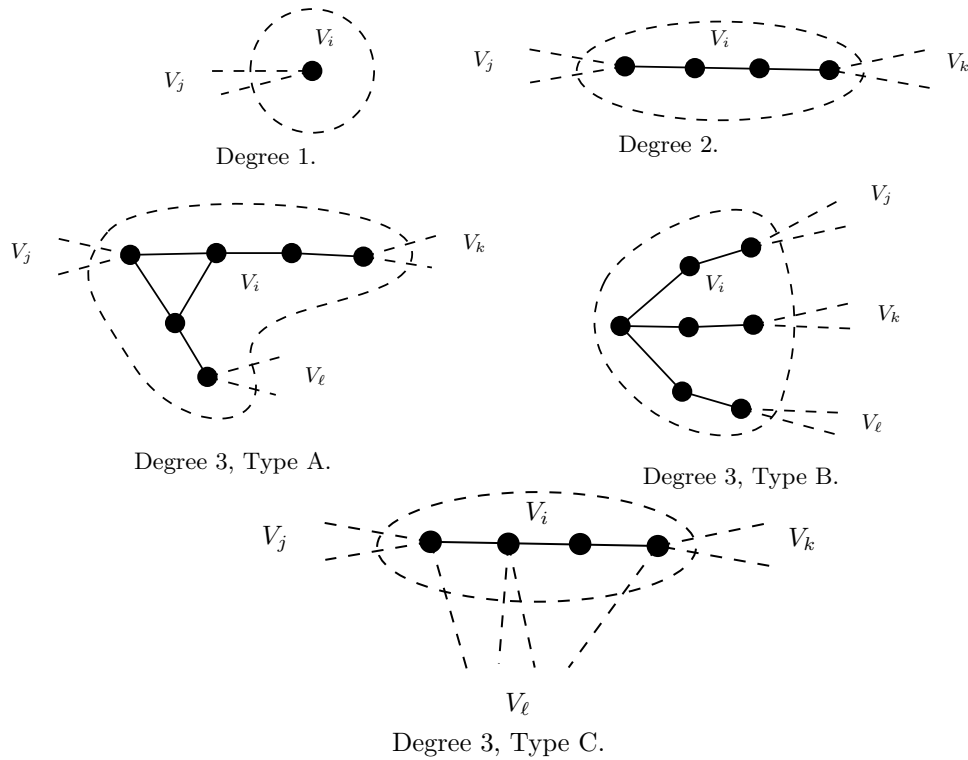
- 974 1. *If the degree of h_i in H is one, then V_i is reduced to a single vertex.*
- 975 2. *If the degree of h_i in H is two, then V_i is reduced to a single path P and, if h_j, h_k are*
 976 *the two neighbors of h_i then exactly one endpoint of P is connected to V_j , the other is*
 977 *connected to V_k , and all the other vertices of P are neither connected to V_j nor V_k .*
- 978 3. *If the degree of h_i in H is three, then the subgraph induced by V_i is of one of the three*
 979 *following types:*
 - 980 – *Type A: the subgraph is a triangle with a path attached to each of the three corners*
 981 *(which might be reduced to a single vertex) where the other endpoint of the path is*
 982 *attached to a V_j , and no other vertex is attached to V_j .*
 - 983 – *Type B: the subgraph is an induced subdivided star where only the last vertex of each*
 984 *branch is connected to a set V_j , and in that case it is connected to exactly one V_j .*
 - 985 – *Type C. the subgraph is a path, and there exists j, k such that the only connections with*
 986 *V_j and V_k are on the endpoints of the path. Any connection is possible for the vertices*
 987 *of the path with the last set V_ℓ .*

988 **Proof.** 1. If the degree of h_i is one, then a Steiner tree only needs the vertex that is connected
 989 to the rest of the model, so V_i has only one vertex.

990 2. If the degree is two, then any Steiner tree contains a path between a vertex in V_j and a
 991 vertex in V_k , and the shortest such paths is an induced path, thus the subgraph induced
 992 by V_i must be an induced path, with only the endpoints connected to the rest of the
 993 graph.

994 3. For degree 3, there are several cases.

- 995 – If V_i contains a cycle, then by minimality the removal of any vertex on this cycle
 996 disconnects V_i from another branch. It follows that V_i has type A.
- 997 – If V_i does not have a cycle, it has at most three leaves. If it has exactly three leaves
 998 then it has type B.
- 999 – Otherwise, V_i is a path, and by the degree-2 case, only the endpoints can connect to
 1000 some sets V_j and V_k , but the connections to the third set V_ℓ are not controlled. This
 1001 is type C. ◀



■ **Figure 8** The types of the V_i 's in Corollary 40

1002 9.2 H with an isolated vertex and extension

1003 ► **Theorem 41.** *Let H be a graph on 5 vertices containing an isolated vertex. We can certify*
 1004 *H -minor-free graphs with certificates of size $O(\log n)$.*

1005 The rest of this section is devoted to the proof of Theorem 41.

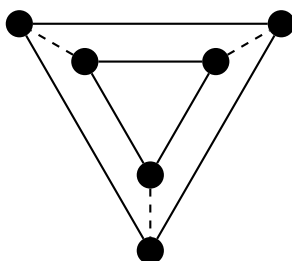
1006 Let H' be the graph H where an isolated vertex has been removed. A H -minor-free graph
 1007 is either a H' -minor-free graph, or it is a graph G such that all the models of H' contain
 1008 all the vertices of G . Since H' -minor-free graphs can be certified within $O(\log n)$ bits by
 1009 Theorem 2, we can assume that G is a connected H' -minimal graph.

1010 The core of the proof relies on the following lemma.

1011 ► **Lemma 42.** *For every graph H' on four vertices, any H' -minimal graph is in one of the*
 1012 *following categories:*

- 1013 1. *subdivided copies of H' , that are graphs obtained from H' by replacing each edge by some*
 1014 *path,*
- 1015 2. *graphs of size 4,*
- 1016 3. *induced cycles,*
- 1017 4. *induced cycle plus a vertex,*
- 1018 5. *the graphs of the type of Figure 9,*
- 1019 6. *graphs with at most six vertices of degree larger than 2,*

1020 Before proving the lemma, let us use it to conclude the proof of Theorem 41.



■ **Figure 9** This drawing represents a class of graphs built by taking two vertex-disjoint triangles, and linking pairs of corners of the triangles by vertex-disjoint paths.

1021 **Proof of Theorem 41.** Observe that all the classes of Lemma 42 can be certified with
 1022 certificates of size $O(\log n)$, and in these classes it is easy to certify H' -minimality. This is
 1023 because in all these classes there is a constant number of special vertices: the vertices before
 1024 the subdivision for Item 1, the additional vertex for Item 4, the corners of the triangles for
 1025 Item 5, the vertices of degree larger than 2 in Item 6, and none for Items 2 and 3. The
 1026 vertices that are not special have degree 2. Certifying these classes boils down to having
 1027 spanning trees pointing to the special vertices, and having a certification of every path of
 1028 non-special vertices, to transfer the knowledge of the endpoints from one side of the paths to
 1029 the other. Then basic consistency checks verify the certification. Because the structure is so
 1030 constrained, it is easy also to check whether the graph is H' -minimal. ◀

1031 Since the graph G is connected, the same proof holds for a graph H' plus a single vertex
 1032 of degree one as long as all the vertices of H' are equivalent. A graph is *vertex transitive* if
 1033 for every pair of vertices (u, v) , there exists an automorphism of H mapping u to v . We thus
 1034 get the following.

1035 ▶ **Lemma 43.** *Let H be a graph on 5 vertices obtained by adding a pending edge to a vertex*
 1036 *transitive graph. Then H -minor-free graph can be certified with $O(\log n)$ bits.*

1037 We end this section with the proof of Lemma 42. The proof is quite technical, so we split
 1038 it into several claims. First let v_1, \dots, v_4 be the vertices of H' , fix an H' -minimal graph G
 1039 and denote by V_1, \dots, V_4 a model of H' in G .

1040 ▷ **Claim 44.** If H' is acyclic, then G has size at most 4 (Item 2).

1041 **Proof.** If H' is acyclic, then it is either a star with 3 leaves or a disjoint union of paths. Using
 1042 Corollary 40 in the latter case, observe that if G contains H' as a minor, it also contains H'
 1043 as a subgraph. In particular, if G has more than 4 vertices, then G is not H' -minimal, since
 1044 one can remove a vertex while keeping H' as a subgraph. ◀

1045 ▷ **Claim 45.** If H' has at most one degree-3 vertex, then G either has size 4, or is an induced
 1046 cycle plus at most one vertex (Items 2, 3 and 4).

1047 **Proof.** By the previous claim, we may assume that H' contains a cycle, which is C_3 or C_4
 1048 due to the size of H' .

1049 If $H' = C_4$, it means that G contains a cycle of size at least 4 and that every such cycle
 1050 contains all the vertices of G . In other words, either G has size exactly four or is an induced
 1051 cycle.

1052 If H' is a triangle plus a vertex, then we claim that G is an induced cycle plus a unique
 1053 vertex. Indeed, since H' has at most one degree 3 vertex, the vertex not in the triangle has

1054 degree at most one. Thus, for any cycle C of G , the cycle plus any vertex incident to C is a
 1055 H' -minor. Since G is H' -minimal, the graph G is an induced cycle plus a vertex. ◀

1056 Using the three claims above, it only remains to handle the cases where H is K_4 or
 1057 K_4 minus an edge (namely, a diamond). In both cases, assume that V_3, V_4 (at least) are
 1058 associated to degree 3 vertices of H' . If both V_3 and V_4 have type A or B, there is a unique
 1059 vertex $x \in V_3$ incident to a vertex y in V_4 . If we add y to V_3 , and remove it from V_4 , then the
 1060 size of V_4 is decreasing, and the V_i 's still form a model of H' . We can repeat this operation
 1061 until V_4 does not have type A or B. Therefore, we can assume from now on that V_3 or V_4
 1062 has type C.

1063 ▷ **Claim 46.** If H' is a diamond, then $G = K_4$ or G is a subdivided diamond (Items 2 and 1).

1064 **Proof.** In H' , v_1 and v_2 have degree 2, and by Corollary 40, the subsets V_1 and V_2 are paths.
 1065 Moreover, if there is an edge between them, and any of V_1, V_2, V_3 or V_4 has two vertices,
 1066 we could remove one of these vertices, and still have a model of H' . Therefore, each V_i is
 1067 reduced to a single vertex (otherwise G is not H' -minimal) hence G is K_4 .

1068 We may thus assume that there is no edge between V_1 and V_2 . If V_3 or V_4 has type A,
 1069 then $G \setminus V_1$ contains a diamond as a minor, a contradiction since G is H' -minimal. Moreover,
 1070 as we already observed, at least one of V_3, V_4 , say V_3 , has type C. We now have two cases:
 1071 V_4 has either type B or C.

1072 Assume that V_4 has type B, and let $u \in V_4$ be the vertex of V_4 adjacent to V_3 . If u sees
 1073 two vertices of V_3 , then replacing (V_3, V_4) by $(V_3 \cup \{u\}, V_4 \setminus u)$ gives a model where $V_3 \cup \{u\}$
 1074 has type A, a contradiction. Therefore, u sees a unique vertex of V_3 and G is a subdivided
 1075 diamond.

1076 Otherwise, V_4 has type C, hence V_3 and V_4 induce two paths (with maybe edges between
 1077 them). There cannot be two edges between V_3 and V_4 , otherwise, $G \setminus V_1$ contains a diamond-
 1078 minor, a contradiction. Therefore, there is only one edge between V_3 and V_4 and G is again
 1079 a subdivision of a diamond. ◀

1080 We now assume that $H' = K_4$. Let us first prove the following claim:

1081 ▷ **Claim 47.** If G contains two vertex-disjoint cycles C_1, C_2 such that $G \setminus C_1$ is connected
 1082 and with three pairwise non-incident edges between C_1 and $G \setminus C_1$, then G is the graph
 1083 depicted on Figure 9.

1084 **Proof.** Let a_1b_1, a_2b_2, a_3b_3 be the edges from the statement, with $a_i \in C_1$.

1085 Assume first that C_2 is not a triangle. Then we can remove a vertex of $G \setminus C_1$ in such a
 1086 way it remains connected and still contains the b_i 's. This gives a K_4 -model, a contradiction.
 1087 Hence, we assume that C_2 is a triangle.

1088 We say that $u \in C_2$ has a private path to one of the a_i 's if it has such a path that avoids
 1089 $C_2 \setminus \{u\}$. If some vertex $u \in C_2$ has no such path, then $(G \setminus C_1) \setminus \{u\}$ is connected, hence
 1090 $G \setminus u$ contains a K_4 minor, a contradiction. Moreover, if two vertices $u, v \in C_2$ have a private
 1091 path to the same a_i , then we get a K_4 minor avoiding some other a_j . Therefore, each vertex
 1092 of C_2 is associated with a unique a_i by considering private paths. Observe that there is
 1093 exactly one path for each of the three choice of endpoints (since if there were two paths, one
 1094 could remove a vertex which lies in one path but not in the other and get a K_4 minor).

1095 It remains to show that C_1 is a triangle. To this end, observe that the structure we found
 1096 on $G \setminus C_2$ ensures that the hypotheses of the statement are still met when exchanging C_1
 1097 with C_2 , and the first argument of the proof shows that C_1 is a triangle. ◀

1098 Recall that we can assume that V_3 or V_4 has type C . Applying the same argument to
 1099 each pair V_i, V_j ensures that all but at most one set V_i (say V_4) are of type C . We now do a
 1100 case analysis on the type of V_4 .

1101 \triangleright **Claim 48.** If V_4 has type A , then G is a graph from Figure 9 (Item 5).

1102 **Proof.** Since V_4 has type A , it contains a triangle C . Moreover, $G \setminus C$ contains a cycle since
 1103 it contains V_1, V_2, V_3 which are pairwise connected. We then conclude using Claim 47. \blacktriangleleft

1104 \triangleright **Claim 49.** If V_4 has type B , then G is either a subdivided K_4 or a graph from Figure 9
 1105 (Items 1 and 5).

1106 **Proof.** If V_4 has type B , then it is a subdivided star with three branches. Let x be the vertex
 1107 of V_4 of degree three and a_1, a_2, a_3 be the endpoints of the subdivided star rooted in x (note
 1108 that the a_i 's are indeed distinct from x). Without loss of generality, each a_i is connected
 1109 to V_i (and not to some other V_j since otherwise $G \setminus a_j$ contains a K_4 -minor). If some a_i
 1110 is connected to at least two vertices of V_i , then $G \setminus (\{a_i\} \cup V_i)$ contains a cycle as well as
 1111 $V_i \cup \{a_i\}$ with the conditions of Claim 47. So the G is the graph of Figure 9. Therefore, each
 1112 a_i is connected to exactly one vertex of V_i and G is a subdivided K_4 . \blacktriangleleft

1113 \triangleright **Claim 50.** If V_4 has type C , then G is a graph from Figure 9, (a subgraph of) a wheel, a
 1114 subdivided K_4 , or has at most five vertices of degree more than 2 (Items 5, 4, 1 and 6).

1115 **Proof.** If V_4 has type C , then all four sets have type C . We may assume that V_1 is maximal,
 1116 that is there is no model of K_4 where one of the sets strictly contains V_1 . In particular, any
 1117 addition of a neighbor of a vertex of V_j to V_1 does not keep a model.

1118 Now we claim that $V_2 \cup V_3 \cup V_4$ is a cycle C . Indeed, it must contain a cycle C , since
 1119 V_2, V_3, V_4 is a model of the triangle. If w is not in C , either it is adjacent to V_1 and then can
 1120 be added to V_1 (a contradiction) or it is not, and then $G \setminus w$ still contains C and V_1 , hence
 1121 has a model of K_4 so G is not H' -minimal.

1122 Let C be the cycle containing all the vertices of V_2, V_3, V_4 and X_1, X_2, X_3 the neighbors
 1123 of V_1 in respectively V_2, V_3, V_4 .

1124 **Case 1: C is not induced.**

1125 Any chord of the cycle separates the cycle into two sides. If there is a chord e of C that
 1126 leaves one side of the cycle with at least one element of each of X_1, X_2, X_3 , then we can
 1127 remove a vertex on the opposite side of the cycle and still have a K_4 -minor, a contradiction
 1128 with the H -minimality of G . So, without loss of generality, e separates X_1 on one side and at
 1129 least one element from X_2, X_3 on the other side. Let P, P' be the two parts of C separated
 1130 by e where P' only contains X_1 . In this case, we can apply Claim 47 with the cycle $e + P'$,
 1131 and a cycle using V_1 , a part of P and edges between V_1 and X_2, X_3 . Hence, this case again
 1132 boils down to the graphs of Figure 9.

1133 **Case 2: C is induced.**

1134 If V_1 is reduced to a single vertex, then the graph G is a wheel. So we can assume that
 1135 V_1 has at least two vertices. And it is a path since it has type C and both endpoints of the
 1136 path have neighbors in C (otherwise the model is not minimal). Since we have a K_4 -model,
 1137 we need the whole set V_1 to have at least three different neighbors on C .

1138 First, note that every vertex of V_1 has at most 2 neighbors on C (otherwise, we have
 1139 a $K_4 + K_1$ model since V_1 contains at least two vertices). More generally, if a subpath of
 1140 V_1 has at least three neighbors on C , we have a contradiction. Consider an endpoint v of
 1141 V_1 . If all its neighbors in C have another neighbor in V_1 , then the subpath $V_1 \setminus v$ has three

1142 neighbors in C . Therefore, we can assume that both endpoints of V_1 have a private neighbor
 1143 in C (that is a neighbor in C with no other neighbor in V_1). Similarly, V_1 is adjacent to at
 1144 most 4 vertices in C (otherwise V_1 minus one endpoint has three neighbors in C).

1145 **Case 2a: V_1 has four neighbors in C .**

1146 We claim that only endpoints of V_1 have neighbors in C and each endpoint has exactly two
 1147 neighbors. Let us denote by v_1, \dots, v_ℓ the vertices of V_1 with neighbors in C (in that order
 1148 in the path V_1). Recall that v_1 and v_ℓ both have a private neighbor in C . For each $j \neq 1, \ell$,
 1149 we have $N(v_j) \subset N(v_1)$ (otherwise $|N(V_1 \setminus v_\ell) \cap C| \geq 3$), and similarly $N(v_j) \subset N(v_\ell)$, so
 1150 $N(V_1) \subset N(v_1) \cup N(v_\ell)$. If $\ell > 2$, then $N(v_1) \cap N(v_\ell) \neq \emptyset$ so $|N(C)| \leq |N(v_1) \cap N(v_\ell)| \leq 3$,
 1151 which is impossible.

1152 So $\ell = 2$, and v_1, v_2 and their (at most four) neighbors in C are the only vertices of degree
 1153 more than 2 in G , which concludes.

1154 **Case 2b: V_1 has 3 neighbors in C .**

1155 Let v_1, \dots, v_ℓ be the vertices of V_1 with neighbors in C . Since v_1 and v_ℓ have private
 1156 neighbors on C , all the other vertices are adjacent to the same vertex w of C . Let us denote
 1157 by a and b respectively the private neighbors of v_1 and v_ℓ . Note that $v_\ell v_1 v_1 a P_{ab} b v_\ell$ is an
 1158 induced cycle C' (where P_{ab} is the subpath of C from a to b avoiding w). If the only vertex
 1159 outside of C' is w , then G is a subgraph of a wheel. If w sees only one v_i , then the graph
 1160 is a subdivided K_4 . Otherwise, there are at least 4 paths starting from w to C' and one of
 1161 them is subdivided. The removal of a vertex in a subdivided path still leaves a graph with a
 1162 K_4 minor, a contradiction. \blacktriangleleft

1163 **10 Lower bounds**

1164 In this section, we show logarithmic lower bounds for H -minor-freeness for every 2-connected
 1165 graph H . These results generalize the lower bounds of [28] for K_k and $K_{p,q}$. Our technique
 1166 is a simple reduction from the certification of paths, via a local simulation. In contrast, the
 1167 proofs of [27] were ad-hoc adaptations of the constructions of [38] and [26], with explicit
 1168 counting arguments. Moreover, our lower bounds apply in the stronger model of locally
 1169 checkable proofs, where the verifier can look at a constant distance, see [38].

1170 **► Theorem 51.** *For every 2-connected graph H with at least 3 vertices, certifying H -minor-*
 1171 *freeness requires $\Omega(\log n)$ bits.*

1172 Let us start by proving a couple of lemmas. Let H be a 2-connected graph, and let $e = uv$
 1173 be an arbitrary edge of H . Let H^- be the graph $H \setminus e$. Note that H^- is connected. We are
 1174 going to consider copies of H^- , that we index as H_i^- 's, and where the copies of the vertices
 1175 u and v will be called u_i and v_i . Let \mathcal{P} be the class of all the graphs that can be made by
 1176 taking some k copies of H^- , and by identifying for every $i \in [1, k-1]$, v_i with u_{i+1} . In other
 1177 words, \mathcal{P} is the set of paths, where every edge is a copy of H^- . The class \mathcal{C} is the same as \mathcal{P}
 1178 except that we close the paths into cycles, that is, we identify v_k with u_1 .

1179 **► Lemma 52.** *The graphs of \mathcal{P} are all H -minor-free, and the graphs of \mathcal{C} all contain H as a*
 1180 *minor.*

1181 **Proof.** Let G be a graph of \mathcal{P} . Note that every vertex v_i (identified with u_{i+1}) for $i \in$
 1182 $\{1, \dots, k-1\}$, is a cut vertex of G . Therefore, since H is 2-connected, a model of H can
 1183 only appear between two such vertices. By construction this cannot happen, as the graphs
 1184 between the cut vertices are all H^- . Thus G is H -minor-free.

1185 Now let G be a graph of \mathcal{C} . We claim that G contains H as a minor. Consider the
 1186 following model of H . Any H_i^- is a model of H except for the edge uv . Since we have made
 1187 a cycle of H_i^- 's, there is a path between v_i and u_i outside H_i^- , and this path finishes the
 1188 model of H . ◀

1189 ▶ **Lemma 53.** *Let H be a 2-connected graph. If there is a certification with $O(f(n))$ bits for
 1190 H -minor-free graphs, then there is a $O(f(n))$ certification for paths.*

1191 **Proof.** Suppose there exists a certification with $O(f(n))$ bits for H -minor-free graphs. The
 1192 certification of paths boils down to differentiate between paths and cycles, since the vertices
 1193 can locally check that they have degree 2. Consider the following certification of paths. The
 1194 idea is that the vertices of the path (or cycle) will simulate the computation they would do
 1195 if instead of being linked by edges, they were linked by copies of H^- . The prover will give
 1196 to every vertex the certificates of H -minor-freeness for these simulated graphs, that is, for
 1197 every vertex the certificates of the two copies of H^- adjacent to it in the simulated graph.
 1198 Every vertex will check with its neighbor in the real graph that they have been given the
 1199 same certificates for these virtual H^- . Then every vertex will run the verification algorithm
 1200 for H -minor-freeness in the simulated graph.

1201 By construction, the simulated graph is either in \mathcal{P} or in \mathcal{C} . Thus, if the verification
 1202 algorithm accepts, that is, if the simulated graph is H -minor-free, then the graph is in \mathcal{P} ,
 1203 and then the real graph is a path. If the verification algorithm rejects, that is if the simulated
 1204 graph is not H -minor-free then the graph is in \mathcal{C} , and then the real graph is a cycle. In other
 1205 words we have designed a local certification for paths, with certificates of size $O(f(n))$. ◀

1206 **Proof of Theorem 51.** Now Theorem 51 follows from the fact that paths cannot be certified
 1207 with $o(\log n)$ bits [38, 42]. Note that the proof applies in the locally checkable proof setting,
 1208 as soon as the number of copies of H^- is large enough, since the lower bound for paths also
 1209 applies to locally checkable proofs. ◀

1210 11 Discussion

1211 Milestones to go further

1212 In this paper, we develop several tools and use them to show that some minor closed graph
 1213 classes can be certified with $O(\log n)$ bits. One can probably use the tools we developed to
 1214 certify new classes, we simply wanted to illustrate the interest of these tools. Let us now
 1215 discuss the tools that are missing in order to tackle the general question on H -minor-freeness
 1216 and which steps can be interesting to tackle it.

1217 First, as we explained in Section 9, certification of H -minor-free classes seems easier when
 1218 H is sparse. One first question that might be interested to look at is the following:

1219 ▶ **Question 54.** *Let T be a tree. Can T -minor-free graphs be certified with $O(\log n)$ bits?*

1220 The answer to this question for small graphs H (up to 5 vertices) is not very interesting,
 1221 since the number of vertices of degree at least 3 is bounded (and then the whole structure
 1222 of the graph is "simple"). Even if it remains simple for any H , there is no trivial argument
 1223 allowing us to certify these vertices with $O(\log n)$ bits. In the light of the recent results that
 1224 establish that $O(\log n)$ bits is doable for paths minors [29], and $O(\log^2 n)$ is doable for planar
 1225 minors [32], Question 54 seems to be the simplest open question.

1226 A natural approach to tackle Conjecture 1 would consist in an induction on the size
 1227 of the excluded minor H . Indeed, knowing how to certify $(H \setminus x)$ -minor-freeness for any

1228 possible x may help to certify H -minor-freeness. The basic idea would consist in separating
 1229 two cases. 1) When H is not heavily connected where we can heavily use the fact that
 1230 $(H \setminus x)$ -minor-freeness can be certified. And 2) when H is heavily connected, try to use
 1231 a more general argument. A first step toward step 1) would consist in proving that if
 1232 H -minor-freeness can be certified then so is $H + K_1$ -minor-freeness³. We proved it for five
 1233 vertices in Theorem 41, but the proof heavily uses the structure of the graphs on four vertices.
 1234 One can then naturally ask the following general question:

1235 ► **Question 55.** *Let H be a graph. Can $(H + K_1)$ -minor-free graphs be certified with $O(\log n)$*
 1236 *bits when H can be certified with $O(\log n)$ bits?*

1237 As in the proof of Theorem 41, we know that we can assume that G is H -minimal. Even
 1238 if most of the techniques for Lemma 41 are specific, Corollary 40 gives some (basics) general
 1239 properties of H -minimal graphs which might be useful to tackle this question.

1240 In structural graph theory, a particular class of H -minimal graphs received a considerable
 1241 attention which are minimally non-planar graphs, in other words, graphs G that are minimal
 1242 and that contains either a K_5 or a $K_{3,3}$ as a minor. It might be interesting to determine if
 1243 minimally non-planar graphs can be certified with $O(\log n)$ bits.

1244 Note that if we can answer positively Question 55 positively, the second step would
 1245 consist in proving the conjecture when we add to H a vertex attached to a single vertex of
 1246 H . Proving this case would, in particular, imply a positive answer to Question 54.

1247 If we want to consider dense graphs, the questions seem to become even harder. In
 1248 particular, one of the first main complicated H -minor class to deal with is probably the
 1249 class of K_5 -minor-free graphs. There are several reasons for that. First, it is the smallest
 1250 4-connected graph and the hardness to certify seem to be highly related to the connectivity
 1251 of the graph that is forbidden as a minor. The second reason is that it is the smallest graph
 1252 for which H -minor-free graphs is a super class of planar graphs. In other words, we cannot
 1253 take advantage of the “planarity” of the graph (formally or informally) to certify the graph
 1254 class. We then ask the following question:

1255 ► **Question 56.** *Can K_5 -minor-free graphs be certified with $O(\log n)$ bits?*

1256 Wagner proved in [53] that a graph is K_5 -minor-free if and only if it can be built from
 1257 planar graphs and from a special graph V_8 by repeated clique sums. A *clique sum* consists in
 1258 taking two graphs of the class and gluing them on a clique and then (potentially) remove
 1259 edges of that clique. While it should have been easy to certify this sum if we keep the edges
 1260 of the clique, the fact that they might disappear makes the work much more complicated for
 1261 certification.

1262 More generally, many decompositions are using the fact that we replace a subgraph by
 1263 a smaller structure (a single vertex or an edge for instance) only connected to the initial
 1264 neighbors of that structure in the graph. Certifying such structures is a challenging question
 1265 whose positive answer can probably permit to break several of the current hardest cases.

1266 Obstacles towards lower bounds

1267 There are also several obstacles preventing us to prove extra-logarithmic lower bounds for
 1268 the certificate size of H -minor-free graphs. Basically, the only techniques we know consist
 1269 in (explicit or implicit) reductions to communication complexity. In particular [38] and [9]

³ $H + K_1$ is the graph H plus an isolated vertex.

1270 designed lower bounds for respectively non-3-colorable graphs and bounded diameter graphs
 1271 as reductions from the disjointness problem in non-deterministic communication complexity.

1272 Let us remind what these reductions look like. In such a reduction, one considers a family
 1273 of graphs with two vertex sets A and B , with few edges in between. These graphs are defined
 1274 in such a way that the input of Alice for the disjointness problem can be encoded in the
 1275 edges of A and the input of Bob in the edges of B . Then, given a certification scheme, Alice
 1276 and Bob can basically simulate the verification algorithm, and deduce an answer for the
 1277 disjointness problem. If a certification with small labels existed for the property at hand,
 1278 then the communication protocol would contradict known lower bounds which proves a lower
 1279 bound for certification.

1280 The difficulty of using this proof for H -minor-free graphs comes from the fact that it is
 1281 difficult to control where a minor can appear, that is, to control the models of H . For example,
 1282 it is difficult to control that if H appears in the graph, then the vertices V_i associated with
 1283 some vertex i of H are on Alice's side. As a comparison, for proving properties on the
 1284 diameter, [9] used a construction where all the longest paths in the graph had to start from
 1285 Alice side and finish in Bob side, but such a property seems difficult to obtain for minors.

1286 Connectivity questions

1287 A large part of the paper is devoted to certify connectivity and related notions that are of
 1288 independent importance, for instance to certify the robustness of a network. For these, we
 1289 do not have lower bounds, and leave the following question open.

1290 ► **Question 57.** *Does the certification of k -connectivity require $\Omega(\log n)$ bits?*

1291 For this question it is tempting to try a construction close to the one we have used
 1292 for H -minor-free graphs. For example, one could think that the vertices of the path/cycle
 1293 could simulate the k -th power of the graph which is k -connected if and only if the graph
 1294 is a cycle. But this does not work: we want the *yes*-instances for the property (*e.g.* the
 1295 k -connected graphs) to be in mapped to *yes*-instances for acyclicity (*e.g.* paths), and not
 1296 with the *no*-instances, which are the cycles.

1297 An interesting open problem about k -connectivity also is on the positive side:

1298 ► **Question 58.** *Can k -connectivity be certified with $O(\log n)$ bits for any $k \geq 4$?*

1299 Beyond the question of certifying the connectivity itself, we would like to be able to
 1300 decompose graphs based on k -connected components, like what we did with the block-cut tree
 1301 for 2-connectivity. Such decomposition are more complicated and less studied than block-cut
 1302 trees, but for 3-connectivity such a tool is SPQR trees [4]. Unfortunately, similarly to the
 1303 clique sum operation we mentioned earlier, some steps of the SPQR tree construction are
 1304 based on edges that can be removed in later steps, making it hard to certify this structure.

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