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Meta-Schrödinger transformations

Stoimen Stoimenov and Malte Henkel

Abstract Meta-Schrödinger transformations are the dynamical symmetries of equations of combined ballistic and diffusive transport. Their Lie algebra is derived in $1 + 1$ space dimensions and the infinite-dimensional generalisation is constructed. Representations without time-translation-invariance are given and the co-variant two-point functions are computed.

1 Introduction

While $2D$ conformal invariance [4] is an essential ingredient in string theory [35] or equilibrium critical phenomena [19, 23], non-equilibrium statistical mechanics furnishes different examples where a naturally realised dilatation-symmetry can be extended to larger (eventually infinite-dimensional) symmetry algebra. Besides the conformal algebra $\mathfrak{vir} \oplus \mathfrak{vir}$ itself, see e.g. [5, 8, 9, 13] and also the example of generalised hydrodynamics after quantum quenches [6, 11, 14, 34, 36], the most simple example are systems described by an underlying simple diffusion equation whose dynamical symmetry was recognised by Jacobi [29] and Lie [30] and is called today *Schrödinger algebra* $\mathfrak{sch}(d)$ [33]. Its infinite-dimensional extension is the *Schrödinger-Virasoro algebra* $\mathfrak{sv}(d)$ [22]. *Contra* a wide-held belief, the Schrödinger algebra is *not* the non-relativistic limit obtained from the conformal algebra by a Lie algebra contraction. Rather this procedure leads to the *conformal*

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Galilean algebra, e.g. [21, 24, 16, 17, 32, 1] (including its infinite-dimensional extension), which also arises independently in gravitational physics [7, 2, 3]. On the other hand, the dynamical symmetries of simple ballistic transport equations has recently been identified as the new *meta-conformal algebra* $\text{meta}(1, d)$ [28], whose infinite-dimensional extensions are isomorphic to the direct sum of two Virasoro algebras in $d = 1$ spatial dimensions and of three Virasoro algebras for $d = 2$. Again, the non-relativistic limit of these are the conformal Galilean algebras.

The physical context of these application is physical ageing: a many-body system is prepared in a disordered state and then quenched to either a critical point or else into the ordered phase where several equivalent equilibrium states coexist. After the quench, such systems (i) relax slowly, (ii) break time-translation-invariance and (iii) obey dynamical scaling, which are the three defining properties of *physical ageing* [26]. Schrödinger-invariance is realised in quenches to the ordered phase, see [26] for a detailed review. Meta-conformal invariance may be realised if the underlying microscopic dynamics has a directional bias [20, 28].

Here, we shall describe the new ‘*meta-Schrödinger algebra*’ $\text{metasch}(1, 1)$ [38]. It arises for combined ballistic and diffusive transport in different spatial directions, see eq. (2) below. Such equations also arise in the study of driven diffusive systems, see [37] for a classic review. Section 2 outlines the construction, including of the infinite-dimensional extension, called *meta-Schrödinger-Virasoro algebra* $\text{msv}(1, 1)$. We prove the semi-direct sum [38]

$$\text{msv}(1, 1) \cong (\text{vir} \oplus \text{vir}) \ltimes \text{gal}(1) \cong \text{vir} \ltimes \mathfrak{sv}(1) \quad (1)$$

(where $\text{gal}(1)$ is the infinite-dimensional algebra of generalised Galilei transformations in the y -direction). Section 3 considers the necessary generalisations for physical ageing, when the time-translation generator $-\partial_t$ must be modified. Section 4 lists the co-variant two-point functions of quasi-primary scaling operators.

2 Construction of the meta-Schrödinger algebra

Definition: *The meta-Schrödinger algebra $\text{metasch}(1, 1)$ acts as dynamical symmetry algebra of the following biased evolution equation*

$$\mathcal{S}\Phi(t, x, y) := \left(\partial_t - S_1 \partial_x - S_2 \partial_y^2 \right) \Phi(t, x, y) = 0. \quad (2)$$

Following [33], infinitesimal symmetries of (2) are written in the form

$$X = -A(t, x, y)\partial_t - B(t, x, y)\partial_x - C(t, x, y)\partial_y - D(t, x, y), \quad (3)$$

where the functions A, B, C, D depend all on t, x, y and must satisfy

$$[\mathcal{S}, X]\Phi(t, x, y) = \lambda(t, x, y)\mathcal{S}\Phi(t, x, y) \quad (4)$$

for an arbitrary field $\Phi = \Phi(t, x, y)$. The functions A, B, C, D must obey the system¹

$$\begin{aligned} S_2 A_{yy} + S_1 A_x - \dot{A} - \lambda &= 0, & S_2 B_{yy} + S_1 B_x - \dot{B} + \lambda S_1 &= 0 \\ 2S_2 C_y + \lambda S_2 &= 0, & S_2 C_{yy} + S_1 C_x - \dot{C} + 2S_2 D_y &= 0 \\ S_2 D_{yy} + S_1 D_x - \dot{D} &= 0, & 2S_2 A_y &= 0, & 2S_2 B_y &= 0 \end{aligned} \quad (5)$$

The solution of (5) is aided by considering two sub-algebras.

1. For fields $\Phi = \Phi(t, x)$, eq. (2) becomes the ballistic transport equation in the spatial x -direction. Its dynamical symmetry is the meta-conformal algebra $\text{meta}(1, 1)$.
2. For fields $\Phi = \Phi(t, y)$, eq. (2) becomes the diffusion equation in the spatial y -direction. Its dynamical symmetry is the Schrödinger algebra $\text{sch}(1)$.

Both should be sub-algebras of the sought Lie algebra $\text{metasch}(1, 1)$. A natural starting point for the construction of $\text{metasch}(1, 1)$ will be representations of the meta-conformal algebra which obey the condition (4) with the Schrödinger operator (2). Recall the algebraic structure of the meta-conformal and Schrödinger algebras

$$\text{meta}(1, 1) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}), \quad \text{sch}(1) \cong \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{hei}(1) \quad (6)$$

(the Heisenberg algebra $\mathfrak{hei}(1)$ includes the central extension). We start from

Ansatz: Representations of $\text{metasch}(1, 1)$ are given by semi-direct sums of representations of the meta-conformal algebra $\text{meta}(1, 1)$, with known action in the spatial direction x and the Heisenberg algebra $\mathfrak{hei}(1)$, with known action in the spatial direction y .

We write $X_{-1} = -\partial_t, Y_{-1}^x = -\partial_x, Y_{-\frac{1}{2}}^y = -\partial_y, M_0 = -\mathcal{M}$ and

$$\begin{aligned} X_0 &= -t\partial_t - x\partial_x - \frac{1}{2}y\partial_y - \delta \\ X_1 &= -(t^2 + \alpha x^2)\partial_t - (2tx + \beta x^2)\partial_x - C^{X_1}(t, x, y)\partial_y - D^{X_1}(t, x, y) - 2\delta t - 2\gamma x \\ Y_0^x &= -\alpha x\partial_t - (t + \beta x)\partial_x - C^{Y_0^x}(t, x, y)\partial_y - D^{Y_0^x}(t, x, y) - \gamma \\ Y_1^x &= -\alpha(2tx + \beta x^2)\partial_t - ((t + \beta x)^2 + \alpha x^2)\partial_x - C^{Y_1^x}(t, x, y)\partial_y - D^{Y_1^x}(t, x, y) \\ &\quad - 2\gamma t - 2(\alpha\delta + \beta\gamma)x \\ Y_{\frac{1}{2}}^y &= -A^{Y_{\frac{1}{2}}^y}(t, x)\partial_t - B^{Y_{\frac{1}{2}}^y}(t, x)\partial_x - C^{Y_{\frac{1}{2}}^y}(t, x, y)\partial_y - D^{Y_{\frac{1}{2}}^y}(t, x, y) - \mathcal{M}y \end{aligned} \quad (7)$$

where α, β are constants. To find A, B, C, D of each generator, use the sub-algebras:

- the meta-conformal algebra $\text{meta}(1, 1) = \langle X_n, Y_n^x \rangle_{n \in \mathbb{Z}}$, with the commutators

$$\begin{aligned} [X_n, X_m] &= (n - m)X_{n+m}, & [X_n, Y_m^x] &= (n - m)Y_{n+m}^x \\ [Y_n^x, Y_m^x] &= (n - m)(\alpha X_{n+m} + \beta Y_{n+m}^x) \end{aligned} \quad (8)$$

- the Schrödinger sub-algebra $\text{sch}(1) = \langle X_{0, \pm 1}, Y_{\pm \frac{1}{2}}^y, M_0 \rangle$ and

¹ We use the notations $\dot{A} = \partial_t A(t, x, y)$, $A_{xy} = \partial_x \partial_y A(t, x, y)$ etc.

$$\begin{aligned} [X_n, X_m] &= (n-m)X_{n+m}, & [X_n, Y_p^y] &= \left(\frac{n}{2} - p\right)Y_{n+p}^y, \\ [Y_p^y, Y_q^y] &= (p-q)M_{p+q}, & [X_n, M_m] &= mM_{n+m} \end{aligned} \quad (9)$$

For $n \in \mathbb{Z}$ and $p \in \mathbb{Z} + \frac{1}{2}$, one has the infinite-dimensional algebras $\text{metab}(1, 1) = \langle X_n, Y_n^x \rangle$ and $\mathfrak{sb}(1) = \langle X_n, Y_n^x, Y_p^y, M_n \rangle$.

We should find $\mathfrak{sl}(2, \mathbb{R}) = \langle X_{0, \pm 1} \rangle$ as sub-algebra acting on both x and y , but such that the commutator $[Y_n^x, Y_p^y]$ closes into the algebra. All unknown functions in the generators (7) are found from the above commutator relations (8, 9) and the equations (5). The first equation of the system (5) gives $\lambda^{X_1} = -2t + 2\alpha S_1 x$. Upon substitution into the second equation (5), this leads to a quadratic equation for S_1

$$\alpha S_1^2 + \beta S_1 - 1 = 0. \quad (10)$$

Set $c := -\alpha S_1$, then $\alpha = c(c - \beta) \neq 0$. It is enough to construct X_1 explicitly.

2.1 The general case: $\alpha \neq 0$

The algebra $\text{metasch}(1, 1)$ is spanned by the generators [38]:

$$\begin{aligned} X_{-1} &= -\partial_t, & X_0 &= -t\partial_t - x\partial_x - \frac{y}{2}\partial_y - \delta \\ X_1 &= -(t^2 + \alpha x^2)\partial_t - (2tx + \beta x^2)\partial_x - (t + cx)y\partial_y - 2t\delta - 2\gamma x - \frac{\mathcal{M}}{2}y^2 \\ Y_{-1}^x &= -\partial_x, & Y_0^x &= -\alpha x\partial_t - (t + \beta x)\partial_x - \frac{c}{2}y\partial_y - \gamma \\ Y_1^x &= -\alpha(2tx + \beta x^2)\partial_t - (t^2 + 2\beta tx + (\alpha + \beta^2)x^2)\partial_x - (ct + (\alpha + \beta c)x)y\partial_y \\ &\quad - 2\gamma t - 2(\alpha\delta + \beta\gamma)x - \frac{c\mathcal{M}}{2}y^2 \\ Y_{\frac{1}{2}}^y &= -\partial_y, & Y_{\frac{1}{2}}^y &= -(t + cx)\partial_y - \mathcal{M}y, & M_0 &= -\mathcal{M}, \end{aligned} \quad (11)$$

with the non-vanishing commutation relations, with $n, m = \pm 1, 0$ and $p = \pm \frac{1}{2}$

$$\begin{aligned} [X_n, X_m] &= (n-m)X_{n+m}, & [X_n, Y_m^x] &= (n-m)Y_{n+m}^x \\ [X_n, Y_p^y] &= \left(\frac{n}{2} - p\right)Y_{n+p}^y, & [Y_n^x, Y_m^x] &= (n-m)(\alpha X_{n+m} + \beta Y_{n+m}^x) \\ [Y_n^x, Y_p^y] &= c\left(\frac{n}{2} - p\right)Y_{n+p}^y, & [Y_{1/2}^y, Y_{-1/2}^y] &= M_0. \end{aligned} \quad (12)$$

Next, if we let

$$S_1 = -\frac{c}{\alpha} = -\frac{1}{c-\beta}, \quad S_2 = \frac{1}{2\mathcal{M}} \frac{2c-\beta}{c-\beta}, \quad \text{and} \quad \gamma = \frac{2c-\beta}{4} + (\beta-c)\delta \quad (13)$$

then the Schrödinger operator (2) becomes $\mathcal{S} = \partial_t + \frac{1}{c-\beta} \partial_x - \frac{1}{2\mathcal{M}} \frac{2c-\beta}{c-\beta} \partial_y^2$. It is readily checked that all symmetry conditions (4) are obeyed [38]. Notice that the representation (11) and all consequences are valid only for $c \neq \beta$ and $c \neq \beta/2$.

2.2 Infinite-dimensional extension

The infinite-dimensional extension of the representation (11) is constructed as follows [38]. First, in terms of the of the variable $\rho = t + cx$, the infinite-dimensional extension of the Heisenberg algebra is

$$Y_p^y = -\rho^{p+1/2} \partial_y - (p+1/2) \mathcal{M} \rho^{p-1/2} y, \quad M_n = -\mathcal{M} \rho^n \quad (14)$$

such that for $p, q \in \mathbb{Z} + \frac{1}{2}$ we have the commutator $[Y_p^y, Y_q^y] = (p-q)M_{p+q}$. Next, following [28], define a new family of generators $\mathcal{Y}_n := \mathcal{N} (aX_n + Y_n^x)$ whose normalisation \mathcal{N} will be fixed shortly. The new generators satisfy

$$[\mathcal{Y}_n, \mathcal{Y}_m] = (n-m)(2a + \beta) \mathcal{N} \mathcal{Y}_{n+m} \quad (15)$$

provided that a satisfies $a^2 + \beta a - c(c-\beta) = 0$. The two solutions $a_{1,2}$ of this quadratic equation, namely $a_1 = -c$ and $a_2 = c - \beta$, produce two distinct forms, denoted $\mathcal{Y}_n^{(1,2)}$, of the generators. We then obtain

$$\mathcal{Y}_n^{(1)} = \mathcal{N}^{(1)} (-cX_n + Y_n^x), \quad \mathcal{Y}_n^{(2)} = \mathcal{N}^{(2)} ((c-\beta)X_n + Y_n^x) \quad (16)$$

which both satisfy the commutator (15). We now fix the normalisations from the requirements $(\beta - 2c) \mathcal{N}^{(1)} \stackrel{!}{=} 1 \stackrel{!}{=} (2c - \beta) \mathcal{N}^{(2)}$.

Analogously, we construct

$$\mathcal{A}_n := X_n + b\mathcal{Y}_n; \quad \text{with } [\mathcal{A}_n, \mathcal{A}_m] = (n-m)\mathcal{A}_{n+m}. \quad (17)$$

Since a has the admissible values $a_{1,2}$, it follows that either $b = 0$ or $b = -1$ [38]. Then three distinct forms of the \mathcal{A}_n are possible, namely

$$\left. \begin{aligned} \mathcal{A}_n^{(0)} = X_n & \quad \text{with } [\mathcal{A}_n^{(0)}, \mathcal{Y}_m^{(1,2)}] = (n-m)\mathcal{Y}_{n+m}^{(1,2)} \quad \text{if } b = 0 \\ \mathcal{A}_n^{(1)} = \frac{(c-\beta)X_n + Y_n^x}{2c-\beta} = \mathcal{Y}_n^{(2)} \\ \mathcal{A}_n^{(2)} = -\frac{-cX_n + Y_n^x}{2c-\beta} = \mathcal{Y}_n^{(1)} \end{aligned} \right\} \text{with } [\mathcal{A}_n^{(0)}, \mathcal{Y}_m^{(1,2)}] = 0 \quad \text{if } b = -1 \quad (18)$$

This construction is valid² if $\beta \neq 2c$ and $\beta \neq c$. Then the maximal finite-dimensional sub-algebra (11) is the dynamical symmetry of the equation (2).

² In the limit $\beta - 2c \rightarrow 0$, a Lie algebra contraction should lead to representations related to the conformal Galilean algebra.

We write down the generators explicitly³ for $b = -1$. Because of (18), we have $\mathcal{A}_n^{(1)} = \mathcal{Y}_n^{(2)}$, and $\mathcal{A}_n^{(2)} = \mathcal{Y}_n^{(1)}$. Hence, these possibilities are not independent. Let $\mathcal{A}_n = \mathcal{A}_n^{(1)}$ and $\mathcal{Y}_n = \mathcal{Y}_n^{(1)}$. They are readily obtained from the explicit expressions for X_n, Y_n^x in (11), but working with light-cone-like variables $\sigma = t + (\beta - c)x$ and $\rho = (t + cx)$ leads to the more elegant form

$$\mathcal{Y}_n = -\sigma^{n+1} \partial_\sigma + (n+1) \frac{c\delta - \gamma}{2c - \beta} \sigma^n \quad (19a)$$

$$\mathcal{A}_n = -\rho^{n+1} \partial_\rho - (n+1) \left(\frac{(c - \beta)\delta + \gamma}{2c - \beta} + \frac{y}{2} \partial_y \right) \rho^n - \frac{n(n+1)}{4} \mathcal{M} y^2 \rho^{n-1} \quad (19b)$$

Hence the Lie algebra $\langle \mathcal{A}_n, \mathcal{Y}_n, Y_p^y, M_n \rangle$ has the non-vanishing commutators, for $n, m \in \mathbb{Z}$ and $p, q \in \mathbb{Z} + \frac{1}{2}$, using (14)

$$\begin{aligned} [\mathcal{A}_n, \mathcal{A}_m] &= (n - m) \mathcal{A}_{n+m}, & [\mathcal{Y}_n, \mathcal{Y}_m] &= (n - m) \mathcal{Y}_{n+m} & (20) \\ [\mathcal{A}_n, Y_p^y] &= \left(\frac{n}{2} - p \right) Y_{n+p}^y, & [Y_p^y, Y_q^y] &= (p - q) M_{p+q}, & [\mathcal{A}_n, M_m] &= -m M_{n+m} \end{aligned}$$

which are those of the meta-Schrödinger-Virasoro Lie algebra (1). The Schrödinger-Virasoro algebra $\mathfrak{sv}(1) = \langle \mathcal{A}_n^B, Y_p^y, M_n \rangle \subset \mathfrak{msv}(1, 1)$ is an obvious sub-algebra.

Since in light-cone coordinates, $\mathcal{S} = \left(\frac{2c - \beta}{c - \beta} \partial_\rho - \frac{1}{2\mathcal{M}} \partial_y^2 \right)$, the dynamical symmetry follows from, if $\gamma = \frac{2c - \beta}{4} + (\beta - c)\delta$,

$$\begin{aligned} [\mathcal{S}, \mathcal{A}_n] &= \frac{2c - \beta}{c - \beta} ((n+1)\rho^n \mathcal{S} \\ &\quad + n(n+1)\rho^{n-1} \left(\frac{(c - \beta)\delta + \gamma}{2c - \beta} - \frac{1}{4} \right) + \frac{n^3 - n}{4} \rho^{n-2} \mathcal{M} y^2) & (21) \\ [\mathcal{S}, Y_p^y] &= \frac{2c - \beta}{c - \beta} \mathcal{M} \left(p - \frac{1}{2} \right) \left(p + \frac{1}{2} \right) \rho^{p-3/2} y \quad ; \quad [\mathcal{S}, \mathcal{Y}_n] = [\mathcal{S}, M_0] = 0 \end{aligned}$$

such that either the maximal finite-dimensional sub-algebra $\mathfrak{metasch}(1, 1)$ or else $\langle \mathcal{A}_{\pm 1, 0}, \mathcal{Y}_n, Y_{\pm \frac{1}{2}}, M_0 \rangle$ leave the solution space of (2) invariant.

2.3 The special case $\alpha = 0$

If $\alpha = 0$, one has $\lambda^{X_1} = -2t$ and from (10) $S_1 = 1/\beta$. Since we had before $\alpha = c(c - \beta)$, the results for $\alpha = 0$ can be obtained from those of the previous sub-section by setting $c = 0$ (but not $c = \beta$!) in the generators (11) as well as in commutation relations (12). This directly produces an infinite-dimensional representation of $\mathfrak{msv}(1, 1)$ whose commutators follow from (12) with $\alpha = c = 0$. In the

³ The case $b = 0$ gives the same algebra, up to a change of basis [38].

light-cone variables $\tau = t, v = t + \beta x, y$, an elegant form is

$$\begin{aligned} A_n &:= X_n - \frac{1}{\beta} Y_n^x = -\tau^{n+1} \partial_\tau - (n+1) \left(\frac{1}{2} \tau^n y \partial_y + \left(\delta - \frac{\gamma}{\beta} \right) \tau^n + \frac{n}{4} \mathcal{M} \tau^{n-1} y^2 \right) \\ Y_n^x &= -\beta v^{n+1} \partial_v - (n+1) \gamma v^n, \\ Y_p^y &= -\tau^{p+\frac{1}{2}} \partial_y - \left(p + \frac{1}{2} \right) \mathcal{M} \tau^{p-\frac{1}{2}} y, \quad M_n = -\mathcal{M} \tau^n, \end{aligned} \quad (22)$$

whose non-vanishing commutators ($n, m \in \mathbb{Z}$ and $p, q \in \mathbb{Z} + \frac{1}{2}$)

$$\begin{aligned} [A_n, A_m] &= (n-m) A_{n+m}, & [A_n, Y_p^y] &= \left(\frac{n}{2} - p \right) Y_{n+p}^y, \\ [Y_n^x, Y_m^x] &= (n-m) \beta Y_{n+m}^x, & [Y_p^y, Y_q^y] &= (p-q) M_{p+q} \end{aligned} \quad (23)$$

again reproduce (1). In light-cone variables the Schrödinger operator (2) simplifies into $\mathcal{S} = \partial_t - \frac{1}{\beta} \partial_x - \frac{1}{2\mathcal{M}} \partial_y^2 = \partial_\tau - \frac{1}{2\mathcal{M}} \partial_y^2$ and if $\frac{\gamma}{\beta} = \delta - \frac{1}{4}$, the dynamical symmetries are obvious, as before [38].

2.4 Representations without time-translation-invariance

A system undergoing physical ageing is brought out of equilibrium by a quench in its thermodynamic parameters. It cannot be at equilibrium which suggests that the generator $X_{-1} = -\partial_t$ of time-translations should not be part of the symmetry algebra [25]. However, for the meta-Schrödinger algebra, such a restrictive prescription is not adequate, because eq. (12) shows that without the generator X_{-1} of time-translations, the conformal algebra of the generators Y_n^x does not close, especially

$$[Y_0^x, Y_{-1}^x] = \alpha X_{-1} + \beta Y_{-1}^x. \quad (24)$$

For the Schrödinger-Virasoro algebra, it can be shown how to generalise [31] the defining representation such that (a) time-translation-invariance is broken and (b) the full Lie algebra $\mathfrak{sv}(d)$ is kept [27]. This procedure can be applied to the meta-Schrödinger-Virasoro algebra as well. In the special case $\alpha = 0$, to which we restrict here, we find for the generator $A_n = X_n - \frac{1}{\beta} Y_n^x$, in light cone variables [38]

$$A_n = -\tau^{n+1} \partial_\tau - \frac{n+1}{2} \tau^n y \partial_y - (n+1) \left(\delta - \frac{\gamma}{\beta} \right) \tau^n - n\xi \tau^n - \frac{n(n+1)}{4} \mathcal{M} \tau^{n-1} y^2 \quad (25)$$

All other generators maintain their form stated in (23). The only new element is a ‘second scaling dimension’ ξ . Together with the ‘first scaling dimension’ δ and the ‘rapidity’ γ , it can be used to characterise scaling operators out of stationary states.

In light-cone variables, the invariant Schrödinger operator becomes $\mathcal{S} = \partial_\tau - \frac{1}{2\mathcal{M}} \partial_y^2 + \frac{\xi}{\tau}$. For checking the symmetry, all commutators of the time-translation-invariant case can be taken over. The only exception is

$$[\mathcal{S}, A_1] = -2t\mathcal{S} + (2\mathcal{M}S_2 - 1)y\partial_y - 2\left(\delta + \xi - \frac{\gamma}{\beta} - \frac{1}{4}\right) \quad (26)$$

Hence the conditions for a symmetry are now $S_2 = \frac{1}{2\mathcal{M}}$ and $\frac{\gamma}{\beta} = \delta + \xi - \frac{1}{4}$.

3 Covariant two-point functions

As an application, we consider the form of the co-variant two point-function

$$F(t_1, t_2, x_1, x_2, y_1, y_2) := \langle \Phi_1(t_1, x_1, y_1) \Phi_2(t_2, x_2, y_2) \rangle \quad (27)$$

where Φ_1, Φ_2 are quasi-primary scaling operators of $\text{metasch}(1, 1)$. From the representations constructed before, the scaling operators are characterised by the several parameters introduced. Physically, we distinguish between the stationary case, which is time-translation-invariant and the ageing case, which is not [38].

3.1 Stationary case

In the time-translation-invariant case we consider scaling operators Φ_i ($i = 1, 2$), characterised by the parameters $(\delta_i, \frac{\gamma_i}{\beta_i}, \mathcal{M}_i, \beta_i)$, transforming covariantly under the representation with $\alpha = 0$ (called case A). Lifting the single-body representation of section 2 to a two-body representation, this produces a set of Ward identities which fix the form of $F(t, x, y) = G^{(A)}(t, t + \beta x, y)$. Because of the translation-invariance, F will only depend on the differences $t := t_1 - t_2$, $x := x_1 - x_2$ and $y := y_1 - y_2$ and it can also be shown that $\beta_1 = \beta_2 = \beta$. Letting $v = t + \beta x$, the final result is

$$G^{(A)}(t, v, y) = G_0 \delta_{\mathcal{M}_1 + \mathcal{M}_2, 0} \delta_{\delta_1, \delta_2} \delta_{\gamma_1, \gamma_2} t^{-2\delta_1 + 2\gamma_1/\beta} v^{-2\gamma_1/\beta} \exp\left(-\frac{\mathcal{M}_1 y^2}{2t}\right). \quad (28)$$

and the normalisation constant G_0 . This form combines aspects of Schrödinger-invariance in the transverse coordinate y and of meta-conformal invariance in the parallel coordinate x .

Similarly, if $\alpha \neq 0$ (called case B) one works with the coordinates $\rho = t + cx$ and $\sigma = t + (\beta - c)x$. Again, $\beta := \beta_1 = \beta_2$ and $c := c_1 = c_2$. Using the definitions $\Gamma := \frac{c\delta - \gamma}{2c - \beta}$ and $\Delta := \delta$ and writing $F(t, x, y) = G^{(B)}(\rho, \sigma, y)$, we find

$$G^{(B)}(\rho, \sigma, y) = G_0 \delta_{\mathcal{M}_1 + \mathcal{M}_2, 0} \delta_{\Delta_1, \Delta_2} \delta_{\Gamma_1, \Gamma_2} \rho^{-2\Delta_1 + 2\Gamma_1} \sigma^{-2\Gamma_1} \exp\left(-\frac{\mathcal{M}_1 y^2}{2\rho}\right) \quad (29)$$

and where we also have $\rho = \rho_1 - \rho_2$, $\sigma = \sigma_1 - \sigma_2$ and $y = y_1 - y_2$.

3.2 Ageing case

For non-equilibrium dynamics, as it occurs in ageing systems, time-translation-invariance does not hold and the scaling operators $\Phi_{1,2}$ are quasi-primary with respect to the representations derived in section 3. The time variables are now $t = t_1 - t_2$ and $u = t_1/t_2$. We confine ourselves to the case $\alpha = 0$ (case A) and find, with $F(t, u, x, y) = G^{(A)}(t, u, v, y)$ and $v = t + \beta x$

$$G^{(A)}(t, u, v, y) = G_0 \delta_{\mathcal{M}_1 + \mathcal{M}_2, 0} \delta_{\gamma_1, \gamma_2} \delta_{\delta_1 + \xi_1, \delta_2 + \xi_2} \quad (30)$$

$$\times t^{-\delta_1 - \delta_2 + 2\gamma_1/\beta} u^{\xi_1} (u-1)^{-\xi_1 - \xi_2} v^{-2\gamma_1/\beta} \exp\left(-\frac{\mathcal{M}_1 y^2}{2t}\right)$$

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