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# Looking for equivalences of the services between left and right continuity in the Network Calculus theory

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**Abstract**—The Network Calculus theory allows to compute upper bounds on delays and memory usage in data networks. The mathematical foundations rely on piecewise-continuous functions being, depending on the authors, either left- or right-continuous. It is then impossible to use all existing results in a single analysis, since the hypotheses are incompatibles. This paper shows how, under some reasonable assumptions, it is possible to project property from left-continuous world to right-continuous world, and vice-versa.

**Index Terms**—Network Calculus, Continuity

## 1 INTRODUCTION

Nowadays, the Network Calculus theory, based on the Min-plus algebra, is used since 20 years to analyse the worst-case performances in data networks [1] and several studies continue to analyse new networks [2].

The basic objects in the theory are functions representing the amount of data sent by a source in an interval of time. Such functions are called cumulative curves (see Section 3.1). An assumption made to perform most of the results is that these cumulative curves are left-continuous functions (cf. [1, Definition 1.1] and the related discussion for details). Indeed, this assumption provides useful properties as it will be presented in Section 6. We will call the theory developed with the left-continuous cumulative curves the left world.

However, some notions are more difficult to apprehend in the left world like the decomposition of the flows into packets. Thus, some studies are developed considering the cumulative curves as right-continuous functions [3] where the idea is to link the Network Calculus theory with CPA (Compositional Performance Analysis [4]). Contrary to the left-continuous cumulative curves, the right-continuous ones make the decomposition of the flows into packets easier [5]. Also, some results are proven with the two assumptions as [6] shows. But, the common properties and theorems of the Network Calculus theory are not proven with this assumption of right-continuous cumulative curves. The theory and results developed with the right-continuous cumulative curves will be regrouped under the right world.

To sum up, most results of the Network Calculus theory are developed with the assumption of left-continuous cumulative curves but interesting results exist with the right-continuous assumption. To use both results, two choices are

possible: prove all the results developed in the left world in the right world and conversely (as in [6] or proving equivalences between the two worlds to transfer results from one world to the other.

In the following, the aim is to prove the equivalences between the two models developed around the two assumptions.

To do that, we will work with the set of left-continuous functions ( $\mathcal{L}$ ) and the set of right-continuous functions ( $\mathcal{R}$ ). Then, we need to introduce two operators to pass from a set to another: the left and right extensions represented by  $(\cdot)_l$  and  $(\cdot)_r$  respectively, as illustrated in Figure 1.

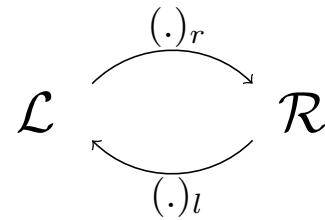


Fig. 1: Graph with the set of left-continuous functions ( $\mathcal{L}$ ) and the set of right-continuous functions ( $\mathcal{R}$ ) and the operators to pass from one set to another: the left ( $(\cdot)_l$ ) and the right ( $(\cdot)_r$ ) extension.

The contribution of this paper is to show which properties are preserved passing from a world to another. To do that, as the Network Calculus theory is based on the  $(\min, +)$  dioid, itself based on the piecewise-continuous functions from the non negative reals to the reals ( $\mathcal{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ ), we will work on three layers:

- 1) the properties on the sets of functions (Section 2.3),
- 2) how the Min-plus convolution supports the left/right sets (Section 2.4),
- 3) see the Network Calculus properties which stay to be valid passing from a world to another as the notions of cumulative curves (Section 3.1), the arrival curves (Section 3.2) and the service curves (with the simple, maximal, shaper and strict service curves) (Section 4).

## 2 MATHEMATICAL BACKGROUND

### 2.1 Notations

First, let  $\mathbb{R}$ , and respectively  $\mathbb{R}^+$  denote the set of reals and non-negative reals respectively.  $\mathbb{N}$  denotes the set of integers.  $\&$  represents the logical and.

Let  $\lfloor \cdot \rfloor$ , respectively  $\lceil \cdot \rceil$  denote the floor and ceiling functions such that  $\forall x \in \mathbb{R}$ ,  $\lfloor x \rfloor \in \mathbb{N}$ ,  $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$  and  $\lceil x \rceil \in \mathbb{N}$ ,  $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ .

Also, we denote by  $\wedge$ , respectively  $\vee$ , the minimum and the maximum operator i.e.  $a \wedge b \stackrel{\text{def}}{=} \min(a, b)$  and  $a \vee b \stackrel{\text{def}}{=} \max(a, b)$ .  $[\cdot]^+$  represents the non-negative closure:  $\forall f \in \mathcal{F}$ ,  $[f]^+ \stackrel{\text{def}}{=} f \vee 0$ .

Finally, we will use the limit of functions and note  $f(t+) \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} f(t + \varepsilon)$  the right limit at  $t$  of the function  $f$  if it exists and  $f(t-) \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} f(t - \varepsilon)$  the left limit at  $t$  of the function  $f$  if it exists.

### 2.2 Sets and operations

After introducing the notations, we will define some specific sets of functions according to the Min-plus theory (as said in the introduction) and to work with left and right world, some operators to pass from one to another: the left and right extensions.

First, let define what is a piecewise continuous function.

**Definition 1** (Piecewise continuous function). A function  $f$  is said to be piecewise continuous if it has a finite number of discontinuities on any finite interval.

Then, we can introduce the useful sets of functions.

**Definition 2** (Function sets:  $\mathcal{F}$ ,  $\mathcal{R}$ ,  $\mathcal{L}$  and subsets).  $\mathcal{F}$  is the set of functions from  $\mathbb{R}^+$  to  $\mathbb{R}$ , piecewise-continuous.

$\mathcal{R}$ , (respectively  $\mathcal{L}$ ) is the set of right-continuous functions, (respectively left-continuous functions).

Let  $X$  be a subset of  $\mathcal{F}$ . Then

$X^\uparrow \subset X$  is the subset of the non-decreasing functions of  $X$ ,

$X_0 \subset X^\uparrow$  is the subset of functions of  $X$  such that  $\forall f \in X_0$ ,  $f(0) = 0$ ,

$X_{0+} \subset X_0^\uparrow$  is the subset of functions of  $X_0$  such that  $\forall f \in X_{0+}$ ,  $f(0+) = 0$ .

Also,  $X_0^\uparrow = X_0 \cap X^\uparrow$  is the subset of the non-decreasing functions of  $X$  such that  $\forall f \in X_0$ ,  $f(0) = 0$ ,

$X_{0+}^\uparrow = X_{0+} \cap X^\uparrow$  is the subset of the non-decreasing functions of  $X$  such that  $\forall f \in X_{0+}$ ,  $f(0+) = 0$ .

**Remark 1.** Considering the set  $\mathcal{R}_0^\uparrow$ , note that a function  $f \in \mathcal{R}_0^\uparrow$  can't be null at a singular point  $t = 0$  because  $f \in \mathcal{R}$  then  $f(0+) = f(0) = 0$ . Consequently,  $\mathcal{R}_0^\uparrow = \mathcal{R}_{0+}^\uparrow$ . Also, note that  $X_{0+}^\uparrow \subset X_0^\uparrow$ .

The following property is an alternative expression of the limits for non-decreasing functions.

**Property 1** (Alternative expression for the right/left limit). Let  $f \in \mathcal{F}^\uparrow$ . Then

$$f(t+) = \inf_{\varepsilon > 0} f(t + \varepsilon) \quad (1)$$

$$f(t-) = \sup_{\varepsilon > 0} f(t - \varepsilon) \quad (2)$$

The proof is given in Appendix 8.1.

**Definition 3** (Right and Left extensions).  $(\cdot)_r$  defines the right extension operator such that

$$\begin{aligned} (\cdot)_r : \mathcal{F} &\rightarrow \mathcal{R} \\ &: f \mapsto (f_r : t \mapsto f(t+)) \end{aligned} \quad (3)$$

Correspondingly,  $(\cdot)_l$  defines the left extension operator such that

$$\begin{aligned} (\cdot)_l : \mathcal{F} &\rightarrow \mathcal{L} \\ &: f \mapsto (f_l : t \mapsto f(t-)) \end{aligned} \quad (4)$$

Also, we can apply these extensions to a set  $X \subset \mathcal{F}$  as

$$(X)_l \stackrel{\text{def}}{=} \{f_l \mid f \in X\} \quad (5)$$

$$(X)_r \stackrel{\text{def}}{=} \{f_r \mid f \in X\}. \quad (6)$$

**Remark 2.** when there is no ambiguity, the parenthesis will be remove in the extensions, i.e.  $(f)_l$  and  $f_l$  represent the same: the left extension of  $f$  and  $(f)_r$  and  $f_r$  represent the right extension of  $f$ .

A useful property, is the monotony of the extensions as shown in the following property, Property 2.

**Property 2** (Monotony of the extensions). Let  $f, g \in \mathcal{F}^\uparrow$  such that  $f \leq g$ . Then,

$$f_r \leq g_r \quad (7)$$

$$f_l \leq g_l \quad (8)$$

*Proof.* Let  $f, g \in \mathcal{F}^\uparrow$  such that  $f \leq g$ . Then, according to Property 1,  $\forall t \in \mathbb{R}^+$ ,

$$\inf_{\varepsilon > 0} f(t + \varepsilon) \leq \inf_{\varepsilon > 0} g(t + \varepsilon)$$

and

$$\sup_{\varepsilon > 0} f(t + \varepsilon) \leq \sup_{\varepsilon > 0} g(t + \varepsilon).$$

□

### 2.3 Stability of the subsets of $\mathcal{F}$

Now, after introducing the useful notations, sets and operators, we will see which properties of the subsets of  $\mathcal{F}$  are preserved passing from a world to another.

First, let see the impact of the composition of the extensions.

**Property 3** (Composition of the extensions). Let  $f \in \mathcal{F}_{0+}^\uparrow$ . Then

$$(f_r)_l = f_l \quad (9)$$

$$(f_l)_r = f_r \quad (10)$$

*Proof.* Let  $x \in \mathbb{R}^+$ .

Let  $(u_n)_{n \in \mathbb{N}}$  be a decreasing sequence converging to 0. Then,

$$(f_r)_l(x) = \lim_{n \rightarrow \infty} f_r(x - u_n). \quad (11)$$

Let  $(v_m)_{m \in \mathbb{N}}$  be a decreasing sequence converging to 0. Then,

$$(f_r)_l(x) = \lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} f(x - u_n + v_m)). \quad (12)$$

$\forall n \in \mathbb{N}$ , since  $(v_m)$  is decreasing,  $\exists m \in \mathbb{N}$  such that  $u_n - v_m > 0$ . Let note  $\phi(n)$  this number, and define  $(w_k)_{k \in \mathbb{N}}$  by  $w_k = u_k - v_{\phi(k)}$ . We have  $\lim_{k \rightarrow \infty} w_k = 0$ , and  $\forall k : w_k >$

0, so  $\lim_{k \rightarrow \infty} f(x - w_k) = f_l(x)$ . Since any sub-sequence converges to the same limit as the sequence itself (when it exists),  $(f_r)_l(x) = f_l(x)$ .

The same way, we also have  $(f_l)_r = f_r$ .  $\square$

Now, we can work on the stability of the subsets of  $\mathcal{F}$ .

**Property 4** (Stability of non-decreasing subsets of  $\mathcal{F}$ ). *Let  $\mathcal{L}^\uparrow, \mathcal{R}^\uparrow$  defined as in Definition 2 and  $(\cdot)_r, (\cdot)_l$  the left and right extension as defined in Definition 3. Then*

$$(\mathcal{R}^\uparrow)_l \subset \mathcal{L}^\uparrow \quad (\mathcal{L}^\uparrow)_r \subset \mathcal{R}^\uparrow \quad (13)$$

*Proof.*  $\forall f \in \mathcal{F}^\uparrow, \forall x, y \in \mathbb{R}p, x < y, f(x) \leq f(y)$

$$\implies \begin{cases} \inf_{\epsilon > 0} f(x + \epsilon) \leq \inf_{\epsilon > 0} f(y + \epsilon) \Leftrightarrow f_l(x) \leq f_l(y), \\ \sup_{\epsilon > 0} f(x - \epsilon) \leq \sup_{\epsilon > 0} f(y - \epsilon) \Leftrightarrow f_r(x) \leq f_r(y) \end{cases} \quad (14)$$

Thus,  $f_r$  and  $f_l$  are non-decreasing. As  $\mathcal{L}^\uparrow \subset \mathcal{F}^\uparrow$  and  $\mathcal{R}^\uparrow \subset \mathcal{F}^\uparrow$  then,

$$(\mathcal{R}^\uparrow)_l \subset \mathcal{L}^\uparrow \quad (\mathcal{L}^\uparrow)_r \subset \mathcal{R}^\uparrow \quad (15)$$

$\square$

**Property 5** (Stability of the subsets of  $\mathcal{F}$ ). *Let  $\mathcal{R}_0^\uparrow, \mathcal{R}_{0+}^\uparrow, \mathcal{L}_0^\uparrow, \mathcal{L}_{0+}^\uparrow$  defined as in Definition 2 and  $(\cdot)_r, (\cdot)_l$  the left and right extension as defined in Definition 3. Then*

$$(\mathcal{L}_{0+}^\uparrow)_r \subset \mathcal{R}_{0+}^\uparrow \quad (\mathcal{R}_{0+}^\uparrow)_l \subset \mathcal{L}_{0+}^\uparrow \quad (16)$$

$$(\mathcal{R}_0^\uparrow)_l \subset \mathcal{L}_0^\uparrow \quad (\mathcal{L}_0^\uparrow)_r \not\subset \mathcal{R}_0^\uparrow \quad (17)$$

$$((\mathcal{L}_{0+}^\uparrow)_r)_l \subset \mathcal{L}_{0+}^\uparrow \quad ((\mathcal{R}_{0+}^\uparrow)_l)_r \subset \mathcal{R}_{0+}^\uparrow \quad (18)$$

*Proof.* According to Property 4, the non-decreasing aspect of the subsets is preserved. We only need to see if the other properties are preserved *i.e.* if the null point at  $t = 0$  and the limit at  $t = 0$  are preserved.

- proof of Equation (16): We need to see if the property such that the limit of the function when  $t \rightarrow 0+$  is null is preserved.

Let  $f \in \mathcal{R}_{0+}^\uparrow$ . At  $t = 0$ , it is continue so  $f(0) = f_l(0) = 0$  and it exists a vicinity around 0:  $[0; \epsilon[$  such that  $\forall t \in [0; \epsilon[, f(t) = 0$  and it is continuous then  $\forall t \in [0; \epsilon[, f_l(t) = 0$ . Then,  $f_l \in \mathcal{L}_{0+}^\uparrow$  and  $(\mathcal{R}_{0+}^\uparrow)_l \subset \mathcal{L}_{0+}^\uparrow$ . Similarly, we have that  $(\mathcal{L}_{0+}^\uparrow)_r \subset \mathcal{R}_{0+}^\uparrow$ .

- proof of Equation (17):

First, we can easily note that it is not preserved for the right extension of the set  $\mathcal{L}_0^\uparrow$  because  $\exists f \in \mathcal{L}_0^\uparrow \mid f_r \notin \mathcal{R}_0^\uparrow$  with, for instance, the function:

$$f(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0 \end{cases} \quad (19)$$

Indeed,  $f_r(t) = 1, \forall t \in \mathbb{R}^+$  and  $f_r \notin \mathcal{R}_0^\uparrow$ . The property  $f(0) = 0$  is not preserved by the right extension operator. Consequently,  $(\mathcal{L}_0^\uparrow)_r \not\subset \mathcal{R}_0^\uparrow$ .

However,  $\mathcal{R}_0^\uparrow = \mathcal{R}_{0+}^\uparrow, (\mathcal{R}_{0+}^\uparrow)_l \subset \mathcal{L}_{0+}^\uparrow \subset \mathcal{L}_0^\uparrow$ .

- proof of Equation (18): This last property, Property 3, shows that  $((\mathcal{L}_{0+}^\uparrow)_r)_l \subset \mathcal{L}_{0+}^\uparrow$  and  $((\mathcal{R}_{0+}^\uparrow)_l)_r \subset \mathcal{R}_{0+}^\uparrow$ .  $\square$

## 2.4 Stability of the Min-plus convolution

Finally, we want to see the impact of the left and right extension on the Min-plus convolution.

Let, first, remind the Min-plus convolution according to the Network Calculus theory.

**Definition 4** (Min-plus convolution). *\* represents the Min-plus convolution and is defined for two functions  $f, g \in \mathcal{F}$  as*

$$f * g = \inf_{0 \leq s \leq t} \{f(s) + g(t - s)\} \quad (20)$$

$$= \inf_{\substack{0 \leq u, s \\ u+s=t}} \{f(s) + g(u)\} \quad (21)$$

In the following parts, Min-plus convolution or convolution means the same.

Let introduce a new expression for the convolution equivalent to the previous ones (Equation (20) and (21)). This expression will be useful to see the impact of the extension on the convolution.

**Property 6** (Alternative expression of the convolution). *Let  $f, g \in \mathcal{F}$  be two functions. Then*

$$f * g(t) = \inf_{0 \leq s} \{g(t \wedge s) + f([t - s]^+)\} \quad (22)$$

*Proof.* Let  $f, g \in \mathcal{F}$  be two functions.

$$\inf_{0 \leq s} \{g(t \wedge s) + f([t - s]^+)\} \quad (23)$$

$$= \min \begin{cases} \inf_{0 \leq s < t} \{g(t \wedge s) + f([t - s]^+)\} \\ \inf_{s \leq t} \{g(t \wedge s) + f([t - s]^+)\} \end{cases} \quad (24)$$

Note that  $\begin{cases} 0 \leq s < t \implies t \wedge s = s \ \& \ [t - s]^+ = t - s \\ s \leq t \implies t \wedge s = t \ \& \ [t - s]^+ = 0 \end{cases}$ , so

$$= \min \begin{cases} \inf_{0 \leq s < t} \{g(s) + f(t - s)\} \\ \inf_{s \leq t} \{g(t) + f(0)\} \end{cases} \quad (25)$$

$$= \min \begin{cases} \inf_{0 \leq s < t} \{g(s) + f(t - s)\} \\ (g(t) + f(0)) \end{cases} \quad (26)$$

$$= \inf_{0 \leq s \leq t} \{g(s) + f(t - s)\} \quad (27)$$

$$= f * g(t) \quad (28)$$

$\square$

This property can be illustrated with Figure 2 adapted from [1, Fig. 2.1]. The convolution is commonly illustrated with this idea of sliding one function over the other, as represented at top of Figure 2. Property 6 represents exactly this idea: for each  $s$ , let  $h_s : t \mapsto g(t \wedge s) + f([t - s]^+)$ . Then,  $f * g = \inf_{s \geq 0} h_s$ . The relation is illustrated on the bottom of Figure 2, where  $h_s$  is plotted with two values of  $s$ .

**Property 7** (Non-decreasing stability of convolution). *Let  $f, g$  be two non-decreasing functions then  $f * g$  is also a non-decreasing function *i.e.*,*

$$\forall f, g \in \mathcal{F}^\uparrow : f * g \in \mathcal{F}^\uparrow. \quad (29)$$

*Proof.* This property is proven in [1, Lemma 2.3].  $\square$

**Theorem 1** (Left stability of the convolution). *Let  $f, g$  be two non-decreasing left-continuous functions. Then  $f * g$  is also a non-decreasing left-continuous function *i.e.*,*

$$\forall f, g \in \mathcal{L}^\uparrow : f * g \in \mathcal{L}^\uparrow \quad (30)$$

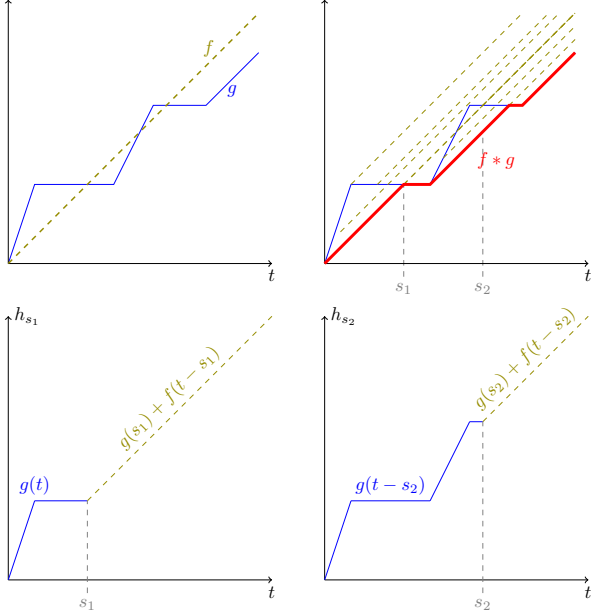


Fig. 2: Up graphs: Illustration of the convolution ( $f * g$ : thick) between  $f$  (dashed) and  $g$  (plain), from [1, Fig. 2.1].  
Down graphs:  $h_s : t \mapsto g(t \wedge s) + f([t - s]^+)$  for two specific values of  $s$ .

*Proof.* This theorem is proven in [1, Proposition 3.11].  $\square$

**Theorem 2** (Convolution and extensions). *Let  $f, f', g \in \mathcal{F}^\uparrow$  such that  $f'(0+) = f'(0)$ . Then*

$$(f * g)_l = f_l * g_l \quad (31)$$

$$(f' * g)_r = f'_r * g \quad (32)$$

*Proof.* of Equation (31) Let  $f, g \in \mathcal{F}^\uparrow$ .

The proof consists in bound the expression  $f_l * g_l$ .

On the one hand, we need to prove that  $f_l * g_l \leq f * g$ . Firstly,  $g_l \leq g \implies f_l * g_l \leq f_l * g$ . Also,  $f_l \leq f \implies f_l * g \leq f * g$ . Then,  $f_l * g_l \leq f * g$ .

On the other hand, we need to prove that  $(f * g)_l \leq f_l * g_l$ . We will use the expression of the convolution and the limit:  $\forall t \in \mathbb{R}^+$ ,

$$(f * g)_l(t) = \sup_{\epsilon > 0} \{f * g(t - \epsilon)\} \quad (33)$$

$$= \sup_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t - \epsilon} \{f(s) + g(t - \epsilon - s)\} \right\} \quad (34)$$

$$= \sup_{\eta > 0} \left\{ \inf_{\eta \leq u \leq t - \eta} \{f(u - \eta) + g(t - \eta - u)\} \right\} \quad (35)$$

with  $\eta = \epsilon/2$  and  $u = s + \eta$

$$(f * g)_l(t) \leq \sup_{\eta > 0} \left\{ \inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \right\} \quad (36)$$

because  $\forall \eta > 0, [\eta; t - \eta] \subset [0; t]$ .

Continuing from Equation (36):  $\forall t \in \mathbb{R}^+, \forall \eta > 0, \forall u \in [0; t]$ ,

$$\inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \quad (37)$$

$$\leq f(u - \eta) + g(t - \eta - u)$$

$$\implies \sup_{\eta > 0} \left\{ \inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \right\} \quad (38)$$

$$\leq \sup_{\eta > 0} \{f(u - \eta) + g(t - \eta - u)\}$$

$$\implies \sup_{\eta > 0} \left\{ \inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \right\} \quad (39)$$

$$\leq \inf_{0 \leq u \leq t} \left\{ \sup_{\eta > 0} \{f(u - \eta) + g(t - \eta - u)\} \right\}$$

$$\implies \sup_{\eta > 0} \left\{ \inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \right\} \quad (40)$$

$$\leq \inf_{0 \leq u \leq t} \{(f)_l(u) + (g)_l(t - u)\}$$

$$\implies \sup_{\eta > 0} \left\{ \inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \right\} \quad (41)$$

$$\leq (f)_l * (g)_l(t)$$

Consequently,

$$(f * g)_l(t) \leq \sup_{\eta > 0} \left\{ \inf_{0 \leq u \leq t} \{f(u - \eta) + g(t - \eta - u)\} \right\} \leq (f)_l * (g)_l(t) \quad (42)$$

Finally, combining the bounds:

$$(f * g)_l \leq f_l * g_l \leq f * g$$

and passing to the left limit:

$$((f * g)_l)_l = (f * g)_l \leq (f_l * g_l)_l = f_l * g_l \leq (f * g)_l$$

. Consequently, Then  $(f * g)_l = f_l * g_l$ .  $\square$

*Proof.* of Equation (32) Let  $f', g \in \mathcal{F}^\uparrow$  be two non-decreasing functions such that  $f'(0+) = f'(0)$ . The proof consists in reducing the both terms (the right side and the left side of the equality) into a common same third term.

Let start from the right side:  $f'_r * g$ ,

$$f'_r * g(t) \stackrel{(22)}{=} \inf_{0 \leq s} \{g(t \wedge s) + (f')_r([t - s]^+)\} \quad (43)$$

$$\stackrel{(3)}{=} \inf_{0 \leq s} \left\{ g(t \wedge s) + \inf_{\epsilon > 0} \{f'([t - s]^+ + \epsilon)\} \right\} \quad (44)$$

$$= \min \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + \inf_{\epsilon > 0} \{f'([t - s]^+ + \epsilon)\}\} \right. \\ \left. \inf_{t < s} \{g(t \wedge s) + \inf_{\epsilon > 0} \{f'([t - s]^+ + \epsilon)\}\} \right\} \quad (45)$$

Note that  $\begin{cases} s \leq t \Rightarrow [t - s]^+ = t - s \\ t < s \Rightarrow [t - s]^+ = 0 \end{cases}$  so,

$$f'_r * g(t) = \min \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + \inf_{\epsilon > 0} \{f'(t - s + \epsilon)\}\} \right. \\ \left. \inf_{t < s} \{g(t \wedge s) + \inf_{\epsilon > 0} \{f'(\epsilon)\}\} \right\} \quad (46)$$

$$\text{Note that } \begin{cases} s \leq t & \Rightarrow t - s + \epsilon > 0 \\ & \Rightarrow t - s + \epsilon = [t - s + \epsilon]^+ \\ t < s & \Rightarrow t \wedge s = t \\ & \Rightarrow \inf_{\epsilon > 0} \{f'(\epsilon)\} = f'(0+) = f'(0) \end{cases} \quad \text{so,}$$

As

$$\begin{aligned} & \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \\ & \leq \inf_{\epsilon > 0} \{g(t) + f'(\epsilon)\} \text{ with } s = t \\ & = g(t) + f'(0) \end{aligned}$$

$$f'_r * g(t) = \min \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + \inf_{\epsilon > 0} \{f'([t - s + \epsilon]^+)\}\} \right. \\ \left. \inf_{t < s} \{g(t) + f'(0)\} \right\} \quad (47)$$

Consequently,

$$f'_r * g(t) = \inf \left\{ \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \right. \\ \left. g(t) + f'(0) \right\} \quad (48)$$

$$(f' * g)_r = \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \quad (53)$$

Note that

$$\begin{aligned} & \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \\ & \leq \inf_{\epsilon > 0} \{g(t) + f'(\epsilon)\} \text{ with } s = t \\ & = g(t) + f'(0) \end{aligned}$$

so,

$$f'_r * g(t) = \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \quad (49)$$

Now, let start from the left side:  $(f' * g)_r(t)$  and reduce it to the same expression, Equation (49).

$$(f' * g)_r(t) \stackrel{(3)}{=} \inf_{\epsilon > 0} \{f' * g(t + \epsilon)\} \quad (50)$$

$$\stackrel{(22)}{=} \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \quad (51)$$

$$= \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \right. \\ \left. \inf_{t < s} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \quad (52)$$

Looking for the second of Equation (52):

$\inf_{t < s} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\}$ , we have

$$t < s \Rightarrow \begin{cases} t \leq (t + \epsilon) \wedge s \leq t + \epsilon \\ 0 \leq [t + \epsilon - s]^+ \leq \epsilon \end{cases}$$

then, as  $f', g$  are non-decreasing, we have

$$\begin{cases} g(t) \leq g((t + \epsilon) \wedge s) \leq g(t + \epsilon) \\ f'(0) \leq f'([t + \epsilon - s]^+) \leq f'(\epsilon) \end{cases}$$

Then, by addition and limit, we can surround the term  $\inf_{t < s} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\}$ :

$$\begin{aligned} & \inf_{\epsilon > 0} \{g(t) + f'(0)\} \\ & \leq \inf_{\epsilon > 0} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \\ & \leq \inf_{\epsilon > 0} \{g(t + \epsilon) + f'(\epsilon)\} \end{aligned}$$

So

$$\begin{aligned} & g(t) + f'(0) \\ & \leq \inf_{\epsilon > 0} \{g((t + \epsilon) \wedge s) + f'([t + \epsilon - s]^+)\} \\ & \leq g(t+) + f'(0) \end{aligned}$$

$$= \inf_{\epsilon > 0} \left\{ \inf_{0 \leq s \leq t} \{g(t \wedge s) + f'([t + \epsilon - s]^+)\} \right\} \quad (54)$$

And,  $f'_r * g = (f * g)_r$ .  $\square$ 

**Corollary 1** (Right absorption of the convolution). *Let  $f$  be a right-continuous non-decreasing function such that  $f(0+) = f(0) = 0$  and  $g$  be a piece-wise non-decreasing function such that  $g(0) = 0$ . Then  $f * g$  is right-continuous non-decreasing and such that  $f * g(0+) = f * g(0) = 0$ , i.e.*

$$\forall f \in \mathcal{R}_{0+}^\uparrow, \forall g \in \mathcal{F}_0^\uparrow, f * g \in \mathcal{R}_{0+}^\uparrow \quad (55)$$

*Proof.* According to the stability of non decreasing functions and using the Theorem 2,  $\forall f \in \mathcal{R}_{0+}^\uparrow (f_r = f)$ , then

$$\forall g \in \mathcal{F}^\uparrow, f * g = f_r * g \stackrel{(32)}{=} (f * g)_r \in \mathcal{R}^\uparrow \quad (56)$$

Moreover,  $f * g(0+) = f * g(0) = 0$  because  $f(0) = g(0) = 0$ .  $\square$

### 3 CONTINUITY AND FLOWS

Now, the aim is to see if the Network Calculus properties are preserved passing from a world to another. First, we will see the impact on the cumulative curves. Then, we will show that the notion of arrival curves is preserved.

#### 3.1 Cumulative curves

In Network Calculus, flows are modelled by cumulative functions  $A \in \mathcal{F}_0^\uparrow$  such that  $A(t)$  counts the total amount of data generated by the flow up to time  $t$ . Since a flow is a cumulative amount of data, it must be a non-decreasing function. The condition on finite number of discontinuity is related to the discrete aspect of computer behavior and simplifies the mathematical part. The condition of null value at 0 is related to the fact that all results in network calculus are based on differences: a bound on  $A(t)$  has to be understood as a bound on  $A(t) - A(0)$ .

However, according to Section 2.3 and to find equivalence between left and right worlds, we will enforce the right continuity at the origin, that is to say consider cumulative curves in the set  $\mathcal{F}_{0+}^\uparrow$  instead of  $\mathcal{F}_0^\uparrow$ . Indeed, we first have that this set is stable using the right/left extension operators and secondly, we have results concerning the Min-plus convolution that tilt the balance in favour of the sets  $\mathcal{R}_{0+}^\uparrow / \mathcal{L}_{0+}^\uparrow$ . Looking the behavior of the system in the real world, an interpretation can be that the network is ready before the applications: the time origin of the network is strictly smaller than the time origin of the applications.

### 3.2 Arrival curves

The notion of arrival curve is used to bound the amount of data sent by a flow on any interval of time. There are two notions: the maximal arrival curve and the minimum arrival curve. By definition,  $(\bar{\alpha}, \underline{\alpha}) \in \mathcal{F}^\dagger$  are maximal and minimal arrival curve of a cumulative curve  $A$  if

$$\forall s, t \in \mathbb{R}^+, s \leq t, A(t) - A(s) \leq \bar{\alpha}(t - s), \quad (57)$$

$$\forall s, t \in \mathbb{R}^+, s \leq t, A(t) - A(s) \geq \underline{\alpha}(t - s). \quad (58)$$

**Property 8.** Let  $\bar{\alpha} \in \mathcal{F}^\dagger$  be a maximal arrival curve of a cumulative curve  $A \in \mathcal{F}_{0+}^\dagger$ . Then,  $\bar{\alpha}$  is a maximal arrival curve of  $A_r$  and  $A_l$ .

*Proof.* Let  $\bar{\alpha} \in \mathcal{F}^\dagger$  be a maximal arrival curve of a cumulative curve  $A \in \mathcal{F}_{0+}^\dagger$ . According to the definition of the maximal arrival curve,  $\forall s, t \in \mathbb{R}^+, s \leq t$ ,

$$A(t) - A(s) \leq \bar{\alpha}(t - s) \quad (59)$$

Passing to the left limit, according to Property 2,

$$A_l(t) - A_l(s) \leq \bar{\alpha}_l(t - s) \leq \bar{\alpha}(t - s). \quad (60)$$

Then,  $\bar{\alpha}$  is a maximal arrival curve of  $A_l$ .

Also,  $\forall \varepsilon > 0$ ,

$$A(t + \varepsilon) - A(s + \varepsilon) \leq \bar{\alpha}(t - s) \quad (61)$$

Passing to the limit when  $\varepsilon \rightarrow 0$ ,

$$A_r(t) - A_r(s) \leq \bar{\alpha}(t - s). \quad (62)$$

Then,  $\bar{\alpha}$  is a maximal arrival curve of  $A_r$ .  $\square$

**Property 9.** Let  $\underline{\alpha} \in \mathcal{F}^\dagger$  be a minimal arrival curve of a cumulative curve  $A \in \mathcal{F}_{0+}^\dagger$ . Then,  $\underline{\alpha}$  is a minimal arrival curve of  $A_r$  and  $A_l$ .

The proof is similar to the previous one.

Consequently, the notion of arrival curves is preserved passing from a world to another.

## 4 CONTINUITY AND SERVERS

Now, we will work on the servers and prove that the servers are preserved passing from a world to another. This section contains the main contributions of the paper.

### 4.1 Defining servers in both worlds

A server  $S$  describes relationships between input and output flows,  $S \subset \mathcal{F}_0^\dagger \times \mathcal{F}_0^\dagger$ . Then  $(A, D) \in S$ , denoted as  $A \xrightarrow{S} D$ , means that a server  $S$  receives an input flow  $A(t)$ , and delivers the output  $D(t)$ . A system  $S$  might be, for example, a single buffer served at a constant rate, a complex communication node, or even a complete network.

An important assumption made with servers  $S$  is that  $D \leq A$ , meaning that data goes out after being entered. Basic network calculus assumes also that there is no loss neither data creation, compression or deflating, even if some extensions to that have been defined in [7].

As previously said, most results in the litterature consider only non-decreasing left continuous arrival curves such that they are null at  $t = 0$  ( $S \subset \mathcal{L}_0^\dagger \times \mathcal{L}_0^\dagger$ ). In the following, we want to consider  $S \subset X \times X$  for different

subsets  $X \subset \mathcal{F}_0^\dagger$ . To do that, we need to rewrite the common definitions related to the servers parameterized by  $X$ .

**Definition 5 (Server).** Let  $X \subset \mathcal{F}_0^\dagger$ . A server  $S^X \subseteq X \times X$  is a right-total relation between flow cumulative functions ( $\forall A \in X, \exists D, (A, D) \in S^X$ ) that satisfies  $(A, D) \in S^X \implies A \geq D$ .

We denote  $A \xrightarrow{S^X} D$  for  $(A, D) \in S^X$ .

**Definition 6 (Min-plus minimal service curve).** Let  $X \subset \mathcal{F}_0^\dagger$ . Let  $S^X$  be a server and  $\beta \in \mathcal{F}^\dagger$ . The server  $S^X$  is said to offer a min-plus minimal service curve  $\beta$  if

$$\forall A, \forall D : A \xrightarrow{S^X} D \implies D \geq A * \beta. \quad (63)$$

Let note

$$\mathcal{S}_{mp}^X(\beta) = \left\{ S^X \mid \forall (A, D) \in S^X, D \geq A * \beta \right\}. \quad (64)$$

**Definition 7 (Maximal service curve).** Let  $X \subset \mathcal{F}_0^\dagger$ . Let  $\beta^M \in \mathcal{F}_0^\dagger$ . A server  $S^X$  offers a maximal service curve  $\beta^M$  if

$$(A, D) \in S^X \implies D \leq A * \beta^M. \quad (65)$$

Let note

$$\mathcal{S}_{max}^X(\beta^M) = \left\{ S^X \mid \forall (A, D) \in S^X, D \leq A * \beta^M \right\}. \quad (66)$$

**Definition 8 (Shaper).** Let  $\sigma \in \mathcal{F}_0^\dagger$  and  $X \subset \mathcal{F}_0^\dagger$ . A server  $S^X$  is a  $\sigma$ -shaper (also said offers a shaping service curve  $\sigma$ ) if

$$(A, D) \in S^X \implies D \leq D * \sigma. \quad (67)$$

Let note

$$\mathcal{S}_{sh}^X(\sigma) = \left\{ S^X \mid \forall (A, D) \in S^X, D \leq D * \sigma \right\}. \quad (68)$$

**Definition 9 (Backlogged period (BP)).** Let  $X \subset \mathcal{F}_0^\dagger$ . Let  $S^X$  be a server and  $(A, D) \in S^X$ . An interval  $I$  is a backlogged period for  $(A, D)$  if

$$\forall t \in I, A(t) - D(t) > 0. \quad (69)$$

**Definition 10 (Start of backlogged period).** Let  $X \subset \mathcal{F}_0^\dagger$ . Let  $S^X$  be a server and  $(A, D) \in S^X$ . The start of backlogged period of time  $t \in \mathbb{R}^+$  is defined by

$$Start_{A,D}(t) \stackrel{\text{def}}{=} \sup\{u \leq t \mid D(u) = A(u)\}. \quad (70)$$

**Remark 3.** Note that with  $X \subset \mathcal{L}_0^\dagger$ ,  $\forall t \in I$ ,  $I$  a backlogged period for  $(A, D) \in S^X$ ,  $A(Start_{A,D}(t)) = D(Start_{A,D}(t))$ . This property is usefull in the proofs of some results in the Network Calculus theory.

**Definition 11 (Strict minimal service curve).** Let  $X \subset \mathcal{F}_0^\dagger$ . A server  $S^X$  offers a strict service curve  $\beta \in \mathcal{F}^\dagger$  if

$$\forall (A, D) \in S^X, \forall (s, t] \text{ backlogged period}, D(t) - D(s) \geq \beta(t - s). \quad (71)$$

Let note

$$\mathcal{S}_{st}^X(\beta) = \left\{ S^X \mid \forall (A, D) \in S^X, \forall (s, t] \text{ backlogged period}, \right. \\ \left. D(t) - D(s) \geq \beta(t - s) \right\}. \quad (72)$$

In the previous definitions, the set  $X$  is, in most results of the Network Calculus, the set of left-continuous non-decreasing functions such that they are null at  $t = 0$  (i.e.  $X = \mathcal{L}_0^\uparrow$ ).

Before looking which properties are preserved between the left and right worlds, we need to define the left/right extensions of a server.

**Definition 12** (Right/Left extension of servers). *Let  $X \subset \mathcal{F}_0^\uparrow$  and  $S^X$  be a server. The right extension of a server is defined by*

$$(S^X)_r = \left\{ (A_r, D_r) \mid \exists (A, D) \in S^X \right\} \quad (73)$$

Reciprocally, the left extension of a server  $S^X$ ,  $X \subset \mathcal{F}$  is

$$(S^X)_l = \left\{ (A_l, D_l) \mid \exists (A, D) \in S^X \right\} \quad (74)$$

**Property 10.** *The right or left extension of a server is a server.*

*Proof.* Let  $X \in \mathcal{F}_0^\uparrow$  and  $S^X$  be a server. We need to prove that  $\forall (A', D') \in (S^X)_r$ ,  $A' \geq D'$ .

Let  $(A', D') \in (S^X)_r$  then  $(A', D') = (A_r, D_r) \mid \exists (A, D) \in S^X$ . But  $A \geq D$  and  $(A, D) \in S^X$  so  $(A, D) \in \mathcal{F}_0^\uparrow$ , according to Property 2,

$$A_r \geq D_r.$$

Then,  $(S^X)_r$  is a server.

The same way,  $(S^X)_l$  is also a server.  $\square$

Now, we want to see if the other servers (the Min-plus minimal server, the maximal server, the shaper and the strict minimal server) are stable under the extensions, i.e. if the properties associated to them are preserved passing from a world to another.

## 4.2 Equivalence of Min-plus minimal services

Let us start with the minimal Min-Plus services. Theorem 3 shows that if a min-plus minimal service with left-continuous arrival curves offers a service curve then, this service curve is also one for the right extension of the server. Reciprocally, if a min-plus minimal service with right-continuous arrival curves offers a service curve then, the left extension of this service curve is also one for the left extension of the server.

**Theorem 3** (Equivalence of Min-plus minimal services). *Let  $S \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}$  be a server offering a min-plus minimal service curve  $\beta \in \mathcal{F}^\uparrow$ . Then  $\beta$  is a Min-plus minimal service curve for  $S_r$ . That is to say*

$$\forall S \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}(\beta), S_r \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}(\beta) \quad (75)$$

Also, let  $S' \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}$  be a server offering a min-plus minimal service curve  $\beta' \in \mathcal{F}^\uparrow$ .

Then  $\beta'_l$  is a Min-plus minimal service curve for  $S'_l$ . That is to say

$$\forall S' \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}(\beta'), S'_l \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}(\beta'_l) \quad (76)$$

*Proof.* The proof is split into two parts:

- 1) Proof of Equation (75): Let  $S \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}$  be a server offering a min-plus minimal service curve  $\beta \in \mathcal{F}^\uparrow$  and  $(A, D) \in S$  then,  $D \geq A * \beta$ .

As  $A, D \in \mathcal{L}_{0+}^\uparrow$ , using the limit, we have

$$D \geq A * \beta \implies D_r \geq (A * \beta)_r \quad (77)$$

According to Theorem 2, we have that  $A_r * \beta = (A * \beta)_r$ . Then,

$$D \geq A * \beta \implies D_r \geq A_r * \beta \quad (78)$$

Consequently,

$$\forall S \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}(\beta), S_r \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}(\beta) \quad (79)$$

- 2) Proof of Equation (76): Let  $S' \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}$  be a server offering a min-plus minimal service curve  $\beta' \in \mathcal{F}^\uparrow$ . and  $(A, D) \in S'$  then,  $D \geq A * \beta'$ .

As  $A, D \in \mathcal{R}_{0+}^\uparrow$ , using the limit, we have

$$D \geq A * \beta' \implies D_l \geq (A * \beta')_l \quad (80)$$

According to Theorem 2,  $(A * \beta')_l = A_l * \beta'_l$ .

So,

$$D_l \geq A_l * \beta'_l \quad (81)$$

Then,

$$\forall S' \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}(\beta'), S'_l \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}(\beta'_l) \quad (82)$$

$\square$

To sum up, we just proved that if  $\beta$  is a Min-plus minimal service curve for a server  $S \in \mathcal{S}_{mp}^{\mathcal{L}_0^\uparrow}$  then it is also for  $S_r$ . Reciprocally, if  $\beta$  is a Min-plus minimal service curve for a server  $S' \in \mathcal{S}_{mp}^{\mathcal{R}_0^\uparrow}$ , then  $(\beta)_l$  also is for  $S'_l$ .

## 4.3 Equivalence of maximal services

Similar to what was done in Section 4.2, we will show the equivalence between the right and left worlds for the maximal services, i.e. the aim is to demonstrate that a maximal service curve for the left servers is a maximal service curve for the right extension server and reciprocally.

**Theorem 4.** *Let  $S \in \mathcal{S}_{max}^{\mathcal{L}_0^\uparrow}(\beta^M)$  be a server offering a maximal service curve  $\beta^M \in \mathcal{F}^\uparrow$ .*

*Then  $\beta^M$  is a maximal service curve for  $S_r$ . That is to say,*

$$\forall S \in \mathcal{S}_{max}^{\mathcal{L}_0^\uparrow}(\beta^M), S_r \in \mathcal{S}_{max}^{\mathcal{R}_0^\uparrow}(\beta^M) \quad (83)$$

Also, let  $S' \in \mathcal{S}_{max}^{\mathcal{R}_0^\uparrow}(\beta'^M)$  be a server offering a maximal service curve  $\beta'^M \in \mathcal{F}^\uparrow$ .

Then  $\beta'^M$  is a maximal service curve for  $S'_l$ . That is to say,

$$\forall S' \in \mathcal{S}_{max}^{\mathcal{R}_0^\uparrow}(\beta'^M), S'_l \in \mathcal{S}_{max}^{\mathcal{L}_0^\uparrow}(\beta'^M) \quad (84)$$

*Proof.* The proof is split into two parts:

- 1) Proof of Equation (83): Let  $S \in \mathcal{S}_{max}^{\mathcal{L}_0^\uparrow}(\beta^M)$  be a server offering a maximal service curve  $\beta^M \in \mathcal{F}^\uparrow$  and  $(A, D) \in S$  then,  $D \leq A * \beta^M$ . As  $A, D \in \mathcal{L}_{0+}^\uparrow$ , using the limit, we have

$$D \leq A * \beta^M \implies D_r \leq (A * \beta^M)_r \quad (85)$$

According to Theorem 2, we have that  $(A * \beta^M)_r = A_r * \beta^M$ . Then,

$$D \leq A * \beta^M \implies D_r \leq A_r * \beta^M \quad (86)$$



Consequently,

$$\forall S \in \mathcal{S}_{max}^{\mathcal{L}_{0+}^{\uparrow}}(\beta^M), S_r \in \mathcal{S}_{max}^{\mathcal{R}_{0+}^{\uparrow}}(\beta^M) \quad (87)$$

2) Proof of Equation (84): Let  $S' \in \mathcal{S}_{max}^{\mathcal{R}_{0+}^{\uparrow}}(\beta'^M)$  be a server offering a maximal service curve  $\beta'^M \in \mathcal{F}^{\uparrow}$ . and  $(A, D) \in S'$  then,  $D \leq A * \beta'^M$ .

As  $A, D \in \mathcal{R}_{0+}^{\uparrow}$ , using the limit, we have

$$D \leq A * \beta'^M \implies D_l \leq (A * \beta'^M)_l \leq A_l * \beta'^M \quad (88)$$

Then,

$$\forall S' \in \mathcal{S}_{max}^{\mathcal{R}_{0+}^{\uparrow}}(\beta'^M), S'_l \in \mathcal{S}_{max}^{\mathcal{L}_{0+}^{\uparrow}}(\beta'^M) \quad (89)$$

□

#### 4.4 Equivalences of shapers

Now, we will look for the equivalence between the right and left world for the shapers, *i.e.* the aim is to demonstrate that a shaping curve for the left servers is a shaping curve for the right extension server and reciprocally.

**Theorem 5.** Let  $S \in \mathcal{S}_{sh}^{\mathcal{L}_{0+}^{\uparrow}}(\sigma)$  be a server offering a shaping service curve  $\sigma \in \mathcal{F}^{\uparrow}$ .

Then  $\sigma$  is a shaping service curve for  $S_r$ . That is to say,

$$\forall S \in \mathcal{S}_{sh}^{\mathcal{L}_{0+}^{\uparrow}}(\sigma), S_r \in \mathcal{S}_{sh}^{\mathcal{R}_{0+}^{\uparrow}}(\sigma) \quad (90)$$

Also, let  $S' \in \mathcal{S}_{sh}^{\mathcal{R}_{0+}^{\uparrow}}(\sigma')$  be a server offering a shaping service curve  $\sigma' \in \mathcal{F}^{\uparrow}$ .

Then  $\sigma'$  is a shaping service curve for  $S'_l$ . That is to say,

$$\forall S' \in \mathcal{S}_{sh}^{\mathcal{R}_{0+}^{\uparrow}}(\sigma'), S'_l \in \mathcal{S}_{sh}^{\mathcal{L}_{0+}^{\uparrow}}(\sigma') \quad (91)$$

The proof is similar to the one of Theorem 4 and is given in Appendix.

#### 4.5 No direct equivalence of strict minimal services

We may expect a similar result to the Min-plus minimal service for the strict Min-plus service.

As in the definition of the strict minimal service, the curve is a strict minimal service curve for any  $(s, t)$  backlogged period (Definition 11), we first need to study these backlogged periods. First, let start with this latter notion: the backlogged periods and show that it is not preserved passing from a world to another with an illustration. Thus, it is not possible to have an equivalence for the strict minimal services and it is shown with a counter-example. Next, preliminary work are suggested to adapt the theory in order to get an equivalence and also preserved this property of strict minimal service.

##### 4.5.1 Strict service is not preserved between left/right worlds

Let take an example to illustrate that the strict minimal service is not preserved by continuity change. Consider a

system  $A \xrightarrow{S} B$  and  $B \xrightarrow{S'} C$  and its right extension  $A_r \xrightarrow{S_r} B_r$  and  $B_r \xrightarrow{S'_r} C_r$  with

$$A(t) = A_r(t) = t \wedge 3 \quad (92)$$

$$B(t) = \lfloor t \rfloor \wedge 3 \quad (93)$$

$$B_r(t) = (\lfloor t \rfloor - 1) \wedge 3 \quad (94)$$

$$C(t) = C_r(t) = [(t - 1) \wedge 3]^+ \quad (95)$$

The top side of Figure 3 represents these systems and the botside of Figure 3 shows the cumulative curves  $A, B, B_r$  and  $C$  (left side shows the left continuous ones ( $A, B$  and  $C$ ) and right side shows the right continuous ones ( $A, B_r$  and  $C$ )).

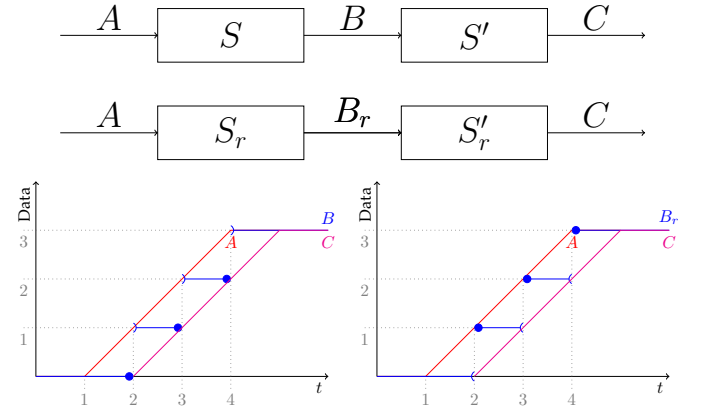


Fig. 3: Illustration of the backlog with the current definition (Definition 9).

If we study the backlogged periods of  $(A, B)$  and his right extension  $(A, B_r)$  then, with the current definition of the backlogged period (Definition 9), we see that  $[1 ; 4]$  is a backlogged period for the left-continuous functions  $(A, B)$ . However,  $[1 ; 4]$  isn't one for  $(A, B_r)$ , the right-extension of  $(A, B)$  because *e.g.*  $A(2) = B(2)$ . Consequently, the backlogged period for the left-continuous functions is not in general one for the right-extensions.

Regarding to this analysis, we show that, with the current definition of the backlogged period (Definition 9), there is no equivalence between left and right worlds. Consequently, we can't have equivalence between the strict Min-plus service curve, illustrated by a counter example: let  $\beta(t) = 2 \cdot [t - 1]^+$ . Indeed,  $\forall (s, t)$  backlogged period of  $(B, C)$ ,  $C(t) - C(s) \geq \beta(t - s)$  because  $\forall (s, t)$  backlogged period,  $t - s < 1 \implies \beta(t - s) = 0$ . However,  $(2, 3.9)$  is a backlogged period for  $(B_r, C)$  and  $C(3.9) - C(2) = 1 \leq 1.8 = \beta(3.9 - 2)$ . Then,  $S' \in \mathcal{S}_{st}^{\mathcal{L}_{0+}^{\uparrow}}(\beta)$  but  $S'_r \notin \mathcal{S}_{st}^{\mathcal{L}_{0+}^{\uparrow}}(\beta)$ . Figure 4 illustrates this service curve  $\beta$ .

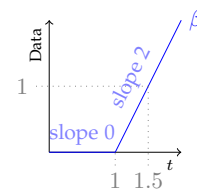


Fig. 4: Graph of a service curve  $\beta(t) = 2 \cdot [t - 1]^+$ .

Then, this example shows that, with the current definition of the backlogged period, we can't have any equivalence of the strict minimal service between the left and the right world.

Now going from right to left we can also construct a similar example showing that the equivalence is not possible according to the current definition of the backlogged period.

Consequently, the strict minimal service is not preserved passing from a world to another.

#### 4.5.2 Looking for a stable backlogged period

However, the difference between the backlogged periods seems to be due to singular points: the discontinuities of the cumulative curves. Then, the idea is to introduce new definitions like stable backlogged periods to include these singular points and preserved these definitions of backlogged period passing from a world to another. Here, it is presented some preliminary results concerning the equivalence of the strict minimal service using new definitions of the backlogged period. Let first suggest new definition of the backlogged period and some related properties.

**Definition 13** (Stable backlogged period 1 (SBP1)). *Let  $S^X$  be a server and  $(A, D) \in S^X$ . An interval  $I$  is a stable backlogged period 1 (SBP1) for  $(A, D)$  if*

$$\forall t \in I, \begin{cases} A(t) - D(t) > 0 \\ \text{or} \\ \exists \varepsilon > 0 \mid \{u \in [t - \varepsilon; t + \varepsilon] \mid A(u) - D(u) = 0\} = \{t\} \end{cases} \quad (96)$$

**Definition 14** (Stable backlogged period 2 (SBP2)). *Let  $S^X$  be a server and  $(A, D) \in S^X$ . An interval  $I$  is a stable backlogged period 2 (SBP2) for  $(A, D)$  if*

$$\begin{aligned} \exists n \in \mathbb{N}, (t_1, \dots, t_n) \in I^n \text{ such as} \\ \forall t \in I \setminus \{t_1, \dots, t_n\}, A(t) - D(t) > 0 \end{aligned} \quad (97)$$

**Property 11** (Equivalence for open intervals of the SBP1/SBP2). *Let  $(A, D) \in S^{\mathcal{F}_0^+}$ . If  $(s, t)$  is a stable backlogged period (SBP1 or SBP2) for  $(A, D)$ , then so is a stable backlogged period for  $(A_l, D_l)$  and for  $(A_r, D_r)$ .*

*Proof.* Let  $(A, D) \in S^{\mathcal{F}_0^+}$ .  $A$  and  $D$  are piecewise-continuous, then it exists a finite number of discontinuities in any bounded interval. Consequently,  $\forall (a; b)$  stable backlogged period (SBP1 or SBP2), it exists a finite number of points such that  $A_l = D_l$  or  $A_r = D_r$  in any bounded interval *i.e.* the discontinuities of  $(A, D)$ .

Consequently,  $(a, b)$  is a stable backlogged period (SBP1 or SBP2) for  $(A_l, D_l)$  and  $(A_r, D_r)$ .  $\square$

**Remark 4.** *Note that the SBP2 is more general. Indeed, the equivalence concerns also the closed and semi-closed intervals. Let  $(A, D) \in S^{\mathcal{F}_0^+}$ .  $\forall |s, t|$  stable backlogged period 2 for  $(A, D)$  (with  $|$  denotes either closed or open bound),  $|s, t|$  is a backlogged period for  $(A_l, D_l)$  and for  $(A_r, D_r)$ . This is due to the fact that the stable backlogged period 2 doesn't take into account the bound of the interval, *i.e.*  $A$  and  $D$  can be equal at the bound even if it is not an isolated point.*

**Remark 5.** *First, we can see a hierarchy with the different backlogged periods such as*

$$BP \implies SBP1 \implies SBP2. \quad (98)$$

*Secondly, the idea of the backlogged period is to catch each time interval where the server needs to work. Does a singular point  $t$  such that  $A(t) = D(t)$ ,  $(A, D) \in S^X$  really mean that the server stop to work? Indeed can he stop to work during a null interval of time? If the servers works during  $|t_1 ; t_2|$  and  $|t_2 ; t_3|$ , he probably works physically during  $|t_1 ; t_3|$ . This is why we suggest other definitions for the backlogged period which include the singularities.*

*Finally, we suggest two new definitions because the first one is such that if there is no singularities in the backlogged period, the current definition and the stable one are equivalent. However, due to the bound of the intervals, those two definitions are not equivalent in the left and right models. The last one, the stable backlogged period 2 owns this property because it can be a closed interval including the bounds even if  $A(t) = D(t)$ ,  $t$  a bound of the interval.*

To illustrate this remark, Figure 5 shows a sample with three different couples  $(A, D)$  and a maximal backlogged periods bounded by  $s$  and  $t$ , the cases are:

- 1) On the first column,  $A$  is continuous and  $D$  is discontinuous on  $t$ ,
- 2) On the second one,  $A$  is discontinuous on  $s$  and  $D$  continuous,
- 3) On the last one,  $A$  and  $D$  are continuous.

Also, the up line of the figure illustrates the left extensions of these functions and the down line illustrates the right extensions.

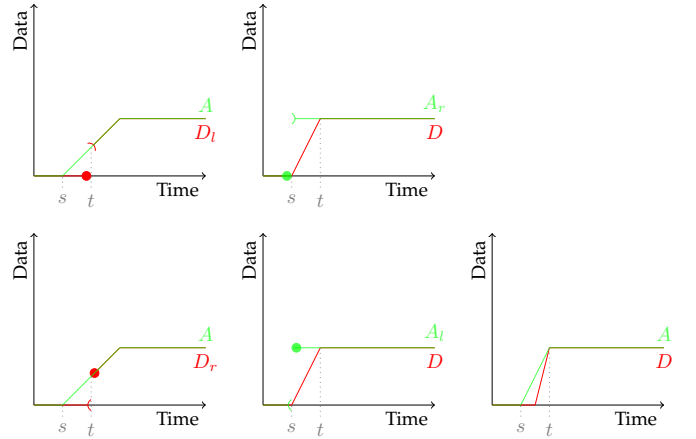


Fig. 5: Illustration of the differences between the backlogged periods.

This example doesn't show the singularities because it is taken into account in the same way for the two stable backlogged periods. The aim is to see the difference between them regarding the bounds.

A backlogged period associated to Figure 5 is presented in Table 1 in the same order for the three definition of the backlogged periods.

As it is explain in the remark, all the stable backlogged periods 2 are equal but different from the current backlogged periods. The stable backlogged periods 1 are equal to the current backlogged periods but are not equal to each others.

TABLE 1: Recapitulative table of the maximal backlogged periods of Figure 5.

BP: ]s; t[	BP: ]s; t[	
SBP1: ]s; t[	SBP1: ]s; t[	
SBP2: ]s; t[	SBP2: ]s; t[	
BP: ]s; t[	BP: ]s; t[	BP: ]s; t[
SBP1: ]s; t[	SBP1: ]s; t[	SBP1: ]s; t[
SBP2: ]s; t[	SBP2: ]s; t[	SBP2: ]s; t[

#### 4.5.3 The strict non nap services

Finally, let use the new definitions of the backlogged periods to introduce a 'new' strict Min-plus service called strict non nap service and then prove the equivalence between the left and right models with this new definition.

**Definition 15** (Strict non nap services). Let  $X \subset \mathcal{F}_0^\uparrow$ . A server  $\tilde{S}_1^X$  offers a strict non nap service of type 1 and curve  $\tilde{\beta} \in \mathcal{F}^\uparrow$  if

$$\forall (A, D) \in \tilde{S}_1^X \mid \forall (s, t) \text{ stable backlogged period 1 (SBP1)}, \\ D(t) - D(s) \geq \tilde{\beta}(t - s). \quad (99)$$

A server  $\tilde{S}_2^X$  offers a strict non nap service of type 2 and curve  $\tilde{\beta} \in \mathcal{F}^\uparrow$  if

$$\forall (A, D) \in \tilde{S}_2^X \mid \forall (s, t) \text{ stable backlogged period 2 (SBP2)}, \\ D(t) - D(s) \geq \tilde{\beta}(t - s) \quad (100)$$

with  $\mid$  can be ) or ] .

For  $i \in \{1; 2\}$ , let note

$$\tilde{S}_{nn-i}^X(\tilde{\beta}) = \left\{ S^X \mid \forall (A, D) \in S^X, \forall ]s, t[ \text{ SBP } i, \\ D(t) - D(s) \geq \tilde{\beta}(t - s) \right\}. \quad (101)$$

**Property 12.** Let  $i \in \{1; 2\}$ . Let  $\tilde{S} \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{L}^\uparrow}$  be a server offering a strict non nap service curve  $\tilde{\beta} \in \mathcal{F}^\uparrow$ . Then  $\tilde{\beta}$  is a strict non nap service curve for  $\tilde{S}_r$ . That is to say,

$$\forall S \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{L}^\uparrow}(\tilde{\beta}), S_r \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{R}^\uparrow}(\tilde{\beta}) \quad (102)$$

Also, let  $\tilde{S}' \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{R}^\uparrow}$  be a server offering a strict non nap service curve  $\tilde{\beta}' \in \mathcal{F}^\uparrow$ . Then  $\tilde{\beta}'_l$  is a strict non nap service curve for  $\tilde{S}'_l$ . That is to say,

$$\forall S' \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{R}^\uparrow}(\tilde{\beta}'), S'_l \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{L}^\uparrow}(\tilde{\beta}'_l) \quad (103)$$

**Remark 6.** As explain in [8], there is a choice with the bound of the interval  $(s; t]$ ,  $\mid$  can be open or closed.

With the generalized backlogged period 2 (Definition 14), the choice is still open because for any interval (open, closed are semi-closed)  $]s; t]$ , there is equivalence between left and right models.

However, the bounds are important in the generalized backlogged period 1 (Definition 13) because only the open intervals are equivalent. Then,  $]s; t[$  seems to be the only choice for this definition of the backlogged periods.

*Proof.* The proof is split into two parts:

- 1) proof of Equation (102) Let  $i \in \{1; 2\}$ .

Let  $S \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{L}^\uparrow}(\beta)$  be a server offering a strict non nap service curve  $\beta \in \mathcal{F}^\uparrow$ . Then,  $\forall (A, D) \in S \mid \forall (s, t) \mid \subset \text{ SBP } i, D(t) - D(s) \geq \beta(t - s)$  where  $\mid$  depends on  $i$ .

According Properties 11,  $\forall (s; t]$  stable backlogged period  $i$  of  $(A, D) \in S$  then,  $(s; t]$  is a stable backlogged period  $i$  of  $(A, D) \in S_r$ .

Also,  $\forall (A, D) \in S, \forall (s, t]$  stable backlogged period  $i$ ,

$$D(t) - D(s) \geq \beta(t - s) \implies D_r(t) - D_r(s) \geq \beta_r(t - s) \geq \beta(t - s) \quad (104)$$

Then,  $\beta$  is a strict minimal service curve for  $(A_r, D_r)$ . Consequently,

$$\forall S \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{L}^\uparrow}(\beta), S_r \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{R}^\uparrow}(\beta) \quad (105)$$

- 2) proof of Equation (103) Let  $i \in \{1; 2\}$ .

Let  $S' \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{R}^\uparrow}(\beta')$  be a server offering a strict non nap service curve  $\beta' \in \mathcal{F}^\uparrow$ . Then,  $\forall (A', D') \in S' \mid \forall (s, t) \mid \subset \text{ SBP } i, D'(t) - D'(s) \geq \beta'(t - s)$  where  $\mid$  depends on  $i$ .

According Properties 11,  $\forall (s; t]$  stable backlogged period  $i$  of  $(A', D') \in S'$  then,  $(s; t]$  is a stable backlogged period  $i$  of  $(A', D') \in S'_l$ .

Also,  $\forall (A', D') \in S', \forall (s, t]$  stable backlogged period  $i$ ,

$$D'(t) - D'(s) \geq \beta'(t - s) \implies D'_l(t) - D'_l(s) \geq \beta'_l(t - s) \quad (106)$$

Consequently,

$$\forall S' \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{R}^\uparrow}(\beta'), S'_l \in \tilde{\mathcal{S}}_{nn-i}^{\mathcal{L}^\uparrow}(\beta'_l) \quad (107)$$

□

## 5 USAGE OF THE RESULTS

In this section, we will show on an example to see how to pass from a world to another using the previous results. Consider a system  $S$  shared by two flows  $A, A'$ , followed by a packetizer  $P$ , as in Figure 6.

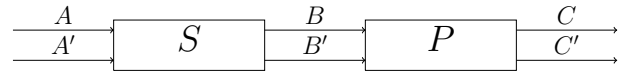


Fig. 6: Illustration of a network  $\mathcal{N}$  composed of a server ( $S$ ) and a packetizer ( $P$ ).

Assume that  $S$  uses a Static Priority Preemptive policy (SPP) and offers a strict minimal service of curve  $\beta$ . Using the most common approach, we assume that the cumulative curves are left continuous and  $(A, A')$  are  $(L, L')$ -packetized. Also,  $A$  has higher priority than  $A'$ .

Let do the two first steps in the left world. According to [1, Theorem 7.6],  $[\beta - \alpha_A]^\uparrow$  is a strict minimal service curve for  $A'$ .

And, according to [9, Prop. 1.3.5], a strict minimal service implies a minplus minimal service.

Now, using Theorem 3, we can transfer this service from the left to the right world and state that  $[\beta - \alpha_A]^\uparrow$  is a minplus minimal service curve for  $(A_r, B_r) \in S_r$ .

Now, in the right world, according to [9, Theorem 1.7.1],  $\beta_{A'} = [([\beta - \alpha_A]^\uparrow - L'^M)]^\uparrow$  (with  $L'^M$  the maximum packet

length of  $A'_r$ ) is a minimal service curve for the sequence  $S; P$  that maps  $A'_r$  into  $C'_r$ . (notice that  $[(\beta - \alpha_A)_\uparrow^+ - L'^M]_\uparrow^+ = [\beta - \alpha_A - L'^M]_\uparrow^+$ ).

Finally, using Theorem 3, we can transfer this service from the right to the left world, i.e.  $(\beta_{A'})_l$  is a minplus service for  $S; P$ . as illustrated in Figure 7.

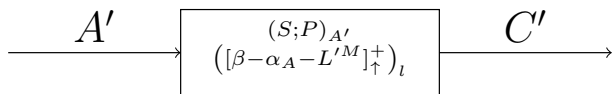


Fig. 7: Illustration of the network  $\mathcal{N}$  reduced according to the flow  $(A, C)$ .

This is an exemple of the usage of the results presented in this paper. However, the aim is to use more results to improve analysis as, for instance, the link with CPA theory as [3] suggests it.

## 6 STATE OF THE ART

As we previously said, most of the theory is based on the assumption of the left-continuous cumulative curves as, justified in [1, §1.3] and [9, §1.2.1]. In particular, this property implies that for any backlog instant  $t$ , it exists a specific instant  $s$ , the start of the backlog such that  $A(s) = D(s)$ . This property is used in most proofs related to the derivation of the residual services. But, as noted in the previous part, some notions are difficult in the left world. In fact, it was already noticed in [9, §1.1.1] that “It would be nice to stick to either left or right continuous functions. However, depending on the model, there is no best choice”. An example is the system which waits for the reception of all bits of a frame before transmitting it: the packetizer. A definition in the left world is quite complex (e.g. [1, Def. 8.2]) whereas it is simple in the right world (e.g. [9, Def. 1.7.3] and [3, Def. 10])

By definition, it is right-continuous and that creates some errors or difficulties as [1] reveals in *Section 8.3 bibliographic notes*. As it is naturally right-continuous, it is an advantage to develop the theory with the right world.

Also, works on the both worlds are know as for instance [6]. It is demonstrates that the delay and the backlog are not influenced by the continuity of the cumulative curves. Moreover, we have the same hierarchy in the server (strict minimal service implies minPlus minimal services) and the expression of the FIFO and Static Priority services are almost identical (‘almost’ due to continuity) and lead to the same numerical results.

Concerning the right world, [10] deals with the right and left continuity concerning the operators of the Network Calculus. Focusing on the assumption of right-continuous cumulative curves, [3] developed the link between the Network Calculus and the Compositional Performance Analysis (CPA) theory. The idea is to pass from the quantity of data of the network calculus ( $A$ ) to the number of events in the CPA theory ( $E$ ) using the packetization  $P$ , and the link is  $E = P(A)$ . But, this paper is developed using the right-continuous assumption and this choice is explained in [11, §4.4].

All this previous works show a new point of view in the theory of Network Calculus and other results are develop changing some assumptions. To continue in this idea, [12] is another extension to the theory inverting the amount of data and the time that leads to the max-plus theory and other results.

## 7 CONCLUSION

Network calculus represents flow behaviour with piecewise continuous functions. For some results, it is easier to consider left-continuous functions, whereas for other, right-continuous are more convenient. But using in the same analysis results developed from incompatible hypotheses is not sound. Then, to use results developed with different hypotheses, the only solution was, up to now, to chose one hypothesis and to redevelop the missing results (as done in [6]). But this is fastidious and error-prone, whereas the engineer intuition was that theses continuity problems do not really matter. This paper shows that, under reasonable assumptions, the minplus service can ignore continuity modelling problems.

To do so, this paper has developed some results related to the minplus convolution, one of the fundamental operator of the Network Calculus, regarding either the right-continuous and the left-continuous functions. It also has given a new expression of the convolution that may be usefull in other contexts. Also, the main result is that the properties linked to the convolution are preserved. Thus, the equivalences between the left and the right world is proven for the arrival curves, the minplus minimal service curves, the maximal service curves and the shaping curves. Conversely, the strict minimal service is not preserved due to the backlogged periods.

First a counter-example showed that there is no equivalence with the current definition but new definitions of the backlogged periods (the stable backlogged periods) were suggested and allow to have an equivalence between the two worlds. However, we can’t use, currently those new definition, it is a preliminary work because the start of the backlogged period need to be adapted and the proofs using these results need to be checked with the new definitions: the stable backlogged period and the strict non nap service.

In particular, we would like to use these results to improve the analysis results by combining it with CPA theory, as proposed in [3], [13].

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## 8 ANNEXE

### 8.1 Proof of Property 1

The proof come from [https://encyclopediaofmath.org/wiki/Increasing\\_function](https://encyclopediaofmath.org/wiki/Increasing_function).

*Proof.* A real-valued function  $f$  defined on a certain set  $E$  of real numbers such that the condition

$$x' < x'', x', x'' \in E$$

implies

$$f(x') < f(x'').$$

Such functions are sometimes called strictly increasing functions, the term "increasing functions" being reserved for functions which, for such given  $x'$  and  $x''$ , merely satisfy the condition

$$f(x') \leq f(x'')$$

(non-decreasing functions). The inverse function of any strictly increasing function is single-valued and is also strictly increasing. If  $x_0$  is a right-sided (or left-sided) limit point of the set  $E$  (cf. Limit point of a set), if  $f$  is a non-decreasing function and if the set  $A = \{y : y = f(x), x > x_0, x \in E\}$  is bounded from below - or if  $\{y : y = f(x), x < x_0, x \in E\}$  is bounded from above - then, as  $x \rightarrow x_0+$  ( or, correspondingly, as  $x \rightarrow x_0-$ ,  $x \in E$ , the values  $f(x)$  will have a finite limit; if the set is not bounded from below (or, correspondingly, from above), the values  $f(x)$  have an infinite limit equal to  $-\infty$  ( or, correspondingly, to  $+\infty$ ).

Comments If  $f$  is non-decreasing on  $E$  and  $x_0 \in E$ , then the set  $A$  referred to above is automatically bounded from below by  $f(x_0)$ , unless it is empty. If, in addition,  $x_0$  is a limit point of  $\{x \in E : x > x_0\}$ , then the right-hand limit of  $f$  at  $x_0$  is simply the infimum of  $A$ :

$$\lim_{x \rightarrow x_0-} f(x) = \inf A.$$

Similar remarks hold for left-hand limits. □