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Exact controllability to eigensolutions of the bilinear heat equation on compact networks

Piermarco Cannarsa

University of Rome Tor Vergata
Via della Ricerca Scientifica 1, 00133 Roma
`cannarsa@mat.uniroma2.it`

Alessandro Duca

Université Paris-Saclay, UVSQ, Laboratoire de Mathématiques de Versailles.
45 avenue des États-Unis 78035 Versailles cedex, France
`alessandro.duca@uvsq.fr`

Cristina Urbani

University of Rome Tor Vergata
Via della Ricerca Scientifica 1, 00133 Roma
`urbani@mat.uniroma2.it`

Abstract

Partial differential equation on networks have been widely investigated in the last decades in view of their application to quantum mechanics (Schrödinger type equations) or to the analysis of flexible structures (wave type equations). Nevertheless, very few results are available for diffusive models despite an increasing demand arising from life sciences such as neurobiology. This paper analyzes the controllability properties of the heat equation on a compact network under the action of a single input bilinear control.

By adapting a recent method due to [Alabau-Boussouira, Cannarsa, Urbani, Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control], an exact controllability result to the eigensolutions of the uncontrolled problem is obtained in this work. A crucial step has been the construction of a suitable biorthogonal family under a non-uniform gap condition of the eigenvalues of the Laplacian on a graph. Application to star graphs and tadpole graphs are included.

Keywords: bilinear control, heat equations, compact graph, biorthogonal family

2010 MSC: 93B05, 35R02

1 Introduction

In recent years, networks have attracted interest from a larger and larger part of the scientific community. This is certainly due to an increasing demand coming from numerous applications, but also to the new theoretical issues raised by the interaction between dynamics and network structure.

Applications of network type models, also called graphs, can be found, for instance, in quantum mechanics to study the dynamics of free electrons in organic molecules (see the seminal work [32] and also [34, 35]). In material science, they are often used to analyze the superconductivity in granular and artificial materials [4], or even in physics for acoustic and electromagnetic wave-guides networks (see [26, 30]).

Another relevant application can be found in the study of structures composed by flexible/elastic strings or bars. When the elements of the structure are sufficiently thin, the system can be studied by considering wave type equations on graph domains. Many references on the subject are contained in the book [20], where also various spectral and controllability results are provided (see also [31, 18]).

Controllability of wave type equations on networks has been widely investigated in the last decades mostly for what concerns locally distributed or boundary controls. A non-exhaustive list on the subject is [7, 8, 19, 21].

We recall that a network, or graph, is a structure composed by edges connected in some points which are called vertices. The vertices are classified as external or internal: a vertex is external if it is the endpoint of only one edge, and it is internal otherwise. On such a structure it is possible to define a Hilbert space and a self-adjoint Laplace operator which satisfies the so-called Neumann-Kirchhoff boundary conditions at the

internal vertices and Dirichlet or Neumann boundary conditions at the external vertices. For a wide analysis on graphs, we refer to the book [14].

In this paper, we are interested in the bilinear control of networks – or graphs – of diffusive type. In other words, we consider parabolic type equations on each edge of the graph and we study the controllability of the system via suitable multiplicative controls. It is well known that, even on classical domains, important applications motivate the introduction of such kind of control actions that are more realistic in certain situations. For instance, in [28] several examples are given where the potential coefficient is used as a control.

A model described by our setting comes, for instance, from a neurobiological problem. From the seminal work by A. L. Hodgkin and A. F. Huxley [27] it is known that the spread of electric current in neurons can be modeled by diffusive evolution equations. Therefore, Rall in [33] proposed a cable model for a dendritic tree which consists of a cylindrical trunk and cylindrical branch components. This formulation leads to consider a cable equation of the form

$$(1) \quad C_m \frac{\partial}{\partial t} v = \frac{a}{2R_i} \frac{\partial^2}{\partial x^2} v - gv$$

on a network structure. Here R_i, C_m, a and g are positive constants representing, respectively, axoplasmic resistivity, membrane capacitance per unit area, fiber radius and membrane conductance. The variable v represents the membrane potential at position x along the cable at time t . It is assumed that at the nodes, where branching occurs, the potential is continuous and there is conservation of current flowing through the node (Neumann-Kirchhoff conditions). At the terminals of the dendritic tree it is possible to impose open-end (Dirichlet) or closed-end (Neumann) conditions (see [1]). Equation (1) describes the case of a passive cable with constant membrane conductance. However, to model realistic synaptic inputs we must allow the membrane capacity to depend on both space and time [1, Paragraph 7]. Thus, the cable equation on the edges of the network becomes

$$\frac{\partial}{\partial t} v = \frac{\partial^2}{\partial x^2} v - g(x, t)v,$$

which is exactly of the form of our problem.

Bilinear control problems on networks were already studied for the Schrödinger equation in [5, 6, 24, 25]. However, fewer controllability results are available for diffusive networks. For instance, in [10] the authors study the null controllability properties of a parabolic equation on networks, by means of additive controls. Nevertheless, there is a lack of controllability results for multiplicative controls and, in particular, for controls that are only a scalar function of time, even in standard domains. This is certainly due to a structural obstacle, described in [9], to control these systems *along* a reference trajectory and also to the impossibility to apply methods based on the inverse mapping theorem developed for bilinear hyperbolic problems (see [11, 12]).

On the other hand, the negative result in [9] does not prevent us from controlling parabolic systems *to* a reference trajectory. Let us consider the abstract evolution equation

$$(2) \quad \begin{cases} y'(t) + Ay(t) + u(t)By(t) = 0, & t \in (0, T) \\ y(0) = y_0 \end{cases}$$

on a Hilbert space X . When $A : D(A) \subseteq X \rightarrow X$ is a densely defined, self-adjoint linear operator with compact resolvent, B a bounded linear operator and u is a bilinear control, the controllability of (2) was studied in [2, 3, 17]. Denoting by $\{\lambda_k\}_{k \in \mathbb{N}^*}$ the eigenvalues of the operator A , and by $\{\phi_k\}_{k \in \mathbb{N}^*}$ the corresponding normalized eigenfunctions, it is easy to see that the functions $\varphi_j(t) = e^{-\lambda_j t} \phi_j$ are solutions of the free dynamics ($u \equiv 0$) with initial condition $y_0 = \phi_j$. Such trajectories are called eigensolutions of problem (2). In [2] it is proved that under the gap condition

$$(3) \quad \sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \gamma > 0$$

and the spreading assumption

$$(4) \quad \langle B\phi_j, \phi_j \rangle \neq 0, \quad |\lambda_k - \lambda_j|^q |\langle B\phi_j, \phi_k \rangle| \geq b, \quad \forall k \neq j$$

with $p, q > 0$, system (2) is locally controllable to the j -th eigensolution for any time $T > 0$. Observe that, because of the gap condition (3), the result is mostly applicable to low dimensional problems.

Nevertheless, since graphs are, essentially, one-dimensional domains, one can hope for an extension of the previous result to these structures. Therefore, in this work we will focus our attention on the following control problem

$$(5) \quad \begin{cases} \partial_t \psi(t) - \Delta \psi(t) + u(t)B\psi(t) = 0, & t \in (0, T) \\ \psi(0) = \psi_0, \end{cases}$$

defined on a graph, where $-\Delta$ is a Dirichlet-Neumann Laplacian equipped with suitable boundary conditions, B is a bounded linear operator and the coefficient $u(\cdot)$ is a bilinear control.

However, when studying similar dynamics on a graph, one soon realizes that condition (3) fails. Indeed, what is always possible to prove for a Dirichlet-Neumann Laplacian on a general compact graph (see [20]) is a uniform gap condition among blocks of eigenvalues, namely,

$$\sqrt{\lambda_{k+M}} - \sqrt{\lambda_k} \geq \gamma > 0,$$

for some $M \in \mathbb{N}^*$. Furthermore, for graphs with a specific structure (tadpole graphs, star graphs, double-ring graphs etc) it has been proved in [24] that, under suitable hypotheses on the length of the edges, also a weak gap condition between consecutive eigenvalues holds:

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq a_k,$$

with $a_k \asymp k^{-2}$.

Even so, we were able to prove the following controllability result for general compact graphs.

Theorem 1.1. *Let $T > 0$, \mathcal{G} be a compact graph and $-\Delta$ be a Dirichlet-Neumann Laplacian on $L^2(\mathcal{G}, \mathbb{R})$. Denoted by $\{\lambda_k\}_{k \in \mathbb{N}}$ the eigenvalues of $-\Delta$, and by $\{\phi_k\}_{k \in \mathbb{N}^*}$ the associated normalized eigenfunctions, assume that the following gap condition is satisfied*

$$(6) \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq a_k$$

where $\{a_k\}_{k \in \mathbb{N}^*}$ is a sequence of positive real numbers such that $a_k \asymp k^{-p}$, with $p > 0$, for $k \in \mathbb{N}^*$.

Let $B : X \rightarrow X$ be a bounded linear operator on \mathcal{G} such that, for some $j \in \mathbb{N}^*$

$$(7) \quad \langle B\phi_j, \phi_1 \rangle \neq 0, \text{ and } \exists b, q > 0 : \lambda_k^q |\langle B\phi_j, \phi_k \rangle| \geq b, \quad \forall k \neq 1.$$

Then, system (5) is locally controllable to the j -th eigensolution $\varphi_j(t) = e^{-\lambda_k t} \phi_j$ in any positive time $T > 0$, that is, there exists $u \in L^2(0, T)$ such that $\psi(T) = \varphi_j(T)$. Furthermore,

$$\|u\|_{L^2(0, T)} \leq \frac{e^{-\pi^2 C/T}}{e^{2\pi^2 C/(3T)} - 1},$$

for a suitable positive constant C .

In Theorem 1.1, the sequence $\{a_k\}_{k \in \mathbb{N}^*}$ is assumed to be polynomially decreasing. However, it is not difficult to prove that the same result is also valid for a wider class of spectral gaps as, for instance, $a_k \asymp e^{-\sqrt{\lambda_k}}$. Nevertheless, the weak gap condition of the explicit networks analyzed in this work is satisfied with $a_k \asymp k^{-p}$.

Another model giving rise to a fading gap condition is the one studied in [13]. In this work the authors study the minimal null control time for problem with additive controls. They could overcome the phenomenon of condensation of eigenvalues in the moment problem they solve by a block resolution technique.

Even if the proof of our result is based on a linearization argument and the resolution of a moment problem as well, we are unable to use the approach of [13] because no estimate on the cost of control with respect to control time T is there provided.

This paper is organized as follows.

- In Section 2 we introduce graph theory and recall some useful spectral properties of the Laplace operator on such structures,
- In Section 3, we introduce the notion of controllability to eigensolutions. Then, we prove Theorem 1.1 and we state two semi-global controllability results.
- Section 4 is devoted to explicit applications to specific graphs. Moreover we give a flavour of how to treat the case of multiple eigenvalues.

2 Preliminaries

2.1 Introduction to graph theory

We denote by graph a structure composed by segments or half-lines (named edges) connected each other in some points (named vertices). When the graph is composed only by a finite number of edges with finite length, it is called compact graph.

Let \mathcal{G} be a compact graph composed by $N \in \mathbb{N}^*$ edges $\{e_j\}_{j \leq N}$ of lengths $\{L_j\}_{j \leq N}$ and $M \in \mathbb{N}^*$ vertices $\{v_j\}_{j \leq M}$. For every vertex v of the graph \mathcal{G} , we denote

$$N(v) := \{l \in \{1, \dots, N\} \mid v \in e_l\}.$$

We call V_e and V_i the external and the internal vertices of the graph \mathcal{G} (see Figure 1).

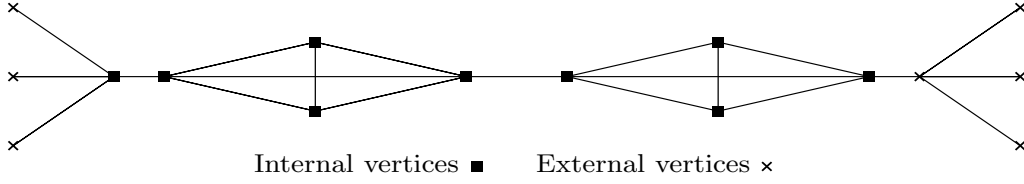


Figure 1: Internal and external vertices in a compact graph.

We equip the graph with a metric parametrizing each edge e_j with a coordinate going from 0 to its length L_j . We consider functions $f := (f^1, \dots, f^N)$ with domain the graph \mathcal{G} such that

$$f^j : (0, L_j) \rightarrow \mathbb{R}, \quad j \leq N.$$

We introduce now the Hilbert space $L^2(\mathcal{G}) := L^2(\mathcal{G}, \mathbb{R}) = \prod_{j \leq N} L^2((0, L_j), \mathbb{R})$ equipped with the norm $\|\cdot\|_{L^2}$ induced by the scalar product

$$\langle \psi, \varphi \rangle_{L^2} := \sum_{j \leq N} \langle \psi^j, \varphi^j \rangle_{L^2(e_j, \mathbb{R})} = \sum_{j \leq N} \int_{e_j} \psi^j(x) \varphi^j(x) dx, \quad \forall \psi, \varphi \in L^2(\mathcal{G}, \mathbb{R}).$$

Let $f \in H^1$ and v be a vertex of \mathcal{G} connected once to an edge e_j with $j \leq N$. When the coordinate parametrizing e_j in the vertex v is equal to 0 (resp. L_j), we denote

$$(8) \quad \partial_x f^j(v) = \partial_x f^j(0), \quad (\text{resp. } \partial_x f^j(v) = -\partial_x f^j(L_j)).$$

When e_j is a loop and it is connected to v in both of its extremes, we use the notation

$$(9) \quad \partial_x f^j(v) = \partial_x f^j(0) - \partial_x f^j(L_j).$$

When v is an external vertex and e_j is the only edge connected to v , we call $\partial_x f(v) = \partial_x f^j(v)$.

Now, we are finally ready to characterize the self-adjoint Laplacian $-\Delta$ appearing in equation (5), but before, we need to introduce the following boundary conditions on the vertices of the graph.

Vertex boundary conditions.

- (\mathcal{D}) A vertex $v \in V_e$ is equipped with Dirichlet boundary conditions when $f(v) = 0$ for every $f \in D(-\Delta)$.
- (\mathcal{N}) A vertex $v \in V_e$ is equipped with Neumann boundary conditions when $\partial_x f(v) = 0$ for every $f \in D(-\Delta)$.
- (\mathcal{NK}) A vertex $v \in V_i$ is equipped with Neumann-Kirchhoff boundary conditions when every $f \in D(-\Delta)$ is continuous in v and $\sum_{j \in N(v)} \partial_x f^j(v) = 0$ (according to the notations (8) and (9)).

Let us introduce the different definitions of $D(-\Delta)$ considered in this work.

Operator boundary conditions.

- The operator $-\Delta$ on \mathcal{G} is said to be a Dirichlet Laplacian when

$$D(-\Delta) = \{\psi \in H^2 : \psi \text{ satisfies } (\mathcal{NK}) \text{ in every } v \in V_i \text{ and } (\mathcal{D}) \text{ in each } v \in V_e\}.$$

- The operator $-\Delta$ on \mathcal{G} is said to be a Neumann Laplacian when

$$D(-\Delta) = \{\psi \in H^2 : \psi \text{ satisfies } (\mathcal{NK}) \text{ in every } v \in V_i \text{ and } (\mathcal{N}) \text{ in each } v \in V_e\}.$$

- The operator $-\Delta$ on \mathcal{G} is said to be a Dirichlet-Neumann Laplacian when

$$D(-\Delta) = \{\psi \in H^2 : \psi \text{ satisfies } (\mathcal{NK}) \text{ in every } v \in V_i \text{ and each } v \in V_e \text{ verifies } (\mathcal{D}) \text{ or } (\mathcal{N})\}.$$

Remark 2.1. The boundary conditions described above imply that the Laplacian $-\Delta$ is self-adjoint (see [29, Theorem 3]) and admits compact resolvent (see [29, Theorem 18]).

2.2 Some spectral properties

We denote by $\{\lambda_k\}_{k \in \mathbb{N}^*}$ the ordered sequence of eigenvalues of $-\Delta$ and $\{\phi_k\}_{k \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\mathcal{G}, \mathbb{R})$ composed by the corresponding eigenfunctions.

The following Lemma rephrases [24, Lemma 6] (arXiv version [24, Lemma 2.3]) which is a consequence of [14, Theorem 3.1.8] and [14, Theorem 3.1.10].

Lemma 2.2. [24, Lemma 6] (arXiv version [24, Lemma 2.3]) *Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be the eigenvalues of a Dirichlet-Neumann Laplacian $-\Delta$ defined on a compact graph. There holds $\lambda_k \asymp k^2$ for $k \geq 2$, i.e. there exist $C_1, C_2 > 0$ such that*

$$C_1 k^2 \leq \lambda_k \leq C_2 k^2, \quad \forall k \geq 2.$$

When $-\Delta$ is not a Neumann Laplacian, we have $0 \notin \sigma(-\Delta)$ (the spectrum of $-\Delta$), while when A is a Neumann Laplacian we have $0 \in \sigma(-\Delta)$ and $\lambda_1 = 0$.

From [20, Proposition 6.2], we have the following lemma.

Lemma 2.3. [20, Proposition 6.2] *Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be the ordered sequence of eigenvalues of a Dirichlet-Neumann Laplacian $-\Delta$ defined on a compact graph. There exist $\delta > 0$ and $M \in \mathbb{N}^*$ such that*

$$\inf_{k \in \mathbb{N}^*} \left| \sqrt{\lambda_{k+M}} - \sqrt{\lambda_k} \right| \geq \delta M.$$

Now, we present a spectral result valid for specific structures of graphs. The statement for the tadpole, the two-tails tadpole, the double-rings graph and the star graph with at most 4 edges follows from [24, Proposition 8] (arXiv version [24, Proposition 2.5]). The case of the star graph with $N \in \mathbb{N}^*$ edges is proved in Appendix A by using the techniques developed in [25, Section 3].

Proposition 2.4. *Let \mathcal{G} be either a tadpole, a two-tails tadpole, a double-rings graph or a star graph. Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be the ordered sequence of eigenvalues of a Dirichlet-Neumann Laplacian $-\Delta$ defined on \mathcal{G} . If the set of the lengths of the edges $\{L_j\}_{j \leq N} \in (\mathbb{R}^+)^N$ is so that $\{1, L_1, \dots, L_N\}$ are linearly independent over \mathbb{Q} and all the ratios L_k/L_j are algebraic irrational numbers, then, for every $\epsilon > 0$, there exists $C > 0$ such that*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq C k^{-1-\epsilon}, \quad \forall k \in \mathbb{N}^*$$

Proof. See Appendix A. □

3 Abstract controllability results

3.1 Introduction to controllability to eigensolutions

System (5) is a specific case of a more general control problem defined on a separable Hilbert space. Now, we recall the abstract result from [2] of local exact controllability of a bilinear parabolic problem of the form

$$(10) \quad \begin{cases} y'(t) + Ay(t) + u(t)By(t) = 0, & t > 0, \\ y(0) = y_0. \end{cases}$$

Let $(X, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be a separable Hilbert space and let $A : D(A) \subseteq X \rightarrow X$ be a densely defined linear operator such that

$$(11) \quad \begin{aligned} (a) & \quad A \text{ is self-adjoint,} \\ (b) & \quad \exists \sigma \geq 0 : \langle Ax, x \rangle \geq -\sigma \|x\|^2, \quad \forall x \in D(A), \\ (c) & \quad \exists \lambda > -\sigma \text{ such that } (\lambda I + A)^{-1} : X \rightarrow X \text{ is compact.} \end{aligned}$$

When the assumptions above are verified, the spectrum of A consists of a sequence of real numbers $\{\lambda_k\}_{k \in \mathbb{N}^*}$ which can be ordered, without loss of generality, as $-\sigma \leq \lambda_k \leq \lambda_{k+1} \rightarrow \infty$ as $k \rightarrow \infty$. We denote by $\{\phi_k\}_{k \in \mathbb{N}^*}$ the associated eigenfunctions, that is, $A\phi_k = \lambda_k \phi_k$ with $\|\phi_k\| = 1, \forall k \in \mathbb{N}^*$. Furthermore, $-A$ generates a strongly continuous semigroup denoted by e^{-tA} . The well-posedness of (10) is a classical result (see, i.e. [9]).

Proposition 3.1. *Let $T > 0$. Let $A : D(A) \subseteq X \rightarrow X$ satisfy (11), $B : X \rightarrow X$ be a bounded linear operator, $u \in L^2(0, T)$. Then, for any $y_0 \in X$ there exists a unique mild solution y of (10), that is, a function $y \in C^0([0, T]; X)$ which satisfies*

$$y(t; y_0, u) = e^{-tA}y_0 - \int_0^t e^{-(t-s)A}u(s)By(s; y_0, u)ds, \quad \forall t \in [0, T].$$

Moreover, there exists a constant $C(T) > 0$ such that

$$\sup_{t \in [0, T]} \|y(t; y_0, u)\| \leq C(T) \|y_0\|.$$

Fixed any $j \in \mathbb{N}^*$, we introduce the j -th *eigensolution* of problem (10), that is the function $\varphi_j(t) := e^{-\lambda_j t} \phi_j$, which solves (10) for $u \equiv 0$ and $y_0 = \phi_j$.

We recall that problem (10) is said to be locally controllable to the j -th eigensolution in time $T > 0$ if there exists $R_T > 0$ such that, for any $y_0 \in X$ with $\|y_0 - \varphi_j\| < R_T$, there exists a control $u \in L^2(0, T)$ for which the solution of (10) verifies $y(T; y_0, u) = \varphi_j(T)$.

Given $T > 0$ and $j \in \mathbb{N}^*$, we introduce the linear problem

$$(12) \quad \begin{cases} z'(t) + Az(t) + u(t)B\phi_j = 0, & t \in (0, T), \\ z(0) = z_0. \end{cases}$$

We denote by $z(\cdot; z_0, u)$ the solution of (12) associated to the initial condition z_0 and control u . We say that the pair $\{A, B\}$ is j -null controllable if there exists a constant $K(T) > 0$ such that for any $z_0 \in X$ there exist a control $u \in L^2(0, T)$ for which the solution of (12) satisfies $z(T; z_0, u) = 0$ and moreover $\|u\|_{L^2(0, T)} \leq K(T) \|z_0\|$. In particular, we call

$$K(T) := \sup_{\|z_0\|=1} \inf \left\{ \|u\|_{L^2(0, T)} : z(T; z_0, u) = 0 \right\}$$

the *control cost*. When assumption (11), (3) and (4) on the pair $\{A, B\}$ are verified, the authors of [2] proved the following controllability result.

Theorem 3.2. *Let $A : D(A) \subset X \rightarrow X$ be a densely defined linear operator such that (11) holds and let $B : X \rightarrow X$ be a bounded linear operator such that (4) holds. Let $\{A, B\}$ be j -null controllable in any $T > 0$ for some $j \in \mathbb{N}^*$ and such that*

$$(13) \quad K(\tau) \leq e^{\nu/\tau}, \quad \forall 0 < \tau \leq T_0,$$

for some constants $\nu, T_0 > 0$.

Then, the system (10) is locally controllable to the j -th eigensolution in any positive time $T > 0$. Moreover, the following estimate holds

$$(14) \quad \|u\|_{L^2(0, T)} \leq \frac{e^{-\pi^2 C/T}}{e^{2\pi^2 C/(3T)} - 1},$$

for a suitable positive constant C .

3.2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let $T > 0$, \mathcal{G} be a compact graph and let $X = L^2(\mathcal{G}, \mathbb{R})$. We shall prove the controllability of the following problem

$$(15) \quad \begin{cases} \partial_t \psi(t) - \Delta \psi(t) + u(t)B\psi(t) = 0, & t \in (0, T), \\ \psi(0) = \psi_0. \end{cases}$$

to the j -th eigensolution $\varphi_j(t)$ in time T .

The proof of Theorem 1.1 is based on the following propositions.

Proposition 3.3. *Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be a sequence of ordered non-negative real numbers such that $\sum_{k=2}^{\infty} \lambda_k^{-1} < +\infty$. Assume that there exist $M \in \mathbb{N}^*$, $\gamma > 0$ and an ordered sequence of decreasing positive real numbers $\{a_k\}_{k \in \mathbb{N}^*}$ verifying, for every $k \in \mathbb{N}^*$,*

$$(16) \quad \begin{cases} \sqrt{\lambda_{k+M}} - \sqrt{\lambda_k} \geq \gamma, \\ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq a_k. \end{cases}$$

Then, there exists a sequence of functions $\{\sigma_k\}_{k \in \mathbb{N}^*}$ which is biorthogonal to the family of exponentials $\{e^{\lambda_k t}\}_{k \in \mathbb{N}^*}$ in $L^2(0, T)$:

$$\int_0^T \sigma_k(t) e^{\lambda_j t} dt = \delta_{k,j}, \quad \forall k, j \in \mathbb{N}^*.$$

Moreover, the biorthogonal family $\{\sigma_k\}_{k \in \mathbb{N}^*}$ satisfies

$$(17) \quad \|\sigma_k\|_{L^2(0, T)}^2 \leq C \left(1 + \frac{\gamma^2}{a_k(a_k + 2\sqrt{\lambda_1})} \right)^{2M} e^{-2\lambda_k T} e^{C/(T\gamma^2)} e^{C\sqrt{\lambda_k}/\gamma} B(T, \gamma),$$

where

$$B(T, \gamma) := \begin{cases} \frac{1}{T} + \frac{1}{T^2\gamma^2}, & T \leq \frac{1}{\gamma^2}, \\ C\gamma^2, & T > \frac{1}{\gamma^2}, \end{cases}$$

Proof. See Appendix B. □

Notice that under the hypotheses of Theorem 1.1, the conditions in (16) are satisfied. Indeed, the first inequality in (16) is always valid for a Dirichlet-Neumann Laplacian $-\Delta$ on a graph thanks to Lemma 2.3.

Proposition 3.4. *Let $-\Delta$ be a Dirichlet-Neumann Laplacian on X . Let $B : X \rightarrow X$ be a linear bounded operator on \mathcal{G} such that (7) holds.*

Let $p > 0$ and $j \in \mathbb{N}^$, we define the following function*

$$(18) \quad G_{R,j}(T) := \frac{R}{T^2} e^{R/T} \sum_{k=1}^{\infty} \lambda_k^p \frac{e^{-2\lambda_k T + R\sqrt{\lambda_k}}}{| \langle B\phi_j, \phi_k \rangle |^2}.$$

Then, for any $R, T > 0$, the series in (18) is convergent and there exists a positive constant C_j such that

$$(19) \quad G_{R,j}(T) \leq e^{2C_j/T}, \quad \forall 0 < T \leq 1.$$

Proof. For any $\lambda \geq 0$, we set $f_1(\lambda) = e^{-2\lambda T + R\sqrt{\lambda}}$ and $f_2(\lambda) = e^{-\lambda T + R\sqrt{\lambda}}$. Then, the maximums of f_1 and f_2 is respectively attained for $\lambda = \left(\frac{R}{4T}\right)^2$ and $\lambda = \left(\frac{R}{2T}\right)^2$. From this remark and from assumption (7), we deduce that

$$(20) \quad \begin{aligned} G_{R,j}(T) &\leq \frac{R}{T^2} e^{R/T} \left(\frac{\lambda_1^p e^{R^2/(8T)}}{| \langle B\phi_j, \phi_1 \rangle |^2} + \frac{e^{R^2/4T}}{b^2} \sum_{k=2}^{\infty} \lambda_k^{p+2q} e^{-\lambda_k T} \right) \\ &\leq \frac{R}{T^2} e^{R/T} \left(\frac{\lambda_1^p e^{R^2/(8T)}}{| \langle B\phi_j, \phi_1 \rangle |^2} + \frac{e^{R^2/4T}}{b^2} \sum_{k=1}^{\infty} \lambda_k^{p+2q} e^{-\lambda_k T} \right). \end{aligned}$$

Now, for any $\lambda \geq 0$, we set $g(\lambda) = \lambda^{p+2q} e^{-\lambda T}$. Then, we have

$$g(\lambda) \text{ is } \begin{cases} \text{increasing} & \text{if } 0 \leq \lambda < (p+2q)/T, \\ \text{decreasing} & \text{if } \lambda \geq (p+2q)/T, \end{cases}$$

and attains its maximum for $\lambda = (p+2q)/T$. We define the following index:

$$k_1 := \sup \{k \in \mathbb{N}^* : \lambda_k \leq (p+2q)/T\}.$$

and we divide the sum in (20) as follows

$$(21) \quad \sum_{k=1}^{\infty} \lambda_k^{p+2q} e^{-\lambda_k T} = \sum_{k \leq k_1 - M} \lambda_k^{p+2q} e^{-\lambda_k T} + \sum_{k_1 - M + 1 \leq k \leq k_1 + M} \lambda_k^{p+2q} e^{-\lambda_k T} + \sum_{k \geq k_1 + M + 1} \lambda_k^{p+2q} e^{-\lambda_k T},$$

where M has been introduced in Lemma 2.3. Of course, if $k_1 \leq M$, we consider the first sum in (21) equal to 0, and if $k_1 \leq M - 1$ the second sum starts from $k = 1$.

For any $k \leq k_1 - M$, from the properties of g , we have

$$\int_{\lambda_k}^{\lambda_{k+M}} \lambda^{p+2q} e^{-\lambda T} d\lambda \geq (\lambda_{k+M} - \lambda_k) \lambda_k^{p+2q} e^{-\lambda_k T} \geq \gamma \lambda_k^{p+2q} e^{-\lambda_k T},$$

and for any $k \geq k_1 + M + 1$,

$$\int_{\lambda_{k-M}}^{\lambda_k} \lambda^{p+2q} e^{-\lambda T} d\lambda \geq (\lambda_k - \lambda_{k-M}) \lambda_k^{p+2q} e^{-\lambda_k T} \geq \gamma \lambda_k^{p+2q} e^{-\lambda_k T}.$$

Therefore, using the estimates above in (21), we deduce that

$$(22) \quad \sum_{k=1}^{\infty} \lambda_k^{p+2q} e^{-\lambda_k T} \leq \frac{2M}{\gamma} \int_0^{\infty} \lambda^{p+2q} e^{-\lambda T} d\lambda + \sum_{k_1 - M + 1 \leq k \leq k_1 + M} \lambda_k^{p+2q} e^{-\lambda_k T}.$$

Moreover, since g has a maximum at $\lambda = (p + 2q)/T$, we have

$$(23) \quad k = k_1 - M - 1, k_1 - M + 2, \dots, k_1 + M \implies \lambda_k^{p+2q} e^{-\lambda_k T} \leq ((p + 2q)/T)^{p+2q} e^{-(p+2q)}.$$

The integral term of (22) can be rewritten as

$$(24) \quad \int_0^\infty \lambda^{p+2q} e^{-\lambda T} d\lambda = \frac{1}{T} \int_0^\infty \left(\frac{s}{T}\right)^{p+2q} e^{-s} ds = \frac{1}{T^{1+p+2q}} \int_0^\infty s^{p+2q} e^{-s} ds = \frac{\Gamma(1+p+2q)}{T^{1+p+2q}},$$

where $\Gamma(\cdot)$ is the Euler integral of second kind. Hence, from (23) and (24) we conclude that there exist two positive constants $C_{q,M}, C_{q,M,\gamma}$ such that

$$\sum_{k=1}^\infty \lambda_k^{p+2q} e^{-\lambda_k T} \leq \frac{C_{q,M}}{T^{p+2q}} + \frac{C_{q,M,\gamma}}{T^{p+2q+1}}.$$

This last estimate implies the existence of a constant $C_j > 0$ such that

$$G_{R,j}(T) \leq \frac{R}{T^2} e^{R/T} \left[\frac{\lambda_1^p e^{R^2/(8T)}}{|\langle B\phi_j, \phi_1 \rangle|^2} + \frac{e^{R^2/4T}}{b^2} \left(\frac{C_{q,M}}{T^{p+2q}} + \frac{C_{q,M,\gamma}}{T^{p+2q+1}} \right) \right] \leq e^{2C_j/T}, \quad \forall 0 < T \leq 1.$$

□

Proposition 3.5. *Let $-\Delta$ be a Dirichlet-Neumann Laplacian on X such that its eigenvalues satisfy (6). Let $B : X \rightarrow X$ be a linear bounded operator on \mathcal{G} such that (7) holds.*

Then, the pair $\{-\Delta, B\}$ is j -null controllable in any positive time $T > 0$.

Proof. Our aim is to show that, for all $T > 0$, there exists a constant $K(T) > 0$ such that for any initial condition $z_0 \in X$ there exists a control $u \in L^2(0, T)$ which steers the solution z of the linear system (12) (with $A = -\Delta$) to 0 in a time T , that is, (12) is null controllable. To this purpose, we rewrite the null controllability condition by using the expansion in Fourier series of the solution z of (12):

$$0 = z(T; z_0, u) = \sum_{k \in \mathbb{N}^*} e^{-\lambda_k T} \langle z_0, \phi_k \rangle \phi_k - \int_0^T u(s) \sum_{k \in \mathbb{N}^*} e^{-\lambda_k(T-s)} \langle B\phi_j, \phi_k \rangle \phi_k ds.$$

Thanks to the orthonormality of the family of eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}^*}$, we obtain that the following relations have to be satisfied by u

$$\langle z_0, \phi_k \rangle = \int_0^T e^{\lambda_k s} u(s) \langle B\phi_j, \phi_k \rangle ds, \quad \forall k \in \mathbb{N}^*,$$

which can be rewritten in compact form as

$$(25) \quad \int_0^T e^{\lambda_k s} u(s) ds = d_k, \quad \forall k \in \mathbb{N}^*,$$

where the coefficients $d_k := \frac{\langle z_0, \phi_k \rangle}{\langle B\phi_j, \phi_k \rangle}$, $\forall k \in \mathbb{N}^*$, are well defined thanks to hypothesis (7).

Since the eigenvalues of the Laplacian on \mathcal{G} with (\mathcal{D}) , (\mathcal{N}) or (\mathcal{NK}) satisfy the hypotheses of Proposition 3.3, we use the biorthogonal family $\{\sigma_k\}_{k \in \mathbb{N}}$ to build our control, so that u satisfies (25):

$$u(t) := \sum_{k \in \mathbb{N}} d_k \sigma_k(t).$$

To conclude the proof, it remains to show that $u \in L^2(0, T)$. To this purpose, we use the bound (17) for the biorthogonal family $\{\sigma_k\}_{k \in \mathbb{N}}$:

$$\begin{aligned} \|u\|_{L^2(0,T)} &= \sum_{k \in \mathbb{N}^*} \left| \frac{\langle z_0, \phi_k \rangle}{\langle B\phi_j, \phi_k \rangle} \right| \|\sigma_k\|_{L^2(0,T)} \leq \|z_0\| \left(\sum_{k \in \mathbb{N}^*} \frac{\|\sigma_k\|_{L^2(0,T)}^2}{|\langle B\phi_j, \phi_k \rangle|^2} \right)^{1/2} \\ &\leq \|z_0\| \left(C e^{C/(T\gamma^2)} B(T, \gamma) \sum_{k \in \mathbb{N}^*} \left(1 + \frac{\gamma^2}{a_k(a_k + 2\sqrt{\lambda_1})} \right)^{2M} \frac{e^{-2\lambda_k T} e^{C\sqrt{\lambda_k}/\gamma}}{|\langle B\phi_j, \phi_k \rangle|^2} \right)^{1/2}. \end{aligned}$$

We recall that $\lambda_k \asymp k^2$ and $a_k \asymp k^{-p}$ for $k \geq 2$. Thus, the right-hand side of the above inequality can be bounded by

$$\|z_0\| \left(C e^{C/(T\gamma^2)} B(T, \gamma) \sum_{k \in \mathbb{N}^*} \lambda_k^{2pM} \frac{e^{-2\lambda_k T + C\sqrt{\lambda_k}/\gamma}}{|\langle B\phi_j, \phi_k \rangle|^2} \right)^{1/2}.$$

From Proposition 3.4, we know that the series in the formula above is convergent for any $T > 0$. Hence $u \in L^2(0, T)$ and we have shown that there exists a constant $K(T) > 0$ such that

$$\|u\|_{L^2(0, T)} \leq K(T) \|z_0\|.$$

□

Now we can proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. From Proposition 3.5 we deduce that the pair $\{-\Delta, B\}$ is j -null controllable in any time $T > 0$. What remains to prove in order to use Theorem 3.2 is to show that there exist $T_0, \nu > 0$ such that

$$K(T) = \left(C e^{C/(T\gamma^2)} B(T, \gamma) \sum_{k \in \mathbb{N}^*} \lambda_k^{2pM} \frac{e^{-2\lambda_k T} e^{C\sqrt{\lambda_k}/\gamma}}{|\langle B\phi_j, \phi_k \rangle|^2} \right)^{1/2}$$

satisfies (13).

One can check that there exists a constant $R > 0$ such that

$$K(T) \leq \left(C e^{C/(T\gamma^2)} \frac{1}{T\gamma^2} \sum_{k \in \mathbb{N}^*} \lambda_k^{2pM} \frac{e^{-2\lambda_k T + C\sqrt{\lambda_k}/\gamma}}{|\langle B\phi_j, \phi_k \rangle|^2} \right)^{1/2} \leq \|z_0\| G_{R,j}(T)^{1/2}, \quad \forall T \leq \min \left\{ 1, \frac{1}{\gamma^2} \right\},$$

where the function $G_{R,j}(\cdot)$ is defined in (18). Thus, from estimate (19), we infer that

$$K(T) \leq e^{C_j/T}, \quad \forall 0 < T \leq T_0,$$

with $T_0 := \min \left\{ 1, \frac{1}{\gamma^2} \right\}$ and C_j a suitable positive constant independent of T .

Therefore, we can apply Theorem 3.2 and conclude that (15) is locally controllable to the j -th eigensolutions in any time $T > 0$ by means of a control $u \in L^2(0, T)$ which satisfies (14). □

3.3 Semi-global controllability results

We have proved in Proposition 3.5 that if the eigenvalues of the Dirichlet-Neumann Laplacian satisfy (6) and the bounded linear operator B fulfills the spreading property (7), then the pair $\{-\Delta, B\}$ is j -null controllable. Furthermore, we have shown in the proof of Theorem 1.1 that, under the same assumptions on the operators $-\Delta$ and B , the control cost behaves as (13).

Thus, it is possible to apply [2, Theorem 1.4] and [2, Theorem 1.5] and deduce the following semi-global controllability results.

Theorem 3.6. *Let $-\Delta$ be a Dirichlet-Neumann Laplacian on $X = L^2(\mathcal{G}, \mathbb{R})$ such that the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*}$ satisfy (6). Let $B : X \rightarrow X$ be a bounded linear operator on \mathcal{G} such that (7) holds.*

Then, there exists $r_1 > 0$ such that for every $R > 0$ there exists $T_R > 0$ such that for all $y_0 \in X$ with

$$(26) \quad |\langle y_0, \phi_1 \rangle - 1| < r_1, \quad \|y_0 - \langle y_0, \phi_1 \rangle \phi_1\| \leq R,$$

problem (15) is controllable to the first eigensolution φ_1 in time T_R .

The interpretation of Theorem 3.6 is the following: if the initial datum lies on a (infinite) strip, defined in (26), then we can steer the solution of (15) to the first eigensolution in time T_R (see (a) in Figure 2 below). The idea to prove this semi-global result is to let the semigroup of the Laplacian evolve freely till the solution of (15) enters a suitable neighborhood of the first eigenfunction, and then to apply our local controllability result (see [2, Theorem 1.4] for the proof).

Our second semi-global result below, Theorem 3.7, ensures the controllability in time T_R of any initial condition $y_0 \in X \setminus \phi_1^\perp$ to the evolution of its orthogonal projection along the first eigensolution $\varphi_1(t)$, that is the following trajectory

$$\zeta_1(t) = \langle y_0, \phi_1 \rangle \varphi_1(t).$$

Theorem 3.7. *Let $-\Delta$ be a Dirichlet-Neumann Laplacian on $X = L^2(\mathcal{G}, \mathbb{R})$ such that the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*}$ satisfy (6). Let $B : X \rightarrow X$ be a bounded linear operator on \mathcal{G} such that (7) holds.*

Then, for any $R > 0$ there exists $T_R > 0$ such that for all $y_0 \in X$ with

$$\|y_0 - \langle y_0, \phi_1 \rangle \phi_1\| \leq R |\langle y_0, \phi_1 \rangle|$$

problem (15) is controllable to the trajectory

$$\zeta_1(t) = \langle y_0, \phi_1 \rangle \varphi_1(t)$$

in time T_R .

For the proof, see [2, Theorem 1.5].

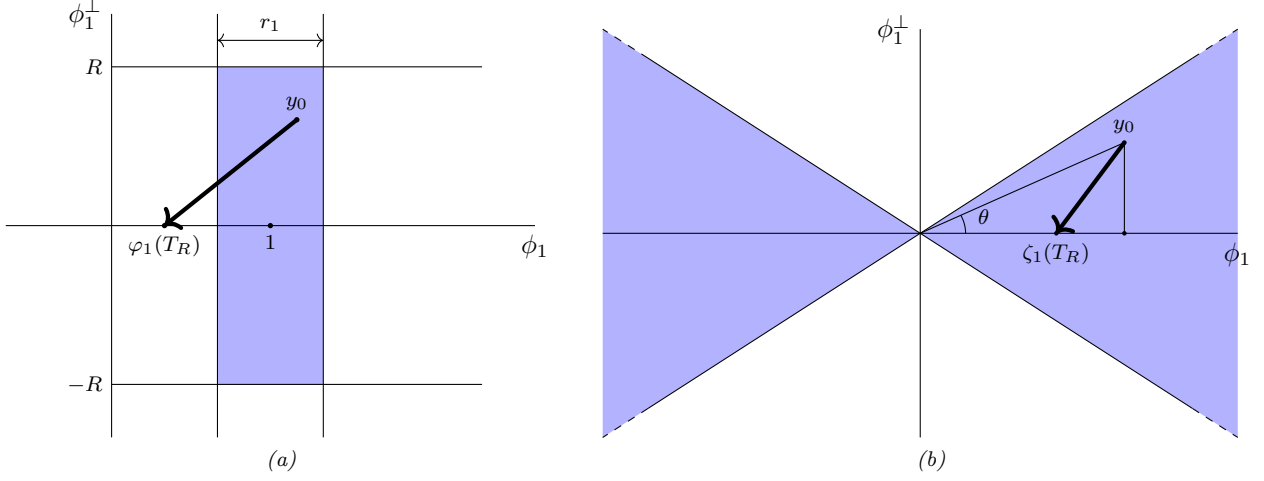


Figure 2: Figure (a) illustrates the result of Theorem 3.6: the solution of problem (15) with initial condition lying in the blue region can be driven to the first eigensolution φ_1 in time T_R (which is uniform for any y_0 in the strip). In figure (b) we highlighted the cone of amplitude $2 \arctan(R)$ of initial conditions which can be steered to the trajectory ζ_1 in time T_R . Since R is arbitrary, we are able to apply Theorem 3.7 for any $y_0 \in X \setminus \phi_1^\perp$.

4 Some explicit applications

4.1 Star graph

Let us consider a star-graph graph \mathcal{G} composed by 3 edges $\{e_1, e_2, e_3\}$ connected in one vertex v . We parametrize each e_j with a coordinate going from 0 to its length L_j in v (see Figure 3).

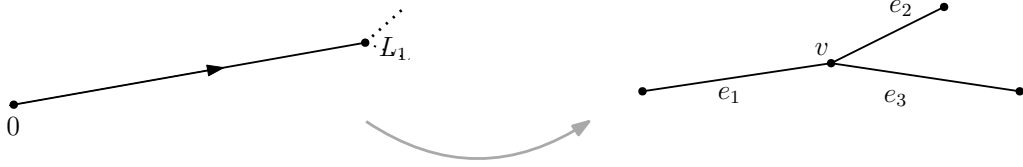


Figure 3: The figure shows the parametrization of a star graph with 3 edges.

In such a framework, $\psi \in L^2(\mathcal{G}, \mathbb{R})$ defined on \mathcal{G} is composed by three components (ψ^1, ψ^2, ψ^3) where each $\psi^j \in L^2(e_j, \mathbb{R})$. We assume that $-\Delta$ is a Neumann Laplacian and

$$(27) \quad u \in L^2((0, T), \mathbb{R}), \quad B : \psi = (\psi^1, \psi^2, \psi^3) \in L^2(\mathcal{G}, \mathbb{R}) \mapsto (\mu(x)\psi^1, 0, 0).$$

where μ is a sufficiently regular function. The bilinear system (5) becomes the problem

$$(28) \quad \begin{cases} \partial_t \psi^1(t, x) - \partial_x^2 \psi^1(t, x) + u(t)\mu(x)\psi^1(t, x) = 0, & t \in (0, T), x \in (0, L_1), \\ \partial_t \psi^2(t, x) - \partial_x^2 \psi^2(t, x) = 0, & t \in (0, T), x \in (0, L_2), \\ \partial_t \psi^3(t, x) - \partial_x^2 \psi^3(t, x) = 0, & t \in (0, T), x \in (0, L_3), \end{cases}$$

endowed with the following boundary conditions

$$(29) \quad \begin{aligned} \psi^1(L_1) = \psi^2(L_2) = \psi^3(L_3), \quad \partial_x \psi^1(L_1) + \partial_x \psi^2(L_2) + \partial_x \psi^3(L_3) = 0, \\ \partial_x \psi^1(0) = \partial_x \psi^2(0) = \partial_x \psi^3(0) = 0. \end{aligned}$$

We notice that the first eigenvalue $\lambda_1 = 0$ corresponds to the following eigenfunction

$$\phi_1 = \frac{1}{\sqrt{L_1 + L_2 + L_3}}(1, 1, 1).$$

The Neumann boundary conditions on the external vertices of \mathcal{G} imply that each other ϕ_j , with $j \geq 2$, satisfies $\phi_j^l(0) = 0$ for every $l \leq 3$ which yields

$$(30) \quad \phi_j(x) = (a_j^1 \cos(x\sqrt{\lambda_j}), a_j^2 \cos(x\sqrt{\lambda_j}), a_j^3 \cos(x\sqrt{\lambda_j})),$$

with $\{a_j^l\}_{l \leq 3} \subset \mathbb{R}$. Now, the (\mathcal{NK}) conditions in V_i lead to

$$(31) \quad a_j^1 \cos(\sqrt{\lambda_j} L_1) = \dots = a_j^3 \cos(\sqrt{\lambda_j} L_3), \quad \sum_{l \leq 3} a_j^l \sin(\sqrt{\lambda_j} L_l) = 0,$$

When, for instance, all the lengths of the edges of \mathcal{G} are equal to L , it is easy to see that the sequence of non-repeated eigenvalues is obtained by reordering

$$(32) \quad \left\{ \frac{(2j-1)^2 \pi^2}{4L^2} \right\}_{j \in \mathbb{N}^*}, \quad \left\{ \frac{j^2 \pi^2}{L^2} \right\}_{j \in \mathbb{N}},$$

which is $\left\{ \frac{j^2 \pi^2}{4L^2} \right\}_{j \in \mathbb{N}}$. Notice that the eigenvalues $\left\{ \frac{(2j-1)^2 \pi^2}{4L^2} \right\}_{j \in \mathbb{N}^*}$ have multiplicity $N-1$ and then they are doubles in this specific case. When instead the lengths are such that each $L_j/L_k \notin \mathbb{Q}$, the relations (31) yield the identities

$$(33) \quad \sum_{l \leq 3} \tan(\sqrt{\lambda_j} L_l) = 0, \quad \sum_{l \leq 3} |a_j^l|^2 \sin(L_l \sqrt{\lambda_j}) \cos(L_l \sqrt{\lambda_j}) = 0.$$

As a consequence of (33), the numbers $\sqrt{\lambda_j}$ with $j \geq 2$ are solutions of the transcendental equation

$$\sum_{l \leq 3} \tan(x L_l) = 0.$$

The corresponding eigenfunctions are deduced by (30) with $\{a_j^l\}_{l \leq 3} \subset \mathbb{R}$. We choose these coefficients so that $\{\phi_j\}_{j \in \mathbb{N}^*}$ forms a Hilbert basis of $L^2(\mathcal{G}, \mathbb{R})$. The orthonormality condition yields

$$(34) \quad 1 = \sum_{l \leq 3} \int_0^{L_l} (a_j^l)^2 \cos^2(x \sqrt{\lambda_j}) dx = \sum_{l \leq 3} (a_j^l)^2 \left(\frac{L_l}{2} + \frac{\cos(L_l \sqrt{\lambda_j}) \sin(L_l \sqrt{\lambda_j})}{2\sqrt{\lambda_j}} \right).$$

Now, we use (33) in (34) which ensures $1 = \sum_{l=1}^3 (a_j^l)^2 L_l / 2$. The continuity relation in the Neumann-Kichhoff conditions in V_i implies $a_j^l = a_j^k \frac{\cos(\sqrt{\lambda_j} L_k)}{\cos(\sqrt{\lambda_j} L_l)}$ for $l \neq k$ and then

$$(35) \quad (a_j^k)^2 \left(L_k + \sum_{l \in \{1,2,3\} \setminus \{k\}} L_l \frac{\cos^2(\sqrt{\lambda_j} L_k)}{\cos^2(\sqrt{\lambda_j} L_l)} \right) = 2.$$

The relation (35) allows us to identify the parameters $\{a_j^l\}_{l \leq 3} \subset \mathbb{R}$ up to their signs as

$$(36) \quad |a_j^k| = \sqrt{\frac{2 \prod_{m \neq k} \cos^2(\sqrt{\lambda_j} L_m)}{\sum_{l=1}^3 L_l \prod_{m \neq l} \cos^2(\sqrt{\lambda_j} L_m)}}.$$

Theorem 4.1. *Let $\mu(x) = \cos\left(\frac{x\pi}{2L_1}\right)$ with $x \in (0, L_1)$. If $\{L_j\}_{j \leq 3} \in (\mathbb{R}^+)^N$ is so that $\{1, L_1, \dots, L_3\}$ are linearly independent over \mathbb{Q} and all the ratios L_k/L_j are algebraic irrational numbers, then system (28) is locally controllable to the first eigensolution in any time $T > 0$. Moreover, the control u satisfies (14).*

Proof. The result is ensured by referring to Theorem 1.1. First, the validity of the spectral gap follows by Lemma 2.2 and Proposition 2.4. Second, by direct computations it is possible to deduce that

$$\langle B\phi_1, \phi_1 \rangle = \langle \mu\phi_1^1, \phi_1^1 \rangle_{L^2(0, L_1)} = \frac{2L_1}{\pi(L_1 + L_2 + L_3)} \neq 0$$

and, for every $k \geq 2$,

$$(37) \quad \langle B\phi_1, \phi_k \rangle = \langle \mu\phi_1^1, \phi_k^1 \rangle_{L^2(0, L_1)} = a_k^1 \frac{2L_1 \pi \cos(\sqrt{\lambda_k} L_1)}{\pi^2 - 4L_1^2 \lambda_k}.$$

Thanks to [24, Proposition 27] (arXiv version [24, Proposition A.2]), for every $\epsilon > 0$, there exists $C_1 > 0$ such that $|\cos(\sqrt{\lambda_k} L_l)| \geq C_1 \lambda_k^{\frac{-1-\epsilon}{2}}$ for every $l \leq 3$. This identity and (36) ensure the existence of $C_2, C_3 > 0$ such that

$$(38) \quad |a_k^1| \geq \sqrt{\frac{2}{\sum_{l=1}^3 L_l \cos^{-2}(\sqrt{\lambda_k} L_l)}} \geq \sqrt{\frac{2}{C_2 \lambda_k^{1+\epsilon} \sum_{l=1}^3 L_l}} \geq \frac{C_3}{\lambda_k^{\frac{1+\epsilon}{2}}}, \quad \forall k \geq 2.$$

We use again $|\cos(\sqrt{\lambda_k} L_1)| \geq C_1 \lambda_k^{\frac{-1-\epsilon}{2}}$ in (37). Thanks to (38), there exists $C_4 > 0$ such that

$$|\langle B\phi_1, \phi_k \rangle| \geq \frac{C_4}{\lambda_k^{\frac{1+\epsilon}{2}}}, \quad \forall k \geq 2.$$

The result finally follows from Theorem 1.1. □

4.2 Tadpole graph

Let \mathcal{G} be a tadpole graph composed by two edges $\{e_1, e_2\}$ connected in an internal vertex v . The edge e_1 is self-closing and parametrized in the clockwise direction with a coordinate going from 0 to L_1 (the length e_1). On the ‘‘tail’’ e_2 , we consider a coordinate going from 0 in the to L_2 and we associate the 0 to the external vertex \tilde{v} (see Figure 4).

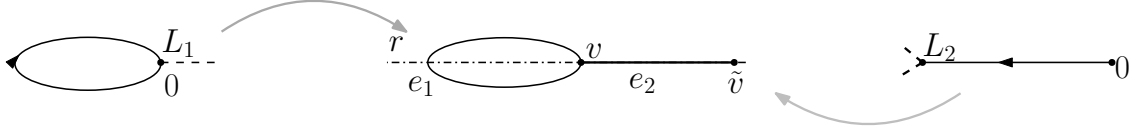


Figure 4: The parametrization of the tadpole graph and its symmetry axis r .

We consider the Hilbert space $L^2(\mathcal{G}, \mathbb{R})$ composed by functions $\psi = (\psi^1, \psi^2)$ such that each $\psi^l \in L^2(e_l, \mathbb{R})$. We assume that $-\Delta$ is a Neumann Laplacian and

$$(39) \quad u \in L^2((0, T), \mathbb{R}), \quad B : \psi = (\psi^1, \psi^2) \in L^2(\mathcal{G}, \mathbb{R}) \mapsto (\mu(x)\psi^1, 0),$$

where μ is a sufficiently regular function. The bilinear system (5) reads as

$$(40) \quad \begin{cases} \partial_t \psi^1(t, x) - \partial_x^2 \psi^1(t, x) + u(t)\mu(x)\psi^1(t, x) = 0, & t \in (0, T), x \in (0, L_1), \\ \partial_t \psi^2(t, x) - \partial_x^2 \psi^2(t, x) = 0, & t \in (0, T), x \in (0, L_2), \end{cases}$$

endowed with the following boundary conditions

$$(41) \quad \psi^1(L_1) = \psi^1(0) = \psi^2(L_2), \quad \partial_x \psi^1(L_1) - \partial_x \psi^1(0) + \partial_x \psi^2(L_2) = 0, \quad \partial_x \psi^2(0) = 0.$$

Now, we refer to the proof of [24, Theorem 15] (arXiv version [24, Theorem 3.7]) and we notice that the sequence of eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}^*}$ of $-\Delta$ can be assumed to be only composed by symmetric or skew-symmetric functions with respect to natural symmetry axis r appearing in \mathcal{G} (see Figure 4). If $\phi_k = (\phi_k^1, \phi_k^2)$ is skew-symmetric, then

$$\phi_k^2 \equiv 0, \quad \phi_k^1(0) = \phi_k^1(L_1/2) = \phi_k^1(L_1) = 0, \quad \partial_x \phi_k^1(0) = \partial_x \phi_k^1(L_1).$$

As a consequence, the eigenvalues corresponding to the skew-symmetric eigenfunctions are $\left\{ \frac{4k^2\pi^2}{L_1^2} \right\}_{k \in \mathbb{N}^*}$ and such eigenfunctions are

$$\left\{ \left(\sqrt{\frac{2}{L_1}} \sin\left(x \frac{2k\pi}{L_1}\right), 0 \right) \right\}_{k \in \mathbb{N}^*}.$$

Afterwards, if $\phi_k = (\phi_k^1, \phi_k^2)$ is symmetric, then we have

$$\partial_x \phi_k^1(L_1/2) = 0, \quad \phi_k^1(\cdot) = \phi_k^1(L_1 - \cdot).$$

Firstly, we notice that the first eigenvalue $\lambda_1 = 0$ corresponds to a symmetric eigenfunction

$$\phi_1 = \frac{1}{\sqrt{L_1 + L_2}}(1, 1).$$

Secondly, the (\mathcal{N}) condition on \tilde{v} implies that each other symmetric eigenfunction associated to the eigenvalue λ_k have the following formulation

$$\left\{ \left(a_k^1 \cos\left(\sqrt{\lambda_k}\left(x - \frac{L_1}{2}\right)\right), a_k^2 \cos(\sqrt{\lambda_k}x) \right) \right\}_{k \in \mathbb{N}^*}, \quad \text{for } \{(a_k^1, a_k^2)\}_{k \in \mathbb{N}^*} \subset \mathbb{R}^2.$$

The (\mathcal{NK}) conditions in v ensure that $a_k^1 \cos(\sqrt{\lambda_k}(L_1/2)) = a_k^2 \cos(\sqrt{\lambda_k}L_2)$ and

$$2a_k^1 \sin(\sqrt{\lambda_k}(L_1/2)) + a_k^2 \sin(\sqrt{\lambda_k}L_2) = 0.$$

Similarly to what we noticed for the star graph, such eigenvalues can be made explicit when $L_1 = L_2$. While if $L_1/L_2 \notin \mathbb{Q}$, then $\sqrt{\lambda_k}$ is a solution of the transcendental equation

$$(42) \quad 2 \tan(x(L_1/2)) + \tan(xL_2) = 0.$$

The boundary conditions also yield the following identities, for every $k \geq 2$,

$$(43) \quad (a_k^1)^2 \sin(L_1 \sqrt{\lambda_k}) + (a_k^2)^2 \sin(L_2 \sqrt{\lambda_k}) \cos(L_2 \sqrt{\lambda_k}) = 0.$$

As for the star graph, we choose the coefficients $\{a_k^l\}_{l \leq 2} \subset \mathbb{R}$ so that $\{\phi_k\}_{k \in \mathbb{N}^*}$ forms a Hilbert basis of $L^2(\mathcal{G}, \mathbb{R})$. The orthonormality condition yields that

$$(44) \quad 1 = (a_k^1)^2 \left(\frac{L_1}{2} + \frac{\sin(L_1 \sqrt{\lambda_k})}{2\sqrt{\lambda_k}} \right) + (a_k^2)^2 \left(\frac{L_2}{2} + \frac{\sin(L_2 \sqrt{\lambda_k}) \cos(L_2 \sqrt{\lambda_k})}{2\sqrt{\lambda_k}} \right).$$

Now, we use (43) in (44) which ensures $2 = (a_k^1)^2 L_1 + (a_k^2)^2 L_2$. As for the star graph, the continuity relation in the Neumann-Kichhoff conditions in v allows us to identify the parameters $\{a_k^l\}_{l \leq 2} \subset \mathbb{R}$, up to their sign, as

$$(45) \quad |a_k^1| = \sqrt{\frac{2 \cos^2(\sqrt{\lambda_k} L_2)}{L_1 \cos^2(\sqrt{\lambda_k} L_m) + L_2 \cos^2(\sqrt{\lambda_k} L_1/2)}}, \quad |a_k^1| = \sqrt{\frac{2 \cos^2(\sqrt{\lambda_k} L_1/2)}{L_1 \cos^2(\sqrt{\lambda_k} L_m) + L_2 \cos^2(\sqrt{\lambda_k} L_1/2)}}.$$

Theorem 4.2. *Let $\mu(x) = x$ with $x \in (0, L_1)$. If $\{L_1, L_2\} \in (\mathbb{R}^+)^2$ is so that $\{1, L_1, L_2\}$ are linearly independent over \mathbb{Q} and the ratio L_1/L_2 is an algebraic irrational number, then system (40) is locally controllable to the first eigensolution in any positive time $T > 0$. Moreover, the control u satisfies (14).*

Proof. The proof follows the one of Theorem 4.1. First, the spectral gap is valid thanks to Lemma 2.2 and Proposition 2.4. Second, we can compute

$$\langle B\phi_1, \phi_1 \rangle = \langle \mu\phi_1^1, \phi_1^1 \rangle_{L^2(0, L_1)} = \frac{L_1}{2(L_1 + L_2)} \neq 0.$$

We consider a symmetric eigenfunction ϕ_k and

$$(46) \quad \langle B\phi_1, \phi_k \rangle = \langle \mu\phi_1^1, \phi_k^1 \rangle_{L^2(0, L_1)} = a_k^1 \frac{L_1 \sin(\sqrt{\lambda_k} L_1/2)}{\sqrt{\lambda_k}}.$$

Now, from (42), we have the following relation

$$2 \cos(\sqrt{\lambda_k} L_2) \sin(\sqrt{\lambda_k} L_1/2) + \cos(\sqrt{\lambda_k} L_1/2) \sin(\sqrt{\lambda_k} L_2) = 0.$$

We consider [24, Remark 28] (arXiv version [24, Proposition A.3]) with $\{L_1/2, L_2\}$ and the previous relation yields that, for every $\epsilon > 0$, there exists $C_1 > 0$ such that

$$(47) \quad \begin{aligned} |\sin(\sqrt{\lambda_k} L_1/2)| &\geq \frac{C_1}{\sqrt{\lambda_k}^{1+\epsilon}}, & |\sin(\sqrt{\lambda_k} L_2)| &\geq \frac{C_1}{\sqrt{\lambda_k}^{1+\epsilon}}, \\ |\cos(\sqrt{\lambda_k} L_1/2)| &\geq \frac{C_1}{\sqrt{\lambda_k}^{1+\epsilon}}, & |\cos(\sqrt{\lambda_k} L_2)| &\geq \frac{C_1}{\sqrt{\lambda_k}^{1+\epsilon}}. \end{aligned}$$

If we use the second line of (47) in (45), then we deduce the existence of $C_2 > 0$ such that

$$|a_k^1| \geq \frac{C_2}{\sqrt{\lambda_k}^{1+\epsilon}}.$$

We use now the first line of (47) in (46). The previous relation implies that, for every $\epsilon > 0$, there exists $C_3 > 0$ such that,

$$|\langle B\phi_1, \phi_k \rangle| \geq \frac{C_3}{\sqrt{\lambda_k}^{3+2\epsilon}}.$$

When we consider ϕ_k as a skew-symmetric eigenfunction, we have

$$|\langle B\phi_1, \phi_k \rangle| \geq \frac{L_1}{j\pi}.$$

The result finally follows from Theorem 1.1. □

4.3 Space filtering in presence of multiple eigenvalues

In this section, we discuss how to deal with systems with multiple eigenvalues for which the spectral gap (6) is not satisfied. Examples of this situation are the star graphs and the tadpoles with some edges of same length. For instance, in the specific case of a star graph with 3 edges of equal length, the spectrum is given by (32), where multiples eigenvalues appear.

We apply the technique of filtration of the Hilbert space $L^2(\mathcal{G}, \mathbb{R})$ in order to obtain simple spectrum. The main idea is to investigate the existence of a subspace \mathcal{H} , preserved by the dynamics of (5), where the spectrum of the Laplacian turns out to be simple. In such case, controllability can be studied in \mathcal{H} under suitable assumptions on B . This idea was already adopted in [25, Section 6] and [5, 6] for the bilinear controllability of the Schrödinger equation. The application to our framework is straightforward and, here, we just present the main steps in an explicit example.

Let \mathcal{G} be a star graph with four edges parametrized as in Section 4.1. We assume $L_1 = L_2 = L_3 = 1$ and $L_4 = L > 0$. We consider a Dirichlet Laplacian $-\Delta$ on \mathcal{G} and its spectrum contains the eigenvalues

$$\tilde{\lambda}_k = k^2 \pi^2, \quad \forall k \in \mathbb{N}^*.$$

We notice that $\{\tilde{\lambda}_k\}_{k \in \mathbb{N}^*}$ are multiple eigenvalues (if $L \in \mathbb{R} \setminus \mathbb{Q}$, then they are actually double eigenvalues). Two families of corresponding orthonormalized eigenfunctions are $\{f_k\}_{k \in \mathbb{N}^*}$ and $\{g_k\}_{k \in \mathbb{N}^*}$ such that

$$f_k = (\sin(k\pi x), -\sin(k\pi x), 0, 0), \quad \forall k \in \mathbb{N}^*,$$

$$g_k = \left(\sqrt{\frac{1}{2}} \sin(k\pi x), \sqrt{\frac{1}{2}} \sin(k\pi x), -\sin(k\pi x), 0 \right), \quad \forall k \in \mathbb{N}^*.$$

Now, we introduce the space $\mathcal{H} = \overline{\text{span}\{f_k : k \in \mathbb{N}^*\}}^{L^2}$ and we notice that

$$\mathcal{H} = \{(\tilde{\psi}, -\tilde{\psi}, 0, 0) : \tilde{\psi} \in L^2((0, 1), \mathbb{R})\}.$$

The spectrum of the Dirichlet Laplacian $-\Delta$ in \mathcal{H} is composed by the eigenvalues $\{\tilde{\lambda}_k\}_{k \in \mathbb{N}^*}$ which are now simple. Thus, Theorem 1.1 can be applied in this space \mathcal{H} for a suitable operator B defined as follows.

- We consider B so that, for any initial data in \mathcal{H} , the corresponding solution of (5) remains in \mathcal{H} for any time. A possible choice of B is

$$B : \psi = (\psi^1, \psi^2, \psi^3, \psi^4) \in L^2(\mathcal{G}, \mathbb{R}) \longmapsto B\psi = (\mu\psi^1, \mu\psi^2, \mu_1\psi^3, \mu_2\psi^4),$$

where μ, μ_1 and μ_2 are bounded functions. Indeed,

$$\text{for any } \psi = (\tilde{\psi}, -\tilde{\psi}, 0, 0) \in \mathcal{H}, \text{ we have } B\psi = (\mu\tilde{\psi}, -\mu\tilde{\psi}, 0, 0) \in \mathcal{H}.$$

- We choose B so that, conditions (7) are verified with respect to $\{f_k\}_{k \in \mathbb{N}^*}$. In our case, one can identify \mathcal{H} with $L^2(0, 1)$ and B with the multiplication operator by a function μ in $L^2(0, 1)$. This is possible because each element in \mathcal{H} is uniquely determined by its first component which belongs to $L^2(e_1) \equiv L^2(0, 1)$. So, our problem boils down to the controllability of a one-dimensional control system in the interval $(0, 1)$ (see the examples in [2, 3, 17]). A possible choice of the function μ is $\mu(x) = x^2$ as explained in [2, Section 6.1]. Finally, the operator B defined by

$$B\psi = (x^2\psi^1, x^2\psi^2, \mu_1\psi^3, \mu_2\psi^4),$$

with μ_1, μ_2 any bounded functions, allows to locally control our original problem in \mathcal{H} to any eigensolution

$$(e^{-j^2\pi^2 t} \sin(j\pi x), -e^{-j^2\pi^2 t} \sin(j\pi x), 0, 0).$$

This strategy is not only valid on compact graphs in presence multiple eigenvalues, but also on infinite graphs. In such a framework, it is possible to construct eigenfunctions for the Laplacian in presence of suitable substructures of the graph. It is true even though there are edges of infinite length and the Laplacian has not compact resolvent. In the span of these eigenfunctions, one can study the controllability of the system as explained above. For further details, we refer to [5, 6] where this idea is applied for the Schrödinger equation.

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A Appendix: Proof of Proposition 2.4

In the current appendix, we use the techniques developed in [25, Section 3] in order to prove the validity of the spectral gap of Proposition 2.4.

As in [25, Section 3], we start by considering an ordered sequence of pairwise distinct numbers $\mathbf{v} = (\nu_k)_{k \in \mathbb{Z}^*} \subset \mathbb{R}$ such that there exist $M \in \mathbb{N}^*$ and $\delta > 0$ such that

$$(48) \quad \inf_{\{k \in \mathbb{Z}^* : k+M \neq 0\}} |\nu_{k+M} - \nu_k| \geq \delta M.$$

This property implies that it does not exist M consecutive indices $k, k+1 \in \mathbb{Z}^*$ such that $|\nu_{k+1} - \nu_k| < \delta$ and then, there exist some $j \in \mathbb{Z}^* \setminus \{-1\}$ such that $|\nu_{j+1} - \nu_j| \geq \delta$. This distribution of the numbers \mathbf{v} allows us to define a partition of \mathbb{Z}^* in subsets $\{E_m\}_{m \in \mathbb{Z}^*}$ (see [25, Section 3.3] for further details on this partition). The partition of \mathbb{Z}^* in subsets $\{E_m\}_{m \in \mathbb{Z}^*}$ also defines an equivalence relation in \mathbb{Z}^* and $\{E_m\}_{m \in \mathbb{Z}^*}$ are the equivalence classes corresponding to such relation. Clearly, $|E_m| \leq M$ thanks to (48). Let $s(m)$ be the smallest element of E_m . For every $\mathbf{x} := \{x_k\}_{k \in \mathbb{Z}^*} \subset \mathbb{R}$ and $m \in \mathbb{Z}^*$, we define \mathbf{x}^m as the vector in $\mathbb{R}^{|E_m|}$ composed by those elements of \mathbf{x} with indices in E_m , *i.e.*

$$\mathbf{x}^m := \{x_l^m\}_{l \in |E_m|}, \quad : \quad x_l^m = x_{s(m)+(l-1)}, \quad \forall l \leq |E_m|.$$

We are finally ready to introduce the matrix $F_m(\mathbf{v}^m) : \mathbb{R}^{|E_m|} \rightarrow \mathbb{R}^{|E_m|}$ with components

$$F_{m;j,k}(\mathbf{v}^m) := \begin{cases} \prod_{\substack{l \neq j \\ l \leq k}} (\nu_j^m - \nu_l^m)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k, \end{cases} \quad \forall j, k \leq |E_m|.$$

For each $k \in \mathbb{Z}^*$, there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$, while $s(m(k))$ represents the smallest element of $E_{m(k)}$. In the following Lemma, we state [25, Lemma 3.16] which characterizes the matrix $F_{m;j,k}(\mathbf{v}^m)$.

Lemma A.1. [25, Lemma 3.16] *Let $\mathbf{v} := (\nu_k)_{k \in \mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying (48). Let G be an entire function such that $G \in L^\infty(\mathbb{R}, \mathbb{R})$. Let $J, I > 0$ be such that*

$$|G(z)| \leq J e^{I|z|}, \quad \forall z \in \mathbb{C}.$$

If $(\nu_k)_{k \in \mathbb{Z}^}$ are simple zeros of G such that there exist $\tilde{d} \geq 0, C > 0$ such that*

$$|G'(\nu_k)| \geq \frac{C}{|k|^{1+\tilde{d}}}, \quad \forall k \in \mathbb{Z}^*,$$

then there exists $C > 0$ so that

$$\text{Tr}\left(F_m(\mathbf{v}^m)^* F_m(\mathbf{v}^m)\right) \leq C \min\{|l| \in E_m\}^{2(1+\tilde{d})}, \quad \forall m \in \mathbb{Z}^*.$$

As a consequence of the previous Lemma, we have the following one.

Lemma A.2. *Let $\mathbf{v} := (\nu_k)_{k \in \mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying (48). Let G be an entire function such that $G \in L^\infty(\mathbb{R}, \mathbb{R})$. Let $J, I > 0$ be such that*

$$|G(z)| \leq J e^{I|z|}, \quad \forall z \in \mathbb{C}.$$

If $(\nu_k)_{k \in \mathbb{Z}^}$ are simple zeros of G such that there exist $\tilde{d} \geq 0, C > 0$ such that*

$$|G'(\nu_k)| \geq \frac{C}{|k|^{1+\tilde{d}}}, \quad \forall k \in \mathbb{Z}^*,$$

then, for every $\epsilon > 0$, there exists $C > 0$ such that

$$|\nu_{k+1} - \nu_k| \geq C |k|^{-1-\epsilon}, \quad \forall k \in \mathbb{Z}^* \setminus \{-1\}$$

Proof. As introduced before, from (48), it does not exist M consecutive $k, k+1 \in \mathbb{Z}^*$ such that $|\nu_{k+1} - \nu_k| < \delta$. It is clear that if $k \in E_{m(k)}$ and $k+1 \in E_{m(k)+1}$, then $E_{m(k+1)} = E_{m(k)+1}$ and $|\nu_{k+1} - \nu_k| > \delta$. Let us consider now the case such that $k, k+1 \in E_{m(k)}$ which yields $E_{m(k)} = E_{m(k+1)}$. First, we notice that, for every $l, n \in E_{m(k)}$, we have

$$(49) \quad |\nu_l - \nu_n| < g := (M-1)\delta,$$

where g clearly does not depend on k . Second, thanks to Lemma A.1, there exist $C_1, C_2 > 0$ so that

$$(50) \quad \prod_{\substack{l \in E_{m(k)} \\ l < k+1}} (\nu_{k+1} - \nu_l)^{-2} \leq 1 + \sum_{n \in E_{m(k)} \setminus \{s(m(k))\}} \prod_{\substack{l \in E_{m(k)} \\ l < n}} (\nu_{k+1} - \nu_l)^{-2} = \text{Tr} \left(F_{m(k)} (\mathbf{v}^{m(k)})^* F_{m(k)} (\mathbf{v}^{m(k)}) \right) \\ \leq C_1 \min\{|l| \in E_{m(k)}\}^{2(1+\bar{d})} \leq C_1 (|k| - M + 1)^{2(1+\bar{d})} \leq C_2 |k|^{2(1+\bar{d})}.$$

Third, due to (49), for $C_3 := \min\{1, g^{-M+2}\}$, it yields that

$$(\nu_{k+1} - \nu_k)^2 = (\nu_{k+1} - \nu_k)^2 \frac{\prod_{\substack{l \in E_{m(k)} \setminus \{k\} \\ l < k+1}} (\nu_{k+1} - \nu_l)^2}{\prod_{\substack{l \in E_{m(k)} \setminus \{k\} \\ l < k+1}} (\nu_{k+1} - \nu_l)^2} \geq C_3 \prod_{\substack{l \in E_{m(k)} \\ l < k+1}} (\nu_{k+1} - \nu_l)^2.$$

The statement follows by gathering the last relation with the identity (50). \square

We are finally ready to prove Proposition 2.4.

Proof of Proposition 2.4. The statement for the cases where \mathcal{G} is either a tadpole, a two-tails tadpole or a double-rings graph is provided as [24, Proposition 8] (arXiv version [24, Proposition 2.5]). We now prove the result when \mathcal{G} is a N edges star graph. We assume $\lambda_1 \neq 0$. When $\lambda_1 = 0$, it enough to separate the first eigenvalue from the rest of the sequence. Let us denote by $\mathbf{v} := (\nu_k)_{k \in \mathbb{Z}^*}$ the sequence with elements

$$\nu_k = \sqrt{\lambda_k}, \quad \forall k > 0; \quad \nu_k = -\sqrt{\lambda_{-k}}, \quad \forall k < 0.$$

Let $I_1 \subseteq \{1, \dots, N\}$ be the set of indices of those edges containing an external vertex equipped with Neumann boundary condition and $I_2 := \{1, \dots, N\} \setminus I_1$. We introduce the entire map

$$G(x) := \prod_{l \in I_2} \sin(xL_l) \prod_{l \in I_1} \cos(xL_l) \left(\sum_{l \in I_2} \cot(xL_l) + \sum_{l \in I_1} \tan(xL_l) \right).$$

The map $G(x)$ verifies the hypotheses of Lemma A.2, as explained in the proof of [25, Theorem 5.1], which gives the claim. \square

B Appendix: Proof of Proposition 3.3

The aim of this appendix is to adapt to the current setting the techniques developed in the works [15] and [16] in order to prove Proposition 3.3. The mentioned paper considers the biorthogonal family to a family of exponentials $\{e^{\lambda_k t}\}_{k \in \mathbb{N}^*}$ where $\{\lambda_k\}_{k \in \mathbb{N}^*}$ is a sequence of ordered positive real numbers such that there exist $0 < \gamma < \gamma^*$ verifying, for every $k \in \mathbb{N}^*$,

$$\begin{cases} \sqrt{\lambda_{2k}} - \sqrt{\lambda_{2k-1}} \geq \gamma, \\ \sqrt{\lambda_{2k+1}} - \sqrt{\lambda_{2k}} \geq \gamma^*. \end{cases}$$

The authors provide an upper bound for the L^2 -norm of each element of such biorthogonal family.

Let $[r]$ be the entire part of $r \in \mathbb{R}$. We introduce the notation, for $\rho > 0$,

$$N_k(\rho) := \text{card}\{m \in \mathbb{N}^* : 0 < |\lambda_m - \lambda_k| \leq \rho\}.$$

Lemma B.1. *Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be a sequence of ordered non-negative real numbers. Assume that there exist $M \in \mathbb{N}^*$, $\gamma > 0$ and an order sequence of decreasing positive real numbers $\{a_k\}_{k \in \mathbb{N}^*}$ verifying, for every $k \in \mathbb{N}^*$,*

$$(51) \quad \begin{cases} \sqrt{\lambda_{k+M}} - \sqrt{\lambda_k} \geq \gamma, \\ \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq a_k. \end{cases}$$

Then, for all $k \in \mathbb{N}^$, we have $N_k(\rho) = 0$ for $\rho \in (0, a_k(a_k + 2\sqrt{\lambda_1}))$. In addition, for every $k \in \mathbb{N}^*$ and $\rho > 0$*

$$N_m(\rho) \leq 2 \left(\frac{\sqrt{\rho}}{\gamma} + 1 \right) M.$$

Proof. The first relation is proved as follows. For $k \in \mathbb{N}^* \setminus \{1\}$, we notice

$$\lambda_k - \lambda_{k-1} = \left(\sqrt{\lambda_k} - \sqrt{\lambda_{k-1}}\right) \left(\sqrt{\lambda_k} + \sqrt{\lambda_{k-1}}\right) \geq a_{k-1} \left(a_{k-1} + 2\sqrt{\lambda_{k-1}}\right) \geq a_{k-1} \left(a_{k-1} + 2\sqrt{\lambda_1}\right)$$

A similar inequality holds when we consider the gap $\lambda_{k+1} - \lambda_k$. Now, the closest element to λ_k belonging to $\{\lambda_k\}_{k \in \mathbb{N}^*}$ is either λ_{k+1} , or λ_{k-1} , which yields the first relation. The estimation for $N_1(\cdot)$ follows equivalently.

The last relation is a consequence of the first hypothesis in (51). First, we notice that for every $k, m \in \mathbb{N}^*$ such that $|\lambda_k - \lambda_m| \leq \rho$, we have

$$(52) \quad \left|\sqrt{\lambda_k} - \sqrt{\lambda_m}\right| \leq \sqrt{|\lambda_k - \lambda_m|} \leq \sqrt{\rho}.$$

Second, we assume $|k - m| > M$ and (51) yields

$$\left|\sqrt{\lambda_k} - \sqrt{\lambda_m}\right| \geq \left\lfloor \frac{|k - m|}{M} \right\rfloor \gamma \geq \left(\frac{|k - m|}{M} - 1\right) \gamma.$$

Thanks to the last relation and (52), we obtain

$$(53) \quad |k - m| \leq M \left(\frac{\left|\sqrt{\lambda_k} - \sqrt{\lambda_m}\right|}{\gamma} + 1\right) \leq \left(\frac{\sqrt{\rho}}{\gamma} + 1\right) M.$$

Obviously, such relation is also valid when $|k - m| \leq M$. Let k_* be the smallest natural number such that $\lambda_k - \lambda_{k_*} \leq \rho$. Let k^* be the largest natural number such that $\lambda_{k^*} - \lambda_k \leq \rho$. We conclude the proof by observing that, thanks to (53), we have

$$N_k(\rho) = k^* - k_* = k^* - k + k - k_* \leq 2 \left(\frac{\sqrt{\rho}}{\gamma} + 1\right) M.$$

□

In what follows, we assume that $\{\lambda_k\}_{k \in \mathbb{N}^*}$ is a sequence of ordered positive real numbers such that $\sum_{k \in \mathbb{N}^*} \lambda_k^{-1} < +\infty$. Such choice allows us to define the following sequence of functions

$$F_k(z) := \prod_{m \in \mathbb{N}^* \setminus \{k\}} \left(1 - \frac{iz - \lambda_k}{\lambda_m - \lambda_k}\right), \quad \forall k \in \mathbb{N}^*.$$

We refer to [15, Section 3.3] for further details on their well-posedness .

Lemma B.2. *Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be a sequence of ordered non-negative real numbers such that $\sum_{k \in \mathbb{N}^*} \lambda_k^{-1} < +\infty$. Assume that the hypotheses of Lemma B.1 are verified. For every $k \in \mathbb{N}^*$, we have the following estimate*

$$\ln |F_k(z - i\lambda_k)| \leq \int_0^{+\infty} N_k(\rho) \frac{|z|}{\rho^2 + \rho|k|} d\rho.$$

Proof. The result is proved exactly as [15, Lemma 3.3].

□

Lemma B.3. *Assume that the hypotheses of Lemma B.2 are verified. For every $k \in \mathbb{N}^*$, we have the following estimate*

$$|F_k(z)| \leq \left(1 + \frac{|z| + \lambda_k}{a_k (a_k + 2\sqrt{\lambda_1})}\right)^M e^{\frac{2M\pi}{\gamma} \sqrt{|z| + \lambda_k}}.$$

Proof. The proof follows by adapting to our context the techniques developed in the proof of [15, Lemma 3.2]. Let us focus on the case $k \in \mathbb{N}^* \setminus \{1\}$. The proof for $k = 1$ follows from the same reasons by using the specific relation from Lemma B.1. First, thanks to Lemma B.1 and Lemma B.2, we have

$$\ln |F_k(z - i\lambda_k)| \leq \frac{2M}{\gamma} \int_0^{+\infty} \frac{\sqrt{\rho}|z|}{\rho^2 + \rho|k|} d\rho + M \int_{a_k(a_k + 2\sqrt{\lambda_1})}^{+\infty} \frac{|z|}{\rho^2 + \rho|k|} d\rho, \quad \forall k \in \mathbb{N}^* \setminus \{1\}.$$

As it has been proved in the aforementioned work, the first integral appearing in the previous relations is bounded by $\pi\sqrt{|z|}$, while the second one by $\ln\left(1 + \frac{|z|}{a_k(a_k + 2\sqrt{\lambda_1})}\right)$. Therefore, we have

$$\ln |F_k(z - i\lambda_k)| \leq \frac{2M\pi}{\gamma} \sqrt{|z|} + M \ln \left(1 + \frac{|z|}{a_k(a_k + 2\sqrt{\lambda_1})}\right), \quad \forall k \in \mathbb{N}^* \setminus \{1\},$$

which implies the result by acting the change of variable $\tilde{z} = z - i\lambda_k$ in the following resulting inequality

$$|F_k(z - i\lambda_k)| \leq \left(1 + \frac{|z|}{a_k(a_k + 2\sqrt{\lambda_1})}\right)^M e^{\frac{2M\pi}{\gamma}\sqrt{|z|}}, \quad \forall k \in \mathbb{N}^* \setminus \{1\}.$$

□

Lemma B.4. *Assume that the hypotheses of Lemma B.2 are verified. There exists $C > 0$ independent of $k \in \mathbb{N}^*$, $\{a_k\}_{k \in \mathbb{N}^*}$, $\gamma > 0$ and $z \in \mathbb{C}$ such that*

$$|F_k(z)| \leq C \left(1 + \frac{\gamma^2}{a_k(a_k + 2\sqrt{\lambda_1})}\right)^M e^{\frac{4M\pi}{\gamma}\sqrt{|z|+\lambda_k}}, \quad \forall k \in \mathbb{N}^*.$$

Proof. The above property follows from Lemma B.3 and from the inequality $\frac{|z|+\lambda}{\gamma^2} \leq e^{\frac{\sqrt{|z|+\lambda}}{\gamma}}$. □

The remaining part of the proof of Proposition 3.3 is the same of the corresponding one of [15, Theorem 2.3] and [16, Theorem 2.4]. Here, we just underline the main strategy and we refer to [16, Section 4.4 and Section 4.5] for further details. The core of this part is to construct the biorthogonal family fixed $T > 0$. Let $N' \geq 2$. We define

$$T' := \min\{T, M^2\gamma^{-1}\}, \quad C_{N',T'} := \frac{T'}{2 \sum_{k=N'}^{\infty} \frac{1}{k^3}},$$

$$a_k := \frac{C_{N',T'}}{k^2}, \quad P_{N',T'}(z) := e^{iz\frac{T'}{2}} \sum_{k=N'}^{\infty} \cos(a_k z).$$

Lemma B.5. *The function $P_{N',T'}$ is entire over \mathbb{C} and satisfies*

- $P_{N',T'}(0) = 1$;
- for every $z \in \mathbb{C}$ such that $\text{Im}(z) \geq 0$, we have $|P_{N',T'}(z)| \leq 1$;
- for every $z \in \mathbb{C}$, we have $|e^{-i\frac{T'}{2}z} P_{N',T'}(z)| \leq e^{|z|\frac{T'}{2}}$.

Moreover, there exists $\theta_0 > 0$ and $\theta_1 > 0$, both independent of N' and T' , such that $P_{N',T'}$ satisfies

- if $\left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{\frac{1}{2}} + 1 \geq N'$, then $\ln |P_{N',T'}(x)| \leq -\frac{\theta_1}{2^3} \left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{\frac{1}{2}}$;
- if $\left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{\frac{1}{2}} + 1 \leq N'$, then $\ln |P_{N',T'}(x)| \leq -\frac{\theta_1}{N'^3} \left(\frac{C_{N',T'}|x|}{\theta_0}\right)^2$.

Proof. See the proof of [16, Lemma 4.3]. □

Under the validity of Lemma B.5, we can consider now N' such that

$$N' \geq 2 + \frac{M^2\theta}{\gamma^2 T'}, \quad \theta := 2^{11} \pi^2 \frac{\theta_0}{\theta_1^2}.$$

In order to construct the biorthogonal family of Proposition 3.3, we introduce the following functions

$$\forall m \in \mathbb{N}, \forall z \in \mathbb{C}, \quad f_{m,N',T'}(z) = F_m(z) \frac{P_{N',T'}(-z)}{P_{N',T'}(i\lambda_m)}.$$

Now, each $f_{m,N',T'}$ verifies [16, Lemma 4.4] and we define the Fourier transform of $f_{m,N',T'}(-x)e^{ix\frac{T'}{2}}$:

$$\psi_{m,N',T'}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} f_{m,N',T'}(-x) e^{-ix\frac{T'}{2}} e^{-ix\xi} dx.$$

We use the functions $\psi_{m,N',T'}$ to build the biorthogonal family as explained in the following Lemma.

Lemma B.6. *Let $T > 0$. Let T' and N' as above. The functions*

$$\sigma_{m,N',T'}^+(t) = \psi_{m,N',T'}\left(\frac{T}{2} - t\right) e^{-\lambda_m T}, \quad \forall m \in \mathbb{N}^*$$

form a biorthogonal family to the exponentials $\{e^{-\lambda_m t}\}_{m \in \mathbb{N}^}$ verifying the inequality (17).*

Proof. The proof follows exactly the one of [16, Lemma 4.5] by using the inequality provided in Lemma B.4. □

Proof of Proposition 3.3. The claim follows directly from Lemma B.6 by noticing that the upper bounds provided for the elements of the biorthogonal family do not depend on the intermediate parameters N' and T' . □

References

- [1] L. F. Abbott, E. Farhi and S. Gutmann. The path integral for dendritic trees. *Biological cybernetics*, 66(1):49–60, 1991.
- [2] F. Alabau-Boussouira, P. Cannarsa and C. Urbani. Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control. *Preprint available on arXiv:1811.08806*.
- [3] F. Alabau-Boussouira, P. Cannarsa and C. Urbani. Superexponential stabilizability of evolution equations of parabolic type via bilinear control. *Journal of Evolution Equations*, 20(1):941–967, 2020.
- [4] Alexander, S. Superconductivity of networks. A percolation approach to the effects of disorder. *Physical Review B*, 27(3):1541–1557, 1983.
- [5] K. Ammari and A. Duca. Controllability of localized quantum states on infinite graphs through bilinear control fields. *Internat. J. Control*, 94(7):1824–1837, 2021.
- [6] K. Ammari and A. Duca. Controllability of periodic bilinear quantum systems on infinite graphs. *Journal of Mathematical Physics*, 61(10):101507, 2020.
- [7] S.A. Avdonin and W. Moran. Simultaneous controllability of elastic systems and beams. *Systems and Control Letters*, 44(2):147–155, 2001.
- [8] S.A. Avdonin and M. Tucsnark. Simultaneous controllability in sharp time for two elastic strings. *ESAIM Control Optim. Calc. Var.*, 6:259–273, 2001.
- [9] J.M. Ball, J. Marsden and E M. Slemrod. Controllability for distributed bilinear systems. *SIAM Journal on Control and Optimization*, 20(4):575–597, 1982.
- [10] J.A. Bárcena-Petisco, M. Cavalcante, G. Coclite, N. de Nitti and E. Zuazua. Control of hyperbolic and parabolic equations on networks and singular limits. *Preprint available on hal-03233211*.
- [11] K. Beauchard. Local controllability and non-controllability for a 1D wave equation with bilinear control. *Journal of Differential Equations*, 250(4):2064–2098, 2011.
- [12] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *Journal de mathématiques pures et appliquées*, 94(5):520–554, 2010.
- [13] A. Benabdallah, F. Boyer and M. Morancey. A block moment method to handle spectral condensation phenomenon in parabolic control problems. *Ann. H. Lebesgue*, 3:717–793, 2020.
- [14] G. Berkolaiko and P. Kuchment. *Introduction to quantum graphs*, volume 186 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2013.
- [15] P. Cannarsa, P. Martinez and J. Vancostenoble. Sharp Estimate of the Cost of Controllability for a Degenerate Parabolic Equation with Interior Degeneracy. *Minimax Theory Appl*, 6(2):251–280, 2021.
- [16] P. Cannarsa, P. Martinez and J. Vancostenoble. The cost of controlling weakly degenerate parabolic equations by boundary controls. *Math. Control Relat. Fields*, 7(2):171–211, 2017.
- [17] P. Cannarsa and C. Urbani. Superexponential stabilizability of degenerate parabolic equations via bilinear control. *Inverse Problems and Related Topics*, 31–45, Springer Singapore, 2020.
- [18] C. Castro and E. Zuazua. A hybrid system consisting of two flexible beams connected by a point mass: spectral analysis and well-posedness in asymmetric spaces. In *"Elasticité, Viscoélasticité et Contrôle Optimal"*, *ESAIM Proceedings*, 2:17–54, 1997.
- [19] R. Dáger. Observation and control of vibrations in tree-shaped networks of strings. *SIAM J. Control Optim.*, 43(2):590–623, 2004.
- [20] R. Dáger and E. Zuazua. *Wave propagation, observation and control in 1-d flexible multi-structures*. Springer-Verlag, Berlin, 2006.
- [21] R. Dáger and E. Zuazua. Spectral boundary controllability of networks of strings. *C. R. Acad. Sci. Paris, Série I*, 334(7):545–550, 2002.
- [22] B. Dekoninck and S. Nicaise. Control of networks of Euler-Bernoulli beams. *ESAIM:COCV*, 4:57–81, 1999.

- [23] B. Dekoninck and S. Nicaise. The eigenvalue problem for networks of beams. *Linear Algebra Appl.*, 314(1–3):165–189, 2000.
- [24] A. Duca. Bilinear quantum systems on compact graphs: well-posedness and global exact controllability. *Automatica J. IFAC*, 123:109324, 2021.
- [25] A. Duca. Global exact controllability of bilinear quantum systems on compact graphs and energetic controllability. *SIAM J. Control Optim.*, 58(6):3092–3129, 2020.
- [26] Flesia, C., Johnston, R., & Kunz, H. Strong localization of classical waves: A numerical study. *Europhysics Letters (EPL)*, 3(4), 497–502, 1987.
- [27] Hodgkin, A. L. and A. F. Huxley. A quantitative description of membrane current and its application to conduction and excitation in nerve. *J. Physic*, 500-544, 1952.
- [28] A.Y. Khapalov. *Controllability of partial differential equations governed by multiplicative controls*, Lecture Notes in Mathematics, Springer-Verlag, Berlin , 1995: XVI+284, 2010.
- [29] P. Kuchment. Quantum graphs. I. Some basic structures. *Waves Random Media*, 14(1):107–128, 2004.
- [30] Mitra, R., & Lee, S. W. *Analytical Techniques in the Theory of Guided Waves*. New York: Macmillan, 1971.
- [31] D. Mugnolo *Parabolic systems and evolution equations on networks*. Universität Ulm, Zusammenfassung der Habilitation, 2010.
- [32] Pauling, L. The diamagnetic anisotropy of aromatic molecules. *The Journal of Chemical Physics*, 4(10):673–677, 1936.
- [33] W. Rall. Branching dendritic trees and motoneuron membrane resistivity. *Experimental neurology*, 1(5):491–527, 1959.
- [34] Richardson, M. J., & Balazs, N. L. On the network model of molecules and solids. *Annals of Physics*, 73(2):308–325, 1972.
- [35] Ruedenberg, K., & Scherr, C. W. Free-electron network model for conjugated systems. I. Theory. *The Journal of Chemical Physics*, 21(9):1565–1581, 1953.