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# DIRECTED DEGENERACY MAPS FOR PRECUBICAL SETS

PHILIPPE GAUCHER

ABSTRACT. Transverse (symmetric precubical) sets were introduced to make the construction of the parallel product with synchronization for process algebras functorial. It is proved that one can do directed homotopy on transverse sets in the following sense. A  $q$ -realization functor from transverse sets to flows is introduced using a  $q$ -cofibrant replacement functor of flows. By topologizing the cotransverse maps, the cotransverse topological cube is constructed. It can be regarded both as a cotransverse topological space and as a cotransverse Lawvere metric space. A natural realization functor from transverse sets to flows is introduced using Raussen's notion of natural  $d$ -path extended to transverse sets thanks to their structure of Lawvere metric space. It is proved that these two realization functors are homotopy equivalent by using the fact that the small category defining transverse sets is  $c$ -Reedy in Shulman's sense. This generalizes to transverse sets results previously obtained for precubical sets. Note that a precubical set and the free transverse set generated by it have isomorphic realizations as flows.

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## 1. INTRODUCTION

**Presentation.** Precubical sets are an important combinatorial model for directed homotopy [7]. The  $n$ -cube represents the concurrent execution of  $n$  actions. It has been known for a long time that the usual degeneracy maps used in non-directed homotopy theory are not convenient for directed homotopy. The purpose of this paper is to introduce a convenient notion of degeneracy map for doing directed homotopy.

In fact, this paper is the second paper, and not the first one, about *transverse (symmetric precubical) sets*. This notion is introduced in [11] to make the construction of the parallel product with synchronization of process algebras functorial. It is proved in [11,

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Theorem 3.1.15] that is the *only* solution to achieve such a goal. A transverse symmetric precubical set is a presheaf on the category  $\widehat{\square}$  generated by the posets  $[n] = \{0 < 1\}^n$  for  $n \geq 0$  and by all *cotransverse maps* which are the strictly increasing maps preserving adjacency (Definition 3.3). Note that to avoid cumbersome and inconsistent terminology, the terminologies *adjacency-preserving map* and *transverse symmetric precubical set* of [11] are replaced in this paper by *cotransverse map* and *transverse set* respectively.

In [15], two realization functors from precubical sets to flows, the q-realization functor using a q-cofibrant replacement functor of the q-model structure of flows (recalled in Section 6) and the natural realization functor using Raussen's notion of natural  $d$ -path, are compared and are proved to be homotopy equivalent. The technical goal of this paper is to generalize these results to transverse sets.

Two obstacles must be overcome before reaching this goal. The first obstacle is to find a way of topologizing the cotransverse maps. It is the purpose of Proposition 4.1. We obtain the *cotransverse topological cube* which is a cotransverse object both in the category of topological spaces and in the category of Lawvere metric spaces. The point is that the cotransverse maps are quasi-isometric. They preserve finite distances indeed. It implies that the topological version of the cotransverse maps takes natural  $d$ -paths of the topological cube to natural  $d$ -paths. It enables us to define natural  $d$ -paths for transverse sets in Section 7. We then obtain the natural realization functor of a transverse set as a flow similarly to [15, Section 5]. The q-realization functor of a transverse set as a flow is already defined in [11, Definition 2.2.9]. The second obstacle is that the small category  $\widehat{\square}$  is not Reedy. It is not even Reedy in Berger-Moerdijk's sense [3, Definition 1.1] or in Cisinski's sense [5, Definition 8.1.1]. However, it is c-Reedy in Shulman's sense [27, Definition 8.25]. It is the key point to compare the q-realization and the natural realization of a transverse set as a flow. We give now the main result of the paper.

**Theorem.** (Theorem 8.3) *The natural realization functor  $| - |_{nat}$  from transverse sets to flow defined in Definition 8.2 is a  $m$ -realization functor. It is homotopy equivalent to the q-realization functor  $| - |_q$  of Theorem 6.10 in the following sense. There exists a  $m$ -realization functor  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  and two natural transformations inducing bijections on the sets of states*

$$| - |_q \Leftarrow F(-) \Rightarrow | - |_{nat}$$

*such that for all transverse sets  $K$  and all  $(\alpha, \beta) \in K_0 \times K_0$ , there is the zig-zag of natural homotopy equivalences between  $m$ -cofibrant topological spaces*

$$\mathbb{P}_{\alpha, \beta} | K |_q \xleftarrow{\simeq} \mathbb{P}_{\alpha, \beta} F(K) \xrightarrow{\simeq} \mathbb{P}_{\alpha, \beta} | K |_{nat} .$$

The link between the precubical and transverse settings is described as follows. For a given q-cofibrant replacement functor of the q-model structure of flows, the q-realization of a precubical set is *isomorphic* to the q-realization of the free transverse set generated by the precubical set (see Notation 3.13) and the natural realization of a precubical set is *isomorphic* to the natural realization of the free transverse set generated by the precubical set.

Subsequent papers will study directed homotopy on transverse sets and will show how it is possible to extend most of the results proved for precubical sets.

**Outline of the paper.** Section 2 recalls some basic facts about Lawvere metric spaces. The Lawvere metric  $\overrightarrow{d}_1$  plays an important role in many places of the paper, in particular for defining the natural  $d$ -paths of a transverse set as some particular kind of quasi-isometry in Section 7. Section 3 recalls some basic facts about precubical sets and transverse (symmetric precubical) sets. It also expounds in Proposition 3.15 a missing argument in the proof of [11, Corollary 2.2.11]. Section 4 starts from the observation made in Proposition 4.1 to topologize the cotransverse maps and proves some useful properties about them. It culminates with Theorem 4.9 which expounds the cotransverse topological cube and Theorem 4.14 which expounds the cotransverse Lawvere cube. Section 5 proves that the category of cotransverse objects of a model category satisfying some mild conditions has a structure of a c-Reedy model category and that it coincides with the projective model structure. It enables us to give a necessary and sufficient condition for a cotransverse object to be projective cofibrant in Theorem 5.10. The latter condition is used in Proposition 6.6 to prove that the projective r-cofibrant replacement of the cotransverse flow associated with a r-realization functor gives rise to a r-realization functor. It is the key fact to prove Theorem 6.8 and Theorem 8.3. Section 6 defines the notions of q-realization, m-realization and h-realization of a transverse set as a flow. Theorem 6.10 provides an example of a q-realization functor from transverse sets to flows. Section 7 is devoted to defining the notion of natural  $d$ -path of a transverse set. It requires to recall what is the underlying topological space of a Lawvere metric space and to make some calculations about the cubes and more generally about the transverse sets. Geometrically, the natural  $d$ -paths of a transverse set are locally, on each cube, a quasi-isometry for the Lawvere metric. Section 8 concludes this paper by defining the natural realization of a transverse set in Definition 8.2 and by proving in Theorem 8.3 that it is a m-realization functor which is homotopy equivalent to the q-realization functor from transverse sets to flows of Theorem 6.10.

**Prerequisites and notations.** All necessary reminders are made throughout the paper. We refer to [1] for locally presentable categories, to [25] for combinatorial model categories. We refer to [20] and to [19] for more general model categories. We work with the category **Top** of  $\Delta$ -generated spaces or of  $\Delta$ -Hausdorff  $\Delta$ -generated spaces (cf. [13, Section 2 and Appendix B]). The category **Top** is equipped with its q-model structure (we use the terminology of [23]). The m-model structure [6] and the h-model structure [2] of **Top** are also used in various places of the paper. The set of maps from  $X$  to  $Y$  of a category  $\mathcal{C}$  is denoted by  $\mathcal{C}(X, Y)$ . **Set** is the category of sets with all set maps.  $\mathcal{C}^I$  is the category of functor from a small category  $I$  to a category  $\mathcal{C}$  together with the natural transformations.  $\cong$  means isomorphism,  $\simeq$  means weak equivalence or homotopy equivalence, depending on the context.

## 2. THE LAWVERE DIRECTED $n$ -CUBE

Since there are several variants of the notion of metric space in the mathematical literature, the one which is used in this paper is recalled. The symmetric version will have to be recalled in Section 7.

**2.1. Definition.** [21] *A Lawvere metric space  $(X, d)$  is a set  $X$  equipped with a map  $d : X \times X \rightarrow [0, \infty]$  called a (Lawvere) metric such that:*

- $\forall x \in X, d(x, x) = 0$
- $\forall (x, y, z) \in X \times X \times X, d(x, y) \leq d(x, z) + d(z, y)$ .

A map  $f : (X, d) \rightarrow (Y, d)$  of Lawvere metric spaces is a set map  $f : X \rightarrow Y$  which is short, i.e.  $\forall (x, y) \in X \times X, d(f(x), f(y)) \leq d(x, y)$ . The category of Lawvere metric spaces is denoted by **LvMet**. A short map  $f : (X, d) \rightarrow (Y, d)$  is quasi-isometric if  $\forall (x, y) \in X \times X, d(x, y) < \infty \Rightarrow d(f(x), f(y)) = d(x, y)$ .

The category of Lawvere metric spaces is bicomplete since it is the category of enriched small categories over a bicomplete closed symmetric monoidal structure on  $[0, \infty]$  [21].

**2.2. Notation.** Let  $[0] = \{()\}$  and  $[n] = \{0, 1\}^n$  for  $n \geq 1$ . By convention, one has  $\{0, 1\}^0 = [0] = \{()\}$ . In the sequel, for all  $n \geq 1$ , both the sets  $[n]$  and  $[0, 1]^n$  are equipped with the product order. By convention,  $[0, 1]^0$  is a singleton.

**2.3. Definition.** Let  $x = (x_1, \dots, x_n)$  and  $x' = (x'_1, \dots, x'_n)$  be two elements of  $[0, 1]^n$  with  $n \geq 1$ . Let  $\vec{d}_1 : [0, 1]^n \times [0, 1]^n \rightarrow [0, \infty]$  be the set map defined by

$$\vec{d}_1(x, x') = \begin{cases} \sum_{i=1}^n |x_i - x'_i| & \text{if } x \leq x' \\ \infty & \text{otherwise.} \end{cases}$$

**2.4. Proposition.** Let  $n \geq 0$ . The set map  $\vec{d}_1 : [0, 1]^n \times [0, 1]^n \rightarrow [0, \infty]$  is a Lawvere metric. It restricts to a Lawvere metric on  $\{0, 1\}^n$ .

*Proof.* Let  $x, y, z \in [0, 1]^n$ . If  $\vec{d}_1(x, z) + \vec{d}_1(z, y)$  is finite, then  $x \leq z \leq y$ , which implies that  $\vec{d}_1(x, y)$  is finite and that  $\vec{d}_1(x, y) = \vec{d}_1(x, z) + \vec{d}_1(z, y)$ . If  $\vec{d}_1(x, z) + \vec{d}_1(z, y)$  is infinite, then the inequality  $\vec{d}_1(x, y) \leq \vec{d}_1(x, z) + \vec{d}_1(z, y)$  always holds.  $\square$

### 3. PRECUBICAL AND TRANSVERSE SET

**3.1. Notation.** Let  $A \subset \{1, \dots, n\}$ . Denote by  $\epsilon_A$  the tuple  $(\epsilon_1, \dots, \epsilon_n)$  with  $\epsilon_i = 0$  if  $i \notin A$  and  $\epsilon_i = 1$  if  $i \in A$ . Let  $0_n = \epsilon_\emptyset$  and  $1_n = \epsilon_{\{1, \dots, n\}}$ .

Let  $\delta_i^\alpha : [n-1] \rightarrow [n]$  be the coface map defined for  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  by  $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$ . The small category  $\square$  is by definition the subcategory of the category of posets with the set of objects  $\{[n], n \geq 0\}$  and generated by the morphisms  $\delta_i^\alpha$ . The maps of  $\square$  are called the *cocubical* maps.

**3.2. Definition.** [4] The category of presheaves over  $\square$ , denoted by  $\square^{op}\mathbf{Set}$ , is called the category of precubical sets. Let  $\square[n] := \square(-, [n])$ . For  $K \in \square^{op}\mathbf{Set}$ , denote by  $K_n = K([n])$  the set of  $n$ -cubes of  $K$ . Let  $f : [m] \rightarrow [n]$  be a cocubical map. It gives rise to a set map denote by  $f^* : K_n \rightarrow K_m$ . An element of  $K_0$  is called a vertex of  $K$ .

The following definition is equivalent to [11, Definition 2.1.5].

**3.3. Definition.** A set map  $f : [m] \rightarrow [n]$  is cotransverse if it is strictly increasing and if  $\forall x, y \in [m], \vec{d}_1(x, y) = 1$  implies  $\vec{d}_1(f(x), f(y)) = 1$ .

The adjective adjacency-preserving is used in [11] instead. The word cotransverse is preferred because it is consistent with the terminology of transverse (symmetric precubical) sets <sup>1</sup>.

<sup>1</sup>Unlike in [11], the words symmetric and precubical are omitted.

By [11, Proposition 2.1.6], for any  $n \geq 1$ , the coface map  $\delta_i^\alpha : [n-1] \rightarrow [n]$  is cotransverse and any strictly increasing map from  $[n]$  to itself is cotransverse as well. Let  $\sigma_i : [n] \rightarrow [n]$  be the set map defined for  $1 \leq i \leq n-1$  and  $n \geq 2$  by  $\sigma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \dots, \epsilon_n)$ . These maps are called the symmetry maps [17]. The symmetry maps are clearly cotransverse.

**3.4. Notation.** Let  $\widehat{\square}$  be the small subcategory of the category of posets generated by the cotransverse maps.

The following proposition is crucial in many places of this paper.

**3.5. Proposition.** [11, Proposition 3.1.14] Let  $0 \leq m \leq n$ . Every cotransverse (resp. cotransverse one-to-one) map  $f : [m] \rightarrow [n]$  factors uniquely as a composite  $[m] \xrightarrow{\psi} [m] \xrightarrow{\phi} [n]$  with  $\phi \in \square$  and  $\psi$  cotransverse (resp. cotransverse one-to-one).

By a cardinality argument, if  $\psi : [m] \rightarrow [m]$  is one-to-one, then it is bijective and therefore it is a symmetry map. Thus the one-to-one cotransverse maps are composites of coface maps and symmetry maps.

**3.6. Definition.** [11, Definition 2.1.13] The category of presheaves over  $\widehat{\square}$ , denoted by  $\widehat{\square}^{op}\mathbf{Set}$ , is called the category of transverse sets. Let  $\widehat{\square}[n] := \widehat{\square}(-, [n])$ . For  $K \in \widehat{\square}^{op}\mathbf{Set}$ , denote by  $K_n = K([n])$  the set of  $n$ -cubes of  $K$ . Let  $f : [m] \rightarrow [n]$  be a cotransverse map. It gives rise to a set map denote by  $f^* : K_n \rightarrow K_m$ . An element of  $K_0$  is called a vertex of  $K$ .

**3.7. Definition.** Let  $\mathcal{C}$  be a category. A cotransverse object of  $\mathcal{C}$  is a functor  $\widehat{\square} \rightarrow \mathcal{C}$ .

There is the elementary proposition:

**3.8. Proposition.** Let  $\mathcal{C}$  be a cocomplete category. Let  $X : \widehat{\square} \rightarrow \mathcal{C}$  be a cotransverse object of  $\mathcal{C}$ . Let

$$\widehat{X}(K) = \int^{[n] \in \widehat{\square}} K_n \cdot X([n])$$

The mapping  $X \mapsto \widehat{X}$  induces an equivalence of categories between the category of cotransverse objects of  $\mathcal{C}$  and the colimit-preserving preserving functors from  $\widehat{\square}^{op}\mathbf{Set}$  to  $\mathcal{C}$ .

**3.9. Notation.** For the sequel, the cotransverse object associated with a colimit-preserving functor  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathcal{C}$  is denoted by  $F(\widehat{\square}[*])$ .

*Proof.* Let us denote by  $\widetilde{F} = F(\widehat{\square}[*])$  for this proof only the cotransverse object of  $\mathcal{C}$  associated with a colimit-preserving functor  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathcal{C}$ . Since  $F$  is colimit-preserving, one has the isomorphisms

$$\widetilde{F}(K) = \int^{[n] \in \widehat{\square}} K_n \cdot F(\widehat{\square}[n]) \cong F\left(\int^{[n] \in \widehat{\square}} K_n \cdot \widehat{\square}[n]\right) \cong F(K)$$

and the isomorphisms

$$\widetilde{G} = \widehat{G}(\widehat{\square}[*]) = \int^{[n] \in \widehat{\square}} \widehat{\square}([n], *) \cdot G([n]) \cong G([*]).$$

Finally, if  $K : i \mapsto K^i$  is a small diagram of transverse sets over a small category  $I$ , one has

$$\begin{aligned} \widehat{X}(\varinjlim K^i) &= \int^{[n] \in \widehat{\square}} (\varinjlim K_n^i).X([n]) \\ &\cong \int^{[n] \in \widehat{\square}} \varinjlim (K_n^i.X([n])) \cong \varinjlim \left( \int^{[n] \in \widehat{\square}} K_n^i.X([n]) \right) = \varinjlim \widehat{X}(K^i), \end{aligned}$$

the first equality by definition of  $\widehat{X}$  and since colimits are calculated objectwise in  $\widehat{\square}^{op}\mathbf{Set}$ , the first isomorphism because of the adjunction  $\mathcal{C}(S.X, Y) \cong \mathbf{Set}(S, \mathcal{C}(X, Y))$ , the second isomorphism by commuting the colimits, and finally the last equality by definition of  $\widehat{X}$ .  $\square$

**3.10. Notation.** Let  $n \geq 1$ . Let  $h : [0, 1]^n \rightarrow [0, n]$  be the continuous map defined by

$$h(x_1, \dots, x_n) = \sum_{i=1}^n x_i.$$

Note that for all  $x, y \in [0, 1]^n$ ,  $x \leq y$  implies  $h(x) \leq h(y)$  and that  $x \leq y$  and  $h(x) = h(y)$  implies  $x = y$ .

**3.11. Proposition.** Let  $n \geq 1$ . Let  $f : [n] \rightarrow [n]$  be a cotransverse map. Then for all  $(\epsilon_1, \dots, \epsilon_n) \in [n]$ , one has  $h(\epsilon_1, \dots, \epsilon_n) = h(f(\epsilon_1, \dots, \epsilon_n))$ .

*Proof.* We proceed by induction on  $h(\epsilon_1, \dots, \epsilon_n)$ . Consider the increasing sequence

$$\epsilon_\emptyset < \epsilon_{\{1\}} < \epsilon_{\{1,2\}} < \dots < \epsilon_{\{1,2,\dots,n\}}$$

of elements of  $[n]$ . The map  $f$  being cotransverse by hypothesis, one has

$$\begin{aligned} \overrightarrow{d}_1(f(\epsilon_\emptyset), f(\epsilon_{\{1\}})) &= 1, \\ \overrightarrow{d}_1(f(\epsilon_{\{1\}}), f(\epsilon_{\{1,2\}})) &= 1, \\ &\dots \\ \overrightarrow{d}_1(f(\epsilon_{\{1,\dots,n-1\}}), f(\epsilon_{\{1,\dots,n\}})) &= 1. \end{aligned}$$

Since  $f : [n] \rightarrow [n]$  is strictly increasing, we obtain  $f(\epsilon_\emptyset) = \epsilon_\emptyset$  and  $f(\epsilon_{\{1,\dots,n\}}) = \epsilon_{\{1,\dots,n\}}$ . We deduce that  $h(\epsilon_\emptyset) = hf(\epsilon_\emptyset)$  and that  $h(\epsilon_{\{1,\dots,n\}}) = hf(\epsilon_{\{1,\dots,n\}})$ . The formula is therefore proved for  $h(\epsilon_1, \dots, \epsilon_n) = 0$  (and also for  $h(\epsilon_1, \dots, \epsilon_n) = n$ ). Suppose the formula proved for all  $(\epsilon_1, \dots, \epsilon_n) \in [n]$  such that  $h(\epsilon_1, \dots, \epsilon_n) \leq H < n$ . Let  $(\epsilon_1, \dots, \epsilon_n) \in [n]$  such that  $h(\epsilon_1, \dots, \epsilon_n) = H+1 \geq 1$ . There exists  $(\epsilon'_1, \dots, \epsilon'_n) \in [n]$  with  $h(\epsilon'_1, \dots, \epsilon'_n) = H$  and  $(\epsilon'_1, \dots, \epsilon'_n) < (\epsilon_1, \dots, \epsilon_n)$ . We deduce that  $\overrightarrow{d}_1((\epsilon'_1, \dots, \epsilon'_n), (\epsilon_1, \dots, \epsilon_n)) = 1$ . The map  $f$  being cotransverse, we obtain  $\overrightarrow{d}_1(f(\epsilon'_1, \dots, \epsilon'_n), f(\epsilon_1, \dots, \epsilon_n)) = 1$ . We obtain the equalities  $h(f(\epsilon_1, \dots, \epsilon_n)) = h(f(\epsilon'_1, \dots, \epsilon'_n)) + 1 = H + 1$ , the first equality by definition of  $d_1$  and the second equality by induction hypothesis.  $\square$

As a corollary, we obtain the following proposition.

**3.12. Proposition.** Let  $\psi : [m] \rightarrow [n]$  be a cotransverse map. Then  $\psi$  induces a map of Lawvere metric spaces from  $[m]$  to  $[n]$  which is quasi-isometric.

A cotransverse map is not necessarily an isometry. For example, the map  $\gamma_1 : [2] \rightarrow [2]$  defined by  $\gamma_1(\epsilon_1, \epsilon_2) = (\max(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_2))$  is cotransverse and  $\gamma_1(1, 0) = \gamma_1(0, 1) = (1, 0)$ . Note that  $\overrightarrow{d}_1((0, 1), (1, 0)) = \infty$ .

**3.13. Notation.** The inclusion of small categories  $\square \subset \widehat{\square}$  induces a forgetful functor  $\widehat{\omega} : \widehat{\square}^{op}\mathbf{Set} \rightarrow \square^{op}\mathbf{Set}$  which has a left adjoint  $\widehat{\mathcal{L}} : \square^{op}\mathbf{Set} \rightarrow \widehat{\square}^{op}\mathbf{Set}$  which is called the free transverse set generated by a precubical set.

**3.14. Notation.** For a precubical (transverse resp.) set  $K$ , let  $K_{\leq n}$  be the precubical (transverse resp.) set defined by

$$(K_{\leq n})_p = \begin{cases} K_p & \text{if } p \leq n \\ \emptyset & \text{if } p > n. \end{cases}$$

This definition makes sense because  $\square([m], [n]) = \widehat{\square}([m], [n]) = \emptyset$  when  $m > n$ . Let  $\partial\square[n] = \square[n]_{\leq n-1}$  and  $\partial\widehat{\square}[n] = \widehat{\square}[n]_{\leq n-1}$  for all  $n \geq 0$ .

**3.15. Proposition.** For all  $n \geq 0$ , one has the isomorphism of transverse sets  $\widehat{\mathcal{L}}(\square[n]) \cong \widehat{\square}[n]$ . There is the isomorphism of transverse sets  $\widehat{\mathcal{L}}(\partial\square[n]) \cong \partial\widehat{\square}[n]$  for all  $n \geq 0$ .

*Proof.* The first statement is [11, Proposition 2.1.14]. The short argument is repeated for the ease of the reader. For every transverse set  $K$ , one has  $K_n = (\widehat{\omega}K)_n$  for all  $n \geq 0$ . Since the functor  $\square \subset \widehat{\square}$  is the identity on objects, we obtain for all  $n \geq 0$  the bijections

$$\widehat{\square}^{op}\mathbf{Set}(\widehat{\mathcal{L}}(\square[n]), K) \cong \square^{op}\mathbf{Set}(\square[n], \widehat{\omega}K) = (\widehat{\omega}K)_n = K_n = \widehat{\square}^{op}\mathbf{Set}(\widehat{\square}[n], K).$$

By the Yoneda lemma, one obtains the isomorphism  $\widehat{\mathcal{L}}(\square[n]) \cong \widehat{\square}[n]$  for all  $n \geq 0$ . The second statement is stated with an incorrect argument in the proof of [11, Corollary 2.2.11]. The missing argument is explained now. Consider the small category  $J'$  such that the objects are the coface maps  $[p] \rightarrow [n] \in \square$  with  $p < n$  and such that the morphisms of  $J'$  are the commutative squares of the form

$$\begin{array}{ccc} [p] & \longrightarrow & [n] \\ \in \square \downarrow & & \parallel \\ [q] & \longrightarrow & [n] \end{array}$$

Since  $\square([p], \square[n]) = \emptyset$  for  $p > n$  and since  $\widehat{\mathcal{L}} : \square^{op}\mathbf{Set} \rightarrow \widehat{\square}^{op}\mathbf{Set}$  is colimit-preserving, we obtain the isomorphism of transverse sets

$$\varinjlim_{[p] \rightarrow [n] \in J'} \widehat{\square}[p] \cong \widehat{\mathcal{L}}(\partial\square[n]).$$

Consider the small category  $J$  such that the objects are the maps  $[p] \rightarrow [n] \in \widehat{\square}$  with  $p < n$  and such that the morphisms are the commutative squares of the form

$$\begin{array}{ccc} [p] & \longrightarrow & [n] \\ \in \widehat{\square} \downarrow & & \parallel \\ [q] & \longrightarrow & [n] \end{array}$$

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Since  $\widehat{\square}([p], \square[n]) = \emptyset$  for  $p > n$ , we obtain the isomorphism of transverse sets

$$\varinjlim_{[p] \rightarrow [n] \in J} \widehat{\square}[p] \cong \partial \widehat{\square}[n].$$

Consider the inclusion functor  $L : J' \rightarrow J$ . It induces a map of transverse sets

$$\widehat{\mathcal{L}}(\partial \square[n]) \longrightarrow \partial \widehat{\square}[n].$$

By [22, Theorem 1 p. 213], it suffices to prove that the comma category  $(k \downarrow L)$  is nonempty and connected for all objects  $k$  of  $J$  to complete the proof. Let  $k : [p] \rightarrow [n]$  be an object of  $J$ . We see immediately that the comma category  $(k \downarrow L)$  is nonempty because it contains the commutative square

$$\begin{array}{ccc} [p] & \xrightarrow{\psi} & [p] \\ k \downarrow & & \downarrow \psi' \in \square \\ [n] & \xlongequal{\quad} & [n] \end{array}$$

where the top map  $\psi : [p] \rightarrow [p]$  is given by the unique factorization given by Proposition 3.5 of  $k : [p] \rightarrow [n]$  as the composite of a map of  $\widehat{\square}([p], [p])$  followed by a coface map  $\psi'$ . Consider another object

$$\begin{array}{ccc} [p] & \xrightarrow{\psi_1} & [r] \\ k \downarrow & & \downarrow \psi_3 \in \square \\ [n] & \xrightarrow{\psi_2} & [s] \end{array}$$

of the comma category  $(k \downarrow L)$ . Consider the following diagram of  $\widehat{\square}$ :

$$\begin{array}{ccccc} [p] & \xrightarrow{\psi} & [p] & \xrightarrow{\psi' \in \square} & [n] \\ \parallel & & \downarrow g & & \downarrow \psi_2 \\ [p] & \xrightarrow{g\psi} & [p] & \xrightarrow{g' \in \square} & [s] \\ & \curvearrowright h & & & \parallel \\ [p] & \xrightarrow{\psi_1} & [r] & \xrightarrow{\psi_3 \in \square} & [s] \\ & & \downarrow h' \in \square & & \parallel \end{array}$$

where the factorizations  $\psi_2 \psi' = g' g$  and  $\psi_1 = h' h$  are given by the factorization of Proposition 3.5. We obtain  $(\psi_3 h') h = \psi_3 \psi_1 = \psi_2 \psi' \psi = g'(g\psi)$ . By uniqueness of the factorization of Proposition 3.5, we deduce that  $\psi_3 h' = g'$  and  $h = g\psi$ . We deduce the map of  $(k \downarrow L)$  depicted in Figure 1. We conclude that the comma category  $(k \downarrow L)$  is connected.  $\square$

**3.16. Remark.** *In fact, we could prove that the comma category  $(k \downarrow L)$  has an initial object given by the factorization of  $k$  using Proposition 3.5.*

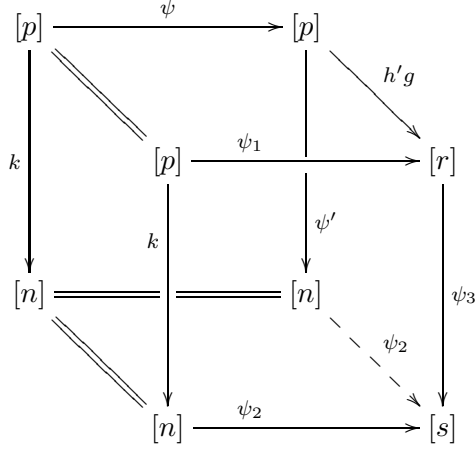


FIGURE 1.  $(k \downarrow L)$  is connected

#### 4. COTRANSVERSE TOPOLOGICAL CUBE

The purpose of this section is to topologize the cotransverse maps, i.e. to extend any cotransverse map  $f : [m] \rightarrow [n]$  to a map of Lawvere metric spaces  $T(f)$  from  $([0, 1]^m, \vec{d}_1)$  to  $([0, 1]^n, \vec{d}_1)$  which is quasi-isometric. The starting point is the following observation.

**4.1. Proposition.** *Let  $n \geq 1$ . Let  $f = (f_1, \dots, f_n) : [n] \rightarrow [n]$  be a cotransverse map. Then there is the equality*

$$f_i(x_1, \dots, x_n) = \max_{(\epsilon_1, \dots, \epsilon_n) \in f_i^{-1}(1)} \min\{x_k \mid \epsilon_k = 1\}$$

for all  $1 \leq i \leq n$ .

*Proof.* There are two mutually exclusive cases:  $f_i(x_1, \dots, x_n) = 0$  or  $f_i(x_1, \dots, x_n) = 1$ . Let us treat the case  $f_i(x_1, \dots, x_n) = 0$  at first. For all  $(\epsilon_1, \dots, \epsilon_n) \in f_i^{-1}(1)$ ,  $\min\{x_k \mid \epsilon_k = 1\} = 1$  implies  $(x_1, \dots, x_n) \geq (\epsilon_1, \dots, \epsilon_n)$ , which implies  $f_i(x_1, \dots, x_n) = 1$ : contradiction. Thus  $f_i(x_1, \dots, x_n) = 0$  implies that for all  $(\epsilon_1, \dots, \epsilon_n) \in f_i^{-1}(1)$ , one has  $\min\{x_k \mid \epsilon_k = 1\} = 0$ . Assume now that  $f_i(x_1, \dots, x_n) = 1$ . Then  $(x_1, \dots, x_n) \in f_i^{-1}(1)$ . Since  $\min\{x_i \mid x_i = 1\} = 1$ , the proof is complete.  $\square$

To give the reader the intuition of Proposition 4.1, consider the cotransverse map  $f : [3] \rightarrow [3]$  described in Figure 2 (it is the example [11, Figure 5]). Let  $f = (f_1, f_2, f_3)$ . The reader must keep in mind that, for boolean values, there are the equalities

$$\min(x, y) = x \text{ and } y, \max(x, y) = x \text{ or } y.$$

If  $x_1 = 1$  and  $x_3 = 1$ , or  $x_1 = 1$  and  $x_2 = 1$  and  $x_3 = 1$ , then  $f_1(x_1, x_2, x_3) = 1$ . Thus

$$f_1(x_1, x_2, x_3) = \max(\min(x_1, x_3), \min(x_1, x_2, x_3)).$$

If  $x_1 = 1$  and  $x_2 = 1$ , or  $x_2 = 1$  and  $x_3 = 1$ , or  $x_1 = 1$  and  $x_2 = 1$  and  $x_3 = 1$ , then  $f_2(x_1, x_2, x_3) = 1$ . Thus

$$f_2(x_1, x_2, x_3) = \max(\min(x_1, x_2), \min(x_2, x_3), \min(x_1, x_2, x_3)).$$

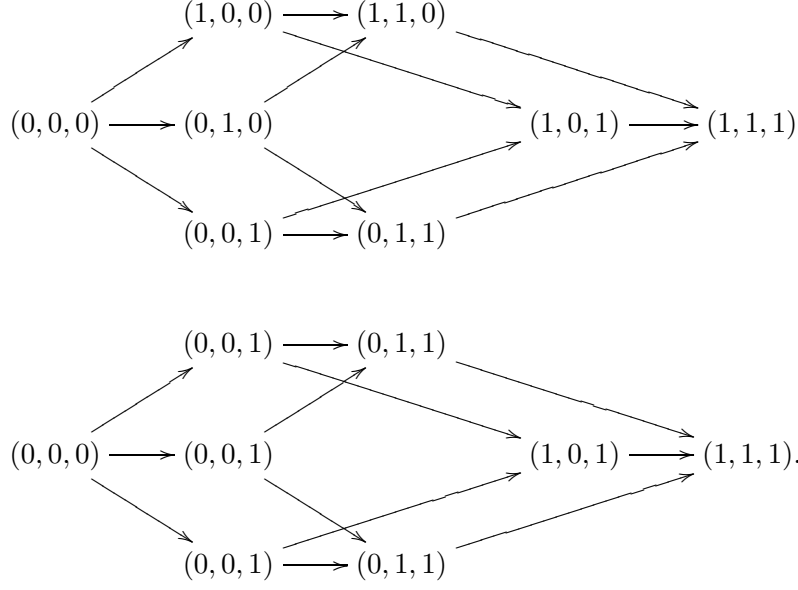


FIGURE 2. The cotransverse map  $f : [3] \rightarrow [3]$

Finally, if  $x_1 = 1$  and  $x_2 = 1$ , or  $x_1 = 1$  and  $x_3 = 1$ , or  $x_2 = 1$  and  $x_3 = 1$ , or  $x_1 = 1$  and  $x_2 = 1$  and  $x_3 = 1$ , then  $f_3(x_1, x_2, x_3) = 1$ . Thus

$$f_3(x_1, x_2, x_3) = \max(\min(x_1, x_2), \min(x_1, x_3), \min(x_2, x_3), \min(x_1, x_2, x_3)).$$

**4.2. Definition.** Let  $f = (f_1, \dots, f_n) : [n] \rightarrow [n]$  be a cotransverse map. Let

$$\mathbb{T}(f) : [0, 1]^n \rightarrow [0, 1]^n$$

be the set map defined by

$$\mathbb{T}(f)(x_1, \dots, x_n) = (\mathbb{T}(f)_1(x_1, \dots, x_n), \dots, \mathbb{T}(f)_n(x_1, \dots, x_n))$$

with

$$\mathbb{T}(f)_i(x_1, \dots, x_n) = \max_{(\epsilon_1, \dots, \epsilon_n) \in f_i^{-1}(1)} \min\{x_k \mid \epsilon_k = 1\}$$

for all  $1 \leq i \leq n$ .

**4.3. Proposition.** Let  $n \geq 1$ . For all  $x \in [n] \subset [0, 1]^n$ , one has  $\mathbb{T}(f)(x) = f(x)$ .

*Proof.* It is a consequence of Proposition 4.1. □

**4.4. Proposition.** For all cotransverse maps  $f : [n] \rightarrow [n]$ , the set map

$$\mathbb{T}(f) : [0, 1]^n \rightarrow [0, 1]^n$$

is continuous and strictly increasing. Moreover it satisfies the properties

$$\forall (x_1, \dots, x_n) \in [0, 1]^n, h(x_1, \dots, x_n) = h(\mathbb{T}(f)(x_1, \dots, x_n)).$$

*Proof.* By Proposition 4.3 and Proposition 3.11, the theorem holds for  $(x_1, \dots, x_n) \in [n] \subset [0, 1]^n$ . From the fact that each projection map  $(x_1, \dots, x_n) \mapsto x_k$  from  $[0, 1]^n$  equipped with the product order to  $[0, 1]$  is continuous and nondecreasing, we deduce the first sentence. Consider a tuple  $(x_1, \dots, x_n) \in [0, 1]^n$ . There exists a permutation  $\sigma$  of

$\{1, \dots, n\}$  such that  $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$ . Using Proposition 3.11 again, write

$$\begin{aligned} f(\epsilon_{\{\sigma(1)\}}) &= \epsilon_{\{\sigma'(1)\}}, \\ f(\epsilon_{\{\sigma(1), \sigma(2)\}}) &= \epsilon_{\{\sigma'(1), \sigma'(2)\}}, \\ &\dots \\ f(\epsilon_{\{\sigma(1), \dots, \sigma(n)\}}) &= \epsilon_{\{\sigma'(1), \dots, \sigma'(n)\}}. \end{aligned}$$

From the permutation  $\sigma$  of  $\{1, \dots, n\}$ , we therefore obtain a new permutation  $\sigma'$  of  $\{1, \dots, n\}$ . One has  $\epsilon_{\{\sigma(1)\}} \in f_{\sigma'(1)}^{-1}(1)$ . It means that  $T(f)_{\sigma'(1)}(x_1, \dots, x_n) = x_{\sigma(1)}$  because  $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$ . One then has  $\epsilon_{\{\sigma(1), \sigma(2)\}} \in f_{\sigma'(1)}^{-1}(1)$ . It means that  $T(f)_{\sigma'(2)}(x_1, \dots, x_n) = x_{\sigma(2)}$  because  $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$ . By repeating a finitely number of times the same argument, we obtain the equality  $T(f)_{\sigma'(i)}(x_1, \dots, x_n) = x_{\sigma(i)}$  for all  $1 \leq i \leq n$ . It implies that  $T(f)(x_1, \dots, x_n) = (x_{\sigma\sigma'^{-1}(1)}, \dots, x_{\sigma\sigma'^{-1}(n)})$ . It means that

$$h(T(f)(x_1, \dots, x_n)) = x_{\sigma\sigma'^{-1}(1)} + \dots + x_{\sigma\sigma'^{-1}(n)} = h(x_1, \dots, x_n),$$

the first equality by definition of  $h$  and the second equality since  $\sigma\sigma'^{-1}$  is a permutation of  $\{1, \dots, n\}$ .  $\square$

**4.5. Proposition.** *Let  $f : [n] \rightarrow [n]$  and  $g : [n] \rightarrow [n]$  be two cotransverse maps. Then there is the equality*

$$T(fg) = T(f)T(g).$$

*Proof.* Consider a tuple  $(x_1, \dots, x_n) \in [0, 1]^n$ . We want to prove that

$$T(fg)(x_1, \dots, x_n) = T(f)T(g)(x_1, \dots, x_n).$$

We use the calculation made in the proof of Proposition 4.4. Let  $\sigma$  be a permutation of  $\{1, \dots, n\}$  such that  $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$ . Using Proposition 3.11, write

$$\begin{aligned} g(\epsilon_{\{\sigma(1)\}}) &= \epsilon_{\{\sigma'(1)\}}, \\ g(\epsilon_{\{\sigma(1), \sigma(2)\}}) &= \epsilon_{\{\sigma'(1), \sigma'(2)\}}, \\ &\dots \\ g(\epsilon_{\{\sigma(1), \dots, \sigma(n)\}}) &= \epsilon_{\{\sigma'(1), \dots, \sigma'(n)\}} \end{aligned}$$

for some permutation  $\sigma'$  of  $\{1, \dots, n\}$ . There is the equality

$$T(g)(x_1, \dots, x_n) = (x_{\sigma\sigma'^{-1}(1)}, \dots, x_{\sigma\sigma'^{-1}(n)}).$$

One has  $y_{\sigma'(1)} \geq \dots \geq y_{\sigma'(n)}$  because  $y_{\sigma'(i)} = x_{\sigma(i)}$  for all  $1 \leq i \leq n$ . Using Proposition 3.11 again, write

$$\begin{aligned} f(\epsilon_{\{\sigma'(1)\}}) &= \epsilon_{\{\sigma''(1)\}}, \\ f(\epsilon_{\{\sigma'(1), \sigma'(2)\}}) &= \epsilon_{\{\sigma''(1), \sigma''(2)\}}, \\ &\dots \\ f(\epsilon_{\{\sigma'(1), \dots, \sigma'(n)\}}) &= \epsilon_{\{\sigma''(1), \dots, \sigma''(n)\}} \end{aligned}$$

for some permutation  $\sigma''$  of  $\{1, \dots, n\}$ . There is the equality

$$T(f)(y_1, \dots, y_n) = (y_{\sigma'\sigma''^{-1}(1)}, \dots, y_{\sigma'\sigma''^{-1}(n)}) = (x_{\sigma\sigma''^{-1}(1)}, \dots, x_{\sigma\sigma''^{-1}(n)}),$$

the left-hand equality by the calculation made in the proof of Proposition 4.4, the right-hand equality by definition of  $y_i$ . Since we have

$$\begin{aligned} fg(\epsilon_{\{\sigma(1)\}}) &= f(\epsilon_{\{\sigma'(1)\}}) = \epsilon_{\{\sigma''(1)\}}, \\ fg(\epsilon_{\{\sigma(1),\sigma(2)\}}) &= f(\epsilon_{\{\sigma'(1),\sigma'(2)\}}) = \epsilon_{\{\sigma''(1),\sigma''(2)\}}, \\ &\dots \\ fg(\epsilon_{\{\sigma(1),\dots,\sigma(n)\}}) &= f(\epsilon_{\{\sigma'(1),\dots,\sigma'(n)\}}) = \epsilon_{\{\sigma''(1),\dots,\sigma''(n)\}}, \end{aligned}$$

we obtain using the calculation made in the proof of Proposition 4.4 that

$$T(f)T(g)(x_1, \dots, x_n) = T(f)(y_1, \dots, y_n) = T(fg)(x_1, \dots, x_n).$$

□

**4.6. Notation.** For  $\delta_i^\alpha : [n-1] \rightarrow [n] \in \square$ , let

$$T(\delta_i^\alpha) = \begin{cases} [0, 1]^{n-1} \rightarrow [0, 1]^n \\ (\epsilon_1, \dots, \epsilon_{n-1}) \mapsto (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1}) \end{cases}$$

for all  $n \geq 1$  and  $\alpha \in \{0, 1\}$ .

**4.7. Proposition.** Let  $f : [n] \rightarrow [p]$  and  $g : [m] \rightarrow [n]$  be two cotransverse maps with  $f \in \square$  or  $g \in \square$ . Then there is the equality

$$T(fg) = T(f)T(g).$$

*Proof.* It is well known if both  $f$  and  $g$  belong to  $\square$ . If only one of the two maps  $f$  or  $g$  belongs to  $\square$ , we use Definition 4.2 of  $T(f)$  or  $T(g)$  for the map not belonging to  $\square$  and we add 0 or 1 to the other coordinates, depending on the coface map. □

**4.8. Proposition.** Let  $f : [n] \rightarrow [p]$  and  $g : [m] \rightarrow [n]$  be two cotransverse maps. Then there is the equality

$$T(fg) = T(f)T(g).$$

*Proof.* Consider the commutative diagram of  $\widehat{\square}$  (the vertical maps are coface maps)

$$\begin{array}{ccccc} [m] & \xrightarrow{g} & [n] & \xrightarrow{f} & [p] \\ \parallel & & \uparrow \delta & & \uparrow \delta' \\ [m] & \xrightarrow{g'} & [m] & \xrightarrow{f'} & [m] \end{array}$$

where the factorizations  $g = \delta g'$  and  $f = \delta' f'$  are given by Proposition 3.5. Then there is the sequence of equalities (by repeatedly using Proposition 4.5 and Proposition 4.7)

$$\begin{aligned} T(fg) &= T(\delta' f' g') = T(\delta')T(f' g') = T(\delta')T(f')T(g') \\ &= T(\delta' f')T(g') = T(f\delta)T(g') = T(f)T(\delta)T(g') = T(f)T(\delta g') = T(f)T(g). \end{aligned}$$

□

4.9. **Theorem.** *The mappings*

$$\begin{aligned} [n] &\mapsto [0, 1]^n \text{ for all } n \geq 0 \\ f : [n] &\rightarrow [n] \in \widehat{\square} \mapsto T(f) \text{ for all } n \geq 1 \\ \delta_i^\alpha : [n-1] &\rightarrow [n] \mapsto T(\delta_i^\alpha) \text{ for all } n \geq 1 \end{aligned}$$

give rise to a cotransverse topological space called the cotransverse topological cube and denoted by  $|\widehat{\square}[*]|_{geom}$ .

*Proof.* The functoriality is a consequence of Proposition 4.8.  $\square$

Proposition 3.8 and Theorem 4.9 lead to the following definition:

4.10. **Definition.** *Let  $K$  be a transverse set. Let*

$$|K|_{geom} = \int^{[n] \in \widehat{\square}} K_n \cdot |\widehat{\square}[n]|_{geom}$$

*This gives rise to a colimit-preserving functor  $|-|_{geom} : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Top}$ .*

One has  $|\widehat{\square}[n]|_{geom} \cong |\square[n]|_{geom}$  for all  $n \geq 0$  where  $|\square[n]|_{geom}$  is the geometric realization of the precubical set  $\square[n]$  [15, Notation 4.1]. Since all involved functors are colimit-preserving, one obtains the natural homeomorphism  $|\widehat{\mathcal{L}}K|_{geom} \cong |K|_{geom}$  for all precubical sets  $K$ . By Proposition 3.15, we deduce the natural homeomorphism  $|\partial\widehat{\square}[n]|_{geom} \cong |\partial\square[n]|_{geom}$  for all  $n \geq 0$ . For all  $n \geq 0$ , there is the pushout diagram of topological spaces

$$\begin{array}{ccc} \coprod_{x \in K_{n+1}} |\partial\widehat{\square}[n+1]|_{geom} & \xrightarrow{\coprod_{x \in K_{n+1}} |x|_{geom}} & |K_{\leq n}|_{geom} \\ \downarrow & & \downarrow \\ \coprod_{x \in K_{n+1}} |\widehat{\square}[n+1]|_{geom} & \longrightarrow & |K_{\leq n+1}|_{geom} \end{array}$$

It implies that the topological space  $|K|_{geom}$  is a CW-complex for all transverse sets  $K$ , and therefore that it is  $\Delta$ -generated and Hausdorff.

4.11. **Notation.** *Let  $|\widehat{\square}[n]|_{\vec{d}_1}$  be the Lawvere metric space  $([0, 1]^n, \vec{d}_1)$  for all  $n \geq 0$ .*

There is a topological version of Proposition 3.12:

4.12. **Proposition.** *Let  $n \geq 1$ . Let  $f : [n] \rightarrow [n]$  be a cotransverse map. Then  $T(f) : [0, 1]^n \rightarrow [0, 1]^n$  yields a map of Lawvere metric spaces from  $|\widehat{\square}[n]|_{\vec{d}_1}$  to itself which is quasi-isometric.*

*Proof.* Let  $x, y \in [0, 1]^n$ . Suppose first that  $x \leq y$  are comparable. Then  $h(x) \leq h(y)$ . By Proposition 4.4, there is the inequality  $h(T(f)(x)) = h(x) \leq h(y) = h(T(f)(y))$ . We deduce that  $\vec{d}_1(x, y) = h(y) - h(x) = h(T(f)(y)) - h(T(f)(x)) = d_1(T(f)(x), T(f)(y))$ , the first equality by definition of  $\vec{d}_1$ , the second equality by the previous remark, and the last equality by definition of  $\vec{d}_1$  and since  $T(f)$  is strictly increasing. Now suppose

that  $x \leq y$  is false. It means that  $\vec{d}_1(x, y) = \infty$ . It implies that  $\vec{d}_1(T(f)(x), T(f)(y)) \leq \vec{d}_1(x, y)$ . Thus,  $T(f) : [0, 1]^n \rightarrow [0, 1]^n$  is a map of Lawvere metric spaces.  $\square$

**4.13. Corollary.** *Let  $f : [m] \rightarrow [n]$  be a cotransverse map. The induced map  $T(f) : |\widehat{\square}[m]|_{\vec{d}_1} \rightarrow |\widehat{\square}[n]|_{\vec{d}_1}$  is a map of Lawvere metric spaces which is also quasi-isometric.*

This leads to the theorem:

**4.14. Theorem.** *The mappings*

$$\begin{aligned} [n] &\mapsto [0, 1]^n \text{ for all } n \geq 0 \\ f : [n] \rightarrow [n] \in \widehat{\square} &\mapsto T(f) \text{ for all } n \geq 1 \\ \delta_i^\alpha : [n-1] \rightarrow [n] &\mapsto T(\delta_i^\alpha) \text{ for all } n \geq 1 \end{aligned}$$

*give rise to a cotransverse Lawvere metric space called the cotransverse Lawvere cube and denoted by  $|\widehat{\square}[*]|_{\vec{d}_1}$ .*

Proposition 3.8 and Theorem 4.14 lead to the following definition:

**4.15. Definition.** *Let  $K$  be a transverse set. Let*

$$|K|_{\vec{d}_1} = \int^{[n] \in \widehat{\square}} K_n \cdot |\widehat{\square}[n]|_{\vec{d}_1}.$$

*This gives rise to a colimit-preserving functor  $|-|_{\vec{d}_1} : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{LvMet}$ .*

## 5. THE C-REEDY MODEL STRUCTURE OF COTRANSVERSE OBJECTS

In this section,  $\mathcal{M}$  is a model category such that the projective model structure on  $\mathcal{M}^{\widehat{\square}=n}$  exists for all  $n \geq 0$  where  $\widehat{\square}=n$  is the full subcategory of  $\widehat{\square}$  having one object  $[n]$  (see Proposition 5.2). It is the case if  $\mathcal{M}$  is an accessible model category in the sense of [26, Definition 5.1] or [18, Definition 3.1.6] by [18, Theorem 3.4.1] or a cofibrantly generated model category by [19, Theorem 11.6.1].

We want to give a necessary and sufficient condition for a cotransverse object of  $\mathcal{M}$  to be projective cofibrant in Theorem 5.10. The latter theorem is the generalization of [9, Proposition 2.3.1] from the setting of precubical sets to the one of transverse sets. The key fact used in [9] is that the small category  $\square$  is a direct Reedy category, which implies that the projective model structure on cocubical objects exists and that it coincides with the Reedy model structure for all model categories. The small category  $\widehat{\square}$  is not Reedy, not even Reedy in Berger-Moerdijk's sense [3, Definition 1.1] or in Cisinski's sense [5, Definition 8.1.1] because the factorization of a map is not unique up to isomorphism; however the category of factorizations of a map has a final object by Proposition 5.3. The small category  $\widehat{\square}$  turns out to be c-Reedy in Shulman's sense [27, Definition 8.25].

**5.1. Notation.** *Let*

$$\begin{aligned} \vec{\square} &= \widehat{\square} \\ \overleftrightarrow{\square} &= \overleftarrow{\square} = \coprod_{n \geq 0} \{f : [n] \rightarrow [n] \mid f \in \widehat{\square}\} \end{aligned}$$

*We consider the degree function  $d([n]) = n$  for all  $n \geq 0$ .*

Every morphism  $f : [m] \rightarrow [n]$  of  $\widehat{\square}$  is basic in the sense of [27, Definition 6.12] since every factorization of  $f$  as a composite  $[m] \rightarrow [p] \rightarrow [n]$  implies that  $m \leq p \leq n$ , and therefore that every factorization is not fundamental in the sense of [27, Definition 6.12]:  $p < \min(m, n) = m$  is impossible indeed. Hence the following proposition:

**5.2. Proposition.** *Let  $n \geq 0$ . The subcategory  $\widehat{\square}_{=n}$  of  $\widehat{\square}$  generated by the objects of degree  $n$  and the basic morphisms in the sense of [27, Definition 6.12] between them is the full subcategory of  $\widehat{\square}$  having one object  $[n]$ . In particular, one has*

$$\widehat{\square}_{=n}([n], [n]) = \widehat{\square}([n], [n]).$$

**5.3. Proposition.** *Let  $f$  be a map of  $\widehat{\square}$ . Consider the category of factorizations of  $f$ : its objects are the pairs of morphisms  $(h, g)$  such that  $hg = f$  with  $h \in \overrightarrow{\square}$  and  $g \in \overleftarrow{\square}$  and its morphisms  $k : (h, g) \rightarrow (h', g')$  are morphisms  $k$  (which are called connecting morphisms) such that there is a commutative diagram*

$$\begin{array}{ccccc} \bullet & \xrightarrow{g} & \bullet & \xrightarrow{h} & \bullet \\ \parallel & & \uparrow k & & \parallel \\ \bullet & \xrightarrow{g'} & \bullet & \xrightarrow{h'} & \bullet \end{array}$$

*Note that  $k$  is necessarily degree-preserving. The category of factorizations of  $f$  has a final object.*

*Proof.* Let  $f$  be a map of  $\widehat{\square}$ . Consider the factorization  $(h, g)$  given by Proposition 3.5: in particular,  $h \in \square$ . Consider another factorization  $(h', g')$  of  $f$ . Consider the commutative diagram of solid arrows of  $\widehat{\square}$

$$\begin{array}{ccccc} \bullet & \xrightarrow{g} & \bullet & \xrightarrow{h} & \bullet \\ \parallel & & \uparrow k & & \parallel \\ \bullet & \xrightarrow{g'} & \bullet & \xrightarrow{h'} & \bullet \end{array}$$

Proposition 3.5 yields the factorization  $h' = h''k$  with  $h'' \in \square$  and  $k$  which are unique. We obtain  $hg = h'g' = h''kg'$ . By uniqueness of the factorization of  $f=hg$  given by Proposition 3.5, we obtain  $h = h''$  and  $g = kg'$ , and therefore  $h' = hk$ .  $\square$

**5.4. Proposition.** *The small category  $\widehat{\square}$  is a c-Reedy category in the sense of [27, Definition 8.25].*

*Proof.* One has  $\overleftrightarrow{\square} \subset \overrightarrow{\square} \cap \overleftarrow{\square}$  (first axiom). Every morphism of  $\overleftrightarrow{\square}$  is degree-preserving (second axiom). Every morphism of  $\overrightarrow{\square} \setminus \overleftrightarrow{\square}$  strictly raises degree and every morphism of  $\overleftarrow{\square} \setminus \overleftrightarrow{\square} = \emptyset$  strictly lowers degree (third axiom). The category of factorizations of  $f$  with connecting maps in  $\overleftrightarrow{\square}$  is connected by Proposition 5.3 (fourth axiom). For every  $n \geq 0$ , and any degree  $m < n$ , the functor  $\overleftarrow{\square}([n], -) : \widehat{\square}_{=m} \rightarrow \mathbf{Set}$  is an (empty) coproduct of retracts of representables (fifth axiom).  $\square$

**5.5. Notation.** *Let  $\widehat{\square}_{<n}$  be the full category of  $\widehat{\square}$  containing the objects  $[0], \dots, [n-1]$  <sup>2</sup>.*

<sup>2</sup>This category should be denoted by  $\widehat{\square}_n$  with the notation of [27]; I find this notation a bit confusing.



5.6. **Notation.** Let  $n \geq 0$ . Following the notations of [27, page 37], let

$$\partial_n \widehat{\square}([p], [q]) = \int^{[m] \in \widehat{\square}_{<n}} \widehat{\square}([m], [q]) \times \widehat{\square}([p], [m])$$

The latching and matching object functors  $L_n, M_n : \mathcal{M}^{\widehat{\square}} \rightarrow \mathcal{M}^{\widehat{\square}=n}$  are given by

$$(M_n A)_{[n]} = \int_{[m] \in \widehat{\square}} A([m])^{\partial_n \widehat{\square}([n], [m])}$$

$$(L_n A)_{[n]} = \int^{[p] \in \widehat{\square}} \partial_n \widehat{\square}([p], [n]) \cdot A([p])$$

We obtain:

5.7. **Theorem.** Suppose that the projective model structure on  $\mathcal{M}^{\widehat{\square}=n}$  exists for all  $n \geq 0$ . There exists a unique model structure on  $\mathcal{M}^{\widehat{\square}}$  such that

- The weak equivalences are objectwise.
- A map  $A \rightarrow B$  of  $\mathcal{M}^{\widehat{\square}}$  is a fibration (trivial fibration resp.) if for all  $n \geq 0$ , the map  $A([n]) \rightarrow (M_n A)_{[n]} \sqcup_{(M_n B)_{[n]}} B([n])$  is a fibration (trivial fibration resp.) of  $\mathcal{M}$ .
- A map  $A \rightarrow B$  of  $\mathcal{M}^{\widehat{\square}}$  is a cofibration (trivial cofibration resp.) if for all  $n \geq 0$ ,  $L_n B \sqcup_{L_n A} A \rightarrow B$  is a projective cofibration (trivial cofibration resp.) of the projective model structure of  $\mathcal{M}^{\widehat{\square}=n}$ .

This model structure is called the c-Reedy model structure of  $\mathcal{M}^{\widehat{\square}}$ .

*Proof.* By Proposition 5.4 and [27, Theorem 8.26], the small category  $\widehat{\square}$  is almost c-Reedy in the sense of [27, Definition 8.8]. The proof is complete thanks to [27, Theorem 8.9].  $\square$

5.8. **Proposition.** One has

$$\partial_n \widehat{\square}([p], [q]) = \begin{cases} \emptyset & \text{if } p > q \text{ or } n \leq p \\ \widehat{\square}([p], [q]) & \text{if } p \leq q \text{ and } p < n \end{cases}$$

*Proof.* The composition induces a set map  $\widehat{\square}([m], [q]) \times \widehat{\square}([p], [m]) \rightarrow \widehat{\square}([p], [q])$ . If  $p > q$ , then  $\widehat{\square}([p], [q]) = \emptyset$ , which implies that  $\widehat{\square}([m], [q]) \times \widehat{\square}([p], [m]) = \emptyset$  for all  $[m] \in \widehat{\square}_{<n}$ . If  $n \leq p$ , then  $n - 1 < p$ . It means that for all  $[m] \in \widehat{\square}_n$ , one has  $\widehat{\square}([p], [m]) = \emptyset$ , which implies that  $\widehat{\square}([m], [q]) \times \widehat{\square}([p], [m]) = \emptyset$  for all  $[m] \in \widehat{\square}_{<n}$  as well. Assume now that  $p \leq q$  and  $p < n$ . The set  $\partial_n \widehat{\square}([p], [q])$  is the quotient of

$$\coprod_{m < n} \widehat{\square}([m], [q]) \times \widehat{\square}([p], [m])$$

by the equivalence relation generated by identifying two pairs  $(h, g)$  and  $(h', g')$  such that  $hg = h'g'$  related by a connecting map, i.e. such that there exists a commutative diagram of  $\widehat{\square}$  of the form

$$\begin{array}{ccccc} [p] & \xrightarrow{g} & \bullet & \xrightarrow{h} & [q] \\ & & \wedge & & \\ & & k \downarrow & & \\ [p] & \xrightarrow{g'} & \bullet & \xrightarrow{h'} & [q] \end{array}$$

Consider such a pair  $(h, g)$ . By applying Proposition 3.5 to  $g : [p] \rightarrow [m]$ , we obtain a commutative diagram of  $\widehat{\square}$  of the form

$$\begin{array}{ccccc} [p] & \xrightarrow{\in \widehat{\square}} & [p] & \xrightarrow{hk} & [q] \\ \parallel & & \downarrow k \in \square & & \parallel \\ [p] & \xrightarrow{g} & [m] & \xrightarrow{h} & [q] \end{array}$$

It means that in  $\partial_n \widehat{\square}([p], [q])$ , every element of  $\widehat{\square}([m], [q]) \times \widehat{\square}([p], [m])$  is equivalent to an element of  $\widehat{\square}([p], [q]) \times \widehat{\square}([p], [p])$ . Consider  $(h, g) \in \widehat{\square}([p], [q]) \times \widehat{\square}([p], [p])$ . By applying Proposition 3.5 to  $h$ , we obtain a commutative diagram of  $\widehat{\square}$  of the form

$$\begin{array}{ccccc} [p] & \xrightarrow{kg} & [p] & \xrightarrow{\in \square} & [q] \\ \parallel & & \uparrow k & & \parallel \\ [p] & \xrightarrow{g} & [p] & \xrightarrow{h} & [q]. \end{array}$$

It means that every element of  $\widehat{\square}([p], [q]) \times \widehat{\square}([p], [p])$  is equivalent in  $\partial_n \widehat{\square}([p], [q])$  to an element of  $\square([p], [q]) \times \widehat{\square}([p], [p])$ . It means that  $\partial_n \widehat{\square}([p], [q])$  is the quotient of  $\square([p], [q]) \times \widehat{\square}([p], [p])$  by the equivalence relation. If  $(h, g)$  and  $(h', g')$  are two equivalent elements of  $\square([p], [q]) \times \widehat{\square}([p], [p])$  in  $\partial_n \widehat{\square}([p], [q])$ , then it implies in particular that  $hg = h'g'$ . By the uniqueness of the factorization given by Proposition 3.5, it implies that  $h = h'$  and  $g = g'$ . We obtain

$$\partial_n \widehat{\square}([p], [q]) \cong \square([p], [q]) \times \widehat{\square}([p], [p]) \cong \widehat{\square}([p], [q]),$$

the first isomorphism since the equivalence relation on  $\square([p], [q]) \times \widehat{\square}([p], [p])$  restricts to the equality by the previous arguments, the second isomorphism by the uniqueness of Proposition 3.5.  $\square$

**5.9. Proposition.** *For all  $n \geq 0$ , there is the isomorphism of transverse sets*

$$\partial \widehat{\square}[n] \cong \int^{[p] \in \widehat{\square} < n} \widehat{\square}([p], [n]).\widehat{\square}[p]$$

*Proof.* This is due to the fact that  $\widehat{\square}([m], [n]) = \emptyset$  when  $m > n$ .  $\square$

**5.10. Theorem.** *Suppose that the projective model structure on  $\mathcal{M}^{\widehat{\square}=n}$  exists for all  $n \geq 0$ . Then the projective model structure on  $\mathcal{M}^{\widehat{\square}}$  exists and coincides with the c-Reedy model structure. Let  $A : \widehat{\square} \rightarrow \mathcal{M}$  be a cotransverse object of  $\mathcal{M}$ . It is projective cofibrant if and only if for all  $n \geq 0$ , the map  $\widehat{A}(\partial \widehat{\square}[-]) \rightarrow \widehat{A}(\widehat{\square}[-])$  is a projective cofibration of  $\mathcal{M}^{\widehat{\square}=n}$ .*

Note that  $\square$  is also a c-Reedy category since it is a Reedy category. In this case, there is the isomorphism of categories  $\mathbf{Flow} \cong \mathbf{Flow}^{\widehat{\square}=n}$  for all  $n \geq 0$  and we recover [9, Proposition 2.3.1] of the precubical setting.

*Proof.* The matching object functor  $M_n : \mathcal{M}^{\widehat{\square}} \rightarrow \mathcal{M}^{\widehat{\square}=n}$  for all  $n \geq 0$  can be calculated as follows. There is the sequence of isomorphisms of  $\mathcal{M}$

$$(M_n A)_{[n]} \cong \int_{[m] \in \widehat{\square}} A([m])^{\partial_n \widehat{\square}([n], [m])} \cong \int_{[m] \in \widehat{\square}} A([m])^{\emptyset} \cong \mathbf{1}$$

where  $\mathbf{1}$  is the terminal object of  $\mathcal{M}$ , the first isomorphism by definition of the matching object functor (Notation 5.6), the second isomorphism by Proposition 5.8, and the third isomorphism by definition of the weighted limit. Thus, the c-Reedy model structure of Theorem 5.7 on  $\mathcal{M}^{\widehat{\square}}$  coincides with the projective model structure which therefore exists. There is the sequence of isomorphisms of  $\mathcal{M}$

$$(L_n A)_{[n]} \cong \int^{[p] \in \widehat{\square}} \widehat{\square}([p], [n]).A([p]) \cong \int^{[p] \in \widehat{\square} < n} \widehat{\square}([p], [n]).A([p]) \cong \widehat{A}(\partial \widehat{\square}[n]),$$

the first isomorphism by definition of the latching object functor (Notation 5.6), the second isomorphism by definition of the weighted colimit and since  $\partial_n \widehat{\square}([p], [n]) = \emptyset$  for  $p \geq n$  by Proposition 5.8, and finally the third isomorphism by Proposition 5.9 and since  $\widehat{A}$  is colimit-preserving. By Theorem 5.7, the cotransverse object  $A$  is projective cofibrant if and only if for all  $n \geq 0$ , the map  $L_n A \rightarrow A$  is a projective cofibration of the projective model structure of  $\mathcal{M}^{\widehat{\square} = n}$ . Since  $A([n]) = \widehat{A}(\widehat{\square}[n])$  by definition of  $\widehat{A}$ , the proof is complete.  $\square$

## 6. REALIZING A TRANSVERSE SET AS A FLOW

**6.1. Definition.** [8, Definition 4.11] *A flow is a small semicategory enriched over the closed monoidal category  $(\mathbf{Top}, \times)$ . The corresponding category is denoted by  $\mathbf{Flow}$ .*

A flow  $X$  consists of a topological space  $\mathbb{P}X$  of *execution paths*, a discrete space  $X^0$  of *states*, two continuous maps  $s$  and  $t$  from  $\mathbb{P}X$  to  $X^0$  called the source and target map respectively, and a continuous and associative map  $* : \{(x, y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \rightarrow \mathbb{P}X$  such that  $s(x * y) = s(x)$  and  $t(x * y) = t(y)$ . Let  $\mathbb{P}_{\alpha, \beta} X = \{x \in \mathbb{P}X \mid s(x) = \alpha \text{ and } t(x) = \beta\}$ : it is the space of execution paths from  $\alpha$  to  $\beta$ ,  $\alpha$  is called the initial state and  $\beta$  is called the final state. Note that the composition is denoted by  $x * y$ , not by  $y \circ x$ . The category  $\mathbf{Flow}$  is locally presentable by [12, Theorem 6.11].

**6.2. Theorem.** [14, Theorem 7.4] *With  $r \in \{q, m, h\}$ . Then there exists a unique model structure on  $\mathbf{Flow}$  such that:*

- *A map of flows  $f : X \rightarrow Y$  is a weak equivalence if and only if  $f^0 : X^0 \rightarrow Y^0$  is a bijection and for all  $(\alpha, \beta) \in X^0 \times X^0$ , the continuous map  $\mathbb{P}_{\alpha, \beta} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$  is a weak equivalence of the  $r$ -model structure of  $\mathbf{Top}$ .*
- *A map of flows  $f : X \rightarrow Y$  is a fibration if and only if for all  $(\alpha, \beta) \in X^0 \times X^0$ , the continuous map  $\mathbb{P}_{\alpha, \beta} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$  is a fibration of the  $r$ -model structure of  $\mathbf{Top}$ .*

*This model structure is accessible and all objects are fibrant. It is called the  $r$ -model structure of  $\mathbf{Flow}$ .*

By [14, Theorem 7.7], the  $m$ -model structure is the mixing of the  $q$ -model structure and the  $h$ -model structure in the sense of [6, Theorem 2.1]. Every  $q$ -cofibration of flows is a  $m$ -cofibration and every  $m$ -cofibration of flows is a  $h$ -cofibration by [6, Proposition 3.6]. Note that all involved model categories being accessible, the projective and injective  $r$ -model structures on  $\mathbf{Flow}^I$  exists for all small categories  $I$  by [18, Theorem 3.4.1].

**6.3. Proposition.** *With  $r \in \{q, m, h\}$ . Let  $f : X \rightarrow Y$  be a weak equivalence of the  $r$ -model structure of flows between  $r$ -cofibrant flows. Then for all  $(\alpha, \beta) \in X^0 \times X^0$ , the*

continuous map  $\mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  is a homotopy equivalence between  $r$ -cofibrant spaces. In particular,  $f$  is a weak equivalence of the  $h$ -model structure of flows.

*Proof.* If  $r = h$ , then the map  $\mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  is a homotopy equivalence of spaces by definition of the weak equivalences of the  $h$ -model structure of flows. If  $r = q$ , then the spaces  $\mathbb{P}_{\alpha,\beta}X$  and  $\mathbb{P}_{f(\alpha),f(\beta)}Y$  are  $q$ -cofibrant by [13, Theorem 5.7]. Using Whitehead [19, Theorem 7.5.10], we deduce that the map  $\mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  is a homotopy equivalence of spaces. It remains the case  $r = m$ . The spaces  $\mathbb{P}_{\alpha,\beta}X$  and  $\mathbb{P}_{f(\alpha),f(\beta)}Y$  are  $m$ -cofibrant by [14, Theorem 8.7]. By [6, Corollary 3.4], we deduce that the weak homotopy equivalence  $\mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  is a homotopy equivalence of spaces as well.  $\square$

**6.4. Definition.** With  $r \in \{q, m, h\}$ . A functor  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  is a  $r$ -realization functor if it satisfies the following properties:

- $F$  is colimit-preserving.
- For all  $n \geq 0$ , the map of diagrams of flows  $F(\partial\widehat{\square}[n]) \rightarrow F(\widehat{\square}[n])$  is a  $r$ -cofibration of  $\mathbf{Flow}$ .
- There is an objectwise weak equivalence of cotransverse flows  $F(\widehat{\square}[*]) \rightarrow \{0 < 1\}^*$  in the  $r$ -model structure of  $\mathbf{Flow}$ .

**6.5. Lemma.** (well-known<sup>3</sup>) Let  $I$  be a small category. Let  $\mathcal{M}$  be a model category (not necessarily cofibrantly generated) such that both the projective model structure  $(\mathcal{M}^I)_{proj}$  and the injective model structure  $(\mathcal{M}^I)_{inj}$  exist. Then the identity of  $\mathcal{M}$  yields a left Quillen functor  $(\mathcal{M}^I)_{proj} \rightarrow (\mathcal{M}^I)_{inj}$ . In particular, every projective cofibration is an injective cofibration.

*Proof.* Let  $A \rightarrow B$  be a projective cofibration of  $\mathcal{M}^I$ . Let  $C \rightarrow D$  be a trivial fibration of  $\mathcal{M}$ . Let  $i \in I$ . Consider a commutative diagram of  $\mathcal{M}$  of the form

$$\begin{array}{ccc} A(i) & \longrightarrow & C \\ \downarrow & \nearrow \ell & \downarrow \\ B(i) & \longrightarrow & D \end{array}$$

The lift  $\ell$  exists if and only, by adjunction, the lift  $\bar{\ell}$  exists in the commutative diagram of  $\mathcal{M}^I$

$$\begin{array}{ccc} A & \longrightarrow & C^{I(-,i)} \\ \downarrow & \nearrow \bar{\ell} & \downarrow \\ B & \longrightarrow & D^{I(-,i)} \end{array}$$

The point is that  $C^{I(-,i)} \rightarrow D^{I(-,i)}$  is a projective trivial fibration. Thus the lift  $\bar{\ell}$  exists, and so does the lift  $\ell$ . We have proved that  $A \rightarrow B$  is an injective cofibration and the proof is complete.  $\square$

**6.6. Proposition.** With  $r \in \{q, m, h\}$ . Let  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  be a  $r$ -realization functor. Then for all transverse sets  $K$ , the flow  $F(K)$  is  $r$ -cofibrant and there is a natural bijection

<sup>3</sup>I am unable to find a textbook expounding this elementary result.

$K_0 \cong F(K)^0$ . A projective  $r$ -cofibrant replacement of the cotransverse flow  $F(\widehat{\square}[*])$ , let us denote it by  $F^{cof}(\widehat{\square}[*])$ , gives rise to a  $r$ -realization functor as well.

*Proof.* Let  $K$  be a transverse set. Let  $n \geq 0$ . There is the pushout diagram of transverse sets

$$\begin{array}{ccc} \coprod_{x \in K_{n+1}} \widehat{\square}[n+1]_{\leq n} & \xrightarrow{\coprod_{x \in K_{n+1}} x} & K_{\leq n} \\ \downarrow & & \downarrow \\ \coprod_{x \in K_{n+1}} \widehat{\square}[n+1] & \xrightarrow{\quad} & K_{\leq n+1} \end{array}$$

because  $\partial \widehat{\square}[n+1] = \widehat{\square}[n+1]_{\leq n}$  for all  $n \geq 0$ . By applying  $F$ , we see that the canonical map  $\emptyset \rightarrow F(K) = \varinjlim F(K_{\leq n})$  is a transfinite composition of  $r$ -cofibrations of flows. Hence,  $F(K)$  is  $r$ -cofibrant. From the objectwise weak equivalence of cocubical flows  $F(\widehat{\square}[*]) \rightarrow \{0 < 1\}^*$ , we deduce the objectwise bijection of cotransverse sets  $F(\widehat{\square}[*])^0 \cong \{0, 1\}^* \cong \widehat{\square}[*]_0$ . We obtain the natural bijection  $F(K)^0 \cong K_0$ . By Theorem 5.10, the map  $F^{cof}(\partial \widehat{\square}[-]) \rightarrow F^{cof}(\widehat{\square}[-])$  is a projective  $r$ -cofibration of  $\mathbf{Flow}^{\widehat{\square}=-n}$  for all  $n \geq 0$ . By Lemma 6.5, the map  $F^{cof}(\partial \widehat{\square}[-]) \rightarrow F^{cof}(\widehat{\square}[-])$  is an injective  $r$ -cofibration of  $\mathbf{Flow}^{\widehat{\square}=-n}$  for all  $n \geq 0$ . Since the composite map  $F^{cof}(\widehat{\square}[*]) \rightarrow F(\widehat{\square}[*]) \rightarrow \{0 < 1\}^*$  is a weak equivalence in the projective  $r$ -model structure of  $\mathbf{Flow}^{\widehat{\square}}$ , the proof is complete.  $\square$

**6.7. Theorem.** *With  $r \in \{q, m, h\}$ . Let  $F_1, F_2 : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  be two  $r$ -realization functors. Suppose that there exists a commutative diagram of cotransverse flows*

$$\begin{array}{ccc} F_1(\widehat{\square}[*]) & \longrightarrow & F_2(\widehat{\square}[*]) \\ \downarrow & & \downarrow \\ \{0 < 1\}^* & \xlongequal{\quad} & \{0 < 1\}^* \end{array}$$

*Then for all transverse sets  $K$ , the natural map of flows  $F_1(K) \rightarrow F_2(K)$  is a weak equivalence of the  $r$ -model structure of  $\mathbf{Flow}$  between  $r$ -cofibrant flows. Moreover, for all  $(\alpha, \beta) \in K_0 \times K_0$ , the continuous map  $\mathbb{P}_{\alpha, \beta} F_1(K) \rightarrow \mathbb{P}_{\alpha, \beta} F_2(K)$  is a homotopy equivalence between  $r$ -cofibrant topological spaces.*

*Proof.* By the two-out-of-three property, for all  $n \geq 0$ , the map of flows  $F_1(\widehat{\square}[n]) \rightarrow F_2(\widehat{\square}[n])$  is a weak equivalence of the  $r$ -model structure of  $\mathbf{Flow}$ . By Proposition 3.8, the hypotheses yield a natural transformation  $\mu : F_1 \Rightarrow F_2$ . Let us prove by induction on  $n \geq 0$  that the canonical map  $F_1(K_{\leq n}) \rightarrow F_2(K_{\leq n})$  is a weak equivalence of the  $r$ -model structure of flows between  $r$ -cofibrant flows for all transverse sets  $K$ . The case  $n = 0$  is a consequence of Proposition 6.6. Let  $n \geq 0$ . There is the pushout diagram of transverse

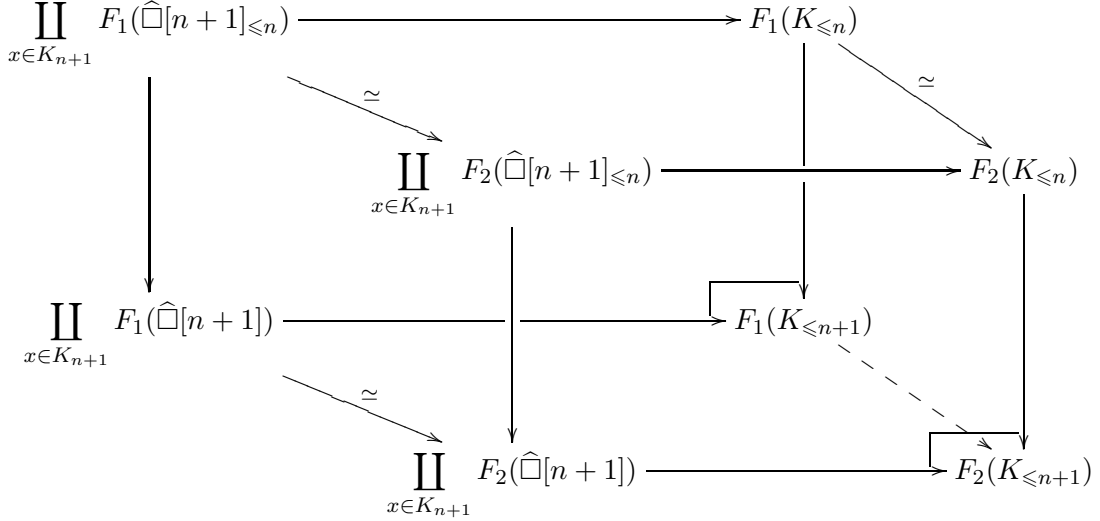
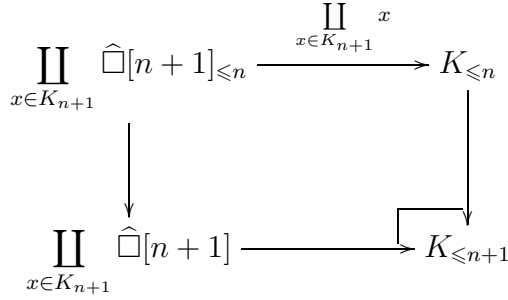


FIGURE 3. From  $n$  to  $n + 1$

sets



because  $\partial\widehat{\square}[n+1] = \widehat{\square}[n+1]_{\leq n}$  for all  $n \geq 0$ . Using the existence of the natural transformation  $F_1 \Rightarrow F_2$ , the passage from  $n$  to  $n + 1$  can be depicted by the following diagram of flows of Figure 3. By the induction hypothesis, the maps of flows  $F_1(\widehat{\square}[n+1]_{\leq n}) \rightarrow F_2(\widehat{\square}[n+1]_{\leq n})$  and  $F_1(K_{\leq n}) \rightarrow F_2(K_{\leq n})$  are weak equivalences of the r-model structure of flows between r-cofibrant flows. We have already seen above that the map of flows  $F_1(\widehat{\square}[n+1]) \rightarrow F_2(\widehat{\square}[n+1])$  is also a weak equivalence of the r-model structure of flows. By definition of a r-realization functor, we can apply the cube lemma [19, Proposition 15.10.10] [20, Lemma 5.2.6] in the r-model structure of **Flow** to conclude that the map  $F_1(K_{\leq n+1}) \rightarrow F_2(K_{\leq n+1})$  is a weak equivalence of the r-model structure of **Flow** between r-cofibrant flows. Since the colimits  $\varinjlim F_1(K_{\leq n})$  and  $\varinjlim F_2(K_{\leq n})$  are colimits of towers of r-cofibrations between r-cofibrant flows, they are homotopy colimits by [19, Proposition 15.10.12]. We conclude that the map of flows  $F_1(K) \rightarrow F_2(K)$  is a weak equivalence of the r-model structure of **Flow** between r-cofibrant flows. The proof is complete thanks to Proposition 6.3.  $\square$

**6.8. Theorem.** *With  $r \in \{q, m, h\}$ . Consider two r-realization functors*

$$F_1, F_2 : \widehat{\square}^{op}\mathbf{Set} \longrightarrow \mathbf{Flow}.$$

*Then there exists a r-realization functor  $F_3$  and a zigzag of natural transformations*

$$F_1 \leftarrow F_3 \Rightarrow F_2$$

such that there is a commutative diagram of cotransverse flows

$$\begin{array}{ccccc}
F_1(\widehat{\square}[*]) & \longleftarrow & F_3(\widehat{\square}[*]) & \longrightarrow & F_2(\widehat{\square}[*]) \\
\downarrow & & \downarrow & & \downarrow \\
\{0 < 1\}^* & \longequal{\quad} & \{0 < 1\}^* & \longequal{\quad} & \{0 < 1\}^*
\end{array}$$

and such that for all transverse sets  $K$ , the maps  $F_3(K) \rightarrow F_1(K)$  and  $F_3(K) \rightarrow F_2(K)$  natural with respect to  $K$  are weak equivalences of the  $r$ -model structure of  $\mathbf{Flow}$  between  $r$ -cofibrant flows. Moreover, for all  $(\alpha, \beta) \in K_0 \times K_0$ , the natural maps  $\mathbb{P}_{\alpha, \beta} F_3(K) \xrightarrow{\simeq} \mathbb{P}_{\alpha, \beta} F_1(K)$  and  $\mathbb{P}_{\alpha, \beta} F_3(K) \xrightarrow{\simeq} \mathbb{P}_{\alpha, \beta} F_2(K)$  are homotopy equivalences between  $r$ -cofibrant topological spaces.

*Proof.* Consider a projective  $r$ -cofibrant replacement  $F_3$  of  $F_1$  in the projective  $r$ -model structure of  $\mathbf{Flow}^{\widehat{\square}}$ . By Proposition 6.6,  $F_3$  is a  $r$ -realization functor. Consider the diagram of solid arrows of  $\mathbf{Flow}^{\widehat{\square}}$

$$\begin{array}{ccc}
F_3(\widehat{\square}[*]) & \xrightarrow{-\mu-} & F_2(\widehat{\square}[*]) \\
\downarrow & & \downarrow \\
F_1(\widehat{\square}[*]) & \longrightarrow & \{0 < 1\}^*
\end{array}$$

Since all spaces of execution paths of  $\{0 < 1\}^*$  are discrete, the right vertical map is a trivial projective  $r$ -fibration of  $\mathbf{Flow}^{\widehat{\square}}$ . Thus, there exists a map of cotransverse flows  $\mu : F_3(\widehat{\square}[*]) \rightarrow F_2(\widehat{\square}[*])$  making commutative the diagram above. The proof is complete thanks to Theorem 6.7.  $\square$

Note the difference with the precubical case of [15, Theorem 3.8]. There is, in general, in the transverse setting, no natural transformation from  $F_1$  to  $F_2$ . The point is that, in the precubical setting, the category  $\square_{=n}$  (see Proposition 5.2) is the terminal category for all  $n \geq 0$ . Thus, in the precubical setting, every  $r$ -realization functor corresponds to a projective  $r$ -cofibrant cocubical flow. It is not true in the transverse setting since the set  $\widehat{\square}([n], [n])$  is not a singleton for all  $n \geq 2$ .

**6.9. Proposition.** *Every  $q$ -realization functor is a  $m$ -realization functor. Every  $m$ -realization functor is a  $h$ -realization functor.*

*Proof.* Every  $q$ -realization functor is a  $m$ -realization functor because every  $q$ -cofibration of flows is a  $m$ -cofibration of flows by [14, Proposition 7.8] and because the weak equivalences are the same in the two model structures. Let  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  be a  $m$ -realization functor. Then for all  $n \geq 0$ , the map of flows  $F(\partial\widehat{\square}[n]) \rightarrow F(\widehat{\square}[n])$  is a  $h$ -cofibration by [6, Proposition 3.6]. The map of flows  $F(\widehat{\square}[n]) \rightarrow \{0 < 1\}^n$  is a weak equivalence of the  $h$ -model structure of flows by Proposition 6.3. We have proved that  $F$  is a  $h$ -realization functor.  $\square$

**6.10. Theorem.** *There exists a  $q$ -realization functor  $|-|_q : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$ .*

*Proof.* Let  $(-)^{cof}$  be a q-cofibrant replacement functor of **Flow**. Let

$$|K|_q = \int^{[n] \in \widehat{\square}} K_n \cdot (\{0 < 1\}^n)^{cof}$$

By [11, Proposition 2.2.10], there is the isomorphism  $|K|_q \cong |\widehat{\mathcal{L}}K|_q$  for all precubical sets  $K$  where the left-hand term is the q-realization of the precubical set  $K$  with the same q-cofibrant replacement functor. Using Proposition 3.15, we deduce the isomorphisms of flows  $|\square[n]|_q \cong |\widehat{\square}[n]|_q$  and  $|\partial\square[n]|_q \cong |\partial\widehat{\square}[n]|_q$  for all  $n \geq 0$ . The proof is complete thanks to [10, Proposition 7.4].  $\square$

## 7. NATURAL $d$ -PATH OF A TRANSVERSE SET

It is necessary to consider the symmetric version of the notion of Lawvere metric space to obtain a convenient notion of the underlying topological space of a Lawvere metric space.

**7.1. Definition.** A pseudometric space  $(X, d)$  is a set  $X$  equipped with a map  $d : X \times X \rightarrow [0, \infty]$  called a pseudometric such that:

- $\forall x \in X, d(x, x) = 0$
- $\forall (x, y) \in X \times X, d(x, y) = d(y, x)$  (symmetry axiom)
- $\forall (x, y, z) \in X \times X \times X, d(x, y) \leq d(x, z) + d(z, y)$ .

A map  $f : (X, d) \rightarrow (Y, d)$  of pseudometric spaces is a set map  $f : X \rightarrow Y$  which is short, i.e.  $\forall (x, y) \in X \times X, d(f(x), f(y)) \leq d(x, y)$ . The category of pseudometric spaces is denoted by **PseudoMet**.

The family of balls  $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$  of a pseudometric space  $(X, d)$  with  $x \in X$  and  $\epsilon > 0$  generates a topology called the underlying topology of  $(X, d)$ . This construction gives rise to a functor from pseudometric spaces to general topological spaces which is not colimit-preserving by [16, Remark 3.30]. The category of pseudometric spaces is bicomplete, being a reflective full subcategory of the bicomplete category of Lawvere metric spaces by [16, Proposition 3.21]<sup>4</sup>. Start from a Lawvere metric space  $(X, d)$ . The image by the reflection is the pseudometric space  $(X, d^\wedge)$  defined for all  $(x, y) \in X \times X$  by

$$d^\wedge(x, y) = \min_{n \geq 0} \min_{x=x_0, x_1, \dots, x_{n+2}=y} \left( \sum_{i=0}^n d(x_{i+1}, x_i) + d(x_{i+1}, x_{i+2}) \right)$$

With  $(x_0, x_1, x_2) = (x, x, y)$ , we obtain  $d(x, y) = d(x, x) + d(x, y) \geq d^\wedge(x, y)$  for all  $(x, y) \in X \times X$ .

**7.2. Definition.** The underlying topological space of a Lawvere metric space  $(X, d)$  is by definition the underlying topological space of the pseudometric space  $(X, d^\wedge)$ .

**7.3. Notation.** Let  $K$  be a transverse set. The underlying set of  $|K|_{\vec{d}_1}$  equipped with the pseudometric  $\vec{d}_1^\wedge$  gives rise to a pseudometric space denoted by  $|K|_{d_1^\wedge}$ . One has

$$|K|_{d_1^\wedge} \cong \int^{[n] \in \widehat{\square}} K_n \cdot |\widehat{\square}[n]|_{d_1^\wedge}.$$

<sup>4</sup>Note that there is a typo in the statement of [16, Proposition 3.21] which is corrected here.



This gives rise to a colimit-preserving functor  $|-|_{\vec{d}_1^\wedge} : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{PseudoMet}$ . The underlying topological space of the pseudometric space  $|K|_{d_1^\wedge}$  is denoted by  $|K|_{d_1}$ .

**7.4. Notation.** Let  $n \geq 1$ . Let  $(x_1, \dots, x_n), (x'_1, \dots, x'_n) \in [0, 1]^n$ . Let

$$d_1((x_1, \dots, x_n), (x'_1, \dots, x'_n)) = \sum_{i=1}^n |x_i - x'_i|.$$

**7.5. Proposition.** Let  $n \geq 1$ . For all  $x, y \in [0, 1]^n$ , there is the equality

$$\vec{d}_1^\wedge(x, y) = d_1(x, y).$$

*Proof.* By definition,  $\vec{d}_1^\wedge(x, y)$  is the minimum of the sums of the form

$$\left(\vec{d}_1(x_1, x_0) + \vec{d}_1(x_1, x_2)\right) + \left(\vec{d}_1(x_2, x_1) + \vec{d}_1(x_2, x_3)\right) + \dots + \left(\vec{d}_1(x_{n+1}, x_n) + \vec{d}_1(x_{n+1}, x_{n+2})\right)$$

with  $n \geq 0$  and  $x_0 = x$  and  $x_{n+2} = y$ . To have a finite sum, the only possibility is that

$$x_1 = x_2 = \dots = x_n = x_{n+1} = z, z \leq x, z \leq y.$$

Consequently, one has

$$\vec{d}_1^\wedge(x, y) = \min_{\substack{z \leq x \\ z \leq y}} \left(\vec{d}_1(z, x) + \vec{d}_1(z, y)\right) = \min_{\substack{z \leq x \\ z \leq y}} \left(d_1(z, x) + d_1(z, y)\right).$$

From the triangular inequality, we obtain  $d_1(x, y) \leq \vec{d}_1^\wedge(x, y)$ . Write  $x = x_0 + x_1$  and  $y = y_0 + y_1$  with  $x_0 \leq y_0$  and  $y_1 \leq x_1$ . Let  $z = x_0 + y_1$ . Then one has

$$\vec{d}_1(z, x) + \vec{d}_1(z, y) = \left(h(x_1) - h(y_1)\right) + \left(h(y_0) - h(x_0)\right) = d_1(x, y).$$

We deduce the inequality  $\vec{d}_1^\wedge(x, y) \leq d_1(x, y)$ . □

**7.6. Corollary.** For all  $n \geq 0$ , there is the homeomorphism  $|\widehat{\square}[n]|_{d_1} \cong [0, 1^n]$ .

**7.7. Proposition.** For all transverse sets  $K$ , the underlying sets of the topological spaces  $|K|_{geom}$  and  $|K|_{d_1}$  are equal. Moreover, the identity of the underlying set of  $|K|_{geom}$  yields a continuous map from  $|K|_{geom}$  to  $|K|_{d_1}$  and the topological space  $|K|_{d_1}$  is Hausdorff.

*Proof.* Assume at first that **Top** is the category of  $\Delta$ -generated spaces. The topology of  $|K|_{geom}$  is given by the final topology on the colimit of the underlying set of the  $|\widehat{\square}[n]|_{geom}$ . The forgetful functor  $\mathbf{PseudoMet} \rightarrow \mathbf{Set}$  from pseudometric spaces to sets has a right adjoint given by taking a set  $S$  to the pseudometric space  $(S, d_0)$  with  $d_0(x, y) = 0$  for all  $x, y \in S$ . Thus, the forgetful functor  $\mathbf{PseudoMet} \rightarrow \mathbf{Set}$  is colimit-preserving. We deduce that the underlying sets of  $|K|_{geom}$ , of  $|K|_{\vec{d}_1^\wedge}$  and therefore of  $|K|_{d_1}$  are equal. From Corollary 7.6, we obtain the homeomorphism  $|\widehat{\square}[n]|_{geom} \cong |\widehat{\square}[n]|_{d_1}$ . For each  $c \in K_n$ , we obtain a composite continuous map

$$|\widehat{\square}[n]|_{geom} \cong |\widehat{\square}[n]|_{d_1} \xrightarrow{|c|_{d_1}} |K|_{d_1}$$

and, by the universal property of the colimit, a continuous map  $|K|_{geom} \rightarrow |K|_{d_1}$ . It is easy to see that the pseudometric of  $|K|_{d_1^\wedge}$  restricts to a metric on each path-connected component thanks to the homeomorphisms  $|\widehat{\square}[n]|_{d_1} \cong [0, 1^n]$  for all  $n \geq 0$ . Thus, the

topological space  $|K|_{d_1}$  is Hausdorff. Since  $|K|_{geom}$  is Hausdorff as well, the proof is valid for  $\Delta$ -Hausdorff  $\Delta$ -generated spaces and the proof is complete.  $\square$

Let  $U$  be a topological space. A (Moore) path of  $U$  consists of a continuous map  $[0, \ell] \rightarrow U$  with  $\ell > 0$ . Let  $\gamma_1 : [0, \ell_1] \rightarrow U$  and  $\gamma_2 : [0, \ell_2] \rightarrow U$  be two paths of a topological space  $U$  such that  $\gamma_1(\ell_1) = \gamma_2(0)$ . The Moore composition  $\gamma_1 * \gamma_2 : [0, \ell_1 + \ell_2] \rightarrow U$  is the Moore path defined by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0, \ell_1] \\ \gamma_2(t - \ell_1) & \text{for } t \in [\ell_1, \ell_1 + \ell_2]. \end{cases}$$

The Moore composition of Moore paths is strictly associative.

Let  $n \geq 1$ . A  $d$ -path of  $|\widehat{\square}[n]|_{geom} = [0, 1]^n$  is a nonconstant continuous map  $\gamma : [0, \ell] \rightarrow [0, 1]^n$  with  $\ell > 0$  such that  $\gamma(0), \gamma(\ell) \in \{0, 1\}^n$  and such that  $\gamma$  is nondecreasing with respect to each axis of coordinates. Let  $K$  be a general transverse set. A  $d$ -path of  $K$  is a path  $[0, \ell] \rightarrow |K|_{geom}$  which is the Moore composition  $\gamma_1 * \dots * \gamma_n$  of  $d$ -paths of cubes of  $|K|_{geom}$ .  $\gamma(0) \in K_0$  is called the initial state of  $\gamma$  and  $\gamma(\ell) \in K_0$  is called the final state of  $\gamma$ . This definition makes sense because for all cotransverse maps  $f : [m] \rightarrow [n]$ , the continuous map  $T(f) : [0, 1]^m \rightarrow [0, 1]^n$  is nondecreasing by Proposition 4.4.

**7.8. Remark.** All  $d$ -paths of a transverse set  $K$  start and end at a vertex of  $K$ .

**7.9. Definition.** Let  $n \geq 1$ . A natural  $d$ -path of the topological  $n$ -cube  $[0, 1]^n$  is a  $d$ -path  $\gamma = (\gamma_1, \dots, \gamma_n) : [0, n] \rightarrow [0, 1]^n$  such that for all  $t \in [0, n]$ , one has  $t = \gamma_1(t) + \dots + \gamma_n(t)$ . The set of natural  $d$ -paths of  $[0, 1]^n$  is denoted by  $N_n$ . It is equipped with the compact-open topology. By [15, Proposition 4.10], the space  $N_n$  is  $\Delta$ -generated and  $\Delta$ -Hausdorff.

Another way to formulate this definition is as follows:

**7.10. Proposition.** Equip  $([0, n], \leq)$  with the metric  $\vec{d}_1 : [0, n] \times [0, n] \rightarrow [0, \infty]$  defined by

$$\vec{d}_1(x, y) = \begin{cases} y - x & \text{if } x \leq y \\ \infty & \text{if } x > y. \end{cases}$$

A set map  $\gamma : [0, n] \rightarrow [0, 1]^n$  is a natural  $d$ -path if and only if it is a quasi-isometry for  $\vec{d}_1$ .

*Proof.* The equality  $t = \gamma_1(t) + \dots + \gamma_n(t)$  for all  $t \in [0, n]$  implies that any natural  $d$ -path is a quasi-isometry for  $\vec{d}_1$ . Conversely, suppose that the set map  $\gamma : [0, n] \rightarrow [0, 1]^n$  is a quasi-isometry for  $\vec{d}_1$ . Then by Corollary 7.6, it is continuous for  $[0, 1]^n$  equipped with the standard topology. And being a quasi-isometry, it satisfies  $t = \vec{d}_1(0, t) = \vec{d}_1(\gamma(0), \gamma(t)) = \gamma_1(t) + \dots + \gamma_n(t)$  for all  $t \in [0, n]$ . Consequently, the continuous map  $\gamma : [0, n] \rightarrow [0, 1]^n$  is a natural  $d$ -path.  $\square$

We can now generalize to transverse set the notion of natural  $d$ -path introduced by Raussen in [24, Definition 2.14] for precubical sets. Let  $K$  be a general transverse set. A natural  $d$ -path of  $K$  is a path  $[0, \ell] \rightarrow |K|_{geom}$  with  $\ell > 0$  which is the Moore composition  $\gamma_1 * \dots * \gamma_n$  of natural  $d$ -paths of cubes of  $|K|_{geom}$ .  $\gamma(0) \in K_0$  is called the initial state of  $\gamma$  and  $\gamma(\ell) \in K_0$  is called the final state of  $\gamma$ . This definition makes sense because the identity induces a continuous map from  $|K|_{geom}$  to the underlying topological space  $|K|_{d_1}$

of the Lawvere metric space  $|K|_{\vec{d}_1}$  by Proposition 7.7 and because for all cotransverse maps  $f : [m] \rightarrow [n]$ , the map  $T(f) : [0, 1]^m \rightarrow [0, 1]^n$  is a quasi-isometry by Corollary 4.13.

## 8. NATURAL REALIZATION OF A TRANSVERSE SET

We want to use the notion of natural  $d$ -path of a transverse set introduced in Section 7 to build the natural realization functor from transverse sets to flows, exactly as we proceed in [15, Section 5]. The definition is almost a copy-pasting. However, the verification of the functoriality is a little bit more complicated than in the precubical setting: see Proposition 8.1.

We define a flow  $|\widehat{\square}[n]|_{nat}$  for  $n \geq 0$  called *the natural  $n$ -cube* as follows. The set of states is  $\{0, 1\}^n$ . Let  $n \geq 1$  and  $\alpha, \beta \in \{0, 1\}^n$ . Assume that  $\vec{d}_1(\alpha, \beta) = m \geq 1$ , which implies that  $\alpha < \beta$ . Then let  $\mathbb{P}_{\alpha, \beta}|\widehat{\square}[n]|_{nat} = N_m$  which must be understood as the space of natural  $d$ -paths in the  $m$ -subcube from  $\alpha$  to  $\beta$ . Assume that  $\alpha \geq \beta$ . Let  $\mathbb{P}_{\alpha, \beta}|\widehat{\square}[n]|_{nat} = \emptyset$ . The composition law is defined by the Moore composition of natural  $d$ -paths, which is still a natural  $d$ -path.

Let  $f : [m] \rightarrow [n]$  be a cotransverse map. Let  $\alpha, \beta \in \{0, 1\}^m$ . Assume that  $k = \vec{d}_1(\alpha, \beta) \geq 1$ . There exists a unique coface map  $\delta : [k] \rightarrow [m]$  with takes  $0_k$  to  $\alpha$  and  $1_k$  to  $\beta$ . Consider the commutative diagram of  $\widehat{\square}$  where the vertical maps are coface maps

$$\begin{array}{ccc} [m] & \xrightarrow{f} & [n] \\ \delta \uparrow & & \uparrow \delta' \\ [k] & \xrightarrow{[f]_{\alpha, \beta}} & [k] \end{array}$$

obtained by applying Proposition 3.5 to  $f\delta$ . Then the continuous map  $\mathbb{P}_{\alpha, \beta}|\widehat{\square}[m]|_{nat} \rightarrow \mathbb{P}_{f(\alpha), f(\beta)}|\widehat{\square}[n]|_{nat}$  induced by  $f$  is the continuous map  $T([f]_{\alpha, \beta}) : N_k \rightarrow N_k$ .

**8.1. Proposition.** *We obtain a well-defined cotransverse flow  $|\widehat{\square}[*]|_{nat}$ .*

*Proof.* Let  $f : [m] \rightarrow [n]$  and  $g : [n] \rightarrow [p]$  be two cotransverse maps. Let  $\alpha, \beta \in \{0, 1\}^m$ . Assume that  $k = \vec{d}_1(\alpha, \beta) \geq 1$ . Consider the commutative diagram of  $\widehat{\square}$  where the vertical maps are coface maps:

$$\begin{array}{ccccc} [m] & \xrightarrow{f} & [n] & \xrightarrow{g} & [p] \\ \delta \uparrow & & \uparrow \delta' & & \uparrow \delta'' \\ [k] & \xrightarrow{[f]_{\alpha, \beta}} & [k] & \xrightarrow{[g]_{f(\alpha), f(\beta)}} & [k] \end{array}$$

Because of the uniqueness of the factorization given by Proposition 3.5, we have

$$[gf]_{\alpha, \beta} = [g]_{f(\alpha), f(\beta)}[f]_{\alpha, \beta}.$$

We obtain

$$T([gf]_{\alpha, \beta}) = T([g]_{f(\alpha), f(\beta)})T([f]_{\alpha, \beta})$$

by Proposition 4.8. □

Using Proposition 3.8 and Proposition 8.1, we obtain:

8.2. **Definition.** Let  $K$  be a transverse set. Consider the colimit-preserving functor

$$|K|_{nat} = \int^{[n] \in \widehat{\square}} K_n \cdot |\widehat{\square}[n]|_{nat}.$$

It is called the natural realization of  $K$  as a flow.

The following theorem concludes the paper.

8.3. **Theorem.** The natural realization functor  $|-|_{nat}$  from transverse sets to flow defined in Definition 8.2 is a  $m$ -realization functor. It is homotopy equivalent to the  $q$ -realization functor  $|-|_q$  of Theorem 6.10 in the following sense. There exists a  $m$ -realization functor  $F : \widehat{\square}^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  and two natural transformations inducing bijections on the sets of states

$$|-|_q \Leftarrow F(-) \Rightarrow |-|_{nat}$$

such that for all transverse sets  $K$  and all  $(\alpha, \beta) \in K_0 \times K_0$ , there is the zig-zag of natural homotopy equivalences between  $m$ -cofibrant topological spaces

$$\mathbb{P}_{\alpha, \beta} |K|_q \xleftarrow{\cong} \mathbb{P}_{\alpha, \beta} F(K) \xrightarrow{\cong} \mathbb{P}_{\alpha, \beta} |K|_{nat} .$$

Note that in fact,  $\mathbb{P}_{\alpha, \beta} |K|_q$  is even  $q$ -cofibrant for all  $(\alpha, \beta) \in K_0 \times K_0$ .

*Proof.* One has  $|\square[n]|_{nat} = |\widehat{\square}[n]|_{nat}$ : the natural realization of the precubical set  $\square[n]$  is equal to the natural realization of the transverse set  $\widehat{\square}[n]$  because it is exactly the same definition. Using Proposition 3.15, we deduce for all  $n \geq 0$  the natural isomorphism  $|\square[n]|_{nat} \cong |\widehat{\mathcal{L}}(\square[n])|_{nat}$ . Since all involved functors are colimit-preserving, we obtain for all precubical sets  $K$  the isomorphism of flows  $|K|_{nat} \cong |\widehat{\mathcal{L}}(K)|_{nat}$ . Using Proposition 3.15, we deduce the isomorphism of flows  $|\partial\square[n]|_{nat} \cong |\partial\widehat{\square}[n]|_{nat}$  for all  $n \geq 0$ . Thus the natural realization functor from transverse sets to flow is a  $m$ -realization functor because the natural realization functor of precubical sets as a flow is a  $m$ -realization functor by [15, Theorem 5.9]. Every  $q$ -realization functor is a  $m$ -realization functor by Proposition 6.9. The proof is complete thanks to Theorem 6.8.  $\square$

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