



**HAL**  
open science

# Minimization of differential equations and algebraic values of $E$ -functions

Alin Bostan, Tanguy Rivoal, Bruno Salvy

► **To cite this version:**

Alin Bostan, Tanguy Rivoal, Bruno Salvy. Minimization of differential equations and algebraic values of  $E$ -functions. *Mathematics of Computation*, 2024, 93, pp.1427-1472. 10.1090/mcom/3912. hal-03771150v2

**HAL Id: hal-03771150**

**<https://hal.science/hal-03771150v2>**

Submitted on 18 Jul 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MINIMIZATION OF DIFFERENTIAL EQUATIONS AND ALGEBRAIC VALUES OF $E$ -FUNCTIONS

ALIN BOSTAN, TANGUY RIVOAL, AND BRUNO SALVY

ABSTRACT. A power series being given as the solution of a linear differential equation with appropriate initial conditions, minimization consists in finding a non-trivial linear differential equation of minimal order having this power series as a solution. This problem exists in both homogeneous and inhomogeneous variants; it is distinct from, but related to, the classical problem of factorization of differential operators. Recently, minimization has found applications in Transcendental Number Theory, more specifically in the computation of non-zero algebraic points where Siegel's  $E$ -functions take algebraic values. We present algorithms and implementations for these questions, and discuss examples and experiments.

## 1. INTRODUCTION

1.1. **Minimization.** A linear differential equation (LDE)

$$\mathcal{L}(y(z)) := a_r(z)y^{(r)}(z) + \cdots + a_0(z)y(z) = 0 \quad (1)$$

with polynomial coefficients  $a_i(z)$  in  $\mathbb{Q}[z]$  is given, together with initial conditions specifying uniquely a formal power series solution  $S \in \mathbb{Q}[[z]]$ , i.e.  $\mathcal{L}(S(z)) = 0$ . In its homogeneous variant, the problem of *minimization* is to find a homogeneous linear differential equation

$$\mathcal{M}(y(z)) := b_m(z)y^{(m)}(z) + \cdots + b_0(z)y(z) = 0 \quad (2)$$

of minimal possible order  $m$ , with polynomial coefficients  $b_j(z)$  in  $\mathbb{Q}[z]$  and also having  $S$  as a solution. In the inhomogeneous version, the input is the same, but in the output, a non-zero polynomial right-hand side in  $\mathbb{Q}[z]$  is also possible, which may allow for the existence of an equation of even smaller order.

Both these problems exist for other fields of coefficients and a large part of our discussion extends to such situations; we focus here on the case of the field  $\mathbb{Q}$  to keep the discussion simple and reflect more closely the capabilities of our implementation.

When the origin is an *ordinary point* of the input equation  $\mathcal{L}(y(z)) = 0$ , i.e.  $a_r(0) \neq 0$ , initial conditions are given as the vector of values  $(S(0), \dots, S^{(r-1)}(0))$  in  $\mathbb{Q}^r$ . Otherwise, the origin is called a *singularity* of the equation (1). Equation (1) may still have power series solutions even in the singular case. The definition of initial conditions in this case is

---

*Date:* July 18, 2023.

*2010 Mathematics Subject Classification.* 68W30, 11J81, 16S32, 34M15, 33F10.

*Key words and phrases.* Linear differential operators, Minimization, Factorization, Desingularization,  $E$ -functions, Siegel-Shidlovskii theorem, Beukers' algorithm, Adamczewski-Rivoal algorithm.

more delicate; it is discussed in Section 2.1. Interestingly, when the origin is singular, it is even possible that (1) possesses a basis of solutions in  $\mathbb{Q}[[z]]$ ; in this case the origin is called an *apparent singularity*.

**1.2. Algebraic values of  $E$ -functions.** Minimization has a nice application to transcendence theory, more precisely to the determination of algebraic values taken by  $E$ -functions at algebraic points.  $E$ -functions are entire functions with special arithmetic properties; in particular, they satisfy linear differential equations over  $\overline{\mathbb{Q}}(z)$  and their Taylor series at 0 belong to  $\overline{\mathbb{Q}}[[z]]$ . They have been introduced by Siegel in 1929 as a generalization of the exponential function, and have been studied in depth by Siegel, Shidlovskii, Nesterenko, André, Beukers and others. We refer to Section 3 for their precise definition, statements of results and bibliographic references.

Adamczewski and Rivoal have given an algorithm [AR18] that determines the *finite* list of algebraic numbers  $\alpha$  such that  $f(\alpha)$  is also algebraic, given as input an  $E$ -function  $f(z)$  (represented by a linear differential equation and sufficiently many initial terms). The first two steps of this algorithm rely on the computation of a minimal homogeneous linear differential equation and of a minimal linear differential inhomogeneous equation for  $f$ .

Another question in number theory concerns the algebraicity of special values of analytic functions whose Taylor series at 0 belongs to  $\overline{\mathbb{Q}}[[z]]$  and satisfies a Mahler equation over  $\overline{\mathbb{Q}}(z)$ , i.e., an equation  $\sum_{j=0}^d p_j(z)f(z^{r^j}) = 0$  where  $p_j(z) \in \overline{\mathbb{Q}}[z]$ . There exists an approach based on a similar but different type of minimization [AF17, AF18], that is not considered here.

**1.3. Relation of minimization to factorization of linear differential operators.** The problems of factorization and minimization are closely related since they are both concerned with finding (right) factors of linear differential operators. But they are different problems. For instance, an equation can be minimal even when the operator factors. A simple example is given by the equation  $(1-z)y'' - y' = 0$  and its power series solution  $S = \ln(1-z)$ . The corresponding operator clearly has  $\partial_z := \frac{d}{dz}$  as a right factor of order 1, but no homogeneous equation of order 1 can have a solution with a logarithmic singularity. Also, in general, a linear differential operator may have infinitely many factorizations and the problem of minimization is to find a minimal (not necessarily irreducible) right factor that vanishes at the solution  $S$ . Thus one cannot simply use an existing implementation of a factorization algorithm in order to solve the minimization problem. Even in the case of finitely many factorizations, minimization can be much simpler than factorization; this is illustrated by an example in Section 5.1.

Still, factorization and minimization share many algorithmic tools. Indeed, the algorithm we present in Section 2 is obtained by combining sub-algorithms of van Hoeij's description of his factorization algorithms [vH97a, vH97b], exploiting the fact that the situation of minimization is made easier by the extra information provided by the input power series  $S$ . Furthermore, in applications, it is sometimes possible to take advantage

of further structure that the minimal operator is known to possess, such as being Fuchsian (i.e., having only regular singularities, see §2.3.1), or having at most one irregular singularity, e.g. at infinity as happens for  $E$ -functions (see Section 3).

**1.4. Notes on the history of factorization algorithms.** The tools used in factorization and minimization algorithms have a convoluted history. Fabry’s doctoral thesis [Fab85] is well-known for having completed the classification of the general form that can be taken by formal solutions of linear differential equations at their singularities. Quite surprisingly, it seems mostly forgotten that the last part of his thesis (from Section 32 page 86, to the end, page 105), Fabry uses the classification result to design a factorization algorithm for LDEs with rational function coefficients. The principle is to perform a local analysis at each singularity and try all possible choices of local behaviors for the factor under construction. For instance, at any regular singular point, the roots of the indicial equations of the factor have to be roots of the indicial equations of the equations (see Lemma 2.1). At irregular singular points, the exponentials must also be chosen from those of the formal solutions. Then, what remains is to find the apparent singularities. In the case of a Fuchsian factor (Sections 32–34), Fabry uses Fuchs’ relation (see §2.3.3) to determine possible roots of the indicial equations at the apparent singularities and then a local expansion of the logarithmic derivative of the Wronskian of local solutions at a well-chosen singular point reconstructs the required information. This is not completely general as a well-chosen point has to exist; otherwise Fabry introduces undetermined coefficients that will have to be determined later as solutions of a polynomial system. In the non-Fuchsian case (Sections 35–36), he does not have a complete solution in his thesis, but a reduction result showing that finding a right factor amounts to bounding the number of apparent singularities of that factor. A few years later [Fab88a, Fab88b], he addresses the non-Fuchsian case by observing that the relevant information can be found in the expansion of the logarithmic derivative of the Wronskian at infinity. This anticipates by one century the generalized Fuchs relation of Bertrand and Beukers [BB85, Th. 3], and its more precise form by Bertrand and Laumon [Ber88, A.2] described in Section 2.3. Again, to make the algorithm complete, it may be the case that undetermined coefficients are needed. To summarize, Fabry’s works contain the first proof that factoring LDEs is algorithmically decidable.

Another approach leading to a general and complete algorithm was started by Markov [Mar91a]<sup>1</sup> and developed by Bendixson [Ben92] and Beke [Bek94]<sup>2</sup>. The idea is to construct a  $k$ th exterior power of the linear differential operator to be factored. This operator must cancel the logarithmic derivative of the Wronskian of the solutions of a factor of order  $k$  (other linear differential equations are constructed for the next coefficients). One then looks for “hyperexponential solutions” (that is, for solutions  $y$

<sup>1</sup>Markov [Mar91a] is aware of Fabry’s note [Fab88b]; he writes “It seems to me that this question can be solved by simpler considerations than those of Mr. Fabry.”

<sup>2</sup>Most modern references call this approach “Beke’s algorithm”. Beke [Bek94] credits Bendixson [Ben92] in the footnote at the end of his article. Both Bendixson and Beke seem to be unaware of Fabry’s and Markov’s works.

such that  $y'/y$  is rational; these correspond to right factors of order 1). This leads to looking for a rational solution of the associated Riccati equation. As in Fabry’s method<sup>3</sup>, this is solved by finding finitely many possibilities at each singularity, gluing them in all possible ways and finding the missing part as a rational solution of a linear differential equation [vdPS03, Prop. 4.9].

General bounds on the degree of the coefficients of the factors can be deduced from both methods; this has been done by Grigoriev for the Markov–Bendixson–Beke algorithm [Gri90, Theorem 1.2] (note however that the bound is only asymptotic) and by Singer for the Fabry-type approach [Sin93, Lemma 3.7] via the generalized Fuchs relation in [BB85, Ber88] (a more precise version can be found in [BRS21, Theorem 1]). The only algorithmic complexity result we are aware of in the area of factorization algorithms is due to Grigoriev [Gri90, Theorem 1.1]; it is deduced by studying in depth (improvements of) the Markov–Bendixson–Beke approach.

On the algorithmic front, van Hoeij [vH97a] showed that in many cases, using bounds instead of trying all possible combinations leads to an improvement to Fabry’s approach (which he rediscovered). This does not give a complete algorithm and the missing cases are handled by Beke’s algorithm. A precise exposition of the computation of these bounds is the topic of our Section 2.3. As for the Markov–Bendixson–Beke approach, various practical improvements have been proposed in the literature [Sch89, Bro92, Bro94, Tsa94, vH97a, CvH04, JKM13]. Even with these improvements, the Markov–Bendixson–Beke algorithm is not competitive for operators of order larger than 5. In practice, it is outperformed by Fabry-type algorithms such as van Hoeij’s [vH97a]. (Another practical, but incomplete, algorithm is the *eigenring method*, a Berlekamp-style algorithm introduced by Singer [Sin96] and improved by van Hoeij [vH96].)

For additional historical information on early contributions on factoring (by Painlevé, Fabry, Markov, Bendixson, Beke, etc.) the reader is invited to consult the following additional references: [Sch97, Chap. III and Chap. IV, §176, §177] for details of (a variant of) the Markov–Bendixson–Beke algorithm; [Hil15, Chap. II, §10] for historical aspects; [Ogi67, Letters IX–XII] for letters sent by Hermite to Markov and comments about them; [Sch09, p. 61–123] for a remarkable bibliography covering the “golden age” (1865–1907) of the theory of linear differential equations.

**1.5. Previous work on minimization.** In contrast to factorization, it is much more difficult to locate a similar algorithm for minimization in the literature.

In [vH97a, §6], van Hoeij describes an algorithm (called “Construct  $R$ ”) that solves the following problem: given a linear differential operator  $S$  with coefficients in  $\mathbb{Q}((z))$ , find a non-trivial right factor of an operator  $\mathcal{L}$  with coefficients in  $\mathbb{Q}(z)$  known to be a left multiple of  $S$ . Our minimization algorithm is very close in spirit to that algorithm. From this perspective, we could say that the first minimization algorithm we are aware of lies “between the lines” of [vH97a, §6].

---

<sup>3</sup>This approach to finding hyperexponential solutions was proposed by Markov [Mar91c, Mar91b] and detailed by Bendixson [Ben92] and Beke [Bek94]; a different and more general method is claimed by Painlevé [Pai91].

A different, symbolic-numeric, approach to factoring LDEs was proposed by van der Hoeven [vdH07]. Although minimization is never explicitly considered in this article, one could consider that it is implicitly solvable by concatenating several statements from its §3.3 and §3.4. In the same spirit, the more recent work by Chyzak, Goyer and Mezzarobba [CGM22, §4] makes this much more explicit, for Fuchsian input operators  $\mathcal{L}$ . For instance, a numerical criterion of minimality is provided in the Fuchsian case by a numerical computation of the monodromy matrices, see lines 1 and 2 of Algorithm 1 in [CGM22].

The proof of [BBMKM16, Proposition 8.4] contains a minimization proof on an explicit and challenging example; this proof was the original inspiration of our paper.

Adamczewski and Rivoal proposed in [AR18, §3] a method for minimization based on two ingredients: (i) a priori degree bounds for right factors of  $\mathcal{L}$  and (ii) bounds on the order at  $z = 0$  of linear combinations with coefficients in  $\mathbb{Q}[z]$  of  $S$  and its derivatives. For (ii), they use multiplicity estimates due to Bertrand and Beukers [BB85], and further refined and made completely explicit by Bertrand, Chirskii and Yebbou [BCY04]. For (i), they use Grigoriev’s estimates from [Gri90, Theorem 1.2]. However, Grigoriev’s result is only asymptotic. We gave an explicit and effective bound in [BRS21]. It is fair to say that the combination of [AR18, §3] and [BRS21, Theorem 1] provides the first complete proof that minimization is algorithmically decidable. However, the corresponding algorithm is highly inefficient in theory, to the point of being completely impractical (see the example in §2.2.3). Our paper can thus be seen as the first one providing an efficient general algorithm for minimization.

**1.6. Relation to “order-degree curves”.** To each left multiple<sup>4</sup> with polynomial coefficients of a linear differential operator  $\mathcal{L}$ , one associates the point  $(r', d')$  where  $r'$  is the order and  $d'$  the degree. The (discrete) *order-degree curve*  $\mathcal{C}_{\mathcal{L}}$  is obtained by keeping those points that lie on the lower part of the convex hull of this set of points. In several cases, this curve has been shown to be well approximated by a hyperbola for  $r$  sufficiently large [BCL<sup>+</sup>07, BCCL10, CK12b, CK12a, CJKS13, Kau14]. This curve has also been studied in relation to the *desingularisation*  $\mathcal{D}$  of  $\mathcal{L}$ , which is the lowest-order left multiple of  $\mathcal{L}$  with no apparent singularities [CS98, Tsa00, LVO00, ABvH06, CKS16].

Both problems have in common the remarkable role played by apparent singularities, however computing  $\mathcal{D}$  and computing  $\mathcal{M}$  are unrelated problems. The first one is about computing a left *multiple* of  $\mathcal{L}$  with a special property, the second one is about computing a right *factor*  $\mathcal{M}$  of  $\mathcal{L}$  with a minimality property. Although the curve  $\mathcal{C}_{\mathcal{M}}$  lies below the curve  $\mathcal{C}_{\mathcal{L}}$ , bounds on the degree of  $\mathcal{M}$  are not accessible via the tools developed up to now for  $\mathcal{C}_{\mathcal{L}}$ . Moreover, the degree of  $\mathcal{M}$  can be exponentially large with respect to the size of  $\mathcal{L}$ . This is illustrated by all classical families  $(P_n(z))$  of orthogonal polynomials, as they satisfy a small linear differential equation  $\mathcal{L}(y) = 0$  of order 2 with coefficients that are polynomial in  $n$  (and thus of bit-size logarithmic in  $n$ ), but their

---

<sup>4</sup>Left multiples are obtained by multiplication on the left with linear differential operators whose coefficients belong to  $\mathbb{Q}(z)$ .



minimal LDE  $P_n(z)y'(z) - P'_n(z)y(z) = 0$  has degree  $n$ , which is exponential in the bit-size of  $\mathcal{L}$ .

**1.7. Contributions.** We give a new minimization algorithm that is efficient in practice. The necessary tools are presented in detail, with degree bounds obtained as solutions of explicit linear programming problems, that we did not find in the literature. The algorithm for the inhomogeneous case is also new; it reduces the problem to that of finding rational solutions of the adjoint operator to the minimal homogeneous one.

Both these minimization algorithms make the necessary computations for  $E$ -functions proposed by Adamczewski and Rivoal [AR18] accessible in practice when the coefficients are rational. The final step in their method is based on a process of desingularization of differential systems, due to Beukers, that is of independent interest. We make it more explicit, with a detailed proof. A new canonical decomposition of  $E$ -functions is presented, together with corresponding algorithms. Explicit families of interesting “trivial” evaluations of hypergeometric  ${}_1F_1$  functions at algebraic points are deduced.

All these algorithms are practical and an efficient implementation in the computer algebra system Maple vindicates them<sup>5</sup>.

**1.8. Structure of the article.** Section 2 describes our minimization algorithm, both in its homogeneous form (Section 2.2) and in its inhomogeneous variant (Section 2.4). Both crucially rely on computations of degree bounds, described in Section 2.3. Section 3 discusses the application of the minimization algorithms to  $E$ -functions over  $\mathbb{Q}$ . The new algorithm is a practical variant of the Adamczewski-Rivoal algorithm recalled in Section 3.1, itself based on Beukers’ desingularization procedure described and enhanced in Section 3.2. The algorithm from Section 3.1 is then applied in Section 3.3 to an effective decomposition of  $E$ -functions over  $\mathbb{Q}$ . Extensions to  $E$ -functions with coefficients in a number field and to  $E$ -functions in Siegel’s original sense are discussed in Sections 3.4 and 3.5. In Section 4 we present two infinite families of  ${}_1F_1$  hypergeometric functions that take algebraic values at non-trivial algebraic points (and a similar family of  ${}_2F_1$  evaluations, even though they are not  $E$ -functions). Finally, Section 5 describes our implementation of the algorithms and illustrates it with a few timings.

## 2. MINIMIZATION ALGORITHM

**2.1. Power series solutions.** We recall properties of linear differential equations that can be found in the classical treatises of Ince [Inc56, Chap. XVI, XVII] or Poole [Poo60, Chap. V]. Moreover, the presentation is specialized to the case of coefficients  $a_i$  of Eq. (1) that are polynomials rather than formal power series.

Given an operator  $\mathcal{L}$  as in (1), the image by  $\mathcal{L}$  of a monomial  $z^s$  with  $s \in \mathbb{N}$  is a polynomial

$$f(s, z) = z^{s+g_{\mathcal{L}}}(p_0(s) + p_1(s)z + \cdots + p_t(s)z^t), \quad -r \leq g_{\mathcal{L}}, \quad 0 \leq t, \quad (3)$$

<sup>5</sup>Minimization is available as part of the gfun package at <https://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/>; the code for exceptional algebraic values of  $E$ -functions can also be downloaded from that page, together with a worksheet of examples.

with polynomials  $p_i(s)$  of degree at most  $r$  whose coefficients depend on those of the  $a_i$  and  $p_0 \neq 0$ . The polynomial  $p_0$  is called the *indicial polynomial* of  $\mathcal{L}$  at 0; we also denote it by  $\text{ind}_{\mathcal{L}}$  and write the identity above as

$$\mathcal{L}(z^s) \sim \text{ind}_{\mathcal{L}}(s)z^{s+g_{\mathcal{L}}}, \quad z \rightarrow 0. \quad (4)$$

By linear combination, the image by  $\mathcal{L}$  of a formal power series  $S(z) = \sum_{i \geq 0} c_i z^i$  is the formal power series

$$\mathcal{L}(S) = \sum_{i \geq 0} c_i f(i, z).$$

The coefficients of  $z^k$  for  $k = g, g+1, \dots$  in  $\mathcal{L}(S) = 0$  give the equations

$$c_0 p_0(0) = 0, \quad c_0 p_1(0) + c_1 p_0(1) = 0, \dots, \quad c_0 p_{t-1}(0) + \dots + c_{t-1} p_0(t-1) = 0, \quad (5)$$

and the linear recurrence of order  $t$

$$c_i p_t(i) + \dots + c_{t+i} p_0(t+i) = 0, \quad i \geq 0. \quad (6)$$

These equations imply that the valuation of  $S$  (the index of its first non-zero coefficient) is a zero of the indicial polynomial  $p_0$ . Let

$$\mathcal{Z}_{\mathcal{L}} = \{k \in \mathbb{N} \mid p_0(k) = 0\}$$

be the set of nonnegative integer roots of the indicial polynomial of  $\mathcal{L}$  at 0. For all  $i \notin \mathcal{Z}_{\mathcal{L}}$ , the coefficient  $c_i$  is determined from the previous ones by the  $(i+1)$ th equation of the infinite system (5)–(6). For this reason, the *initial conditions* of the differential equation (1) are the values of  $y^{(i)}(0)$  for  $i \in \mathcal{Z}_{\mathcal{L}}$ , as all the other ones are determined by the system (5)–(6). In the non-singular case when  $a_r(0) \neq 0$ , the indicial polynomial is  $p_0(s) = s(s-1)\dots(s-r+1)$  and then  $\mathcal{Z}_{\mathcal{L}} = \{0, 1, \dots, r-1\}$ , recovering the usual definition. This discussion leads to the following result that will be used to find right factors of  $\mathcal{L}$ . (See Prop. 4.3 and Section 4.3 in [BLS17] for similar considerations.)

**Lemma 2.1.** *With the notation above, let  $S$  be a power series solution of  $\mathcal{L}$  and  $\mathcal{M}$  be a right factor of  $\mathcal{L}$ . If there exists a polynomial  $T$  such that  $T^{(i)}(0) = S^{(i)}(0)$  for all  $i \in \mathcal{Z}_{\mathcal{L}}$  and  $\mathcal{M}(T) = O(z^{\max \mathcal{Z}_{\mathcal{L}} + g_{\mathcal{L}} + 1})$ , then  $\mathcal{M}(S) = 0$ .*

*Proof.* We first show that  $\mathcal{Z}_{\mathcal{M}} \subset \mathcal{Z}_{\mathcal{L}}$ . Indeed, if  $\mathcal{L} = \mathcal{A}\mathcal{M}$ , then

$$\mathcal{L}(z^s) = \mathcal{A}(\mathcal{M}(z^s)) \sim \mathcal{A}(\text{ind}_{\mathcal{M}}(s)z^{s+g_{\mathcal{M}}}) \sim \text{ind}_{\mathcal{M}}(s) \text{ind}_{\mathcal{A}}(s+g_{\mathcal{M}})z^{s+g_{\mathcal{M}}+g_{\mathcal{A}}}, \quad z \rightarrow 0,$$

which, in combination with (4), implies that

$$\text{ind}_{\mathcal{L}}(s) = \text{ind}_{\mathcal{M}}(s) \text{ind}_{\mathcal{A}}(s+g_{\mathcal{M}}) \quad \text{and} \quad g_{\mathcal{L}} = g_{\mathcal{M}} + g_{\mathcal{A}}.$$

In particular, the indicial polynomial of  $\mathcal{M}$  divides that of  $\mathcal{L}$ , and hence  $\mathcal{Z}_{\mathcal{M}} \subset \mathcal{Z}_{\mathcal{L}}$ .

Applying the discussion above to  $\mathcal{M}$  shows that the coefficients of  $T$  satisfy the first  $\max \mathcal{Z}_{\mathcal{L}} + 1$  equations of the system (5)–(6). It follows that  $T$  can be extended to a unique power series solution of  $\mathcal{M}$ . As  $\mathcal{M}$  is a right factor of  $\mathcal{L}$ , this power series is also a solution of  $\mathcal{L}$ . Since it has the same initial conditions as  $S$ , they coincide.  $\square$



**2.2. Homogeneous minimization.** Since initial conditions are given for the power series  $S$  solution of the linear differential equation, it is possible to compute arbitrarily many coefficients of  $S$ . The algorithm relies on the computation of upper bounds on the degree of the coefficients of right factors of the linear differential operator of a given order. Given such bounds and sufficiently many coefficients of  $S$ , it is easy to set up a (structured) linear system whose solutions are the possible coefficients of a right factor, or only 0 if no such factor exists. When a non-zero solution is found, one takes its greatest common right divisor with the original linear differential operator and checks it using Lemma 2.1. This approach is described in Algorithm 1.

---

**Algorithm 1** Minimal right factor

---

*Input:*  $\mathcal{L} = a_r(z)\partial_z^r + \dots + a_0(z)$  in  $\mathbb{Q}[z]\langle\partial_z\rangle$ ;  
     ini:  $S_0$  a truncated power series at precision  $\geq \max \mathcal{Z}_{\mathcal{L}}$   
         specifying a unique solution  $S \in \mathbb{Q}[[z]]$  of  $\mathcal{L}(S) = 0$ .  
*Output:* a right factor of  $\mathcal{L}$  in  $\mathbb{Q}[z]\langle\partial_z\rangle$  of minimal order that vanishes at  $S$

- 1:  $\mathcal{M} := \mathcal{L}$ ;  $T := S_0$ ;  $m := r$ ;  $p := \max \mathcal{Z}_{\mathcal{L}} + r$ ;
- 2: **while**  $m > 1$  **do**
- 3:      $m := m - 1$
- 4:     **if**  $N := \text{BOUNDDEGREECOEFFS}(\mathcal{L}, m) \neq \text{FAIL}$  **then**
- 5:         **while true do**
- 6:              $T := \text{SERIESOLUTION}(\mathcal{L}, T, p + m)$ ;  $k := \lfloor p/(m + 1) \rfloor$ ;
- 7:              $\mathcal{H} := \text{APPROXIMANTBASIS}(T, T', \dots, T^{(m)}; k, \dots, k; p)$ ;
- 8:             **if**  $\mathcal{H} = \emptyset$  and  $p \geq (m + 1)(N + 1)$  **then break** // No right factor of order  $m$
- 9:             **if**  $\mathcal{H} \neq \emptyset$  **then** //  $\mathcal{H}$  contains at least a candidate factor  $h$
- 10:                  $\mathcal{G} := \text{GREATESTCOMMONRIGHTDIVISOR}(\mathcal{L}, h)$ ;
- 11:                 **if**  $\mathcal{G}(T) = O(z^{\max \mathcal{Z}_{\mathcal{L}} + g_{\mathcal{L}} + 1})$  **then**  $\mathcal{M} := \mathcal{G}$ ;  $m := \text{ord } \mathcal{M}$ ; **break**
- 12:              $p := 2p$
- 13: **return**  $\mathcal{M}$

---

It relies on several other algorithms that we now review.

### 2.2.1. Sub-algorithms.

**SERIESOLUTION.** Takes as input a linear differential operator, a truncated power series solution of it, and a target precision  $p$ . It returns the power series solution of the operator up to  $O(z^p)$ , obtained either by truncating the power series given as input, or by extending it using the linear recurrence deduced from the differential equation.

**APPROXIMANTBASIS.** Takes as input  $k$  power series  $(S_1, \dots, S_k)$  that are the truncations at precision  $p$  of the successive derivatives of  $S$ ;  $k$  nonnegative integers  $(s_1, \dots, s_k)$  and the precision  $p$ . It first computes a basis  $B(z) \in \mathbb{Q}[z]^{k \times k}$  of the  $\mathbb{Q}[z]$ -module

$$\mathcal{A}_p := \{(p_1, \dots, p_k) \mid p_1 S_1 + \dots + p_k S_k = O(z^p)\} \quad (7)$$

in *shifted Popov form* [Pop72, VBB92, BLV99, JNV20] with shift vector  $(-s_1, \dots, -s_k)$ . This implies that any element  $P$  of  $\mathcal{A}_p$  with degrees bounded by  $(s_1, \dots, s_k)$  is a linear

combination of the rows of  $B$  whose index  $i$  satisfies  $\deg B_{ii} \leq s_i$ . Those are the rows returned by APPROXIMANTBASIS. As the  $S_i$  are successive derivatives, these rows can be interpreted as linear differential operators  $p_1 + p_2\partial + \dots + p_k\partial^{k-1}$ . Efficient algorithms to compute such bases are known [JNV20].

GREATESTCOMMONRIGHTDIVISOR. Computes the monic greatest common right divisor (gcrd) of two linear differential operators with coefficients in  $\mathbb{Q}(z)$ . This is classically achieved by a non-commutative version of Euclid's algorithm [Ore33] and more efficient methods are known [Gri90, vdH16].

BOUNDDEGREECOEFFS. This is the heart of the algorithmic work, described in Section 2.3. It takes as input an operator of order  $r$  and a positive integer  $m < r$ . It returns either FAIL when it has proved that no right factor of order  $m$  with polynomial coefficients exist; or an upper bound on the degree of each of the coefficients such a factor would have.

**Theorem 2.2.** *Given a linear differential operator  $\mathcal{L} \in \mathbb{Q}[z]\langle\partial_z\rangle$  and a truncated power series specifying a unique solution  $S \in \mathbb{Q}[[z]]$  of  $\mathcal{L}(S) = 0$ , Algorithm 1 computes a non-zero right factor  $\mathcal{M}$  of  $\mathcal{L}$  of minimal order such that  $\mathcal{M}(S) = 0$ .*

*Proof.* 1. (Correctness assuming termination.) Since  $T$  is expanded at precision  $p + m$  in Line 6 and  $p > \max \mathcal{Z}_{\mathcal{L}}$  from Lines 1 and 12, it satisfies  $T^{(i)}(0) = S^{(i)}(0)$  for  $i \in \mathcal{Z}_{\mathcal{L}}$ . In Line 7, all series  $T, T', \dots, T^{(m)}$  are known at precision  $p$ . It follows that if the basis returned by APPROXIMANTBASIS is empty with the given bounds on the degrees of the coefficients in Line 8, there is no right-factor of  $\mathcal{L}$  of order  $m$ . Otherwise, taking  $\mathcal{G}$  a gcrd of  $\mathcal{L}$  and an element  $h$  of  $\mathcal{H}$  gives a right factor of  $\mathcal{L}$  to which Lemma 2.1 applies, showing that  $\mathcal{M}(S) = 0$  if the condition on Line 11 holds. The loop on  $m$  makes the algorithm terminate on a right factor of minimal order.

2. (Termination.) The only possible source on non-termination in the algorithm is the loop where  $p$  is doubled every time  $\mathcal{G}$  fails to cancel  $T$  to sufficient precision. Let  $V_p$  be the  $\mathbb{Q}$ -vector space generated by the approximants of the modules  $\mathcal{A}_{p'}$  from Eq. (7) for all  $p' \geq p$ . Since the approximants have degrees bounded by  $(N, \dots, N)$ , these are finite-dimensional vector spaces and  $V_{p+1} \subset V_p$ . Thus there exists  $p_0$  such that  $V_{p_0}$  is the intersection of all  $V_p$  for  $p \geq p_0$ . Any approximant  $h = (h_0, \dots, h_m)$  in  $\mathcal{H}$  in Line 7 for  $p \geq p_0$  has the property that  $h_0S + \dots + h_mS^{(m)} = O(z^k)$  for all  $k \geq p$  and thus annihilates  $S$ , and therefore so does its gcrd with  $\mathcal{L}$ , making the algorithm terminate.  $\square$

2.2.2. *Comparison with van Hoeij's algorithm.* Van Hoeij's Algorithm "Construct  $R$ " [vH97a, p. 552] follows a similar pattern. Our termination proof is essentially his. The difference is that instead of looking for an arbitrary right factor of  $\mathcal{L}$ , we need to make sure that the factor returned by the algorithm cancels the power series  $S$ . This is ensured by the test in Line 11.

2.2.3. *Example.* Consider the sequence

$$u_n = \sum_{k=0}^n \frac{n!(n+k)!}{k!^4(n-k)!^3}.$$

Zeilberger's creative telescoping algorithm [Zei91] shows that  $u_n$  satisfies a linear recurrence of order 4 with coefficients that are polynomials in  $n$  of degree at most 10:

$$(29412n^4 + 224352n^3 + 632931n^2 + 781692n + 356309)(n+3)^2(n+4)^4 u_{n+4} \\ + \cdots + 4(29412n^4 + 342000n^3 + 1482459n^2 + 2838258n + 2024696)(n+1)^2 u_n = 0.$$

This recurrence translates into a linear differential operator of order 10 annihilating the generating function  $S(z) = \sum_{n \geq 0} u_n z^n$ , with coefficients of degree at most 8:

$$\mathcal{L} = 29412z^8 \partial_z^{10} - 684z^7(688z - 1489) \partial_z^9 - 21z^6(156864z^2 + 742368z - 588707) \partial_z^8 + \cdots \\ + (99370416z^3 - 1926228512z^2 - 19342508z + 8500) \partial_z + 4(2024696z^2 - 3141504z - 32725). \quad (8)$$

The only integer roots of the indicial polynomial of  $\mathcal{L}$  at 0 are in  $\mathcal{Z}_{\mathcal{L}} = \{0, 1\}$  so that the initial conditions specifying  $S$  uniquely are  $S(0) = u_0 = 1, S'(0) = u_1 = 3$ . The differential operator  $\mathcal{L}$  is not minimal for  $S$ . There are two stages in the execution of the algorithm: first, a right factor is sought; next, its minimality is proved.

In the first stage, tight bounds on the degrees of coefficients of right factors are not needed. One can compute more and more coefficients of the series expansion of the solution and try to reconstruct a factor by computing an approximant basis. When a non-trivial factor exists, it will be discovered.

In the second stage, or if no non-trivial factor exists, i.e., if  $\mathcal{L}$  is minimal, then one has to certify this minimality. This is where tight bounds are useful. In this example, the bound on the degree of the coefficients that follows from the work of Bertrand, Chirskii and Yebbou [BCY04, Lemma 3.1] is larger than

$$10^{10^{10^{33}}}.$$

This makes it a purely theoretical result that cannot be used in a computation. Indeed, with current implementations and hardware, already bounds on degrees of order  $10^7$  become too large for practical computations.

*Computation of a right factor.* Not knowing in advance that  $\mathcal{L}$  is not minimal, our algorithm first computes bounds on the degrees of the coefficients of right factors. During this computation of bounds, it discovers that  $\mathcal{L}$  does not have any right factor of order 9, 8, or 7. For order 6, a bound 30 for the degrees of the coefficients is found. With this bound, a candidate linear differential operator of order 6 and degree 8 is found:

$$\mathcal{M} = z^4(1882368z^4 - 2206584z^3 + 1703460z^2 + 67815z + 272) \partial_z^6 + \cdots + \\ + 2(3764736z^6 - 41001696z^5 + 157022376z^4 - 184937064z^3 - 6917519z^2 - 3408891z - 41888).$$

The computation of the greatest common right factor stops at its first step, discovering that this differential operator is a right factor of  $\mathcal{L}$ .

*Proof of minimality.* This is the stage where good degree bounds are useful. At order 5, the bound given by our previous work [BRS21, Thm. 1], using the generalized exponents and the slopes of the Newton polygons of  $\mathcal{L}$ , is only 87, to be compared with the purely theoretical bound above. This means that proving minimality reduces to the computation of  $88 \times 6 + 1 = 529$  coefficients of the power series followed by a computation of an

approximant basis. This is a quantity that is manageable, but motivates the quest for tight degree bounds.

At this same order 5, our algorithm computes the better bound  $N = 15$  (instead of 87). Thus, with  $p + 1$  coefficients of  $S$ , where  $p = 96 = 6 \times 16$ , the computation of APPROXIMANTBASIS shows that there is no non-zero operator  $h$  of order 5 with coefficients of degree at most  $N$  such that  $h(S(z)) = O(z^p)$  and therefore no right factor of  $\mathcal{L}$  of order 5 annihilating  $S$ .

Next, the bound on the degrees of the coefficients of a right factor of order 4 is smaller than 15, so that if a right factor of that order existed, it would have been obtained for order 5. Finally, the computations of bounds for orders 3, 2, 1 show that no factor of these orders exist. This concludes the proof of minimality of the operator  $\mathcal{M}$  for  $S$ .

**2.3. Degree bounds.** The computation of degree bounds for a factor of a given order is a key step in van Hoeij's factorization algorithm [vH97a, §9]. We recall the ingredients here. Compared to our earlier work [BRS21] where we have obtained universal bounds, the bounds computed here are tailored to the equation under study, rather than depending only on its order, degree and height. This allows for smaller bounds and more efficient computations. As shown in the example above, having good bounds is important when certifying the minimality of a right factor. In this work, this is achieved by setting up explicit integer linear programming problems that do not appear in the earlier literature.

**2.3.1. Singularities of the factors.** Dividing  $\mathcal{L}$  by its leading coefficient  $a_r$  gives a monic operator with rational function coefficients. In this form, the singularities of  $\mathcal{L}$  are the poles of its coefficients. A singularity  $\alpha$  of  $\mathcal{L}$  is called *regular* if the indicial polynomial of  $\mathcal{L}$  at  $\alpha$  has degree equal to the order  $r$  of  $\mathcal{L}$ , and it is called *irregular* otherwise. The right factors will be searched in the same monic form. Recall that the valuation  $\text{val}_\alpha(r)$  of a rational function  $r$  at  $\alpha$  is the exponent of the leading term of the Laurent expansion of  $r$  at  $\alpha$  (and  $\text{val}_\alpha(0) = \infty$ ). At a regular singularity the valuation of each coefficient  $a_i$  of  $\mathcal{L}$  in monic form is at least  $i - r$ . Bounds on the degrees of the coefficients of factors are obtained by bounding the valuations of their coefficients in monic form at each singularity and at infinity, and by bounding the number of *apparent* singularities. Apparent singularities are poles of the coefficients where the operator has a basis of  $r$  formal power series solutions; they are regular. All these notions are classical and can be found for instance in Ince's book [Inc56].

**2.3.2. Newton polygons and valuations of the coefficients of the factors.** The Newton polygon of the operator  $\mathcal{L}$  from Eq. (1) at 0 is the convex hull of the union of the quadrants  $(i, \text{val}_0(a_i) - i) + (\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0})$ . The knowledge of the Newton polygon of  $\mathcal{L}$  at 0 gives lower bounds on the valuations of its coefficients. The main property of relevance here is that the Newton polygon of a product of operators is the (Minkowski) sum of their Newton polygons ([Mal79, Lemme 1.4.1]). For instance, when 0 is an ordinary point or a regular singularity of  $\mathcal{L}$ , the only slope of the lower part of its Newton polygon is 0 and this is therefore a property of the Newton polygons of the monic factors of  $\mathcal{L}$ , which reflects the fact that they are regular at 0 in that case.

More generally, let  $(x_0, y_0) = (0, y_0), \dots, (x_k, y_k) = (r, y_k)$  be the points on the lower part of the Newton polygon of  $\mathcal{L}$  and let  $((n_1, d_1), \dots, (n_k, d_k))$  with  $(n_i, d_i) = (x_i - x_{i-1}, y_i - y_{i-1})$  be the tuple of segments of the Newton polygon of  $\mathcal{L}$  sorted by increasing slope. Then the lowest possible Newton polygon for a monic factor of order  $m$  is obtained from the solution of the “0-1 knapsack problem”

$$\min \sum_{i=1}^k c_i n_i \quad \text{subject to} \quad \sum_{i=1}^k c_i d_i = m \quad \text{and} \quad c_i \in \{0, 1\}, \quad i = 1, \dots, k,$$

where  $c_i$  is either 1 or 0 depending on whether or not the slope  $(n_i, d_i)$  is used. This solution allows one to obtain lower bounds on the valuations at 0 of the coefficients of monic factors of  $\mathcal{L}$  of order  $m$ . The 0-1 knapsack problem is NP-hard but lower bounds can be found efficiently if needed [Vaz01, Ch. 8]. In practice, this has never been a costly step in our computations and an optimal value may lead to a better degree bound which saves computation time in other steps of the minimization algorithm.

The same process can be performed at every irregular singularity  $\alpha$  of  $\mathcal{L}$  by considering the Newton polygon formed from  $\text{val}_\alpha$  instead of  $\text{val}_0$ . Thus lower bounds on the valuations of the coefficients of a factor are found at each singularity, from the Newton polygon of  $\mathcal{L}$  and the order of the factor. Applying the same process at  $\infty$  (for instance by changing  $z$  into  $1/z$  and working at 0) gives bounds on the valuation at infinity of these coefficients.

**2.3.3. Fuchs’ relation and apparent singularities of the factors.** Let  $\mathcal{M}$  denote a monic right-factor of order  $m$  of the operator  $\mathcal{L}$  to be minimized. The study of the Newton polygons of  $\mathcal{L}$  provides lower bounds on the valuations of the coefficients of  $\mathcal{M}$  at the singularities of  $\mathcal{L}$ . We now show how to obtain an upper bound on the number of apparent singularities of  $\mathcal{M}$ ; together with the lower bounds on valuations, this will provide upper bounds on the degrees of (the polynomial version of)  $\mathcal{M}$ .

We first recall the principle of the method in the case where  $\mathcal{M}$  is Fuchsian, that is, if all its singularities (including  $\infty$ ) are regular. Fuchs’ relation [vdPS03, p. 138] states that

$$\sum_{\rho \in \text{Sing}(\mathcal{M})} S_\rho(\mathcal{M}) = -m(m-1), \quad (9)$$

where  $\text{Sing}(\mathcal{M})$  is the set of singularities of  $\mathcal{M}$ , including the apparent ones and infinity, and where

$$S_\rho(\mathcal{M}) := \sum_{j=1}^m e_j(\rho) - \frac{m(m-1)}{2}, \quad (10)$$

the numbers  $e_j(\rho)$  being the local exponents of  $\mathcal{M}$  at the point  $\rho$  (they are the roots of the indicial polynomial at  $\rho$ ). At an apparent singularity  $\rho$ , the quantity  $S_\rho(\mathcal{M})$  is a positive integer, so that the number of apparent singularities is upper bounded by

$$\#\text{App}(\mathcal{M}) \leq -m(m-1) - \sum_{\rho \in \sigma(\mathcal{M})} S_\rho(\mathcal{M}), \quad (11)$$

with  $\sigma(\mathcal{M})$  the subset of  $\text{Sing}(\mathcal{M})$  formed by the singularities of  $\mathcal{M}$  that are not apparent.

Since  $\mathcal{M}$  is a right factor of  $\mathcal{L}$ , the set  $\sigma(\mathcal{M})$  is a subset of  $\sigma(\mathcal{L})$ . The set  $\sigma(\mathcal{L})$  is known, since it corresponds to the roots of the leading coefficient  $a_r(z)$  that are not apparent singularities of  $\mathcal{L}$ , plus possibly  $\infty$ . Let  $\mu_1, \dots, \mu_s$  be the irreducible factors of  $a_r(z)$  corresponding to non-apparent singularities of  $\mathcal{L}$  and by convention let  $\mu_0 = z$ . At a finite  $\rho \in \sigma(\mathcal{L})$ , given by its minimal polynomial  $\mu_i$ , the indicial polynomial  $\text{ind}_\rho^{\mathcal{L}}(\theta) \in \mathbb{Q}(\rho)[\theta]$  is easily computed. Then the unknown indicial polynomial  $\text{ind}_\rho^{\mathcal{M}}(\theta) \in \mathbb{Q}(\rho)[\theta]$  has to be a factor of  $\text{ind}_\rho^{\mathcal{L}}(\theta)$  of degree exactly  $m$ . Let  $I_{i,j}(\theta)$ ,  $j = 1, \dots, k_i \leq r$  be the irreducible factors of  $\text{ind}_\rho^{\mathcal{L}}(\theta)$  in  $\mathbb{Q}(\rho)[\theta]$ , repeated with their multiplicity (and similarly  $I_{0,j}(\theta)$  denote the factors of the indicial polynomial of  $\mathcal{L}$  at infinity). The sum of the roots of  $I_{i,j}$  lies in  $\mathbb{Q}(\rho)$  and its sum over all roots of  $\mu_i$  is a rational number  $e_{i,j}$ . A bound on the number of apparent singularities is therefore obtained by solving the following integer linear programming problem [Sch86, Part VI]

$$\begin{aligned} \max A \quad \text{subject to} \quad & A = -m(m-1) - \sum_{i=0}^s \deg \mu_i \left( \sum_{j=1}^{k_i} c_{i,j} e_{i,j} - \frac{m(m-1)}{2} \right) \in \mathbb{N} \\ & \text{and for all } i \in \{0, \dots, s\}, \quad \sum_{j=1}^{k_i} c_{i,j} \deg I_{i,j} = m, \quad c_{i,j} \in \{0, 1\}. \end{aligned}$$

The constraints express the fact that there should be  $m$  exponents at each root of  $a_r$ , be them singular or ordinary for  $\mathcal{M}$ .

This process can be used whenever the right factor  $\mathcal{M}$  to be found is known to be Fuchsian, thus in particular when  $\mathcal{L}$  itself is Fuchsian.

Note that if there is no choice of  $c_{i,j}$  for which  $A \in \mathbb{N}$ , then there is no right factor of order  $m$ . As for the previous optimization problem, this is potentially a computationally expensive step. Simple upper bounds can be obtained by solving the relaxed linear programming problem where the constraints  $0 \leq c_{i,j} \leq 1$  replace the binary variables.

**2.3.4. Generalized Fuchs relation.** To an irregular singular point  $\rho$  of  $\mathcal{L}$  is associated a set of *exponential parts*. If  $\rho$  is finite, these are polynomials  $w(z)$  in some rational power  $1/r$  of  $z$  ( $r \in \mathbb{N}_{>0}$ ) such that  $\mathcal{L}$  admits a formal solution of the form

$$\exp\left(\int \frac{w(1/(z-\rho))}{z-\rho} dz\right) S(z), \quad S \in \mathbb{Q}[[ (z-\rho)^{1/r} ]][\log(z-\rho)], \quad \text{val}_{z=\rho} S = 0.$$

The case when  $\rho = \infty$  is obtained by changing  $z$  into  $1/z$  in the equation. That a full basis of formal solutions can be obtained in this way goes back to Fabry's classification [Fab85]. Algorithms for the computation of the list of exponential parts at a point  $\rho$  have been introduced early in computer algebra [Mal79, Thm. 4.2.1 and Cor. 4.3.1], [DDDT82, vH97b]. When  $\rho$  is a regular singular point, the exponential parts are constants that coincide with the roots of the indicial polynomial. In general, the *generalized local exponents* are the constant coefficients of the exponential parts. If each of them is counted with multiplicity  $r$ , their number is exactly the order of the operator.



When  $\mathcal{M}$  is a right factor of  $\mathcal{L}$ , its exponential parts at  $\rho$  form a subset of those of  $\mathcal{L}$ . If the order of  $\mathcal{M}$  is  $m$ , the Fuchs relation (9) generalizes as

$$\sum_{\rho \in \text{Sing}(\mathcal{M})} \left( S_\rho(\mathcal{M}) - \frac{1}{2} I_\rho(\mathcal{M}) \right) = -m(m-1), \quad (12)$$

where  $S_\rho$  is as in Eq. (10), with the generalized local exponents taking the place of the local exponents and

$$I_\rho(\mathcal{M}) := 2 \sum_{1 \leq i < j \leq m} \deg(w_i - w_j),$$

where the  $w_i$  are the exponential parts at  $\rho$ , see [Ber99, Thm. 2 and §5], [Ber88, p. 84]. As the  $w_i$  are polynomials in a fractional power of  $z$ , their degree here is a rational number. (The quantity  $I_\rho(\mathcal{M})$  is related to Malgrange's irregularity of  $\mathcal{M}$  at  $\rho$ ; see [BRS21, §2.2] for details which are not essential here.)

Thus, the analogue of Eq. (11) is

$$\#\text{App}(\mathcal{M}) \leq -m(m-1) - \sum_{\rho \in \sigma(\mathcal{M})} \left( S_\rho(\mathcal{M}) - \frac{1}{2} I_\rho(\mathcal{M}) \right). \quad (13)$$

The corresponding optimization problem is slightly more involved. As in the Fuchsian situation,  $\sigma(\mathcal{M}) \subset \sigma(\mathcal{L})$  and we denote by  $\mu_1, \dots, \mu_s$  the irreducible factors of the leading coefficient  $a_r(z)$  that correspond to non-apparent singularities of  $\mathcal{L}$  and  $\mu_0 = z$ . At a finite  $\rho \in \sigma(\mathcal{L})$ , given by its minimal polynomial  $\mu_i$ , the exponential parts are given as

$$w_{i1}((z - \rho)^{1/r_{i1}}), \dots, w_{ik_i}((z - \rho)^{1/r_{ik_i}})$$

with minimal  $r_{ij}$ . Each contributes  $r_{ij}$  times its constant coefficient to the set of generalized local exponents of  $\mathcal{L}$  at  $\rho$ , so that  $\sum r_{ij} = \text{ord}(\mathcal{L})$ . The exponential parts of  $\mathcal{M}$  at  $\rho$  form a subset of those of  $\mathcal{L}$ . This property, combined with the generalized Fuchs relation (12), leads to the following integer linear programming problem

$$\begin{aligned} & \max A \quad \text{subject to} \\ A &= -m(m-1) - \sum_{i=0}^s \sum_{\mu_i(\rho)=0} \left( \sum_{j=1}^{k_i} c_{i,j} (r_{i,j} w_{i,j}(0) - \frac{r_{i,j}(r_{i,j}-1)}{2} \deg w_{i,j}) - \frac{m(m-1)}{2} \right) \\ & \quad + \sum_{i=0}^s \deg \mu_i \sum_{j=1}^{k_i} \sum_{1 \leq k \neq j \leq k_i} d_{i,\{j,k\}} \deg(w_{ij} - w_{ik}) \in \mathbb{N} \\ & \text{and for all } i \in \{0, \dots, s\}, \quad \sum_{j=1}^{k_i} c_{i,j} r_{i,j} = m, \quad c_{i,j} \in \{0, 1\}, \\ & \text{and for all } (i, j), \quad \sum_{k \neq j} d_{i,\{j,k\}} = c_{i,j}(m-1), \quad d_{i,\{j,k\}} \in \{0, 1\}. \end{aligned}$$

The last set of constraints consists in adding one variable for each pair of  $(w_{ij}, w_{ik})$  and forcing the sum of these variables for fixed  $i$  to be the number of pairs, namely  $m - 1$ , an idea taken from [DAS15].

2.3.5. *Example.* Consider the equation

$$zy'' + (1 - 6z)y' + (z - 3)y = 0,$$

with initial condition  $y(0) = 1$ , which specifies a unique power series solution  $S(z) = 1 + 3z + 13z^2/2 + \dots$ . It has two singular points, at 0 and  $\infty$ . The point 0 is regular, with exponents 0, 0. The point  $\infty$  is irregular, with exponential parts  $w_{\pm} = \alpha_{\pm}z + 1/2$ , where  $\alpha_{\pm} = -3 \pm 2\sqrt{2}$ , corresponding to formal solutions  $\exp(-\alpha_{\pm}z)/\sqrt{z}$  at infinity and both generalized exponents are equal to 1/2. In the notation above, we have

$$s = 1, \quad \mu_0 = \mu_1 = z, \quad k_0 = k_1 = 2, \quad r_{0,1} = r_{0,2} = r_{1,1} = r_{1,2} = 1, \\ w_{0,1} = w_+, w_{0,2} = w_-, w_{1,1} = w_{1,2} = 0.$$

Looking for a right factor of order  $m$  leads to maximizing  $A \in \mathbb{N}$  such that

$$A = -m(m - 1) - (c_{0,1}/2 + c_{0,2}/2 - m(m - 1)/2) + d_{0,\{1,2\}},$$

with the constraints

$$c_{0,1} + c_{0,2} = m, \quad d_{0,\{1,2\}} = c_{0,1}(m - 1) = c_{0,2}(m - 1), \\ c_{0,1}, c_{0,2}, d_{0,\{1,2\}} \text{ in } \{0, 1\}.$$

For the order  $m = 1$  of a right factor, the last constraints force  $d_{0,\{1,2\}} = 0$ ,  $c_{0,1} + c_{0,2} = 1$ , which makes  $A < 0$ , showing that there is no solution and thus no factorization with a right factor of order 1; the equation is minimal.

2.3.6. *Example.* We show in more detail the computation for the order 10 differential equation (8) of Section 2.2.3.

There are two singularities: 0 and infinity. The point 0 is regular, with exponents

$$0, 0, 0, 0, 1, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4,$$

with  $\alpha_i$  the four roots of the irreducible polynomial

$$P_{\alpha} = 29412x^4 - 246240x^3 + 764259x^2 - 1042332x + 527381.$$

The point  $\infty$  is *irregular*. Its exponential parts are

$$1, 1, \beta_i \ (i = 1, \dots, 4), \gamma_i x + 3/2 \ (i = 1, \dots, 4),$$

with  $\beta_i$  and  $\gamma_i$  roots of the irreducible polynomials

$$P_{\beta} = 29412x^4 - 342000x^3 + 1482459x^2 - 2838258x + 2024696, \quad P_{\gamma} = x^4 + 16x^3 - 112x^2 + 284x + 4.$$

Thus, for this equation,

$$S_0(\mathcal{L}) = 2 + \sum_i \alpha_i - 45 = -\frac{1489}{43}, \quad S_{\infty}(\mathcal{L}) = 2 + \sum_i \beta_i + 4\frac{3}{2} - 45 = -\frac{1901}{43}, \quad I_{\infty}(\mathcal{L}) = 60$$

and the generalized Fuchs equation reduces to

$$-\frac{1489}{43} - \frac{1901}{43} - 30 = -90.$$

For lower orders  $m$ , the optimization problem to be solved is

$$\begin{aligned} & \max A \quad \text{subject to} \\ & A = -m(m-1) - \left( c_{0,1} + c_{0,2} + c_{0,\{3,4,5,6\}} \frac{500}{43} + c_{0,\{7,8,9,10\}} 4 \frac{3}{2} - \frac{m(m-1)}{2} \right) \\ & \quad - \left( c_{1,5} + c_{1,6} + c_{1,\{7,8,9,10\}} \frac{360}{43} - \frac{m(m-1)}{2} \right) \\ & + 4d_{0,\{1,\{7,8,9,10\}\}} + 4d_{0,\{2,\{7,8,9,10\}\}} + 16d_{0,\{\{3,4,5,6\},\{7,8,9,10\}\}} + 6d_{0,\{\{7,8,9,10\},\{7,8,9,10\}\}} \in \mathbb{N} \end{aligned}$$

with constraints

$$\begin{aligned} c_{0,1} + c_{0,2} + 4c_{0,\{3,4,5,6\}} + 4c_{0,\{7,8,9,10\}} &= c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + c_{1,6} + 4c_{1,\{7,8,9,10\}} = m \\ d_{0,\{1,2\}} + 4d_{0,\{1,\{3,4,5,6\}\}} + 4d_{0,\{1,\{7,8,9,10\}\}} &= c_{0,1}(m-1), \\ d_{0,\{1,2\}} + 4d_{0,\{2,\{3,4,5,6\}\}} + 4d_{0,\{2,\{7,8,9,10\}\}} &= c_{0,2}(m-1), \\ d_{0,\{1,\{3,4,5,6\}\}} + d_{0,\{2,\{3,4,5,6\}\}} + 3d_{0,\{3,4,5,6\},\{3,4,5,6\}} + 4d_{0,\{\{3,4,5,6\},\{7,8,9,10\}\}} &= (m-1)c_{0,\{3,4,5,6\}}, \\ d_{0,\{1,\{7,8,9,10\}\}} + d_{0,\{2,\{7,8,9,10\}\}} + 4d_{0,\{\{3,4,5,6\},\{7,8,9,10\}\}} + 3d_{0,\{7,8,9,10\},\{7,8,9,10\}} &= (m-1)c_{0,\{7,8,9,10\}}, \\ & \text{and for all } (i, j, k), \quad c_{i,j} \in \{0, 1\}, \quad d_{i,\{j,k\}} \in \{0, 1\}. \end{aligned}$$

Integrality of  $A$  forces  $c_{0,\{3,4,5,6\}} = c_{1,\{7,8,9,10\}}$ . If they are both equal to 1, the first two lines of the constraint on  $A$  give a quantity that is at most  $-20$ . Making  $A \geq 0$  then requires  $d_{0,\{3,4,5,6\},\{7,8,9,10\}} = 1$ . The last constraint then makes  $c_{0,\{7,8,9,10\}} = 1$ , which turns the constraint on  $A$  into

$$-20 - 6 + 16 + 4d_{0,\{1,\{7,8,9,10\}\}} + 4d_{0,\{2,\{7,8,9,10\}\}} + 6d_{0,\{\{7,8,9,10\},\{7,8,9,10\}\}} \geq 0.$$

Therefore  $d_{0,\{\{7,8,9,10\},\{7,8,9,10\}\}} = 1$  and at least one of  $d_{0,\{1,\{7,8,9,10\}\}}$  and  $d_{0,\{2,\{7,8,9,10\}\}}$  is 1 too. The last constraint then shows that  $m = 9$  or  $m = 10$  depending on whether one or two of them are 1. We know that  $m = 10$  is possible: it is the original equation. If  $m = 9$ , then the first constraint gives  $c_{0,1} + c_{0,2} = 1$ . Injecting into the constraint for  $A$  makes  $A < 0$ , a contradiction.

We have thus proved that for a strict factor of  $\mathcal{A}$ ,  $c_{0,\{3,4,5,6\}} = c_{1,\{7,8,9,10\}} = 0$ . This makes all variables 0 in the left-hand side of the penultimate constraint. The last constraint becomes

$$d_{0,\{1,\{7,8,9,10\}\}} + d_{0,\{2,\{7,8,9,10\}\}} + 3d_{0,\{7,8,9,10\},\{7,8,9,10\}} = (m-1)c_{0,\{7,8,9,10\}}.$$

If  $c_{0,\{7,8,9,10\}}$  was equal to 0, then the constraint on  $A$  would give  $c_{0,1} = c_{0,2} = 0$  too, which would give  $m = 0$  in the second one, a contradiction. Therefore  $c_{0,\{7,8,9,10\}} = 1$ . The

remaining constraints are

$$\begin{aligned}
A &= -(c_{0,1} + c_{0,2} + 6 + c_{1,5} + c_{1,6}) \\
&\quad + 4d_{0,\{1,\{7,8,9,10\}\}} + 4d_{0,\{2,\{7,8,9,10\}\}} + 6d_{0,\{\{7,8,9,10\},\{7,8,9,10\}\}} \geq 0, \\
c_{0,1} + c_{0,2} + 4 &= c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + c_{1,6} = m, \\
d_{0,\{1,2\}} + 4d_{0,\{1,\{7,8,9,10\}\}} &= c_{0,1}(m - 1), \\
d_{0,\{1,2\}} + 4d_{0,\{2,\{7,8,9,10\}\}} &= c_{0,2}(m - 1), \\
d_{0,\{1,\{7,8,9,10\}\}} + d_{0,\{2,\{7,8,9,10\}\}} + 3d_{0,\{7,8,9,10\},\{7,8,9,10\}} &= m - 1.
\end{aligned}$$

The second one then implies that *the order of a strict right factor of  $\mathcal{L}$  can only be one of  $\{4, 5, 6\}$ .*

With  $m = 6$ , there is only one solution (meaning that no optimization is needed), with all the remaining variables equal to 1 and the bound  $A$  on the number of apparent singularities equal to 4. There are therefore at most 5 regular singularities (these four and 0, which is a regular singularity of  $\mathcal{L}$ ). Such a factor can be written

$$\mathcal{M} = \partial_z^6 + \frac{a_5}{A} \partial_z^5 + \cdots + \frac{a_0}{A^6},$$

with  $A$  of degree at most 5. The Newton polygon of  $\mathcal{L}$  at infinity has for vertices  $(0, 0)$ ,  $(6, 0)$ ,  $(10, 4)$ . The largest possibility for  $\mathcal{M}$  is therefore  $(0, 0)$ ,  $(2, 0)$ ,  $(6, 4)$ , leading to the following bounds on the degree of the numerators  $a_i$ :  $(\deg A - 1, \deg A^2 - 2, \deg A^3 - 4, \deg A^4 - 4, \deg A^5 - 4, \deg A^6 - 4)$ . Reducing to the same denominator gives the bounds  $(30, 29, 28, 26, 26, 26, 26)$  on the degrees of the coefficients of  $(\partial^6, \dots, 1)$ . This is the bound used in Example 2.2.3, leading to the discovery of the factor  $\mathcal{M}$ .

With  $m = 5$ , there are several solutions, which are as in the case when  $m = 6$ , but with one of  $c_{0,1}$  and  $c_{0,2}$  equal to 0 and consequently  $d_{0,\{1,2\}} = 0$  and one of  $d_{0,\{1,\{7,8,9,10\}\}}$ ,  $d_{0,\{2,\{7,8,9,10\}\}}$  equals 0, leading to a bound  $A \leq 6 + 4 - (6 + 1 + 1) = 2$  on the number of apparent singularities. By the same reasoning as above, this leads to the bounds  $(15, 14, 13, 13, 13, 13)$  on the degrees of the coefficients of a factor of order 5. Using about 90 coefficients of the series shows that such a factor does not exist.

Finally, with  $m = 4$  the only of the remaining  $d$  variables that is not 0 is  $d_{0,\{7,8,9,10\},\{7,8,9,10\}}$  and the bound on  $A$  becomes  $6 - (6) = 0$ . Again, a computation with degree bounds  $(5, 4, 3, 3, 3)$  proves that no factor of degree 4 exists.

*Note on the relaxed problem.* For efficiency reasons, one may prefer solving the relaxed optimization problem where the variables  $c_{i,j}$  and  $d_{i,\{j,k\}}$  are not restricted to the set  $\{0, 1\}$ , but can be real numbers in the interval  $[0, 1]$ . Also, during the optimization,  $A$  is not restricted to be an integer. Then what happens in this example is that the absence of a factor of order 9 is not detected. Instead, the optimization finds a solution with  $A = 4.5$ , which leads to a bound of 4 on the number of apparent singularities and thus of 45 on the degree of the coefficients. With this bound, sufficiently many coefficients are computed so that the computation of an approximant basis finds a candidate operator. This turns out to be the factor  $\mathcal{M}$  of order 6 above. The next stage is to prove its minimality. For order 5, the optimization of the relaxed problem gives a bound equal to 5 for  $A$  (to be compared

with 2 above), leading to a bound of 30 on the degree of the coefficients. Using about 190 coefficients of the series (instead of 90 above) shows that no such factor exists. For lower orders, the relaxed problem does not have any solution, concluding the computation.

**2.4. Inhomogeneous minimization.** Again, we consider an equation like Eq. (1) and initial conditions for a unique formal power series  $S$  solution of it. Using the method of the previous section, we can assume that it has minimal order. The problem of inhomogeneous minimization is to find an equation

$$\mathcal{M}(y(z)) = B(z), \quad \text{with } \mathcal{M} = b_s(z)\partial_z^s + \cdots + b_0(z),$$

with  $s < r$  and rational function coefficients  $b_0, \dots, b_s, B$  ( $b_s \neq 0$ ), having  $S$  as a solution. When such an equation exists with  $B \neq 0$ , applying  $B\partial_z - B'$  on both sides of the equation yields a homogeneous linear differential equation of order  $s + 1$  satisfied by  $S$ , so that minimality of  $\mathcal{L}$  implies  $s = r - 1$ . Without loss of generality (up to replacing  $\mathcal{M}$  by  $\frac{1}{B}\mathcal{M}$ ) one can assume  $B(z) = 1$  and then differentiation implies

$$\partial\mathcal{M} = R(z)\mathcal{L}$$

for some non-zero rational function  $R$ , which is therefore an *integrating factor* of  $\mathcal{L}$ . This implies that  $R$  is a rational function solution of the adjoint equation [Poo60, Chap. III.§10]

$$\mathcal{L}^*(R) = 0. \tag{14}$$

Finding rational solutions of linear differential equations is a classical problem, whose solution can be found by an algorithm due to Abramov [Abr89, AK91], with roots in Liouville's work [Lio33]. This algorithm returns a basis of rational solutions of Eq. (14). This is a decision algorithm: if no non-zero rational solution is found, this proves that there is no inhomogeneous linear differential equation of order smaller than  $r$  satisfied by the power series  $S$ . Otherwise, minimality implies that the basis consists of one solution  $R(z)$ . The operator  $\mathcal{M}$  (known as the *bilinear concomitant* [Poo60]) can be reconstructed coefficient by coefficient (this is equivalent to [AR18, p. 703]). Then by design,  $\mathcal{M}(S)$  is a constant  $c$ , which can be computed from the first coefficients of the power series  $S$ , completing the computation of the minimal inhomogeneous equation  $\mathcal{M}(y) = c$  satisfied by  $S$ . This computation is summarized in Algorithm 2.

### 3. EFFICIENT COMPUTATION OF THE SET OF ALGEBRAIC VALUES TAKEN BY $E$ -FUNCTIONS AT ALGEBRAIC POINTS

**3.1. The Adamczewski-Rivoal algorithm.** In this section, we consider mainly  $E$ -functions with Taylor coefficients in  $\mathbb{Q}$ . An  $E$ -function over  $\mathbb{Q}$  is a power series

$$f(z) := \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \text{ in } \mathbb{Q}[[z]]$$

with  $a_n \in \mathbb{Q}$  and such that there exists  $C > 0$  with the following properties:

- (i)  $f$  satisfies a homogeneous linear differential equation with coefficients in  $\mathbb{Q}(z)$ ;
- (ii) for any  $n \geq 0$ ,  $|a_n| \leq C^{n+1}$ ;

**Algorithm 2** Minimal inhomogeneous linear differential equation

---

*Input:*  $\mathcal{L} = a_r(z)\partial_z^r + \cdots + a_0(z)$ ,  
a linear operator of minimal order that vanishes at  $S$ ;  
ini:  $S_0$  a truncated power series at precision  $\geq \max \mathcal{Z}_{\mathcal{L}}$

*Output:*  $\mathcal{M} = b_{r-1}(z)\partial_z^{r-1} + \cdots + b_0(z)$  and  $B(z) \in \mathbb{Q}(z)$ , such that  $\mathcal{M}(S) = B$   
or FAIL if no such pair exists.

- 1:  $\mathcal{L}^* := \text{adjoint}(\mathcal{L})$ ;
- 2:  $\mathcal{S} = \text{BASISRATIONALSOLUTIONS}(\mathcal{L}^*)$
- 3: **if**  $\mathcal{S} = \emptyset$  **then return** FAIL
- 4: Let  $R$  be the unique element of  $\mathcal{S}$
- 5:  $b_{r-1} := Ra_r$
- 6: **for**  $j = r - 2, \dots, 0$  **do**  $b_j := Ra_{j+1} - b'_{j+1}$
- 7: Compute  $S$  up to precision  $r - \min_j \text{val}_0(b_j)$
- 8: Let  $B$  be the constant term of  $\mathcal{M}(S)$
- 9: **return**  $\mathcal{M}, B$

---

(iii) for any  $n \geq 0$ , there exists  $d_n \in \mathbb{N} \setminus \{0\}$  such that  $d_n \leq C^{n+1}$  and  $d_n a_m \in \mathbb{Z}$  for all  $0 \leq m \leq n$ .

We shall sometimes simply write ‘‘LDE’’ for ‘‘linear differential equation with coefficients in  $\mathbb{Q}(z)$  or in  $\overline{\mathbb{Q}}(z)$ ’’; unless otherwise stated, an LDE will be assumed to be homogeneous. In the rest of this section,  $E$ -functions over  $\mathbb{Q}$  are simply called  $E$ -functions. This is justified by the fact that most of the discussion applies to more general settings, in particular to  $E$ -functions with Taylor coefficients in  $\overline{\mathbb{Q}}$  and to  $E$ -functions in Siegel’s more general sense, as discussed in Sections 3.4 and 3.5.

$E$ -functions are entire functions (by (ii)). Polynomials in  $\mathbb{Q}[z]$  are trivial examples of  $E$ -functions; all non-polynomial  $E$ -functions are transcendental over  $\mathbb{Q}(z)$ . The class of  $E$ -functions includes the exponential function  $\exp(z)$ , Bessel’s function of the first kind

$$J_0(z) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2} \left(\frac{z}{2}\right)^{2m} = {}_0F_1[\cdot; 1; -z^2/4],$$

and more generally the *hypergeometric  $E$ -functions*, i.e. series of the form

$${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; \lambda z^{q-p+1}] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} \lambda^n z^{(q-p+1)n}$$

with rational parameters  $a_i, b_j$ ,  $q \geq p \geq 0$ ,  $\lambda \in \overline{\mathbb{Q}}^*$  and where  $(\alpha)_0 := 1$ ,  $(\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1)$  for  $n \geq 1$ .  $E$ -functions form a sub-ring of the ring of formal power series in  $\mathbb{Q}[[z]]$ , stable by  $d/dz$  and  $\int_0^z$ ; these properties can be used to construct many examples of  $E$ -functions starting from hypergeometric series. Shidlovskii has proved in [Shi89, p. 184] that any  $E$ -function solution of an LDE of order 1 is of the form  $p(z)e^{\lambda z}$  for some  $p(z) \in \overline{\mathbb{Q}}[z]$  and  $\lambda \in \overline{\mathbb{Q}}$ . Gorelov has proved in [Gor04] that any  $E$ -function solution of an LDE of order 2 is a  $\overline{\mathbb{Q}}(z)$ -linear combination of hypergeometric  $E$ -functions



with  $p = q = 1$  (he had obtained earlier in [Gor00] a similar but more precise result for  $E$ -functions solution of an inhomogeneous LDE of order 1). However, Fresán and Jossen have recently showed in [FJ21] that not all  $E$ -functions are  $\overline{\mathbb{Q}}(z)$ -linear combinations of hypergeometric  $E$ -functions, nor even more generally polynomials in hypergeometric  $E$ -functions with algebraic coefficients.

As of today, no algorithm is known neither for deciding whether a linear differential equation  $\mathcal{L}(y(z)) = 0$  admits solutions that are  $E$ -functions, nor for deciding whether a solution of  $y(z)$  of  $\mathcal{L}$ , uniquely determined by sufficiently many initial conditions, is an  $E$ -function. It is actually not clear whether these questions are decidable or not. Consequently, the algorithm described below relies on the following assumption:

(A) An oracle guarantees that the input  $f$  is an  $E$ -function.

In practice, an  $E$ -function is given by an explicit expression for its Taylor coefficients as a multiple hypergeometric sum and  $\mathcal{L}$  can then be computed for instance by Zeilberger's creative telescoping algorithm [Zei91].

Siegel initiated a program to determine when an  $E$ -function takes a transcendental value at an algebraic point [Sie14]. This culminated with the celebrated Siegel-Shidlovskii theorem: given a vector  $Y$  of  $E$ -functions  $f_1, \dots, f_n$  solution of a differential system  $Y' = AY$  with a matrix  $A$  with elements in  $\overline{\mathbb{Q}}(z)$ , the transcendence degree over  $\overline{\mathbb{Q}}(z)$  of the field generated by  $f_1(z), \dots, f_n(z)$  over  $\overline{\mathbb{Q}}(z)$  is equal to the transcendence degree over  $\overline{\mathbb{Q}}$  of the field generated by  $f_1(\alpha), \dots, f_n(\alpha)$  over  $\overline{\mathbb{Q}}$  for every non-zero algebraic number  $\alpha$  which is not a singularity of  $A$  (i.e., a pole of some element of  $A$ ). In 2006, Beukers [Beu06, Thm. 1.3] refined this theorem by proving that any homogeneous polynomial relation between the values  $f_1(\alpha), \dots, f_n(\alpha)$  with coefficients in  $\overline{\mathbb{Q}}$  is a specialization of a homogeneous polynomial relation between the functions  $f_1(z), \dots, f_n(z)$  with coefficients in  $\overline{\mathbb{Q}}(z)$ , again when  $\alpha$  is not a singularity of  $A$ . A less precise version of this theorem (but for  $E$ -functions in Siegel's more general sense; see Section 3.5 for details) had been obtained in 1996 by Nesterenko and Shidlovskii [NS96], where  $\alpha$  is simply assumed not to lie in a certain finite set  $S$ , depending on the  $f_j$ 's but not specified in their article. A fundamental consequence of their result is that a transcendental  $E$ -function  $f$  takes only finitely many algebraic values when evaluated at algebraic points. To see this, one considers a non-trivial minimal inhomogeneous differential equation with polynomial coefficients  $p + \sum_{j=0}^{\mu} p_j f^{(j)} = 0$  satisfied by  $f$  over  $\overline{\mathbb{Q}}(z)$  and applies the Nesterenko-Shidlovskii theorem to the functions  $f_1 := 1, f_2 := f, \dots, f_{\mu} := f^{(\mu-1)}$ . They are linearly independent over  $\overline{\mathbb{Q}}(z)$ , hence the numbers  $1, f(\alpha), \dots, f^{(\mu-1)}(\alpha)$  are  $\overline{\mathbb{Q}}$ -linearly independent over  $\overline{\mathbb{Q}}$  for all  $\alpha \in \overline{\mathbb{Q}} \setminus S$ ; in particular  $f(\alpha) \notin \overline{\mathbb{Q}}$  for all such  $\alpha$ 's. For  $E$ -functions in the strict sense, we now know thanks to Beukers [Beu06] that if  $\alpha \in \overline{\mathbb{Q}}^*$  is not a root of the leading coefficient  $p_{\mu}$  above, then  $f(\alpha) \notin \overline{\mathbb{Q}}$ .

Thus, in order to completely determine when an  $E$ -function takes a transcendental value at a given non-zero algebraic point, one issue needs to be dealt with: what happens for the (finite number of) algebraic numbers that are roots of  $p_{\mu}$  (in the same setting as above). This was done by Adamczewski and Rivoal [AR18] by pushing further Beukers' ideas from [Beu06]. The end result is an algorithm that takes as input an  $E$ -function  $f$

**Algorithm 3** Algebraic values of  $E$ -functions over  $\mathbb{Q}$ 


---

*Input:*  $\mathcal{L} = a_r(z)\partial_z^r + \dots + a_0(z)$ ;  
 ini:  $f_0$  a truncated power series at precision  $p_0 \geq r$   
 specifying a unique solution  $f \in \mathbb{Q}[[z]]$  of  $\mathcal{L}(f) = 0$ .  
*It is assumed that  $f$  is an  $E$ -function.*

*Output:* Either “ $f$  is a polynomial”,  
 or the finite set of all identities  $f(\alpha) = \beta$  with algebraic  $\alpha$  and  $\beta$ .

$\mathcal{L}_{\min} := \text{MINIMALRIGHTFACTOR}(\mathcal{L}, \text{ini})$  // Algorithm 1  
 $\mathcal{L}_{\text{inhom}}, g := \text{MINIMALINHOMOGENEOUSRIGHTFACTOR}(\mathcal{L}_{\min})$  // Algorithm 2  
**if**  $\text{ord}\mathcal{L}_{\text{inhom}} = 0$  **then return**  $f$  is a polynomial

Define the polynomials  $v_0, \dots, v_{s+1}$  by  $\mathcal{L}_{\text{inhom}}(f) - g = v_0 f^{(s)} - v_1 f^{(s-1)} - \dots - v_s f - v_{s+1}$   
 Form the companion matrix  $M$  s.t.  $(0, f', f'', \dots, f^{(s)})^\top = M \cdot (1, f, f', \dots, f^{(s-1)})^\top$   
 $\mathcal{R} := \{f(0) = f_0(0)\}$   
**for**  $\mu \in \mathbb{Q}[z]$  irreducible factor of  $v_0$  with  $\mu(0) \neq 0$  **do**  
 Write  $\alpha$  for a root of  $\mu$   
 $B := \text{BEUKERSALGO}(M, \alpha)$   
 $(b_1, \dots, b_m) := \text{basis of the left kernel of } B(\alpha)$  // Basis of algebraic relations at  $\alpha$   
**if** there exists  $(\beta, -1, 0, \dots, 0)$  in  $\text{Span}(b_1, \dots, b_m)$  **then**  
 $\mathcal{R} := \mathcal{R} \cup \{f(\alpha) = \beta\}$   
**return**  $\mathcal{R}$

---

and either detects that  $f$  is algebraic (in which case it is even a polynomial), or computes the (finite) list of identities  $f(\alpha) = \beta$  for algebraic values  $\alpha$  and  $\beta$ .

Algorithm 3 gives a version of the algorithm by Adamczewski and Rivoal that benefits from the minimization algorithms of Section 2. It is stated here for the  $E$ -functions over  $\mathbb{Q}$  considered in this section (see the comments in Sections 3.4 and 3.5 below for its extension to more general settings). The algorithm relies on two results due to Beukers [Beu06]:

- (1) If  $F = (f_1, \dots, f_n)^\top$  with  $E$ -functions  $f_i$  is a solution of  $Y' = AY$ , the entries of  $A$  belonging to  $\overline{\mathbb{Q}}(z)$  and if  $f_1(z), \dots, f_n(z)$  are linearly independent over  $\overline{\mathbb{Q}}(z)$ , then for any non-zero  $\alpha$  that is not a pole of  $A$ , the numbers  $f_1(\alpha), \dots, f_n(\alpha)$  are linearly independent over  $\overline{\mathbb{Q}}$  [Beu06, Corollary 1.4];
- (2) Under the same assumptions, there exists a matrix  $M$  with entries in  $\overline{\mathbb{Q}}[z]$  such that  $F = ME$ , and  $E$  is a vector of  $E$ -functions solution to a system  $Y' = BY$  where  $B$  does not have any non-zero pole [Beu06, Theorem 1.5].

Starting from a linear differential operator  $\mathcal{L}$  and initial conditions specifying an  $E$ -function  $f$ , the algorithm first computes a minimal inhomogeneous equation of order  $s$  for  $f$ . (This step also allows to detect and discard a polynomial  $f$ .) By minimality of this equation,  $F = (1, f, f', \dots, f^{(s-1)})$  is a vector of  $\overline{\mathbb{Q}}(z)$ -linearly independent  $E$ -functions solution to a matrix deduced from the equation. Given a matrix  $M$  as in the result (2) above, it follows that the points  $\alpha$  where  $(1, f(\alpha), \dots, f^{(s-1)}(\alpha))$  are linearly dependent

over  $\overline{\mathbb{Q}}$  are non-zero poles of  $A$  where the left-kernel of  $M$  is not reduced to 0. The specific case of  $f(\alpha)$  being algebraic corresponds to the existence of a non-zero vector in that kernel whose first two coordinates only are not zero. The remaining question is the computation of these matrices  $M$ , which is described in Section 3.2.

Note that the first two steps of Algorithm 3, i.e. the calls to `MINIMALRIGHTFACTOR( $\mathcal{L}$ ,ini)` and `MINIMALINHOMOGENEOUSRIGHTFACTOR( $\mathcal{L}_{\min}$ )`, are not specific to  $E$ -functions, and both return an output even when  $f$  is not an  $E$ -function. In that case, `BEUKERSALGO( $M, \alpha$ )` terminates (by design) and it may even output  $\alpha$ 's such that  $f(\alpha) \in \overline{\mathbb{Q}}$ ; but it can no longer be claimed that all such  $\alpha$ 's have been found.

**3.2. Beukers' algorithm and desingularization.** Algorithm 3 concludes with a call to Algorithm `BEUKERSALGO( $M, \alpha$ )` described below. It is a clever desingularization process, which is different from the one developed by Barkatou and Maddah [BM15], in that it does not rely on Moser's reduction [Mos60, Bar95]. The end result is the following (Theorem 1.5 in [Beu06]).

**Theorem 3.1.** *Let  $Y = (f_1, \dots, f_n)^T$  be a vector of  $\overline{\mathbb{Q}}(z)$ -linearly independent  $E$ -functions satisfying  $Y' = AY$ , where  $A$  is an  $n \times n$  matrix with entries in  $\overline{\mathbb{Q}}(z)$ . Then, there exists a vector of  $E$ -functions  $Z = (e_1, \dots, e_n)^T$  solution of  $Z' = BZ$  with  $B$  having entries in  $\overline{\mathbb{Q}}[z, 1/z]$ , and there exists a polynomial matrix  $M$  with entries in  $\overline{\mathbb{Q}}[z]$  and  $\det(M) \neq 0$ , such that  $(f_1, \dots, f_n)^T = M \cdot (e_1, \dots, e_n)^T$ .*

The key properties used in the proof of Theorem 3.1 are the statements **(P1)**, **(P1')** and **(P2)** listed below. Note that for an  $E$ -function  $f \in \overline{\mathbb{Q}}[[z]]$ , we write  $L_f^{\min}$  for the monic linear differential operator in  $\mathbb{Q}(z)\langle \partial_z \rangle$  of minimal order that cancels  $f$ .

**(P1)** For any  $E$ -function  $f \in \overline{\mathbb{Q}}[[z]]$ , the finite non-zero singularities of  $L_f^{\min}$  are apparent;

**(P1')** If  $A$  is an  $n \times n$  matrix with entries in  $\overline{\mathbb{Q}}(z)$ , and if  $F$  is a vector of  $\overline{\mathbb{Q}}(z)$ -linearly independent  $E$ -functions satisfying  $F' = AF$ , then the finite non-zero singularities of the system  $Y' = AY$  are apparent;

**(P2)** If an  $E$ -function  $f$  and  $\alpha \in \overline{\mathbb{Q}}$  are such that  $f(\alpha) \in \overline{\mathbb{Q}}$ , then  $(f(z) - f(\alpha))/(z - \alpha)$  is an  $E$ -function.

Property **(P1)** is *André's theorem* [And00a, Cor. 4.4] and **(P2)** is an important property of  $E$ -functions proved by Beukers [Beu06, Prop. 4.1]. Property **(P1')** is a system version of André's theorem, which is not, to our knowledge, explicitly stated in the literature, although it is implicitly contained in Beukers' proof of his Theorem 1.5 in [Beu06, p. 378]. For completeness, we detail the proof of **(P1')**, which goes along the following lines.

*Proof of (P1').* Let  $\mathcal{G}$  be the differential Galois group of (the Picard-Vessiot field of)  $Y' = AY$ . Let  $V$  be the  $\overline{\mathbb{Q}}$ -vector space generated by the orbit  $\{\sigma(F) \mid \sigma \in \mathcal{G}\}$ , where  $F = (f_1, \dots, f_n)^T$  is a vector of  $E$ -functions satisfying  $F' = AF$ , linearly independent over  $\overline{\mathbb{Q}}(z)$ . The conclusion of **(P1')** clearly follows by combining the following two steps.

**Step 1.** The dimension of  $V$  over  $\overline{\mathbb{Q}}$  is equal to  $n$ , hence one can extract from  $V$  a fundamental matrix of solutions  $\mathcal{F}$ , whose first column is  $F$ .

**Step 2.** All the entries of  $\mathcal{F}$  are holomorphic at all non-zero points  $\alpha \in \mathbb{C} \setminus \{0\}$ .

*Proof of Step 2:* Let  $L_i := L_{f_i}^{\min}$ . Since elements of  $\mathcal{G}$  commute with differentiation, all  $\sigma(f_i)$  are solutions of  $L_i$  for all  $\sigma \in \mathcal{G}$ . By André's theorem (P1),  $\sigma(f_i)$  has no true singularity in  $\mathbb{C} \setminus \{0\}$ . Hence  $\mathcal{F}$  is holomorphic at any  $\alpha \in \mathbb{C} \setminus \{0\}$ .

*Proof of Step 1:* If  $A$  is a companion matrix, then the shape of the system  $Y' = AY$  implies that  $F$  is of the form  $F = (f, f', \dots, f^{(n-1)})^T$ , where  $f$  is the  $E$ -function  $f = f_1$ . The linear independence assumption implies that  $L_f^{\min}$  has order  $n$ . By [vdPS03, Corollary 1.38] (see also [BB85, p. 190, Proposition 3]), the dimension of the  $\overline{\mathbb{Q}}$ -vector space  $\tilde{V}$  generated by the orbit  $\{\sigma(f) \mid \sigma \in \mathcal{G}\}$  is equal to  $n$ . On the one hand, this dimension is upper bounded by the dimension of  $V$ , since any linear relation among the entries of  $V$  yields a linear relation among the elements of the set  $\{\sigma(f) \mid \sigma \in \mathcal{G}\}$ . On the other hand,  $V$  is included in the solution space of  $Y' = AY$ , hence it has dimension at most  $n$ . Therefore,  $\dim_{\overline{\mathbb{Q}}}(V) = n$ , and the assertion is proved in the companion case.

Now, if  $A$  is a general matrix, by the cyclic vector lemma (see e.g. [CK02, Thm 3.11], or [vdPS03, Proposition 2.9]) the system  $Y' = AY$  is “gauge equivalent” to  $Z' = CZ$ , where  $C$  is a companion matrix with entries in  $\overline{\mathbb{Q}}(z)$ . This means that there exists an invertible matrix  $P$  with entries in  $\overline{\mathbb{Q}}(z)$  such that  $Z := P \cdot Y$  satisfies  $Z' = P[A] \cdot Z$ , where  $P[A] := (PA + P')P^{-1}$  is equal to a companion matrix  $C$ . Moreover, by [Cop36, §6] (see also [vdPS03, Lemma 2.10]), the entries of the matrix  $P$  can be chosen to be polynomials in  $\overline{\mathbb{Q}}[z]$ , of degree at most  $n - 1$ . By construction,  $G := P \cdot F$  satisfies  $G' = C \cdot G$ . Hence, the vector  $G$  is necessarily of the form  $G := (g, g', \dots, g^{(n-1)})^T$ , where  $g$  is a  $\overline{\mathbb{Q}}[z]$ -linear combination of the  $E$ -functions  $f_i$ . In particular,  $g$  is itself an  $E$ -function. Moreover,  $L_g^{\min}$  has order  $n$ : indeed, any  $\overline{\mathbb{Q}}[z]$ -linear combination  $0 = \mathbf{v} \cdot G$  between  $g, g', \dots, g^{(n-1)}$  yields a  $\overline{\mathbb{Q}}[z]$ -linear combination  $0 = (\mathbf{v}P) \cdot F$  between the entries of  $F$ ; since these are assumed linearly independent over  $\overline{\mathbb{Q}}(z)$ , and since  $P$  is invertible,  $\mathbf{v}$  is necessarily zero. We are now in position to apply the companion case. Since gauge equivalent systems have the same differential Galois group [Sin09, p. 13], the new companion system  $Z' = CZ$  has differential Galois group  $\mathcal{G}$ . By applying the companion case, we deduce that  $\dim_{\overline{\mathbb{Q}}}(V_C) = n$ , where  $V_C$  is the  $\overline{\mathbb{Q}}$ -vector space generated by the orbit  $\{\sigma(G) \mid \sigma \in \mathcal{G}\}$ . It remains to show that  $\dim_{\overline{\mathbb{Q}}}(V) = \dim_{\overline{\mathbb{Q}}}(V_C)$ . Choose  $\sigma_1, \dots, \sigma_n$  in  $\mathcal{G}$  such that  $\sigma_1(G), \dots, \sigma_n(G)$  are linearly independent over  $\overline{\mathbb{Q}}$ . Then,  $\sigma_1(F), \dots, \sigma_n(F)$  are also linearly independent over  $\overline{\mathbb{Q}}$ , because of the relation  $G = P \cdot F$  and the fact that all elements in  $\mathcal{G}$  leave  $P$  invariant. It follows that  $V$  has dimension at least  $n$ ; since  $V$  is included in the solution space of  $Y' = AY$ , it also has dimension at most  $n$ , therefore  $\dim_{\overline{\mathbb{Q}}}(V) = n$ , which concludes the proof.  $\square$

We now prove Theorem 3.1. Our proof is inspired by that of Beukers in [Beu06]. The main difference is that our proof does not depend on a specific desingularization procedure for linear differential systems.

*Proof of Theorem 3.1.* We make use of a *desingularization lemma* [BM15, Theorem 2]: there exists a polynomial matrix  $M$  with entries in  $\overline{\mathbb{Q}}[z]$  and with  $\det(M) \neq 0$  such that

the finite poles of  $B = M[A] := M^{-1}(AM - M')$  are exactly the true (i.e, non-apparent) singularities of  $Y' = AY$  and such that  $\det(M)$  is a non-zero polynomial whose roots are among the apparent singularities of  $Y' = AY$ . (See also Proposition 3.2 below.)

In our case, by Property **(P1')** above, the entries of  $B$  are in  $\overline{\mathbb{Q}}[z, 1/z]$ .

Define  $Z = (e_1, \dots, e_n)^T := M^{-1} \cdot (f_1, \dots, f_n)^T$ , so that  $(f_1, \dots, f_n)^T = M \cdot (e_1, \dots, e_n)^T$ . A simple computation shows that  $Z' = BZ$ . It remains to prove that all the  $e_i$ 's are  $E$ -functions. The proof relies on Property **(P2)** above.

By definition, each  $e_i$  is equal to  $\frac{1}{\det(M)} \cdot \sum_{j=1}^n p_{i,j} f_j$  for polynomials  $p_{i,j}$  in  $\overline{\mathbb{Q}}[z]$ . Since  $B$  has no non-zero pole, each  $e_i$  is holomorphic at every apparent singularity  $\rho \neq 0$  of  $Y' = AY$ . Therefore,  $\sum_{j=1}^n p_{i,j} f_j$  is an  $E$ -function which vanishes at any root of  $\det(M)$  at an order at least equal to the multiplicity of that root in  $\det(M)$ . By repeated application of Property **(P2)**, it follows that  $e_i$  is an  $E$ -function.  $\square$

Beukers' proof [Beu06, p. 378] of Theorem 3.1 actually contains a general effective desingularization process, which deserves to be stated independently of the context of  $E$ -functions. It is given in Algorithm 4, whose properties are summarized in the following.

**Proposition 3.2.** *Let  $A$  be an  $n \times n$  matrix with entries in  $\mathbb{Q}(z)$  and let  $\alpha \in \overline{\mathbb{Q}}$  be such that a fundamental solution  $\mathcal{Y}$  of  $Y' = AY$  is holomorphic at  $\alpha$ . Then Algorithm 4 computes a matrix of polynomials  $B \in (\mathbb{Q}(\alpha)[z])^{n \times n}$  such that  $\mathcal{Y} = B\mathcal{Z}$  with  $\mathcal{Z}$  a fundamental solution of  $Z' = CZ$  also holomorphic at  $\alpha$  and  $C \in (\mathbb{Q}(\alpha)(z))^{n \times n}$  only has poles where  $A$  does, except at  $\alpha$ , where it is holomorphic.*

*Proof.* We reproduce Beukers' proof, with more details.

By hypothesis, the determinant  $W = \det \mathcal{Y}$  is holomorphic in a neighborhood of  $z = \alpha$ . If  $W(\alpha) \neq 0$ , then  $\mathcal{Y}^{-1}$  is holomorphic in the neighborhood of  $z = \alpha$  and therefore so is  $A = \mathcal{Y}'\mathcal{Y}^{-1}$ . In that case,  $C = A$  and  $B = \text{Id}_n$  gives the result.

Otherwise, as  $W \neq 0$ , there exists  $r \in \mathbb{N}_{>0}$  such that  $W(z) \sim c(z - \alpha)^r$  for  $z \rightarrow \alpha$  with  $c \neq 0$ . Since  $W$  satisfies  $W' = \text{Trace}(A)W$ , it follows that  $r$  is the residue of  $\text{Trace}(A)$  at  $z = \alpha$ . Starting with  $B = \text{Id}_n$ ,  $C = A$  and  $\mathcal{Z} = \mathcal{Y}$ , the algorithm repeats at most  $r$  times an operation that updates  $C$  and  $B$  so that  $\mathcal{Y} = B\mathcal{Z}$  and

- $B$  is a matrix of polynomials;
- $\mathcal{Z}$  is holomorphic at  $\alpha$ ;
- $C := B^{-1}(AB - B')$  has no pole outside those of  $A$ ;
- $\mathcal{Z}$  is a fundamental solution of  $Z' = CZ$ ;
- $\text{val}_{z=\alpha} \det \mathcal{Z} = \text{val}_{z=\alpha} \det \mathcal{Y} - 1$ .

By composing these steps, it is sufficient to prove that one iteration of the loop has these properties.

Each step is centered around the definition of a matrix  $M$  as follows. Let  $k > 0$  be the order of the pole of the matrix  $C$  at  $\alpha$ , let  $i$  be the index of the first row of  $C$  with a pole of order  $k$ , let  $v$  be the constant vector of coefficients of  $(z - \alpha)^{-k}$  in that row,  $D$  be the diagonal matrix  $\text{diag}(1, \dots, 1, z - \alpha, 1, \dots, 1)$  with  $z - \alpha$  in the  $i$ th position and  $M$  be an invertible constant matrix with  $v$  in its  $i$ th row. Then  $\tilde{B} := BM^{-1}D$  possesses the desired properties:

- $\tilde{B}$  is the product of matrices of polynomials;
- $\tilde{Z} := D^{-1}M\mathcal{Z}$  is holomorphic at  $\alpha$ : since both  $\mathcal{Z}$  and  $\mathcal{Z}'$  are holomorphic at  $\alpha$ , the product  $v\mathcal{Z}$  is 0 at  $\alpha$ , making the  $i$ th row of  $M\mathcal{Z}$  a multiple of  $(z - \alpha)$ ;
- $\tilde{C} := D^{-1}M(CM^{-1}D - M^{-1}D')$  has no pole outside those of  $C$ ;
- $\tilde{Z}$  is a fundamental solution of  $Z' = \tilde{C}Z$ ;
- $\det \tilde{Z} = \det M \det \mathcal{Z} / (z - \alpha)$ . □

---

**Algorithm 4** Removal of Singularities (BEUKERSALGO( $M, \alpha$ ))
 

---

*Input:*  $A$ : matrix in  $\mathbb{Q}(z)^{n \times n}$ ;

$\alpha$ : root of the denominator of an entry in  $A$

*Output:*  $B$ : matrix in  $(\mathbb{Q}(\alpha)[z])^{n \times n}$  satisfying Proposition 3.2.

$r := \text{Res}_{z=\alpha} \text{Trace}(A)$

// Residue of the trace

**if**  $r \notin \mathbb{N}_{\geq 0}$  **then error** singularity cannot be removed

$C := A$ ;  $B := \text{Id}_n$

**for**  $m = 1, \dots, r$  **do**

$k :=$  order of the pole of  $C$  at  $\alpha$

**if**  $k = 0$  **then break**

$i :=$  index of the first row of  $C$  with a pole of order  $k$  at  $\alpha$

$v :=$  vector of coefficients of  $(z - \alpha)^{-k}$  in row  $i$  of  $C$

$M :=$  an invertible constant matrix with  $v$  its  $i$ th row; // Complete  $v$  into a basis

$D := \text{diag}(1, \dots, 1, z - \alpha, 1, \dots, 1)$  with  $z - \alpha$  in the  $i$ th position

$B := BM^{-1}D$

$C := D^{-1}MCM^{-1}D - D^{-1}D'$

**return**  $B$

---

When this algorithm is applied in Algorithm 3 and the value of  $r$  computed in the first step of Algorithm 4 is 1, which occurs frequently in practice, then the kernel computed in Algorithm 3 has dimension 1 and its entries are nothing but the values of the coefficients of the differential equation at  $z = \alpha$ . In other words, in this situation, there is only one algebraic relation, which is obtained by evaluating the differential equation at  $z = \alpha$ . It is always possible to obtain an algebraic relation that way; the strength of Algorithm 3 is that it returns a *basis* of all these relations, even when  $r > 1$ .

**3.3. Effective decomposition of  $E$ -functions.** For an  $E$ -function  $f \in \mathbb{Q}[[z]]$  (or more generally in  $\overline{\mathbb{Q}}[[z]]$ ), we call *exceptional values* those (finitely many) non-zero algebraic numbers  $\alpha$  such that  $f(\alpha) \in \overline{\mathbb{Q}}$ . The set of exceptional values of  $f$  is denoted by  $\text{Exc}(f)$ . We call the  $E$ -function  $f$  *purely transcendental* if it has no exceptional values, i.e. if  $\text{Exc}(f) = \emptyset$ .

This subsection deals with the fact that every  $E$ -function is equal to the sum of a polynomial and of a polynomial multiple of a purely transcendental  $E$ -function. The existence of such a decomposition was proved in [Riv16] for general  $E$ -functions. We give an alternative proof in the special case of  $E$ -functions with coefficients in  $\mathbb{Q}$  in



**Theorem 3.4.** Before that, in Proposition 3.3, we define a canonical decomposition in the general case. Decompositions are not unique in general. Indeed, if we have  $f = p + qg$  where  $p, q$  are in  $\overline{\mathbb{Q}}[z]$  and  $g$  is a purely transcendental  $E$ -function, then for any  $u \in \overline{\mathbb{Q}}[z]$ , the identity  $f = p - qu + q(g + u)$  is another admissible decomposition because  $g + u$  is still a purely transcendental  $E$ -function. In particular, the Taylor coefficients of  $f$  and  $g$  may lie in two different number fields. However, we have the following:

**Proposition 3.3.** *Every transcendental  $E$ -function (with coefficients in  $\overline{\mathbb{Q}}$ ) can be written in a unique way as  $f = p + qg$  with  $p, q \in \overline{\mathbb{Q}}[z]$ ,  $q$  monic and  $q(0) \neq 0$ ,  $\deg(p) < \deg(q)$  and  $g$  a purely transcendental  $E$ -function.*

We shall call this decomposition the *canonical decomposition* of  $f$ .

*Proof.* As said above, it is proved in [Riv16] that any transcendental  $E$ -function  $f$  can be written  $P + QF$  where  $P, Q \in \overline{\mathbb{Q}}[z]$  and  $F$  is a purely transcendental  $E$ -function. Note that since  $f$  is transcendental,  $Q$  is not identically zero: we can then write  $Q(z) = z^m \sum_{k=0}^d q_k z^k$  with  $q_0 q_d \neq 0$  and  $d, m \in \mathbb{N}$ . The function  $G(z) := q_d z^m F(z)$  is still a purely transcendental  $E$ -function and  $QF = qG$  where  $q(z) := \sum_{k=0}^d (q_k/q_d) z^k \in \overline{\mathbb{Q}}[z]$  is monic and such that  $q(0) \neq 0$ . We now perform the Euclidean division of  $P$  by  $q$ : we have  $P = rq + p$  for some  $p, r \in \overline{\mathbb{Q}}[z]$  with  $\deg(p) < \deg(q)$ . Defining  $g := G + r$ , which is again a purely transcendental  $E$ -function, we observe that the decomposition  $f = p + qg$  is of the form in the proposition.

We now prove uniqueness of such a decomposition. Consider two decompositions  $p + qg = \tilde{p} + \tilde{q}\tilde{g}$  of an  $E$ -function  $f$  with polynomials  $p, q, \tilde{p}, \tilde{q}$  and  $g, \tilde{g}$  purely transcendental functions as in the statement of the Proposition.

We first prove that  $q = \tilde{q}$ . Obviously, these polynomials share the same set of roots, namely  $\text{Exc}(f)$ . Moreover  $g$  and  $\tilde{g}$  being purely transcendental, we claim that any root of  $q$  and  $\tilde{q}$  has the same multiplicity in  $q$  and in  $\tilde{q}$ , so that  $q$  and  $\tilde{q}$  are equal up to a non-zero constant factor, hence equal because they are both monic. To prove the claim, let  $\rho$  be a root of  $q$  of multiplicity  $m$  and of multiplicity  $\tilde{m}$  for  $\tilde{q}$ : if  $m < \tilde{m}$  then differentiating  $m$  times both sides of  $p + qg = \tilde{p} + \tilde{q}\tilde{g}$  and evaluating at  $z = \rho$ , we obtain that  $g(\rho) \in \overline{\mathbb{Q}}$  which is not possible because  $\rho \neq 0$ , hence by symmetry of the situation we have  $m = \tilde{m}$ .

Finally, since in the decompositions  $p + qg = \tilde{p} + \tilde{q}\tilde{g}$ , we have  $\deg(p) < \deg(q)$  and  $\deg(\tilde{p}) < \deg(\tilde{q})$ , this forces  $p = \tilde{p}$  and  $g = \tilde{g}$ . Indeed,  $g - \tilde{g} = (\tilde{p} - p)/q := u$  is a rational  $E$ -function, i.e., a polynomial in  $\overline{\mathbb{Q}}[z]$ . Hence, consideration of the degree on both sides of  $\tilde{p} - p = uq$  forces  $u = 0$ .  $\square$

Our main contribution in this section is to prove that when  $f$  has coefficients in  $\mathbb{Q}$ , then we can find polynomials and a purely transcendental  $E$ -function involved in the canonical decomposition that also have coefficients in  $\mathbb{Q}$ , and moreover that one can compute this decomposition algorithmically. More precisely, we prove:

**Theorem 3.4.** *Any transcendental  $E$ -function  $f \in \mathbb{Q}[[z]]$  admits a canonical decomposition  $f = p + qg$ , where  $p$  and  $q$  are polynomials in  $\mathbb{Q}[z]$  and  $g \in \mathbb{Q}[[z]]$  is a purely transcendental*

$E$ -function. Moreover, if  $f$  is given by a linear differential equation together with sufficiently many initial terms, then one can effectively determine  $p$  and  $q$ .

Of course, once  $p$  and  $q$  are determined,  $g$  is determined by a linear differential equation together with sufficiently many initial terms, simply because  $g = (f - p)/q$ . Before proceeding to the proof of Theorem 3.4, we state a very useful fact.

**Proposition 3.5.** *Let  $f \in \mathbb{Q}[[z]]$  be an  $E$ -function and let  $\alpha \in \text{Exc}(f)$ . Then,*

- (i)  $f(\alpha) \in \mathbb{Q}(\alpha)$ ;
- (ii) all Galoisian conjugates of  $\alpha$  belong to  $\text{Exc}(f)$ ;
- (iii) for any Galoisian conjugate  $\alpha'$  of  $\alpha$ , the value  $f(\alpha')$  is a Galoisian conjugate of  $f(\alpha)$ .

*Proof.* The three statements are consequences of Algorithm 3. For any given root  $\alpha$  of any irreducible factor  $\mu \in \mathbb{Q}[z]$  of  $v_0$ , the algorithm determines if there exists a vector in  $\overline{\mathbb{Q}}^{s+1}$  of the form  $(\beta, -1, 0, \dots, 0)$  which is in the left kernel of the matrix  $\mathcal{M}(\alpha)$ , whose entries are in  $\mathbb{Q}(\alpha)$ . The existence of this vector is equivalent to  $f(\alpha) = \beta \in \overline{\mathbb{Q}}$ . When it exists, this proves that  $f(\alpha) = \beta \in \mathbb{Q}(\alpha)$ , a fact proved in [FR14, FR16] in the general case (with  $\mathbb{Q}(\alpha)$  replaced by  $\mathbb{K}(\alpha)$  when  $f$  has coefficients in a number field  $\mathbb{K}$ ). Now, such a vector exists if and only there exists a vector of the same form in the left kernel of the matrix  $\mathcal{M}(\alpha')$ , where  $\alpha'$  is any Galoisian conjugate of  $\alpha$  (i.e. any other root of  $\mu$  in this case). It follows that for any conjugate  $\alpha'$  of  $\alpha$ ,  $f(\alpha')$  is a conjugate of  $f(\alpha)$ .  $\square$

The existence of a decomposition as in Theorem 3.4 can be deduced from Proposition 3.3, by letting  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  act on the decomposition delivered by Proposition 3.3 and by using its uniqueness. In order to obtain the polynomials  $p$  and  $q$  effectively, we propose a “rational version” of the proof in [Riv16], which avoids working in algebraic extensions.

*Proof of Theorem 3.4.* The set  $\text{Exc}(f)$  can be computed using Algorithm 3. If  $\text{Exc}(f) = \emptyset$ , then the canonical decomposition of  $f$  is  $f = p + qg$  with  $p = 0$ ,  $q = 1$  and  $g = f$ . From now on, we assume that  $\text{Exc}(f) = \{\alpha_1, \dots, \alpha_k\} \neq \emptyset$ ; by Proposition 3.5,  $\text{Exc}(f)$  can be partitioned into blocks of Galois conjugated values.

Let us first assume that there is only one such block, i.e. that  $\{\alpha_1, \dots, \alpha_k\}$  is the set of roots of a monic irreducible polynomial  $E \in \mathbb{Q}[z]$ . For any  $m \geq 0$ , we consider the  $E$ -adic expansion of  $f$  to order  $m$ :

$$f = p_0 + p_1 E + \dots + p_m E^m + E^{m+1} g_m, \quad (15)$$

with  $p_0(z), \dots, p_m(z) \in \mathbb{C}[z]$  each of degree less than  $k = \deg(E)$ , and  $g_m \in \mathbb{C}[[z]]$ .

We will prove the following claims:

**Claim 1.**  $p_0, \dots, p_m \in \mathbb{Q}[z]$  and  $g_m \in \mathbb{Q}[[z]]$ .

**Claim 2.**  $g_m$  is an  $E$ -function.

**Claim 3.** There exists an  $m \geq 0$  such that  $g_m$  is purely transcendental.

From these claims, the proof of the first part of the theorem follows by taking  $p := p_0 + p_1 E + \dots + p_m E^m$ ,  $q := E^{m+1}$  and  $g := g_m$ .

**Proof of Claim 1.** It is enough to prove it for  $m = 0$ , and then iterate. We have  $f = p_0 + E g_0$  with  $p_0 \in \mathbb{C}[z]$  of degree less than  $k$  and  $g_0 \in \mathbb{C}[[z]]$  and we need to prove that the coefficients of  $p_0$  and  $g_0$  are actually in  $\mathbb{Q}$ . First, we observe that  $p_0(z) = c_0 + c_1 z + \cdots + c_{k-1} z^{k-1}$  is the unique polynomial in  $\mathbb{C}[z]$  such that  $p_0(\alpha_i) = f(\alpha_i)$  for  $1 \leq i \leq k$ . In matrix terms, this rewrites as

$$\begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{k-1} \\ \vdots & & & \vdots \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ \vdots \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} f(\alpha_1) \\ \vdots \\ f(\alpha_{k-1}) \end{pmatrix}$$

and by multiplying this equality on the left by the transpose of the Vandermonde matrix, we get the equivalent identity

$$\begin{pmatrix} k & \sum_i \alpha_i & \cdots & \sum_i \alpha_i^{k-1} \\ \sum_i \alpha_i & \sum_i \alpha_i^2 & \cdots & \sum_i \alpha_i^k \\ \vdots & & & \vdots \\ \sum_i \alpha_i^{k-1} & \sum_i \alpha_i^k & \cdots & \sum_i \alpha_i^{2k-2} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} \sum_i f(\alpha_i) \\ \sum_i \alpha_i f(\alpha_i) \\ \vdots \\ \sum_i \alpha_i^{k-1} f(\alpha_i) \end{pmatrix}. \quad (16)$$

Now, the matrix on left-hand side of Eq. (16) is invertible and with coefficients in  $\mathbb{Q}$ , since it contains the power sums of the roots of the polynomial  $E \in \mathbb{Q}[z]$ . On the other hand, for any  $E$ -function  $g \in \mathbb{Q}[[z]]$ , we have that  $\sum_i g(\alpha_i) \in \mathbb{Q}$ , by Proposition 3.5. Applying this to the  $E$ -functions  $f(z), z f(z), \dots, z^{k-1} f(z)$ , we deduce that the right-hand side of Eq. (16) is a vector in  $\mathbb{Q}^k$ . This implies that the  $c_i$ 's are all in  $\mathbb{Q}$ , hence  $p_0 \in \mathbb{Q}[z]$ . From there it directly follows that  $g_0 \in \mathbb{Q}[[z]]$ .

**Proof of Claim 2.** It is again enough to prove the claim for  $m = 0$ , and then iterate. Indeed, from Eq. (15) it follows that the  $E$ -adic expansion of  $g_{m-1}$  to order 1 is  $g_{m-1} = p_m + E g_m$ , and since  $f = p_0 + E g_0$  is an  $E$ -function one deduces iteratively that  $g_0, g_1, \dots, g_m$  are  $E$ -functions.

It remains to prove that if we have  $f = p + E g$  with  $E \in \overline{\mathbb{Q}}[z]$  and  $p \in \overline{\mathbb{Q}}[z]$  of degree less than  $k = \deg(E)$  and  $g \in \overline{\mathbb{Q}}[[z]]$ , then  $g$  is an  $E$ -function. This is done by induction on  $k \geq 1$ . For  $k = 1$ , this is precisely Property (P2). Assume the property is proved for any  $E$  of degree  $k - 1 \geq 1$  and any  $p$  of degree less than  $k - 1$ . Assume we have  $f = p + E g$  with  $E \in \overline{\mathbb{Q}}[z]$  of degree  $k$ ,  $p \in \overline{\mathbb{Q}}[z]$  of degree less than  $k$  and  $g \in \overline{\mathbb{Q}}[[z]]$ . Let  $\beta$  be one of the roots of  $E$  and write  $p(z) = \sum_{j=0}^{k-1} p_j (z - \beta)^j$ . Then  $f(\beta) = p_0$  and by Property (P2),

$$\frac{f(z) - f(\beta)}{z - \beta} = \sum_{j=0}^{k-2} p_{j+1} (z - \beta)^j + \frac{E(z)}{z - \beta} g(z)$$

is an  $E$ -function. Since  $E(z)/(z - \beta)$  is a polynomial of degree  $k - 1$  and  $\sum_{j=0}^{k-2} p_{j+1} (z - \beta)^j$  is of degree less than  $k - 1$ , we deduce that  $g$  is an  $E$ -function by the induction hypothesis.

**Proof of Claim 3.** From Eq. (15) it follows that the only exceptional values of the  $g_m$ 's are necessarily contained in the set  $\text{Exc}(f) = \{\alpha_1, \dots, \alpha_k\}$ .

We will show that there exists an  $m$  such that  $g_m$  does not have any of the  $\alpha_j$ 's as an exceptional value, and therefore  $g_m$  is purely transcendental. By Proposition 3.5, this is equivalent to proving that the  $g_m$ 's cannot all share  $\alpha := \alpha_1$  as an exceptional value.

Setting  $g_{-1} = f$ , it follows from Eq. (15) (with  $m$  replaced by  $m - 1$ , and then by differentiating  $m$  times) that  $1, f^{(m)}(\alpha)$  and  $g_{m-1}(\alpha)$  are linearly dependent over  $\overline{\mathbb{Q}}$  for all  $m \geq 0$  by a relation of the form  $f^{(m)}(\alpha) = u_m + v_m g_{m-1}(\alpha)$  with  $u_m, v_m \in \overline{\mathbb{Q}}$  and  $v_m \neq 0$ . Hence,

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}}(f(\alpha), \dots, f^{(m)}(\alpha)) = \operatorname{trdeg}_{\overline{\mathbb{Q}}}(g_{-1}(\alpha), \dots, g_{m-1}(\alpha)) \quad \text{for all } m \geq 0.$$

By contradiction, let us now assume that  $g_m(\alpha) \in \overline{\mathbb{Q}}$  for all  $m \geq -1$ . Then we have

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}}(f(\alpha), \dots, f^{(m)}(\alpha)) = 0$$

for all  $m \geq 0$ . Now by Property (P1) it follows that  $f$  satisfies an LDE, of some order  $\mu \geq 1$ , having only  $z = 0$  as finite singularity. By considering the corresponding companion system  $Y' = AY$  where  $f$  is the first element of the column vector  $Y$ , the matrix  $A$  has Laurent polynomial entries in  $z$ , hence the Siegel-Shidlovskii theorem ensures that

$$0 = \operatorname{trdeg}_{\overline{\mathbb{Q}}}(f(\alpha), \dots, f^{(\mu-1)}(\alpha)) = \operatorname{trdeg}_{\overline{\mathbb{Q}}(z)}(f(z), \dots, f^{(\mu-1)}(z)) \geq 1,$$

a contradiction.

On the effective side, note that one can compute the  $E$ -adic expansion (15) of  $f$  to any order  $m$ , for instance using linear algebra. Then, to compute the needed decomposition, one may, for increasing values  $m = 0, 1, \dots$ , compute a linear differential equation for  $g_m$  as in (15) together with sufficiently many initial terms, and test using Algorithm 3 whether  $\operatorname{Exc}(g_m)$  is empty or not. This procedure will eventually terminate.

We now treat the general case, where  $\operatorname{Exc}(f)$  contains several blocks  $B_1, \dots, B_p$ , each block containing conjugated exceptional values. Denote by  $E_j(z)$  the minimal polynomial  $\prod_{\alpha \in B_j} (z - \alpha)$  of the elements in  $B_j$ . By the reasoning used in the case of a single block, one first finds a decomposition  $f = p_1 + q_1 g_1$  with  $p_1, q_1$  in  $\mathbb{Q}[z]$  and  $g_1 \in \mathbb{Q}[[z]]$  an  $E$ -function such that  $\operatorname{Exc}(g_1) = \operatorname{Exc}(f) \setminus B_1$ . Then, one applies the same to the  $E$ -function  $g_1$ , and writes it as  $g_1 = p_2 + q_2 g_2$ , and thus  $f = (p_1 + q_1 p_2) + (q_1 q_2) g_2$ , with  $p_2, q_2$  in  $\mathbb{Q}[z]$  and  $g_2 \in \mathbb{Q}[[z]]$  an  $E$ -function such that  $\operatorname{Exc}(g_2) = \operatorname{Exc}(f) \setminus (B_1 \cup B_2)$ . Continuing the same way  $p$  times, we end up with a decomposition  $f = p + qg$ , with  $p, q$  in  $\mathbb{Q}[z]$  and  $g \in \mathbb{Q}[[z]]$  an  $E$ -function such that  $\operatorname{Exc}(g) = \operatorname{Exc}(f) \setminus (B_1 \cup \dots \cup B_p) = \emptyset$ . Moreover, by construction we have that  $q$  monic,  $q(0) \neq 0$  and  $\deg(p) < \deg(q)$ . This concludes the proof.  $\square$

**3.4.  $E$ -functions with coefficients in a number field.** In general, an  $E$ -function is a power series

$$f(z) := \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \quad \text{in } \overline{\mathbb{Q}}[[z]]$$

such that

(i)  $f(z)$  satisfies a homogeneous linear differential equation with coefficients in  $\overline{\mathbb{Q}}(z)$ ; there exists  $C > 0$  such that

- (ii) for any  $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and any  $n \geq 0$ ,  $|\sigma(a_n)| \leq C^{n+1}$ ;
- (iii) for any  $n \geq 0$ , there exists  $d_n \in \mathbb{N} \setminus \{0\}$  such that  $d_n \leq C^{n+1}$  and  $d_n a_m \in \mathcal{O}_{\overline{\mathbb{Q}}}$  for all  $0 \leq m \leq n$ .

In particular (ii) with  $\sigma = \text{id}$  implies that  $f(z)$  is an entire function. Moreover, (i) implies that the coefficients  $a_n$  all live in a certain number field, so that there are only finitely many Galoisian conjugates to consider in (ii); if  $a_n \in \mathbb{Q}$ , this definition reduces to that of Section 3.1.

The Adamczewski-Rivoal algorithm applies to these more general situations. The version stated in Section 3.1 also applies. Indeed, all the tools it uses work more generally. This is obviously the case for Beukers' desingularization, it is also the case for the algorithms used by minimization: greatest common right divisors, Hermite-Padé approximants, series solutions and the computation of bounds on the degrees of factors (see [BRS21]).

**3.5. Siegel's original definition.**  $E$ -functions with algebraic coefficients have been first defined by Siegel [Sie14] in 1929 in a more general way: in (ii) and (iii) above, the upper bounds  $(\dots) \leq C^{n+1}$  for all  $n \geq 0$  are replaced by: for all  $\varepsilon > 0$ , there exists  $N(\varepsilon)$  such that  $(\dots) \leq n!^\varepsilon$  for all  $n \geq N(\varepsilon)$ .  $E$ -functions considered above are sometimes denoted  $E^*$ -functions or called " $E$ -functions in the strict sense": since André's work [And00a, And00b], it has become standard (though improper) to call them simply " $E$ -functions" as well.

The Siegel-Shidlovskii and Nesterenko-Shidlovskii theorems both hold in that setting. The latter was refined by Beukers for  $E$ -functions in the strict sense only. Then, André generalized Beukers' lifting theorem to  $E$ -functions in Siegel's sense by a completely different method [And14]; another proof was later given by Lepetit [Lep21] by a (non-trivial) adaptation of Beukers' original method.

We note here that Lepetit [Lep21] also generalized the Adamczewski-Rivoal algorithm to the case of  $E$ -functions in Siegel's original sense: he showed that all the steps in this algorithm work exactly the same *mutatis mutandis*, so that in fact our more efficient algorithm described here applies as well if the input is an  $E$ -function in Siegel's sense with rational coefficients. Moreover, the decomposition  $f = p + qg$  studied in Section 3.3, holds in Siegel's sense, in particular Theorem 3.4. However, it is conjectured that the classes of  $E$ -functions in Siegel's sense and of  $E$ -functions in the strict sense are the same (see [And00a, p. 715]). This implies that the distinction is in practice illusory because all known examples of  $E$ -functions satisfy all the conditions to be  $E$ -functions in the strict sense.

## 4. EXAMPLES

**4.1. The Lorch-Muldoon example.** In a special case of a result due to Lorch and Muldoon [LM95], the starting point is the following equation satisfied by the fourth derivative of Bessel's  $J_0$  function:

$$z(z^2 - 3)^2 y''(z) + (z^2 - 15)(z^2 - 3)y'(z) + z(z^4 - 10z^2 + 45)y(z) = 0,$$

with initial conditions  $y(0) = 3/8, y'(0) = 0$ . With this input, Algorithm 3 returns

$$y(\pm\sqrt{3}) = 0,$$

showing that the only non-zero algebraic points where the  $E$ -function  $J_0^{(4)}$  is algebraic are  $\pm\sqrt{3}$ , where it vanishes. Moreover, the algorithm described in Section 3.3 provides the canonical decomposition  $J_0^{(4)} = p + qg$ , where

$$p(z) = 0, \quad q(z) = z^2 - 3 \quad \text{and} \quad g(z) = -J_2(z)/z^2. \quad (17)$$

Here, the purely transcendental<sup>6</sup>  $E$ -function  $g$  is given by the differential equation

$$y''(z) + 5y'(z) + zy(z) = 0,$$

with initial conditions  $y(0) = -1/8, y'(0) = 0$ . The decomposition (17) explains the *a priori* unexpected fact that  $\text{Exc}(J_0^{(4)}) = \{\pm\sqrt{3}\}$ .

It is easy to construct  $E$ -functions that take algebraic values at certain chosen algebraic points: consider  $p + qg$  where  $p, q \in \mathbb{Q}[z]$  and  $g$  is any  $E$ -function in  $\mathbb{Q}[[z]]$ . Conversely, as we have seen in Theorem 3.4, any  $E$ -function  $f \in \mathbb{Q}[[z]]$  can be written  $f = p + qg$  where  $p, q \in \mathbb{Q}[z]$  and  $g$  is a purely transcendental  $E$ -function.

It turns out to be difficult to find an  $E$ -function which takes an algebraic value at a non-zero algebraic point and which is not obviously of the form  $p + qg$  as above. The goal of the next two sections is to provide two infinite families of  $E$ -functions for which we believe it is difficult to guess *a priori* Propositions 4.1 and 4.3 below. Their proof is inspired in part by that of the evaluation  $J_0^{(4)}(\pm\sqrt{3}) = 0$  above. Besides their theoretical interest, we used these propositions to check the correctness of various routines of our algorithms.

**4.2. A first family of  $E$ -functions.** We start with a result on the exceptional values of an infinite family of  ${}_1F_1$  functions.

**Proposition 4.1.** *Let  $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  and  $d \in \mathbb{N}$ . Then:*

- (i)  $R(z) := \sum_{k=0}^d \binom{d}{k} (a)_k z^{d-k}$  has  $d$  simple roots;
- (ii)  $\text{Exc}({}_1F_1[d+1; a+d+1; -z])$  coincides with the set of roots of  $R$ ;
- (iii) for any root  $\rho$  of  $R$ , the following identity holds:

$${}_1F_1[d+1; a+d+1; -\rho] = -\frac{(a)_{d+1}}{\rho R'(\rho)}. \quad (18)$$

*Proof.* Let us first treat the case  $d = 0$ . Then (i) and (iii) trivially hold since  $R = 1$ . On the other hand, by [Shi89, p. 185, Theorem 1], we have that  $\text{Exc}({}_1F_1[1; a+1; -z]) = \emptyset$  for all  $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , and this proves (ii).

We assume in the rest of the proof that  $d \in \mathbb{N} \setminus \{0\}$ . From the differential equation  $zy''(z) + (a+z+d+1)y'(z) + (d+1)y(z) = 0$  satisfied by  ${}_1F_1[d+1; a+d+1; -z]$ , Algorithm 2 computes its adjoint  $zy''(z) - (z+a+d-1)y'(z) + dy(z) = 0$  and discovers that it admits  $R$  as a non-zero polynomial solution. From there, it computes  $b_1 = zR(z)$

<sup>6</sup>Siegel first proved that Bessel's function  $J_2$  is purely transcendental in [Sie14, pp. 31-32, §4].



and  $b_0 = (a + z + d + 1)R(z) - b'_1(z)$ , with the property that  ${}_1F_1[d + 1; a + d + 1; -z]$  is a solution of the inhomogeneous differential equation

$$b_1(z)y'(z) + b_0(z)y(z) = a(a + 1) \cdots (a + d). \quad (19)$$

It follows from (19) that  $R$  has only simple roots, since if  $R(\rho) = R'(\rho) = 0$ , then  $b_1(\rho) = b_0(\rho) = 0$ , hence  $a \in \{0, -1, \dots, -d\}$ , which is impossible. This proves (i).

Next, Algorithm 3 evaluates (19) at the roots  $\rho$  of  $R$ . Since  $b_1(\rho) = 0$ , it follows that  ${}_1F_1[d + 1; a + d + 1; -\rho] = a(a + 1) \cdots (a + d)/b_0(\rho)$ , and hence (iii) holds.

To prove (ii), note that  $f(z) := {}_1F_1[d + 1; a + d + 1; -z]$  is a transcendental  $E$ -function such that  $1, f, f'$  are linearly dependent over  $\mathbb{Q}(z)$  (as Eq. (19) shows), and  $(f, f')^T$  is solution of a differential system with only 0 as singularity. In particular, by the Siegel-Shidlovskii theorem, for any  $\alpha \in \overline{\mathbb{Q}}^*$  such that  $f(\alpha) \in \overline{\mathbb{Q}}$  we have  $f'(\alpha) \notin \overline{\mathbb{Q}}$ , and consequently the differential equation (19) shows that  $\text{Exc}({}_1F_1[d + 1; a + d + 1; -z])$  coincides with the set of roots of  $R$ , proving (ii).  $\square$

**Remark 4.2.** Let us now make several remarks on Proposition 4.1.

- (a) We do not know if the evaluation (18) is available in the (very rich) literature on special functions. It is remarkable that it was discovered (and proved) using our algorithms. Note that Eq. (18) holds more generally for all  $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ .
- (b) The polynomial  $R(z)$  in Proposition 4.1 is equal to  $(a)_d \cdot {}_1F_1[-d; 1 - a - d; z]$ , thus it can be expressed in terms of generalized Laguerre polynomials as

$$R(z) = (-1)^d d! \cdot L_d^{(-a-d)}(z).$$

As proved by Schur [Sch31], the discriminant of  $R$  is equal to  $\prod_{j=2}^d j^j (j - a - d)^{j-1}$ , which is non-zero since  $a \notin \mathbb{Z}_{\leq 0}$ ; this yields a different proof that  $R$  has only single roots.

- (c) If  $a \in \mathbb{Z}_{\leq 0}$ , then the situation is simpler and well understood. Indeed, formula [07.20.03.0007.01](#) on Wolfram's mathematical functions site implies

$${}_1F_1[d + 1; a + d + 1; -z] = \frac{(-1)^{-a} (-a)!}{(-d)_{-a}} \cdot e^{-z} \cdot L_{-a}^{(d+a)}(z)$$

and in particular

$${}_1F_1[d + 1; d; -z] = e^{-z} \cdot \frac{d - z}{d} \quad \text{for any } d \neq 0$$

and

$${}_1F_1[d + 1; d - 1; -z] = e^{-z} \cdot \frac{(d - z)^2 - d}{d(d - 1)} \quad \text{for any } d \notin \{0, 1\}.$$

In turn, these functional identities induce *infinite* families of numerical identities, such as

$${}_1F_1[d + 1; d; -d] = 0 \quad \text{for any } d \in \mathbb{N} \setminus \{0\} \quad (20)$$

and

$${}_1F_1[d^2 + 1; d^2 - 1; d - d^2] = {}_1F_1[d^2 + 1; d^2 - 1; -d - d^2] = 0 \quad \text{for any } d \in \mathbb{N} \setminus \{0, 1\}. \quad (21)$$

More generally,  $\text{Exc}({}_1F_1[d + 1; a + d + 1; -z])$  coincides with the set of roots of  $L_{-a}^{(d+a)}(z)$ .

- (d) When  $d = 1$ , the rational canonical decomposition of  $f(z) := {}_1F_1[2; a + 2; -z]$  given by Theorem 3.4 is  $f = p + qg$  with  $p = a + 1$ ,  $q = z + a$  and  $g = -{}_1F_1[1; a + 2; -z]$  (note that  $g$  is purely transcendental by remark (2) above).

Decompositions of  $f(z) := {}_1F_1[d + 1; a + d + 1; -z]$  can easily be written down when  $d \geq 2$  but they are neither as explicit nor necessarily canonical. Since  $b_0$  and  $b_1$  (in the proof of Proposition 4.1) are coprime, there exist  $u, v \in \mathbb{Q}[z]$  such that  $b_1u + b_0v = 1$ . Then we have the decomposition  $f = (a)_{d+1}v + Rg$ , where  $g(z) := z(u(z)f(z) - v(z)f'(z))$  is purely transcendental. Indeed, the decomposition is immediate to check and let  $\alpha \in \overline{\mathbb{Q}}^*$  be such that  $g(\alpha) \in \overline{\mathbb{Q}}$ . Then  $f(\alpha) \in \overline{\mathbb{Q}}$  as well, hence  $b_1(\alpha) = 0$  by (ii) in Proposition 4.1, so that  $v(\alpha) \neq 0$  by the relation  $b_1u + b_0v = 1$ . Therefore  $g(\alpha) \notin \overline{\mathbb{Q}}$  because  $f'(\alpha) \notin \overline{\mathbb{Q}}$ . This contradiction proves that there is no such  $\alpha$ .

When  $d = 1$ , this procedure provides an alternative way to obtain the above canonical decomposition of  $f(z) := {}_1F_1[2; a + 2; -z]$ , with  $g$  represented as  $g(z) = z/(a(a + 1)) \cdot ((z + a - 1)f(z) + (z - 1)f'(z)) - 1$ .

When  $d = 2$ , it provides a decomposition of  $f(z) := {}_1F_1[3; a + 3; -z]$  as  $f = p + qg$ , where  $p(z) = z^2/2 + (a - 2)z/2 + 1$ ,  $q(z) = z^2 + 2az + a(a + 1)$  and  $g(z) = -z/(2a(a + 1)(a + 2)) \cdot ((z^2 + 2(a - 1)z + a^2 + 2)f + (z^2 + (a - 2)z + 2)f')$ . The canonical decomposition of  $f$  is then readily obtained as  $f = \tilde{p} + q\tilde{g}$ , where  $\tilde{p} := p - q/2$  and  $\tilde{g} := g + 1/2$ .

- (e) When  $d = 2$ , Proposition 4.1 implies the following evaluation:

$$e^{a \mp i\sqrt{a}} {}_1F_1[a; a + 3; -a \pm i\sqrt{a}] = {}_1F_1[3; a + 3; a \mp i\sqrt{a}] = (a + 2)(1 \mp i\sqrt{a})/2. \quad (22)$$

The left-hand side is a special case of Kummer's identity  $e^{-z} {}_1F_1[a; b; z] = {}_1F_1[b - a; b; -z]$ . The right-hand side follows from the fact that the roots of  $R(z) = z^2 + 2az + a(a + 1)$  are  $\{-a \pm i\sqrt{a}\}$ , since then Proposition 4.1 implies  ${}_1F_1[3; a + 3; \rho] = -a(a + 1)(a + 2)/(\rho R'(\rho)) = (a + 1)(a + 2)/(2(\rho + a + 1))$ .

- (f) Generalized Laguerre polynomials are most of the time irreducible in  $\mathbb{Q}[z]$ , but not always. Filaseta and Lam [FLO2, Thm. 1] proved that if  $\alpha \notin \mathbb{Z}_{<0}$ , then  $L_d^{(\alpha)}(z)$  is irreducible in  $\mathbb{Q}[z]$  for sufficiently large  $d$ . However, for some values  $d, \alpha$ , the polynomial  $L_d^{(\alpha)}(z)$  can be reducible. This is so e.g. for  $d = 5, \alpha = 7/5$ , for which  $L_d^{(\alpha)}(z)$  admits the linear factor  $z - 12/5$ . This observation leads to simple particular

cases of (18) such as:

$${}_1F_1[4; 703/725; -312/725] = -20999/525625, \quad (23)$$

$${}_1F_1[5; -113/3; -140/3] = -30073/27, \quad (24)$$

$${}_1F_1[6; -2/5; -12/5] = 1309/625, \quad (25)$$

$${}_1F_1[6; 314/63; 20/63] = 365707/250047, \quad (26)$$

$${}_1F_1[8; 48/7; 6/7] = 45305/16807. \quad (27)$$

Classifying all pairs  $(d, a) \in \mathbb{N} \times \mathbb{Q}$  such that the  $E$ -function  ${}_1F_1[d+1; a+d+1; -z]$  takes algebraic values at *rational* points  $z$  is a non-trivial task, since by Proposition 4.1 this is equivalent to finding  $(d, a) \in \mathbb{N} \times \mathbb{Q}$  such that  $L_d^{(-a-d)}(z)$  admits a rational root.

For  $2 \leq d \leq 10$ , we systematically searched for such “rational evaluations” arising from *reducible* Laguerre polynomials  $L_d^{(-a-d)}$ . In particular, for all  $2 \leq d \leq 10$ , we looked for  $z_0 \in \mathbb{Q}$  with numerator and denominator between  $-1000$  and  $1000$ , and such that  $L_d^{(-a-d)}(z_0) \in \mathbb{Q}[a]$  has a root  $a_0 \in \mathbb{Q}$ . Each pair  $(a_0, z_0)$  then yields a rational  ${}_1F_1$ -evaluation as above.

With  $d \in \{2, 3, 4\}$ , we could find many such identities, for instance (23) and (24). By contrast, for  $d \geq 5$ , these rational evaluations become quite rare. Of course, there are still infinite families of *trivial* identities such as  ${}_1F_1[d+1; a; 0] = 1$  or of *simple* identities such as (20) and (21). But other rational evaluations are rare with  $d \geq 5$ . For instance, with  $d = 5$  we only found seven non-trivial identities, of which (25) is the simplest and (26) the most complex. For  $6 \leq d \leq 10$ , the only non-trivial rational evaluation we found is (27).

For  $d = 2$ , all rational identities that we have found belong to the following infinite family:

$${}_1F_1 \left[ 3; \frac{11}{4} - q^2 - q; -\frac{(2q+3)(2q+1)}{4} \right] = \frac{(2q-1)(4q^2+4q-7)}{16}, \quad q \in \mathbb{Q}. \quad (28)$$

Note that this identity is a particular case of (22).

For  $d = 3$ , all rational identities that we have found belong to the following parametric identity, which specializes to (23) for  $q = 13/5$ :

$${}_1F_1 \left[ 4; \frac{q^3 - 12q + 8}{2 - 3q}; \frac{q(1-q)(2-q)}{2 - 3q} \right] = -\frac{(q+2)(q^2+2q-2)(q^3-9q+6)}{6(2-3q)^2}. \quad (29)$$

A unified way to prove Eqs. (28)–(29) is by using that for  $d \in \{2, 3\}$ , the curve (in  $a$  and  $z$ ) defined by the generalized Laguerre polynomial  $L_d^{(a)}(z)$  has genus 0, and by using the connection of the  $L_d^{(a)}$ 's to our  ${}_1F_1$ 's. E.g., for  $d = 3$  we have the parametrization

$$\left\{ a = (q^3 - 9q + 6)/(3q - 2), z = q(q-1)(q-2)/(3q-2) \right\} \quad \text{for} \quad L_d^{(a)}(z) = 0$$

from which the above evaluation (29) follows.

- (g) When  $d = 4$ , there is a nice connection between rational evaluations and elliptic curves. The polynomial  $R(z)$  in Proposition 4.1 is equal to  $R(z) = 4! \cdot L_4^{(-a-4)}(z)$ . Thus,  $R(-z - a) = z^4 + 6az^2 - 8az + 3a(a + 2)$  defines an elliptic curve ( $\mathcal{E}$ ) whose Weierstrass form is ( $\mathcal{W}$ )  $z^2 = a^3 - 76/3a + 3440/27$ . The Mordell–Weil group of ( $\mathcal{W}$ ) is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ , with generators

$$\langle P_0 = (-20/3, 0), P_1 = (34/3, 36) \rangle = \langle P_0, P_2 = (-2/3, 12) \rangle,$$

where  $P_0$  is a torsion point of order 2, and  $P_1 + P_2 = P_0$ . Each rational point on the elliptic curve ( $\mathcal{W}$ ) gives rise to a non-trivial evaluation such as (24). The points  $P_0$  and  $P_1$  themselves yield the trivial evaluations  ${}_1F_1[5; 3; -2] = 0$  and  ${}_1F_1[5; 2; 0] = 1$ , while the point  $P_2$  yields the undefined evaluation  ${}_1F_1[5; -4; -6]$ . However, their multiples yield interesting evaluations. For instance, the rational point  $-4.P_1 = (-53/12, 99/8)$  on ( $\mathcal{W}$ ) yields the rational point  $(a, z) = (-128/3, -4)$  on ( $\mathcal{E}$ ), which in turn provides identity (24).

Similarly, the points  $2.P_1 = (7/3, 9)$ ,  $-P_1 = (34/3, -36)$ ,  $-3.P_1 = (-14/3, -12)$  and  $2.P_2 = (7/3, -9)$  on ( $\mathcal{W}$ ) yield the points  $(-8/3, 2)$ ,  $(-1/3, 1)$ ,  $(-27/25, 3/5)$  and  $(-24/25, 6/5)$  on ( $\mathcal{E}$ ), which in turn yield the rational evaluations

$$\begin{aligned} {}_1F_1[5; 7/3; -2/3] &= 5/27, \\ {}_1F_1[5; 14/3; 2/3] &= 55/27, \\ {}_1F_1[5; 98/25; -12/25] &= 1679/3125, \\ {}_1F_1[5; 101/25; 6/25] &= 4199/3125. \end{aligned} \tag{30}$$

Mordell's theorem [Mor22] (see also [HS00, Part C]) implies that there are infinitely many (non-trivial) rational evaluations of the form  ${}_1F_1[5; \alpha; \beta] = \gamma$  with  $\alpha, \beta, \gamma \in \mathbb{Q}$ , such as the five evaluations in (24) and (30).

On the other hand, for any  $d \geq 0$ , the genus of the curve (in  $a$  and  $z$ ) defined by the generalized Laguerre polynomial  $L_d^{(-a-d)}(z)$  is equal to  $\lfloor (d/2 - 1)^2 \rfloor$  [Won05] and hence at least 2 for  $d \geq 5$  (see also [HW06, Prop. 4]). It follows from Proposition 4.1, from Remark 4.2(b) and from Faltings' theorem [Fal83] (see also [HS00, §E.1]) that, for any  $d \geq 5$ , there are finitely many evaluations of the form  ${}_1F_1[d + 1; \alpha; \beta] = \gamma$  with  $\alpha, \beta, \gamma \in \mathbb{Q}$ , such that  $\beta \neq 0$  and  $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq d+1}$ . Similarly, from Remark 4.2(c) it follows that there are finitely many evaluations of the form  ${}_1F_1[d + 1; d - \alpha; \beta] = \gamma$  with  $\beta, \gamma \in \mathbb{Q}$ , such that  $\beta \neq 0$  and  $\alpha \in \mathbb{Z}_{\geq 4}$ .

- (h) With other choices such as  $d = 4$ ,  $\alpha = 12/5$ , the polynomial  $L_d^{(\alpha)}$  has non-linear factors. In particular, we obtain the quadratic irrational evaluation

$${}_1F_1[5; -7/5; (6\sqrt{15} - 42)/5] = 11/5 + 66\sqrt{15}/125, \tag{31}$$

which is not a particular case of (22).

A similarly looking identity, and perhaps even more striking, is the quartic evaluation

$${}_1F_1 \left[ 6; \frac{23 - \sqrt{725 - 20\sqrt{985}}}{10}; -\frac{6}{5} \right] = \frac{111\sqrt{985} - 3\sqrt{3353450 - 106670\sqrt{985}} - 3533}{2500}, \quad (32)$$

although in (32) the  ${}_1F_1$  function on the left is not an  $E$ -function anymore.

Another quartic evaluation, this time involving an  $E$ -function again, is

$${}_1F_1 \left[ 6; \frac{22}{5}; -\frac{2}{5}\alpha \right] = \frac{17}{6255}\beta, \quad (33)$$

where  $\alpha \approx 5.15$  satisfies  $\alpha^4 - 21\alpha^3 + 81\alpha^2 - 21\alpha + 126 = 0$  and  $\beta \approx 1.59$  satisfies  $\beta^4 - 62\beta^3 + 1584\beta^2 - 2338\beta - 49 = 0$ .

We were unable to locate in previous works any of the identities (22)–(33), including in online encyclopedias such as [Wolfram's mathematical functions site](#) and the [Digital Library of Mathematical Functions](#). Given how vast the literature on special functions is, we would not be surprised that some of these identities were already tabulated.

- (i) Equation (18) can also be proved directly starting from the relation between these  ${}_1F_1$  and the incomplete gamma function [OOL+22, 13.6.5], [AS64, 13.6.10]:

$$f(z) := \frac{1}{a} {}_1F_1[1; a + 1; -z] = (-z)^{-a} e^{-z} \gamma(a, -z).$$

Successive differentiation of the hypergeometric series shows that

$$f^{(d)}(z) = \frac{(-1)^d d!}{(a)_d} {}_1F_1[d + 1; a + d + 1; -z].$$

On the other hand, we have  $(\gamma(a, -z))' = (-z)^a e^z / z$  by [OOL+22, 8.1], [AS64, 6.5.2]. Thus, by induction, there are two families of polynomials  $(R_d)$  and  $(Q_d)$  such that

$$\frac{(-1)^d d!}{(a)_d} {}_1F_1[d + 1; a + d + 1; -z] = R_d(z) e^{-z} (-z)^{-a-d} \gamma(a, -z) + (-z)^{-d} Q_d(z) \quad (34)$$

with

$$R_d(z) = e^z (-z)^{a+d} \left( (-z)^{-a} e^{-z} \right)^{(d)}, \quad Q_{d+1} = dQ_d - zQ'_d + R_d, \quad Q_0 = 0.$$

The polynomial  $R_d$  is exactly the Laguerre polynomial  $R$  from before.

Thus, Eq. (18) boils down to an evaluation of this more general formula at a root  $\rho$  of  $R_d$ , giving

$${}_1F_1[d + 1; a + d + 1; -\rho] = \frac{(a)_d Q_d(\rho)}{d! \rho^d}.$$

This is not exactly the same formula as above. The proof is completed by proving by induction that both  $Q_d$  and  $R_d$  satisfy the same recurrence  $u_{d+1} = (z+d+a)u_d - zdu_{d-1}$ ,

giving an explicit evaluation of the determinant

$$\begin{vmatrix} Q_{d+1} & R_{d+1} \\ Q_d & R_d \end{vmatrix} = \begin{vmatrix} z+d+a & -zd \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} Q_d & R_d \\ Q_{d-1} & R_{d-1} \end{vmatrix} = \cdots = d!z^d.$$

Evaluating at  $z = \rho$  gives  $Q_d(\rho)/(d!\rho^d) = -1/R_{d+1}(\rho)$ . This gives another simple expression for the right-hand side of Eq. (18), which follows from  $R_{d+1}(z) = (z+d+a)R_d(z) - zR'_d(z)$ .

**4.3. A second family of  $E$ -functions.** The next result considers exceptional values of second derivatives of products of  ${}_1F_1$  with the exponential function. We recall that  $J_0(-iz/2) = e^{-z/2}{}_1F_1[1/2; 1; z]$  is such a product.

**Proposition 4.3.** *Let  $c \in \overline{\mathbb{Q}}^*$  and  $a, b \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$  with  $a - b \notin \mathbb{N}$ . Let  $F(z) := e^{-cz}{}_1F_1[a; b; z]$ . Then  $\text{Exc}(F'') = \emptyset$ , except in the following (disjoint) cases:*

1. if  $b = a(2c - 1)/c^2$ , then  $\text{Exc}(F'') = \{-a/c^2\}$  and  $F''(-a/c^2) = 0$ ;
2. if  $c = 1$  and  $b = a + 1$ , then  $\text{Exc}(F'') = \{-a \pm i\sqrt{a}\}$  and  $F''(-a \pm i\sqrt{a}) = 1/(1 \pm i\sqrt{a})$ .

*Proof.* With the assumptions on  $a, b, c$ , we prove below that:

**Fact 1.**  $F$  and  $F'$  are linearly independent over  $\overline{\mathbb{Q}}(z)$ , i.e.  $F$  does not satisfy any homogeneous LDE of order less than 2.

**Fact 2.**  $F$  satisfies the second-order LDE

$$zF''(z) = (z - 2cz - b)F'(z) + (cz + a - c^2z - cb)F(z), \quad (35)$$

and it does not satisfy any inhomogeneous LDE of order 1, unless  $c = 1$  and  $b = a + 1$ .

Postponing for a moment the proof of these facts, we distinguish two cases.

**Case 1.** We first assume that either  $c \neq 1$  or  $b \neq a + 1$ . By **Fact 1**, the function  $F$  is a non-polynomial  $E$ -function (hence a transcendental one). By **Fact 2** and by Beukers' Corollary 1.4 of [Beu06], it follows that the numbers 1,  $F(\xi)$  and  $F'(\xi)$  are linearly independent over  $\overline{\mathbb{Q}}$  for any  $\xi \in \overline{\mathbb{Q}}^*$ .

Assume now that  $b \neq (2ac - a)/c^2$ . Then, for any  $\xi \in \mathbb{C}$ , the numbers  $\xi - 2c\xi - b$  and  $c\xi + a - \xi c^2 - cb$  cannot be simultaneously equal to 0. Since the numbers 1,  $F(\xi)$  and  $F'(\xi)$  are linearly independent over  $\overline{\mathbb{Q}}$  for any  $\xi \in \overline{\mathbb{Q}}^*$ , it follows that

$$F''(\xi) = \frac{1}{\xi} ((\xi - 2c\xi - b)F'(\xi) + (c\xi + a - \xi c^2 - cb)F(\xi)) \notin \overline{\mathbb{Q}}.$$

Hence  $\text{Exc}(F'') = \emptyset$  when  $b \neq (2ac - a)/c^2$ .

It remains to treat the sub-case  $b := (2ac - a)/c^2$ . With  $z := -a/c^2$ , we see that

$$z - 2cz - b = cz + a - c^2z - cb = 0,$$

so that  $F''(-a/c^2) = 0$ . Since  $z - 2cz - b$  and  $cz + a - c^2z - cb$  vanish simultaneously for no other value of  $z$  and since the numbers 1,  $F(\xi)$  and  $F'(\xi)$  are linearly independent over  $\overline{\mathbb{Q}}$  for any  $\xi \in \overline{\mathbb{Q}} \setminus \{0, -a/c^2\}$ , we deduce that  $F''(\xi) \notin \overline{\mathbb{Q}}$  for such  $\xi$ . Hence  $\text{Exc}(F'') = \{-a/c^2\}$  when  $b = (2ac - a)/c^2$ . The proposition is thus proved in **Case 1**.

**Case 2.** We assume that  $c = 1$  and  $b = a + 1$ . Then,  $F''(z) = \frac{2}{(a+1)(a+2)}e^{-z}{}_1F_1[a; a+3; z]$ , so  $\text{Exc}(F'') = \text{Exc}(e^{-z}{}_1F_1[a; a+3; z]) = \{-a \pm i\sqrt{a}\}$ . The last equality is a consequence of Proposition 4.1 with  $d = 2$ . The equality  $F''(-a \pm i\sqrt{a}) = 1/(1 \pm i\sqrt{a})$  follows from (22).

**Proof of Fact 1.** When  $a, b \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  and  $a - b \notin \mathbb{N}$ , the asymptotic behavior of  ${}_1F_1[a; b; z]$  as  $z \rightarrow \infty$  (with  $-\pi/2 < \arg(z) < 3\pi/2$ ) is given in [AS64, p. 508, Eq. 13.5.1], [OOL<sup>+</sup>22, 13.7.2]. In the particular cases  $z \rightarrow \pm\infty$ , it reads

$${}_1F_1[a; b; z] \sim_{z \rightarrow +\infty} \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \quad \text{and} \quad {}_1F_1[a; b; z] \sim_{z \rightarrow -\infty} \frac{\Gamma(b)}{\Gamma(b-a)} e^{i\pi a} z^a.$$

This rules out the possibility that  $e^{-cz}{}_1F_1[a; b; z]$  satisfies a differential equation of order 1 over  $\overline{\mathbb{Q}}(z)$ .

Note that a different (purely algebraic) proof is possible, based on a reasoning similar to the one in the statement and proof of [BBH88, Lemma 4.2].

**Proof of Fact 2.** Since  ${}_1F_1[a; b; z]$  satisfies  $zy''(z) + (b-z)y'(z) - ay(z) = 0$  and  $e^{-cz}$  satisfies  $y'(z) + cy(z) = 0$ , it follows by a simple computation that  $F$  satisfies (35). By Fact 1, (35) is the minimal-order homogeneous LDE satisfied by  $F$ .

Assume now that  $F$  satisfies an inhomogeneous LDE of order 1. We will follow the reasoning in Algorithm 2, and show that the adjoint of (35) does not possess any non-zero rational solutions in  $\overline{\mathbb{Q}}(z)$ , unless  $c = 1$  and  $b = a + 1$ .

The adjoint equation of (35) writes

$$zy''(z) + ((1-2c)z - b + 2)y'(z) + (c(c-1)z + bc - a - 2c + 1)y(z) = 0. \quad (36)$$

If it admits a rational solution  $R(z) \in \overline{\mathbb{Q}}(z)$ , then the only potential pole of  $R$  can be located at  $z = 0$ . The indicial equation of (36) at  $z = 0$  is  $s(s - b + 1)$ . Hence, the possible valuations at  $z = 0$  of  $R$  are 0 and  $b - 1$ . Since  $b \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , this implies that  $R$  is actually a polynomial solution in  $\overline{\mathbb{Q}}[z]$  of (36). If  $c \neq 1$ , then the indicial polynomial at infinity of (36) is a non-zero constant, equal to  $c^2 - c$ ; therefore in that case,  $R$  cannot be a polynomial solution. It follows that  $c = 1$ . Now, the indicial polynomial at infinity of (36) is  $s - a + b - 1$ , hence the only possible degree of  $R$  is  $a - b + 1$ . Since  $a - b \notin \mathbb{N}$ , this implies that  $b = a + 1$  and that  $R$  is a constant in  $\overline{\mathbb{Q}}$ . In this case, (36) admits the rational solution  $y(z) = 1$ , and  $F$  satisfies  $zF'(z) + (z + a)F(z) = a$ .  $\square$

Note that, in the spirit of Proposition 4.3, the following examples that can be treated along the same lines: the third derivatives of

$$e^{-z/9}{}_2F_2[1/144, 1/144; -7/16, -7/16; z] \quad \text{and} \quad e^{-z/3}{}_2F_2[1/4, 3/4; 5/4, -9/4; z]$$

vanish at  $z = 3/16$  and  $z = -9/4$ , respectively. Our minimization algorithm finds their minimal differential equations, which are too big to be written here: their (order, degree) are (3, 8) and (2, 7), respectively. Algorithm 3 then shows that they are transcendental functions and that  $\{3/16\}$  and  $\{-9/4\}$  are the exceptional values sets in each case. A complete classification as in Proposition 4.3 seems to be currently out of reach though.



Identities such as

$$e^{-z} {}_2F_2[1/2, 1/3; -1/2, -2/3; z] = 1 - z/2 + 3z^2$$

(implying that the third derivative of the left-hand side is identically zero) show that an assumption corresponding to the assumption  $a - b \notin \mathbb{N}$  in Proposition 4.3 is obviously necessary on the rational parameters of the  ${}_2F_2[a, b; c, d; z]$ : to avoid trivial situations, besides the fact that the parameters  $a, b, c, d$  must all be in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , we must not have  $a - c \in \mathbb{N}$  and  $b - d \in \mathbb{N}$ , or  $a - d \in \mathbb{N}$  and  $b - c \in \mathbb{N}$  (note that in the second example above, we have  $3/4 - (-9/4) \in \mathbb{N}$  but  $1/4 - 5/4 \notin \mathbb{N}$ , while  $1/4 - (-9/4) \notin \mathbb{N}$  and  $3/4 - 5/4 \notin \mathbb{N}$ ).

**4.4. An example with Gauss' hypergeometric function.** The approach leading to special evaluations is very general and not restricted to  $E$ -functions. For instance, Gauss' hypergeometric function  ${}_2F_1[a, b; c; z]$  satisfies a differential equation whose adjoint is solved by  $R(z) := {}_2F_1[1 - a, 1 - b; 2 - c; z]$ . The approach from Section 2.4 then deduces that the hypergeometric function satisfies a first order *inhomogeneous* equation, with coefficients that are not polynomials in general, namely

$$z(z - 1)R(z)y'(z) + (z(1 - z)R'(z) + ((a + b - 1)z + 1 - c)R(z))y(z) + c - 1 = 0.$$

It follows that if  $\rho$  is a simple zero of  $R(z)$  different from 0,1, one gets the special evaluation

$${}_2F_1[a, b; c; \rho] = \frac{1 - c}{\rho(1 - \rho)R'(\rho)}.$$

The special case  $c = a + k + 1$  ( $k \in \mathbb{N}$ ) gives a nice analogue of Proposition 4.1. To state it, recall that the  $k$ th *Jacobi polynomial*  $P_k^{(\alpha, \beta)}$  with parameters  $\alpha, \beta \in \mathbb{C}$  is defined by

$$P_k^{(\alpha, \beta)}(z) := 2^{-k} \cdot \sum_{j=0}^k \binom{k + \alpha}{k - j} \binom{k + \beta}{j} (z - 1)^j (z + 1)^{k - j}.$$

It is classical [Sze75, §6.72] that  $P_k^{(\alpha, \beta)}$  has only simple roots (which are even real and in the interval  $(-1, 1)$  if  $\alpha$  and  $\beta$  are both real and greater than  $-1$ ), with the notable exception of  $\pm 1$  which is a multiple root of  $P_k^{(\alpha, \beta)}$  if one of the parameters  $\alpha, \beta$  is in  $\{-1, \dots, -k\}$ .

**Proposition 4.4.** *Let  $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ ,  $b \in \mathbb{Q}$  and  $k \in \mathbb{N}$ . If  $\rho \in \overline{\mathbb{Q}} \setminus \{0, 1\}$  is a root of the polynomial  $P_k^{(-k-a, b-k-1)}(1 - 2z)$ , then*

$${}_2F_1[a, b; a + k + 1; \rho] = \frac{(-1)^k a \binom{a+k}{k} (1 - \rho)^{k-b}}{(k + a - b) P_k^{(-k-a, b-k)}(1 - 2\rho)}. \quad (37)$$

*Proof.* With  $c = a + k + 1$ , the hypergeometric function  $R(z)$  becomes

$$R(z) = {}_2F_1[1 - a, 1 - b; 1 - a - k; z] = \frac{(-1)^k k!}{(a)_k} (1 - z)^{b-k-1} P_k^{(-k-a, b-k-1)}(1 - 2z).$$

Its roots different from 0,1 are the roots of  $P_k^{(-k-a, b-k-1)}(z)$  different from  $-1, 1$ , which are all simple. The formula for the denominator comes from the derivative

$$R'(z) = -\frac{(-1)^k}{z^{\binom{a+k-1}{k}}} \left( (k+a-b)(1-z)^{b-k-1} P_k^{(-k-a, b-k)}(1-2z) \right. \\ \left. + ((a-1)z + b - a - k)(1-z)^{b-k-2} P_k^{(-k-a, b-k-1)}(1-2z) \right). \quad \square$$

**Remark 4.5.** Let us conclude with a few remarks on Proposition 4.4.

- (1) If none of  $a, b, a-b$  is an integer, then  $f(z) = {}_2F_1[a, b; a+k+1; z]$  is a transcendental function [Vid07]. Therefore, the evaluation in Eq. (37) provides very simple particular cases of algebraic values taken by transcendental  $G$ -functions at algebraic points. Note that (37) holds more generally for  $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$  and  $b \in \mathbb{C}$ .
- (2) As in the case of Proposition 4.1, another proof of Proposition 4.4 relies on a relation analogous to Eq. (34) between  ${}_2F_1[a, b; a+k+1; z]$ ,  ${}_2F_1[a, b; a; z] = (1-z)^{-b}$  and

$${}_2F_1[a, b; a+1; z] = az^{-a} B_z(a, 1-b) = az^{-a} \int_0^z z^{a-1} (1-z)^b dz,$$

an incomplete beta function. The relation is obtained from those two by repeated use of a contiguity relation.

- (3) As in Remark 4.2, nice special cases of (37) can be obtained by studying triples  $(a, b, k)$  for which the Jacobi polynomial  $P_k^{(-k-a, b-k-1)}$  factors non-trivially. For instance, the triples  $(2/5, 3/5, 5)$  and  $(2/3, 7/3, 3)$  yield the evaluations

$${}_2F_1 \left[ \frac{2}{5}, \frac{3}{5}; \frac{32}{5}; \frac{1}{2} \right] = \frac{1683}{2500} \sqrt[5]{8}$$

and

$${}_2F_1 \left[ \frac{2}{3}, \frac{7}{3}; \frac{14}{3}; \frac{3\sqrt{5}-5}{2} \right] = \frac{44}{27\sqrt[3]{28-12\sqrt{5}}}.$$

(Here, by item (1) above, both hypergeometric functions are transcendental.)

## 5. IMPLEMENTATION

**5.1. Minimization is simpler than factorization.** The following simple example illustrates the difference between minimization and factorization. Take

$$A = z^2 \partial_z + 3, \quad B = (z-10) \partial_z + z^5$$

and their product

$$C = AB = z^2(z-10) \partial_z^2 + (z^7 + z^2 + 3z - 30) \partial_z + z^5(5z+3).$$

The computation of a bound on the degree of the coefficients of a factor of order 1 of  $C$  gives  $10^5 + 2$ . (More generally, changing 10 into a large  $N$  leads to a bound  $N^5 + 2$ .) This

leads to a large computation when trying to factor  $C$  without further information. With the extra knowledge that we are looking for a solution of  $C$  with initial condition  $y(0) = 1$ , we easily compute the first 20 coefficients of the unique series solution  $S$  of  $C$  with  $S(0) = 1$  and then compute an approximant basis (a Hermite-Padé approximant) of  $(S, S')$  at order 20. This recovers the operator  $B$ . It is easily checked that  $B$  is a right divisor of  $C$  by Euclidean division. It follows that all solutions of  $B(y) = 0$  such that  $y(0) = 1$  are solutions of  $C(y) = 0$  and by uniqueness this proves that  $B(y) = 0$  is the minimal homogeneous differential equation for  $S$ . This takes less than a second with our implementation. On this example, Maple's factorization routine `DEtools[DFactor]` has to be killed after running for 1 hour, a further indication that factorization is more complex than minimization.

**5.2. Implementation aspects.** The main difficulty is to avoid the computation of high-order expansions of power series with rational coefficients. A first gain is achieved by performing most of the computation modulo a sufficiently large prime number (we take a 31-bit long prime number). When a factor is found with modular coefficients, then the actual degree bounds from that factor are used to determine how many rational coefficients of the power series have to be computed and then obtain the differential operator with rational coefficients.

Another place where time can be saved is in the optimization problems. The computation of an approximant basis returns a linear differential operator of small order if one exists with the given degree bounds. Thus the computation of a tight bound on the number of apparent singularities is only useful if it leads to a bound on the degrees that is smaller than a previously known one. One can therefore add an extra inequality to the optimization problem so that the solver does not waste time in computing an optimum which is larger than what is already known.

In the computation of algebraic values of  $E$ -functions by Algorithm 3, the matrix returned by Beukers' Algorithm 4 is not needed in full, as only its value at  $z = \alpha$  is of interest. Thus, one should instead execute Algorithm 4 over Laurent expansions in powers of  $(z - \alpha)$ , increasing the precision of intermediate computations until the result is found.

**5.3. Timings.** Experimental results<sup>7</sup> on the family of power series

$$f_{m,p}(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k}^m \binom{n+k}{k}^p \right) \frac{z^n}{n!}$$

are reported in Table 1. These power series are exponential generating functions of Apéry-like sequences, hence they are  $E$ -functions by design. The case  $(m, p) = (2, 1)$  was considered by Adamczewski and Rivoal in [AR18, p. 706], who proved that  $f_{2,1}$  is a purely transcendental  $E$ -function. We used our algorithms to reprove this result and to extend it to other values of  $m$  and  $p$ , see Table 1.

For each  $(m, p)$ , we indicate the order and the degree of the (minimal) recurrence computed by Zeilberger's algorithm, the order and the degree of the differential equation deduced from this recurrence, and those of the minimal-order (homogeneous) differential

<sup>7</sup>The timings were obtained with Maple2021 on a 2018 Mac mini.

$(m, p)$	(ord,deg) rec	(ord,deg) original diff.eq.	(ord,deg) minimal diff.eq.	number modular terms	number rational terms	total time (s.)	time cert.
(1,5)	(6,32)	(32,29)	(12,29)	2461	408	13.	26%
(2,4)	(5,26)	(26,24)	(11,24)	1501	317	7.2	26%
(2,5)	(6,44)	(44,41)	(15,41)	5992	693	52.	21%
(3,3)	(6,28)	(28,25)	(12,29)	2461	408	13.	25%
(3,4)	(7,51)	(51,47)	(16,47)	8288	838	92.	22%
(3,5)	(8,76)	(76,72)	(20,72)	20702	1559	358.	30%
(4,2)	(5,27)	(27,24)	(12,24)	1927	343	10.	27%
(4,3)	(6,41)	(41,38)	(15,38)	5362	645	35.	29%
(4,4)	(6,46)	(46,43)	(18,47)	9634	936	109.	24%
(4,5)	(8,92)	(92,88)	(24,88)	37228	2255	983.	21%
(5,1)	(6,32)	(32,29)	(12,29)	2461	408	14.	22%
(5,2)	(7,51)	(51,47)	(16,47)	2288	838	92.	24%
(5,3)	(8,76)	(76,72)	(20,72)	20702	1559	477.	27%
(5,4)	(9,109)	(109,104)	(25,104)	50064	2761	1534.	23%
(5,5)	(10,134)	(134,129)	(30,145)	103024	4562	5216.	23%

TABLE 1. Experimental results (experiments with smaller values of  $(m, p)$  that complete under 3 sec. are not listed)

equation obtained by our implementation when run on this differential equation. We also give the number of coefficients of the sequence that were computed modulo a prime number and the number of rational coefficients of the sequence that were computed. The next column contains the time in seconds spent in the whole computation (homogeneous and inhomogeneous minimization and proof that there are no exceptional values). Finally, the last column gives the proportion of that time spent in certifying the minimality of the differential equation that has been computed.

In more detail, during the computations, the most time-consuming operation is that of the approximant basis over rational coefficients, i.e., the reconstruction of the minimal differential equation itself. This takes between  $1/3$  and  $1/2$  of the time, depending on the examples. Next comes the computation of approximant bases modulo a prime number: even though the order of the power series is much larger in the certification phase, modular computation makes it faster. This takes between  $1/5$  and  $1/3$  of the time. Finally, the third most expensive part is the computation of the rational coefficients needed to reconstruct the operator. In the last case  $(m, p) = (5, 5)$ , this is even more expensive than the other two steps. The other operations needed in these tests, such as computing gcds, finding a minimal inhomogeneous equation, computing Beukers' matrix and its kernel, are all negligible compared to these three.

The time spent in the certification of the minimality that is displayed in the last column is consistently between 20% and 30% of the total time. This overlaps with the proportions given above since this time contains the computation of a large number of modular coefficients and the computation of an approximant basis.

On the basis of these experimental results, we ask the following questions on the family  $f_{m,p}(z)$  and leave them for further research. The data in Table 1, plus a few more experiments (not included in Table 1), are in favor of positive answers to all these questions.

**Question 5.1.** *Is the  $E$ -function  $f_{m,p}$  purely transcendental for any  $m \geq 1$  and  $p \geq 1$ ?*

**Question 5.2.** *Is it true that for any odd  $m$ , the minimal-order linear differential equation  $L_{m,p}^{\min}(y) = 0$  satisfied by  $f_{m,p}$  has order  $\lfloor (N+1)^2/4 \rfloor$  and degree  $\lfloor N(2N^2 - 3N + 4)/12 \rfloor$ , where  $N = m + p$ ? In particular, when both  $m$  and  $p$  are odd, is it true that  $\text{ord}(L_{m,p}^{\min}) = \text{ord}(L_{p,m}^{\min})$  and  $\text{deg}(L_{m,p}^{\min}) = \text{deg}(L_{p,m}^{\min})$ ?*

#### ACKNOWLEDGEMENTS.

This work has been supported by the French ANR grant **DeRerumNatura**, ANR-19-CE40-0018. We thank Frits Beukers, Lucia Di Vizio, Marc Mezzarobba, Gaël Rémond and Michael Singer for many interesting discussions. We are also grateful to the anonymous reviewers for their useful and constructive comments.

#### REFERENCES

- [Abr89] S. A. Abramov, *Rational solutions of linear differential and difference equations with polynomial coefficients*, Zh. Vychisl. Mat. i Mat. Fiz. **29** (1989), no. 11, 1611–1620, 1757.
- [ABvH06] S. A. Abramov, M. A. Barkatou, and M. van Hoeij, *Apparent singularities of linear difference equations with polynomial coefficients*, Appl. Algebra Engrg. Comm. Comput. **17** (2006), no. 2, 117–133.
- [AF17] Boris Adamczewski and Colin Faverjon, *Méthode de Mahler: relations linéaires, transcendance et applications aux nombres automatiques*, Proc. Lond. Math. Soc. (3) **115** (2017), no. 1, 55–90.
- [AF18] ———, *Méthode de Mahler, transcendance et relations linéaires: aspects effectifs*, J. Théor. Nombres Bordeaux **30** (2018), no. 2, 557–573.
- [AK91] S. A. Abramov and K. Yu. Kvensenko, *Fast algorithms for the search of the rational solutions of linear differential equations with polynomial coefficients*, ISSAC'91—Proceedings of the 16th International Symposium on Symbolic and Algebraic Computation, ACM Press, 1991, pp. 267–270.
- [And00a] Yves André, *Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité*, Ann. of Math. (2) **151** (2000), no. 2, 705–740.
- [And00b] ———, *Séries Gevrey de type arithmétique. II. Transcendance sans transcendance*, Ann. of Math. (2) **151** (2000), no. 2, 741–756.
- [And14] ———, *Solution algebras of differential equations and quasi-homogeneous varieties: a new differential Galois correspondence*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 2, 449–467.
- [AR18] Boris Adamczewski and Tanguy Rivoal, *Exceptional values of  $E$ -functions at algebraic points*, Bull. Lond. Math. Soc. **50** (2018), no. 4, 697–708.

- [AS64] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, D.C., 1964.
- [Bar95] A. Barkatou, *A rational version of Moser's algorithm*, ISSAC'95—Proceedings of the 20th International Symposium on Symbolic and Algebraic Computation, ACM Press, 1995, p. 297–302.
- [BB85] Daniel Bertrand and Frits Beukers, *Équations différentielles linéaires et majorations de multiplicités*, Ann. Sci. Éc. Norm. Supér. (4) **18** (1985), no. 1, 181–192.
- [BBH88] Frits Beukers, W. Dale Brownawell, and Gert Heckman, *Siegel normality*, Ann. of Math. (2) **127** (1988), no. 2, 279–308.
- [BBMKM16] Alin Bostan, Mireille Bousquet-Mélou, Manuel Kauers, and Stephen Melczer, *On 3-dimensional lattice walks confined to the positive octant*, Ann. Comb. **20** (2016), no. 4, 661–704.
- [BCCL10] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, and Ziming Li, *Complexity of creative telescoping for bivariate rational functions*, ISSAC'10—Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2010, pp. 203–210.
- [BCL<sup>+</sup>07] Alin Bostan, Frédéric Chyzak, Grégoire Lecerf, Bruno Salvy, and Éric Schost, *Differential equations for algebraic functions*, ISSAC'07—Proceedings of the 32nd International Symposium on Symbolic and Algebraic Computation, ACM Press, 2007, pp. 25–32.
- [BCY04] Daniel Bertrand, Vladimir Chirskii, and Johan Yebbou, *Effective estimates for global relations on Euler-type series*, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), no. 2, 241–260.
- [Bek94] Emanuel Beke, *Die Irreducibilität der homogenen linearen Differentialgleichungen*, Math. Ann. **45** (1894), no. 2, 278–294.
- [Ben92] Ivar Bendixson, *Sur les équations différentielles linéaires homogènes*, Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar **49** (1892), 91–105.
- [Ber88] Daniel Bertrand, *Exposants des systèmes différentiels, vecteurs cycliques et majorations de multiplicités*, Équations différentielles dans le champ complexe, Vol. I (Strasbourg, 1985), Publ. Inst. Rech. Math. Av., Univ. Louis Pasteur, Strasbourg, 1988, pp. 61–85.
- [Ber99] ———, *On André's proof of the Siegel-Shidlovsky theorem*, Colloque Franco-Japonais: Théorie des Nombres Transcendants (Tokyo, 1998), Sem. Math. Sci., vol. 27, Keio Univ., Yokohama, 1999, pp. 51–63.
- [Beu06] F. Beukers, *A refined version of the Siegel-Shidlovskii theorem*, Ann. of Math. (2) **163** (2006), no. 1, 369–379.
- [BLS17] Alin Bostan, Pierre Lairez, and Bruno Salvy, *Multiple binomial sums*, J. Symbolic Comput. **80** (2017), no. part 2, 351–386.
- [BLV99] Bernhard Beckermann, George Labahn, and Gilles Villard, *Shifted normal forms of polynomial matrices*, ISSAC'99—Proceedings of the 24th International Symposium on Symbolic and Algebraic Computation, ACM Press, 1999, pp. 189–196.
- [BM15] Moulay A. Barkatou and Suzy S. Maddah, *Removing apparent singularities of systems of linear differential equations with rational function coefficients*, ISSAC'15—Proceedings of the 40th International Symposium on Symbolic and Algebraic Computation, ACM Press, 2015, pp. 53–60.
- [Bro92] Manuel Bronstein, *Linear ordinary differential equations: Breaking through the order 2 barrier*, ISSAC'92—Proceedings of the 17th International Symposium on Symbolic and Algebraic Computation, ACM Press, 1992, pp. 42–48.
- [Bro94] ———, *An improved algorithm for factoring linear ordinary differential operators*, ISSAC'94—Proceedings of the 19th International Symposium on Symbolic and Algebraic Computation, ACM Press, 1994, pp. 336–340.
- [BRS21] Alin Bostan, Tanguy Rivoal, and Bruno Salvy, *Explicit degree bounds for right factors of linear differential operators*, Bull. Lond. Math. Soc. **53** (2021), no. 1, 53–62.



- [CGM22] Frédéric Chyzak, Alexandre Goyer, and Marc Mezzarobba, *Symbolic-numeric factorization of differential operators*, ISSAC '22—Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation, ACM, New York, [2022] ©2022, pp. 73–82.
- [CJKS13] Shaoshi Chen, Maximilian Jaroschek, Manuel Kauers, and Michael F. Singer, *Desingularization explains order-degree curves for Ore operators*, ISSAC'13—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2013, pp. 157–164.
- [CK02] R. C. Churchill and Jerald J. Kovacic, *Cyclic vectors*, Differential algebra and related topics (Newark, NJ, 2000), World Sci. Publ., River Edge, NJ, 2002, pp. 191–218.
- [CK12a] Shaoshi Chen and Manuel Kauers, *Order-degree curves for hypergeometric creative telescoping*, ISSAC'12—Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2012, pp. 122–129.
- [CK12b] ———, *Trading order for degree in creative telescoping*, J. Symbolic Comput. **47** (2012), no. 8, 968–995.
- [CKS16] Shaoshi Chen, Manuel Kauers, and Michael F. Singer, *Desingularization of Ore operators*, J. Symbolic Comput. **74** (2016), 617–626.
- [Cop36] Frances Thorndike Cope, *Formal Solutions of Irregular Linear Differential Equations. Part II*, Amer. J. Math. **58** (1936), no. 1, 130–140.
- [CS98] Frédéric Chyzak and Bruno Salvy, *Non-commutative elimination in Ore algebras proves multivariate identities*, J. Symbolic Comput. **26** (1998), no. 2, 187–227.
- [CvH04] Thomas Cluzeau and Mark van Hoeij, *A modular algorithm for computing the exponential solutions of a linear differential operator*, J. Symbolic Comput. **38** (2004), no. 3, 1043–1076.
- [DAS15] A. Dehghan, S. M. Assari, and M. Shah, *Gmmcp tracker: Globally optimal generalized maximum multi clique problem for multiple object tracking*, 2015 IEEE Conference on Computer Vision and Pattern Recognition (CVPR), 2015, pp. 4091–4099.
- [DDDT82] J. Della Dora, Cl. Dicrescenzo, and E. Tournier, *An algorithm to obtain formal solutions of a linear homogeneous differential equation at an irregular singular point*, Computer algebra (Marseille, 1982), Lecture Notes in Comput. Sci., vol. 144, Springer, Berlin-New York, 1982, pp. 273–280.
- [Fab85] E. Fabry, *Sur les intégrales des équations différentielles linéaires à coefficients rationnels*, Thèse de doctorat ès sciences mathématiques, Faculté des Sciences de Paris, July 1885.
- [Fab88a] Fabry, *Réductibilité des équations différentielles linéaires*, Bull. Soc. Math. France **16** (1888), 135–142.
- [Fab88b] E. Fabry, *Réductibilité des équations différentielles linéaires*, C. R. Acad. Sci., Paris **106** (1888), 732–734.
- [Fal83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.
- [FJ21] Javier Fresán and Peter Jossen, *A non-hypergeometric  $E$ -function*, Ann. of Math. (2) **194** (2021), no. 3, 903–942.
- [FL02] M. Filaseta and T.-Y. Lam, *On the irreducibility of the generalized Laguerre polynomials*, Acta Arith. **105** (2002), no. 2, 177–182.
- [FR14] Stéphane Fischler and Tanguy Rivoal, *On the values of  $G$ -functions*, Comment. Math. Helv. **89** (2014), no. 2, 313–341 (English).
- [FR16] ———, *Arithmetic theory of  $E$ -operators*, J. Éc. Polytech., Math. **3** (2016), 31–65 (English).
- [Gor00] V. A. Gorelov, *Algebraic independence of the values of  $E$ -functions at singular points and Siegel's conjecture*, Math. Notes **67** (2000), no. 2, 138–151 (English).
- [Gor04] ———, *On the Siegel conjecture for second-order homogeneous linear differential equations*, Math. Notes **75** (2004), no. 4, 513–529 (English).
- [Gri90] D. Yu. Grigoriev, *Complexity of factoring and calculating the GCD of linear ordinary differential operators*, J. Symbolic Comput. **10** (1990), no. 1, 7–37.

- [Hil15] E. Hilb, *Lineare Differentialgleichungen im komplexen Gebiet*, pp. 471–562, 1915, Chapter II.B.5 of “B. Analysis der komplexen Größen” in *Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen: Zweiter Band in Drei Teilen Analysis*, Vieweg+Teubner Verlag, 1921.
- [HS00] Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000, An introduction.
- [HW06] Farshid Hajir and Siman Wong, *Specializations of one-parameter families of polynomials*, *Ann. Inst. Fourier (Grenoble)* **56** (2006), no. 4, 1127–1163.
- [Inc56] E. L. Ince, *Ordinary differential equations*, Dover Publications, New York, 1956, Reprint of the 1926 edition.
- [JKM13] Fredrik Johansson, Manuel Kauers, and Marc Mezzarobba, *Finding hyperexponential solutions of linear ODEs by numerical evaluation*, ISSAC’13—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2013, pp. 211–218.
- [JNV20] Claude-Pierre Jeannerod, Vincent Neiger, and Gilles Villard, *Fast computation of approximant bases in canonical form*, *J. Symbolic Comput.* **98** (2020), 192–224.
- [Kau14] Manuel Kauers, *Bounds for  $D$ -finite closure properties*, ISSAC’14—Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2014, pp. 288–295.
- [Lep21] Gabriel Lepetit, *Le théorème d’André-Chudnovsky-Katz au sens large*, *North-W. Eur. J. of Math.* **7** (2021), 83–149.
- [Lio33] Joseph Liouville, *Second mémoire sur la détermination des intégrales dont la valeur est algébrique*, *J. Éc. Polytech.* **14** (1833), 149–193.
- [LM95] Lee Lorch and Martin E. Muldoon, *Transcendentality of zeros of higher derivatives of functions involving Bessel functions*, *Internat. J. Math. Math. Sci.* **18** (1995), no. 3, 551–560.
- [LVO00] Huishi Li and Freddy Van Oystaeyen, *Elimination of variables in linear solvable polynomial algebras and  $\partial$ -holonomicity*, *J. Algebra* **234** (2000), no. 1, 101–127.
- [Mal79] B. Malgrange, *Sur la réduction formelle des équations différentielles à singularités irrégulières*, Preprint, 1979.
- [Mar91a] André Markoff, *Sur la théorie des équations différentielles linéaires*, *C. R. Acad. Sci., Paris* **113** (1891), 790–791.
- [Mar91b] ———, *Sur la théorie des équations différentielles linéaires*, *C. R. Acad. Sci., Paris* **113** (1891), 1024–1025.
- [Mar91c] ———, *Sur les équations différentielles linéaires*, *C. R. Acad. Sci., Paris* **113** (1891), 685–688.
- [Mor22] L. J. Mordell, *On the rational solutions of the indeterminate equations of the third and fourth degrees*, *Math. Proc. Cambridge Philos. Soc.* **21** (1922), 179—192.
- [Mos60] Jürgen Moser, *The order of a singularity in Fuchs’ theory*, *Math. Z* **72** (1959/1960), 379–398.
- [NS96] Yu. V. Nesterenko and A. B. Shidlovskii, *On the linear independence of values of  $E$ -functions*, *Mat. Sb.* **187** (1996), no. 8, 93–108.
- [Ogi67] H. Oigigova, *Les lettres de Ch. Hermite à A. Markoff, 1885–1899*, *Rev. Histoire Sci. Appl.* **20** (1967), 2–32.
- [OOL<sup>+</sup>22] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and eds. M. A. McClain, *NIST digital library of mathematical functions*, <http://dlmf.nist.gov/>, 2022.
- [Ore33] Oystein Ore, *Theory of non-commutative polynomials*, *Ann. of Math.* **34** (1933), 480–508.
- [Pai91] P. Painlevé, *Remarque sur une Communication de M. Markoff, relative à des équations différentielles linéaires*, *C. R. Acad. Sci., Paris* **113** (1891), 739–740 (French).
- [Poo60] E. G. C. Poole, *Introduction to the theory of linear differential equations*, Dover Publications Inc., New York, 1960.

- [Pop72] V. M. Popov, *Invariant description of linear, time-invariant controllable systems*, SIAM J. Control **10** (1972), 252–264.
- [Riv16] T. Rivoal, *Valeurs algébriques de  $E$ -fonctions aux points algébriques*, Unpublished note (2016), 4 pages.
- [Sch97] Ludwig Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen. In zwei Bänden. Band II, Theil 1*, Teubner, 1897.
- [Sch09] L. Schlesinger, *Bericht über die Entwicklung der Theorie der linearen Differentialgleichungen seit 1865*, Jahresber. Deutsch. Math.-Verein. **18** (1909), 133–266.
- [Sch31] J. Schur, *Affektlose Gleichungen in der Theorie der Laguerreschen und Hermiteschen Polynome*, J. Reine Angew. Math. **165** (1931), 52–58.
- [Sch86] Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986, A Wiley-Interscience Publication.
- [Sch89] F. Schwarz, *A factorization algorithm for linear ordinary differential equations*, ISSAC'89—Proceedings of the 14th International Symposium on Symbolic and Algebraic Computation, ACM Press, 1989, pp. 17–25.
- [Shi89] Andrei Borisovich Shidlovskii, *Transcendental numbers*, De Gruyter Studies in Mathematics, vol. 12, Walter de Gruyter & Co., Berlin, 1989, Translated from the Russian by Neal Koblitz, with a foreword by W. Dale Brownawell.
- [Sie14] Carl L. Siegel, *Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1]*, On some applications of Diophantine approximations, Quad./Monogr., vol. 2, Ed. Norm., Pisa, 2014, pp. 81–138.
- [Sin93] Michael F. Singer, *Moduli of linear differential equations on the Riemann sphere with fixed Galois groups*, Pacific J. Math. **160** (1993), no. 2, 343–395.
- [Sin96] ———, *Testing reducibility of linear differential operators: a group-theoretic perspective*, Appl. Algebra Engrg. Comm. Comput. **7** (1996), no. 2, 77–104.
- [Sin09] ———, *Introduction to the Galois theory of linear differential equations*, Algebraic theory of differential equations, London Math. Soc. Lecture Note Ser., vol. 357, Cambridge Univ. Press, Cambridge, 2009, pp. 1–82.
- [Sze75] Gábor Szegő, *Orthogonal polynomials*, fourth ed., American Mathematical Society Colloquium Publications, Vol. XXIII, American Mathematical Society, Providence, R.I., 1975.
- [Tsa94] Sergey P. Tsarev, *Problems that appear during factorization of ordinary linear differential operators*, Program. Comput. Softw. **20** (1994), no. 1, 67–75.
- [Tsa00] Harrison Tsai, *Weyl closure of a linear differential operator*, vol. 29, 2000, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998), pp. 747–775.
- [Vaz01] Vijay V. Vazirani, *Approximation algorithms*, Springer-Verlag, Berlin, 2001.
- [VBB92] M. Van Barel and A. Bultheel, *A general module-theoretic framework for vector  $M$ -Padé and matrix rational interpolation*, Numer. Algorithms **3** (1992), no. 1-4, 451–461.
- [vdH07] Joris van der Hoeven, *Around the numeric-symbolic computation of differential Galois groups*, J. Symbolic Comput. **42** (2007), no. 1-2, 236–264.
- [vdH16] ———, *On the complexity of skew arithmetic*, Appl. Algebra Engrg. Comm. Comput. **27** (2016), no. 2, 105–122.
- [vdPS03] Marius van der Put and Michael F. Singer, *Galois theory of linear differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 328, Springer-Verlag, Berlin, 2003.
- [vH96] Mark van Hoeij, *Rational solutions of the mixed differential equation and its application to factorization of differential operators*, ISSAC'96—Proceedings of the 21st International Symposium on Symbolic and Algebraic Computation, ACM Press, 1996, pp. 219–225.

- [vH97a] ———, *Factorization of differential operators with rational functions coefficients*, J. Symbolic Comput. **24** (1997), no. 5, 537–561.
- [vH97b] ———, *Formal solutions and factorization of differential operators with power series coefficients*, J. Symbolic Comput. **24** (1997), no. 1, 1–30.
- [Vid07] Raimundas Vidūnas, *Degenerate Gauss hypergeometric functions*, Kyushu J. Math. **61** (2007), no. 1, 109–135.
- [Won05] Siman Wong, *On the genus of generalized Laguerre polynomials*, J. Algebra **288** (2005), no. 2, 392–399.
- [Zei91] Doron Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), no. 3, 195–204.

ALIN BOSTAN, INRIA, UNIVERSITÉ PARIS-SACLAY, 1 RUE HONORÉ D’ESTIENNE D’ORVES, 91120 PALAISEAU, FRANCE.

*Email address:* alin.bostan@inria.fr

TANGUY RIVOAL, INSTITUT FOURIER, CNRS ET UNIVERSITÉ GRENOBLE ALPES, CS 40700, 38058 GRENOBLE CEDEX 9.

*Email address:* tanguy.rivoal@univ-grenoble-alpes.fr

BRUNO SALVY, UNIV LYON, ENSL, UCBL, CNRS, INRIA, LIP, F-69342, LYON CEDEX 07, FRANCE.

*Email address:* bruno.salvy@inria.fr