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# Systematic Design Method of Passive Ladder Filters using a Generalised Variable

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**Abstract**—This paper focuses on the design of passive filters with a ladder structure. Based on convex optimisation, we propose an approach to extend the traditional *LC*-ladder filter design method to lossless passive components. To achieve this, a generalised variable  $T(s)$  is introduced, leading to a new convex formulation of the design problem. The approach is applied on an academic design problem: the design of a *LC*-bandpass ladder filter. This provides a first and crucial step before designing filters of practical interest.

**Index Terms**—Frequency Filter Design, Passive Components, Ladder Structure

## I. INTRODUCTION

Despite the rise of digital filters, analog filters are still of importance for electronics applications. In particular, they are used in high frequency circuits or low power applications, where their digital counterparts are too costly or requires too much power [1]. In RadioFrequency applications, analog passive filters, based on acoustic resonators (such as SAW/BAW), are especially appreciated for their performance, low-power consumption and high-quality factor.

In order to face future challenges, new sophisticated components with better characteristics (filtering performance, power consumption, integration,...) are emerging [2]. The resulting system consequently becomes increasingly complex and traditional design methods cannot efficiently solve the design problems. In order to improve the design flow and to solve highly complex design problems, new *systematic* design methods are required. Convex optimisation is a natural candidate to develop such methods. It is a powerful framework, endowed with generic solvers, able to optimally solve a large variety of engineering problems [3]. Furthermore, many analysis and design problems can be formulated as an Linear Matrix Inequality (LMI) optimisation problem, which is an important

class of convex optimisation known to be solvable in a *decent* computational time [4].

A specificity of analog filters is the configuration the elements are set in, known as the *structure* of the filter. Some structure may demonstrate substantial benefits. For example, *LC* filters with a *ladder* structure are known to be robust to component value variability [5]. However, the introduction of a structure makes the resulting design problem harder to solve. In the System Design community, it is well known that optimisation algorithms for design problems of structured systems have a prohibitive computational complexity [6].

The usual design method of passive filters involves two steps. First, a transfer function  $s_{21}$ , whose frequency-response is under modulus constraints, is computed. When  $s_{21}$  is a standard transfer function, depending on the Laplace variable  $s$ , the synthesis problem can be transformed into an LMI optimisation problem [7]. Second, the resulting transfer function  $s_{21}$  needs to verify realisability constraints, in order to be implementable in a given structure. These constraints are complex to find, and even harder to check, for a general structure. In fact, they generally lead to a non-convex formulation of the problem and require a good initial point to prevent convergence issues [1], [8]. However, for a *LC*-ladder filter, i.e. a ladder filter where inductances  $L$  and capacitances  $C$  are alternatively set in the serial and the parallel arms, these conditions appear to be simple and can be easily included into the LMI optimisation problem of the synthesis of  $s_{21}$  [5].

In the Control community, a paradigm has recently emerged to formulate analysis and design problems of structured systems as LMI optimisation problems. The underlying idea is the introduction of a generalised variable which encompasses the information pertaining to the structure. In the last decade, this approach has been successful in many areas of applications as diverse as formation flying [9], micro-Electronics [10], [11], biological networks [12]. Our purpose is to apply this idea to the design of structured passive filters.

Recently, the synthesis problem of a transfer function  $s_{21}(T(s))$  depending on a generalised variable  $T(s)$  has been formulated as an LMI optimisation problem [13]. The main contribution of this paper is then to provide realisability conditions that can be included in this formulation, in the case where  $T(j\omega)^* = T(j\omega)$ . The resulting circuit has then a ladder structure with serial elements impedance of the form  $Z(T(j\omega)) = \alpha \cdot T(j\omega)$  and parallel elements admittance of the form  $Y(T(j\omega)) = \beta \cdot T(j\omega)$ . This is applied to the direct synthesis of  $LC$ -bandpass filter with a ladder structure. This result may be viewed as an intermediary, but crucial, result before tackling design problems of practical interest.

### A. Structure of the paper

This paper is organised as follows. Section II presents the design problem of  $LC$ -ladder filters. Section III introduces and solves the extended design problem of  $T$ -elements ladder filters. The approach is illustrated in Section IV on the design of a  $LC$ -bandpass ladder filter. Section V concludes the paper.

### B. Notations and Definitions

The letter  $s$  stands for the Laplace variable and  $j\bar{\mathcal{R}}$  for the entire imaginary axis.  $z^*$  and  $\Re\{z\}$  respectively denote the complex conjugate and the real part of the complex number  $z$ . A  $T$ -element refers to an element described by an impedance  $Z$  [resp. an admittance  $Y$ ] of the form  $Z(T(j\omega)) = \alpha \cdot T(j\omega)$  [resp.  $Y(T(j\omega)) = \alpha \cdot T(j\omega)$ ], where  $\alpha$  is a real, positive scalar.

**Definition 1 (Passive Element).** An element with an impedance  $Z$  is said to be passive if  $Z$  satisfies:  $\forall \omega \in \mathbb{R}, Z(j\omega) + Z(j\omega)^* \geq 0$ . It is said to be lossless-passive if  $Z(j\omega) + Z(j\omega)^* = 0$ .

**Definition 2 ( $T$ -positive-real).** A rational function  $f(T(s))$  of a complex variable  $T(s)$  is said to be  $T$ -positive-real if  $f(T(s))$  is real for  $T(s)$  real and

$$\Re\{f(T(s))\} > 0 \quad \text{for} \quad \Re\{T(s)\} > 0$$

**Definition 3 ( $T$ -bounded-real).** Any real rational function  $h(T(s))$  that satisfies the conditions:

- 1)  $h(T(s))$  has no poles on the closed right-half plane.
- 2)  $\forall T(j\omega) \in j\bar{\mathcal{R}}, |h(T(j\omega))| \leq 1$

is said to be a  $T$ -bounded-real rational function.

## II. LC-LADDER FILTER DESIGN

### A. Background about Passive Filters

A passive filter is traditionally considered as a two-port network  $Q$ , composed of *analog* passive elements, inserted between a generator ( $E_g, R_g$ ) and a resistive load  $R_l$  (Fig. 1). The input-output power transfer of a passive two-port  $Q$  is characterised by the scattering matrix  $S$ :

$$\begin{bmatrix} \frac{U_1 - R_g I_1}{2\sqrt{R_g}} \\ \frac{U_2 - R_l I_2}{2\sqrt{R_l}} \end{bmatrix} = \underbrace{\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}}_{:=S} \begin{bmatrix} \frac{U_1 + R_g I_1}{2\sqrt{R_g}} \\ \frac{U_2 + R_l I_2}{2\sqrt{R_l}} \end{bmatrix}$$

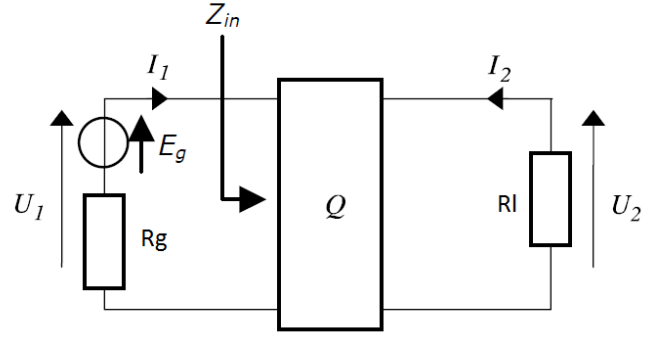


Fig. 1: An Analog Filter

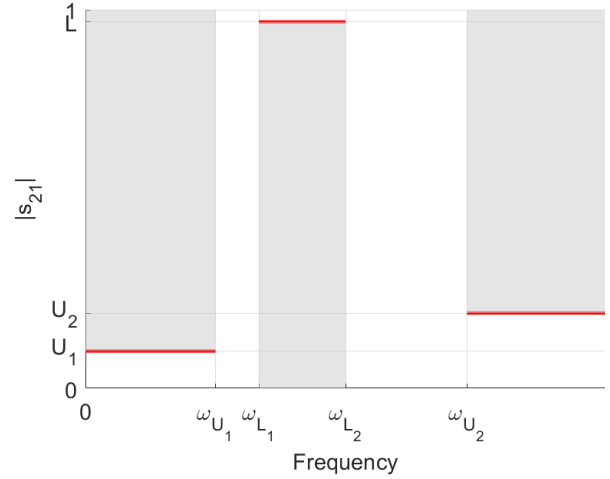


Fig. 2: Example of a spectral mask for bandpass requirements

where  $U_1$  and  $I_1$  are respectively the *complex* input voltage and current. For a *lossless*-passive network  $Q$ , the scattering parameters are linked by the relation:

$$|s_{21}(j\omega)|^2 = 1 - |s_{11}(j\omega)|^2 \quad (1)$$

Frequency-filter constraints are set on the squared magnitude of the scattering parameter  $|s_{21}(j\omega)|^2$ , as it represents the power gain between the generator and the source [14]. These frequency-domain constraints will be referred to as the *spectral mask* in the sequel.

**Example 1.** Typical spectral mask for passive filters are bandpass constraints, illustrated in Fig. 2 and correspond to:

$$\forall \omega \in \mathbb{R}, |s_{21}(j\omega)| \leq 1 \quad (2)$$

$$\forall \omega \in [0; \omega_{U_1}], |s_{21}(j\omega)| \leq U_1 \quad (3)$$

$$\forall \omega \in [\omega_{L_1}; \omega_{L_2}], |s_{21}(j\omega)| \geq L \quad (4)$$

$$\forall \omega \in [\omega_{U_2}; \infty[, |s_{21}(j\omega)| \leq U_2 \quad (5)$$

The behaviour of the two-port  $Q$  terminated on  $R_l$  can be represented by the *input impedance*  $z_{in}$ , defined by:

$z_{in} = U_1/I_1$ . Then, simple calculation provides the relation:

$$z_{in} = R_g \cdot \frac{1 + s_{11}}{1 - s_{11}} \quad (6)$$

### B. Ladder Structure

Passive filters are mostly designed in a *ladder* form. Historically the most studied in circuit theory, *ladder* passive filters are known to have low sensitivity to component values variations [5], [15]. In addition, ladder filters have the property that all the transmission zeros are included in the anti-resonant frequencies of the series arms and the resonant frequencies of the parallel arms [16], [17].

A ladder topology for four elements is illustrated in Fig. 3. This results in a specific decomposition for  $z_{in}$ :

$$z_{in} = Z_1 + \frac{1}{Y_2 + \frac{1}{Z_3 + \frac{1}{Y_4 + \frac{1}{R_l}}}} \quad (7)$$

When  $Z_{2i-1}(s) = \alpha_{2i-1} \cdot s$  and  $Y_{2i}(s) = \alpha_{2i} \cdot s$ ,  $i \in \{1, 2\}$ , the decomposition of (7) is referred to as a *continued fractional expansion*.

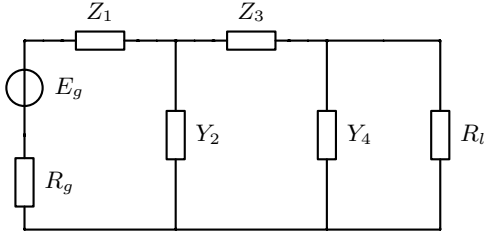


Fig. 3: Example of a Ladder Filter

### C. LC-Ladder Filter Design Problem

In a *LC*-ladder filter, serial and parallel elements are respectively inductances  $L$  and capacitances  $C$ . As far as the authors know, this is the only filter design problem, with a prescribed structure, which has a convex formulation.

This problem is usually decoupled into two sub-problems: 1) the synthesis of the scattering parameter  $s_{21}$  under frequency modulus constraints and 2) the satisfaction of the *LC*-ladder realisability conditions [5].

1) *Synthesis Problem*: The first sub-problem deals with the synthesis of the scattering parameter  $s_{21}$ , which should satisfy a spectral mask  $SM$ , such as for instance bandpass requirements.

**Problem 1 (Standard Synthesis Problem).** Given a spectral mask  $SM$ ,

Compute  $a_0, \dots, a_n$  and  $b_0, \dots, b_m$  such that

$$s_{21}(s) = \frac{b_0 + b_1 \cdot s + \dots + b_m \cdot s^m}{a_0 + a_1 \cdot s + \dots + a_n \cdot s^n}$$

and  $|s_{21}(j\omega)|$  satisfies  $SM$ .

2) *LC-Ladder Realisability Conditions*: The second sub-problem is about finding the conditions such that the filter, resulting from Problem 1, is in a *LC*-ladder form. One has then to find the conditions such that  $z_{in}$  can be decomposed as a continued-fractional expansion in  $s$ .

**Problem 2 (LC-Ladder Realisability Conditions).**

Given  $s_{21}$  (or  $z_{in}$ ),

Find conditions such that  $z_{in}$  admits a continued fractional expansion in  $s$ :

$$z_{in}(s) = \alpha_1 \cdot s + \frac{1}{\alpha_2 \cdot s + \frac{1}{\ddots \frac{1}{\alpha_n \cdot s + \frac{1}{R_l}}}} \quad (8)$$

with  $\alpha_1 > 0, \alpha_2 > 0, \dots, \alpha_n > 0$ .

### D. LC-Ladder Filter Design Methods

Structural realisability conditions are in general difficult to formulate as an LMI optimisation problem. However, those for *LC*-ladder filters are rather simple. It appears that sufficient conditions are that  $s_{21}(s)$  is  $T$ -bounded-real, with  $T(s) = s$ , and is a stable all-pole function, i.e.  $s_{21}(s) = \frac{1}{g(s)}$  with  $g$  a Hurwitz polynomial [5].

In the traditional approach, solutions of Problem 1 are calculated using the Butterworth or the Chebyshev approximations. A *prototype lowpass* filter is then obtained [18]. As these approximations always satisfy the *LC*-ladder realisability conditions, a prototype *LC*-ladder filter is synthesised. Using element and frequency transformations, other standard filters may be obtained [5]. However, the resulting impedances of the transformed elements may not represent components of practical interest.

In [7], it is shown how Problem 1 can be formulated as an LMI optimisation problem. By adding the realisability conditions, one can formulate the *LC*-ladder filter design problem as an LMI optimisation problem. In the next section, it is shown how this approach can be extended to ladder filter with other lossless-passive elements. To achieve this, a generalised variable  $T(s)$  is introduced. The resulting design problem remains an LMI optimisation problem.

## III. GENERALISED VARIABLE APPROACH

### A. Introduction of a Generalised Variable

Design problems with a prescribed system structure are known to be generally complex to solve [6]. An idea has recently emerged in order to take the internal structure into account, via the introduction of a generalised variable  $T$ . From a System theory perspective, these systems are viewed as the interconnection of identical subsystems [12]. The resulting design problem is then the design of the interconnection of these subsystems [10].

Considering ladder filters, this paradigm can be applied as follows. Serial elements should have an impedance as  $Z_{2i-1}(T(j\omega)) = \alpha_{2i-1} \cdot T(j\omega)$ , while the parallel elements should have an admittance as  $Y_{2i}(T(j\omega)) = \alpha_{2i} \cdot T(j\omega)$ .

The resulting decision variables are then the parameters  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Here, the variable  $T$  represents the internal structure of the linear model of the components. The next example illustrates this idea.

**Example 2.** Usual LC serial and parallel arms for ladder bandpass filters are given in Fig. 4. The serial impedance  $Z_s$

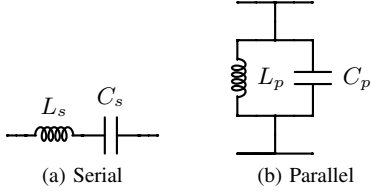


Fig. 4: Usual serial/parallel arms for LC bandpass filters

and the parallel admittance  $Y_p$  are given by:

$$Z_s(j\omega) = L_s \cdot \frac{\omega_{0_s}^2 - \omega^2}{j\omega} \quad Y_p(j\omega) = C_p \cdot \frac{\omega_{0_p}^2 - \omega^2}{j\omega}$$

with  $\omega_{0_s}^2 = 1/L_s C_s$  and  $\omega_{0_p}^2 = 1/L_p C_p$ . In the particular case where  $\omega_{0_s}^2 = \omega_{0_p}^2 = \omega_0^2$ , the impedance  $Z_s$  and the admittance  $Y_p$  can be expressed as:

$$Z_s(T(j\omega)) = L_s \cdot T(j\omega) \quad Y_p(T(j\omega)) = C_p \cdot T(j\omega)$$

where  $T$  will be the generalised variable and is defined as

$$T(s) := \frac{\omega_0^2 + s^2}{s} \quad (9)$$

**Remark 1.** In this example,  $\omega_{0_s}$  and  $\omega_{0_p}$  need to be equal to a fixed, chosen  $\omega_0$ . This constraint is not specific to our approach and also appears in the standard design methodology. For typical bandpass constraints of (2)-(5),  $\omega_0$  is usually set to  $\omega_0 = \sqrt{\omega_{L_1} \omega_{L_2}}$  [18].

**Remark 2.** Standard LC-lowpass, LC-highpass and LC-bandstop ladder filters may be considered by respectively defining  $T(s) := s$ ,  $T(s) := 1/s$  and  $T(s) := s/(\omega_0^2 + s^2)$ .

### B. Filter Synthesis with a Generalised Variable

This subsection considers the extension of the standard synthesis problem of Problem 1 to the synthesis problem of the scattering parameter  $s_{21}$  as a rational function of the generalised variable  $T$ .

#### Problem 3 (Generalised Variable Synthesis Problem).

Given a spectral mask  $SM$ ,

Compute  $a_0, \dots, a_n$  and  $b_0, \dots, b_m$  such that

$$s_{21}(T(s)) = \frac{b_0 + b_1 \cdot T(s) + \dots + b_m \cdot T^m(s)}{a_0 + a_1 \cdot T(s) + \dots + a_n \cdot T^n(s)}$$

and  $|s_{21}(T(j\omega))|$  satisfies  $SM$ .

Based on the ideas of [19], [20], the approach developed in [7] has recently been extended [13] to solve Problem 3. This leads to an equivalent LMI optimisation problem, which can then be efficiently solved.

### C. T-Elements Ladder Realisability

#### 1) New Problem Formulation:

#### Problem 4 (T-Elements Ladder Realisability Conditions).

Given  $s_{21}$  (or  $z_{in}$ ),

Find conditions such that  $z_{in}$  admits a continued fraction in  $T(s)$ :

$$z_{in}(T(s)) = \alpha_1 \cdot T(s) + \frac{1}{\alpha_2 \cdot T(s) + \frac{1}{\dots \frac{1}{\alpha_n \cdot T(s) + \frac{1}{R_l} \text{ (or } R_l)}}} \quad (10)$$

with  $\alpha_1 > 0, \alpha_2 > 0, \dots, \alpha_n > 0$ .

Having a decomposition of  $z_{in}$  as in (10), the serial elements are given by  $Z_{2i-1}(T(j\omega)) = \alpha_{2i-1} \cdot T(j\omega)$  and parallel elements by  $Y_{2i}(T(j\omega)) = \alpha_{2i} \cdot T(j\omega)$ .

2) Extended Ladder Realisability condition: The next theorem directly extends the realisability conditions from LC-ladder filters to T-elements ladder filters. Refer to Appendix A for the proof.

**Theorem 1.** Assume that  $T(s) = -T(-s)$ .

Let  $z_{in}$  be a T-positive-real rational function written as

$$z_{in}(T(s)) = \frac{N(T(s))}{D(T(s))}$$

Suppose that

$$\frac{1}{2} (z_{in}(T) + z_{in}(-T)) = \frac{1}{D(-T)D(T)} \quad (11)$$

Then  $z_{in}(T(s))$  admits a continued fraction expansion such as in (10), with  $\alpha_1 > 0, \alpha_2 > 0, \dots, \alpha_n > 0$ .

**Remark 3.** When  $T(s) = s$ , the usual realisability conditions are found [5].

**Remark 4.** Evaluated on the  $j\omega$ -axis, the assumption on  $T$  of Theorem 1 becomes  $T(j\omega) + T(j\omega)^* = 0$ . Thus, this theorem is available for all the lossless passive elements with impedance [resp. admittance]  $Z(T(j\omega)) = \alpha \cdot T(j\omega)$  [resp.  $Y(T(j\omega)) = \alpha \cdot T(j\omega)$ ]. In particular, this includes any components modelled as the interconnection of inductances and capacitances.

Conditions of Theorem 1 seem hard to connect to the convex optimisation problem of the synthesis problem of  $s_{21}$ . The aim of Corollary 1 is to provide this connection.

**Corollary 1.** Assume that  $T(s) = -T(-s)$ .

Let  $s_{21}(T(s))$  be a T bounded-real rational function written as

$$s_{21}(T(s)) = \frac{N(T(s))}{D(T(s))}$$

Suppose that

$$s_{21}(T(s))s_{21}(-T(s)) = \frac{1}{D(T(s))D(-T(s))}$$

Then the resulting  $z_{in}(T(s))$  admits a continued fraction expansion such as in (10), with  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ , ...,  $\alpha_n > 0$ .

$T$ -elements ladder realisability conditions of Corollary 1 are that  $s_{21}$  is  $T$ -bounded-real and has a constant numerator. These conditions can be easily added to the LMI optimisation problem of the synthesis of  $s_{21}(T)$ . The global design problem is then an LMI optimisation problem.

#### IV. APPLICATION

In this section, our approach is illustrated on an academic example: the design problem of a  $LC$ -bandpass ladder filter. The constraints are those of (2)-(5) with  $\omega_{U_1} = 0.75$  rad/s,  $\omega_{L_1} = 1$  rad/s,  $\omega_{L_2} = 1.5$  rad/s and  $\omega_{U_2} = 2.2$  rad/s, and  $U_1 = 0.1$ ,  $L = 0.96$  and  $U_2 = 0.2$ . The generalised variable  $T$  is defined as in (9) with  $\omega_0 = \sqrt{\omega_{L_1} \cdot \omega_{L_2}}$ .

In order to get the maximum power transmitted (*matching impedance*), the internal resistance of the generator and the resistive load are generally considered to be equal to the same value  $R$ :  $R_g = R_l = R$ . The value of  $R$  can be normalised to  $R = 1\Omega$ , as impedance scaling may be achieved subsequently [5].

First, the squared magnitude  $|s_{21}(T(j\omega))|^2$  is computed using the Robust Control toolbox of Matlab, by applying the LMI optimisation-based approach described in [13]:

$$|s_{21}(T)|^2 = \frac{0.05402}{T^8 + 0.08091 \cdot T^6 + 0.009036 \cdot T^4 - 0.00008729 \cdot T^2 + 0.05402}$$

which satisfies the condition of Corollary 1. Recalling that  $T(j\omega) = -T(-j\omega)$ , by (1) it comes that  $|s_{11}(T(j\omega))|^2 = 1 - |s_{21}(T(j\omega))|^2$ . Then, using the spectral factorisation technique [13],  $s_{11}(T)$  is calculated:

$$s_{11}(T) = \frac{T^4 + 0.5784 \cdot T^3 + 0.2079 \cdot T^2 + 0.02954 \cdot T}{T^4 + 1.78 \cdot T^3 + 1.625 \cdot T^2 + 0.8693 \cdot T + 0.2324}$$

Thus, using (6),  $z_{in}(T)$  is obtained:

$$z_{in}(T) = \frac{1.665 \cdot T^4 + 1.963 \cdot T^3 + 1.525 \cdot T^2 + 0.7481 \cdot T + 0.1934}{T^3 + 1.179 \cdot T^2 + 0.699 \cdot T + 0.1934}$$

Finally, by decomposing  $z_{in}(T)$  as in (10), the ladder circuit of Fig. 5 is obtained, with:

$$\begin{aligned} L_1 &\approx 1.665 \text{ H} & C_1 &= \frac{1}{L_1 \cdot \omega_0^2} \approx 0.4004 \text{ F} \\ C_2 &\approx 2.767 \text{ F} & L_2 &= \frac{1}{C_2 \cdot \omega_0^2} \approx 0.2409 \text{ H} \\ L_3 &\approx 2.204 \text{ H} & C_3 &= \frac{1}{L_3 \cdot \omega_0^2} \approx 0.3025 \text{ F} \\ C_4 &\approx 0.848 \text{ F} & L_4 &= \frac{1}{C_4 \cdot \omega_0^2} \approx 0.7862 \text{ H} \end{aligned}$$

#### V. CONCLUSION

In this paper, we have focused on the design of analog filter with lossless-passive elements set in a ladder structure. To achieve this, a generalised variable  $T(s)$  has been introduced. We have extended the  $LC$ -ladder filter design method to the design of  $T$ -elements ladder filters. We finally applied this approach to the direct design of  $LC$ -bandpass ladder filter.

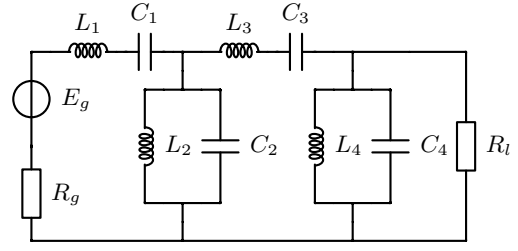


Fig. 5: Illustrative Example - a  $LC$  bandpass ladder filter

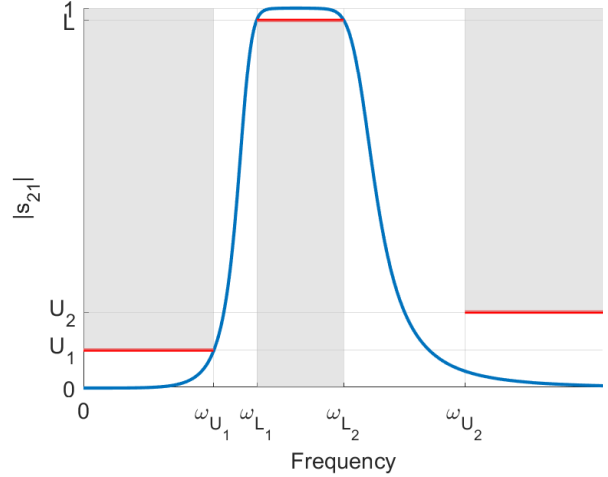


Fig. 6: Synthesis of a bandpass filter

Future research directions include the application of the developed approach to other ladder filters made of components modelled as interconnected inductances and capacitances. This approach may also be extended to lossy passive components.

Modern RadioFrequency filters are made of two kinds of acoustic resonators set in a ladder form. Serial components have impedances as  $Z_s(j\omega) = \alpha_s \cdot \frac{1}{j\omega} \cdot \frac{\omega^2 - \omega_{r_s}^2}{\omega^2 - \omega_{a_s}^2}$  and paral-

lel  $Z_p(j\omega) = \alpha_p \cdot \frac{1}{j\omega} \cdot \frac{\omega^2 - \omega_{r_p}^2}{\omega^2 - \omega_{a_p}^2}$ . This leads to the introduction of two generalised variables  $T_s(s)$  and  $T_p(s)$  such that  $Z_s(T_s(j\omega)) = \alpha_s \cdot T_s(j\omega)$  and  $Z_p(T_p(j\omega)) = \alpha_p \cdot T_p(j\omega)$ . Therefore, extending the approach to two generalised variables would be another direction of practical interest.

#### APPENDIX

##### PROOFS OF THEOREM 1 AND COROLLARY 1

Let first state and demonstrate the following lemma.

**Lemma 1.** *Let  $z$  be a  $T$ -positive-real rational function. Then, if it exists, a pole at infinity must be simple and have real positive residue.*

*Proof of Lemma 1.* Let  $T(s)$  be a complex variable such that  $\Re\{T(s)\} > 0$ . Let be a pole at infinity of order  $n$  of the rational function  $z$ . In the neighbourhood of this pole, using a Laurent expansion, we may write

that the dominant part of this expansion is  $a_n \cdot T^n(s)$ , where  $a_n$  is called a residue. Using a polar form, one can write  $a_n = |a_n|e^{j\theta_1}$  and  $T(s) = |T(s)|e^{j\theta_2(s)}$ , where  $\theta_2(s) \in (-\frac{\pi}{2}; \frac{\pi}{2})$ , as  $\Re\{T(s)\} > 0$ . Then

$$\Re\{a_n \cdot T^n(s)\} = |a_n||T(s)|^n \cos(\theta_1 + n\theta_2(s))$$

which is strictly positive for all  $\theta_2(s)$  in  $(-\frac{\pi}{2}; \frac{\pi}{2})$ , by definition. This implies that  $\theta_1 = 0$  and  $n \in \{0, 1\}$ . Therefore, a pole at infinity is at most simple and  $a_n > 0$ .  $\square$

Let demonstrate now Theorem 1.

*Proof of Theorem 1.* Without loss of generality, let assumes that the degree  $n$  of the numerator  $N$  is greater than or equal to the degree  $d$  of the denominator  $D$ :  $n \geq d$ .

By Lemma 1, as  $z_{in}$  is  $T$ -positive-real rational, a pole at infinity, if it exists, must be simple and then  $n$  and  $d$  may differ by at most 1:

$$|n - d| \leq 1$$

Furthermore, by (11),  $N$  and  $D$  cannot have same degree, leading to  $|n - d| = 1$ . Then, using the partial fractional expansion technique, this leads to:

$$z_{in}(T(s)) = \alpha_1 \cdot T(s) + z_2(T(s))$$

with the residue  $\alpha_1$  positive  $\alpha_1 > 0$ . Moreover, as

$$\forall T(j\omega) \in j\bar{\mathcal{R}} \quad \Re\{z_{in}(T(j\omega))\} = \Re\{z_2(T(j\omega))\}$$

using the *minimum-real part* theorem [5],  $z_2$  is also a  $T$ -positive-real rational function. Furthermore, as

$$\begin{aligned} \frac{1}{2} (z_2(T(s)) + z_2(-T(s))) &= \frac{1}{2} (z_{in}(T(s)) + z_{in}(-T(s))) \\ &= \frac{K}{D(T(s))D(-T(s))} \end{aligned}$$

then  $y_2 = 1/z_2$  has also a pole at infinity:  $y_2(T(s)) = \alpha_2 \cdot T(s) + y_3(T(s))$ . The process can then be iterated until the value of the resistance  $R_l$  is obtained. This occurs after a number of infinity pole extractions equal to the degree of the input impedance  $z_{in}$ .  $\square$

Finally, let demonstrate Corollary 1.

*Proof of Corollary 1.* As  $s_{21}$  is a  $T$ -bounded-real rational function, by (1),  $s_{11}$  is also a  $T$ -bounded-real rational function. Mere calculation provides:  $\forall T(j\omega) \in j\bar{\mathcal{R}}$ ,

$$\Re\{z_{in}(T(j\omega))\} = R_g \cdot \frac{1 - |s_{11}(T(j\omega))|^2}{|1 - s_{11}(T(j\omega))|^2} \geq 0$$

Moreover, as  $s_{11}$  has no pole on the closed right-half plane, according to the maximum modulus theorem,  $|s_{11}(T(s))|$  has its maximum value for  $T(s) \in j\bar{\mathcal{R}}$ . So  $\Re\{z_{in}(T(s))\} > 0$  in  $\Re\{T(s)\} > 0$ . Thus  $z_{in}$  is a  $T$  positive-real rational function.  $\square$

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