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A Riemann Hilbert approach to the study of the generating function associated to Pearcey process

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Abstract

Using Riemann-Hilbert problem methods, we establish a Tracy-Widom like formula for generating function of occupancy numbers of the Pearcey process. This formula is linked to a coupled vector differential equation of order 3. We also obtained a non linear coupled heat equation. Combining these two equations leads to a PDE for the logarithm of the Fredholm determinant for the Pearcey process.

1 Introduction

The Pearcey process is a universal determinantal point process associated to the Pearcey kernel (see (6) for the definition of the kernel). It appeared for the first time in the study of spectrum of random matrices with external sources [1]. If $2N$ is the size of matrices, the Pearcey process describes the behaviour, when $N \rightarrow \infty$, of eigenvalues near a point where the density of states admits a cusp-like gap. Another model linked with Pearcey process is a $2N$ 1-dimension non intersecting Brownian motions model [2] starting from 0 at $t = 0$ and ending at $\pm a$ (half particules at a and the others at $-a$) at time $t = 1$. For this model, there exists a time t_c such that, for $t < t_c$ the distribution of particles is supported in an interval and for $t > t_c$ it splits in two disjoint intervals. The distribution of particles of the Brownian motion for t close to t_c is described by the Pearcey process. A last example which reveals the universality of the Pearcey process is the one of random skew plane partition [3]. Studying the limit shape associated to this model leads to different processes: Beta process, extended Airy process, and extended Pearcey process.

As for the Airy process, it is possible to express probabilities associated to the Pearcey process in terms of some partial derivative equations by studying the Fredholm determinant of the Pearcey kernel operator (6). Studying Brownian motion model at several times and deriving the kernel associated to the Pearcey process, Tracy and Widom [4], obtained partial derivatives equations for the distribution associated to the Pearcey process. Using KP-tau functions and Hirota bilinear equations for the study of random matrix with external sources, Adler and van Moerbeke [5] introduced a non linear 4th order PDE for the Pearcey process. With Riemann-Hilbert methods Bertola and Cafasso [6] obtained same PDE's as Adler and van Moerbeke and introduced a new one for the gap probability for the Pearcey process.

The aim of this article is to present a Tracy-Widom formula for the generating function of the Pearcey process linked with two vector valued functions satisfying a system of coupled non linear third order differential equations and of non linear heat equation.

Let $N > 1$ be an integer, $\vec{\mathbf{a}} := (a_1, \dots, a_N)$ with $a_1 < \dots < a_N$ and $\vec{\mathbf{k}} = (k_0, k_1, \dots, k_{N-1}, k_N)$ such that $k_0 = k_N = 0$, $k_j \in [0, 1]$, $j = 1, \dots, N - 1$ and $k_j \neq k_{j+1}$, $j = 0, \dots, N - 1$.

Consider the Pearcey process (which depends on an additional positive parameter τ) and define its generating function

$$F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) := \mathbb{E} \left(\prod_{j=1}^{N-1} (1 - k_j)^{\sharp(a_j, a_{j+1})} \right) \quad (1)$$

where $\sharp I$ is the random variable counting the number of points of the process in the interval I .

This generating function is well defined for $(a_i)_{1 \leq i \leq N}$ finites.

Computing derivatives with respect to k_j 's of $F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}})$ allows us to express joint probability of occupancy numbers of particles:

$$\mathbb{P} \left(\bigcap_{j=1}^{N-1} \sharp(a_j, a_{j+1}) = m_j \right) = \frac{(-1)^{m_1 + \dots + m_{N-1}}}{m_1! \dots m_{N-1}!} \frac{\partial^{m_1 + \dots + m_{N-1}}}{\partial k_1^{m_1} \dots \partial k_{N-1}^{m_{N-1}}} F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) \Big|_{\vec{\mathbf{k}}=(1, \dots, 1)} \quad (2)$$

(see for instance [7] for a similar computation).

Remark 1. Since for all $B \subset \mathbb{R}$ bounded borelian $\sharp B < \infty$, $F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) > 0$.

Right, with Jensen's inequality,

$$F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) \geq \exp \left(\sum_{j=1}^{N-1} \log(1 - k_j) \mathbb{E}(\sharp(a_j, a_{j+1})) \right)$$

and $\forall 1 \leq j \leq N - 1$, $\log(1 - k_j) \mathbb{E}(\sharp(a_j, a_{j+1})) > -\infty$.

Charlier and Moreillon studied generating function of the Pearcey process [8] on intervals of the form $(-x_j, x_j)$. They considered a parameter r of dilatation of their intervals and obtained an expression for the generating function of the Pearcey process linked with an hamiltonian. Recently, Kimura and Zahabi [9] introduced higher order Pearcey kernels and obtained an hamiltonian structure for the level spacing distribution by studying a different Lax Pair for the Pearcey process on intervals of the form $[-s, s]$.

In this article, we follow a different way. We consider generating function not necessarily on symmetric intervals and we use a parameter s of translation of intervals instead of dilatation. This work is inspired by works of Claeys and Doeraene [10] and Charlier and Doeraene [11] where they studied generating function for the Airy process and for the Bessel process. Recently Cafasso and Tarricone [12] obtained a Tracy-Widom type formula for generating function associated to the higher order Airy process.

The generating function associated to the Pearcey process satisfies the following:

Theorem 2. Let $s \in \mathbb{R}$, $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ be as (1) where $\vec{\mathbf{a}} + s = (a_1 + s, \dots, a_N + s)$, then

$$\frac{\partial^2}{\partial s^2} \log \left(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) \right) = p^T(s)q(s)$$

with $(p(s), q(s)) = (p(s, \tau, \vec{\mathbf{a}}, \vec{\mathbf{k}}), q(s, \tau, \vec{\mathbf{a}}, \vec{\mathbf{k}}))$ vectors of size N satisfying:
a coupled third order differential equation

$$\begin{cases} \partial_{sss} p^T + 3(\partial_s p^T) q p^T + 3p^T q (\partial_s p^T) - \tau \partial_s p^T + p^T D_{s, \vec{\mathbf{a}}} = 0 \\ \partial_{sss} q + 3(\partial_s q) p^T q + 3q p^T (\partial_s q) - \tau \partial_s q - D_{s, \vec{\mathbf{a}}} q = 0 \end{cases} \quad (3)$$

where $D_{s, \vec{\mathbf{a}}} := \text{Diag}(a_1 + s, \dots, a_N + s)$
and a coupled non linear heat equation

$$\begin{cases} -\frac{1}{2} \partial_{ss} p^T - \partial_\tau p^T = p^T q p^T \\ -\frac{1}{2} \partial_{ss} q + \partial_\tau q = q p^T q \end{cases} \quad (4)$$

Defining $u(s, \tau) := \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}))$, u satisfies

$$\frac{\partial^2}{\partial \tau^2} u(s, \tau) + \frac{1}{2} \left(\frac{\partial^2}{\partial s^2} u(s, \tau) \right)^2 + \frac{1}{12} \frac{\partial^4}{\partial s^4} u(s, \tau) - \frac{1}{3} \tau \frac{\partial^2}{\partial s^2} u(s, \tau) = 0 \quad (5)$$

Remark 3. The equation (5) was already known for the gap probability for the Pearcey process (see for instance equation (1.10) in [13]).

The paper is organised as follows: in Section 2 we establish a link between $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and a Fredholm determinant of an integrable operator in the sense of IKS [14] and introduce the Riemann-Hilbert problem associated to this integrable operator. In Section 3 we present a Lax pair associated to the Riemann-Hilbert problem for the Pearcey kernel and obtain a coupled vector differential equation with respect to the variable s and a coupled non linear heat equation with respect to s and τ for elements of the Riemann-Hilbert problem. Finally, in Section 4 we compute the logarithmic derivative of $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ with respect to s and prove Theorem 2 with results of Sections 2 and 3.

2 From generating function of Pearcey process to a Riemann-Hilbert Problem

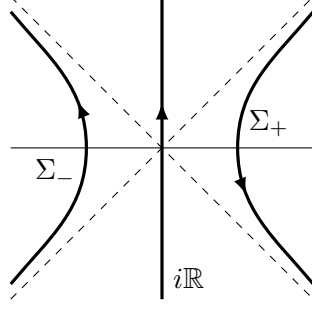
In this section we establish a link between the generating function $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of an integrable operator and introduce the Riemann-Hilbert Problem (RHP) associated to this integrable operator.

The Pearcey kernel operator

The Pearcey process is a determinantal point process associated to the Pearcey kernel operator. For $(x, \tau) \in \mathbb{R}^2$ define $\theta_x(\mu) := \frac{\mu^4}{4} - \frac{\tau \mu^2}{2} - x\mu$ and introduce the Pearcey Kernel

$$K_P(x, y; \tau) := \frac{1}{(2i\pi)^2} \int_{\Sigma} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu \quad (6)$$

where $\Sigma := \Sigma_- \cup \Sigma_+$ is as in the following figure:



Introduce $\mathbb{K}_P : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with kernel K_P . According to Theorem 2 of [15], if s and a_i 's are all finites, then the generating function $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of the operator \mathbb{K}_P are linked in the following way:

$$F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) = \det \left(1 - \sum_{j=1}^{N-1} k_j \chi_{(a_j+s, a_{j+1}+s)} \mathbb{K}_P \right) \quad (7)$$

where χ_I is the characteristic function of the interval I .

From Pearcey kernel operator to an integrable operator

The kernel of the operator \mathbb{K}_P defined as (6) is not the kernel of an integrable operator. Therefore we link this Fredholm determinant with the one of an integrable operator in the sense of [14] so that we can study $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ with Riemann-Hilbert problem associated to integrable operator.

Define

$$\tilde{f}(\mu) = \frac{1}{2i\pi} \begin{pmatrix} \frac{e^{\frac{1}{2}\theta_0(\mu)} \chi_{\Sigma}(\mu)}{\sqrt{k_1 - k_0} e^{-\frac{1}{2}\theta_0(\mu) + (a_1+s)\mu} \chi_{i\mathbb{R}}(\mu)} \\ \vdots \\ \frac{e^{-\frac{1}{2}\theta_0(\mu)} \chi_{i\mathbb{R}}(\mu)}{\sqrt{k_N - k_{N-1}} e^{-\frac{1}{2}\theta_0(\mu) + (a_N+s)\mu} \chi_{i\mathbb{R}}(\mu)} \end{pmatrix} \quad (8)$$

$$\tilde{g}(\mu) = \begin{pmatrix} \frac{e^{-\frac{1}{2}\theta_0(\mu)} \chi_{i\mathbb{R}}(\mu)}{\sqrt{k_1 - k_0} e^{\frac{1}{2}\theta_0(\mu) - (a_1+s)\mu} \chi_{\Sigma}(\mu)} \\ \vdots \\ \frac{e^{\frac{1}{2}\theta_0(\mu)} \chi_{\Sigma}(\mu)}{\sqrt{k_N - k_{N-1}} e^{\frac{1}{2}\theta_0(\mu) - (a_N+s)\mu} \chi_{\Sigma}(\mu)} \end{pmatrix} \quad (9)$$

We decompose \tilde{f} and \tilde{g} in a block 1×1 and a vector of size N because in what follows we will study a Riemann-Hilbert problem of size $(N+1) \times (N+1)$ and matrices will be partitioned in four blocks of size 1×1 , $1 \times N$, $N \times 1$ and $N \times N$.

Define $\mathbb{K} : L^2(\Sigma \cup i\mathbb{R}) \rightarrow L^2(\Sigma \cup i\mathbb{R})$ integrable operator in the sense of [14] with kernel

$$K(u, v) = \frac{\tilde{f}^T(u) \tilde{g}(v)}{u - v} \quad (10)$$

We establish the relation between $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of the operator \mathbb{K} as follow:

Proposition 4. Let $F(\vec{\mathbf{a}} + s, \vec{\mathbf{k}})$ be as in Theorem 2 and \mathbb{K} be as in (10). Then

$$F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) = \det(1 - \mathbb{K})_{L^2(\Sigma \cup i\mathbb{R})}$$

Proof. The idea of the proof is to compose the operator \mathbb{K} with multiplication operator and Fourier operator so that its Fredholm determinant will not change and to obtain an operator defined on $L^2(\mathbb{R})$ with kernel $\sum_{j=1}^{N-1} k_j \chi_{(a_j+s, a_{j+1}+s)}(x) K_P(x, y)$.

The proof of this proposition is similar to the one of Theorem 4.1 in [6] and can be adapted as follows.

Since $i\mathbb{R}$ and Σ are disjoint, you can decompose $L^2(\Sigma \cup i\mathbb{R})$ as $L^2(i\mathbb{R}) \oplus L^2(\Sigma)$ and write the following equality using matrix notation:

$$\det(1 - \mathbb{K})_{L^2(\Sigma \cup i\mathbb{R})} = \det \left(1 - \begin{pmatrix} 0 & \sum_{j=1}^N \mathcal{G}_j \\ \mathcal{F} & 0 \end{pmatrix} \right)_{L^2(i\mathbb{R}) \oplus L^2(\Sigma)} = \det \left(1 - \sum_{j=1}^N \mathcal{G}_j \circ \mathcal{F} \right)_{L^2(i\mathbb{R})}$$

where \mathcal{F} and \mathcal{G}_j are defined below with \mathcal{G}_j depending on k_j 's

$$\begin{aligned} \mathcal{F} : L^2(i\mathbb{R}) &\longrightarrow L^2(\Sigma) \\ f &\longmapsto \frac{e^{\frac{1}{2}\theta_0(\lambda)}}{2i\pi} \int_{\Sigma} \frac{e^{-\frac{1}{2}\theta_0(\mu)}}{\lambda - \mu} g(\mu) d\mu \end{aligned}$$

$$\begin{aligned} \mathcal{G}_j : L^2(\Sigma) &\longrightarrow L^2(i\mathbb{R}) \\ g &\longmapsto (k_j - k_{j-1}) \frac{e^{-\frac{1}{2}\theta_0(\xi) + \xi(a_j+s)}}{2i\pi} \int_{\Sigma} \frac{e^{\frac{1}{2}\theta_0(\lambda) - \lambda(a_j+s)}}{\xi - \lambda} g(\lambda) d\lambda \end{aligned}$$

Then composing with a multiplication operator \mathcal{M} and using a Fourier operator \mathcal{T} (for example as in [16]) it is possible to relate Fredholm determinants of \mathbb{K} and \mathbb{K}_P . Here the Fourier composition allows to go from an operator on $L^2(i\mathbb{R})$ to one on $L^2(\mathbb{R})$.

With $\mathcal{M} := e^{-\frac{1}{2}\theta_0(\mu)}$ and

$$\begin{aligned} \mathcal{T} : L^2(i\mathbb{R}) &\longrightarrow L^2(\mathbb{R}) \\ f &\longmapsto \frac{1}{\sqrt{2i\pi}} \int_{i\mathbb{R}} e^{-\xi x} f(\xi) d\xi \end{aligned}$$

$\mathcal{T} \circ \mathcal{M}^{-1} \circ \mathcal{G}_j \circ \mathcal{F} \circ \mathcal{M} \circ \mathcal{T}^{-1}$ has kernel

$$\begin{aligned} \mathcal{L}_j(x, y) &= \frac{k_j - k_{j-1}}{(2i\pi)^2} \int_{i\mathbb{R}} e^{\xi(a_j+s-x)} \int_{i\mathbb{R}} \int_{\Sigma} \frac{e^{\theta_{a_j+s}(\lambda) - \theta_y(\mu)}}{(\xi - \lambda)(\lambda - \mu)} d\lambda d\mu \frac{d\xi}{2i\pi} \\ \mathcal{L}_j(x, y) &= \begin{cases} \frac{k_j - k_{j-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_+} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu, & x > a_j + s \\ -\frac{k_j - k_{j-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_-} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu, & x < a_j + s \end{cases} \end{aligned}$$

If $x < a_1 + s$, then $x < a_j + s$ for all j and $\sum_{j=1}^N \mathcal{T} \circ \mathcal{M}^{-1} \circ \mathcal{G}_j \circ \mathcal{F} \circ \mathcal{M} \circ \mathcal{T}^{-1}$ has kernel

$$-\frac{\sum_{j=1}^N k_j - k_{j-1}}{(2i\pi)^2} \int_{\Sigma_-} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu = -\frac{k_N - k_0}{(2i\pi)^2} \int_{\Sigma_-} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu = 0$$

The same holds for $x > a_N + s$.

If $x \in (a_j + s, a_{j+1} + s)$, then

$$\begin{aligned} \sum_{j=1}^N \mathcal{L}_j(x, y) &= \frac{\sum_{\ell=1}^j k_\ell - k_{\ell-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_+} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu - \frac{\sum_{\ell=j+1}^N k_\ell - k_{\ell-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_-} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu \\ &= k_j \chi_{(a_j+s, a_{j+1}+s)}(x) K_P(x, y) \end{aligned}$$

This concludes the proof. \square

We can study $\det(1 - \mathbb{K})_{L^2(\Sigma \cup i\mathbb{R})}$ with the theory of Riemann-Hilbert problem associated to integrable operator. We describe the Riemann-Hilbert problem associated to \mathbb{K} in what follows.

Riemann-Hilbert Problem associated to Pearcey kernel operator

The contours for the Riemann-Hilbert problem associated to $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ is $\Sigma \cup i\mathbb{R}$ oriented as in the previous figure.

For the jump matrix, introduce

$$f(\mu) := \begin{pmatrix} -\sqrt{k_1 - k_0} e^{-\theta_{a_1+s}(\mu)} \\ \vdots \\ -\sqrt{k_N - k_{N-1}} e^{-\theta_{a_N+s}(\mu)} \end{pmatrix} \chi_{i\mathbb{R}}(\mu) \text{ and } g(\mu) := \begin{pmatrix} -\sqrt{k_1 - k_0} e^{\theta_{a_1+s}(\mu)} \\ \vdots \\ -\sqrt{k_N - k_{N-1}} e^{\theta_{a_N+s}(\mu)} \end{pmatrix} \chi_{\Sigma}(\mu)$$

Riemann-Hilbert Problem 5 (RHP for Γ). *We consider the Riemann-Hilbert problem with contours $\Sigma \cup i\mathbb{R}$ and jump matrix $J(\mu; \vec{\mathbf{a}}, \tau, s) := \left(\begin{array}{c|c} 1 & g^T(\mu) \\ \hline f(\mu) & I_n \end{array} \right) = I_{N+1} - 2i\pi \tilde{f}(\mu) \tilde{g}^T(\mu)$.*

We search a matrix valued function $\Gamma(\mu) = \Gamma(\mu; \vec{\mathbf{a}}, \tau, s)$ such that:

- $\Gamma : \mathbb{C} \setminus \Sigma \cup i\mathbb{R} \rightarrow \mathcal{GL}_{N+1}(\mathbb{C})$ is analytic
- $\Gamma_+(\mu) = \Gamma_-(\mu) J(\mu)$, $\mu \in \Sigma \cup i\mathbb{R}$ where Γ is continuous up to boundary of the contours and $\Gamma_{\pm}(\mu)$ are non-tangential limits approaching μ from left(+) or right(-).
-

$$\Gamma(\mu) = I_{N+1} + \sum_{j \geq 1} \frac{\Gamma_j}{\mu^j} = I_{N+1} + \frac{1}{\mu} \left(\begin{array}{c|c} -\delta(\vec{\mathbf{a}}, \tau, s) & p^T(\vec{\mathbf{a}}, \tau, s) \\ \hline q(\vec{\mathbf{a}}, \tau, s) & \Delta(\vec{\mathbf{a}}, \tau, s) \end{array} \right) + \dots \text{ as } \mu \rightarrow \infty \quad (11)$$

Remark 6. *If Γ satisfies the previous RHP, then $\det(\Gamma)$ is entire and according to the asymptotic $\det(\Gamma) \equiv 1$. Since $\det(\Gamma) \equiv 1$, the previous RHP has an unique solution (if it exists), $\text{Tr}(\Gamma_1) = 0$ and $\delta = \text{Tr}(\Delta)$.*

3 Study of the Riemann-Hilbert Problem associated to Pearcey kernel operator

In this section, we obtain a Lax Pair associated to the previous RHP which leads to a system of coupled vector differential equation for p^T and q (see equation (11) for the definition of p^T and q). Studying the derivative with respect to τ yields to a coupled non linear heat equation satisfied by p^T and q .

A Lax Pair for Ψ

Let Γ be a solution of RHP 5 and T be the gauge transformation:

$$T(\mu) := e^{\frac{1}{N+1} \sum_{j=1}^N \theta_{a_j+s}(\mu)} \text{Diag} \left(1, e^{-\theta_{a_1+s}(\mu)}, \dots, e^{-\theta_{a_N+s}(\mu)} \right).$$

Defining $\Psi(\mu) := \Gamma(\mu)T(\mu)$, the following result holds for Ψ .

Proposition 7 (Lax pair for Ψ). *If Γ is solution of RHP 5 then Ψ satisfy a system of partial differential equation polynomial in μ .*

$$\begin{cases} \partial_s \Psi(\mu) = A(\mu) \Psi(\mu) \\ \partial_\mu \Psi(\mu) = B(\mu) \Psi(\mu) \\ \partial_\tau \Psi(\mu) = C(\mu) \Psi(\mu) \end{cases} \quad (12)$$

where

$$A(\mu) = \mu A_1 + A_0 = \frac{\mu}{N+1} \left(\begin{array}{c|c} -N & (0) \\ \hline (0) & I_N \end{array} \right) + \left(\begin{array}{c|c} 0 & p^T \\ \hline -q & (0) \end{array} \right) \quad (13)$$

$$B(\mu) = \mu^3 \tilde{B}_3 + \mu^2 B_2 + \mu(\tilde{B}_1 + B_1) + \tilde{B}_0 + B_0 \quad (14)$$

with B_j 's depending on Γ_j 's and $\tilde{B}_3 = -A_1$, $\tilde{B}_1 = \tau A_1$

$$\tilde{B}_0 = s A_1 + \frac{1}{N+1} \text{Diag} \left(-\sum_{j=1}^N a_j, N a_1 - \sum_{j \neq 1} a_j, \dots, N a_N - \sum_{j \neq N} a_j \right)$$

and

$$C(\mu) = \mu^2 \tilde{C}_2 + \mu C_1 + C_0 = \frac{\mu^2}{2} A_1 + \mu C_1 + C_0 \quad (15)$$

with C_1 and C_0 depending on Γ_j 's.

Proof. It is easy to show that if Γ satisfies the previous RHP then Ψ satisfies a RHP with jump matrix which does not depend on $\vec{\mathbf{a}}, \tau, s$ and μ (the jump only depends on $k_j, j = 0, \dots, N$).

Then Ψ and $\partial_s \Psi$ have same jump on the contours. From this fact we deduce that $A(\mu) := \partial_s \Psi(\mu) \Psi(\mu)^{-1}$ is entire. Using Liouville's theorem with asymptotics of Ψ and $\partial_s \Psi$ from there

RHP we conclude A is a polynomial of degree 1 in μ . More precisely, we compute A and obtain:

$$A(\mu) = \mu A_1 + A_0 = \frac{\mu}{N+1} \left(\begin{array}{c|c} -N & (0) \\ \hline (0) & I_N \end{array} \right) + \left(\begin{array}{c|c} 0 & p^T \\ \hline -q & (0) \end{array} \right)$$

Using same methods we conclude that $B(\mu) := \partial_\mu \Psi(\mu) \Psi(\mu)^{-1}$ (respectively $C(\mu) := \partial_\tau \Psi(\mu) \Psi(\mu)^{-1}$) is a polynomial of degree 3 (respectively degree 2). We will not do all computations for B and C with the asymptotic as we did for A : we precise what we will do.

Computations with the asymptotic involve T , its partial derivative with respect to μ (respectively τ) and $(\Gamma_j)_{j \geq 1}$. For now we will only compute terms in B (respectively C) which does not depend on $(\Gamma_j)_{j \geq 1}$.

We start with B and write it as:

$$B(\mu) = \mu^3 \tilde{B}_3 + \mu^2 B_2 + \mu(\tilde{B}_1 + B_1) + \tilde{B}_0 + B_0$$

where only $(B_j)_{j \in \{0,1,2\}}$ (respectively $(\tilde{B}_j)_{j \in \{0,1,3\}}$) depends (respectively does not depend) on $(\Gamma_j)_{j \geq 1}$.

$$B(\mu) = (\Gamma_\mu T + \Gamma T_\mu) T^{-1} \Gamma^{-1} = \Gamma_\mu \Gamma^{-1} + \Gamma T_\mu T^{-1} \Gamma^{-1}$$

$\Gamma_\mu \Gamma^{-1} = \mathcal{O}(\mu^{-2})$ as $\mu \rightarrow \infty$, then

$$B(\mu) \sim \left(I_{N+1} + \frac{\Gamma_1}{\mu} + \dots \right) T_\mu T^{-1} \left(I_{N+1} - \frac{\Gamma_1}{\mu} + \dots \right)$$

$B(\mu) = T_\mu T^{-1} + \sum_{j=0}^2 \mu^j B_j$ because of Liouville's theorem. We compute $T_\mu T^{-1}$ and obtain:

$$\tilde{B}_3 = -A_1, \quad \tilde{B}_1 = \tau A_1$$

$$\tilde{B}_0 = s A_1 + \frac{1}{N+1} \text{Diag} \left(-\sum_{j=1}^N a_j, N a_1 - \sum_{j \neq 1} a_j, \dots, N a_N - \sum_{j \neq N} a_j \right)$$

Similarly $C(\mu) = T_\tau T^{-1} + \sum_{j=0}^1 \mu^j C_j$ and $T_\tau T^{-1} = \frac{\mu^2}{2} A_1$.

□

Remark 8. For δ , p^T and q as in (11) the following holds: $\partial_s \delta = -p^T q$.

Actually, computing the term in $1/\mu$ in the asymptotic of A , we obtain $\partial_s \Gamma_1 + [\Gamma_2, A_1] - [\Gamma_1, A_1] \Gamma_1$. Then, because of Liouville's theorem this term is 0 and the block 1×1 on the diagonal block matrix leads to the equation. This equation will be useful later when we will compute the logarithmic derivative for $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$.

Proposition 9. Let p^T and q be as in (11). Then they satisfy the following coupled vector 3^d

order differential equation and non linear coupled heat equation:

$$\begin{cases} \partial_{sss}p^T + 3(\partial_s p^T)qp^T + 3p^T q(\partial_s p^T) - \tau \partial_s p^T + p^T D_{s, \vec{\alpha}} = 0 \\ \partial_{sss}q + 3(\partial_s q)p^T q + 3qp^T(\partial_s q) - \tau \partial_s q - D_{s, \vec{\alpha}} q = 0 \end{cases} \quad (16)$$

$$\begin{cases} -\frac{1}{2}\partial_{ss}p^T - \partial_\tau p^T = p^T qp^T \\ -\frac{1}{2}\partial_{ss}q + \partial_\tau q = qp^T q \end{cases} \quad (17)$$

Proof. The compatibility condition for the Lax pair of Ψ leads to the equation

$$\partial_s B - \partial_\mu A = [A, B] \quad (18)$$

We use the same approach as in [17] and [12]. If we write B_j^{ki} blocs of matrix B_j where B_j^{11} is a scalar, B_j^{12} is a row of size N , B_j^{21} a column of size N and B_j^{22} a $N \times N$ matrix. Then (18) gives a polynomial equation in μ and we obtain an equation for every monomial. This leads to the following equations:

$$B_2^{12} = -p^T, \quad B_2^{21} = q \quad (19)$$

$$\begin{cases} \partial_s B_j^{11} = p^T B_j^{21} + B_j^{12} q \\ \partial_s B_j^{12} = -B_{j-1}^{12} \delta_{j \neq 0} + p^T B_j^{22} - B_j^{11} p^T + \tau p^T \delta_{j,1} + p^T D_{s, \vec{\alpha}} \delta_{j,0} \\ \partial_s B_j^{21} = B_{j-1}^{12} \delta_{j \neq 0} - q B_j^{11} + B_j^{22} q + \tau q \delta_{j,1} + D_{s, \vec{\alpha}} q \delta_{j,0} \\ \partial_s B_j^{22} = -q B_j^{12} - B_j^{21} p^T \end{cases} \quad (20)$$

where $D_{s, \vec{\alpha}} = \text{Diag}(a_1 + s, \dots, a_N + s)$ and $\delta_{i,j}$ is the Kronecker delta.

We define formally the operator ∂_s^{-1} such that $\partial_s^{-1} \partial_s = 1$. From the first and the last equation of (20) we obtain $B_j^{11} = \partial_s^{-1} (p^T B_j^{21} + B_j^{12} q)$ and $B_j^{22} = -\partial_s^{-1} (q B_j^{12} + B_j^{21} p^T)$. With $j = 2$,

$$B_2^{11} = \partial_s^{-1}(0) = c_2^{11}, \quad B_2^{22} = \partial_s^{-1}((0)) = c_2^{22}$$

with c_2^{11} and c_2^{22} independent of s . Actually with the asymptotics we obtain: $B_2 = -A_0$. Then $c_0^{11} = 0$ and $c_0^{22} = (0)$. Same for B_1 , even if it depends on Γ_2 , diagonal terms only depend on Γ_1 . The asymptotic leads to

$$B_1^{11} = p^T q, \quad B_1^{22} = -qp^T$$

Using second and third equations of (20) we compute B_1^{12} and B_1^{21} with $j = 2$ then B_0^{12} and B_0^{21} with $j = 1$.

$$\begin{aligned} B_1^{12} &= \partial_s p^T, \quad B_1^{21} = \partial_s q \\ B_0^{12} &= -\partial_{ss} p^T - 2p^T qp^T + \tau p^T, \quad B_0^{21} = \partial_{ss} q + 2qp^T q - \tau q \end{aligned}$$

Using first and last equations of (20) with $j = 0$ and integrating, we compute B_0^{11} and B_0^{22} .

$$B_0^{11} = p^T (\partial_s q) - (\partial_s p^T) q$$

$$B_0^{22} = q (\partial_s p^T) - (\partial_s q) p^T$$

Finally, it remains two equations (second and third of (20) with $j = 0$). Replacing B_0^{ki} in these

equations we obtain a system of equations satisfied by p^T and q :

$$\begin{cases} \partial_{sss}p^T + 3(\partial_s p^T)qp^T + 3p^T q(\partial_s p^T) - \tau \partial_s p^T + p^T D_{s, \vec{\alpha}} = 0 \\ \partial_{sss}q + 3(\partial_s q)p^T q + 3qp^T(\partial_s q) - \tau \partial_s q - D_{s, \vec{\alpha}} q = 0 \end{cases} \quad (21)$$

Studying the compatibility condition of A and C the same way as for A and B , p^T and q satisfied a coupled non-linear heat equation:

$$\begin{cases} -\frac{1}{2}\partial_{ss}p^T - \partial_\tau p^T = p^T qp^T \\ -\frac{1}{2}\partial_{ss}q + \partial_\tau q = qp^T q \end{cases} \quad (22)$$

□

In appendix A of [9], doing formal computation on an interval semi infinite, Kimura and Zahabi obtained a scalar version of the system of coupled differential equation (21).

Similar equations as in Proposition 9 appeared in the study of limiting one-point distribution of periodic TASEP [18]. The authors obtained coupled mKdV equations and coupled non linear heat equations. Combining these two equations, they proved the second log-derivative of the Fredholm determinant they studied satisfied the second Kadomtsev-Petviashvili equation. It is possible to do similar computations to obtain a PDE for the second log-derivative of the Fredholm determinant of \mathbb{K} . The next section will be partially devoted to this computation.

4 The logarithmic derivative of F and proof of Theorem 2

Finally we prove Theorem 2.

From Remark 1 and Proposition 4, the Fredholm determinant of $(1 - \mathbb{K})$ is different of 0 and $1 - \mathbb{K}$ is invertible.

Defining

$$\tilde{F} := (1 - \mathbb{K})^{-1} \tilde{f}, \quad \text{with } \tilde{f} \text{ as in equation (8)}, \quad (23)$$

if we write $\tilde{F} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_N \end{pmatrix}$ and $\tilde{g} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}$ (where \tilde{g} is defined in equation (9)), we have the

following result.

Lemma 10. *Let Δ be as in (11) and F_i 's, g_i 's as above. Then:*

$$\Delta = \int_{\Sigma \cup i\mathbb{R}} \begin{pmatrix} F_1(\mu) \\ \vdots \\ F_N(\mu) \end{pmatrix} (g_1(\mu), \dots, g_N(\mu)) d\mu$$

Proof. According to theory of Riemann-Hilbert problem, since $(1 - \mathbb{K})$ is invertible, the resolvent of \mathbb{K} and the unique solution to RHP 5 are linked and $\tilde{F} = \Gamma_+ \tilde{f}$.

With this last equality we obtain:

$$\Gamma(\xi) = I_{N+1} - \int_{\Sigma \cup i\mathbb{R}} \frac{\tilde{F}(\mu)\tilde{g}^T(\mu)}{\mu - \xi} d\mu$$

Expanding $\frac{1}{\mu - \xi}$ we express Γ_1 in function of \tilde{F} and \tilde{g} .

$$\Gamma_1 = \int_{\Sigma \cup i\mathbb{R}} \tilde{F}(\mu)\tilde{g}^T(\mu) d\mu$$

According to this previous equality and because of the decomposition by blocks of Γ_1 in (5), with

$$\tilde{F} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_N \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}, \quad (24)$$

$$\Delta = \int_{\Sigma \cup i\mathbb{R}} \begin{pmatrix} F_1(\mu) \\ \vdots \\ F_N(\mu) \end{pmatrix} (g_1(\mu), \dots, g_N(\mu)) d\mu.$$

□

Proposition 11. *Let $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ be as in Theorem 2 and δ as in (11). The following holds:*

$$\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = -\delta$$

Proof.

$$\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = \frac{\partial}{\partial s} \log(\det(1 - \mathbb{K})) = \frac{\partial}{\partial s} \text{Tr}(\log(1 - \mathbb{K})) = -\text{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K})$$

Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\Sigma \cup i\mathbb{R})$.

$$\text{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K}) = \sum_{n \in \mathbb{N}} \langle (1 - \mathbb{K})^{-1} \partial_s \mathbb{K} e_n, e_n \rangle$$

But $\partial_s \mathbb{K}$ has kernel $\chi_{i\mathbb{R}}(u) \tilde{f}^T(u) \tilde{g}(v) \chi_\Sigma(v)$. Then

$$\text{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K}) = \sum_{n \in \mathbb{N}} \langle (1 - \mathbb{K})^{-1} (\chi_{i\mathbb{R}} \tilde{f}^T), e_n \rangle \langle \chi_\Sigma \tilde{g}, e_n \rangle$$

According to (24), definitions of \tilde{f} (8), \tilde{g} (9) and \tilde{F} (23),

$$\text{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K}) = \sum_{n \in \mathbb{N}} (\langle F_1, e_n \rangle, \dots, \langle F_N, e_n \rangle) \begin{pmatrix} \langle g_1, e_n \rangle \\ \vdots \\ \langle g_N, e_n \rangle \end{pmatrix} = \text{Tr}(\Delta)$$

The last equation is a consequence of Lemma 10 and the fact that e_n is an orthonormal basis.

Finally, because of $-\delta + \text{Tr}(\Delta) = \text{Tr}(\Gamma_1) = 0$ (see Remark 6),

$$\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = -\delta$$

□

We use the previous proposition and the discussion on the Lax Pair to prove Theorem 2.

Proof of Theorem 2. Using the previous proposition we derive $\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}))$ with respect to s .

$$\frac{\partial^2}{\partial s^2} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = -\partial_s \delta = p^T(s)q(s) \quad (25)$$

because $\partial_s \delta = -p^T q$ (see Remark 8). From the Proposition 9 p^T and q satisfy equations (3) and (4).

We now prove equation (5).

Define $u(s, \tau) := \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}))$ and $v(s, \tau) := \frac{\partial^2}{\partial s^2} u(s, \tau)$. According to Proposition 11, $v(s, \tau) = p^T(s)q(s)$. Deriving v with respect to τ and using equation (22) to express $\partial_\tau p^T$ and $\partial_\tau q$, we obtain:

$$\partial_\tau v = \frac{1}{2} \partial_s (p^T \partial_s q - \partial_s p^T q) \quad (26)$$

Deriving a second time v with respect to τ (again using equation (22)) yields to

$$\partial_{\tau\tau} v = \frac{1}{2} \left(2v \partial_s v + \frac{1}{2} (p^T \partial_{sss} q + \partial_{sss} p^T q) - \frac{1}{2} \partial_s (\partial_s p^T \partial_s q) \right) \quad (27)$$

Recall $v(s, \tau) = p^T(s)q(s)$, then deriving three times with respect to s the following equation holds:

$$\partial_s (\partial_s p^T \partial_s q) = \frac{1}{3} (\partial_{sss} v - (p^T \partial_{sss} q + \partial_{sss} p^T q)) \quad (28)$$

Replacing $\partial_s (\partial_s p^T \partial_s q)$ in equation (27) and using equation (21) we obtain:

$$\partial_{\tau\tau} v = \partial_s \left(-v \partial_s v - \frac{1}{12} \partial_{sss} v + \frac{1}{3} \tau \partial_s v \right). \quad (29)$$

Replacing v by $\partial_{ss} u$ and integrating two times with respect to s we prove u satisfies equation (5). □

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References

- [1] Edouard Brézin and Shinobu Hikami. Level spacing of random matrices in an external source. *Physical Review E*, 58(6):7176–7185, Dec 1998.
- [2] Pavel M. Bleher and Arno B. J. Kuijlaars. Large n limit of Gaussian Random Matrices with External Source, Part III: Double Scaling Limit. *Communications in Mathematical Physics*, 270(2):481–517, Dec 2006.
- [3] Andrei Okounkov and Nicolai Reshetikhin. Random skew plane partitions and the pearcey process. *Communications in Mathematical Physics*, 269, 04 2005.
- [4] Craig A. Tracy and Harold Widom. The Pearcey Process. *Communications in Mathematical Physics*, 263(2):381–400, Feb 2006.
- [5] Mark Adler and Pierre Moerbeke. Pdes for the gaussian ensemble with external source and the pearcey distribution. *Communications on Pure and Applied Mathematics*, 60:1261 – 1292, 09 2007.
- [6] Marco Bertola and Mattia Cafasso. The Transition between the Gap Probabilities from the Pearcey to the Airy Process - a Riemann-Hilbert Approach. *International Mathematics Research Notices*, 2012(7):1519 – 1568, 2012.
- [7] Kurt Johansson. Discrete orthogonal polynomial ensembles and the plancherel measure. *Annals of Mathematics*, 153(1):259–296, 2001.
- [8] Christophe Charlier and Philippe Moreillon. On generating function of the Pearcey process. *arXiv:2107.01859 [math-ph]*, 2021.
- [9] Taro Kimura and Ali Zahabi. Universal cusp scaling in random partitions. *arXiv:2208.07288 [math-ph]*, 2022.
- [10] Tom Claeys and Antoine Doeraene. The Generating Function for the Airy Point Processes and a System of Coupled Painlevé II Equations. *Studies in Applied Mathematics*, 2018.
- [11] Christophe Charlier and Antoine Doeraene. The generating function for the bessel point process and a system of coupled painlevé v equations. *Random Matrices: Theory and Applications*, 08, 09 2017.
- [12] Mattia Cafasso and Sofia Tarricone. The Riemann-Hilbert approach to the generating function of the higher order Airy point processes. *arXiv:2111.09200 [math-ph]*, 2021.
- [13] M. Adler, M. Cafasso, and P. van Moerbeke. Non-linear PDEs for gap probabilities in random matrices and KP theory. *Physica D: Nonlinear Phenomena*, 241(23):2265–2284, 2012. Integrable Systems in Pure and Applied Mathematics.
- [14] A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavnov. Differential Equations for Quantum Correlation Functions. *International Journal of Modern Physics B*, 4(5):1003–1037, January 1990.

- [15] Alexander Soshnikov. Determinantal random point fields. *Russian Mathematical Surveys*, 55(5):923–975, Oct 2000.
- [16] Manuela Girotti. Gap Probabilities for the Generalized Bessel Process: A Riemann-Hilbert Approach. *Mathematical Physics, Analysis and Geometry*, 17(1–2):183–211, Jun 2014.
- [17] Oliver Warren and John Elgin. The vector nonlinear Schrödinger hierarchy. *Physica D-Nonlinear Phenomena - PHYSICA D*, 228:166–171, 04 2007.
- [18] Jinho Baik, Zhipeng Liu, and Guilherme L. F. Silva. Limiting one-point distribution of periodic TASEP. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 58(1):248 – 302, 2022.