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# Piecewise differential equations: Theory, methods and applications

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## Abstract

Across many real-world issues, crossover tendencies are seen. As these operators are built on kernels that exhibit some qualities that arise in nature, some of these could be captured using the idea of piecewise differentiation and integration. Power-law processes, fading memory processes, and processes that mimic the generalized Mittag-Leffler function are a few examples. The use of piecewise differential and integral operators, however, cannot be applied to all processes involving crossovers. For instance, when groundwater overabstraction causes it to flow from confined to unconfined, a considerable alteration eventually manifests. The idea of piecewise differential equations, which can be thought of as an extension of the piecewise function to the framework of differential equations, is introduced in this work. While we concentrated on ordinary, it is important to note that partial differential equations can also be solved using the same technique. For both integer and noninteger instances, piecewise differential equations were introduced. We explained the usage of the Laplace transform for the linear case and demonstrated how a new class of Bode diagrams could be produced. We provided some examples of numerical solutions as well as conditions for the existence and uniqueness of their solutions. We discussed a few scenarios in which we used chaos and straightforward ordinary differential equations to produce novel varieties of chaos. We think that this idea could lead to some significant conclusions in the future.

Keywords: Piecewise differential equations, Laplace transform, Bode diagram, numerical scheme, new chaotic attractors

## 1 Introduction

Chaos behaviors can be seen throughout nature, and some of them have recently been replicated using mathematical models. Existing models, in particular chaotic processes with crossover characteristics, nevertheless fail to account for some phenomena. When the idea of piecewise differentiation and

integration was first presented, this category of chaotic processes received very little attention. With this idea, a differential operator in a chaotic model is changed by replacing it with a piecewise derivative that entails defining an alternative differential operator in an interval. The use of this idea has, in fact, opened up more research directions toward a revolution in the study of chaos, and new chaotic attractors have been found. But in this instance, the only thing that changes is differentiation. Thus, crossovers are caused by the characteristics of the kernels utilized, which can be the power-law, generalized Mittag-Leffler, exponential decay function, or the Delta-Dirac function. A novel concept is required because some crossover phenomena cannot be described using piecewise differential operators. People can now envisage worlds that may exist but are not present in their daily lives thanks to the cinema industry. For instance, a horse with a human head, people with long tails, like in the movie Avatar, and several other examples. Perhaps the core sciences could benefit from using this creativity to combine two concepts into one new thought. This idea could be used to address a particular procedure in the actual world. To address a particular real-world process that could not be recreated using current theories, such a concept might be put forth. This ought to apply to the study of mathematics and its applications. For instance, one may consider a high level of crossover that could consist of describing the chaos processes at the various intervals to capture crossings that cannot be replicated using piecewise differential operators. Piecewise differential equations will be used to refer to this idea, where the process's governing function fluctuates with time. This idea can be seen as a piecewise function application, which has various uses in simulating human behavior and other real-world issues. Piecewise functions have only been utilized in differential equations to change parameters and source procedures. Although there are references to these processes in the literature, no overarching theory has yet to be created. The issue of the movement of groundwater provides a vivid illustration. This problem has been described as a piecewise differential equation system, where the first portion takes into account flow in a confined aquifer and the second part takes into account flow in an open aquifer. It is important to note that this approach differs greatly from the notion of an equation system in which all of the variables behave within the same time interval and with the same sub-governing functions. In this essay, we'll look into several practical uses for this family of differential equations.

## 2 Definition of Cauchy piecewise problem

### Case 1: Classical Cauchy piecewise problem

Let  $f_1, f_2, \dots, f_n$  be linear or nonlinear functions bounded and continuous. A piecewise ordinary differential equation is defined as

$$\frac{d}{dt}u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ f_2(t, u(t)) & \text{if } t_1 \leq t \leq t_2 \\ \vdots & \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} . \quad (1)$$

### Case 2: Cauchy piecewise problem with power-law

Let  $f_1, f_2, \dots, f_n$  be linear or nonlinear functions bounded and continuous, the function  $u(t)$  is differentiable. A piecewise problem with power-law[1] is represented by

$${}_0^C D_t^\alpha u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ f_2(t, u(t)) & \text{if } t_1 \leq t \leq t_2 \\ \vdots & \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} . \quad (2)$$

**Case 3: Cauchy piecewise problem with Mittag-Leffler**

Let  $f_1, f_2, \dots, f_n$  be linear or nonlinear functions bounded and continuous, the function  $u(t)$  is differentiable. A piecewise problem with Mittag-Leffler[2] is defined by

$${}_0^{ABC} D_t^\alpha u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ f_2(t, u(t)) & \text{if } t_1 \leq t \leq t_2 \\ \vdots & \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} . \quad (3)$$

**Case 4: Cauchy piecewise problem with exponential decay**

Let  $f_1, f_2, \dots, f_n$  be linear or nonlinear functions bounded and continuous, the function  $u(t)$  is differentiable. A piecewise problem with exponential decay[3] can be written as

$${}_0^{CF} D_t^\alpha u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ f_2(t, u(t)) & \text{if } t_1 \leq t \leq t_2 \\ \vdots & \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} . \quad (4)$$

**Remark.** As more general, the following piecewise differential system[4] can be written

$$\begin{cases} \frac{d}{dt} u(t) = f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ {}_0^{CF} D_t^\alpha u(t) = f_2(t, u(t)) & \text{if } t_1 \leq t \leq t_2 \\ {}_0^C D_t^\alpha u(t) = f_3(t, u(t)) & \text{if } t_2 \leq t \leq t_3 \\ {}_0^{ABC} D_t^\alpha u(t) = f_4(t, u(t)) & \text{if } t_3 \leq t \leq t_4 \\ \vdots & \\ Du(t) = f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} . \quad (5)$$

### 3 Laplace transform with examples

When the functions  $f_i$  are linear, the solution can be derived using the Laplace transform. We give an example of course using the same technique for piecewise function. We consider example one of course the other follow.

$$\begin{aligned} \frac{d}{dt} u(t) &= (u(t) - u(t - t_1)) f_1(t, u(t)) \\ &\quad + (u(t - t_2) - u(t - t_3)) f_1(t, u(t)) \\ &\quad + \dots + u(t - t_n) f_n(t, u(t)) . \end{aligned} \quad (6)$$

We can apply the Laplace transform on both sides to obtain

$$s\tilde{u}(s) - u(0) = \sum_{j=1}^n \left[ \begin{array}{c} e^{-st_j} L(f_j(t+t_j, u(t+t_j))) \\ -e^{-st_{j-1}} L(f_{j-1}(t+t_{j-1}, u(t+t_{j-1}))) \end{array} \right]. \quad (7)$$

Our next challenge is to find  $L(f_j(t+t_j, u(t+t_j)))$ .

In the case of power-law, we have

$$s^\alpha \tilde{u}(s) - s^{\alpha-1} u(0) = \sum_{j=1}^n \left[ \begin{array}{c} e^{-st_j} L(f_j(t+t_j, u(t+t_j))) \\ -e^{-st_{j-1}} L(f_{j-1}(t+t_{j-1}, u(t+t_{j-1}))) \end{array} \right]. \quad (8)$$

In the case of Mittag-Leffler, we have

$$\frac{1}{1-\alpha} \frac{s^\alpha \tilde{u}(s) - s^{\alpha-1} u(0)}{s^\alpha + \frac{\alpha}{1-\alpha}} = \sum_{j=1}^n \left[ \begin{array}{c} e^{-st_j} L(f_j(t+t_j, u(t+t_j))) \\ -e^{-st_{j-1}} L(f_{j-1}(t+t_{j-1}, u(t+t_{j-1}))) \end{array} \right]. \quad (9)$$

In the case of exponential decay, we have

$$\frac{1}{1-\alpha} \frac{s^\alpha \tilde{u}(s) - u(0)}{s + \frac{\alpha}{1-\alpha}} = \sum_{j=1}^n \left[ \begin{array}{c} e^{-st_j} L(f_j(t+t_j, u(t+t_j))) \\ -e^{-st_{j-1}} L(f_{j-1}(t+t_{j-1}, u(t+t_{j-1}))) \end{array} \right]. \quad (10)$$

We present some simple examples

$$\frac{dy(t)}{dt} = \begin{cases} t+1 & \text{if } t \in [0, T] \\ e^t + t & \text{if } t \in [T, T_1] \\ t^2 & \text{if } t > T_1 \end{cases}. \quad (11)$$

Thus

$$\begin{aligned} \frac{dy(t)}{dt} &= (t+1)[u(t) - u(t-T)] + (e^t + t)[u(t-T) - u(t-T_1)] \\ &\quad + t^2 u(t-T_1). \end{aligned} \quad (12)$$

Applying the Laplace transform yields

$$\begin{aligned} s\tilde{u}(s) - u(0) &= L(t+1) - e^{-sT} L(t+T+1) + e^{-sT} L(e^{t+T} + t+T) \\ &\quad - e^{-sT_1} L(e^{t+T_1} + t+T_1) + e^{-sT_1} L((t+T_1)^2) \\ &= \frac{1}{s} + \frac{1}{s^2} - e^{-sT} \left( \frac{1}{s^2} + \frac{T+1}{s} \right) + e^{-sT} \left( \frac{e^T}{1+s} + \frac{1}{s^2} + \frac{T}{s} \right) \\ &\quad - e^{-sT_1} \left( \frac{e^{T_1}}{1+s} + \frac{1}{s^2} + \frac{T_1}{s} \right) + e^{-sT_1} \left( \frac{T_1^2}{s} + \frac{2T_1}{s^2} + \frac{2}{s^3} \right). \end{aligned} \quad (13)$$

Therefore

$$\begin{aligned} \tilde{u}(s) &= \frac{u(0)}{s} + \frac{1}{s^2} + \frac{1}{s^3} - e^{-sT} \left( \frac{1}{s^3} + \frac{T+1}{s^2} \right) + e^{-sT} \left( \frac{e^T}{s^2+s} + \frac{1}{s^3} + \frac{T}{s^2} \right) \\ &\quad - e^{-sT_1} \left( \frac{e^{T_1}}{s^2+s} + \frac{1}{s^3} + \frac{T_1}{s^2} \right) + e^{-sT_1} \left( \frac{T_1^2}{s^2} + \frac{2T_1}{s^3} + \frac{2}{s^4} \right). \end{aligned} \quad (14)$$

In the case of power-law, we have

$$\begin{aligned} \tilde{u}(s) = & \frac{u(0)}{s^{\alpha-1}} + \frac{1}{s^\alpha} + \frac{1}{s^{\alpha+1}} - e^{-sT} \left( \frac{1}{s^{\alpha+1}} + \frac{T+1}{s^\alpha} \right) + e^{-sT} \left( \frac{e^T}{s^\alpha + s^{\alpha-1}} + \frac{1}{s^{\alpha+1}} + \frac{T}{s^\alpha} \right) \\ & - e^{-sT_1} \left( \frac{e^{T_1}}{s^\alpha + s^{\alpha-1}} + \frac{1}{s^{\alpha+1}} + \frac{T_1}{s^\alpha} \right) + e^{-sT_1} \left( \frac{T_1^2}{s^\alpha} + \frac{2T_1}{s^{\alpha+1}} + \frac{2}{s^{\alpha+2}} \right). \end{aligned} \quad (15)$$

With this example, we can now introduce the Bode diagram and phase diagram associated to this piecewise Laplace. In the case of classical derivative, we consider  $T = 3$ ,  $T_1 = 5$ . The Bode diagram and phase are presented in figure below.

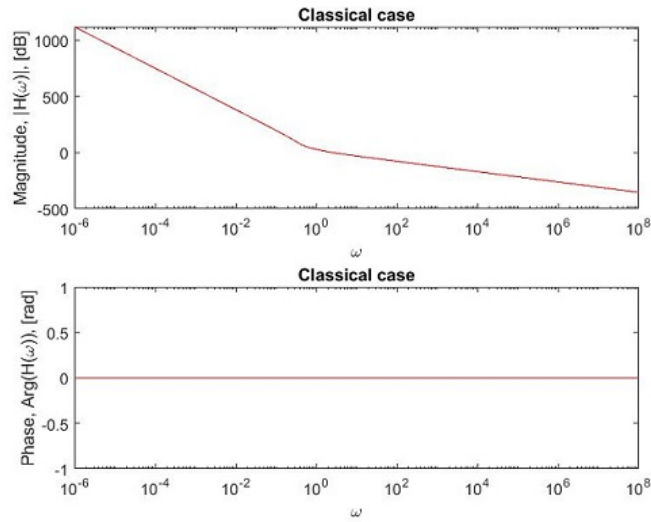


Figure 1. Bode and phase diagram for transfer function in equation (14).

In the case of power-law, we present the following figures for  $\alpha = 0.9, \alpha = 0.7, \alpha = 0.5, \alpha = 0.3$  and  $\alpha = 0.1$ .

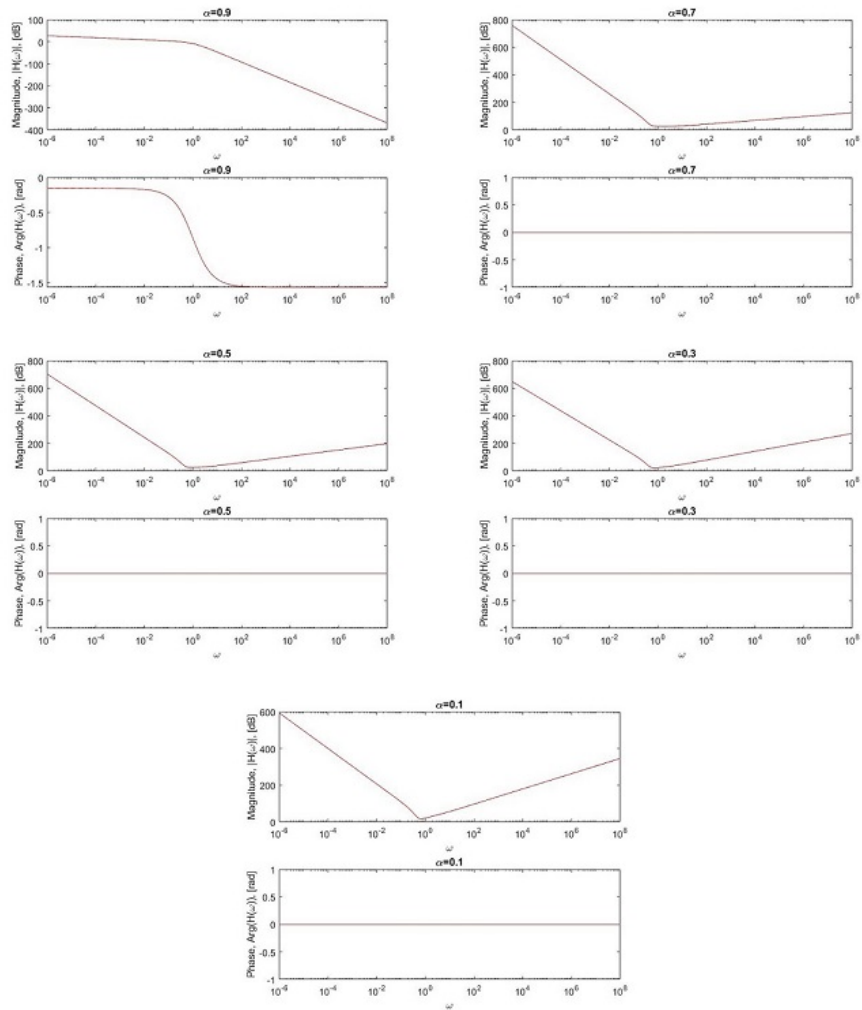


Figure 2. Bode and phase diagram for transfer function in equation (15).

## 4 Some illustrative examples of piecewise differential equations

The translation of observable facts into mathematical models is one of the major limits of mathematical models. A good mathematical model should be thorough and contain practically all of the information found in the actual scenario. Some of these features were frequently left out of mathematical formulations by most mathematicians, and as a result, the conclusions of these models don't always match the data that has been seen, leading non-mathematicians to question the usefulness

of mathematics. There is a need for a new class of ordinary and partial differential equations since differentiation, whether classical or piecewise, cannot capture these subtleties in its mathematical description. We give a few piecewise ordinary and partial differential equations in this section together with their solutions and graphical depictions.

$$\frac{dy}{dt} = \begin{cases} \begin{cases} 2 \cos t - 9y \cos t, & \text{if } t \in [0, 3], \\ y(0) = 0, \end{cases} \\ \begin{cases} \frac{1}{(2t^2+7)^2} - \frac{8ty}{2t^2+7}, & \text{if } t \in [3, 30] \end{cases} \end{cases}, \quad (16)$$

Using green function technique yields

$$y(t) = \begin{cases} \begin{cases} \frac{2}{9} + c \exp(-9 \sin t), & \text{if } t \in [0, 3], \\ y(0) = 0, \end{cases} \\ \begin{cases} \frac{c+t}{2t^2+7}, & \text{if } t \in [3, 30], \\ y(3) = \frac{2}{9} - \frac{2}{9} \exp(-9 \sin 3) \end{cases} \end{cases}. \quad (17)$$

Applying initial conditions, we obtain

$$y(t) = \begin{cases} \frac{2}{9} - \frac{2}{9} \exp(-9 \sin t), & \text{if } t \in [0, 3], \\ \frac{1}{2t^2+7} \left( \frac{23}{9} - \frac{50}{9} \exp(-9 \sin 3) + t \right), & \text{if } t \in [3, 5], \end{cases}. \quad (18)$$

In Figure 3, we present the solution of above Cauchy problem.

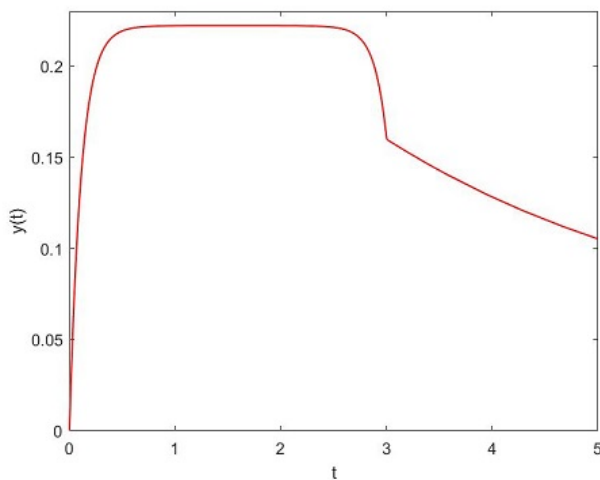


Figure 3. The solution of piecewise Cauchy problem.

We present an exmple with power-law

$${}^C_0 D_t^\alpha y(t) = \begin{cases} -\lambda y(t), & \text{if } t \in [0, 1], \\ \exp(-\gamma t), & \text{if } t \in [1, \infty), \end{cases}. \quad (19)$$

The solution is given as

$$y(t) = \begin{cases} y(0) E_\alpha(-\lambda t^\alpha), & \text{if } t \in [0, 1], \\ \frac{1}{\Gamma(\alpha)} \int_1^\infty e^{-\gamma\tau} (t-\tau)^{\alpha-1} d\tau, & \text{if } t \in [1, \infty), \end{cases} \quad (20)$$

and

$$y(t) = \begin{cases} y(0) E_\alpha(-\lambda t^\alpha), & \text{if } t \in [0, 1], \\ \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \frac{(-\gamma)^j}{j!} \begin{bmatrix} t^{j+\alpha} B(j+1, \alpha) \\ -t^{j+\alpha} B(\frac{1}{t}, j+1, \alpha) \end{bmatrix}, & \text{if } t \in [1, \infty), \end{cases} \quad (21)$$

For  $\gamma = 0$ , we can consider the following problem

$$y(t) = \begin{cases} y(0) E_\alpha(-\lambda t^\alpha), & \text{if } t \in [0, 1], \\ y(0) = 11, \\ \frac{t^\alpha}{\Gamma(\alpha+1)}, & \text{if } t \in [1, 10], \\ y(1) = 3.1 \end{cases} \quad (22)$$

In Figure 4, we present the numerical simulation for the solution of piecewise fractional differential equation.

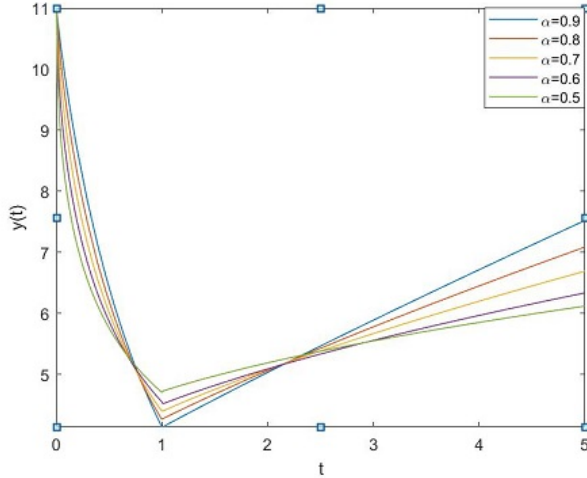


Figure 4. The solution of piecewise Cauchy problem with fractional order for  $\alpha = 0.9$ .

The findings shown in Figure 4 are quite fascinating and instructive since they show two different crossover tendencies. The idea of a piecewise differential equation, whereby Mittag-Leffler decay is observed in the first portion and power-law growth is observed in the second, induces the first crossover behaviours. In fact, this kind of crossover occurs frequently in real-world issues, even in ordinary human existence. The crossovers caused by the kernels are the second crossovers. After a crossover from stretched exponential to short memory power law decay in the first section, there

are two crossings from short memory power law in the second portion. We now consider a piecewise partial differential equation where the first part is heat equation and second part is the advection-diffusion equation.

$$\frac{\partial}{\partial t} u(x, t) = \begin{cases} \begin{cases} k \frac{\partial^2}{\partial x^2} u(x, t), & \text{if } t \in [0, 5], \\ u(x, 0) = 66 \sin \pi x, u(0, t) = u(1, t) = 0 \end{cases} \\ \begin{cases} \frac{\partial^2}{\partial x^2} u(x, t) - a \frac{\partial}{\partial x} u(x, t), & \text{if } t \in [3, 10] \end{cases} \end{cases} \quad (23)$$

where  $k = 1$  and  $a = 1$ . The solution is obtained as

$$u(x, t) = \begin{cases} \begin{cases} 66 \sin \pi x \exp(-\pi^2 t), & \text{if } t \in [0, 4], \\ u(x, 0) = 66 \sin \pi x, u(0, t) = u(2, t) = 0 \end{cases} \\ \begin{cases} 1/33 \exp(2t) (\exp(x) + \exp(-2x)), & \text{if } t \in [4, 10] \\ u(x, 4) = 1/33 e^8 (\exp(x) + \exp(-2x)), \\ u(2, t) = 1/33 \exp(2t) (e^2 + e^{-4}), \\ u(4, t) = 1/33 \exp(2t) (e^4 + e^{-8}) \end{cases} \end{cases} \quad (24)$$

In Figure 5, the numerical simulation is performed for the solution of piecewise partial differential equation.

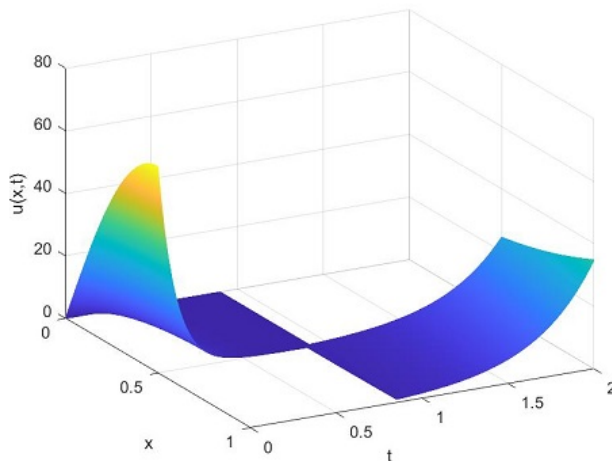


Figure 5. The solution of piecewise partial differential equation.

#### 4.1 Piecewise system: Existence, uniqueness and numerical solution

If  $f_i$  are nonlinear, the analytical methods may not be appropriate for solving these problems. Then, numerical methods are suitable candidates to solve these problems. But some conditions should be satisfied by these functions for existence and uniqueness to be searched. It is required that,  $\forall i \geq 1$ ,  $(f_i)_i$  satisfies the following conditions

1.  $|f_i(t, u(t))|^2 < k_i(1 + |u|^2)$  which is known as growth condition.
2.  $|f_i(t, u(t)) - f_i(t, v(t))|^2 < \tilde{k}_i |u - v|^2$  which is known as Lipschitz condition.

We can now proceed with numerical solution. We first consider piecewise Cauchy problem with classical case

$$\frac{d}{dt}u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ \vdots & \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} \quad (25)$$

Applying the classical integral on both sides leads to

$$u(t) = \begin{cases} u(0) + \int_0^t f_1(\tau, u(\tau)) d\tau, & \text{if } 0 \leq t \leq t_1 \\ \vdots & \\ u(t_{n-1}) + \int_{t_{n-1}}^t f_n(\tau, u(\tau)) d\tau & \text{if } t_{n-1} \leq t \leq t_n \end{cases} \quad (26)$$

Such problem can be solved by Adams-Bashforth method.

$$u(t_{k+1}) = \begin{cases} u(t_{k+1}) + \frac{h}{2} [3f_1(t_k, u(t_k)) - f_1(t_{k-1}, u(t_{k-1}))] \\ \vdots \\ u(t_{k+1}) + \frac{h}{2} [3f_n(t_k, u(t_k)) - f_n(t_{k-1}, u(t_{k-1}))] \end{cases}$$

We consider the case with ABC derivative

$${}_0^{ABC}D_t^\alpha u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ \vdots & \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} \quad (27)$$

We apply the AB integral on both sides to have

$$u(t) = \begin{cases} u(0) + (1 - \alpha) f_1(t, u(t)) + \frac{\alpha}{\Gamma(\alpha)} \int_0^t f_1(\tau, u(\tau)) (t - \tau)^{\alpha-1} d\tau, \\ \text{if } 0 \leq t \leq t_1 \\ \vdots \\ u(t_{n-1}) + (1 - \alpha) f_n(t, u(t)) + \frac{\alpha}{\Gamma(\alpha)} \int_{t_{n-1}}^t f_n(\tau, u(\tau)) (t - \tau)^{\alpha-1} d\tau \\ \text{if } t_{n-1} \leq t \leq t_n \end{cases} \quad (28)$$

At this stage, several methods can be derived. For example, one can approximate these functions using polynomials. The following numerical scheme based on Newton polynomial[5] can be constructed for the piecewise system with ABC derivative

$$u(t_{k+1}) = u(t_{k+1}) = \left\{ \begin{array}{l} \left\{ \begin{array}{l} u(0) + (1 - \alpha) f_1(t_k, u(t_k)) \\ \frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+1)} \sum_{m_1=2}^{l_1} f_1(t_{m_1-2}, u^{m_1-2}) \{(l_1 - m_1 + 1)^\alpha - (l_1 - m_1)^\alpha\} \\ + \frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{m_1=2}^{l_1} [f_1(t_{m_1-1}, u^{m_1-1}) - f_1(t_{m_1-2}, u^{m_1-2})] \\ \times \left\{ \begin{array}{l} (l_1 - m_1 + 1)^\alpha (l_1 - m_1 + 3 + 2\alpha) \\ -(l_1 - m_1)^\alpha (l_1 - m_1 + 3 + 3\alpha) \end{array} \right\} \\ + \frac{\alpha(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{m_1=2}^{l_1} \left[ \begin{array}{l} f_1(t_{m_1}, u^{m_1}) - 2f_1(t_{m_1-1}, u^{m_1-1}) \\ + f_1(t_{m_1-2}, u^{m_1-2}) \end{array} \right] \\ \times \left\{ \begin{array}{l} (l_1 - m_1 + 1)^\alpha \left[ \begin{array}{l} 2(l_1 - m_1)^2 + (3\alpha + 10)(l_1 - m_1) \\ + 2\alpha^2 + 9\alpha + 12 \end{array} \right] \\ -(l_1 - m_1)^\alpha \left[ \begin{array}{l} 2(l_1 - m_1)^2 + (5\alpha + 10)(l_1 - m_1) \\ + 6\alpha^2 + 18\alpha + 12 \end{array} \right] \end{array} \right\} \\ \text{if } 0 \leq t \leq t_1 \\ \vdots \\ u(t_{k-1}) + (1 - \alpha) f_n(t_k, u(t_k)) \\ \frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+1)} \sum_{m_k=l_{(k-1)}+3}^{l_k} f_1(t_{m_k-2}, u^{m_k-2}) \{(l_k - m_k + 1)^\alpha - (l_k - m_k)^\alpha\} \\ + \frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{m_k=l_{(k-1)}+3}^{l_k} [f_1(t_{m_k-1}, u^{m_k-1}) - f_1(t_{m_k-2}, u^{m_k-2})] \\ \times \left\{ \begin{array}{l} (l_k - m_k + 1)^\alpha (l_k - m_k + 3 + 2\alpha) \\ -(l_k - m_k)^\alpha (l_k - m_k + 3 + 3\alpha) \end{array} \right\} \\ + \frac{\alpha(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{m_k=l_{(k-1)}+3}^{l_k} \left[ \begin{array}{l} f_1(t_{m_k}, u^{m_k}) - 2f_1(t_{m_k-1}, u^{m_k-1}) \\ + f_1(t_{m_k-2}, u^{m_k-2}) \end{array} \right] \\ \times \left\{ \begin{array}{l} (l_k - m_k + 1)^\alpha \left[ \begin{array}{l} 2(l_k - m_k)^2 + (3\alpha + 10)(l_k - m_k) \\ + 2\alpha^2 + 9\alpha + 12 \end{array} \right] \\ -(l_k - m_k)^\alpha \left[ \begin{array}{l} 2(l_k - m_k)^2 + (5\alpha + 10)(l_k - m_k) \\ + 6\alpha^2 + 18\alpha + 12 \end{array} \right] \end{array} \right\} \\ \text{if } t_{k-1} \leq t \leq t_k \end{array} \right\} \end{array} \right. \quad (29)$$

We now consider the following system where the derivative is the Caputo-Fabrizio fractional derivative. We now consider the following system with Caputo derivative

$${}^C_0 D_t^\alpha u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ \vdots \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} \quad (30)$$

Putting this polynomial in the original equation and doing some operations[6], the following is obtained

$$u(t_k) = \begin{cases} u(0) + \frac{\alpha \Delta t}{M(\alpha)} \left[ \begin{aligned} & \left( \frac{(l_1-1)^2 - l_1^2}{2} \right) f_1(t_0, u^0) \\ & + \sum_{m_1=1}^{l_1} f_1(t_{m_1}, u^{m_1}) \begin{cases} \frac{1}{2}, m_1 = 0 \\ \frac{(m_1-1)^2 - 2m_1^2 - (m_1+1)^2}{2}, m_1 = 1, 2, \dots, k-1 \end{cases} \end{aligned} \right], \\ \vdots \\ u(t_{k-1}) + \frac{\alpha \Delta t}{M(\alpha)} \left[ \begin{aligned} & \left( \frac{(l_k-1)^2 - l_k^2}{2} \right) f_n(t_{k-1}, u^{k-1}) \\ & + \sum_{m_k=1}^{l_k} f_n(t_{m_k}, u^{m_k}) \begin{cases} \frac{1}{2}, m_k = 0 \\ \frac{(m_k-1)^2 - 2m_k^2 - (m_k+1)^2}{2}, m_k = 1, 2, \dots, k-1 \end{cases} \end{aligned} \right], \end{cases} \quad (31)$$

where

$$\begin{aligned} \alpha_n &= \frac{(k-1)^2 - k^2}{2} \\ \beta_{k,j} &= \begin{cases} \frac{1}{2}, j = 0 \\ \frac{(j-1)^2 - 2j^2 - (j+1)^2}{2}, j = 1, 2, \dots, k-1 \end{cases} \end{aligned} \quad (32)$$

Finally, we consider the following system with Caputo derivative

$${}_0^C D_t^\alpha u(t) = \begin{cases} f_1(t, u(t)) & \text{if } 0 \leq t \leq t_1 \\ \vdots \\ f_n(t, u(t)) & \text{if } t_{n-1} \leq t \leq t_n \end{cases} \quad (33)$$

For above system, we can present numerical scheme based on Lagrange polynomial[7]

$$u(t_{k+1}) = u(t_{k+1}) = \begin{cases} \left\{ \begin{aligned} & u(0) + (1-\alpha) f_1(t_k, u(t_k)) \\ & + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{m_1=1}^{l_1} f_1(t_{m_1}, u^{m_1}) \begin{cases} (l_1 - m_1 + 1)^\alpha (l_1 - m_1 + 2 + \alpha) \\ -(l_1 - m_1)^\alpha (l_1 - m_1 + 2 + 2\alpha) \end{cases} \\ & - \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{m_1=1}^{l_1} [f_1(t_{m_1-1}, u^{m_1-1})] \begin{cases} (l_1 - m_1 + 1)^\alpha \\ -(l_1 - m_1)^\alpha (l_1 - m_1 + 1 + \alpha) \end{cases} \end{aligned} \right\} \\ \vdots \\ \left\{ \begin{aligned} & u(t_{k-1}) + (1-\alpha) f_n(t_k, u(t_k)) \\ & + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{m_k=l_{(k-1)}+2}^{l_k} f_1(t_{m_k}, u^{m_k}) \begin{cases} (l_k - m_k + 1)^\alpha (l_k - m_k + 2 + \alpha) \\ -(l_k - m_k)^\alpha (l_k - m_k + 2 + 2\alpha) \end{cases} \\ & + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{m_k=l_{(k-1)}+2}^{l_k} f_1(t_{m_k-1}, u^{m_k-1}) \begin{cases} (l_k - m_k + 1)^\alpha \\ -(l_k - m_k)^\alpha (l_k - m_k + 1 + \alpha) \end{cases} \end{aligned} \right\} \end{cases} \quad (34)$$

## 5 Applications to mathematical biology

### 5.1 Tumor growth model: Chemotherapy and radiotherapy

Every cancer kind calls for a different treatment plan, thus a precise cancer diagnosis is crucial for the right kind of treatment. Typically, systemic therapy, radiation, and/or surgery are used in treatment (chemotherapy, hormonal treatments, targeted biological therapies). The right choice of a treatment plan takes into account the patient as well as the malignancy. To obtain the anticipated therapeutic outcome, it is critical to complete the treatment procedure within the allotted period. Setting treatment objectives is a crucial first step. Typically, the main objective is to eradicate cancer or significantly extend life. A key objective is enhancing the patient's quality of life. Support for the patient's physical, emotional, and spiritual well-being as well as palliative care throughout the latter phases of cancer treatment might help achieve this. When discovered early and treated in accordance with best standards, some common cancer types, including breast cancer, cervical cancer, oral cancer, and colorectal cancer, have high cure prospects. Even when malignant cells are present in other parts of the body, several cancer types, such as testicular seminoma and other forms of leukemia and lymphoma in children, have excellent cure rates if adequate therapy is given. However, mathematical models have been used to assist in resolving these problems. Again, it remains very difficult to translate what is seen in a particular patient into a mathematical model. This may be because mathematicians do not fully comprehend biological processes or because biologists do not communicate with mathematicians using suitable mathematical words, which makes numerous mathematical models created for these circumstances less effective. We will discuss the potential use of the piecewise ordinary differential equations with a cancer model in this part.

Think about a woman who has breast cancer. Assume that following surgery, chemotherapy and radiation are also used in instances when cancer cells are more prone to spread throughout the body [8,9]. The piecewise system that models the two processes in which chemotherapy is applied in the first process; chemotherapy and radiotherapy are applied in the second process [9] is obtained as follows in order to model the change in cancer cells when these two treatment methods are used consecutively in the patient:

$$\begin{cases} \frac{dT}{dt} = aT(1 - bT) - E_cCT \\ \frac{dE_c}{dt} = d_cE_c(1 - E_c) \\ \frac{dC}{dt} = -\gamma C \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (35)$$

$$\begin{cases} \frac{dT}{dt} = aT(1 - bT) - E_1(E_rDT + E_cCT) \\ \frac{dE_r}{dt} = d_rE_r(1 - E_r) \\ \frac{dE_c}{dt} = d_cE_c(1 - E_c) \\ \frac{dE_1}{dt} = d_1E_1(1 - E_1) \\ \frac{dC}{dt} = -\gamma C \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

Here the function  $T$  is the number of tumor cells,  $a$  is the tumor growth rate, and  $b$  is the reciprocal of the tumor carrying capacity. The function  $E_c$  describes the efficacy of the chemotherapy, the function  $C$  is the concentration of the chemotherapy,  $\gamma$  is the decay rate of the concentration,  $D$  is the amount of radiation,  $E_r$  is the efficacy of the radiotherapy,  $E_1$  is the efficacy of the combined radiotherapy and chemotherapy,  $d_c$ ,  $d_r$  and  $d_1$  are the growth rate of the efficacy.

### 5.1.1 Numerical solution of tumor growth model

The resultant model cannot be solved analytically due to its nonlinearity; thus, a precise numerical method known as the predictor-corrector will be used and modified within the framework of piecewise to solve the model. Figures in this subsection will provide examples of numerical simulations.

$$\begin{cases} \frac{dT}{dt} = aT(1-bT) - E_cCT \\ \frac{dE_c}{dt} = d_cE_c(1-E_c) \\ \frac{dC}{dt} = -\gamma C \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (36)$$

$$\begin{cases} \frac{dT}{dt} = aT(1-bT) - E_1(E_rDT + E_cCT) \\ \frac{dE_r}{dt} = d_rE_r(1-E_r) \\ \frac{dE_c}{dt} = d_cE_c(1-E_c) \\ \frac{dE_1}{dt} = d_1E_1(1-E_1) \\ \frac{dC}{dt} = -\gamma C \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

We shall use some notations for simplicity

$$\begin{aligned} X &= (T_j, E_{cj}, C_j), \\ Y &= (T_j, E_{rj}, E_{cj}, E_j, C_j), \\ F_i^* &= \begin{bmatrix} aT(1-bT) - E_cCT \\ d_cE_c(1-E_c) \\ -\gamma C \end{bmatrix}, \\ G_m^* &= \begin{bmatrix} aT(1-bT) - E_1(E_rDT + E_cCT) \\ d_rE_r(1-E_r) \\ d_cE_c(1-E_c) \\ d_1E_1(1-E_1) \\ -\gamma C \end{bmatrix}, \end{aligned}$$

such that  $i = 1, \dots, 3, m = 1, \dots, 5$ . The numerical solution of above system is obtained as

$$\begin{cases} X(t_{k+1}) = X(t_k) + \frac{h}{2} [F_i^*(t_k, X_k) + F_i^*(t_{k+1}, X_{k+1}^p)], 0 \leq t \leq t_1, \\ Y(t_{k+1}) = Y(t_k) + \frac{h}{2} [G_m^*(t_k, X_k) + G_m^*(t_{k+1}, Y_{k+1}^p)], t_1 \leq t \leq T \end{cases},$$

where

$$\begin{aligned} X^*(t_{k+1}, X_{k+1}^p) &= X(t_k) + hF_i^*(t_k, X_k), \\ Y^*(t_{k+1}, Y_{k+1}^p) &= Y(t_k) + hG_m^*(t_k, Y_k). \end{aligned}$$

In Figure 6, the numerical simulation is given for the following initial data

$$x(0) = 700, y(0) = 0.1, z(0) = 0.1, u(10) = 0, v(10) = 0,$$

and parameters

$$\begin{aligned} a &= 5 \times 10^{-11}, b = 0.0002, D = 100, d_c = 0.18, \\ d_r &= 0.19, d_1 = 0.22, \gamma = 0.009, t_1 = 10, T = 30. \end{aligned}$$

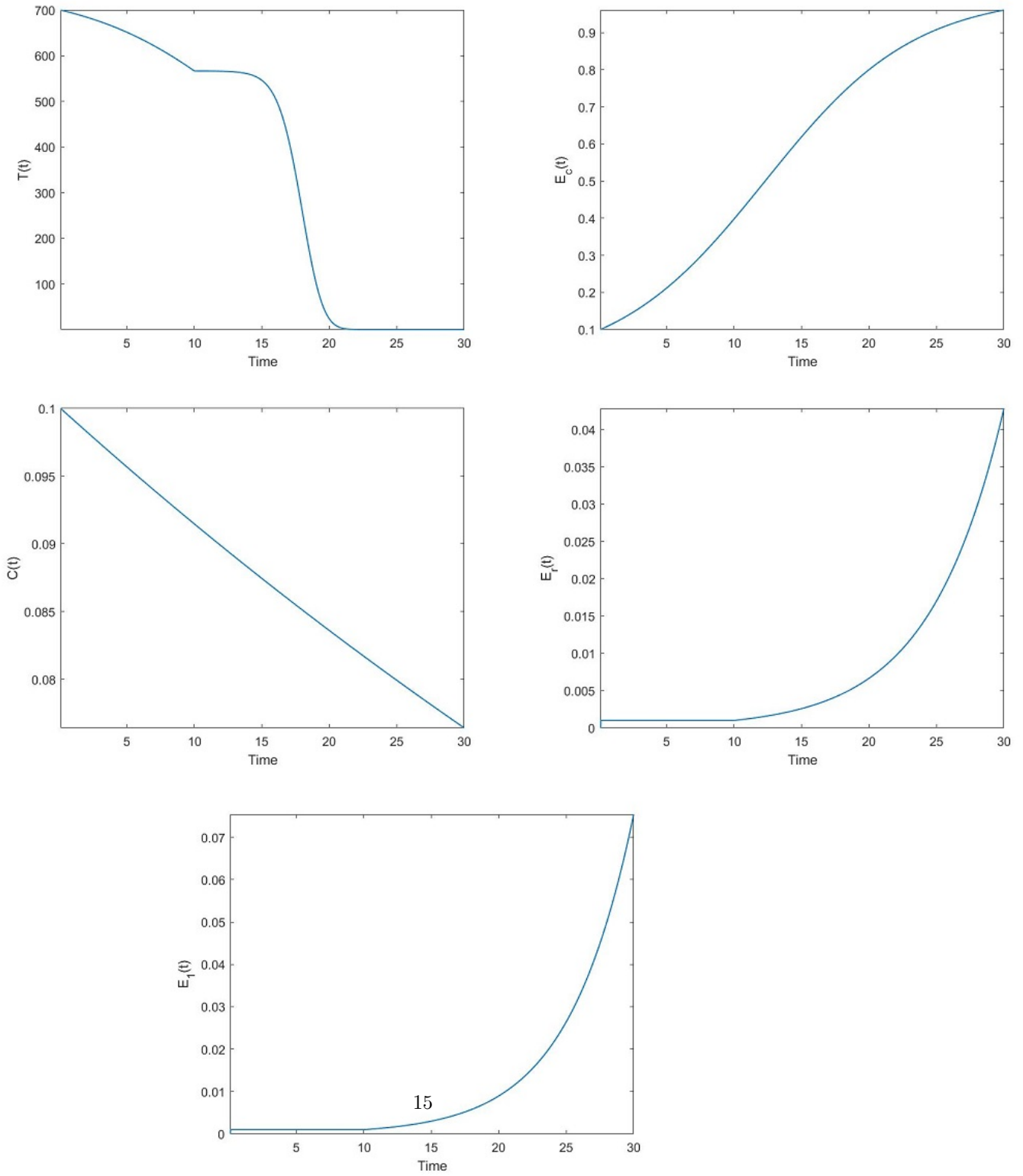


Figure 6. The numerical simulation for Arneodo-Bouali chaotic system.

## 6 Applications to chaos

Nature is filled with chaos, or exponential sensitivity to minor disturbances. Chaos is also expected to serve a variety of functional purposes in biological systems. Recently, with certain restrictions, methods for identifying chaos from empirical observations have been developed. Noting that such techniques need to be a crucial part of the biologist's repertoire. However, conventional algorithms for chaos identification are very sensitive to measurement noise and dissect typical edge situations, making it challenging to find chaos in fields like biology where data are noisy. Mathematical models are intended to simulate these observable phenomena. While some of these processes have been successfully replicated using current mathematical techniques, others still require proper modelling. The piecewise idea is used in this section to illustrate an expanded class.

### 6.1 Hybrid chaos: Arneodo and Bouali chaotic model

We consider a piecewise differential system where the first part is Arneodo chaotic model that describes the dynamics of triple convection [12], the second part is Bouali chaotic system [11].

$$\left\{ \begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = z \\ \frac{dz}{dt} = ax + by + cz - x^3. \end{array} \right. \quad \text{if } 0 \leq t \leq t_1 \quad (37)$$

$$\left\{ \begin{array}{l} \frac{dx}{dt} = a_1x(1-y) - \beta z \\ \frac{dy}{dt} = -b_1y(1-x^2) \\ \frac{dz}{dt} = \mu x \end{array} \right. \quad \text{if } t_1 \leq t \leq T.$$

where initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02 \quad (38)$$

The parameters are  $a = 0.8, b = -1.1, c = -0.45, a_1 = 0.003, \beta = 1.2, b_1 = 0.1, \mu = 0.7$ .

#### 6.1.1 Numerical solution of hybrid chaotic system

In this subsection, we consider hybrid chaotic system which is the combination of Arneodo and Bouali systems.

$$\left\{ \begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = z \\ \frac{dz}{dt} = ax + by + cz - x^3. \end{array} \right. \quad \text{if } 0 \leq t \leq t_1 \quad (39)$$

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1,$$

$$\left\{ \begin{array}{l} \frac{dx}{dt} = px(1-y) - \beta z \\ \frac{dy}{dt} = -by(1-x^2) \\ \frac{dz}{dt} = \mu x \end{array} \right. \quad \text{if } t_1 \leq t \leq T,$$

$$x(t_1) = 0.1, y(t_1) = 0.1, z(t_1) = 0.1.$$

The initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02 \quad (40)$$

The numerical simulations are depicted for the parameters  $p = 0.003, \beta = 1.2, b = 0.1, \mu = 0.7, a = 0.8, b = -1.1, c = -0.45, t_1 = 1000$ .

$$x(t_{n+1}) = \begin{cases} y^n + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} y^{k-2} \Delta t - \frac{4}{3} y^{k-1} \Delta t \\ + \frac{23}{12} y^k \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ x^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (a_1 x^{k-2} (1 - y^{k-2}) - \beta z^{k-2}) \Delta t \\ - \frac{4}{3} (a_1 x^{k-1} (1 - y^{k-1}) - \beta z^{k-1}) \Delta t \\ + \frac{23}{12} (a_1 x^k (1 - y^k) - \beta z^k) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (41)$$

$$y(t_{n+1}) = \begin{cases} z^n + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} z^{k-2} \Delta t - \frac{4}{3} z^{k-1} \Delta t \\ + \frac{23}{12} z^k \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ y^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (-b_1 y^{k-2} (1 - x^{2k-4})) \Delta t \\ - \frac{4}{3} (-b_1 y^{k-1} (1 - x^{2k-2})) \Delta t \\ + \frac{23}{12} (-b_1 y^k (1 - x^{2k})) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (42)$$

$$z(t_{n+1}) = \begin{cases} \begin{array}{l} ax^n + by^n + cz^n - x^{3n} \\ + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} (ax^{k-2} + by^{k-2} + cz^{k-2} - x^{3k-6}) \Delta t \\ - \frac{4}{3} (ax^{k-1} + by^{k-1} + cz^{k-1} - x^{3k-3}) \Delta t \\ + \frac{23}{12} (ax^k + by^k + cz^k - x^{3k}) \Delta t \end{array} \right\} \end{array}, & \text{if } 0 \leq t \leq t_1 \\ \left[ z^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} \mu x^{k-2} \Delta t - \frac{4}{3} \mu x^{k-1} \Delta t \\ + \frac{23}{12} \mu x^k \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (43)$$

In Figure 7, we depict the numerical simulations for above hybrid chaotic model.

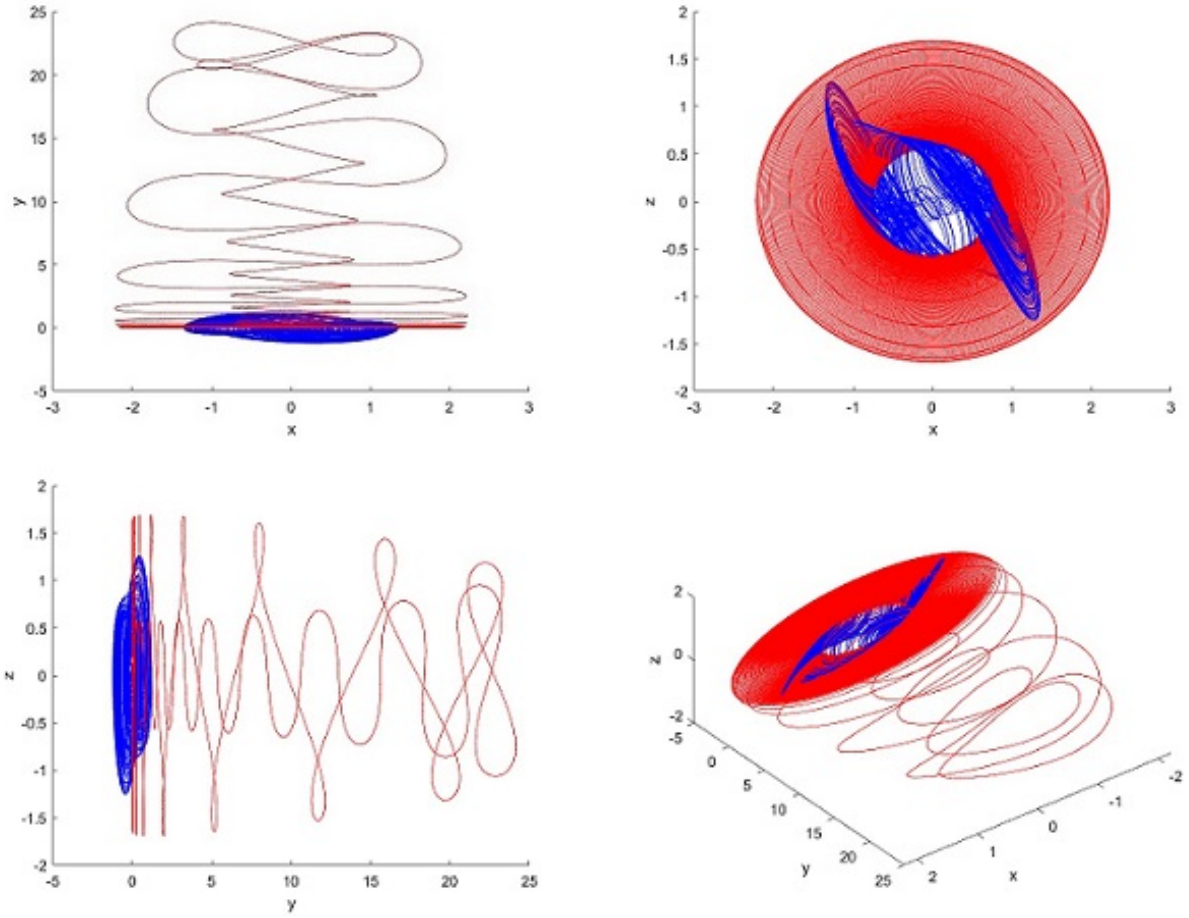


Figure 7. The numerical simulation for Arneodo-Bouali chaotic system.

Replacing the first equation by the second one, the modified piecewise system called as Bouali-Arneodo chaotic can be written as

$$\begin{cases}
\frac{dx}{dt} = px(1-y) - \beta z \\
\frac{dy}{dt} = -b_1y(1-x^2) \\
\frac{dz}{dt} = \mu x
\end{cases} \quad \text{if } 0 \leq t \leq t_1 \tag{44}$$

$$x(0) = 1, y(0) = 1, z(0) = -0.02,$$

$$\begin{cases}
\frac{dx}{dt} = y \\
\frac{dy}{dt} = z \\
\frac{dz}{dt} = ax + by + cz - x^3.
\end{cases} \quad \text{if } t_1 \leq t \leq T,$$

$$x(t_1) = 0.1, y(t_1) = 0.1, z(t_1) = 0.1.$$

In Figure 8, the numerical simulations are depicted for the parameters  $p = 3, \beta = 2.2, b = -1.1, \mu = 0.001, a = 0.3, b_1 = 0.1, c = -0.45, t_1 = 800$ .

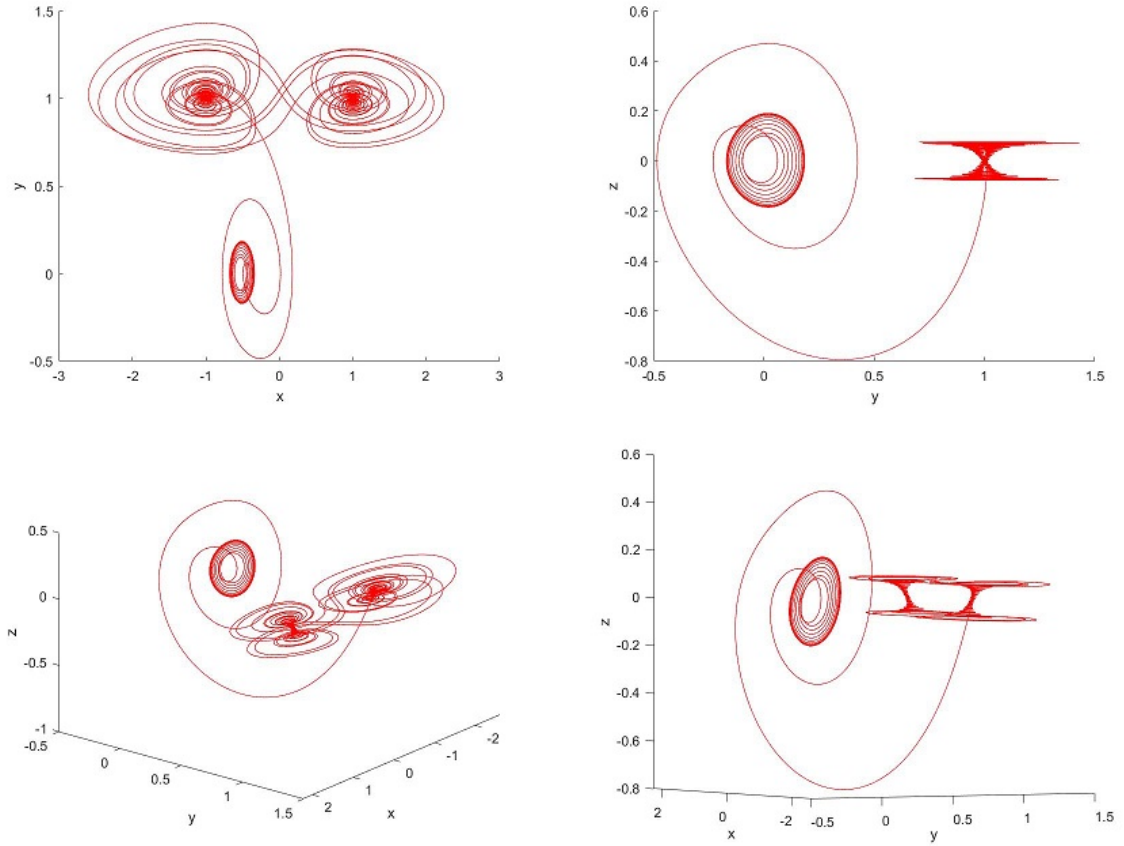


Figure 8. The numerical simulation for Bouali-Arneodo chaotic system.

## 6.2 Hybrid chaos: King-Cobra and Wang-Sun chaotic models

In this subsection, we deal with the combination of King Cobra chaotic system introduced in [13] and Wang-Sun system which exhibits 3-D four-wing smooth autonomous chaotic property [14]

$$\begin{cases} \frac{dx}{dt} = a(y-x) + byz^2 \\ \frac{dy}{dt} = cx + dxz^2 \\ \frac{dz}{dt} = ez + fx \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (45)$$

$$\begin{cases} \frac{dx}{dt} = a_1x + b_1yz \\ \frac{dy}{dt} = c_1x + d_1y - xz \\ \frac{dz}{dt} = e_1xy + f_1z \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

The parameters are  $a = 0.05, b = 1, c = 1, d = 0.5, e = -2, f = -1, a_1 = 0.2, b_1 = 1, c_1 = -7, d_1 = 0.0001, e_1 = 0.7, f_1 = 0.4$ .

### 6.2.1 Numerical solution of hybrid chaotic system

In this subsection, we consider hybrid chaotic system where King-Cobra and Wang-Sun chaotic systems are combined.

$$\begin{cases} \frac{dx}{dt} = a(y-x) + byz^2 \\ \frac{dy}{dt} = cx + dxz^2 \\ \frac{dz}{dt} = ez + fx \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (46)$$

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1,$$

$$\begin{cases} \frac{dx}{dt} = a_1x + b_1yz \\ \frac{dy}{dt} = c_1x + d_1y - xz \\ \frac{dz}{dt} = e_1xy + f_1z \end{cases} \quad \text{if } t_1 \leq t \leq T,$$

$$x(t_1) = 0.1, y(t_1) = 0.1, z(t_1) = 0.1, .$$

The numerical scheme for this model is presented as follows

$$x(t_{n+1}) = \begin{cases} + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} (a(y^{k-2} - x^{k-2}) + by^{k-2}z^{2k-4}) \Delta t \\ -\frac{4}{3} (a(y^{k-1} - x^{k-1}) + by^{k-1}z^{2k-2}) \Delta t \\ + \frac{23}{12} (a(y^k - x^k) + by^kz^{2k}) \Delta t \end{array} \right\}, \text{ if } 0 \leq t \leq t_1 \\ \left[ x^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (a_1x^{k-2} + b_1y^{k-2}z^{k-2}) \Delta t \\ -\frac{4}{3} (a_1x^{k-1} + b_1y^{k-1}z^{k-1}) \Delta t \\ + \frac{23}{12} (a_1x^k + b_1y^kz^k) \Delta t \end{array} \right\} \right], \text{ if } t_1 \leq t \leq T \end{cases} \quad (47)$$

$$y(t_{n+1}) = \begin{cases} +\frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{array}{l} (cx^n + dx^n z^{2n}) \\ \frac{5}{12} (cx^{k-2} + dx^{k-2} z^{2k-4}) \Delta t \\ -\frac{4}{3} (cx^{k-1} + dx^{k-1} z^{2k-2}) \Delta t \\ +\frac{23}{12} (cx^k + dx^k z^{2k}) \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ y^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (c_1 x^{k-2} + d_1 y^{k-2} - x^{k-2} z^{k-2}) \Delta t \\ -\frac{4}{3} (c_1 x^{k-1} + d_1 y^{k-1} - x^{k-1} z^{k-1}) \Delta t \\ +\frac{23}{12} (c_1 x^k + d_1 y^k - x^k z^k) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (48)$$

$$z(t_{n+1}) = \begin{cases} (ez^n + fx^n) + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} (ez^{k-2} + fx^{k-2}) \Delta t \\ -\frac{4}{3} (ez^{k-1} + fx^{k-1}) \Delta t \\ +\frac{23}{12} (ez^k + fx^k) \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ z^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (e_1 x^{k-2} y^{k-2} + f_1 z^{k-2}) \Delta t \\ -\frac{4}{3} (e_1 x^{k-1} y^{k-1} + f_1 z^{k-1}) \Delta t \\ +\frac{23}{12} (e_1 x^k y^k + f_1 z^k) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (49)$$

In Figure 9, we depict the numerical simulations for above hybrid chaotic model.

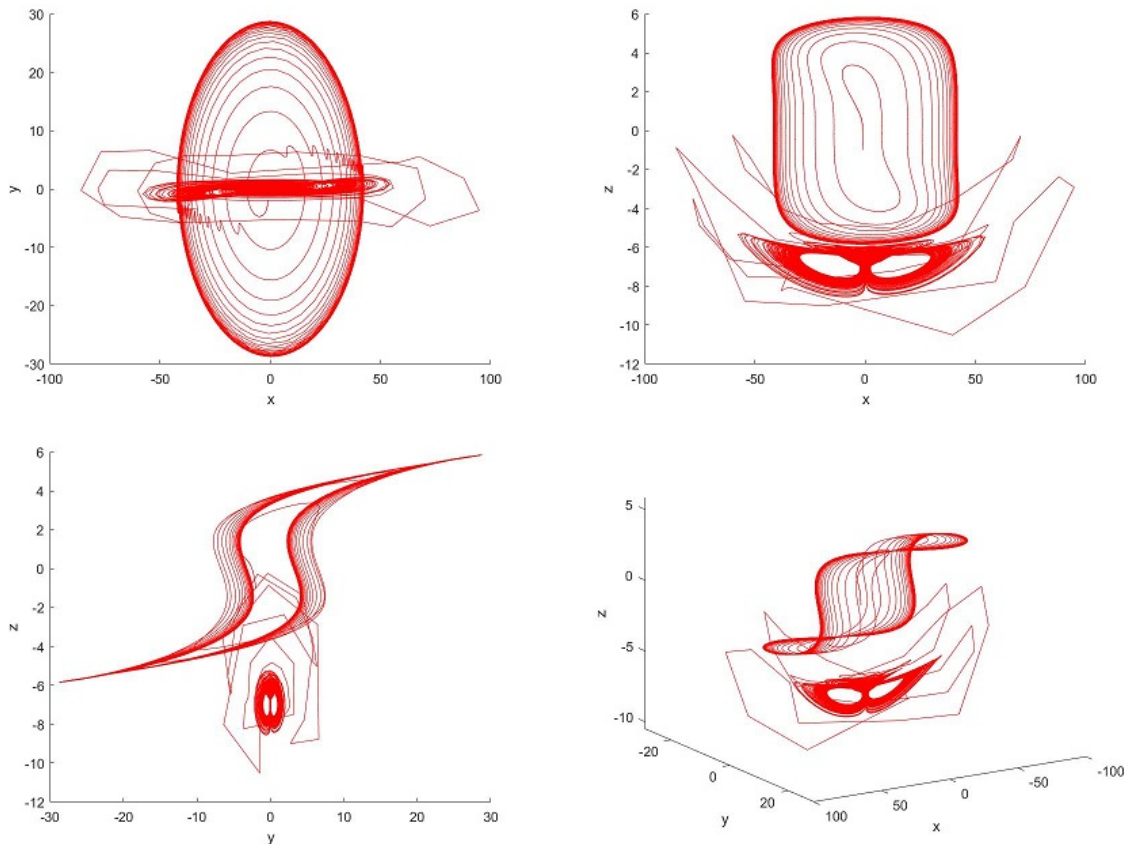


Figure 9. The numerical simulation for King-Cobra-Wang-Sun chaotic system.

### 6.3 Hybrid chaos: Chen-Lee and Labyrinth chaotic models

In this subsection, Chen-Lee system which is based on the Euler equations for the motion of a rigid body [15] and Labyrinth system that describes the movement a particle which moves in a labyrinth under the influence of some external source of energy [16] are represented by the following piecewise differential system

$$\begin{cases} \frac{dx}{dt} = ax - yz \\ \frac{dy}{dt} = by + xz \\ \frac{dz}{dt} = cz + \frac{xy}{3} \end{cases} \text{ if } 0 \leq t \leq t_1, \quad (50)$$

$$\begin{cases} \frac{dx}{dt} = -\sin y(at) - bx(t) \\ \frac{dy}{dt} = -\sin z(at) - by(t) \\ \frac{dz}{dt} = -\sin x(at) - bz(t) \end{cases} \text{ if } t_1 \leq t \leq T.$$

where initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02 \quad (51)$$

The parameters are  $a = 3, \beta = 2.2, b = 1, \mu = 0.001$ .

### 6.3.1 Numerical solution of hybrid chaotic system

In this subsection, we consider hybrid chaotic system which is the combination of Chen-Lee and Labyrinth chaotic systems.

$$\begin{cases} \frac{dx}{dt} = ax - yz \\ \frac{dy}{dt} = by + xz \\ \frac{dz}{dt} = cz + \frac{xy}{3} \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (52)$$

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1,$$

$$\begin{cases} \frac{dx}{dt} = -\sin y(a_1 t) - b_1 x(t) \\ \frac{dy}{dt} = -\sin z(a_1 t) - b_1 y(t) \\ \frac{dz}{dt} = -\sin x(a_1 t) - b_1 z(t) \end{cases} \quad \text{if } t_1 \leq t \leq T,$$

$$x(t_1) = 0.1, y(t_1) = 0.1, z(t_1) = 0.1, .$$

where initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02 \quad (53)$$

The parameters are taken as  $p = 0.003, \beta = 1.2, b = 0.1, \mu = 0.7, a = 0.8, b = -1.1, c = -0.45, t_1 = 1000$ .

$$x(t_{n+1}) = \begin{cases} + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} (ax^{k-2} - y^{k-2} z^{k-2}) \Delta t \\ -\frac{4}{3} (ax^{k-1} - y^{k-1} z^{k-1}) \Delta t \\ + \frac{23}{12} (ax^k - y^k z^k) \Delta t \end{array} \right\}, \text{if } 0 \leq t \leq t_1 \\ \left[ x^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (-\sin y(a_1 t_{k-2}) - b_1 x^{k-2}) \Delta t \\ -\frac{4}{3} (-\sin y(a_1 t_{k-1}) - b_1 x^{k-1}) \Delta t \\ + \frac{23}{12} (-\sin y(a_1 t_k) - b_1 x^k) \Delta t \end{array} \right\} \right], \text{if } t_1 \leq t \leq T \end{cases} \quad (54)$$

$$y(t_{n+1}) = \begin{cases} + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} (bx^{k-2} + x^{k-2} z^{k-2}) \Delta t \\ -\frac{4}{3} (bx^{k-1} + x^{k-1} z^{k-1}) \Delta t \\ + \frac{23}{12} (bx^k + x^k z^k) \Delta t \end{array} \right\}, \text{if } 0 \leq t \leq t_1 \\ \left[ y^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (-\sin z(a_1 t_{k-2}) - b_1 y^{k-2}) \Delta t \\ -\frac{4}{3} (-\sin z(a_1 t_{k-1}) - b_1 y^{k-1}) \Delta t \\ + \frac{23}{12} (-\sin z(a_1 t_k) - b_1 y^k) \Delta t \end{array} \right\} \right], \text{if } t_1 \leq t \leq T \end{cases} \quad (55)$$

$$z(t_{n+1}) = \begin{cases} \left( cz^n + \frac{x^n y^n}{3} \right) + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} \left( cz^{k-2} + \frac{x^{k-2} y^{k-2}}{3} \right) \Delta t \\ -\frac{4}{3} \left( cz^{k-1} + \frac{x^{k-1} y^{k-1}}{3} \right) \Delta t \\ + \frac{23}{12} \left( cz^k + \frac{x^k y^k}{3} \right) \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ z^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} \left( -\sin x (a_1 t_{k-2}) - b_1 z^{k-2} \right) \Delta t \\ -\frac{4}{3} \left( -\sin x (a_1 t_{k-1}) - b_1 z^{k-1} \right) \Delta t \\ + \frac{23}{12} \left( -\sin x (a_1 t_k) - b_1 z^k \right) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (56)$$

In Figure 10, we depict the numerical simulations for above hybrid chaotic model.

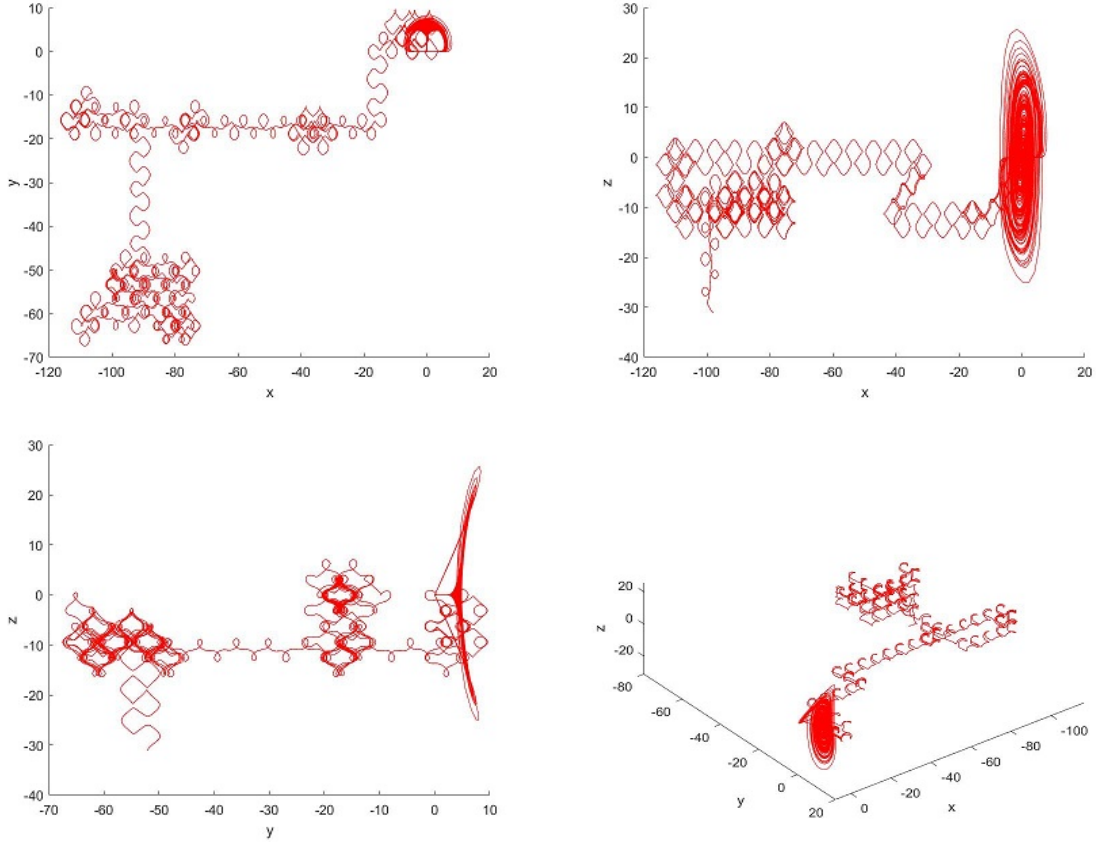


Figure 10. The numerical simulation for Chen-Lee-Labyrinth chaotic system.

#### 6.4 Hybrid chaos: Chen-Lee, Labyrinth and Dequan-Li chaotic models

In this subsection, we combine 3 well-known chaotic systems where the first part is Chen-Lee system, the second part is Labyrinth system and the last part is Dequan-Li system which is three-dimensional quadratic autonomous chaotic system displays an attractor with three scrolls [17].

$$\begin{cases} \frac{dx}{dt} = ax - yz \\ \frac{dy}{dt} = by + xz \\ \frac{dz}{dt} = cz + \frac{xy}{3} \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (57)$$

$$x(0) = 0.1, y(0) = 0.1, z(0) = 0.1,$$

$$\begin{cases} \frac{dx}{dt} = -\sin y(a_1 t) - b_1 x(t) \\ \frac{dy}{dt} = -\sin z(a_1 t) - b_1 y(t) \\ \frac{dz}{dt} = -\sin x(a_1 t) - b_1 z(t) \end{cases} \quad \text{if } t_1 \leq t \leq t_2,$$

$$x(t_1) = 0.1, y(t_1) = 0.1, z(t_1) = 0.1,$$

$$\begin{cases} \frac{dx}{dt} = a_2(y - x) + \delta xz \\ \frac{dy}{dt} = \rho x + \zeta y - xz \\ \frac{dz}{dt} = \gamma z + xy - \varepsilon x^2 \end{cases} \quad \text{if } t_2 \leq t \leq T, .$$

where initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02 \quad (58)$$

The parameters is taken as  $p = 0.003$ ,  $\beta = 1.2$ ,  $b = 0.1$ ,  $\mu = 0.7$ ,  $a = 0.8$ ,  $b = -1.1$ ,  $c = -0.45$ ,  $t_1 = 1000$ .

## 6.5 Numerical solution of hybrid chaotic system

In this subsection, we present the numerical solution of piecewise chaotic system where Chen-Lee, Labyrinth and Dequan-Li chaotic systems.

$$x(t_{n+1}) = \begin{cases} \left[ x^1 + \sum_{k=2}^n \left\{ \begin{array}{l} \frac{5}{12} (ax^{k-2} - y^{k-2}z^{k-2}) \Delta t \\ -\frac{4}{3} (ax^{k-1} - y^{k-1}z^{k-1}) \Delta t \\ +\frac{23}{12} (ax^k - y^kz^k) \Delta t \end{array} \right\}, \text{if } 0 \leq t \leq t_1 \right. \\ \left[ \begin{array}{l} \frac{5}{12} (-\sin y(a_1 t_{k-2}) - b_1 x^{k-2}) \Delta t \\ -\frac{4}{3} (-\sin y(a_1 t_{k-1}) - b_1 x^{k-1}) \Delta t \\ +\frac{23}{12} (-\sin y(a_1 t_k) - b_1 x^k) \Delta t \end{array} \right] \text{, if } t_1 \leq t \leq t_2, \\ \left[ \begin{array}{l} \frac{5}{12} (a_2 (y^{k-2} - x^{k-2}) + \delta x^{k-2} z^{k-2}) \Delta t \\ -\frac{4}{3} (a_2 (y^{k-1} - x^{k-1}) + \delta x^{k-1} z^{k-1}) \Delta t \\ +\frac{23}{12} (a_2 (y^k - x^k) + \delta x^k z^k) \Delta t \end{array} \right] \text{if } t_2 \leq t \leq T, \end{cases} \quad (59)$$

$$y(t_{n+1}) = \begin{cases} + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{array}{l} (bx^n + x^n z^n) \\ \frac{5}{12} (bx^{k-2} + x^{k-2} z^{k-2}) \Delta t \\ -\frac{4}{3} (bx^{k-1} + x^{k-1} z^{k-1}) \Delta t \\ + \frac{23}{12} (bx^k + x^k z^k) \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ y^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (-\sin z (a_1 t_{k-2}) - b_1 y^{k-2}) \Delta t \\ -\frac{4}{3} (-\sin z (a_1 t_{k-1}) - b_1 y^{k-1}) \Delta t \\ + \frac{23}{12} (-\sin z (a_1 t_k) - b_1 y^k) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq t_2, \\ \left[ y^2 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (\rho x^{k-2} + \zeta y^{k-2} - x^{k-2} z^{k-2}) \Delta t \\ -\frac{4}{3} (\rho x^{k-1} + \zeta y^{k-1} - x^{k-1} z^{k-1}) \Delta t \\ + \frac{23}{12} (\rho x^k + \zeta y^k - x^k z^k) \Delta t \end{array} \right\} \right], & \text{if } t_2 \leq t \leq T \end{cases} \quad (60)$$

$$z(t_{n+1}) = \begin{cases} \left( cz^n + \frac{x^n y^n}{3} \right) + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} \left( cz^{k-2} + \frac{x^{k-2} y^{k-2}}{3} \right) \Delta t \\ -\frac{4}{3} \left( cz^{k-1} + \frac{x^{k-1} y^{k-1}}{3} \right) \Delta t \\ + \frac{23}{12} \left( cz^k + \frac{x^k y^k}{3} \right) \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ z^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (-\sin x (a_1 t_{k-2}) - b_1 z^{k-2}) \Delta t \\ -\frac{4}{3} (-\sin x (a_1 t_{k-1}) - b_1 z^{k-1}) \Delta t \\ + \frac{23}{12} (-\sin x (a_1 t_k) - b_1 z^k) \Delta t \end{array} \right\} \right], & \text{if } t_1 \leq t \leq t_2, \\ \left[ z^2 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (\gamma z^{k-2} + x^{k-2} y^{k-2} - \varepsilon x^{2k-4}) \Delta t \\ -\frac{4}{3} (\gamma z^{k-1} + x^{k-1} y^{k-1} - \varepsilon x^{2k-2}) \Delta t \\ + \frac{23}{12} (\gamma z^k + x^k y^k - \varepsilon x^{2k}) \Delta t \end{array} \right\} \right], & \text{if } t_2 \leq t \leq T \end{cases} \quad (61)$$

In Figure 11 and 12, we depict the numerical simulations for above hybrid chaotic model.

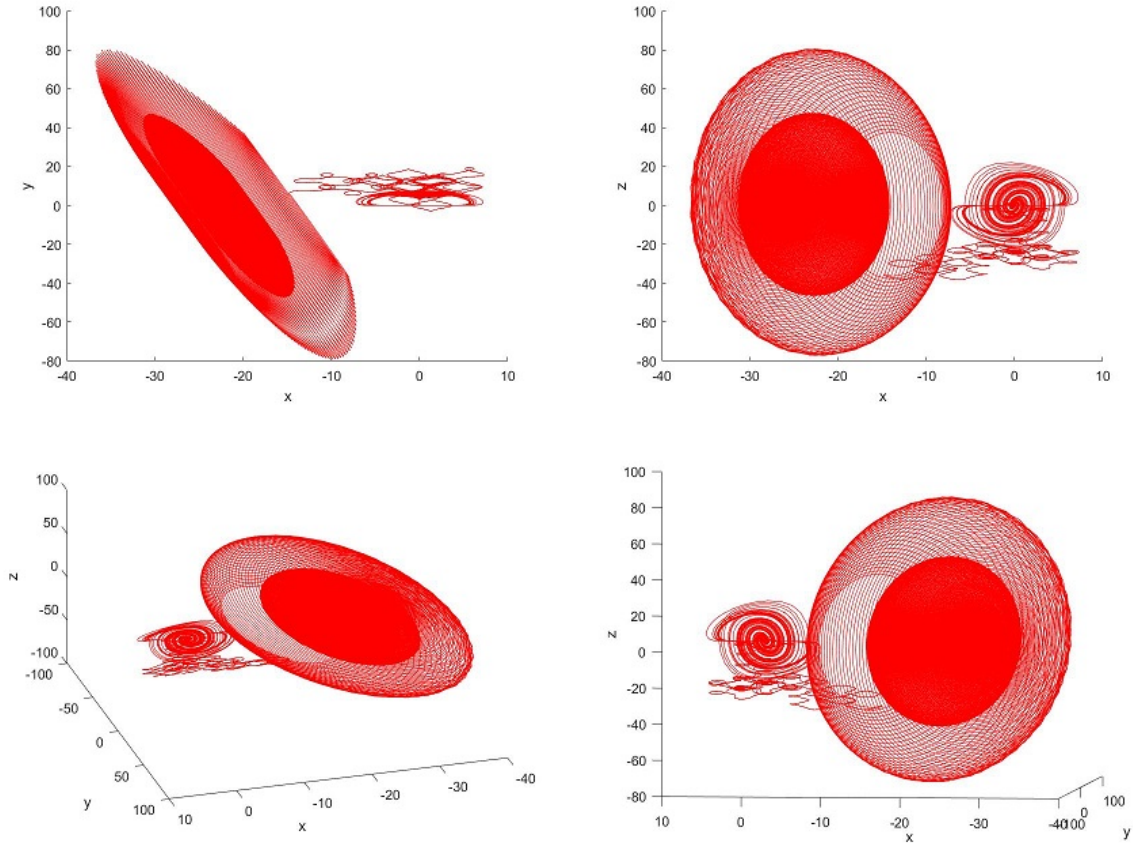


Figure 11. The numerical simulation for Chen-Lee-Labyrinth-Dequan-Li chaotic system.

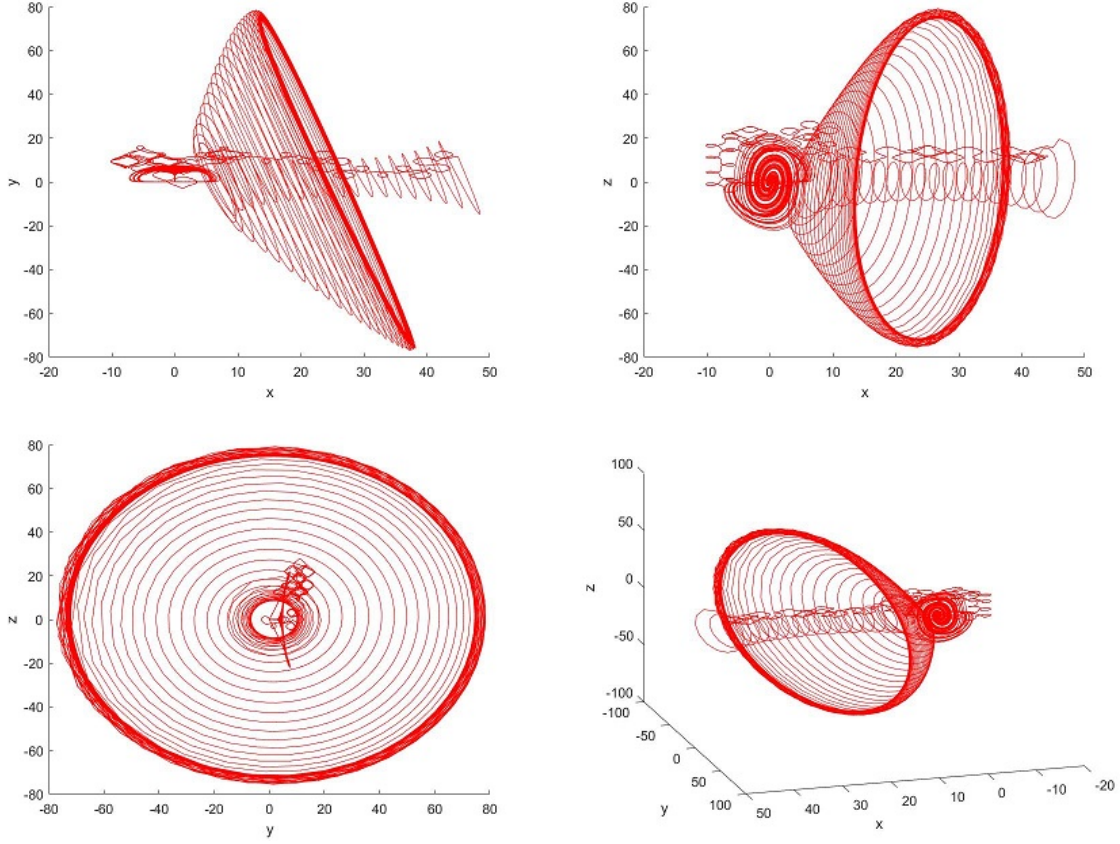


Figure 12. The numerical simulation for Chen-Lee-Labyrinth-Dequan-Li chaotic system.

## 6.6 Deterministic-Stochastic Hybrid chaos: Bouali and Arneodo chaotic models

In this subsection, we suggest the combination 2 chaotic systems where the first part is deterministic Bouali system and the second part is Arneodo chaotic system with stochastic constant.

$$\begin{cases} \frac{dx}{dt} = ax(1-y) - \beta z \\ \frac{dy}{dt} = -by(1-x^2) \\ \frac{dz}{dt} = \mu x \end{cases} \quad \text{if } 0 \leq t \leq t_1 \quad (62)$$

$$\begin{cases} dx = ydt + \sigma_1 x dB_1(t) \\ dy = zdt + \sigma_2 y dB_2(t) \\ dz = \left( \begin{array}{l} ax + by \\ +cz - x^3 \end{array} \right) dt + \sigma_3 z dB_3(t). \end{cases} \quad \text{if } t_1 \leq t \leq T.$$

### 6.6.1 Numerical solution of deterministic-stochastic hybrid chaotic system

In this subsection, we consider hybrid chaotic system which is the combination of deterministic Bouali and Arneodo systems with stochastic constant.

$$\begin{cases} \begin{cases} \frac{dx}{dt} = ax(1-y) - \beta z \\ \frac{dy}{dt} = -by(1-x^2) \\ \frac{dz}{dt} = \mu x \end{cases} & \text{if } 0 \leq t \leq t_1 \end{cases} \quad (63)$$

$$\begin{cases} \begin{cases} dx = ydt + \sigma_1 x dB_1(t) \\ dy = zdt + \sigma_2 y dB_2(t) \\ dz = \begin{pmatrix} ax + by \\ +cz - x^3 \end{pmatrix} dt + \sigma_3 z dB_3(t) \end{cases} & \text{if } t_1 \leq t \leq T. \end{cases}$$

where initial conditions are taken as

$$x(0) = 1, y(0) = 1, z(0) = -0.02 \quad (64)$$

The parameters are taken as  $a = 3, \beta = 2.2, b = 1, \mu = 0.001$ .

$$x(t_{n+1}) = \begin{cases} \left[ \begin{aligned} & + \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (ax^{k-2}(1-y^{k-2}) - \beta z^{k-2}) \Delta t \\ & - \frac{4}{3} (ax^{k-1}(1-y^{k-1}) - \beta z^{k-1}) \Delta t \\ & + \frac{23}{12} (ax^k(1-y^k) - \beta z^k) \Delta t \end{aligned} \right\} \end{aligned} \right], & \text{if } 0 \leq t \leq t_1 \\ \left[ \begin{aligned} & x^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} y^{k-2} \Delta t - \frac{4}{3} y^{k-1} \Delta t \\ & + \frac{23}{12} y^k \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_1(t_{k-1}) - B_1(t_{k-2})) \sigma_1 x^{k-2} \\ & - \frac{4}{3} (B_1(t_k) - B_1(t_{k-1})) \sigma_1 x^{k-1} \\ & + \frac{23}{12} (B_1(t_{k+1}) - B_1(t_k)) \sigma_1 x^k \end{aligned} \right\} \end{aligned} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (65)$$

$$y(t_{n+1}) = \begin{cases} \left[ \begin{aligned} & + \frac{\alpha}{M(\alpha)} \sum_{k=2}^i \left\{ \begin{aligned} & \frac{5}{12} (-by^{k-2}(1-x^{2k-4})) \Delta t \\ & - \frac{4}{3} (-by^{k-1}(1-x^{2k-2})) \Delta t \\ & + \frac{23}{12} (-by^k(1-x^{2k})) \Delta t \end{aligned} \right\} \end{aligned} \right], & \text{if } 0 \leq t \leq t_1 \\ \left[ \begin{aligned} & y^1 + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} z^{k-2} \Delta t - \frac{4}{3} z^{k-1} \Delta t \\ & + \frac{23}{12} z^k \Delta t \end{aligned} \right\} \\ & + \sum_{k=i+3}^n \left\{ \begin{aligned} & \frac{5}{12} (B_2(t_{k-1}) - B_2(t_{k-2})) \sigma_2 y^{k-2} \\ & - \frac{4}{3} (B_2(t_k) - B_2(t_{k-1})) \sigma_2 y^{k-1} \\ & + \frac{23}{12} (B_2(t_{k+1}) - B_2(t_k)) \sigma_2 y^k \end{aligned} \right\} \end{aligned} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (66)$$

$$z(t_{n+1}) = \begin{cases} \mu x^n + \sum_{k=2}^i \left\{ \begin{array}{l} \frac{5}{12} \mu x^{k-2} \Delta t - \frac{4}{3} \mu x^{k-1} \Delta t \\ + \frac{23}{12} \mu x^k \Delta t \end{array} \right\}, & \text{if } 0 \leq t \leq t_1 \\ \left[ \begin{array}{l} z^1 + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (ax^{k-2} + by^{k-2} + cz^{k-2} - x^{3k-6}) \Delta t \\ - \frac{4}{3} (ax^{k-1} + by^{k-1} + cz^{k-1} - x^{3k-3}) \Delta t \\ + \frac{23}{12} (ax^k + by^k + cz^k - x^{3k}) \Delta t \end{array} \right\} \\ + \sum_{k=i+3}^n \left\{ \begin{array}{l} \frac{5}{12} (B_3(t_{k-1}) - B_3(t_{k-2})) \sigma_3 z^{k-2} \\ - \frac{4}{3} (B_3(t_k) - B_3(t_{k-1})) \sigma_3 z^{k-1} \\ + \frac{23}{12} (B_3(t_{k+1}) - B_3(t_k)) \sigma_3 z^k \end{array} \right\} \end{array} \right], & \text{if } t_1 \leq t \leq T \end{cases} \quad (67)$$

In Figure 13, the numerical simulations are presented for above hybrid chaotic model.

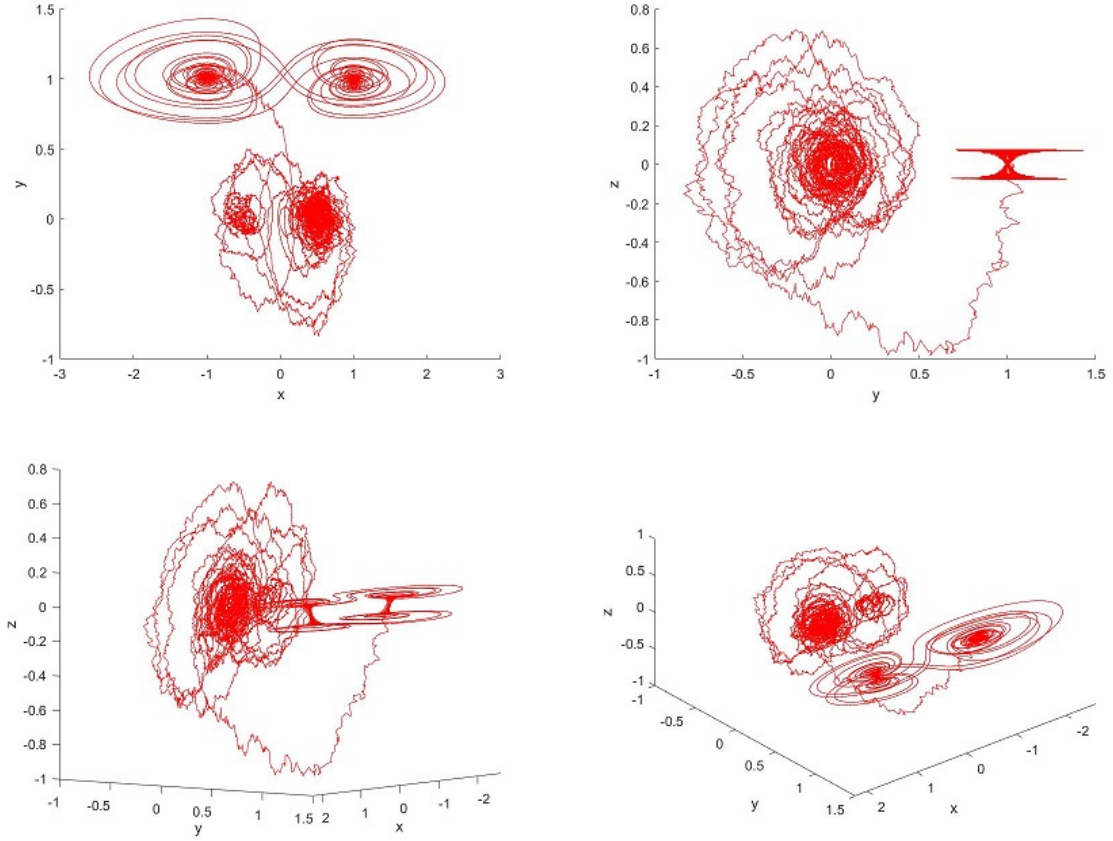


Figure 13. The numerical simulation of deterministic Bouali and stochastic Arneodo system for  $\sigma_1 = 0.0015, \sigma_2 = 0.002, \sigma_3 = 0.0019$ .

## 7 Conclusion

Although many processes in nature cannot be accurately duplicated using current conceptions, existing mathematical concepts have been applied to understand nature with considerable effectiveness. Therefore, researchers concentrate their efforts on developing new terminologies, frameworks, approaches, and ideas that could be utilized to comprehend the complexity of nature. This is one of the main causes for the introduction of various differential operators and the accompanying integrals in recent decades, which have greatly enhanced modeling. The ideas of piecewise differential and integral operators and piecewise functions, on the other hand, were introduced by mathematicians in response to changes that were noticed at various temporal or spatial scales. Although these two ideas, especially piecewise functions, have been theorized about. There are still processes in nature that existing models are unable to faithfully duplicate, despite the fact that these two notions, notably piecewise functions, have been employed in theory and applications to model processes with various patterns. In this work, we propose several piecewise differential equations and provide some significant conclusions that range from theory to application. These results are based on the piecewise derivative notion and piecewise functions. Theoretical and practical avenues will be opened by this extension.

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