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Conditioning diffusion processes with killing rates

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When the unconditioned process is a diffusion submitted to a space-dependent killing rate $k(\vec{x})$, various conditioning constraints can be imposed for a finite time horizon T . We first analyze the conditioned process when one imposes both the surviving distribution at time T and the killing-distribution for the intermediate times $t \in [0, T]$. When the conditioning constraints are less-detailed than these full distributions, we construct the appropriate conditioned processes via the optimization of the dynamical large deviations at Level 2.5 in the presence of the conditioning constraints that one wishes to impose. Finally, we describe various conditioned processes for the infinite horizon $T \rightarrow +\infty$. This general construction is then applied to two illustrative examples in order to generate stochastic trajectories satisfying various types of conditioning constraints : the first example concerns the pure diffusion in dimension d with the quadratic killing rate $k(\vec{x}) = \gamma \vec{x}^2$, while the second example is the Brownian motion with uniform drift submitted to the delta killing rate $k(x) = k\delta(x)$ localized at the origin $x = 0$.

I. INTRODUCTION

The Doob theory of conditioned stochastic processes [1, 2] plays an important role both in mathematics [3–5] and in physics (see the recent review [6] and references therein), with applications in very different fields like ecology [7], finance [8] or nuclear engineering [9, 10]. For diffusion processes, many different conditioning constraints have been studied besides the Brownian Bridge, in particular the Brownian excursion [11, 12], the Brownian meander [13], the taboo processes [14–19], or non-intersecting Brownian bridges [20]. Stochastic bridges have been also studied for many other Markov processes, including various diffusions processes [21–23], discrete-time random walks and Lévy flights [24–26], continuous-time Markov jump processes [26], run-and-tumble trajectories [27], or processes with resetting [28]. The stochastic bridge problem has been also extended to study the conditioning with respect to some global dynamical constraint as measured by a time-additive observable of the stochastic trajectories [29–33].

In the field of diffusion processes with killing rates that appear in many contexts [34–37], the analysis of their conditioning has also a long history [3, 38–43]. However, as discussed in detail in the recent analysis of Schrödinger bridges with unbalanced marginals [44], the literature can be split into two groups based on different frameworks and different assumptions : the first framework involves a Feynman-Kac multiplicative reweighting of the measure, while the second framework considers that one should introduce the appropriate conditioned killing rate. As explained in detail in [44], the second framework is the only one compatible with the optimization of the relative entropy between the conditioned process and the unconditioned process in the presence of the particular conditions that one wishes to impose. This conclusion confirms once again the visionary perspective of E. Schrödinger in his famous paper of 1931 [45] (see the recent english translation [46] with the corresponding detailed commentary), where the conditioning conditions are considered as the atypical result of an experiment concerning a large number N of unconditioned processes. Via this point of view, the theory of Doob conditioning becomes connected to the theories of dynamical large deviations and of stochastic control, as explained in detail in the commentary [46], as well as in the two reviews [47, 48] written from the viewpoint of stochastic control.

The goal of the present paper is thus to revisit the conditioning of diffusion processes with space-dependent killing rates, in order to give a global discussion of the various conditioning constraints that can be imposed for finite horizon T or for infinite horizon $T = +\infty$. It will be also interesting to see the similarities and the differences with the cases where the diffusion process is killed only via an absorbing boundary condition [49–52].

The paper is organized as follows. Section II describes the properties of the unconditioned diffusion processes with killing rates. In section III, we study the conditioning associated to the finite horizon T where one imposes the surviving distribution $P^*(\vec{y}, T)$ at time T and the killing-distribution $K^*(\vec{x}_d, t_d)$ for the intermediate times $t_d \in [0, T]$. In section IV, we analyze the conditioned processes when the conditioning constraints are less-detailed than the full distributions $[P^*(\cdot, T); K^*(\cdot, \cdot)]$ associated to the finite horizon T , and we discuss the consequences for the limit of the infinite horizon $T \rightarrow +\infty$. This general formalism is then applied to the pure diffusion with quadratic killing rate in section V, and to the Brownian motion with uniform drift and delta killing rate in section VI, in order to generate stochastic trajectories satisfying different conditioning constraints. Our conclusions are summarized in section VII. In Appendix A, we describe the link with the dynamical large deviations at Level 2.5 and the stochastic control theory.

Monte Carlo simulations, also presented, illustrate our theoretical findings.

II. UNCONDITIONED PROCESS : DIFFUSION $\vec{X}(t)$ WITH THE KILLING RATE $k(\vec{x})$

In this paper, we assume that the unconditioned process $\vec{X}(t)$ satisfies the Ito Stochastic Differential Equation (SDE) involving the drift $\vec{\mu}(\vec{x})$ and the diffusion coefficient $D(\vec{x})$, but can also be killed with the killing rate $k(\vec{x})$

$$\vec{X}(t+dt) = \begin{cases} \emptyset & \text{with probability } k(\vec{X}(t))dt \\ \vec{X}(t) + \vec{\mu}(\vec{X}(t))dt + \sqrt{2D(\vec{X}(t))}d\vec{W}(t) & \text{with probability } (1 - k(\vec{X}(t))dt) \end{cases} \quad (1)$$

where the components of $\vec{W}(t)$ are d independent Wiener processes. In this section, we recall the properties that will be useful in the other sections to construct conditioned processes.

A. Forward and backward dynamics of the propagator $P(\vec{x}_2, t_2 | \vec{x}_1, t_1)$

The generator \mathcal{F} involving the drift $\vec{\mu}(\vec{x})$, the diffusion coefficient $D(\vec{x})$ and the killing rate $k(\vec{x})$

$$\mathcal{F} \equiv \vec{\mu}(\vec{x}) \cdot \vec{\nabla} + D(\vec{x})\Delta - k(\vec{x}) \quad (2)$$

governs the backward dynamics of the propagator $P(\vec{x}_2, t_2 | \vec{x}, t)$ with respect to its initial variables (\vec{x}, t)

$$\begin{aligned} -\partial_t P(\vec{x}_2, t_2 | \vec{x}, t) &= \mathcal{F}P(\vec{x}_2, t_2 | \vec{x}, t) \\ &= \vec{\mu}(\vec{x}) \cdot \vec{\nabla} P(\vec{x}_2, t_2 | \vec{x}, t) + D(\vec{x})\Delta P(\vec{x}_2, t_2 | \vec{x}, t) - k(\vec{x})P(\vec{x}_2, t_2 | \vec{x}, t) \end{aligned} \quad (3)$$

while the adjoint operator of Eq. 2

$$\mathcal{F}^\dagger = -\vec{\nabla} \cdot \vec{\mu}(\vec{x}) + \Delta D(\vec{x}) - k(\vec{x}) \quad (4)$$

governs the forward dynamics of the propagator $P(\vec{x}, t | \vec{x}_1, t_1)$ with respect to its final variables (\vec{x}, t)

$$\partial_t P(\vec{x}, t | \vec{x}_1, t_1) = -\vec{\nabla} \cdot [\vec{\mu}(\vec{x})P(\vec{x}, t | \vec{x}_1, t_1)] + \Delta [D(\vec{x})P(\vec{x}, t | \vec{x}_1, t_1)] - k(\vec{x})P(\vec{x}, t | \vec{x}_1, t_1) \quad (5)$$

As explained in textbooks (see for instance [53]), whenever the diffusion coefficient $D(\vec{x})$ depends on the position \vec{x} , this forward dynamics for the propagator $P(\vec{x}, t | \vec{x}_1, t_1)$ will be translated into two different Stochastic Differential Equations if one follows the Ito or the Stratonovich prescriptions for the time-discretization. In the present paper, we have chosen the Ito prescription to write the Ito-SDE of Eq. 1 corresponding to Eq. 5. If one wishes to use instead the Stratonovich prescription, one just needs to write the corresponding Stratonovich-SDE that is equivalent to the forward dynamics of Eq. 5 for the propagator, since all the forthcoming analysis will be based on the forward and backward dynamics of Eqs 3 and 5, i.e. on the generator \mathcal{F} and its adjoint \mathcal{F}^\dagger .

B. Survival probability $S(T | \vec{x}, t)$ and killing probability $K(\vec{x}_T, T | \vec{x}, t)$

The survival probability $S(T | \vec{x}, t)$ at time T can be computed via the integration of the propagator $P(\vec{x}_T, T | \vec{x}, t)$ over the final position \vec{x}_T at time T

$$S(T | \vec{x}, t) = \int d^d \vec{x}_T P(\vec{x}_T, T | \vec{x}, t) \quad (6)$$

Using the forward dynamics of Eq. 5, one obtains that its time-decay

$$\begin{aligned} -\partial_T S(T | \vec{x}, t) &= - \int d^d \vec{x}_T \partial_T P(\vec{x}_T, T | \vec{x}, t) \\ &= \int d^d \vec{x}_T k(\vec{x}_T) P(\vec{x}_T, T | \vec{x}, t) \equiv \int d^d \vec{x}_T K(\vec{x}_T, T | \vec{x}, t) \end{aligned} \quad (7)$$

involves the killing probability $K(\vec{x}_T, T|\vec{x}, t)$ at position \vec{x}_T at time T

$$K(\vec{x}_T, T|\vec{x}, t) \equiv k(\vec{x}_T)P(\vec{x}_T, T|\vec{x}, t) \quad (8)$$

Taking into account the initial condition $S(t|\vec{x}, t) = 1$, the survival probability of Eq. 6 can be rewritten

$$S(T|\vec{x}, t) = 1 - \int_t^T dt_d \int d^d \vec{x}_d K(\vec{x}_d, t_d|\vec{x}, t) \quad (9)$$

in terms of the integration of the killing probability $K(\vec{x}_d, t_d|\vec{x}, t)$ over the position \vec{x}_d and over the time $t_d \in [t, T]$.

The survival probability $S(T|\vec{x}, t)$ of Eq. 6 inherits from the propagator $P(\vec{x}_T, T|\vec{x}, t)$ the backward dynamics of Eq. 3 with respect to the initial variables (\vec{x}, t)

$$-\partial_t S(T|\vec{x}, t) = \mathcal{F}S(T|\vec{x}, t) = \vec{\mu}(\vec{x}) \cdot \vec{\nabla} S(T|\vec{x}, t) + D(\vec{x}) \Delta S(T|\vec{x}, t) - k(\vec{x}) S(T|\vec{x}, t) \quad (10)$$

In the limit of the infinite horizon $T \rightarrow +\infty$, the forever survival probability $S(\infty|\vec{x})$ when starting at \vec{x} satisfies the time-independent backward equation

$$0 = \mathcal{F}S(\infty|\vec{x}) = \vec{\mu}(\vec{x}) \cdot \vec{\nabla} S(\infty|\vec{x}) + D(\vec{x}) \Delta S(\infty|\vec{x}) - k(\vec{x}) S(\infty|\vec{x}) \quad (11)$$

III. CONDITIONED PROCESS $\vec{X}^*(t)$ WITH RESPECT TO THE FINITE HORIZON T

A. Conditioning towards the distribution $P^*(\vec{y}, T)$ at T and the killing-distribution $K^*(\vec{x}_d, t_d)$ for $t_d \in [0, T]$

For the unconditioned diffusion process $\vec{X}(t)$ starting at position $\vec{X}(0) = \vec{x}_0$ at time $t = 0$:

(i) the probability to be surviving at time T at the position \vec{y} is given by the unconditioned propagator $P(\vec{y}, T|\vec{x}_0, 0)$, with the corresponding survival probability at time T of Eq. 6

$$S(T|\vec{x}_0, 0) = \int d^d \vec{y} P(\vec{y}, T|\vec{x}_0, 0) \quad (12)$$

(ii) the probability to have been killed at position \vec{x}_d at the time t_d is given by the unconditioned killing probability $K(\vec{x}_d, t_d|\vec{x}_0, 0)$ of Eq. 8 where the corresponding probability to be already dead at time T is complementary to the survival probability of Eq. 12 as explained in Eq. 9

$$\int_0^T dt_d \int d^d \vec{x}_d K(\vec{x}_d, t_d|\vec{x}_0, 0) = 1 - S(T|\vec{x}_0, 0) \quad (13)$$

Now we wish to construct the conditioned diffusion process $X^*(t)$ by imposing instead the following other properties:

(i) another probability $P^*(\vec{y}, T)$ to be surviving at position \vec{y} at time T , whose normalization over \vec{y} corresponds to the conditioned survival probability $S^*(T)$ at time T

$$\int d^d \vec{y} P^*(\vec{y}, T) = S^*(T) \quad (14)$$

(ii) another probability $K^*(\vec{x}_d, t_d)$ to have been killed at position \vec{x}_d at the time t_d , whose normalization over $t_d \in [0, T]$ and over \vec{x}_d is complementary to Eq. 14

$$\int_0^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) = 1 - S^*(T) \quad (15)$$

At any intermediate time $t \in]0, T[$, the conditioned survival probability $S^*(t)$ is given by

$$S^*(t) = 1 - \int_0^t dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \quad (16)$$

B. Conditioned probability $P^*(\vec{x}, t)$ at any intermediate time $t \in]0, T[$

At any intermediate time $t \in]0, T[$, the conditioned probability $P^*(\vec{x}, t)$ to be surviving at position x contains two contributions

$$\begin{aligned} P^*(\vec{x}, t) &= \int_t^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \frac{P(\vec{x}_d, t_d | \vec{x}, t) P(\vec{x}, t | \vec{x}_0, 0)}{P(\vec{x}_d, t_d | \vec{x}_0, 0)} + \int d^d \vec{y} P^*(\vec{y}, T) \frac{P(\vec{y}, T | \vec{x}, t) P(\vec{x}, t | \vec{x}_0, 0)}{P(\vec{y}, T | \vec{x}_0, 0)} \\ &\equiv Q_T(\vec{x}, t) P(\vec{x}, t | \vec{x}_0, 0) \end{aligned} \quad (17)$$

where $P(\vec{x}, t | \vec{x}_0, 0)$ is the unconditioned propagator, while the function $Q_T(\vec{x}, t)$ reads

$$Q_T(\vec{x}, t) \equiv \int_t^T dt_d \int d^d \vec{x}_d \frac{K^*(\vec{x}_d, t_d)}{P(\vec{x}_d, t_d | \vec{x}_0, 0)} P(\vec{x}_d, t_d | \vec{x}, t) + \int d^d \vec{y} \frac{P^*(\vec{y}, T)}{P(\vec{y}, T | \vec{x}_0, 0)} P(\vec{y}, T | \vec{x}, t) \quad (18)$$

The derivative with respect to the time t appearing as the lower boundary of the integral of the first contribution gives the following term involving the killing-time $t_d = t$

$$- \int d^d \vec{x}_d \frac{K^*(\vec{x}_d, t)}{P(\vec{x}_d, t | \vec{x}_0, 0)} P(\vec{x}_d, t | \vec{x}, t) = - \int d^d \vec{x}_d \frac{K^*(\vec{x}_d, t)}{P(\vec{x}_d, t | \vec{x}_0, 0)} \delta^d(\vec{x}_d - \vec{x}) = - \frac{K^*(\vec{x}, t)}{P(\vec{x}, t | \vec{x}_0, 0)} \quad (19)$$

As a consequence, this additional inhomogenous contribution appears in the backward dynamics that the function $Q_T(\vec{x}, t)$ inherits from the unconditioned propagators $P(\vec{x}_d, t_d | \vec{x}, t)$ and $P(\vec{y}, T | \vec{x}, t)$ that satisfy Eq. 3 with respect to their initial variables (\vec{x}, t)

$$\begin{aligned} -\partial_t Q_T(\vec{x}, t) &= \mathcal{F} Q_T(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{P(\vec{x}, t | \vec{x}_0, 0)} \\ &= \vec{\mu}(\vec{x}) \cdot \vec{\nabla} Q_T(\vec{x}, t) + D(\vec{x}) \Delta Q_T(\vec{x}, t) - k(\vec{x}) Q_T(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{P(\vec{x}, t | \vec{x}_0, 0)} \end{aligned} \quad (20)$$

C. Forward dynamics of the conditioned process $X^*(t)$

Using the forward dynamics of Eq. 5 satisfied by the unconditioned propagator $P(\vec{x}, t | \vec{x}_0, 0)$

$$\partial_t P(\vec{x}, t | \vec{x}_0, 0) = -\vec{\nabla} \cdot [\vec{\mu}(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)] + \Delta [D(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)] - k(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0) \quad (21)$$

and the backward dynamics of Eq. 20 satisfied by the function $Q_T(\vec{x}, t)$, one obtains that the time derivative of the conditioned probability of Eq. 17 reads

$$\begin{aligned} \partial_t P^*(\vec{x}, t) &= P(\vec{x}, t | \vec{x}_0, 0) [\partial_t Q_T(\vec{x}, t)] + Q_T(\vec{x}, t) [\partial_t P(\vec{x}, t | \vec{x}_0, 0)] \\ &= -\vec{\nabla} \cdot [\vec{\mu}_T^*(\vec{x}, t) P^*(\vec{x}, t)] + \Delta [D(\vec{x}) P^*(\vec{x}, t)] - K^*(\vec{x}, t) \end{aligned} \quad (22)$$

with the conditioned drift

$$\vec{\mu}_T^*(\vec{x}, t) \equiv \vec{\mu}(\vec{x}) + 2D(\vec{x}) \vec{\nabla} \ln Q_T(\vec{x}, t) \quad (23)$$

while the term involving the diffusion coefficient $D(\vec{x})$ is the same as in the initial dynamics. From the point of view of the killing contribution, let us stress that it is the imposed conditioned killing probability $K^*(\vec{x}, t)$ that appears directly in the conditioned dynamics of Eq. 22. As a consequence, the corresponding conditioned killing rate $k_T^*(\vec{x}, t)$ should be computed from the ratio

$$k_T^*(\vec{x}, t) = \frac{K^*(\vec{x}, t)}{P^*(\vec{x}, t)} = \frac{K^*(\vec{x}, t)}{Q_T(\vec{x}, t) P(\vec{x}, t | \vec{x}_0, 0)} \quad (24)$$

in order to generate stochastic trajectories of the conditioned process $X^*(t)$ via the following Ito SDE with killing

$$\vec{X}^*(t + dt) = \begin{cases} \emptyset & \text{with probability } k_T^*(\vec{X}^*(t), t) dt \\ \vec{X}^*(t) + \vec{\mu}_T^*(\vec{X}^*(t), t) dt + \sqrt{2D(\vec{X}^*(t))} d\vec{W}(t) & \text{with probability } (1 - k_T^*(\vec{X}^*(t), t) dt) \end{cases} \quad (25)$$

IV. CONDITIONING LESS DETAILED THAN THE FULL DISTRIBUTIONS $[P^*(., T); K^*(., .)]$ AT T

In section III, we have constructed the conditioned process $\vec{X}^*(t)$ when the conditioning constraints correspond to the full distributions $[P^*(., T); K^*(., .)]$ associated to the finite horizon T . In the present section, we consider instead the cases where the conditioning constraints are less detailed, and we discuss the limit of the infinite horizon $T \rightarrow +\infty$ when appropriate.

A. Relative entropy cost of the full conditioning conditions $[P^*(., T); K^*(., .)]$ imposed at the time horizon T

Let us now consider a large number N of independent realizations $\vec{X}_n(t)$ of the unconditioned process labelled by $n = 1, 2, \dots, N$ starting all at the same initial condition $\vec{X}_n(0) = \vec{x}_0$ at time $t = 0$. The empirical histogram $\hat{P}(\vec{y}, T)$ at time T of the surviving position $\vec{X}_n(T)$

$$\hat{P}(\vec{y}, T) \equiv \frac{1}{N} \sum_{n=1}^N \delta^d(\vec{X}_n(T) - \vec{y}) \quad (26)$$

and the empirical joint histogram $\hat{K}(\vec{x}, t)$ of the killing events satisfy the global normalization

$$1 = \int d^d \vec{y} \hat{P}(\vec{y}, T) + \int d^d \vec{x} \int_0^T dt \hat{K}(\vec{x}, t) \quad (27)$$

The Sanov theorem concerning the empirical histogram of independent identically distributed variables (see the reviews on large deviations [54–56] and references therein) yields that the joint probability to observe the empirical surviving density $\hat{P}(\vec{y}, T)$ and the empirical killing distribution $\hat{K}(\vec{x}, t)$ satisfy the large deviation form for large N

$$\mathcal{P}_T^{Sanov} [\hat{P}(\cdot, T); \hat{K}(\cdot, \cdot)] \underset{N \rightarrow +\infty}{\simeq} \delta \left(\int d^d \vec{y} \hat{P}(\vec{y}, T) + \int d^d \vec{x} \int_0^T dt \hat{K}(\vec{x}, t) - 1 \right) e^{-N \mathcal{I}_T^{Sanov} [\hat{P}(\cdot, T); \hat{K}(\cdot, \cdot)]} \quad (28)$$

where the delta function imposes the normalization constraint of Eq. 27, while the Sanov rate function

$$\mathcal{I}_T^{Sanov} [\hat{P}(\cdot, T); \hat{K}(\cdot, \cdot)] = \int d^d \vec{y} \hat{P}(\vec{y}, T) \ln \left(\frac{\hat{P}(\vec{y}, T)}{P(\vec{y}, T | \vec{x}_0, 0)} \right) + \int d^d \vec{x} \int_0^T dt \hat{K}(\vec{x}, t) \ln \left(\frac{\hat{K}(\vec{x}, t)}{k(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)} \right) \quad (29)$$

corresponds to the relative entropy of the empirical distributions $[\hat{P}(\vec{y}, T); \hat{K}(\vec{x}, t)]$ with respect to their typical values $[P(\vec{y}, T | \vec{x}_0, 0); k(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)]$.

Following the Schrödinger perspective [45] (see the recent detailed commentary [46] accompanying its english translation, as well as in the two reviews [47, 48] written from the viewpoint of stochastic control), one can interpret the conditioning conditions $P^*(\vec{y}, T)$ and $K^*(\vec{x}, t)$ imposed at the finite horizon T as the empirical results $[\hat{P}(\cdot, T); \hat{K}(\cdot, \cdot)]$ measured in an experiment concerning N independent unconditioned processes, with the following consequences :

(i) the Sanov rate function of Eq. 29 evaluated for the imposed conditions $[P^*(., T); K^*(., .)]$ at the horizon T

$$\mathcal{I}_T^{Sanov} [P^*(., T); K^*(., .)] = \int d^d \vec{y} P^*(\vec{y}, T) \ln \left(\frac{P^*(\vec{y}, T)}{P(\vec{y}, T | \vec{x}_0, 0)} \right) + \int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)} \right) \quad (30)$$

measures how rare it is for large N to see the distributions $[P^*(\vec{y}, T); K^*(\vec{x}, t)]$ different from their typical values distributions $[P(\vec{y}, T | \vec{x}_0, 0); k(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)]$.

(ii) the Sanov rate function $\mathcal{I}_T^{Sanov} [P^*(., T); K^*(., .)]$ of Eq. 30 can be used to give some precise meaning to conditioning conditions that are less detailed than the whole distributions $[P^*(., T); K^*(., .)]$: one needs to optimize the Sanov rate function in the presence of the less detailed conditioning conditions that one wishes to impose in order to construct the appropriate conditioned process. Various simple examples are described in the following subsections.

B. Conditioning towards the surviving distribution $P^*(\vec{y}, T)$ at the horizon T alone

If one wishes to impose only the probability $P^*(\vec{y}, T)$ at time T , together with its corresponding survival probability

$$S^*(T) \equiv \int d^d \vec{y} P^*(\vec{y}, T) \quad (31)$$

one needs to optimize the Sanov rate function $\mathcal{I}_T^{Sanov} [P^*(., T); K^*(., .)]$ of Eq. 30 over the killing probability $K^*(., .)$ normalized to

$$\int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) = 1 - S^*(T) \quad (32)$$

One obtains the optimal solution

$$K^{*opt}(\vec{x}, t) = \left(\frac{1 - S^*(T)}{1 - S(T|\vec{x}_0, 0)} \right) k(\vec{x}) P(\vec{x}, t|\vec{x}_0, 0) \quad (33)$$

and its contribution to the Sanov rate function

$$\int d^d \vec{x} \int_0^T dt K^{*opt}(\vec{x}, t) \ln \left(\frac{K^{*opt}(\vec{x}, t)}{k(\vec{x}) P(\vec{x}, t|\vec{x}_0, 0)} \right) = (1 - S^*(T)) \ln \left(\frac{1 - S^*(T)}{1 - S(T|\vec{x}_0, 0)} \right) \quad (34)$$

So the relative entropy cost of the imposed probability $P^*(\vec{y}, T)$ at time T and of its corresponding survival probability $S^*(T)$ of Eq. 31 reads

$$\begin{aligned} \mathcal{I}_T^{space} [P^*(., T); S^*(T)] &= \mathcal{I}_T^{Sanov} [P^*(., T); K^{*opt}(\vec{x}, t)] \\ &= \int d^d \vec{y} P^*(\vec{y}, T) \ln \left(\frac{P^*(\vec{y}, T)}{P(\vec{y}, T|\vec{x}_0, 0)} \right) + (1 - S^*(T)) \ln \left(\frac{1 - S^*(T)}{1 - S(T|\vec{x}_0, 0)} \right) \end{aligned} \quad (35)$$

In addition, one should use the optimal killing probability $K^{*opt}(\vec{x}, t)$ of Eq. 33, so the corresponding function of Eq. 18 becomes

$$Q_T^{[P^*(., T); S^*(T)]}(\vec{x}, t) = (1 - S^*(T)) \left(\frac{1 - S(T|x, t)}{1 - S(T|\vec{x}_0, 0)} \right) + \int d^d \vec{y} P^*(\vec{y}, T) \frac{P(\vec{y}, T|\vec{x}, t)}{P(\vec{y}, T|\vec{x}_0, 0)} \quad (36)$$

while the corresponding optimal killing rate of Eq. 24 reads

$$k_T^*(\vec{x}, t) = \frac{K^{*opt}(\vec{x}, t)}{P^*(\vec{x}, t)} = \frac{\left(\frac{1 - S^*(T)}{1 - S(T|\vec{x}_0, 0)} \right)}{Q_T^{[P^*(., T); S^*(T)]}(\vec{x}, t)} k(\vec{x}) \quad (37)$$

C. Conditioning towards the killing distribution $K^*(\vec{x}, t)$ for $t \in [0, T]$ alone

If one wishes to impose only the killing distribution $K^*(\vec{x}, t)$ for $t \in [0, T]$ alone, together with its normalization

$$\int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) = 1 - S^*(T) \quad (38)$$

one needs to optimize the Sanov rate function $\mathcal{I}_T^{Sanov} [P^*(., T); K^*(., .)]$ of Eq. 30 over the possible spatial distribution $P^*(\vec{y}, T)$ normalized to $S^*(T)$. One obtains the optimal solution

$$P^{*opt}(\vec{y}, T) = \left(\frac{S^*(T)}{S(T|\vec{x}_0, 0)} \right) P(\vec{y}, T|\vec{x}_0, 0) \quad (39)$$

and corresponding contribution to the Sanov rate function

$$\int d^d \vec{y} P^{*opt}(\vec{y}, T) \ln \left(\frac{P^{*opt}(\vec{y}, T)}{P(\vec{y}, T|\vec{x}_0, 0)} \right) = S^*(T) \ln \left(\frac{S^*(T)}{S(T|\vec{x}_0, 0)} \right) \quad (40)$$

So the relative entropy cost of the killing distribution $K^*(\vec{x}, t)$ and of the corresponding survival probability $S^*(T)$ of Eq. 38 is given by

$$\begin{aligned} \mathcal{I}_T^{killing} [K^*(., .); S^*(T)] &= \mathcal{I}_T^{Sanov} [P^{*opt}(\vec{y}, T); K^*(., .)] \\ &= \int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) P(\vec{x}, t|\vec{x}_0, 0)} \right) + S^*(T) \ln \left(\frac{S^*(T)}{S(T|\vec{x}_0, 0)} \right) \end{aligned} \quad (41)$$

In addition, one should use the optimal solution $P^{*opt}(\vec{y}, T)$ of Eq. 39, so the function of Eq. 18 becomes

$$Q_T^{[K^*(\dots); S^*(T)]}(\vec{x}, t) = \int_t^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \frac{P(\vec{x}_d, t_d | \vec{x}, t)}{P(\vec{x}_d, t_d | \vec{x}_0, 0)} + S^*(T) \left(\frac{S(T | \vec{x}, t)}{S(T | \vec{x}_0, 0)} \right) \quad (42)$$

with the corresponding optimal killing rate of Eq. 24

$$k_T^*(\vec{x}, t) = \frac{K^*(\vec{x}, t)}{Q_T^{[K^*(\dots); S^*(T)]}(\vec{x}, t) P(\vec{x}, t | \vec{x}_0, 0)} \quad (43)$$

Application to the conditioning towards the normalized killing distribution $K^(\vec{x}, t)$ for the infinite horizon $T = +\infty$*

Let us now consider the limit of the infinite horizon $T \rightarrow +\infty$, where one wishes to impose some normalized killing distribution $K^*(\vec{x}, t)$ in Eq. 38

$$\int d^d \vec{x} \int_0^{+\infty} dt K^*(\vec{x}, t) = 1 - S^*(\infty) = 1 \quad (44)$$

so that the conditioned forever-survival probability vanishes $S^*(\infty) = 0$. Then the function of Eq. 42 reduces to

$$Q_\infty^{[K^*(\dots); S^*(\infty)=0]}(\vec{x}, t) = \int_t^{+\infty} dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \frac{P(\vec{x}_d, t_d | \vec{x}, t)}{P(\vec{x}_d, t_d | \vec{x}_0, 0)} \quad (45)$$

while the conditioned killing rate of Eq. 43 reads

$$k_\infty^*(\vec{x}, t) = \frac{K^*(\vec{x}, t)}{Q_\infty^{[K^*(\dots); S^*(\infty)=0]}(\vec{x}, t) P(\vec{x}, t | \vec{x}_0, 0)} \quad (46)$$

D. Conditioning towards the surviving probability $S^*(T)$ at time T alone

If one wishes to impose only the value $S^*(T)$ of the conditioned survival probability at time T , the computations of the two previous subsections IV B and IV C can be used to obtain the following results. The relative entropy cost of imposing the surviving probability $S^*(T)$ at time T alone reduces to

$$\mathcal{I}_T^{surviving} [S^*(T)] = S^*(T) \ln \left(\frac{S^*(T)}{S(T | \vec{x}_0, 0)} \right) + (1 - S^*(T)) \ln \left(\frac{1 - S^*(T)}{1 - S(T | \vec{x}_0, 0)} \right) \quad (47)$$

In addition, the optimal solutions $K^{*opt}(\vec{x}, t)$ of Eq. 33 and $P^{*opt}(\vec{y}, T)$ of Eq. 39 yield that the appropriate function $Q_T(\vec{x}, t)$ reads using Eqs 36 and 42

$$Q_T^{[S^*(T)]}(\vec{x}, t) = \left(\frac{1 - S^*(T)}{1 - S(T | \vec{x}_0, 0)} \right) (1 - S(T | \vec{x}, t)) + \left(\frac{S^*(T)}{S(T | \vec{x}_0, 0)} \right) S(T | \vec{x}, t) \quad (48)$$

with the corresponding optimal killing rate of Eq. 24

$$k_T^*(\vec{x}, t) = \frac{\left(\frac{1 - S^*(T)}{1 - S(T | \vec{x}_0, 0)} \right) k(\vec{x})}{Q_T^{[S^*(T)]}(\vec{x}, t)} \quad (49)$$

1. Special case : conditioning towards the survival probability $S^*(T) = 1$ at the finite horizon T

For the special case where one wishes to impose the full survival $S^*(T) = 1$ at the finite horizon T , the conditioned killing rate of Eq. 49 vanishes as it should

$$k_T^*(\vec{x}, t) = 0 \quad (50)$$

while the function of Eq. 48 reduces to the ratio of survival probabilities $S(T|.,.)$ of the unconditioned process

$$Q_T^{[S^*(T)=1]}(\vec{x}, t) = \frac{S(T|\vec{x}, t)}{S(T|\vec{x}_0, 0)} \quad (51)$$

So the corresponding conditioned drift of Eq. 23 reads in terms of the unconditioned survival probability $S(T|\vec{x}, t)$

$$\vec{\mu}_T^*(\vec{x}, t) = \vec{\mu}(\vec{x}) + 2D(\vec{x})\vec{\nabla} \ln S(T|\vec{x}, t) \quad (52)$$

in agreement with the general formula given in [40], where many illustrative examples can be found, including an application in section 4 to an unconditioned diffusion process on the half-line $[0, +\infty[$, where the drift $\mu(x)$, the diffusion coefficient $D(x)$ and the killing rate $k(x)$ are all linear functions of the position x .

2. Special case : conditioning towards the survival probability $S^*(T) = 0$ at the finite horizon T

For the special case where one wishes to impose no survival $S^*(T) = 0$ at the finite horizon T , the function of Eq. 48 reduces to

$$Q_T^{[S^*(T)=0]}(\vec{x}, t) = \frac{1 - S(T|\vec{x}, t)}{1 - S(T|\vec{x}_0, 0)} \quad (53)$$

So the corresponding conditioned drift of Eq. 23 is given by

$$\vec{\mu}_T^*(\vec{x}, t) = \vec{\mu}(\vec{x}) + 2D(\vec{x})\vec{\nabla} \ln [1 - S(T|\vec{x}, t)] \quad (54)$$

while the conditioned killing rate of Eq. 49 reads

$$k_T^*(\vec{x}, t) = \frac{k(\vec{x})}{[1 - S(T|\vec{x}_0, 0)] Q_T^{[S^*(T)=0]}(\vec{x}, t)} = \frac{k(\vec{x})}{1 - S(T|\vec{x}, t)} \quad (55)$$

E. Limit of the infinite horizon $T = +\infty$: conditioning towards the forever-survival probability $S^*(\infty)$

Let us now consider the limit of the infinite horizon $T \rightarrow +\infty$, where one wishes to impose only the forever-survival probability $S^*(\infty)$. Then the function of Eq. 48 should be computed from the limits

$$Q_\infty^{[S^*(\infty)]}(\vec{x}, t) = (1 - S^*(\infty)) \lim_{T \rightarrow +\infty} \left(\frac{1 - S(T|\vec{x}, t)}{1 - S(T|\vec{x}_0, 0)} \right) + S^*(\infty) \lim_{T \rightarrow +\infty} \left(\frac{S(T|\vec{x}, t)}{S(T|\vec{x}_0, 0)} \right) \quad (56)$$

while the conditioned killing rate of Eq. 49 becomes

$$k_\infty^*(\vec{x}, t) = \frac{\left(\frac{1 - S^*(\infty)}{1 - S(\infty|\vec{x}_0)} \right)}{Q_\infty^{[S^*(\infty)]}(\vec{x}, t)} k(\vec{x}) \quad (57)$$

In order to evaluate the limits of Eq. 56 one needs to distinguish whether the unconditioned survival probability $S(T|.,.)$ vanishes or not for $T \rightarrow +\infty$.

1. Cases where the unconditioned survival probability vanishes $S(+\infty|.) = 0$

When the unconditioned survival probability vanishes $S(+\infty|.) = 0$, the function of Eq. 56

$$Q_\infty^{[S^*(\infty)]}(\vec{x}, t) = (1 - S^*(\infty)) + S^*(\infty) \lim_{T \rightarrow +\infty} \left(\frac{S(T|\vec{x}, t)}{S(T|\vec{x}_0, 0)} \right) \quad (58)$$

involves the limit of the ratio $\left(\frac{S(T|\vec{x}, t)}{S(T|\vec{x}_0, 0)} \right)$ of the two vanishing survival probabilities.

2. Cases where the unconditioned survival probability remains finite $S(+\infty|\cdot) \in]0, 1[$

When the unconditioned survival probability remains finite $S(+\infty|\cdot) \in]0, 1[$, the function of Eq. 56 reduces to

$$Q_\infty^{[S^*(\infty)]}(\vec{x}) = (1 - S^*(\infty)) \left(\frac{1 - S(\infty|\vec{x})}{1 - S(\infty|\vec{x}_0)} \right) + S^*(\infty) \left(\frac{S(\infty|\vec{x})}{S(\infty|\vec{x}_0)} \right) \quad (59)$$

while the conditioned killing rate of Eq. 57 reads

$$k_\infty^*(\vec{x}) = \frac{\left(\frac{1 - S^*(\infty)}{1 - S(\infty|\vec{x}_0)} \right)}{Q_\infty^{[S^*(\infty)]}(\vec{x})} k(\vec{x}) \quad (60)$$

An explicit example where the conditioning is towards zero-survival $S^*(\infty) = 0$ can be found in the book [3] on pages 282-283 for the unconditioned diffusion process on the half-line $[0, +\infty[$ with no drift $\mu(x) = 0$, where the diffusion coefficient $D(x)$ and the killing rate $k(x)$ are given by the space-dependent functions

$$\begin{aligned} D(x) &= \frac{x}{2} \\ k(x) &= \frac{x^2}{2} \end{aligned} \quad (61)$$

F. Conditioning towards the time-killing distribution $K^*(t) = \int_{-\infty}^{+\infty} d^d \vec{x} K^*(\vec{x}, t)$ for $t \in]0, T[$ alone

If one wishes to impose only the time-killing distribution $K^*(t) = \int d^d \vec{x} K^*(\vec{x}, t)$ for $t \in]0, T[$, together with its normalization from Eq. 38

$$\int_0^T dt K^*(t) = 1 - S^*(T) \quad (62)$$

one needs to optimize the rate function $\mathcal{I}_T^{killing} [K^*(\cdot, \cdot); S^*(T)]$ of Eq. 41 over the possible spatial-dependence in x of the killing distributions $K^*(\vec{x}, t)$, with the normalization constraint for each t

$$\int_{-\infty}^{+\infty} d^d \vec{x} K^*(\vec{x}, t) = K^*(t) \quad (63)$$

One obtains the optimal solution

$$K^{*opt}(\vec{x}, t) = \frac{K^*(t)}{K(t|\vec{x}_0, 0)} k(\vec{x}) P(\vec{x}, t|\vec{x}_0, 0) \quad (64)$$

where

$$K(t|\vec{x}_0, 0) \equiv \int d^d \vec{x} k(\vec{x}) P(\vec{x}, t|\vec{x}_0, 0) \quad (65)$$

represents the unconditioned time-killing probability. The corresponding contribution to the rate function of Eq. 41

$$\int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) P(\vec{x}, t|\vec{x}_0, 0)} \right) = \int_0^T dt K^*(t) \ln \left(\frac{K^*(t)}{K(t|\vec{x}_0, 0)} \right) \quad (66)$$

leads to the following relative entropy cost

$$\mathcal{I}_T^{time} [K^*(\cdot); S^*(T)] = \int_0^T dt K^*(t) \ln \left(\frac{K^*(t)}{K(t|\vec{x}_0, 0)} \right) + S^*(T) \ln \left(\frac{S^*(T)}{S(T|\vec{x}_0, 0)} \right) \quad (67)$$

In addition, one should use the optimal solution $K^{*opt}(\vec{x}, t)$ of Eq. 64, so the function of Eq. 42 becomes

$$\begin{aligned} Q_T^{[K^*(\cdot); S^*(T)]}(\vec{x}, t) &= \int_t^T dt_d \frac{K^*(t_d)}{K(t_d|\vec{x}_0, 0)} \int d^d \vec{x}_d k(\vec{x}_d) P(\vec{x}_d, t_d|\vec{x}, t) + \left(\frac{S^*(T)}{S(T|\vec{x}_0, 0)} \right) S(T|\vec{x}, t) \\ &= \int_t^T dt_d \frac{K^*(t_d)}{K(t_d|\vec{x}_0, 0)} K(t_d|\vec{x}, t) + \left(\frac{S^*(T)}{S(T|\vec{x}_0, 0)} \right) S(T|\vec{x}, t) \end{aligned} \quad (68)$$

with the corresponding optimal killing rate of Eq. 43

$$k_T^*(\vec{x}, t) = \frac{K^{*opt}(\vec{x}, t)}{Q_T^{[K^*(\cdot); S^*(T)]}(\vec{x}, t)P(\vec{x}, t|\vec{x}_0, 0)} = \frac{\frac{K^*(t)}{K(t|\vec{x}_0, 0)}}{Q_T^{[K^*(\cdot); S^*(T)]}(\vec{x}, t)} k(\vec{x}) \quad (69)$$

Application to the conditioning towards the normalized time-killing distribution $K^(t)$ for the infinite horizon $T = +\infty$*

Let us now consider the limit of the infinite horizon $T \rightarrow +\infty$, where one wishes to impose some normalized time-killing distribution $K^*(t)$ in Eq. 38

$$\int_0^{+\infty} dt K^*(t) = 1 - S^*(\infty) = 1 \quad (70)$$

so that the conditioned forever-survival probability vanishes $S^*(\infty) = 0$. Then the function of Eq. 68 becomes

$$Q_\infty^{[K^*(\cdot); S^*(\infty)=0]}(\vec{x}, t) = \int_t^{+\infty} dt_d \frac{K^*(t_d)}{K(t_d|\vec{x}_0, 0)} K(t_d|\vec{x}, t) \quad (71)$$

while the conditioned killing rate of Eq. 69 reads

$$k_\infty^*(\vec{x}, t) = \frac{\frac{K^*(t)}{K(t|\vec{x}_0, 0)}}{Q_\infty^{[K^*(\cdot); S^*(\infty)=0]}(\vec{x}, t)} k(\vec{x}) \quad (72)$$

V. APPLICATION TO PURE DIFFUSION WITH QUADRATIC KILLING RATE

In this section, the general framework described previously is applied to the explicit case where the unconditioned process is the pure diffusion in dimension d with the quadratic killing rate $k(\vec{x}) = \gamma \vec{x}^2$.

A. Unconditioned process $\vec{X}(t)$: diffusion coefficient D and quadratic killing rate $k(\vec{x}) = \gamma \vec{x}^2$

The unconditioned process $\vec{X}(t)$ is generated by Eq. 1 with no drift $\vec{\mu}(\vec{x}) = \vec{0}$, with the uniform diffusion coefficient $D(\vec{x}) = D$, and the quadratic killing rate $k(\vec{x}) = \gamma \vec{x}^2$

$$\vec{X}(t+dt) = \begin{cases} \emptyset & \text{with probability } \gamma \vec{X}^2(t) dt \\ \vec{X}(t) + \sqrt{2D} d\vec{W}(t) & \text{with probability } (1 - \gamma \vec{X}^2(t) dt) \end{cases} \quad (73)$$

The self-adjoint generator of Eq. 2

$$\mathcal{F} = \mathcal{F}^\dagger = D\Delta - \gamma \vec{x}^2 \equiv -H \quad (74)$$

corresponds to the quantum Hamiltonian

$$H \equiv -\frac{1}{2m} \Delta + \frac{m\omega^2}{2} \vec{x}^2 \quad (75)$$

of an harmonic oscillator of mass $m = \frac{1}{2D}$ and of frequency

$$\omega \equiv 2\sqrt{D\gamma} \quad (76)$$

As a consequence, the unconditioned propagator $P(\vec{x}_2, t_2|\vec{x}_1, t_1)$ corresponds to the Euclidean propagator of the quantum harmonic oscillator that reads

$$P(\vec{x}_2, t_2|\vec{x}_1, t_1) = \left(\frac{\omega}{4D\pi \sinh[\omega(t_2 - t_1)]} \right)^{\frac{d}{2}} e^{-\frac{\omega}{4D \sinh[\omega(t_2 - t_1)]} [(\vec{x}_2^2 + \vec{x}_1^2) \cosh[\omega(t_2 - t_1)] - 2\vec{x}_2 \cdot \vec{x}_1]} \quad (77)$$

The unconditioned survival probability of Eq. 6 reads

$$S(t_2|\vec{x}_1, t_1) = \int d^d \vec{x}_2 P(\vec{x}_2, t_2|\vec{x}_1, t_1) = \frac{1}{\cosh^{\frac{d}{2}}[\omega(t_2 - t_1)]} e^{-\frac{\omega \tanh[\omega(t_2 - t_1)]}{4D} \vec{x}_1^2} \quad (78)$$

Its decay with respect to t_2 gives the unconditioned killing-time probability

$$\begin{aligned} K(t_2|\vec{x}_1, t_1) &= -\partial_{t_2} S(t_2|\vec{x}_1, t_1) \\ &= \frac{1}{\cosh^{\frac{d}{2}+1}[\omega(t_2 - t_1)]} e^{-\frac{\omega \tanh[\omega(t_2 - t_1)]}{4D} \vec{x}_1^2} \left[\frac{d}{2} \omega \sinh[\omega(t_2 - t_1)] + \frac{\omega^2}{4D \cosh[\omega(t_2 - t_1)]} \vec{x}_1^2 \right] \end{aligned} \quad (79)$$

while the unconditioned space-time killing probability of Eq. 8 reads using Eq. 76 to replace $\gamma = \frac{\omega^2}{4D}$

$$\begin{aligned} K(\vec{x}_2, t_2|\vec{x}_1, t_1) &= k(\vec{x}_2) P(\vec{x}_2, t_2|\vec{x}_1, t_1) \\ &= \frac{\omega^2}{4D} \vec{x}_2^2 \left(\frac{\omega}{4D\pi \sinh[\omega(t_2 - t_1)]} \right)^{\frac{d}{2}} e^{-\frac{\omega}{4D \sinh[\omega(t_2 - t_1)]} [(\vec{x}_2^2 + \vec{x}_1^2) \cosh[\omega(t_2 - t_1)] - 2\vec{x}_2 \cdot \vec{x}_1]} \end{aligned} \quad (80)$$

B. Full conditioning constraints $[P^*(\cdot, T); K^*(\cdot, \cdot)]$ associated to the finite horizon T

Using the propagator of Eq. 77, one obtains that the function $Q_T(\vec{x}, t)$ of Eq. 18 reads

$$\begin{aligned} Q_T(\vec{x}, t) &= \int_t^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \frac{P(\vec{x}_d, t_d|\vec{x}, t)}{P(\vec{x}_d, t_d|\vec{x}_0, 0)} + \int d^d \vec{y} P^*(\vec{y}, T) \frac{P(\vec{y}, T|\vec{x}, t)}{P(\vec{y}, T|\vec{x}_0, 0)} \\ &= \int_t^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \left(\frac{\sinh[\omega t_d]}{\sinh[\omega(t_d - t)]} \right)^{\frac{d}{2}} e^{\frac{\omega(\vec{x}_d^2 + \vec{x}_0^2)}{4D \tanh[\omega t_d]} - \frac{\omega(\vec{x}_d^2 + \vec{x}^2)}{4D \tanh[\omega(t_d - t)]} + \frac{\omega \vec{x}_d \cdot \vec{x}}{2D \sinh[\omega(t_d - t)]} - \frac{\omega \vec{x}_d \cdot \vec{x}_0}{2D \sinh[\omega t_d]} \\ &+ \int d^d \vec{y} P^*(\vec{y}, T) \left(\frac{\sinh[\omega T]}{\sinh[\omega(T - t)]} \right)^{\frac{d}{2}} e^{\frac{\omega(\vec{y}^2 + \vec{x}_0^2)}{4D \tanh[\omega T]} - \frac{\omega(\vec{y}^2 + \vec{x}^2)}{4D \tanh[\omega(T - t)]} + \frac{\omega \vec{y} \cdot \vec{x}}{2D \sinh[\omega(T - t)]} - \frac{\omega \vec{y} \cdot \vec{x}_0}{2D \sinh[\omega T]} \end{aligned} \quad (81)$$

Example of the bridge without being killed : $P^(\vec{y}, T) = \delta^d(\vec{y} - \vec{y}_*)$*

For the case where one imposes the full survival at time T at the single position \vec{y}_*

$$\begin{aligned} P^*(\vec{y}, T) &= \delta^d(\vec{y} - \vec{y}_*) \\ K^*(\vec{x}_d, t_d) &= 0 \end{aligned} \quad (82)$$

Eq. 81 reduces to

$$Q_T(\vec{x}, t) = \left(\frac{\sinh[\omega T]}{\sinh[\omega(T - t)]} \right)^{\frac{d}{2}} e^{\frac{\omega(\vec{y}_*^2 + \vec{x}_0^2)}{4D \tanh[\omega T]} - \frac{\omega(\vec{y}_*^2 + \vec{x}^2)}{4D \tanh[\omega(T - t)]} + \frac{\omega \vec{y}_* \cdot \vec{x}}{2D \sinh[\omega(T - t)]} - \frac{\omega \vec{y}_* \cdot \vec{x}_0}{2D \sinh[\omega T]} \quad (83)$$

The corresponding conditioned drift of Eq. 23 reads

$$\vec{\mu}_T^*(\vec{x}, t) = 2D \vec{\nabla} \ln Q_T(\vec{x}, t) = \frac{\omega}{\sinh[\omega(T - t)]} \vec{y}_* - \frac{\omega}{\tanh[\omega(T - t)]} \vec{x} \quad (84)$$

while the conditioned killing rate of Eq. 24 vanishes $k_T^*(x, t) = 0$. The Brownian bridge is recovered in the limit $\omega \rightarrow 0$ as it should.

In dimension $d = 1$, the conditioned process thus satisfies the Ito stochastic differential equation

$$dX^*(t) = \left(\frac{\omega}{\sinh[\omega(T - t)]} y_* - \frac{\omega}{\tanh[\omega(T - t)]} X^*(t) \right) dt + dW(t) \quad (85)$$

Single bridge without being killed

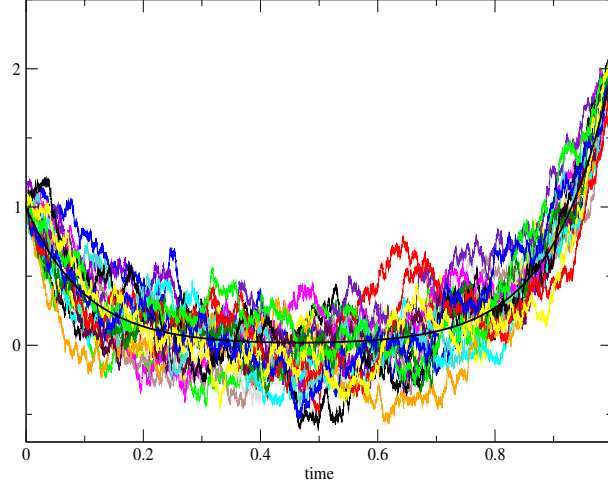


Figure 1: A sample of diffusions satisfying the Ito stochastic differential equation Eq. 85. Each color corresponds to the realization of one process. The thick black curve is the average profile of the stochastic process as given by equation 87. The time step used in the discretization is $dt = 10^{-4}$. Due to the killing rate $\frac{\omega^2}{4D}x^2$, trajectories that are likely to survive spend most of their time near $x = 0$.

Figure 1 shows a set of 20 realizations of the process with parameter $\omega = 10$ as well as the mean trajectory $\langle X^*(t) \rangle$. This last quantity is obtained by averaging the preceding equation over the realizations. Since $\langle dW(t) \rangle = 0$, we have

$$\frac{d\langle X^*(t) \rangle}{dt} = \frac{\omega}{\sinh[\omega(T-t)]} y^* - \frac{\omega}{\tanh[\omega(T-t)]} \langle X^*(t) \rangle \quad (86)$$

Solving the linear Eq. 86 is straightforward and we get

$$\langle X^*(t) \rangle = y^* \frac{\sinh(\omega t)}{\sinh(\omega T)} + x_0 \frac{\sinh(\omega(T-t))}{\sinh(\omega T)} \quad (87)$$

C. Conditioning towards the surviving distribution $P^*(\vec{y}, T)$ at the horizon T alone

If one wishes to impose only the probability $P^*(\vec{y}, T)$ at time T , together with its corresponding survival probability $S^*(T) = \int d^d \vec{y} P^*(\vec{y}, T)$, the function of Eq. 36 reads

$$\begin{aligned} Q_T^{[P^*(\cdot, T); S^*(T)]}(\vec{x}, t) &= [1 - S^*(T)] \left(\frac{1 - S(T|x, t)}{1 - S(T|\vec{x}_0, 0)} \right) + \int d^d \vec{y} P^*(\vec{y}, T) \frac{P(\vec{y}, T|\vec{x}, t)}{P(\vec{y}, T|\vec{x}_0, 0)} \\ &= [1 - S^*(T)] \left(\frac{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega(T-t)]} e^{-\frac{\omega \tanh[\omega(T-t)]}{4D} x^2}}{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega T]} e^{-\frac{\omega \tanh[\omega T]}{4D} \vec{x}_0^2}} \right) \\ &+ \int d^d \vec{y} P^*(\vec{y}, T) \left(\frac{\sinh[\omega T]}{\sinh[\omega(T-t)]} \right)^{\frac{d}{2}} e^{\frac{\omega(\vec{y}^2 + \vec{x}_0^2)}{4D \tanh[\omega T]} - \frac{\omega(\vec{y}^2 + \vec{x}^2)}{4D \tanh[\omega(T-t)]} + \frac{\omega \vec{y} \cdot \vec{x}}{2D \sinh[\omega(T-t)]} - \frac{\omega \vec{y} \cdot \vec{x}_0}{2D \sinh[\omega T]} \end{aligned} \quad (88)$$

D. Conditioning towards the killing distribution $K^*(\vec{x}, t)$ for $t \in [0, T]$ alone

If one wishes to impose only the killing distribution $K^*(\vec{x}, t)$ for $t \in [0, T]$ alone, together with its normalization $\int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) = 1 - S^*(T)$, the function of Eq. 42 reads

$$\begin{aligned}
Q_T^{[K^*(\dots); S^*(T)]}(\vec{x}, t) &= \int_t^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \frac{P(\vec{x}_d, t_d | \vec{x}, t)}{P(\vec{x}_d, t_d | \vec{x}_0, 0)} + S^*(T) \left(\frac{S(T | \vec{x}, t)}{S(T | \vec{x}_0, 0)} \right) \\
&= \int_t^T dt_d \int d^d \vec{x}_d K^*(\vec{x}_d, t_d) \left(\frac{\sinh[\omega t_d]}{\sinh[\omega(t_d - t)]} \right)^{\frac{d}{2}} e^{\frac{\omega(\vec{x}_d^2 + \vec{x}_0^2)}{4D \tanh[\omega t_d]} - \frac{\omega(\vec{x}_d^2 + \vec{x}^2)}{4D \tanh[\omega(t_d - t)]} + \frac{\omega \vec{x}_d \cdot \vec{x}}{2D \sinh[\omega(t_d - t)]} - \frac{\omega \vec{x}_d \cdot \vec{x}_0}{2D \sinh[\omega t_d]} \\
&+ S^*(T) \frac{\cosh^{\frac{d}{2}}[\omega T]}{\cosh^{\frac{d}{2}}[\omega(T - t)]} e^{\frac{\omega \tanh[\omega T]}{4D} \vec{x}_0^2 - \frac{\omega \tanh[\omega(T - t)]}{4D} \vec{x}^2}
\end{aligned} \tag{89}$$

E. Conditioning towards the surviving probability $S^*(T)$ at time T alone

If one wishes to impose only the value $S^*(T)$ of the conditioned survival probability at time T , the function of Eq. 48 reads

$$\begin{aligned}
Q_T^{[S^*(T)]}(\vec{x}, t) &= [1 - S^*(T)] \left(\frac{1 - S(T | x, t)}{1 - S(T | \vec{x}_0, 0)} \right) + S^*(T) \left(\frac{S(T | \vec{x}, t)}{S(T | \vec{x}_0, 0)} \right) \\
&= [1 - S^*(T)] \left(\frac{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega(T - t)]} e^{-\frac{\omega \tanh[\omega(T - t)]}{4D} \vec{x}^2}}{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega T]} e^{-\frac{\omega \tanh[\omega T]}{4D} \vec{x}_0^2}} \right) + S^*(T) \frac{\cosh^{\frac{d}{2}}[\omega T]}{\cosh^{\frac{d}{2}}[\omega(T - t)]} e^{\frac{\omega \tanh[\omega T]}{4D} \vec{x}_0^2 - \frac{\omega \tanh[\omega(T - t)]}{4D} \vec{x}^2}
\end{aligned} \tag{90}$$

1. *Special case : conditioning towards the survival probability $S^*(T) = 1$ at the finite horizon T*

For the special case where one wishes to impose the survival with probability unity $S^*(T) = 1$ at the finite horizon T , the conditioned killing rate of Eq. 49 of course vanishes

$$k_T^*(\vec{x}, t) = 0 \tag{91}$$

while the function of Eq. 90 reduces to

$$Q_T^{[S^*(T)=1]}(\vec{x}, t) = \frac{\cosh^{\frac{d}{2}}[\omega T]}{\cosh^{\frac{d}{2}}[\omega(T - t)]} e^{\frac{\omega \tanh[\omega T]}{4D} \vec{x}_0^2 - \frac{\omega \tanh[\omega(T - t)]}{4D} \vec{x}^2} \tag{92}$$

The corresponding conditioned drift of Eq. 23 reduces to

$$\vec{\mu}_T^*(\vec{x}, t) = 2D \vec{\nabla} \ln Q_T^{[S^*(T)=1]}(\vec{x}, t) = -\omega \tanh[\omega(T - t)] \vec{x} \tag{93}$$

in agreement with [40]. In the limit of the infinite horizon $T \rightarrow +\infty$, the conditioned drift

$$\vec{\mu}_\infty^*(\vec{x}) = -\omega \vec{x} \tag{94}$$

corresponds to the Ornstein-Uhlenbeck process. In one dimension, the conditioned process satisfies the Ito stochastic differential equation

$$dX^*(t) = -\omega \tanh[\omega(T - t)] X^*(t) dt + dW(t) \tag{95}$$

Figure 2 shows a set of 20 realizations of the process with parameter $\omega = 10$ as well as the mean trajectory $\langle X^*(t) \rangle$. This quantity is obtained by averaging the preceding equation over the realizations. Since $\langle dW(t) \rangle = 0$, we have

$$\frac{d\langle X^*(t) \rangle}{dt} = -\omega \tanh[\omega(T - t)] \langle X^*(t) \rangle \tag{96}$$

Solving Eq. 96 is straightforward and we get

$$\langle X^*(t) \rangle = x_0 \frac{\cosh(\omega(T-t))}{\cosh(\omega T)} \quad (97)$$

The conditioning towards $S^*(T) = 1$ is discussed in detail in [40] for the more general case of Gaussian diffusions with quadratic killing rates with time-dependent parameters.

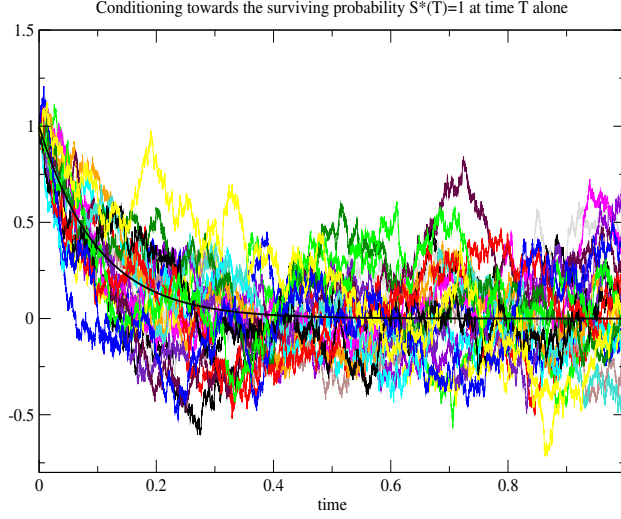


Figure 2: A sample of 20 diffusions satisfying the Ito stochastic differential equation Eq. 95 with $\omega = 10$. Each color corresponds to the realization of one process. The thick black curve is the average profile of the stochastic process as given by equation 87. The time step used in the discretization is $dt = 10^{-4}$.

2. Special case : conditioning towards the survival probability $S^*(T) = 0$ at the finite horizon T

For the special case where one wishes to impose the survival with probability $S^*(T) = 0$ at the finite horizon T , the function of Eq. 90 reduces to

$$Q_T^{[S^*(T)=0]}(\vec{x}, t) = \frac{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega(T-t)]} e^{-\frac{\omega \tanh[\omega(T-t)]}{4D} \vec{x}^2}}{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega T]} e^{-\frac{\omega \tanh[\omega T]}{4D} \vec{x}_0^2}} \quad (98)$$

The corresponding conditioned drift of Eq. 23 reads

$$\vec{\mu}_T^*(\vec{x}, t) = 2D\vec{\nabla} \ln Q_T^{[S^*(T)=0]}(\vec{x}, t) = \frac{\omega \tanh[\omega(T-t)]}{\cosh^{\frac{d}{2}}[\omega(T-t)] e^{-\frac{\omega \tanh[\omega(T-t)]}{4D} \vec{x}^2} - 1} \vec{x} \quad (99)$$

while the conditioned killing rate of Eq. 55 is given by

$$k_T^*(\vec{x}, t) = \frac{k(\vec{x})}{1 - S(T|\vec{x}, t)} = \frac{\frac{\omega^2}{4D} \vec{x}^2}{1 - \frac{1}{\cosh^{\frac{d}{2}}[\omega(T-t)]} e^{-\frac{\omega \tanh[\omega(T-t)]}{4D} \vec{x}^2}} \quad (100)$$

Observe that when $\omega \rightarrow 0$ the conditioned process tends to a limit process whose parameters are given by

$$\lim_{\omega \rightarrow 0} \vec{\mu}_T^*(\vec{x}, t) = \frac{4D\vec{x}}{\vec{x}^2 + Dd(T-t)} \quad (101)$$

and

$$\lim_{\omega \rightarrow 0} k_T^*(\vec{x}, t) = \frac{\vec{x}^2}{(T-t)(\vec{x}^2 + Dd(T-t))} \quad (102)$$

Observe also that when $t \rightarrow T$, the killing rate Eq. 100 diverges as $\sim 1/(T-t)$, ensuring that the process cannot survive at times greater than T .

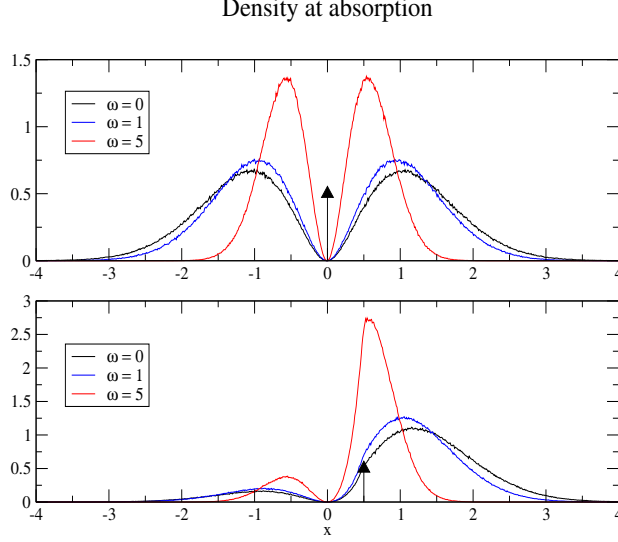


Figure 3: Profile of the density at absorption for the process with drift and killing rate given by the Eqs. 99 and 100 with different intensities of the killing rate. The arrow indicates the starting position of the process: upper figure $x_0 = 0$ (symmetrical case), lower figure $x_0 = 0.5$. Due to the killing rate $\sim x^2$, no absorption occurs at $x = 0$, as expected.

VI. APPLICATION TO THE BROWNIAN WITH UNIFORM DRIFT AND DELTA KILLING RATE

In this section, the general construction is applied to the one-dimensional Brownian with uniform drift when the killing rate is a delta function localized at the origin $x = 0$.

A. Unconditioned process $X(t)$: Brownian with uniform drift $\mu(x) = \mu \geq 0$ and delta killing rate $k(x) = k\delta(x)$

The unconditioned process is the one-dimensional Brownian with uniform drift $\mu(x) = \mu \geq 0$, with uniform diffusion coefficient $D(x) = \frac{1}{2}$, while the killing rate is a delta function of amplitude $k > 0$ localized at the origin $x = 0$

$$k(x) = k\delta(x) \quad (103)$$

The amplitude $k \in]0, +\infty[$ allows to interpolate between the no-killing case $k = 0$ and the absorbing condition at the origin that can be recovered in the limit of infinite amplitude $k \rightarrow +\infty$.

1. Reminder on the unconditioned propagator $P(x, t|x_0, t_0)$

The forward dynamics of Eq. 5 for the propagator $P(x, t|x_0, t_0)$

$$\partial_t P(x, t|x_0, t_0) = -\mu \partial_x P(x, t|x_0, t_0) + \frac{1}{2} \partial_x^2 P(x, t|x_0, t_0) - k\delta(x)P(0, t|x_0, t_0) \quad (104)$$

can be translated for the time Laplace transform

$$\tilde{P}_s(x|x_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} P(x, t|x_0, t_0) \quad (105)$$

into

$$-\delta(x-x_0) + s\tilde{P}_s(x|x_0) = -\mu\partial_x\tilde{P}_s(x|x_0) + \frac{1}{2}\partial_x^2\tilde{P}_s(x|x_0) - k\delta(x)\tilde{P}_s(0|x_0) \quad (106)$$

When there is no killing $k=0$, the free gaussian propagator

$$G(x, t|x_0, t_0) \equiv \frac{1}{\sqrt{2\pi(t-t_0)}} e^{-\frac{[x-x_0-\mu(t-t_0)]^2}{2(t-t_0)}} = \frac{1}{\sqrt{2\pi(t-t_0)}} e^{-\frac{(x-x_0)^2}{2(t-t_0)} + \mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)} \quad (107)$$

has for Laplace transform

$$\begin{aligned} \tilde{G}_s(x|x_0) &\equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G(x, t|x_0, t_0) = \frac{e^{\mu(x-x_0)}}{\sqrt{2\pi}} \int_0^{+\infty} d\tau \tau^{-\frac{1}{2}} e^{-\left(s+\frac{\mu^2}{2}\right)\tau} e^{-\frac{(x-x_0)^2}{2\tau}} \\ &= \frac{e^{\mu(x-x_0) - \sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} \end{aligned} \quad (108)$$

that solves Eq. 106 for $k=0$. For $k \neq 0$, the solution $\tilde{P}_s(x|x_0)$ can be obtained from this free solution $\tilde{G}_s(x|x_0)$ via the resummation of the Dyson perturbative series in the parameter k

$$\begin{aligned} \tilde{P}_s(x|x_0) &= \tilde{G}_s(x|x_0) - k\tilde{G}_s(x|0)\tilde{G}_s(0|x_0) + k^2\tilde{G}_s(x|0)\tilde{G}_s(0|0)\tilde{G}_s(0|x_0) \\ &\quad - k^3\tilde{G}_s(x|0)\tilde{G}_s(0|0)\tilde{G}_s(0|0)\tilde{G}_s(0|x_0) + \dots \\ &= \tilde{G}_s(x|x_0) - \tilde{G}_s(x|0) \frac{k}{1+k\tilde{G}_s(0|0)} \tilde{G}_s(0|x_0) \end{aligned} \quad (109)$$

and thus reads using the explicit expression of Eq. 108

$$\begin{aligned} \tilde{P}_s(x|x_0) &= \frac{e^{\mu(x-x_0) - \sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} - \frac{e^{\mu x - \sqrt{\mu^2+2s}|x|}}{\sqrt{\mu^2+2s}} \left(\frac{k}{1 + \frac{k}{\sqrt{\mu^2+2s}}} \right) \frac{e^{-\mu x_0 - \sqrt{\mu^2+2s}|x_0|}}{\sqrt{\mu^2+2s}} \\ &= e^{\mu(x-x_0)} \left[\frac{e^{-\sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} - \frac{e^{-\sqrt{\mu^2+2s}(|x|+|x_0|)}}{\sqrt{\mu^2+2s}} \left(\frac{k}{k + \sqrt{\mu^2+2s}} \right) \right] \end{aligned} \quad (110)$$

It is useful to write the last factor as an integral over another variable z

$$\begin{aligned} \tilde{P}_s(x|x_0) &= e^{\mu(x-x_0)} \left[\frac{e^{-\sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} - k \frac{e^{-\sqrt{\mu^2+2s}(|x|+|x_0|)}}{\sqrt{\mu^2+2s}} \int_0^{+\infty} dz e^{-(k+\sqrt{\mu^2+2s})z} \right] \\ &= e^{\mu(x-x_0)} \left[\frac{e^{-\sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} - k \int_0^{+\infty} dz e^{-kz} \frac{e^{-\sqrt{\mu^2+2s}(|x|+|x_0|+z)}}{\sqrt{\mu^2+2s}} \right] \end{aligned} \quad (111)$$

in order to rewrite the second contribution of Eq. 110 in terms of the free propagator of Eq. 108 at another argument

$$\tilde{G}_s(|x|+|x_0|+z|0) = \frac{e^{(\mu-\sqrt{\mu^2+2s})(|x|+|x_0|+z)}}{\sqrt{\mu^2+2s}} \quad (112)$$

to obtain

$$\tilde{P}_s(x|x_0) = \tilde{G}_s(x|x_0) - k \int_0^{+\infty} dz e^{-kz} e^{\mu(x-x_0)} e^{-\mu(|x|+|x_0|+z)} \tilde{G}_s(|x|+|x_0|+z|0) \quad (113)$$

The Laplace inversion is now straightforward and one obtains that the propagator $P(x, t|x_0, t_0)$ reads in terms of the Gaussian propagator $G(x, t|x_0, t_0)$ of Eq. 107

$$\begin{aligned}
P(x, t|x_0, t_0) &= G(x, t|x_0, t_0) - ke^{\mu(x-|x|)-\mu(x_0+|x_0|)} \int_0^{+\infty} dz e^{-(k+\mu)z} G(|x| + |x_0| + z, t - t_0|0, 0) \\
&= \frac{e^{-\frac{(x-x_0)^2}{2(t-t_0)} + \mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} - ke^{\mu(x-|x|)-\mu(x_0+|x_0|)} \int_0^{+\infty} dz e^{-(k+\mu)z} \frac{e^{-\frac{(|x|+|x_0|+z)^2}{2(t-t_0)} + \mu(|x|+|x_0|+z) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \\
&= \frac{e^{\mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|x|+|x_0|+z)^2}{2(t-t_0)}} \right] \\
&= \frac{e^{\mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - k \sqrt{\frac{\pi(t-t_0)}{2}} e^{\frac{k}{2}(2(|x|+|x_0|)+k(t-t_0))} \operatorname{erfc} \left(\frac{|x| + |x_0| + k(t-t_0)}{\sqrt{2(t-t_0)}} \right) \right] \quad (114)
\end{aligned}$$

with the complementary Error function $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x du e^{-u^2}$.

2. Survival probability $S(T|x, t)$

The survival probability of Eq. 6 can be obtained via the integration of Eq. 114 over x_T

$$\begin{aligned}
S(T|x, t) &= \int_{-\infty}^{+\infty} dx_T P(x_T, T|x, t) = \int_{-\infty}^{+\infty} dx_T \frac{e^{\mu(x_T-x) - \frac{\mu^2}{2}(T-t)}}{\sqrt{2\pi(T-t)}} \left[e^{-\frac{(x_T-x)^2}{2(T-t)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|x_T|+|x|+z)^2}{2(T-t)}} \right] \\
&= 1 - k \frac{e^{-\mu x - \frac{\mu^2}{2}(T-t)}}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} dx_T e^{\mu x_T} \int_0^{+\infty} dz e^{-kz - \frac{x_T^2}{2(T-t)} - \frac{2|x_T|(|x|+z)}{2(T-t)} - \frac{(|x|+z)^2}{2(T-t)}} \\
&= 1 - \frac{k}{2} e^{-\mu x + k|x| + \frac{1}{2}(k^2 - \mu^2)(T-t)} \int_{-\infty}^{+\infty} dx_T e^{\mu x_T + k|x_T|} \operatorname{erfc} \left(\frac{|x| + |x_T| + k(T-t)}{\sqrt{2(T-t)}} \right) \quad (115)
\end{aligned}$$

It is thus more convenient to return to the propagator Laplace transform of Eq. 110 where the integration over x_T can be explicitly performed in order to obtain the following Laplace transform $\tilde{S}_s(x)$ of the survival probability $S(T|x, t)$

$$\begin{aligned}
\tilde{S}_s(x) &\equiv \int_t^{+\infty} dT e^{-s(T-t)} S(T|x, t) = \int_{-\infty}^{+\infty} dx_T \tilde{P}_s(x_T|x) \\
&= \frac{e^{-\mu x}}{\sqrt{\mu^2 + 2s}} \int_{-\infty}^{+\infty} dx_T e^{\mu x_T - \sqrt{\mu^2 + 2s}|x_T - x|} - \frac{ke^{-\mu x - \sqrt{\mu^2 + 2s}|x|}}{(k + \sqrt{\mu^2 + 2s})\sqrt{\mu^2 + 2s}} \int_{-\infty}^{+\infty} dx_T e^{\mu x_T - \sqrt{\mu^2 + 2s}|x_T|} \\
&= \frac{e^{-\mu x}}{\sqrt{\mu^2 + 2s}} \left[e^{-\sqrt{\mu^2 + 2s}x} \int_{-\infty}^x dx_T e^{(\mu + \sqrt{\mu^2 + 2s})x_T} + e^{\sqrt{\mu^2 + 2s}x} \int_x^{+\infty} dx_T e^{(\mu - \sqrt{\mu^2 + 2s})x_T} \right] \\
&\quad - \frac{ke^{-\mu x - \sqrt{\mu^2 + 2s}|x|}}{(k + \sqrt{\mu^2 + 2s})\sqrt{\mu^2 + 2s}} \left[\int_{-\infty}^0 dx_T e^{(\mu + \sqrt{\mu^2 + 2s})x_T} + \int_0^{+\infty} dx_T e^{(\mu - \sqrt{\mu^2 + 2s})x_T} \right] \\
&= \frac{e^{-\mu x}}{\sqrt{\mu^2 + 2s}} \left[e^{-\sqrt{\mu^2 + 2s}x} \frac{e^{(\mu + \sqrt{\mu^2 + 2s})x}}{\sqrt{\mu^2 + 2s} + \mu} + e^{\sqrt{\mu^2 + 2s}x} \frac{e^{(\mu - \sqrt{\mu^2 + 2s})x}}{\sqrt{\mu^2 + 2s} - \mu} \right] \\
&\quad - \frac{ke^{-\mu x - \sqrt{\mu^2 + 2s}|x|}}{(k + \sqrt{\mu^2 + 2s})\sqrt{\mu^2 + 2s}} \left[\frac{1}{\sqrt{\mu^2 + 2s} + \mu} + \frac{1}{\sqrt{\mu^2 + 2s} - \mu} \right] \\
&= \frac{1}{s} \left[1 - \frac{ke^{-\mu x - \sqrt{\mu^2 + 2s}|x|}}{(k + \sqrt{\mu^2 + 2s})} \right] \quad (116)
\end{aligned}$$

This expression allows to obtain the unconditioned survival probability for large time $T \rightarrow +\infty$ as discussed in the two following subsections.

3. Unconditioned forever-survival probability $S(\infty|x)$

In Eq. 116, the coefficient of $\frac{1}{s}$ as $s \rightarrow 0$ corresponds to the forever survival probability $S(\infty|x)$

$$\tilde{S}_s(x) \underset{s \rightarrow 0}{\simeq} \frac{S(\infty|x)}{s} \quad (117)$$

so one obtains using the positivity of the drift $\mu \geq 0$

$$S(\infty|x) = 1 - \frac{ke^{-\mu(x+|x|)}}{k+\mu} = \begin{cases} 1 - \frac{ke^{-2\mu x}}{k+\mu} & \text{if } x \geq 0 \\ \frac{\mu}{k+\mu} & \text{if } x \leq 0 \end{cases} \quad (118)$$

This result can also be found directly by solving the backward dynamics of Eq. 11

$$0 = \mathcal{F}S(\infty|x) = \mu \partial_x S(\infty|x) + \frac{1}{2} \partial_x^2 S(\infty|x) - k\delta(x)S(\infty|0) \quad (119)$$

The physical meaning of the result of Eq. 118 can be understood as follows :

(i) for vanishing drift $\mu = 0$, the Brownian motion returns an infinite number of times at the origin $x = 0$ where it will be eventually killed, so the forever survival probability vanishes for any starting point x

$$S(\infty|x) = 0 \quad \text{for } \mu = 0 \quad (120)$$

(ii) for strictly positive drift $\mu > 0$, the Brownian motion returns only a finite number of times to the origin $x = 0$ before flowing towards ($x \rightarrow +\infty$), so the forever-survival probability $S(\infty|x)$ remains finite for any starting point x . When the initial condition is negative $x < 0$, the particle has to cross the origin at least once, so the survival probability is simply equal to the survival when one starts from the origin :

$$S(\infty|x < 0) = S(\infty|x = 0) \quad \text{for } \mu > 0 \quad (121)$$

When the initial condition is positive $x > 0$, the particle can escape towards ($+\infty$) without ever touching the origin $x = 0$ with the probability

$$p_{\text{escape}}(x) = 1 - e^{-2\mu x} \quad (122)$$

while it will touch the origin $x = 0$ at least once with the complementary probability $(1 - p_{\text{escape}}(x)) = e^{-2\mu x}$. As a consequence, the result of the survival probability in Eq. 118 can be understood as

$$S(\infty|x > 0) = p_{\text{escape}}(x) + [1 - p_{\text{escape}}(x)] S(\infty|x = 0) \quad \text{for } \mu > 0 \quad (123)$$

4. Unconditioned survival probability $S(T|.,.)$ for large T when the drift vanishes $\mu = 0$

When the drift vanishes $\mu = 0$, the unconditioned survival probability of Eq. 120 vanishes $S(+\infty|.) = 0$. In order to obtain the asymptotic behavior of $S(T|x, t)$ for large time T , one should compute the leading order of the Laplace transform of Eq. 116 for small $s \rightarrow 0^+$

$$\begin{aligned} \tilde{S}_s^{[\mu=0]}(x) &= \frac{1}{s} \left[1 - \frac{ke^{-\sqrt{2s}|x|}}{(k + \sqrt{2s})} \right] = \frac{1}{s} \left[1 - \frac{1}{1 + \frac{\sqrt{2s}}{k}} e^{-\sqrt{2s}|x|} \right] = \frac{1}{s} \left[1 - \left(1 - \frac{\sqrt{2s}}{k} + O(s) \right) \left(1 - \sqrt{2s}|x| + O(s) \right) \right] \\ &= \sqrt{\frac{2}{s}} \left[\frac{1}{k} + |x| \right] + O(s^0) \end{aligned} \quad (124)$$

The Laplace inversion yields that the leading order of the survival probability $S(T|x, t)$ when the time interval $(T - t)$ is large is given by

$$S(T|x, t) \underset{(T-t) \rightarrow +\infty}{\simeq} \sqrt{\frac{2}{\pi(T-t)}} \left[\frac{1}{k} + |x| \right] \quad (125)$$

that generalizes the survival probability for an absorbing condition at the origin that corresponds to $k \rightarrow +\infty$.

B. Full conditioning constraints $[P^*(\cdot, T); K^*(\cdot)]$ associated to the finite horizon T

Using the propagator of Eq. 114, the function $Q_T(x, t)$ of Eq. 18 reads

$$\begin{aligned}
Q_T(x, t) &= \int_t^T dt_d K^*(t_d) \frac{P(0, t_d | x, t)}{P(0, t_d | x_0, 0)} + \int_{-\infty}^{+\infty} dy P^*(y, T) \frac{P(y, T | x, t)}{P(y, T | x_0, 0)} \\
&= \int_t^T dt_d K^*(t_d) \sqrt{\frac{t_d}{t_d - t}} e^{\mu(x_0 - x) + \frac{\mu^2}{2} t} \left[\frac{e^{-\frac{x^2}{2(t_d - t)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|x| + |z|)^2}{2(t_d - t)}}}{e^{-\frac{x_0^2}{2t_d}} - k \int_0^{+\infty} dz_0 e^{-kz_0 - \frac{(|x_0| + |z_0|)^2}{2t_d}}} \right] \\
&+ \int_{-\infty}^{+\infty} dy P^*(y, T) \sqrt{\frac{T}{T - t}} e^{\mu(x_0 - x) + \frac{\mu^2}{2} t} \left[\frac{e^{-\frac{(y-x)^2}{2(T-t)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|y| + |x| + |z|)^2}{2(T-t)}}}{e^{-\frac{(y-x_0)^2}{2T}} - k \int_0^{+\infty} dz_0 e^{-kz_0 - \frac{(|y| + |x_0| + |z_0|)^2}{2T}}} \right] \\
&= \int_t^T dt_d K^*(t_d) \sqrt{\frac{t_d}{t_d - t}} e^{\mu(x_0 - x) + \frac{\mu^2}{2} t} \left[\frac{e^{-\frac{x^2}{2(t_d - t)}} - k \sqrt{\frac{\pi(t_d - t)}{2}} e^{\frac{1}{2} k (2|x| + k(t_d - t))} \operatorname{erfc} \left(\frac{|x| + k(t_d - t)}{\sqrt{2(t_d - t)}} \right)}{e^{-\frac{x_0^2}{2t_d}} - k \sqrt{\frac{\pi t_d}{2}} e^{\frac{1}{2} k (2|x_0| + kt_d)} \operatorname{erfc} \left(\frac{|x_0| + kt_d}{\sqrt{2t_d}} \right)} \right] \\
&+ \int_{-\infty}^{+\infty} dy P^*(y, T) \sqrt{\frac{T}{T - t}} e^{\mu(x_0 - x) + \frac{\mu^2}{2} t} \left[\frac{e^{-\frac{(y-x)^2}{2(T-t)}} - k \sqrt{\frac{\pi(T-t)}{2}} e^{\frac{1}{2} k (2(|x| + |y|) + k(T-t))} \operatorname{erfc} \left(\frac{|x| + |y| + k(T-t)}{\sqrt{2(T-t)}} \right)}{e^{-\frac{(y-x_0)^2}{2T}} - k \sqrt{\frac{\pi T}{2}} e^{\frac{1}{2} k (2(|x_0| + |y|) + kT)} \operatorname{erfc} \left(\frac{|x_0| + |y| + kT}{\sqrt{2T}} \right)} \right]
\end{aligned} \tag{126}$$

Example of the bridge without being killed : $P^(y, T) = \delta(y - y_*)$*

For the case where one imposes the full survival at time T at the single position y_*

$$\begin{aligned}
P^*(y, T) &= \delta(y - y_*) \\
K^*(t_d) &= 0
\end{aligned} \tag{127}$$

Eq. 126 reduces to

$$\begin{aligned}
Q_T(x, t) &= \sqrt{\frac{T}{T - t}} e^{\mu(x_0 - x) + \frac{\mu^2}{2} t} \left[\frac{e^{-\frac{(y_* - x)^2}{2(T-t)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|y_*| + |x| + |z|)^2}{2(T-t)}}}{e^{-\frac{(y_* - x_0)^2}{2T}} - k \int_0^{+\infty} dz_0 e^{-kz_0 - \frac{(|y_*| + |x_0| + |z_0|)^2}{2T}}} \right] \\
&= \sqrt{\frac{T}{T - t}} e^{\mu(x_0 - x) + \frac{\mu^2}{2} t} \left[\frac{e^{-\frac{(y_* - x)^2}{2(T-t)}} - k \sqrt{\frac{\pi}{2(T-t)}} e^{\frac{1}{2} k (2(|y_*| + |x|) + k(T-t))} \operatorname{erfc} \left(\frac{|y_*| + |x| + k(T-t)}{\sqrt{2(T-t)}} \right)}{e^{-\frac{(y_* - x_0)^2}{2T}} - k \sqrt{\frac{\pi}{2T}} e^{\frac{1}{2} k (2(|y_*| + |x_0|) + kT)} \operatorname{erfc} \left(\frac{|y_*| + |x_0| + kT}{\sqrt{2T}} \right)} \right]
\end{aligned} \tag{128}$$

The corresponding conditioned drift of Eq. 23 reads

$$\begin{aligned}
\mu_T^*(x, t) &= \mu + \partial_x \ln Q_T(x, t) = \partial_x \ln \left[e^{-\frac{(y_* - x)^2}{2(T-t)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|y_*| + |x| + |z|)^2}{2(T-t)}} \right] \\
&= \frac{\frac{y_* - x}{T-t} e^{-\frac{(y_* - x)^2}{2(T-t)}} + k \operatorname{sgn}(x) \int_0^{+\infty} dz \frac{|y_*| + |x| + z}{T-t} e^{-kz - \frac{(|y_*| + |x| + |z|)^2}{2(T-t)}}}{e^{-\frac{(y_* - x)^2}{2(T-t)}} - k \int_0^{+\infty} dz e^{-kz - \frac{(|y_*| + |x| + |z|)^2}{2(T-t)}}} \\
&= \frac{\frac{y_* - x}{T-t} e^{-\frac{(y_* - x)^2}{2(T-t)}} + k \operatorname{sgn}(x) \left[e^{-\frac{(|y_*| + |x|)^2}{2(T-t)}} - k \sqrt{\frac{\pi(T-t)}{2}} e^{\frac{1}{2} k (2(|y_*| + |x|) + k(T-t))} \operatorname{erfc} \left(\frac{|y_*| + |x| + k(T-t)}{\sqrt{2(T-t)}} \right) \right]}{e^{-\frac{(y_* - x)^2}{2(T-t)}} - k \sqrt{\frac{\pi}{2(T-t)}} e^{\frac{1}{2} k (2(|y_*| + |x|) + k(T-t))} \operatorname{erfc} \left(\frac{|y_*| + |x| + k(T-t)}{\sqrt{2(T-t)}} \right)}
\end{aligned} \tag{129}$$

C. Conditioning towards the surviving probability $S^*(\infty)$ at the infinite horizon $T = +\infty$

Here one needs to distinguish whether the unconditioned drift μ vanishes or not.

1. Case $\mu = 0$ where the unconditioned forever-survival probability vanishes $S(+\infty|\cdot) = 0$

For vanishing drift $\mu = 0$, the unconditioned survival probability of Eq. 120 vanishes $S(+\infty|\cdot) = 0$, so one should use Eq. 58 with the asymptotic behavior of Eq. 125 to obtain

$$\begin{aligned} Q_\infty^{[S^*(\infty)]}(x, t) &= [1 - S^*(\infty)] + S^*(\infty) \lim_{T \rightarrow +\infty} \left(\frac{S(T|x, t)}{S(T|x_0, 0)} \right) \\ &= [1 - S^*(\infty)] + S^*(\infty) \frac{\frac{1}{k} + |x|}{\frac{1}{k} + |x_0|} \end{aligned} \quad (130)$$

The conditioned drift of Eq. 23 reads

$$\mu_\infty^*(x) = \partial_x \ln Q_\infty^{[S^*(\infty)]}(x) = \operatorname{sgn}(x) \frac{S^*(\infty)}{\frac{1}{k} + |x_0|(1 - S^*(\infty)) + |x|S^*(\infty)} \quad (131)$$

while the conditioned killing rate of Eq. 60 is given by

$$k_\infty^*(x) = \frac{[1 - S^*(\infty)]}{Q_\infty^{[S^*(\infty)]}(0)} k\delta(x) = \frac{[1 - S^*(\infty)]}{[1 - S^*(\infty)] + S^*(\infty) \frac{1}{1+k|x_0|}} k\delta(x) \quad (132)$$

For the special case where the conditioning is towards the forever-survival $S^*(\infty) = 1$, the conditioned killing rate of course vanishes $k^*(x) = 0$, while the drift of Eq. 131 reduces to

$$\mu_\infty^*(x) = \operatorname{sgn}(x) \frac{1}{\frac{1}{k} + |x|} \quad (133)$$

For $x > 0$, one recovers the Bessel drift $\mu_{Bessel}(x) = \frac{1}{x}$ in the limit $k \rightarrow +\infty$.

2. Case $\mu > 0$ where the unconditioned forever-survival probability is finite $S(+\infty|\cdot) > 0$

For strictly positive drift $\mu > 0$, the unconditioned survival probability of Eq. 118 remains finite $S(+\infty|\cdot) > 0$, so one should use Eq. 59 with Eq. 118

$$\begin{aligned} Q_\infty^{[S^*(\infty)]}(x) &= (1 - S^*(\infty)) \left(\frac{1 - S(\infty|x)}{1 - S(\infty|x_0)} \right) + S^*(\infty) \left(\frac{S(\infty|x)}{S(\infty|x_0)} \right) \\ &= (1 - S^*(\infty)) e^{\mu(x_0+|x_0|) - \mu(x+|x|)} + S^*(\infty) \left[\frac{\mu + k(1 - e^{-\mu(x+|x|)})}{\mu + k(1 - e^{-\mu(x_0+|x_0|)})} \right] \end{aligned} \quad (134)$$

The conditioned drift of Eq. 23 reads

$$\begin{aligned} \mu_\infty^*(x) &= \mu + \partial_x \ln Q_\infty^{[S^*(\infty)]}(x) \\ &= \mu - \mu(1 + \operatorname{sgn}(x)) \frac{(1 - S^*(\infty)) e^{\mu(x_0+|x_0|) - \mu(x+|x|)} + S^*(\infty) \left[\frac{-k e^{-\mu(x+|x|)}}{\mu + k(1 - e^{-\mu(x_0+|x_0|)})} \right]}{(1 - S^*(\infty)) e^{\mu(x_0+|x_0|) - \mu(x+|x|)} + S^*(\infty) \left[\frac{\mu + k(1 - e^{-\mu(x+|x|)})}{\mu + k(1 - e^{-\mu(x_0+|x_0|)})} \right]} \\ &= \begin{cases} \mu & \text{if } x < 0 \\ \mu - 2\mu \frac{(1 - S^*(\infty)) e^{\mu(x_0+|x_0|)} + S^*(\infty) \left[\frac{-k}{\mu + k(1 - e^{-\mu(x_0+|x_0|)})} \right]}{(1 - S^*(\infty)) e^{\mu(x_0+|x_0|)} + S^*(\infty) \left[\frac{(\mu+k)e^{2\mu x - k}}{\mu + k(1 - e^{-\mu(x_0+|x_0|)})} \right]} & \text{if } x > 0 \end{cases} \end{aligned} \quad (135)$$

while the conditioned killing rate of Eq. 60 reads

$$k_\infty^*(x) = \frac{\left(\frac{1 - S^*(\infty)}{1 - S(\infty|x_0)} \right)}{Q_\infty^{[S^*(\infty)]}(0)} k\delta(x) = \frac{[1 - S^*(\infty)](k + \mu)}{[1 - S^*(\infty)] + S^*(\infty) \left[\frac{\mu e^{-\mu(x_0+|x_0|)}}{\mu + k(1 - e^{-\mu(x_0+|x_0|)})} \right]} \delta(x) \quad (136)$$

Let us mention the two special cases :

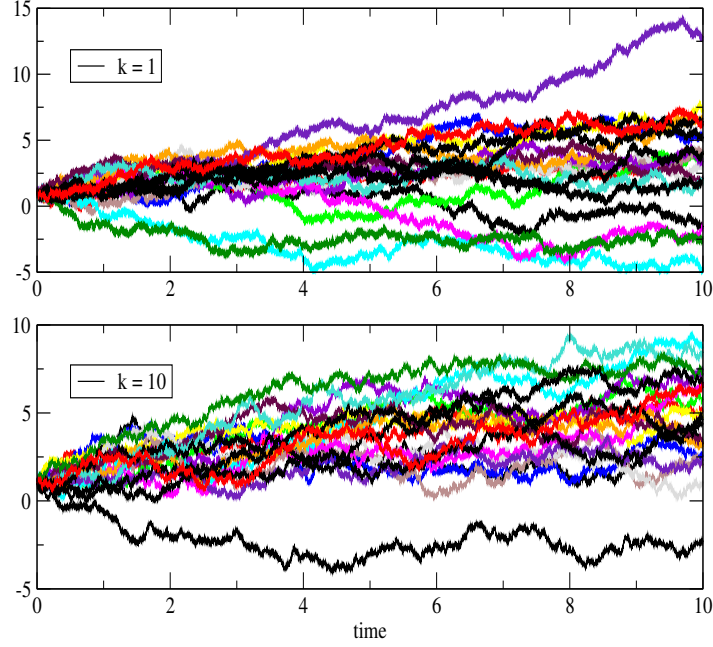


Figure 4: A sample of 20 diffusions satisfying the drift given by Eq. 133 for two different values of the absorbing parameter k . Each color corresponds to the realization of one process. The time step used in the discretization is $dt = 10^{-4}$. For a parameter k close to zero the process behaves like a free Brownian motion (top figure) then as k increases the process behaves more and more like a Bessel process (bottom figure).

- (i) When the conditioning is towards full-survival $S^*(\infty) = 1$, the conditioned drift of Eq. 135 reduces to

$$\mu_{\infty}^*(x) = \mu \left[1 + \frac{k(1 + \text{sgn}(x))}{(\mu + k)e^{\mu(x+|x|)} - k} \right] = \begin{cases} \mu & \text{if } x < 0 \\ \mu \left[1 + \frac{2k}{(\mu+k)e^{2\mu x} - k} \right] & \text{if } x > 0 \end{cases} \quad (137)$$

- (ii) When the conditioning is towards zero-survival $S^*(\infty) = 0$, the conditioned drift of Eq. 135 reduces to

$$\mu_{\infty}^*(x) = -\mu \text{sgn}(x) = \begin{cases} \mu & \text{if } x < 0 \\ -\mu & \text{if } x > 0 \end{cases} \quad (138)$$

while the conditioned killing rate of Eq. 136 reads

$$k_{\infty}^*(x) = (k + \mu)\delta(x) \quad (139)$$

It is worth noticing that the stochastic process described by Eq. 138, without killing, is a Brownian motion with alternating drift or bang-bang process [57, 58] and represents the dynamics of a Brownian particle in a symmetric wedge potential

$$V(x) = \mu|x| \quad \mu > 0 \quad (140)$$

Such a process has been introduced by de Gennes for studying dry friction [59].

VII. CONCLUSION

In this paper, we have revisited the conditioning of diffusion processes with space-dependent killing rates, in order to give a global discussion of the various conditioning constraints that can be imposed for finite horizon T or for infinite

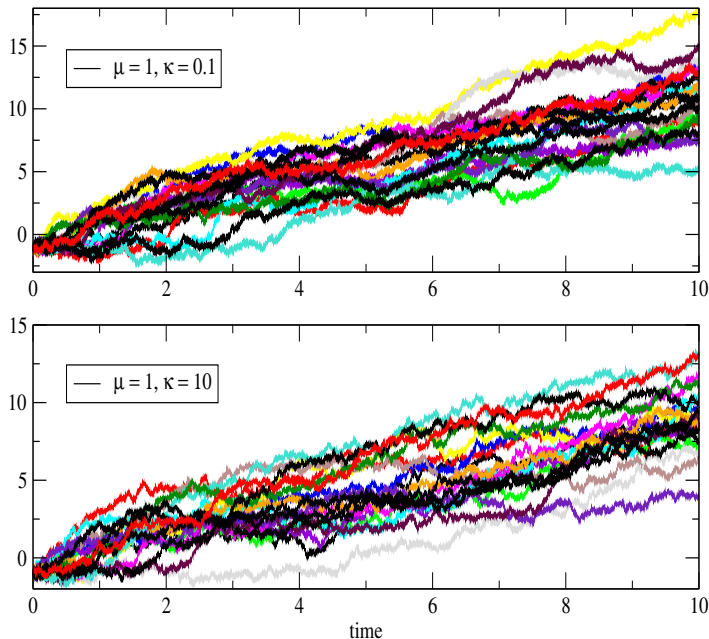


Figure 5: A sample of 20 diffusions satisfying the drift given by Eq. 137 for two different values of the absorbing parameter k . Each color corresponds to the realization of one process. The starting point is -1 and the time step used in the discretization is $dt = 10^{-4}$. Simulations show that the conditioned process is not sensitive to the parameter k (except around $x = 0$). As $X^*(t)$ increases, the process behaves like a simple Brownian motion with drift μ .

horizon $T = +\infty$. Firstly, we have characterized the conditioned process when one imposes both surviving distribution $P^*(\vec{y}, T)$ at time T and the killing-distribution $K^*(\vec{x}_d, t_d)$ for the intermediate times $t_d \in [0, T]$. Secondly, we have focused on cases where the conditioning constraints are less-detailed than these full distributions $[P^*(\cdot, T); K^*(\cdot, \cdot)]$, and we have constructed the appropriate conditioned processes via the optimization of the dynamical large deviations at Level 2.5 in the presence of the conditioning constraints that one wishes to impose. We have also analyzed the consequences for the limit of the infinite horizon $T \rightarrow +\infty$. Finally, we have described the application of this general construction to the pure diffusion in dimension d with the quadratic killing rate $k(\vec{x}) = \gamma \vec{x}^2$, as well as to Brownian motion with uniform drift μ submitted to the delta killing rate $k(x) = k\delta(x)$ localized at the origin $x = 0$, in order to generate stochastic trajectories satisfying various types of conditioning constraints.

Appendix A: Links with the dynamical large deviations at Level 2.5 and the stochastic control theory

In this Appendix, one follows the Schrödinger perspective [45] (see the recent detailed commentary [46] accompanying its english translation as well as in the two reviews [47, 48]). The goal is then to analyze the large deviations properties of a large number N of independent realizations $\vec{X}_n(t)$ of the unconditioned process labelled by $n = 1, 2, \dots, N$ starting all at the same initial condition $\vec{X}_n(0) = \vec{x}_0$.

1. Empirical ensemble-averaged observables for N independent unconditioned processes $X_n(t)$

The ensemble-averaged density $\hat{P}(\vec{x}, t)$ at position \vec{x} and at time t

$$\hat{P}(\vec{x}, t) \equiv \frac{1}{N} \sum_{n=1}^N \delta^d(\vec{X}_n(t) - \vec{x}) \quad (\text{A1})$$

follows some empirical dynamics that can be written as a continuity equation

$$\partial_t \hat{P}(\vec{x}, t) = -\vec{\nabla} \cdot \vec{J}(\vec{x}, t) - \hat{K}(\vec{x}, t) \quad (\text{A2})$$

involving the empirical killing probability $\hat{K}(\vec{x}, t)$ and the empirical current $\vec{J}(\vec{x}, t)$, that can be parametrized in terms of the empirical drift $\vec{\mu}(\vec{x}, t)$, while the diffusion coefficient $D(\vec{x})$ is fixed

$$\vec{J}(\vec{x}, t) = \vec{\mu}(\vec{x}, t) \hat{P}(\vec{x}, t) - \vec{\nabla} \left[D(\vec{x}) \hat{P}(\vec{x}, t) \right] \quad (\text{A3})$$

The normalization of the empirical density $\hat{P}(\vec{x}, t)$ gives the empirical survival probability $\hat{S}(t)$ at time t

$$\hat{S}(t) \equiv \int d^d \vec{x} \hat{P}(\vec{x}, t) \quad (\text{A4})$$

whose time-decay involves the empirical killing probability $\hat{K}(\vec{x}, t)$

$$-\frac{d\hat{S}(t)}{dt} = - \int d^d \vec{x} \partial_t \hat{P}(\vec{x}, t) = \int d^d \vec{x} \hat{K}(\vec{x}, t) \quad (\text{A5})$$

In the thermodynamic limit $N \rightarrow +\infty$, all these empirical observables concentrate on their typical values given by the corresponding observables without hats described in section II of the main text. For large finite N , the dynamical fluctuations around these typical values can be analyzed via the large deviations at Level 2.5, as discussed in the next subsection.

2. Large deviations at Level 2.5 for the empirical dynamics during the time-window $t \in [0, T]$

In the field of large deviation theory (see the reviews [54–56] and references therein), the emergence of the Level 2.5 describing the joint distribution of the empirical density and of the empirical flows has been a major achievement [60–62]. Indeed, in contrast to the Level 2 involving the empirical density alone, the Level 2.5 can be written explicitly for general Markov processes, including discrete-time Markov chains [56, 63–67], continuous-time Markov jump processes [62, 63, 66–80, 82–87] and Diffusion processes [62, 66, 67, 71, 72, 85, 88–90].

In our present setting, the application of the large deviations at Level 2.5 leads to the following conclusion : the joint probability $\mathcal{P}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right]$ to see the empirical density $\hat{P}(\vec{x}, t)$, the empirical current $\vec{J}(\vec{x}, t)$ and the empirical killing distribution $\hat{K}(\vec{x}, t)$ during the time window $0 \leq t \leq T$ follows the large deviation form for large N

$$\mathcal{P}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] \underset{N \rightarrow +\infty}{\simeq} \mathcal{C}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] e^{-N \mathcal{I}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right]} \quad (\text{A6})$$

with the following notations :

(i) the rate function $\mathcal{I}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right]$ at Level 2.5 contains two contributions

$$\begin{aligned} \mathcal{I}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] &= \int_0^T dt \int_{-\infty}^{+\infty} \frac{dx}{4D(\vec{x}) \hat{P}(\vec{x}, t)} \left[\vec{J}(\vec{x}, t) - \vec{\mu}(\vec{x}) \hat{P}(\vec{x}, t) + \vec{\nabla} \left(D(\vec{x}) \hat{P}(\vec{x}, t) \right) \right]^2 \\ &+ \int d^d \vec{x} \int_0^T dt \left[\hat{K}(\vec{x}, t) \ln \left(\frac{\hat{K}(\vec{x}, t)}{k(\vec{x}) \hat{P}(\vec{x}, t)} \right) - \hat{K}(\vec{x}, t) + k(\vec{x}) \hat{P}(\vec{x}, t) \right] \end{aligned} \quad (\text{A7})$$

The first contribution involving the Ito drift $\vec{\mu}(\vec{x})$ and the diffusion coefficient $D(\vec{x})$ corresponds to the usual rate function for diffusion processes [62, 66, 67, 71, 72, 85, 88–90], while the second contribution involving the killing rate $k(\vec{x})$ corresponds to the usual rate function for Markov jump processes [62, 63, 66–80, 82–87]: here the jump empirical flows associated to killing occur from position \vec{x} towards the dead state.

(ii) the constitutive constraints $\mathcal{C}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot) \right]$ at Level 2.5 can be decomposed into the initial condition $\hat{P}(\vec{x}, t = 0) = \delta^d(\vec{x} - \vec{x}_0)$ and the empirical dynamics of Eq. A2

$$\mathcal{C}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{J}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] = \delta(\hat{P}(\vec{x}, 0) - \delta^d(\vec{x} - \vec{x}_0)) \prod_{t \in [0, T]} \prod_{\vec{x}} \left(\partial_t \hat{P}(\vec{x}, t) = -\vec{\nabla} \cdot \vec{J}(\vec{x}, t) - \hat{K}(\vec{x}, t) \right) \quad (\text{A8})$$

The parametrization of Eq. A3 allows to replace the empirical current $\vec{J}(\vec{x}, t)$ by the empirical drift

$$\vec{\mu}(\vec{x}, t) = \frac{\vec{J}(\vec{x}, t) + \vec{\nabla} \left[D(\vec{x}) \hat{P}(\vec{x}, t) \right]}{\hat{P}(\vec{x}, t)} \quad (\text{A9})$$

As a consequence, the large deviations at Level 2.5 of Eq. A6 can be directly translated into the joint probability $\mathcal{P}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right]$ to see the empirical density $\hat{P}(\vec{x}, t)$, the empirical drift $\vec{\mu}(\vec{x}, t)$ and the empirical killing distribution $\hat{K}(\vec{x}, t)$

$$\mathcal{P}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] \underset{N \rightarrow +\infty}{\simeq} \mathcal{C}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] e^{-N \mathcal{I}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right]} \quad (\text{A10})$$

The rate function of Eq. A7 translates into

$$\begin{aligned} \mathcal{I}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] &= \int_0^T dt \int d^d \vec{x} \hat{P}(\vec{x}, t) \frac{\left[\vec{\mu}(\vec{x}, t) - \vec{\mu}(\vec{x}) \right]^2}{4D(\vec{x})} \\ &+ \int d^d \vec{x} \int_0^T dt \left[\hat{K}(\vec{x}, t) \ln \left(\frac{\hat{K}(\vec{x}, t)}{k(\vec{x}) \hat{P}(\vec{x}, t)} \right) - \hat{K}(\vec{x}, t) + k(\vec{x}) \hat{P}(\vec{x}, t) \right] \end{aligned} \quad (\text{A11})$$

while the constitutive constraints of Eq. A8 become

$$\begin{aligned} \mathcal{C}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); \hat{K}(\cdot, \cdot) \right] &= \delta(\hat{P}(\vec{x}, 0) - \delta^d(\vec{x} - \vec{x}_0)) \\ &\times \prod_{t \in [0, T]} \prod_{\vec{x}} \left(\partial_t \hat{P}(\vec{x}, t) + \vec{\nabla} \cdot \left[\vec{\mu}(\vec{x}, t) \hat{P}(\vec{x}, t) \right] - \Delta \left[D(\vec{x}) \hat{P}(\vec{x}, t) \right] + \hat{K}(\vec{x}, t) \right) \end{aligned} \quad (\text{A12})$$

3. Link with the stochastic control theory

In this subsection, one assumes that the empirical density $\hat{P}(\vec{x}, T)$ at time T is given and that the empirical killing probability $\hat{K}(\vec{x}, t)$ is given for $0 \leq t \leq T$

$$\begin{aligned} \hat{P}(\vec{x}, T) &= P^*(\vec{x}, T) \\ \hat{K}(\vec{x}, t) &= K^*(\vec{x}, t) \quad \text{for } t \in [0, T] \end{aligned} \quad (\text{A13})$$

a. Optimization problem

The goal is then to optimize the rate function $\mathcal{I}_{[0,T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); K^*(\cdot, \cdot) \right]$ at Level 2.5 of Eq. A11 over the empirical density $\hat{P}(\vec{x}, t)$ and over the empirical drift $\vec{\mu}(\vec{x}, t)$ at all the intermediate times $t \in]0, T[$, in the presence of the constitutive constraints of Eq. A12 and the supplementary constraints of Eq. A13. These constraints can be separated into :

(i) the boundary conditions for the empirical density $\hat{P}(\cdot, \cdot)$ at the initial time $t = 0$ and at the final time $t = T$

$$\begin{aligned}\hat{P}(\vec{x}, t = 0) &= \delta^d(\vec{x} - \vec{x}_0) \\ \hat{P}(\vec{x}, t = T) &= P^*(\vec{x}, T)\end{aligned}\quad (\text{A14})$$

(ii) the empirical dynamics for $t \in [0, T]$

$$\partial_t \hat{P}(\vec{x}, t) = -\vec{\nabla} \left[\vec{\mu}(\vec{x}, t) \hat{P}(\vec{x}, t) \right] + \Delta \left[D(\vec{x}) \hat{P}(\vec{x}, t) \right] - K^*(\vec{x}, t) \quad (\text{A15})$$

As a consequence, for the intermediate times $t \in]0, T[$, one only needs to optimize the rate function $\mathcal{I}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); K^*(\cdot, \cdot) \right]$ at Level 2.5 of Eq. A11 in the presence of the empirical dynamics (ii), via the introduction of the Lagrangian

$$\mathcal{L} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right] = \mathcal{I}_{[0, T]}^{[2.5]} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot); K^*(\cdot, \cdot) \right] + \mathcal{L}^{Empi} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right] \quad (\text{A16})$$

where the contribution

$$\mathcal{L}^{Empi} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right] \equiv \int_0^T dt \int d^d \vec{x} \psi(\vec{x}, t) \left(\partial_t \hat{P}(\vec{x}, t) + \vec{\nabla} \cdot \left[\vec{\mu}(\vec{x}, t) \hat{P}(\vec{x}, t) \right] - \Delta \left[D(\vec{x}) \hat{P}(\vec{x}, t) \right] + K^*(\vec{x}, t) \right) \quad (\text{A17})$$

contains the Lagrange multiplier $\psi(\vec{x}, t)$ introduced in order to impose the empirical dynamics of Eq. A15.

b. The adjoint-equation method

As usual in stochastic control theory, it is simpler to make some transformation of the Lagrangian before its optimization. In our present case, this amounts to rewrite the Lagrangian contributions of Eq. A17 via integrations by parts, either over time $t \in]0, T[$ using the boundary-conditions of Eq. A14

$$\begin{aligned}\int_0^T dt \psi(\vec{x}, t) \partial_t \hat{P}(\vec{x}, t) &= \left[\psi(\vec{x}, t) \hat{P}(\vec{x}, t) \right]_{t=0}^{t=T} - \int_0^T dt \hat{P}(\vec{x}, t) \partial_t \psi(\vec{x}, t) \\ &= \psi(\vec{x}, T) P^*(\vec{x}, T) - \psi(\vec{x}, 0) \delta^d(\vec{x} - \vec{x}_0) - \int_0^T dt \hat{P}(\vec{x}, t) \partial_t \psi(\vec{x}, t)\end{aligned}\quad (\text{A18})$$

or over space \vec{x}

$$\int d^d \vec{x} \psi(\vec{x}, t) \left(\vec{\nabla} \cdot \left(\vec{\mu}(\vec{x}, t) \hat{P}(\vec{x}, t) \right) - \Delta \left[D(\vec{x}) \hat{P}(\vec{x}, t) \right] \right) = - \int d^d \vec{x} \hat{P}(\vec{x}, t) \left[\vec{\mu}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) \right] \quad (\text{A19})$$

Putting everything together, the Lagrangian of Eq. A17 reads

$$\begin{aligned}\mathcal{L}^{Empi} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right] &= \int d^d \vec{x} \left[\psi(\vec{x}, T) P^*(\vec{x}, T) - \psi(\vec{x}, 0) \delta^d(\vec{x} - \vec{x}_0) - \int_0^T dt \hat{P}(\vec{x}, t) \partial_t \psi(\vec{x}, t) \right] \\ &\quad - \int_0^T dt \int d^d \vec{x} \hat{P}(\vec{x}, t) \left[\vec{\mu}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) \right] + \int_0^T dt \int d^d \vec{x} \psi(\vec{x}, t) K^*(\vec{x}, t) \\ &= - \int_0^T dt \int d^d \vec{x} \hat{P}(\vec{x}, t) \left[\partial_t \psi(\vec{x}, t) + \vec{\mu}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) \right] \\ &\quad + \int_0^T dt \int d^d \vec{x} \psi(\vec{x}, t) K^*(\vec{x}, t) + \int d^d \vec{x} \psi(\vec{x}, t) P^*(\vec{x}, t) - \psi(\vec{x}_0, 0)\end{aligned}\quad (\text{A20})$$

so that the lagrangian of Eq. A16 becomes using the explicit rate function at Level 2.5 of Eq. A11

$$\begin{aligned}\mathcal{L} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right] &= \int d^d \vec{x} \psi(\vec{x}, T) P^*(\vec{x}, T) - \psi(\vec{x}_0, 0) \\ &\quad + \int_0^T dt \int d^d \vec{x} \hat{P}(\vec{x}, t) \left(\frac{\left[\vec{\mu}(\vec{x}, t) - \vec{\mu}(\vec{x}) \right]^2}{4D(\vec{x})} - \left[\partial_t \psi(\vec{x}, t) + \vec{\mu}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) \right] \right) \\ &\quad + \int d^d \vec{x} \int_0^T dt \left[K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) \hat{P}(\vec{x}, t)} \right) - K^*(\vec{x}, t) + k(\vec{x}) \hat{P}(\vec{x}, t) + \psi(\vec{x}, t) K^*(\vec{x}, t) \right]\end{aligned}\quad (\text{A21})$$

The optimization of Eq. A21 over the empirical drift $\vec{\mu}(\vec{x}, t)$

$$0 = \frac{\mathcal{L} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right]}{\partial \vec{\mu}(\vec{x}, t)} = \hat{P}(\vec{x}, t) \left(\frac{\vec{\mu}(\vec{x}, t) - \vec{\mu}(\vec{x})}{2D(\vec{x})} - \vec{\nabla} \psi(\vec{x}, t) \right) \quad (\text{A22})$$

yields the optimal empirical drift $\vec{\mu}^{opt}(\vec{x}, t)$ in terms of the Lagrange multiplier $\psi(\vec{x}, t)$

$$\vec{\mu}^{opt}(\vec{x}, t) = \vec{\mu}(\vec{x}) + 2D(\vec{x}) \vec{\nabla} \psi(\vec{x}, t) \quad (\text{A23})$$

The further optimization of Eq. A21 over the empirical density $\hat{P}(\vec{x}, t)$ reads using the optimal drift of Eq. A23

$$\begin{aligned} 0 &= -\frac{\mathcal{L} \left[\hat{P}(\cdot, \cdot); \vec{\mu}(\cdot, \cdot) \right]}{\partial \hat{P}(\vec{x}, t)} \\ &= -\frac{\left[\vec{\mu}^{opt}(\vec{x}, t) - \vec{\mu}(\vec{x}) \right]^2}{4D(\vec{x})} + \partial_t \psi(\vec{x}, t) + \vec{\mu}^{opt}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{\hat{P}(\vec{x}, t)} - k(\vec{x}) \\ &= -D(\vec{x}) \left[\vec{\nabla} \psi(\vec{x}, t) \right]^2 + \partial_t \psi(\vec{x}, t) + \left(\vec{\mu}(\vec{x}) + 2D(\vec{x}) \vec{\nabla} \psi(\vec{x}, t) \right) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{\hat{P}(\vec{x}, t)} - k(\vec{x}) \\ &= \partial_t \psi(\vec{x}, t) + \vec{\mu}(\vec{x}) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) + D(\vec{x}) \left[\vec{\nabla} \psi(\vec{x}, t) \right]^2 + \frac{K^*(\vec{x}, t)}{\hat{P}(\vec{x}, t)} - k(\vec{x}) \end{aligned} \quad (\text{A24})$$

The change of variables

$$\psi(\vec{x}, t) = \ln q(\vec{x}, t) \quad (\text{A25})$$

transforms the non-linear Eq. A24 for the Lagrange multiplier $\psi(\vec{x}, t)$ into the following linear backward dynamics for the function $q(\vec{x}, t)$

$$-\partial_t q(\vec{x}, t) = \vec{\mu}(\vec{x}) \cdot \vec{\nabla} q(\vec{x}, t) + D(\vec{x}) \Delta q(\vec{x}, t) - k(\vec{x}) q(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{\hat{P}^{opt}(\vec{x}, t)} q(\vec{x}, t) \quad (\text{A26})$$

Using Eq. A25, the optimal empirical drift $\vec{\mu}^{opt}(\vec{x}, t)$ of Eq. A23 becomes

$$\vec{\mu}^{opt}(\vec{x}, t) = \vec{\mu}(\vec{x}) + 2D(\vec{x}) \vec{\nabla} \ln q(\vec{x}, t) \quad (\text{A27})$$

while the optimal empirical density $\hat{P}^{opt}(\vec{x}, t)$ should be the solution of the corresponding empirical forward dynamics of Eq. A15

$$\begin{aligned} \partial_t \hat{P}^{opt}(\vec{x}, t) &= -\vec{\nabla} \cdot \left[\vec{\mu}^{opt}(\vec{x}, t) \hat{P}^{opt}(\vec{x}, t) \right] + \Delta \left[D(\vec{x}) \hat{P}^{opt}(\vec{x}, t) \right] - K^*(\vec{x}, t) \\ &= -\vec{\nabla} \cdot \left[\left(\vec{\mu}(\vec{x}) + 2D(\vec{x}) \vec{\nabla} \ln q(\vec{x}, t) \right) \hat{P}^{opt}(\vec{x}, t) \right] + \Delta \left[D(\vec{x}) \hat{P}^{opt}(\vec{x}, t) \right] - K^*(\vec{x}, t) \end{aligned} \quad (\text{A28})$$

Using the backward dynamics of Eq. A26 for the function $q(\vec{x}, t)$ and the forward optimal dynamics of Eq. A28 for $\hat{P}^{opt}(\vec{x}, t)$, one obtains that the ratio

$$p(\vec{x}, t) \equiv \frac{\hat{P}^{opt}(\vec{x}, t)}{q(\vec{x}, t)} \quad (\text{A29})$$

satisfies the forward unconditioned dynamics involving the unconditioned operator \mathcal{F}^\dagger of Eq. 4

$$\begin{aligned} \partial_t p(\vec{x}, t) &= \frac{1}{q(\vec{x}, t)} \partial_t \hat{P}^{opt}(\vec{x}, t) - \frac{\hat{P}^{opt}(\vec{x}, t)}{q^2(\vec{x}, t)} \partial_t q(\vec{x}, t) \\ &= -\vec{\nabla} \cdot [\vec{\mu}(\vec{x}) p(\vec{x}, t)] + \Delta [D(\vec{x}) p(\vec{x}, t)] - k(\vec{x}) p(\vec{x}, t) = \mathcal{F}^\dagger p(\vec{x}, t) \end{aligned} \quad (\text{A30})$$

c. *Taking into account the time boundary conditions to obtain the final optimal solution*

In summary, the optimal solution $\hat{P}^{opt}(\vec{x}, t)$ is given the product of Eq. A29

$$\hat{P}^{opt}(\vec{x}, t) = q(\vec{x}, t)p(\vec{x}, t) \quad (\text{A31})$$

where $p(\vec{x}, t)$ satisfies the forward unconditioned dynamics of Eq. A30, while $q(\vec{x}, t)$ satisfies the backward dynamics of Eq. A26 that becomes using Eq. A31

$$-\partial_t q(\vec{x}, t) = \vec{\mu}(\vec{x}) \cdot \vec{\nabla} q(\vec{x}, t) + D(\vec{x}) \Delta q(\vec{x}, t) - k(\vec{x})q(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{p(\vec{x}, t)} \quad (\text{A32})$$

In addition, we have to take into account the time-boundary-conditions of Eq. A14 at the initial time $t = 0$ and at the final time $t = T$

$$\begin{aligned} \delta^d(\vec{x} - \vec{x}_0) &= \hat{P}^{opt}(\vec{x}, t = 0) = q(\vec{x}, 0)p(\vec{x}, 0) \\ P^*(\vec{x}, T) &= \hat{P}^{opt}(\vec{x}, t = T) = q(\vec{x}, T)p(\vec{x}, T) \end{aligned} \quad (\text{A33})$$

For the function $p(\vec{x}, t)$, it is natural to choose the unconditioned propagator $P(\vec{x}, t|\vec{x}_0, 0)$ that would be the solution if we were not imposing atypical constraints

$$p(\vec{x}, t) = P(\vec{x}, t|\vec{x}_0, 0) \quad (\text{A34})$$

With this choice, the backward dynamics of Eq. A26 becomes

$$-\partial_t q(\vec{x}, t) = \vec{\mu}(\vec{x}) \cdot \vec{\nabla} q(\vec{x}, t) + D(\vec{x}) \Delta q(\vec{x}, t) - k(\vec{x})q(\vec{x}, t) + \frac{K^*(\vec{x}, t)}{P(\vec{x}, t|\vec{x}_0, 0)} \quad (\text{A35})$$

while Eq. A33 yields the following time boundary conditions for the function $q(\vec{x}, t)$

$$\begin{aligned} q(\vec{x}, t = 0) &= 1 \\ q(\vec{x}, t = T) &= \frac{P^*(\vec{x}, t)}{P(\vec{x}, t|\vec{x}_0, 0)} \end{aligned} \quad (\text{A36})$$

The solution $q(\vec{x}, t)$ of the backward dynamics of Eq. A26 that satisfies the boundary conditions of Eqs A36 thus coincides with the function $Q(\vec{x}, t)$ introduced in Eq. 18 of the main text,

$$q(\vec{x}, t) = Q(\vec{x}, t) \quad (\text{A37})$$

and the optimal solution $\hat{P}^{opt}(\vec{x}, t)$ of Eq. A31 coincides with the conditioned probability $P^*(\vec{x}, t)$ introduced in Eq. 17 in the main text

$$\hat{P}^{opt}(\vec{x}, t) = q(\vec{x}, t)p(\vec{x}, t) = Q(\vec{x}, t)P(\vec{x}, t|\vec{x}_0, 0) = P^*(\vec{x}, t) \quad (\text{A38})$$

d. *Corresponding optimal value of the Lagrangian*

Using the second line of Eq. A24 to replace

$$\frac{[\tilde{\mu}^{opt}(\vec{x}, t) - \vec{\mu}(\vec{x})]^2}{4D(\vec{x})} - \left[\partial_t \psi(\vec{x}, t) + \tilde{\mu}^{opt}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t) \right] = \frac{K^*(\vec{x}, t)}{\hat{P}^{opt}(\vec{x}, t)} - k(\vec{x}) \quad (\text{A39})$$

one obtains the optimal value of the Lagrangian of Eq. A21

$$\begin{aligned}
\mathcal{L} \left[\hat{P}^{opt}(\cdot, \cdot); \tilde{\mu}^{opt}(\cdot, \cdot) \right] &= \int d^d \vec{x} \psi(\vec{x}, T) P^*(\vec{x}, T) - \psi(\vec{x}_0, 0) \\
&+ \int_0^T dt \int d^d \vec{x} \hat{P}^{opt}(\vec{x}, t) \left(\frac{[\tilde{\mu}^{opt}(\vec{x}, t) - \tilde{\mu}(\vec{x})]^2}{4D(\vec{x})} - [\partial_t \psi(\vec{x}, t) + \tilde{\mu}^{opt}(\vec{x}, t) \cdot \vec{\nabla} \psi(\vec{x}, t) + D(\vec{x}) \Delta \psi(\vec{x}, t)] \right) \\
&+ \int d^d \vec{x} \int_0^T dt \left[K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) \hat{P}^{opt}(\vec{x}, t)} \right) - K^*(\vec{x}, t) + k(\vec{x}) \hat{P}^{opt}(\vec{x}, t) + \psi(\vec{x}, t) K^*(\vec{x}, t) \right] \\
&= \int d^d \vec{x} \psi(\vec{x}, t) P^*(\vec{x}, t) - \psi(\vec{x}_0, 0) + \int_0^T dt \int d^d \vec{x} \hat{P}^{opt}(\vec{x}, t) \left(\frac{K^*(\vec{x}, t)}{\hat{P}^{opt}(\vec{x}, t)} - k(\vec{x}) \right) \\
&+ \int d^d \vec{x} \int_0^T dt \left[K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) \hat{P}^{opt}(\vec{x}, t)} \right) - K^*(\vec{x}, t) + k(\vec{x}) \hat{P}^{opt}(\vec{x}, t) + \psi(\vec{x}, t) K^*(\vec{x}, t) \right] \\
&= \int d^d \vec{x} \psi(\vec{x}, t) P^*(\vec{x}, t) - \psi(\vec{x}_0, 0) + \int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) \left[\psi(\vec{x}, t) + \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) \hat{P}^{opt}(\vec{x}, t)} \right) \right] \tag{A40}
\end{aligned}$$

Using Eq. A25 and Eq. A38, the Lagrange multiplier

$$\psi(\vec{x}, t) = \ln q(\vec{x}, t) = \ln Q(\vec{x}, t) = \ln \left(\frac{P^*(\vec{x}, t)}{P(\vec{x}, t | \vec{x}_0, 0)} \right) \tag{A41}$$

and its initial value

$$\psi(\vec{x}_0, 0) = \ln q(\vec{x}_0, 0) = \ln \left(\frac{P^*(\vec{x}_0, 0)}{P(\vec{x}_0, 0 | \vec{x}_0, 0)} \right) = \ln(1) = 0 \tag{A42}$$

can be plugged into Eq. A40 to obtain that the optimal value of the Lagrangian

$$\begin{aligned}
\mathcal{L} \left[\hat{P}^{opt}(\cdot, \cdot); \tilde{\mu}^{opt}(\cdot, \cdot) \right] &= \int d^d \vec{x} P^*(\vec{x}, T) \ln \left(\frac{P^*(\vec{x}, T)}{P(\vec{x}, T | \vec{x}_0, 0)} \right) \\
&+ \int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) \left[\ln \left(\frac{P^*(\vec{x}, t)}{P(\vec{x}, t | \vec{x}_0, 0)} \right) + \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) P^*(\vec{x}, t)} \right) \right] \\
&= \int d^d \vec{x} P^*(\vec{x}, T) \ln \left(\frac{P^*(\vec{x}, T)}{P(\vec{x}, T | \vec{x}_0, 0)} \right) + \int d^d \vec{x} \int_0^T dt K^*(\vec{x}, t) \ln \left(\frac{K^*(\vec{x}, t)}{k(\vec{x}) P(\vec{x}, t | \vec{x}_0, 0)} \right) \tag{A43}
\end{aligned}$$

coincides with the Sanov rate function $\mathcal{I}_T^{Sanov} [P^*(\cdot, T); K^*(\cdot, \cdot)]$ given in Eq. 30 of the main text, as it should for consistency.

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