Acoustic modes attenuation in ducts using CFD with Time-Domain Impedance Boundary Condition

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In the framework of improving aircraft nacelle design to comply international regulation on aircraft noise, a Time-Domain Impedance Boundary Condition (TDIBC) based on the Oscillo-Diffusive Representation is implemented in an industrial Computational Fluid Dynamics (CFD) solver under a characteristic formalism, in order to take into account acoustic liner attenuation. A validation of such TDIBC is carried out, first on a 2D benchmark case, then on a more realistic 3D cylindrical configuration, with focus on the acoustic modal content and its attenuation with and without flow. Unsteady Reynolds-Averaged Navier-Stokes (URANS) computations are performed, with the grazing flow being considered laminar. Good agreement is found between the CFD results and the references (CAA results and experiments), with differences in the order of $\approx 1$ dB for most cases. The formulation chosen for the TDIBC proved to have no significant impact on numerical stability or computational time.

Nomenclature

$A$ = acoustic wave amplitude

$a, b$ = coefficients of EHR (Extended Helmholtz Resonator) model

$c$ = speed of sound

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\( h \) = auxiliary function  
\( I \) = acoustic intensity  
\( i \) = imaginary unit \((\sqrt{-1})\)  
\( k \) = wavenumber in modal decomposition  
\( k_{r, mn} \) = radial wavenumber in modal decomposition  
\( k_{z, mn} \) = axial wavenumber in modal decomposition  
\( l, L_T \) = fictitious 1D space dimension, 1D distance covered in delay advection  
\( \mathcal{L} \) = amplitude of the characteristic convective waves (or fluxes) normal to the boundary  
\( M \) = Mach number  
\( m, n \) = azimuthal, radial order in modal decomposition  
\( n_x, n_y, n_z \) = normal vectors in Cartesian coordinates  
\( N_{\xi}, N_{\eta} \) = number of poles (diffusive, oscillatory)  
\( p \) = pressure  
\( s \) = Laplace variable \((i\omega)\)  
\( s_n, r_n \) = oscillatory poles and weights  
\( t \) = time  
\( \mathbf{r}_1, \mathbf{r}_2 \) = tangential vectors (of the orthonormal plane)  
\( \mathcal{T} \) = amplitude of the characteristic waves (or fluxes) tangent to the boundary  
\( u, v, w \) = Cartesian 3D components of velocity  
\( u_n \) = wall-normal velocity  
\( \mathbf{v} \) = 3D velocity vector (of components \( u, v, w \))  
\( \mathcal{W} \) = characteristic variables vector  
\( z \) = general pole of the Oscillo-Diffusive Representation  
\( Z \) = acoustic impedance  
\( \beta, \beta_\infty \) = reflection coefficient, bulk reflection coefficient  
\( \gamma \) = specific heat ratio  
\( \Gamma_u, \Gamma_T \) = Giles time-filtered relaxation parameter (velocity, temperature)  
\( \partial \) = partial derivative  
\( \lambda \) = characteristic velocity  
\( \xi, \mu \) = diffusive poles and weights  
\( \rho \) = density  
\( \sigma_c, \sigma_p \) = liner’s porosity (cavity, perforations)
\( \sigma_u, \sigma_T \) = NSCBC relaxation parameter (velocity, temperature)

\( \tau \) = delay

\( \varphi \) = auxiliary variable in Oscillo-Diffusive Representation

\( \psi \) = auxiliary delayed variable in Oscillo-Diffusive Representation

\( \Psi \) = wave radial shape in modal decomposition

\( \omega \) = angular frequency

\( \square_\infty \) = prescribed value

\( \square_0 \) = static value

\( \square_{ac} \) = acoustic fluctuation value

\( \tilde{\square} \) = Fourier/Laplace transform, time-filtered value

I. Introduction

In modern times, the environmental impact of civil aerial transports is a crucial topic worldwide. This includes obviously carbon and pollutant emissions, but also other aspects such as acoustic annoyance in the airport surroundings and cabin. With the attention focused on the noise generated by the engine, take-off and landing are two critical phases to be properly tackled due to their high sound levels. One of the most efficient solutions in turbofan acoustic problems is to cover specific engine nacelle parts with sound-absorbing materials called "liners". The typically used layout in aeronautics is the Single Degree Of Freedom (SDOF), consisting of an array of honeycomb cavities backed by a rigid plate on a side and a perforated plate on the opposite one. Variations of this liner have been developed to optimize specific aspects: the Wiremesh SDOF, adding a wiremesh layer to decrease the sensitivity to the grazing flow, or the Double Degree Of Freedom (DDOF) liner, composed by two series of honeycomb cavities of different heights and separated by a porous septum, extending the spectrum range of acoustic absorption. The geometry can be tuned (in terms of cavities' length or width, plate's width and holes' diameter) to obtain a specific response. In order to maximize the noise reduction benefit of the treatment, an accurate numerical approach is fundamental to support its design.

Sound absorption has been traditionally defined and modelled in the frequency domain. Translating this effect in a time-domain formalism would make possible its implementation in classical unsteady Computational Fluid Dynamics (CFD), time-domain Computational AeroAcoustics (CAA) and coupled CFD/CAA tools. Time-domain formulations of acoustical liners are as much appealing as complex to handle. Direct Numerical Simulation (DNS) for real liner geometries mounted in a turbofan model is far from being applicable on an industrial scale, due to the high grid discretization accuracy needed to correctly resolve the unsteady field in the geometric details, as investigated in [1] with the use of Lattice Boltzmann method. Time-Domain Impedance Boundary Conditions (TDIBC) have been
developed in the last two decades in order to numerically model the sound absorption mechanism of a lined wall at a lower numerical cost and to be coupled to a classical time-domain CAA solvers. First attempts emerged with the work of Tam and Auriault [2], whose three-parameters model has been widely used for its implementation simplicity, however backed by a limitation in the range of applicability. Different models have followed, which can be divided into two categories: based on physical parameters or on numerical multi-poles schemes. The first ones are commonly rational multi-parameter, single polynomial or fractional models characterized by coefficients linked to the liner’s physics (resistance, reactance, ...), as in [2, 3]. The second ones are pure numerical models, sum of elementary dynamical systems of first and second order obtained through mathematics approximations. This means that the model’s coefficients are solely function of mathematical expressions and not linked to any physical law. Even though the direct link with their physical meaning is lost, they provide an easier time-domain translation. Some examples are in [4,5]. Their main problem is found to be the need of solving numerically expensive convolutions and storing an accumulator. Models combining physics and multi-poles emerged in the last decade [6, 7]. In particular, the recent work of Monteghetti [8,9] provides a reliable modelling based on a mathematical technique referred to as the Oscillatory-Diffuse Representation (ODR), which adopts auxiliary differential equations to model the lined wall effect and relies directly on the liner’s geometry. This approach is currently validated for linear liners, where the impedance is independent of the variations in acoustic level and flow speed. Preliminary studies from the authors (cf. [10], Section 2.5) have investigated how to extend the model to the nonlinear response of the material, without detailed validation at this stage.

Another point of debate in the literature when speaking of acoustic impedance is which formulation to adopt: impedance $Z$ [2,3,11,12], admittance $Y$ [13–15], or reflection coefficient $\beta$ [4,8,9,16,17]. Gabard et al. [11] suggest that some of the instabilities reported in the literature with the Myers impedance boundary condition might be due to the way the boundary condition was implemented. They show that less instabilities are expected with an implementation based on characteristic waves, as is the case with a $\beta$-formalism. In fact, $\beta$ is a continuous function, without asymptotic-like singularities in Bode’s diagrams, and bounded in amplitude in the range [0,1]. It has been demonstrated in [18] and further confirmed by Monteghetti [8] how using the $\beta$-formulation in a CAA simulation guarantees a CFL stability condition independent on the value of the reflection coefficient, which is not true for a $Z$-formulation, thus making it a reasonable choice for large scale numerical applications. TDIBCs have been initially developed in the framework of CAA, through the simulation of Euler Equations with a Linearized (LEE) or Non-Linearized (NLEE) form, as first studied at ONERA in [19–21]. However, implementing them in a CFD solver is a challenging task. The interest in having a coupled CFD numerical tool with a TDIBC is that it allows to take into account complex flow features while modeling the sound absorption of the liner. Furthermore, a single time-based numerical simulation would be sufficient, without need of separately analyzing the liner’s behaviour with CAA tools, hence drastically reducing the workload by simplifying the coupled CFD/CAA process. A further interest of a TDIBC in a CFD solver is the
possibility to account simultaneously for sound absorption and nonlinear waves propagation in presence of a boundary layer. An industrial application which would profit of all these advantages, is the numerical simulation of the shock waves generated by a transonic rotor and propagated through an acoustically treated 3D inlet. The ODR method provides an attractive approach in this context, thanks to its adaptability to complex liner geometries and broadband frequency applications. From a design perspective of aero-engine and aircraft manufacturers, its capability to simulate the attenuation of nonlinear broadband noise phenomena makes it a preferred choice when building an industrial numerical tool.

First promising results have been achieved in \cite{22} for impedance laws with a simple dynamics, such as the damped Helmholtz oscillator, on a multi-pole model applied to a delayed reflection \cite{16}. A Finite-Volume (FV) approach has been tested in \cite{23}, where a CFD/CAA Linearized Navier-Stokes code and a Large Eddy Simulations (LES) code had implemented the impedance condition of \cite{2}. However, these demand either a cumbersome formulation relying on two solvers or heavy calculation times due to the fine grid requirements of a LES. A successful implementation of the fully broadband ODR model has been recently achieved with a Spectral-Difference CFD code \cite{24, 25}. Its high-order spatial scheme proved to be ideal in the discretization of such time-domain model. The recent work from Shur \textit{et al.} \cite{26, 27} is closely aligned to the present development. A purely numerical multi-pole TDIBC model is derived as extension from Dragna’s \cite{12} in order to include turbulence fluctuations and sound level dependency on the impedance law, avoiding the expensive convolution process by introducing auxiliary functions. Their time-domain impedance model is defined as "dynamic" in respect to the sound pressure level: a linear weighted combination among three discrete impedance values is done throughout the calculation to represent more faithfully the pressure level measured at each point of the liner. In fact, if the sound level is strongly attenuated through the liner or the liner is sufficiently long, a single impedance value calculated at a given SPL would not be realistic of the true SPL on the lined wall. The differences between Shur’s \textit{et al.} work and the present paper can be resumed in two points. First, Dragna’s model does not take into account, in the poles calculation, the delay’s effect created by the back-and-forth propagation of the acoustic waves within the liner’s honeycomb cavities. When considering liners with complex physics, the ODR brings the advantage of explicitly accounting for the delay effect in its model, allowing an impedance discretization with a low number of poles and thus a lower added numerical cost. Second, even though at this stage Shur’s model proved to be more complete with its "dynamic" sound level property, the TDIBC-ODR formulation is intended to be extended to a nonlinear form in the future which will not be discrete as in Shur’s model, but directly responding to any local pressure value on the treated surface. \cite{8}

Further studies on TDIBC are carried on nowadays by Naïr \textit{et al.} in \cite{28} for industrial purpose, with implementation of a purely numerical time-domain impedance model on the time-based CAA Discontinuous-Galerkin solver Actran-DGM. In collaboration with the authors, the same 3D benchmark case here analysed is tackled in \cite{28}, providing valuable additional validation resources for both parts.
In this paper, a further step regarding the TDIBC implementation in a CFD solver is addressed. It represents the continuation of the previous works from the authors [29, 30] to implement the Oscillo-Diffusive Representation (ODR) and its TDIBC model in a CFD code for simulating sound absorption of aeronautical-type liners. In regards to the previous works, herein consolidated results, further analyses and details on the mathematical developments are provided. A full Ph.D. dissertation from the first author [10] provides extensive clarifications on the same subject. A Navier-Stokes Characteristics Boundary Condition (NSCBC) type of acoustic treatment has been implemented with an ODR in the elsA code, jointly owned by ONERA, Safran and Airbus until 2020, and by considering Euler and URANS equations with a structured FV algorithm. The aforementioned ODR-TDIBC is first applied to an academic acoustic benchmark: the GIT case of Jones et al. [31], initially discussed in [29]. Then, an industrial benchmark study on acoustic modes attenuation in a 3D cylindrical geometry is tackled. This was firstly presented by Lavieille et al. [32] and analysed in [30] and [28]. Compared to the previous academic case, this is a more realistic numerical representation of the physical environment of engine nacelles, as also recently studied by Zaabar in [33], where inlet and outlet boundary conditions for modes analyses in axisymmetric ducts are discussed. Similarly to the latter and inspired by the pioneer work of Giles [34] with Euler approximation, a suitable acoustic source term injection is coupled to quasi-non-reflective NSCBC inlet and outlet boundary conditions. The aim is to be capable of introducing different types of acoustic perturbations in a relatively short domain (i.e. without grid stretching zones) without spurious numerical reflections.

The paper is structured as follows. Section II begins with the explanation of the mathematical tools used in the impedance definition under an ODR formalism. Then, the theoretical background on NSCBC is given in order to further discuss the TDIBC implementation, alongside the development of acoustic non-reflecting inlet and outlet conditions. Section III presents consolidated results of the 2D benchmark case previously shown by the authors in [29], particularly in presence of a laminar sheared flow profile. Section IV presents more comprehensive results of the 3D benchmark case shown by the authors in [30]. The conclusive Section V summarizes the main achievements, foreseeing the upcoming work on the topic.

II. Mathematical formulation

This Section focuses on giving the theoretical background and mathematical tools for understanding the present developments. First, Section II.A gives the fundamental equations of the ODR and its associated TDIBC model as derived in [8, 9]. NSCBC theory is introduced in Section II.B based on the original work of [35], followed by the current implementation of TDIBC and acoustic non-reflecting inlet and outlet boundary conditions in Section II.C. Finally, Section II.D gives a short overview on how the acoustic perturbations are introduced.
A. Oscillo-Diffusive Representation

The acoustic concept called impedance characterizes the resistance and reactance of a surface material, hence its capability of reducing sound level. It is defined in the frequency domain as the ratio between Fourier (or Laplace) -transformed acoustic pressure $p_{ac}$ and wall-normal acoustic velocity fluctuations $u_{ac}$:

$$\hat{Z}(\omega) = \frac{\hat{p}_{ac}(\omega)}{\hat{u}_{ac}(\omega)}$$  \hspace{1cm} (1)

In the present work, only locally-reacting materials will be considered. A parameter similar to the impedance, here named "reflection coefficient" ($\beta$), can be derived from (1) and it reads:

$$\hat{\beta}(\omega) = \frac{\hat{Z}(\omega) - \rho_0 c_0}{\hat{Z}(\omega) + \rho_0 c_0}$$  \hspace{1cm} (2)

where the product $\rho_0 c_0$ is the medium characteristic impedance. Regardless its natural belonging to the frequency domain, it is convenient translating the impedance (or reflection coefficient) in the time domain to permit its implementation in classical unsteady CFD and CAA tools. This conversion reveals to be cumbersome, as instantaneous pressure depends on the convolution:

$$p(t) = [Z \star u_n](t)$$  \hspace{1cm} (3)

In order to derive an admissible, physical and numerically stable impedance model in the time-domain, Rienstra [3] suggests that the impedance must be extended in the complex frequency-domain and satisfy necessary conditions of causality, reality and passivity. In a Fourier-transformed formulation (in $\omega$), these conditions are necessary for impedance admissibility, but not sufficient. The recent work of Monteghetti [8] demonstrated how these conditions become sufficient for admissibility property when working with "system theory", using then Laplace-transform instead of Fourier. The correlation between the two is remembered to be given by the identity $s = i\omega$. When working with a reflection coefficient, we can link directly acoustic waves in terms of acoustic pressure $p_{ac}$ and acoustic velocity $u_{ac}$:

$$\left[ u_{ac} - \frac{p_{ac}}{\rho_0 c_0} \right](t) = \beta \star \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right](t)$$  \hspace{1cm} (4)

where the acoustic fluctuation of a variable $\phi_{ac}$ is defined as the difference between its instantaneous value $\phi$ and its time-averaged value $\bar{\phi}$:

$$\phi_{ac} = \phi - \bar{\phi}$$

Various impedance models and their relative time-domain translation have been proposed in the last two decades to either simplify or surround the convolution problem. A slightly modified Extended Helmholtz Resonator (EHR) impedance model proposed by Rienstra in [3] demonstrated to be a suitable impedance model when dealing with
practical SDOF liners:

\[ Z_{EHR}(s) = \frac{1}{\sigma_p} (a_0 + a_1 \sqrt{s} + a_1 s) + \frac{1}{\sigma_c} \coth(b_0 + b_1 \sqrt{s} + b_1 s) \]  

(5)

The parameters \( a_i \) and \( b_i \) are function of the liner’s geometry (perforations and cavities) and can find different formulations. Herein, the coefficient sets proposed by Bruneau [36] and Crandall [37] are used as initial guess for the model, then optimized with a least-square fitting to match the experimental data set and so the ”realistic” impedance law. It has been demonstrated that the impedance model (5) accepts an ODR [8, 9]. With help of (2), the model in (5) is converted into a reflection coefficient formulation, then rewritten as:

\[ \beta(s) = \beta_\infty + h_1(s) + e^{-\tau} h_2(s) \]  

(6)

with \( \beta_\infty \) the bulk reflectivity (i.e. the frequency-independent reflectivity), equal to 1 for a SDOF liner, and \( \tau \) the resonance delay equal to the back-and-forth traveling time inside the liner’s cavity. The functions \( h_1 \) and \( h_2 \) represent respectively the non-delayed and delayed terms, each of them defined as:

\[ h_i = \sum_{n \in \mathbb{N}} r_{i,n} s - s_n + \int_0^\infty \frac{\mu_{i,n}(\xi)}{s + \xi} d\xi \]  

(7)

The same nomenclature as in the original paper [9] has been kept, identifying with \( s_n \) and \( r_n \) respectively the complex conjugated pairs of oscillatory poles and weights and with \( \xi \) and \( \mu \) the real diffusive poles and weights. The link with the EHR model of (5) lies within the calculation of poles and weights, obtained from knowledge of the coefficients \( a_i \) and \( b_i \), which provide exact solution of the same model. The complete formulation after discretization reads:

\[ \beta(s) = \beta_\infty + \sum_{n=1}^{N_s} \frac{r_{1,n}}{s - s_n} + \sum_{k=1}^{N_\xi} \frac{\mu_{1,k}}{s + \xi_k} + e^{-\tau} \left( \sum_{n=1}^{N_s} \frac{r_{2,n}}{s - s_n} + \sum_{k=1}^{N_\xi} \frac{\mu_{2,k}}{s + \xi_k} \right) \]  

(8)

where \( N_s \) and \( N_\xi \) are the number of complex conjugated oscillatory and real diffusive poles, respectively, and \((r_{1,n}, \mu_{1,k})\), \((r_{2,n}, \mu_{2,k})\) the weights for the non-delayed and delayed terms, respectively. Consequently, (8) needs to be converted from Laplace (frequency) to time domain. Instead of solving the convolution in (3), this is done in the ODR method through additional auxiliary functions \( \varphi \). The reflection coefficient operator, combined with the expression of \( \beta \) in (8)
and reminding its relation with acoustic waves in (4), becomes:

$$\left[ u_{ac} - \frac{p_{ac}}{\rho_0 c_0} \right] (t) = \beta \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] (t) = \beta_{\infty} \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] (t) + \sum_{k=1}^{N_e} \left[ \mu_{1,k} \varphi(t, \xi_k) + \mu_{2,k} \varphi(t - \tau, \xi_k) \right] + \sum_{n=1}^{N_s} \left[ r_{1,n} \varphi(t, -s_n) + r_{2,n} \varphi(t - \tau, -s_n) \right]$$

(9)

where the functions $\varphi$ are solution of the ordinary differential equations:

$$\begin{cases}
\frac{\partial \varphi}{\partial t}(t, z) = -z \varphi(t, z) + \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] (t) \\
\varphi(0, z) = 0
\end{cases}$$

(10)

with $t > 0$ and $z$ being either an oscillatory ($-s_n$) or diffusive ($\xi_k$) pole. At last, auxiliary hyperbolic functions $\psi$ are introduced to translate the "delayed effect" in a "transported effect". Such functions solve the three-valued initial boundary problem of a one-dimensional advection of the quantity $\varphi$ at the propagation speed (of sound) $c$ on a length $L_\tau = c\tau$, equal to the back-and-forth distance traveled in the liner’s cavity.

$$\begin{cases}
\frac{\partial \psi}{\partial t}(t, z, l) = c \frac{\partial \psi}{\partial l}(t, z, l) \\
\psi(t, z, 0) = \varphi(t, z) \\
\psi(0, z, l) = 0
\end{cases}$$

(11)

with $l \in (0, L_\tau)$ the one-dimensional spatial coordinate. Therefore, $\psi$ at the last value of the spatial array $l$ (at coordinate $l = L_\tau$) corresponds to the delayed $\varphi$ variable. The discrete impedance model of the convolution of $\beta$ becomes finally:

$$\left[ u_{ac} - \frac{p_{ac}}{\rho_0 c_0} \right] (t) = \beta_{\infty} \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] (t) + \sum_{k=1}^{N_e} \left( \mu_{1,k} \varphi(t, \xi_k) + \mu_{2,k} \varphi(t, \xi_k, L_\tau) \right) + \sum_{n=1}^{N_s} \left[ r_{1,n} \varphi(t, -s_n) + r_{2,n} \varphi(t - \tau, -s_n, L_\tau) \right]$$

(12)

The latter must be resolved simultaneously to the auxiliary variables $\varphi$ and $\psi$, whose boundary and initial conditions are given as in (10) and (11), respectively. This will be done in a characteristic waves formulation in Section II.C.

### B. Navier-Stokes Characteristics Boundary Conditions

Before explaining how the impedance model is implemented in the CFD solver, fundamental theory of NSCBC needs to be given. Firstly introduced by Poinsot and Lele [35] in a CFD framework, NSCBC are a valid alternative for boundary conditions of hyperbolic systems. Specifically, for acoustics problems, Rienstra & Hirschberg [38] suggested the advantage of using characteristics forms, allowing an analytical solution to problems at high sound amplitudes (where nonlinearities appear) and determining optimal discretization schemes and stability conditions. The focus of
NSCBC is on the waves crossing the boundary, solving local 1D Linearized Euler Equations (LEE), also called LODI (Linear One-Dimensional Inviscid) problem. Although this is a crude simplification making the problem non-physical, it can be used practically to manage the outgoing and incoming waves amplitude through a dedicated coupling with the Navier-Stokes equations system. A very similar nomenclature to the one used by Fiévet et al.\cite{24} is used and hence we will often refer to his work for further mathematical derivations and to keep this paper concise. We start our reasoning from Navier-Stokes (NS) equations in conservative form and Cartesian coordinates:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = S$$ \quad (13)

where $S$ denotes the source terms (in the latter considered null) and $U$ is the conservative variables vector:

$$U = [\rho, \rho u, \rho v, \rho w, \rho (e + \frac{1}{2}v^2)]^T \quad (14)$$

with $\rho$ density, $u, v, w$ the three velocity components, $\vec{v} = (u, v, w)$ the three-dimensional velocity vector, $e$ the internal energy. The conservative fluxes ($E, F, G$) can be considered as sum of convective ($E_c, F_c, G_c$) and diffusive ($E_d, F_d, G_d$) fluxes of $U$. Their components are given in detail in [24]. Even though the first NSCBC presentation from Poinsot and Lele\cite{35} was done in Cartesian coordinates, in the present development the boundary is considered as a generic iso-surface $\vec{\chi}(\chi, \eta, \zeta) = \text{constant}$ located at a computational domain end. However, to simplify the reading, we continue our derivation in Cartesian coordinates, and full derivation in generalized coordinates can be found in [24]. A compact form of Navier-Stokes conservation equation (13) can be given in terms of characteristic fluxes:

$$\frac{\partial W}{\partial t} + L + D + T = 0$$ \quad (15)

Fig. 1 NSCBC scheme - adapted from [35].
where:

• \(W\) represents the characteristic variables array, whose time derivative components are defined in terms of primitive variables as:

\[
\begin{align*}
\frac{\partial}{\partial t}W_1 &= n_x \frac{\partial}{\partial t} \rho + n_z \frac{\partial}{\partial t} v - n_y \frac{\partial}{\partial t} w - \frac{1}{c^2} n_x \frac{\partial}{\partial t} p \\
\frac{\partial}{\partial t}W_2 &= n_y \frac{\partial}{\partial t} \rho + n_x \frac{\partial}{\partial t} u - n_z \frac{\partial}{\partial t} v - \frac{1}{c^2} n_y \frac{\partial}{\partial t} p \\
\frac{\partial}{\partial t}W_3 &= n_z \frac{\partial}{\partial t} \rho + n_y \frac{\partial}{\partial t} u - n_x \frac{\partial}{\partial t} v - \frac{1}{c^2} n_z \frac{\partial}{\partial t} p \\
\frac{\partial}{\partial t}W_4 &= (n_x \frac{\partial}{\partial t} u + n_y \frac{\partial}{\partial t} v + n_z \frac{\partial}{\partial t} w) + \frac{1}{\rho c} \frac{\partial}{\partial t} p \\
\frac{\partial}{\partial t}W_5 &= -(n_x \frac{\partial}{\partial t} u + n_y \frac{\partial}{\partial t} v + n_z \frac{\partial}{\partial t} w) + \frac{1}{\rho c} \frac{\partial}{\partial t} p
\end{align*}
\]  

(16)

with \(\rho\) and \(c\) the local density and speed of sound, respectively. Here we use the same normalization of the Jacobian eigenvectors as in Hirsch (cf. [39], p. 178, eq. (16.5.12)).

• \(Y\) is the strength (or amplitude) of the characteristic convective waves.

• \(D\) represents the diffusive fluxes, which are usually expressed by a Dirichlet or Neumann type of boundary condition for calculating the momentum and thermal energy gradients.

• \(T\) represents the tangent (or transverse) fluxes, which can be evaluated \textit{a priori} by knowledge of the inner computational domain cells.

Convective waves \(Y\) can be expressed in different forms: for example, function of time or space derivatives of primitive or conservative variables. After simple but tedious calculations, their expression function of time derivatives of primitive variables is given for a boundary normal to the \(x\) direction (\(n_x = 1\) and \(n_y = n_z = 0\)) as:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + Y_1 + \frac{\rho}{c \sqrt{2}} (Y_4 + Y_5) + T_1 &= 0 \\
\frac{\partial u}{\partial t} + \frac{1}{\sqrt{2}} (Y_4 - Y_5) + T_u &= 0 \\
\frac{\partial v}{\partial t} - Y_3 + T_v &= 0 \\
\frac{\partial w}{\partial t} + Y_2 + T_w &= 0 \\
\frac{\partial p}{\partial t} + \frac{\rho c}{\sqrt{2}} (Y_4 + Y_5) + T_p &= 0
\end{align*}
\]  

(17)

where \(T\) represents the contribution of the tangential flux balance in this primitive variables formulation, known from inner domain solution. As alternative, one can consider it function of space derivative of the primitive variables.
$Q = (\rho, u, v, w, p)$, defined for a boundary normal to the $x$ direction as:

$$
\mathcal{L} = \lambda \mathbf{P}_Q \frac{\partial Q}{\partial x} \iff \begin{cases}
\mathcal{L}_1 &= \lambda_1 \left( \frac{\partial \rho}{\partial x} - \frac{1}{c^2} \frac{\partial p}{\partial x} \right) \\
\mathcal{L}_2 &= \lambda_2 \frac{\partial w}{\partial x} \\
\mathcal{L}_3 &= -\lambda_3 \frac{\partial v}{\partial x} \\
\mathcal{L}_4 &= \lambda_4 \frac{1}{\sqrt{2}} \left( \frac{\partial u}{\partial x} + \frac{1}{\rho c} \frac{\partial p}{\partial x} \right) \\
\mathcal{L}_5 &= \lambda_5 \frac{1}{\sqrt{2}} \left( -\frac{\partial u}{\partial x} + \frac{1}{\rho c} \frac{\partial p}{\partial x} \right)
\end{cases}
$$

with $\mathbf{P}_Q$ the eigenmatrix of the jacobian $\partial \mathbf{E}/\partial Q$, or in other words the transformation matrix from characteristic to primitive variables. A full expression of $\mathcal{L}$ in generalized coordinates is given in Appendix. The array $\lambda$ stands for the characteristic waves velocities, as in Figure [1] when $\lambda$ is positive, the associated $\mathcal{L}$ wave is going outside the computational domain and can be directly computed from (18). When $\lambda$ is negative, the relative wave is entering in the domain and additional boundary conditions are required to calculated it. In the following, similarly to the right boundary of Figure [1]

$$
\begin{cases}
\lambda_1 = \lambda_2 = \lambda_3 = u \\
\lambda_4 = u + c \\
\lambda_5 = u - c
\end{cases}
$$

The characteristic velocities can also be defined as: $\lambda_1$ the entropy wave velocity (or the convection velocity), $\lambda_2$ and $\lambda_3$ the rotational waves velocity, $\lambda_4$ and $\lambda_5$ the acoustic waves velocity propagating in the positive and negative normal directions. Similarly, we can define the amplitudes for respectively the characteristic entropic wave ($\partial W_1$), rotational waves ($\partial W_{2,3}$) and the left- and right-running acoustic waves ($\partial W_{4,5}$). With the last set of equations (18), a simple physical interpretation of $\mathcal{L}_i$ can be given. Take, as example, the Linearized One-Dimensional Inviscid (LODI) NS equations, so that the $T$ and $D$ contributions are neglected, and define the general acoustic wave amplitude as:

$$
\mathcal{A}_{4,5} = u_{ac} \pm \frac{p_{ac}}{\rho_0 c_0}
$$

where the sign $\pm$ defines the wave direction. In a laminar flow or no-flow case, an acoustic fluctuation is defined as the difference between instantaneous and average value:

$$
p = \bar{p} + p_{ac}, \quad u = \bar{u} + u_{ac}
$$
In the case of a turbulent flow, an additional component of hydrodynamic pressure fluctuation would be present at the wall and would have to be separated from the acoustic fluctuation. In Scalo et al. [22], the impedance boundary condition is directly applied to the instantaneous pressure fluctuation, without separating the hydrodynamic and acoustic components. A similar strategy was also used by Sebastian et al. [40] or very recently by Shur et al. [26, 27]. We could indeed question whether such an impedance boundary condition is fully representative of what happens in a lined duct with turbulent flow. Turbulent flows are not subject of this paper, thus this problem is not tackled at this stage and left to future developments. If the fourth and fifth line of (16) are taken for a boundary normal to the $x$ direction:

$$
\begin{align*}
\frac{\partial W_4}{\partial t} &= \frac{\partial u}{\partial t} + \frac{1}{\rho c} \frac{\partial p}{\partial t} \\
\frac{\partial W_5}{\partial t} &= -\frac{\partial u}{\partial t} + \frac{1}{\rho c} \frac{\partial p}{\partial t}
\end{align*}
$$

(22)

and because the time derivative of an average value is null, we can also write:

$$
\frac{\partial W_{4,5}}{\partial t} = \frac{\partial A_{4,5}}{\partial t} = \frac{\partial}{\partial t} \left[ u_{ac} \pm \frac{p_{ac}}{\rho_0 c_0} \right]
$$

(23)

Let us also focus on one wave, say the acoustic wave $L_5$. Then, substituting the fifth line of (16) and the fifth line of (18) in (15), with $D = T = 0$ and aside the constant factors $\sqrt{2}$ needed for considering the effective values in the present code, we find the usual property of waves amplitude conservation along the characteristic line given by the eigenvalue:

$$
\frac{\partial W_5}{\partial t} + L_5 = \left( \frac{\partial u}{\partial t} - \frac{1}{\rho_0 c_0} \frac{\partial p}{\partial t} \right) + \lambda_5 \left( \frac{\partial u}{\partial x} - \frac{1}{\rho_0 c_0} \frac{\partial p}{\partial x} \right) = 0 \iff \frac{\partial A_5}{\partial t} + \lambda_5 \frac{\partial A_5}{\partial x} = 0
$$

(24)

This means that $-L_i$ (with $i \in \{4, 5\}$), represents the time variation of the wave amplitude $A_i$:

$$
\frac{\partial A_{4,5}}{\partial t} = \frac{\partial W_{4,5}}{\partial t} = -L_{4,5}
$$

(25)

This relation between waves’ amplitudes crossing the boundaries and $L_i$ is the great advantage of adopting a characteristic formalism for acoustic problems.

C. NSCBC applications in a lined duct configuration

In this section, different types of NSCBC developed throughout this work are given. To the purpose of the CFD simulations carried out, the following boundary conditions are provided: inflow with specification of velocity and temperature, outflow with prescribed pressure, no-slip wall and acoustically treated wall, the latter being the cardinal point of this work.
1. Subsonic inflow

One typical and necessary boundary condition is the subsonic inflow. In here, the classic NSCBC inflow condition specifying the temperature \(T\) and three velocity components \((u, v, w)\) is given. The prescribed values, i.e. the known thermodynamic properties at the boundary to impose, are defined with the index \(\infty\). Four characteristic waves are entering the domain \((\mathcal{L}_i, i \in \{1, 2, 3, 4\}\) in the left boundary of Figure 1\) and one acoustic wave is leaving the domain \((\mathcal{L}_5)\). If a boundary normal to \(x\) is considered as in (17), we will have:

\[
\begin{align*}
\mathcal{L}_1 &= \frac{\rho}{c\sqrt{2}}(\gamma - 1)(\mathcal{L}_4 + \mathcal{L}_5) - \frac{dT}{dt} - T_T \\
\mathcal{L}_2 &= -\frac{dw}{dt} - T_w \\
\mathcal{L}_3 &= \frac{dv}{dt} - T_v \\
\mathcal{L}_4 &= \mathcal{L}_5 - \sqrt{2}\frac{du}{dt} - T_u
\end{align*}
\]

(26)

where each time derivative is given by:

\[
\begin{align*}
\frac{du}{dt} &= \left(\frac{\partial u}{\partial t}\right)_T + \sigma_u(u - u_\infty) + \Gamma_u(\hat{u} - u_\infty) \\
\frac{dv}{dt} &= \left(\frac{\partial v}{\partial t}\right)_T + \sigma_u(v - v_\infty) + \Gamma_u(\hat{v} - v_\infty) \\
\frac{dw}{dt} &= \left(\frac{\partial w}{\partial t}\right)_T + \sigma_w(w - w_\infty) + \Gamma_w(\hat{w} - w_\infty) \\
\frac{dT}{dt} &= \left(\frac{\partial T}{\partial t}\right)_T + \frac{\rho}{T} (\sigma_T(T - T_\infty) + \Gamma_T(T - T_\infty))
\end{align*}
\]

(27)

In (27), the derivative in \(T\) is the flux conservation term along the tangential direction, taking into consideration the passage from generalized to Cartesian coordinates. Concerning the terms in \(\sigma\) and \(\Gamma\), which are inputs chosen empirically by the user, the following reasoning is made. As introduced in the first NSCBC appearance [35], perfectly non-reflecting boundary conditions are of great interest but difficult to implement in a domain characterised by different types of boundary conditions and can easily lead to ill-posed problems. NSCBC allow to have a weakly-reflecting boundary condition, leading to small levels of reflection but guaranteeing a well-posedness and numerical stability. Using a strong condition at the inlet or outlet, such as imposing the static thermodynamic variables, leads to relatively strong acoustic wave reflections, reason why a soft condition is needed. This is achieved with a relaxation factor \(\sigma\), which allows wave reflections in order to stabilize the instantaneous values around the imposed static value, and have a negligible effect if the computed value is already close enough to it. With this said, \(\sigma_u\) is the classical relaxation parameter linked to the velocity values and applied to the difference between instantaneous \((u)\) and prescribed \((u_\infty)\) values. \(\sigma_T\) is the respective relaxation for the temperature values. \(\Gamma_u\) and \(\Gamma_T\) are the relaxation parameters applied to the difference between time-integrated filtered \((\hat{u})\) and prescribed values, as provided by Giles [34], with modifications
brought by Medida [41] and Hixon [42]. The characteristic filter time used for integrating each variable is chosen of the same order of the acoustic wave period, however not an exact multiple to avoid interactions with the acoustic wave fundamental frequency. If smaller values were taken, stronger reflections at the domain’s ends were observed, hence confirming the latter. Three cases can then be defined:

1) $\sigma = 0$ and $\Gamma = 0$: no relaxation is specified and hence no prescribed value. In this case, the calculation very probably will tend to divergence in terms of mean values.

2) $\sigma \neq 0$ and $\Gamma = 0$: the classical NSCBC relaxation [35] allows to prescribe a value, however non-reflectivity is guaranteed only for plane waves.

3) $\sigma = 0$ and $\Gamma \neq 0$: the relaxation on a Giles time-filtered value allows non-reflecting boundary condition for complex acoustic signatures. In this specific case, the transverse contributions $T$ are neglected since spatial variations are included within the Giles formalism.

In this article, the latter case finds to be ideal for injection of complex acoustic modes with no reflections, and will be used for all numerical simulations.

2. Subsonic outflow

In an identical manner as for the inlet, the outlet condition can be derived. In a subsonic outflow condition, only one acoustic wave is entering the domain ($L_5$ in the right boundary of Figure 1), so one condition needs to be specified for finding this external information, and it is usually linked to the static pressure value, noted $p_\infty$. The characteristic condition will then be:

$$L_5 = -\frac{1}{\rho c \sqrt{2}} \mathcal{T}_p - \frac{dp}{dt}$$

(28)

with:

$$\frac{dp}{dt} = \left( \frac{\partial p}{\partial t} \right)_T + \frac{1}{\rho} (\sigma p (p - p_\infty) + \Gamma p (\hat{p} - p_\infty))$$

(29)

Similarly for the inflow, $\sigma_p = 0$ and $\Gamma_p \neq 0$ is the coefficients choice made in this work for a non-reflecting outlet condition capable of treating complex acoustic signals. In the latter formulation, the unknown ingoing wave ($L_5$) cannot depend on the known outgoing one ($L_4$) or reflections would be generated.

3. Isothermal no-slip wall

One of the most classic boundary condition is the hard (or no-slip) wall. The assumption of isothermal wall is made in the following. At an isotherm no-slip wall, the condition to impose is null velocity components in all directions and constant temperature. In the case of an adiabatic wall (herein not detailed), the temperature equation would be substituted with a null heat flux equation. Therefore, there are three physical inviscid conditions on the normal and tangential velocities and one isothermal condition on the temperature $T_\infty$ to impose. After some lengthy calculations,
for a boundary with $n_z < 0.7$, the characteristic system for the hard wall condition is obtained:

$$
\begin{align}
\frac{1}{\sqrt{2}} (\mathcal{L}_4 - \mathcal{L}_5) &= n_x T_u + n_y T_v + n_z T_w \\
\frac{-n_z}{\sqrt{n_x^2 + n_y^2}} (n_x \mathcal{L}_2 + n_y \mathcal{L}_1) + \frac{\sqrt{n_x^2 + n_y^2}}{n_z} \mathcal{L}_3 &= n_y T_u - n_x T_v \\
\frac{1}{\sqrt{n_x^2 + n_y^2}} (n_x \mathcal{L}_2 - n_y \mathcal{L}_1) &= \frac{-n_x n_z T_u - n_y n_z T_v}{\sqrt{n_x^2 + n_y^2}} + \frac{n_z^2}{n_z^2 + n_y^2} T_w \\
-\frac{1}{\rho} (n_x \mathcal{L}_1 + n_y \mathcal{L}_2 + n_z \mathcal{L}_3) + \frac{1}{c\sqrt{2}} (\gamma - 1) (\mathcal{L}_4 + \mathcal{L}_5) &= \frac{1}{\rho} \sigma \frac{T}{T_0} + \frac{1}{\rho} T_p - \frac{1}{\rho} T_p
\end{align}
$$

(30)

where the first line refers to the null normal velocity term, the second and third line to the null tangential velocity term and the last line to the isothermal condition. The criteria $n_z < 0.7$ has been chosen arbitrarily and a similar derivation can be made for a boundary with $0.7 < n_z < 1$.

4. Time-Domain Impedance Boundary Condition

The objective is now to find the link between two characteristic acoustic waves in presence of an acoustically lined wall. We start our derivation from the previous methodology of a hard wall condition. However, since the objective is to model an acoustic liner, we must take into account that the normal velocity is physically not null, as a mass flow and pressure gradient are present on the liner’s perforations. Such perforations are not directly modelled in the geometry as they are computationally too expensive for industrial applications. Nevertheless, a behaviour as close as possible to reality is looked for with the TDIBC modelling. With the fall of the null normal velocity condition, the values of $\mathcal{L}_i$ (with $i \in \{1, 2, 3\}$) will be computed with use of the other conditions on tangential velocities and temperature. The fourth and missing condition is the relation between the characteristic acoustic waves. We start by reminding from (23) and (24) the link between the time variation of waves amplitude and characteristic variables:

$$
\frac{\partial A_{4.5}}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{p_{ac}}{\rho_0 c_0} \pm u_{ac} \right] = \frac{\partial W_{4.5}}{\partial t}
$$

We remember also the time-domain reflection coefficient formulation (12) linking acoustic waves:

$$
\left[ u_{ac} - \frac{p_{ac}}{\rho_0 c_0} \right] (t) = \beta \star \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] (t) = \beta_{\infty} \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] (t) + \sum_{k=1}^{N_x} \left( \mu_{1,k} \varphi_{\xi_k} + \mu_{2,k} \psi_{\xi_k} \right) + \sum_{n=1}^{N_z} \left( r_{1,n} \varphi_{s_n} + r_{2,n} \psi_{s_n} \right)
$$

We then take the time derivative of the latter reflection coefficient equation:

$$
\frac{\partial}{\partial t} \left[ u_{ac} - \frac{p_{ac}}{\rho_0 c_0} \right] = \beta_{\infty} \frac{\partial}{\partial t} \left[ u_{ac} + \frac{p_{ac}}{\rho_0 c_0} \right] + \sum_{k=1}^{N_x} \left( \mu_{1,k} \bar{\varphi}_{\xi_k} + \mu_{2,k} \bar{\psi}_{\xi_k} \right) + \sum_{n=1}^{N_z} \left( r_{1,n} \bar{\varphi}_{s_n} + r_{2,n} \bar{\psi}_{s_n} \right)
$$

(31)
where the notation \( \overline{\varphi} = \partial \varphi / \partial t \) is introduced and \( \overline{\psi} \) is its delayed counterpart. As direct consequence, we can consider (31) in terms of characteristic variables:

\[
\frac{\partial W_5}{\partial t} = \beta \ast \frac{\partial W_4}{\partial t} \iff \frac{\partial W_5}{\partial t} = \beta_\infty \frac{\partial W_4}{\partial t} + \sum_{k=1}^{N_e} (\mu_{1,k} \overline{\varphi}_k + \mu_{2,k} \overline{\psi}_k) + \sum_{n=1}^{N_\psi} (r_{1,n} \overline{\varphi}_n + r_{2,n} \overline{\psi}_n) \tag{32}
\]

or also, for coming back to the present NSCBC framework and by using (15) in an inviscid case, in terms of characteristic waves:

\[
\mathcal{L}_5 = \beta_\infty (\mathcal{L}_4 + \mathcal{T}_5) - \mathcal{T}_5 + \sum_{k=1}^{N_e} (\mu_{1,k} \overline{\varphi}_k + \mu_{2,k} \overline{\psi}_k) + \sum_{n=1}^{N_\psi} (r_{1,n} \overline{\varphi}_n + r_{2,n} \overline{\psi}_n) \tag{33}
\]

where the tangential terms \( \mathcal{T} \) are remembered to be known from the inner domain. Preliminary computations revealed how the inclusion of tangential terms is essential for a correct solution when the acoustic perturbation is grazing the acoustic wall. The new \( \overline{\varphi} \) variable is solution of the same ODE problem of (10), with the difference that the time derivative of the acoustic wave term is substituted with the characteristic flux balance:

\[
\frac{\partial \overline{\varphi}_{k,n}}{\partial t} = -z \overline{\varphi}_{k,n} + (\mathcal{L}_4 + \mathcal{T}_4) \tag{34}
\]

Being \( \mathcal{L}_4 \) known from the inner computational domain, the three \( \mathcal{L}_i \) (with \( i \in \{1, 2, 3\} \)) and the fifth incoming acoustic wave \( \mathcal{L}_5 \) are now fully determined:

\[
\begin{align*}
\mathcal{L}_3 &= \mathcal{L}_3 = \sigma_u (\nabla \cdot \mathbf{T}_1) + \frac{n_x \mathcal{L}_1 + n_y \mathcal{L}_2}{\sqrt{n_x^2 + n_y^2}} = \sigma_u (\nabla \cdot \mathbf{T}_2) + \frac{n_x \mathcal{L}_1 + n_y \mathcal{L}_2}{\sqrt{n_x^2 + n_y^2}} + \frac{n_z \mathcal{L}_3}{\sqrt{n_z^2 + n_x^2 + n_y^2}} + \frac{\mathcal{L}_5}{\sqrt{n_z^2 + n_x^2 + n_y^2}} + \frac{\mathcal{L}_5}{\sqrt{n_z^2 + n_x^2 + n_y^2}} \\
\mathcal{L}_5 &= \beta_\infty (\mathcal{L}_4 + \mathcal{T}_4) - \mathcal{T}_5 + \sum_{k=1}^{N_e} (\mu_{1,k} \overline{\varphi}_k + \mu_{2,k} \overline{\psi}_k) + \sum_{n=1}^{N_\psi} (r_{1,n} \overline{\varphi}_n + r_{2,n} \overline{\psi}_n) \tag{35}
\end{align*}
\]

It remains to describe how the delayed auxiliary variable \( \overline{\psi} \) in (33) is calculated. This is done within the delay problem previously introduced in Section II.A system (11). To not lose in generality, all the auxiliary variables in the following (\( \overline{\varphi} \) and \( \overline{\psi} \)) are considered for a generic diffusive or oscillatory pole \( z \). As introduced in Section II.A, in order to establish the effect of the back-and-forth traveling of the acoustic wave inside a cavity (as in an acoustic liner), a delayed variable \( \overline{\psi} \) is introduced, function of the auxiliary variable \( \overline{\varphi} \) at the cavity entrance. As proposed by (8), this equals to solve an advection problem in a 1D space dimension (of coordinate \( l \in [0, L_c] \)) propagating at the speed of sound \( c_0 \), as given in (11). The last value of such advection will be allocated to \( \overline{\psi} \). The number of elements \( N_\psi \) in the 1D array is chosen dependent on the maximum frequency \( f_{\text{max}} \) to resolve (at least 2 elements for satisfying Nyquist criterion on the cut-off
frequency) or the resolution wanted in terms of points-per-wavelength $N_\lambda$ at a specific frequency:

$$N_\phi = N_\lambda f_{\text{max}} \tau$$

(36)

$N_\lambda$ and $f_{\text{max}}$ are inputs, so the number of computational cells $N_\phi$ forming the advection domain is known a priori. As a graphical explanation, $N_\phi$ is the number of crosses in Figure 2. The auxiliary variable $\tilde{\phi}$ (yellow cross) is computed at the boundary, and the bottom green cross corresponds to the delayed variable $\tilde{\psi}$. The domain size is twice the cavity length to keep into account the back-and-forth distance, since the delay $\tau$ is defined as the back-and-forth time. It is imperative that the frequency $f_{\text{max}}$ chosen to size the 1D domain is also correctly resolved by the relative impedance model for ensuring numerical stability. In other words, in the frequency range chosen, the reflection coefficient of the impedance model must be a passive function, i.e. bounded within $[0,1]$. Furthermore, it must be reminded that an auxiliary variable exists for each of the $N_\psi + N_\xi$ auxiliary variables, hence a 1D array exists per auxiliary variable and per grid point on the surface treated with the impedance boundary condition. It becomes evident the interest in having a discretizing model requiring the least number of poles possible to limit the computational effort. The ODR satisfies this condition, thanks to the exponential function of the delay term, which demands a limited number of poles for discretizing the impedance model. A 2nd-order upwind scheme in time, in agreement with the global time marching scheme, and a 5th-order polynomial Spectral Difference (SD) scheme in space are used. For the sake of brevity, the Spectral Difference algorithm and its implicit formulation in the code are not herein discussed. A convergence study on the grid and 1D array size resolution was carried out and a full analysis can be found in [10].

D. NSCBC acoustic source

The final development concerned the introduction of an acoustic perturbation (or acoustic source) to any NSCBC. This was done under the form of an additional pressure derivative fluctuation. The simplest example is the outlet...
condition (28), where only the pressure equation is specified as the sole equation needed to enclose this boundary problem. The updated equation with inclusion of acoustic source term $S_{ac}$ is simply:

$$L_5 = -\frac{1}{\rho_c \sqrt{2}} T_p - \frac{dp}{dt} + S_{ac}$$

(37)

where $S_{ac}$, in the simple case of an acoustic plane wave, will be given by:

$$S_{ac} = \frac{\partial p}{\partial t} = A \omega \cos(\omega t)$$

(38)

In the case of an inlet condition as derived in (26), it is remembered that 4 equations are added, respectively for the three velocity components and temperature. Thus, supposing the acoustic wave to be isentropic, the pressure fluctuation is converted into velocity or temperature fluctuation to be included in each equation. More complex sound fields, with dependency on the 3D space directions and not only axial direction as in plane waves, can be considered. A mathematical description of in-duct sound fields can be found in the book of Rienstra and Hirschberg [38]. In an annular or circular duct with hard walls, the acoustic pressure can be expressed as a sum of classical Fourier-Bessel modes. Once the thermodynamic environment is known and the instantaneous pressure measured with microphones or numerical outputs, matrix inversion techniques can provide modes amplitudes. The isentropic pressure fluctuation for a modal amplitude $A_{mn}$ and phase $\phi$ is given function of the cylindrical coordinates ($r, \theta, z$) and the modal structure ($m, n, k_{r,mn}$) as:

$$p_{mn}^{\pm}(r, \theta, z, \phi, t) = A_{mn} \left[ B_{mn} J_m(k_{r,mn} r) + C_{mn} Y_m(k_{r,mn} r) \right] e^{i(\omega t + m \theta - k_{r,mn}^z z + \phi)}$$

(39)

with:

- ($m, n$) the azimuthal and radial orders;
- $J_m$ and $Y_m$ the Bessel functions of the first and second kind, $J'_m$ and $Y'_m$ their derivatives;
- $B_{mn}$ and $C_{mn}$ constants, function of the mode shape;
- $k_{r,mn}$ the radial wavenumber (real positive);
- $k_{z,mn}$ the axial wavenumber, which is real for propagative modes and imaginary for cut-off (evanescent) modes;
- $\pm$ defining if the propagation is on the positive or negative axis.

More detailed expressions are given in Appendix. The time derivative of (39) (and so the acoustic source term) is simply the real part of:

$$S_{ac} = \frac{\partial p}{\partial t} = \Re \left( p_{mn}^{\pm} i \omega e^{i \omega t} \right)$$

(40)
III. 2D results and validation on GIT benchmark

In [29], the authors already showed first results of the methodology described in the previous section applied on a typical 2D acoustic benchmark. Herein, updated results on the same benchmark are proposed with further improvements on the accuracy and efficiency of the TDIBC, especially for the Navier-Stokes computations. For the sake of conciseness, the experimental benchmark from Jones et al. is shortly detailed here and more information can be found in the original paper [31]. As already introduced in [29], this validation was crucial during the present developments, since similar results from the Discontinuous-Galerkin aeroacoustic code (DG-LEE) used by Monteghetti in [8] were available for comparison. The advantage of using these data is that the impedance model was exactly the same adopted herein: the same ODR coefficients (poles and weights) were used (namely \( \tilde{\beta}_D \) and \( \tilde{\beta}_E \) of Table 2 in ref. [8], also reported herein in Table V.A) in Appendix. This permits to determine the correctness of the model implementation in the CFD code without differences due to the impedance modelling. The experimental setup consisted of a rectangular channel with an acoustic liner mounted on top and microphones on the lower wall to capture the instantaneous pressure. Geometrical features are as in the original work. In the experiment, the duct is three-dimensional, but here it is numerically modelled by a 2D rectangular duct. The numerical layout and its boundary conditions are presented in Figure 3.

Fig. 3 Numerical layout and boundary conditions of the 2D case.

A continuous plane wave with a SPL of 130 dB (fluctuation of approximately \( \pm 63 \) Pa) is injected from the left side with an inlet characteristic boundary condition as described by (26) and with a source term as in (38). As in the experiment, the sound level injected is adjusted in a way that, considering the waves reflections generated in the duct (i.e. at the impedance discontinuity at the upstream edge of the liner), the measured level at the entrance is exactly 130 dB. Six frequencies are tested (0.6, 1.0, 1.5, 2.0, 2.5, 3.0 kHz) at two different average Mach numbers (0 and 0.255). In the case of Mach 0, Euler equations are solved, whereas at Mach 0.255 Navier-Stokes equations without turbulence model are solved. This is a first approximation in this framework, opening possibilities to extend the present developments to be applicable with different turbulence models. The plane wave is exiting from the duct’s right side through a non-reflective outlet condition as in (28). Meanwhile the Euler (no flow) case did not pose any particular issue, Navier-Stokes simulations required a particular attention on non-reflecting boundary conditions in presence of a simultaneous acoustic perturbation and sheared boundary layer. Thus, a Giles treatment proved to be essential for ensuring a true non-reflecting property while managing the incoming and outgoing waves with a sheared flow.
The TDIBC covers the central half of the top boundary and is modeled with the ODR previously described. All others boundaries are rigid wall, with slip or no-slip condition depending if solving Euler or Navier-Stokes equations, respectively. Two structured rectangular grids are generated: one for the no-flow (Euler) case and one for the flow (Navier-Stokes) case, the latter requiring a boundary layer refinement. They are shown in Figure 4. The same grid is used for all frequencies, so the low frequencies are theoretically over-meshed. However, convergence studies showed that this grid is well suited for all frequencies considered. At 3 kHz (the highest frequency under consideration), sound waves are captured using 40 points-per-wavelength, which proved sufficient after a preliminary grid convergence study. This corresponds to 300 elements longitudinally. In the Navier-Stokes case, it was found that the inlet and outlet conditions were located too close to the acoustic liner and their interaction had a negative effect on the solution convergence speed. Thus, the total duct length is doubled by extending the hard walled sections at the extremities, resulting in 600 elements in the longitudinal direction. The grid is uniform in the tangential direction, since no refinements are required in proximity of the impedance discontinuity or at the boundaries. In the normal direction, 20 elements are taken for the Euler uniform grid to maintain an almost unitary aspect ratio, and 60 elements for the Navier-Stokes grid with boundary

![Fig. 4 GIT grids: no-flow Euler case (top) and flow Navier-Stokes case (bottom).](image)

![Fig. 5 GIT boundary layer profile: experimental [31], numerical (CFD) and analytical](image)
layer refinement, where approximately 20 elements are devoted to the boundary layer resolution. The flow boundary layer profile imposed at the inlet boundary is the hyperbolic profile proposed in [43], given by the following law:

\[
M_{hy}(h) = M_c \tanh\left(\frac{1 - |h|}{\delta}\right) + M_c \left[1 - \tanh\left(\frac{1}{\delta}\right)\right] \left[\frac{1 + \tanh\left(\frac{1}{\delta}\right)}{\delta} + 1 + |h|\right](1 - |h|)
\] (41)

with \(M_c\) the centerline Mach:

\[
M_c = \frac{M_{avg}}{\delta \log \left(\cosh\left(\frac{1}{\delta}\right)\right) + \left[1 - \tanh\left(\frac{1}{\delta}\right)\right] \left[\frac{1 + \tanh\left(\frac{1}{\delta}\right)}{\delta} + \frac{2}{3}\right]}
\] (42)

In there, \(M_{avg}\) is the average Mach number, \(h\) is the coordinate on the normal distance from the wall (boundary layer height), \(\delta \in [0, 1]\) is the non-dimensional boundary layer thickness. Since the calculation is done with laminar Navier-Stokes equations, the expected boundary layer profile is closer to a hyperbolic law than a fully turbulent profile, as later demonstrated by the CFD result (red curve) in Figure 5.

The liner used in this case is the Ceramic Tubular CT57. Figure 6 shows its ODR against the analytical EHR model [5] and the relative experimental data from [31], at a sound level of 130 dB and for Mach numbers of 0 and 0.255. While the Mach 0 case does not reveal noticeable discrepancies, non-negligible differences between the experimental values and the ODR are found in the case at Mach 0.255. As explained in [8], some difficulties were found when trying to match the ODR with this specific experimental dataset. In particular, a hydrodynamic instability in the data near the resonant frequency at 1 kHz was highlighted in [8]. Since the first step of an ODR is not sensitive itself to the initial guess, provided that the chosen data points cover the anti-resonance, it was chosen to exclude certain data points for obtaining the \(\beta_E\) model (cf. Table I) with only 2 oscillatory poles. If the attention is posed on the reflection coefficient module (bottom-left picture), we can affirm that in the range [0.5 - 1.5] kHz it is underestimated. This will hypothetically bring to a stronger sound reduction at these frequencies, what will be shown confirmed by the CFD computations.

All simulations are run for approximately 30 acoustic periods of total time, with a time resolution of 100 time steps per acoustic periods, chosen after a dedicated convergence study. Pressure signals are extracted at 100 equidistant locations on the lower hard wall, as done in the experiment with microphones. Comparisons in terms of SPL reduction between the experimental values, DG-LEE results and the current CFD implementation are presented in Figure 7 respectively for the six frequencies (from top to bottom) and two Mach numbers (left and right columns). In the abscissa, the longitudinal coordinate is given (i.e. the sound propagation direction), while sound pressure level is in the ordinates. Vertical dashed lines indicate the acoustic liner’s limits. A good accordance is found between the CFD and the reference CAA at all frequencies. Since the impedance ODR model is remembered to be the same for the two solutions, this confirms the correct implementation of the TDIBC in the CFD code. In the flow case at low
Fig. 6 ODR of GIT CT-57 liner, 130dB, Mach 0.0 and Mach 0.255.

frequencies, an underestimation is found for both CFD and CAA against the experiment. This was previously expected in the impedance law discussion, where the reflection coefficient module is also underestimated. In terms of accuracy, the attention should be focused on the no-flow case at 1000 Hz. In there, a strong absorption of almost 60 dB is correctly achieved. If this is converted in Pascal unit measure (as for the primitive variables computed in the NSCBC boundaries in the code), a pressure fluctuation diminution from 63 Pa (130 dB) to 0.063 Pa (70 dB) is obtained. Similarly, in the flow case, a reduction down to 0.2 Pa (80 dB) is obtained solving Navier-Stokes equations. In regards to the added computational cost of the impedance condition, the small grid size of this case did not allow to evaluate any difference. This was a first confirmation that the impedance condition did not have a strong impact on the calculation time and a more detailed commentary on this topic will be made in the next result section.
Fig. 7  SPL comparison of GIT benchmark: experiment, CAA-DG [8] and CFD (elsA) - average Mach of 0 (left) and 0.255 (right).
IV. 3D results on CANNELLE benchmark

Preliminary results by the authors on the modal acoustic attenuation in a 3D cylindrical geometry were shown in [30]. This section aims at providing consolidated results of the latter, with the inclusion of a laminar flow in place of the uniform Euler flow previously considered. Section IV.A presents the experimental bench used to obtain the experimental results of [32]. The geometrical features of the industrial acoustic liners used in this study are undisclosed, but the curves of their impedance laws are given in Section IV.B altogether with their ODR discretization. Section IV.C resumes the numerical choices for the simulations and Section IV.D details the CFD results on the 3D cylindrical domain including a TDIBC, studying different frequencies, azimuthal modes and Mach numbers.

A. CANNELLE test bench description

The CANNELLE test bench was developed by Airbus to experimentally study the sound propagation and attenuation of acoustic modes in a lined cylindrical duct, in view of aircraft nacelle design. Measurements from this bench have been recently used to present an impedance eduction technique [32]. This methodology aimed at being used in multi-modal excitation of the liner, in order to take into account multiple tones and propagating modes. However, results on acoustic modes attenuation achieved with this bench have never been published. The experimental layout is shown in Figure 8. The cylindrical bench is made of different modular parts, of which the central "Test barrel" can be acoustically treated. An acoustic source controlled by 50 loudspeakers is mounted upstream or downstream the test section, in order to simulate a bypass duct or engine nacelle inlet environment, where the acoustic perturbation is propagating along or against the flow, respectively. Figure 9 represents these two wave-traveling cases, where the black arrows indicate the flow direction, the blue waves the acoustic emission and the red surface the liner treatment. Single azimuthal modes (of order noted \( m \)) are excited using suited phase control on the loudspeakers and radial mode orders (\( n \)) are driven by cut-off frequency, hence with no direct control on their generation. During the test campaign, up to 29 modes \((m, n)\) of azimuthal orders comprised within 0 and 9 were generated for 5 frequencies ranging between 1200Hz and 3500Hz and 3 Mach numbers (0, 0.3, 0.6). Microphones were located up- and downstream the treated section to perform modal
detection, allowing to estimate the respective mode amplitudes and then their attenuation in the duct. Different liners were tested in the experimental campaign. Two of them are herein considered for validating the CFD: a bypass SDOF liner Aircelle (ACL) provided by the acoustics department of Safran Nacelles, and a nacelle inlet SDOF liner designed by Airbus (AIB), with addition of a wiremesh layer glued to the perforated sheet.

**B. Impedance ODR of prescribed acoustical liners**

The first step when using an ODR method is to define a reliable fitting of the given impedance law over a determined frequency range. Poles and weights are computed with a MATLAB code developed in [8] for both liners at Mach 0 and 0.3. In Figures 10 and 11, a comparison is shown between the multipole representation (in red and blue lines), the analytical EHR model of (5) in black dashed line and the experimental measurements in white symbols, for respectively the AIB and ACL liners. In both figures, the bottom left and right subfigures represent respectively the reflection coefficient module and phase, i.e. the continuous function that is introduced in the CFD code as representative of the liner’s physics. Three vertical dashed lines correspond to the three frequencies that are investigated in the CFD simulations (1600, 2600 and 3500 Hz), in order to verify that the ODR discretization is accurate at these specific values. Concerning AIB liner, the high resistance level derives from the wiremesh sheet, which is also useful for decreasing the sensitivity of the liner impedance to the grazing flow effects. Indeed, if the Mach 0.3 flow is considered, the reflection coefficient module is practically unchanged. In ACL liner, the Mach 0 case shows a very low resistance and a nonlinear effect with respect to the sound pressure level. This is especially visible near 3000 Hz (black symbols in Figure 11) because of the resonance effect. As seen from null reactance at 3000 Hz, the resonance of the liner is indeed in this range of frequency. This physics can not be captured by the current linear impedance formulation. For simplicity, such nonlinear data have been approximated averaging the resistance level to nearby values (white symbols). This approximation, applied on the range 2800-3500 Hz, has shown negligible effects on the final results. In addition, the no-flow case shows a reflection coefficient close to 1, which makes the liner’s physics hard to be exactly reproduced, being a slight derivation from a hard wall but with a normal velocity component. The grazing flow case presents a
stronger attenuation (lower reflection coefficient in module), masking the nonlinear resonance effect identified in the Mach 0 case. Even if some discrepancies in the resistance level are present, the reflection coefficient module and phase are correctly discretized in the frequency range of interest, hence assuring an appropriate modelling in the CFD calculation. Differences in impedance laws due to SPL variations (up to 140 dB) have shown to be negligible, which means that these liners are linear with respect to the sound level. Thus, only the impedance law at 120 dB is considered. To ensure numerical stability in the event of spurious high-frequency oscillations appearing during the calculation, the impedance discretizations have been computed so to satisfy passivity, reality and causality conditions \[38\] up to 10 kHz. This was done by verifying that the reflection coefficient was bounded between 0 and 1 in the frequency range of interest \([0.5 \text{ kHz}, 10 \text{ kHz}]\) (cf. \[8\], Section 2.2, Proposition 8 for a detailed discussion on the admissibility conditions, which are related to the bounded-real nature of the \(\beta(s)\) function). For both liners and Mach numbers, 6 poles (2 real and 4 complex conjugated) are sufficient for a correct modelling. Their values are given in Appendix in Table 2 and 3.
Fig. 10  ODR of AIB Liner, 120dB, Mach 0.0 and Mach 0.3

Fig. 11  ODR of ACL Liner, 120dB, Mach 0.0 and Mach 0.3
C. CFD simulations setup

Geometrical features retained for the numerical study are as following: a 3D cylinder of diameter 350 mm, an acoustic treatment in the center of 175 mm and 350 mm for AIB and ACL liners respectively, and equally long hard walls of 325mm upstream and downstream the liner. This brings to a ratio “liner length over cylinder diameter” of 0.5 and 1.0 and an overall duct length of 825 mm and 1000 mm, for AIB and ACL configurations respectively. This geometry, which represents faithfully the true bench dimensions, is relatively short for a numerical case involving such complex acoustic content. This was made possible thanks to the non-reflecting boundary conditions previously detailed in Section [1,2]. In this manner, mesh elements were kept to a minimum without need of stretching zones or sponge layers for ensuring the non-reflective property of the boundary conditions. These numerical conditions are an essential tool for reproducing enclosed acoustic configurations at a reasonable computational cost. From this geometry, a structured multi-block grid is created with ANSA pre-processor ensuring minimal accuracy resolution needs: at least 30 points per wavelength in axial direction and 30 points in azimuthal and radial direction. Three different grids have been created to verify the same accuracy level at the three different frequencies. In the Navier-Stokes grid, approximately 25 points in the normal direction are considered for the boundary layer, which is reasonable when solving URANS equations in laminar mode, hence without turbulence modelling. The relative small cell size, however, could yield to the propagation of parasitic oscillations that would normally be cut-off (i.e. much higher than the fundamental frequency). This is a further reason for requiring appropriate non-reflecting boundary conditions in inlet and outlet regions, capable of avoiding spurious fluctuations in their calculation. The boundary layer profile, available from the experimental measurements, is discretized with the hyperbolic law (41) and imposed in the inlet boundary condition (26), symmetric in respect to the azimuthal direction. The flow profile obtained in the CFD with the aforementioned grid hypotheses is shown to be correct, as illustrated in Figure [14]. A total of 12 grids are generated: for the 2 flow cases, 2 liners and 3 frequencies. The grid details are given in Tables 4 and 5 in Appendix.

Fig. 12 CANNELLE structured mesh block strategy (AIB case, where the central liner is half the hard walls).
Two modal detection rings are replicated with two numerical surface outputs at the middle of each hard wall section, represented in Figure 9 by green lines, where instantaneous primitive variables are extracted at each iteration. Thanks to the acoustics toolbox Actran iTM, these values are converted into incident and reflected modes on each surface. It must not mislead the meaning of "incident" here. Taking as example the representation in Figure 15, incident and reflected modes on a surface correspond respectively to $A_L^+$ and $A_L^-$ for (left) Side L, $A_R^+$ and $A_R^-$ for (right) Side R. When talking of incident and transmitted modes, it corresponds to $A_L^+$ and $A_R^+$ if the acoustic waves propagate rightwards. For completion, in the same figure, $R_{L,mn}$ and $R_{R,mn}$ are the modal reflection coefficients (not to confuse with the impedance reflection coefficient $\beta$), expressing the relation between modes of a same section propagating on two opposite directions ($mn$ and $m'n'$), and $T_{L,mn}$, $T_{R,mn}$ are the transmission coefficients, expressing the relation between two modes of two different sections (L and R) propagating in the same direction. Experimental results are given in terms of Transmission Loss (TL), the direct difference in decibels between incident and transmitted modes in the treated case. During the experiment, hard-walled configurations were run for each mode to verify that in this case the TL was null, hence no acoustic losses were found. In the present numerical study, only few configurations were tested in a hard-wall layout for the same verification purpose, which proved confirmed.

As already introduced, loudspeakers could control the azimuthal order injected in the experiment (through a phase controlling process), but not the radial order. For this reason, each experimental run was done at a given azimuthal order only. Modal detection can provide the distribution on all propagative modes (azimuthal and radial) of a given injected signal. However, phase information of the injected signal was not available. To overcome these limitations, a recombination on the actual experimental injected modes ($m,n$) is carried out through a weighting-like post-processing. Practically, CFD simulations are run by imposing a constant pressure amplitude $A_{mn}$ of 120 dB for a specific azimuthal order.

![Image of grids](image_url)

**Fig. 13** CANNELLE Euler (left) and Navier-Stokes (right) grids details - corners smoothing and boundary layer refinement.
and radial mode \((m, n)\). This means having a different injected intensity level for each mode, since the latter does not scale with \(A_{mn}\) but is also function of the modal shape:

\[
< I_{mn} > = \mathcal{A}_{mn}^2 \frac{\pi}{S \rho_0 c_0} \frac{N_{mn} \Psi_{mn}^4}{|k - M k^2_{z,mn}|^2}
\]  

(43)

where \(S\) is the reference surface (i.e. the circular section), \(N_{mn}\) and \(\Psi_{mn}\) are parameters linked to the modal structure (given in Appendix) and "\(< >"\) is the spatially averaged value. Under the assumption of linear acoustics, the intensity levels can then be scaled to those detected in the experiment. In order to satisfy this recombination, it is impossible to run CFD simulations including all radial orders of a single azimuthal order as in the experiment, and each mode has to be injected singularly. In the following, intensity variables \(I\) are considered spatially averaged intensities \(< I >\) as given by (43) and taken in their dimensional form \([W/m^2]\) for avoiding logarithmic expressions. A scaling factor \(I_{(m,n)}^{SF}\) for each injected mode \((m, n)\) is defined as the ratio between the intensity of incident modes in the experiment \(I_{(m,n)}^{exp,inc}\) and the one computed in the CFD \(I_{(m,n)}^{CFD,inc}\):

\[
I_{(m,n)}^{SF} = \frac{I_{(m,n)}^{exp,inc}}{I_{(m,n)}^{CFD,inc}}
\]

(44)

Fig. 14 CANNELLE boundary layer profile.

Fig. 15 Modes decomposition in a cylindrical duct
Thereby, the transmitted intensity level after the treatment is scaled with this factor to be comparable to the experimental acoustic level. For each simulation run at mode \((m, n)\), all the contributions of propagative modes \((m, n')\) are taken into account. As saying, when injecting the mode \((0, 2)\), the transmitted propagative modes \((0, 1)\), \((0, 2)\), \((0, n')\) will be considered in the scaled transmitted intensity \(I_{\text{sc, trn}}^{(m, n)}\):

\[
I_{\text{sc, trn}}^{mn} = I_{\text{SF}}^{mn} \sum_{n'} I_{\text{CFD, trn}}^{mn n'}
\]  (45)

Finally, the Transmission Loss can be computed as the difference (in logarithm) of all propagative modes of a specific azimuthal order:

\[
TL_m = 10 \log_{10} \left( \frac{\sum^n_{n} I_{\text{exp, inc}}^{mn}}{\sum^n_{n} I_{\text{sc, trn}}^{mn}} \right)
\]  (46)

**D. CFD results and comparison to CAA and experiments**

Each simulation is run with a temporal resolution of 100 timesteps per acoustic period for a total time of 40 acoustic periods in the no flow case and approximately 300 periods in the laminar flow case, where convergence was longer to achieve in presence of a sheared boundary layer and simultaneous acoustic injection and attenuation. In all simulations, the last period (i.e. the last 100 iterations) is extracted and post-processed for modal decomposition. A sensitivity study was conducted on the number of periods needed to reach a converged result, and one period showed to be sufficient for most cases, if the simulation reached a sufficient convergence level in terms of acoustic properties.

First, a qualitative analysis in Figure 16 shows converged solutions of modes propagation and attenuation in the cylindrical duct case ACL (cf. Figure 9 for the configuration definition) at 3500 Hz. The spinning acoustic pattern injected on the left side propagates rightwards, it is attenuated in amplitude through the central lined section and leaves the right outlet boundary without numerical reflections. Then, for validating the time-domain impedance condition correctness,
the Transmission Losses (TL) are evaluated. The results herein obtained with elsA CFD software are compared with the experiment [32] and the recent numerical results of Naïr in [28]. In [32], the experimental bench was first presented and an inverse method described for educing the acoustic liners impedance under a multi-modal excitation. The grazing flow was not considered in the education method, and it was proposed as further improvement of these measurements. In [28], the time-based CAA commercial software Actran-DGM is used. It solves Linearized Euler Equations in a Discontinuous-Galerkin formalism, and it considers a prescribed boundary layer as measured in the experiment. The latter and the present CFD solver adopted the same acoustic treatments showed in Figures 10 and 11 but different time-domain impedance models. This allowed a higher fidelity validation of both numerical implementations, rather than solely basing it on the experimental measurements. Although no comprehensive uncertainty study was performed on this very complex bench, the experimental repeatability has been assessed around 1 dB, apart for unexplained instabilities on some high order modes and at high Mach number (Mach 0.6, not repeated numerically). In all the following analyses, the CAA-LEE numerical results will be taken as the reference for two reasons. First, because this CAA code showed to be accurate on several academic and industrial validations. Second and as a reminder of the present development’s objective, to show the capabilities of a CFD software to reproduce what is usually achieved with a CFD/CAA coupled technique.

Figure 17 shows the Transmission Losses for the AIB inlet liner at Mach 0, for frequencies of 1600, 2600 and 3500 Hz. A great match is found between the two codes for most of the modes at all frequencies, with only few cases at high azimuthal order where a slight higher attenuation from the CAA is found (maximum deviation of 2 dB). These cases are also presenting the strongest differences with the experiment (lower than 4 dB), but given the difficulty in reproducing such complex acoustics environment, both experimental and numerical results can be considered a close representation of the reality. The Transmission Losses of the ACL liner case at Mach 0 are shown in Figure 18. Stronger differences are found for high order modes, where elsA presents stronger attenuation levels. However, an overall satisfying accuracy was found in lower order modes between the two codes. A major difference between the experiment and both numerical approaches is that, in this case without flow, the ACL liner presents nonlinearities which are not modelled in neither numerical solvers. However, it must be reminded that this SDOF liner is conceived to always operate in a grazing flow condition in the engine bypass ducts, making experimental and numerical results without flow not comparable to a realistic application.

Figures 19 and 20 present the Transmission Losses with consideration of the sheared laminar flow previously detailed, for respectively the AIB and ACL cases. A good agreement is found for both cases at all frequencies and modes. More in details, in the AIB case, the azimuthal mode \( m0 \) at 1600 Hz in the experiment gave a negative TL, thus it was not considerable for comparison. Nonetheless, the present CFD and CAA from [28] are in good agreement. CFD and CAA are in good agreement for the highest azimuthal orders where the sound attenuation is the strongest, confirming a good modelling of the TDIBC with a grazing flow in the opposite direction than the acoustic propagation.
Similarly, the following comments are made for the ACL case. A better agreement is found for the flow case than the previous no-flow case, reinforcing the idea that calculating sound absorption with this liner without grazing flow is not suitable. Great accordance (lower than 1 dB) was found between the CFD and CAA for high azimuthal modes \(m_4\) and \(m_5\), with slightly higher differences for the mode \(m_6\). Concerning the high azimuthal modes and their strong sound reduction, a proper numerical convergence was hard to achieve. The latter showed to play a significant effect on the calculation of the Transmission Loss, hence confirming that an appropriate numerical method without spurious reflections is essential for accurate modes attenuation analyses. In regards to the added computational time, a hard-wall and a treated configurations were run for comparison for a time of 70 acoustic periods (7000 iterations) with the medium size grid. The treated case took approximately 4% more in wall-clock time, which was established to be acceptable for industrial applications.
Fig. 17  Transmission Losses of AIB liner - Mach 0

Fig. 18  Transmission Losses of ACL liner - Mach 0
Fig. 19  Transmission Losses of AIB liner - Mach 0.3

Fig. 20  Transmission Losses of ACL liner - Mach 0.3
V. Conclusion

The ODR already showed to be a powerful tool in acoustic absorption problems. Its implementation in an industrial CFD code through a Time-Domain Impedance Boundary Condition with a characteristics formalism has been detailed. This article brings in light an extended validation on a classical 2D aeroacoustic benchmark and on high-order acoustic modes attenuation in a 3D case. Regardless the complex numerical layout and the rich acoustics content, the boundary condition showed to be successfully implemented in the CFD code, providing encouraging results in close alignment with an aeroacoustics code and experiment in both benchmarks. The inclusion of a sheared flow proved essential in the impedance behaviour with grazing flow, particularly at high azimuthal orders. Nevertheless, time-domain liner attenuation in a CFD code as alternative to a coupled CFD/CAA process has been proven applicable in complex acoustics environments. A future work extension to a fan inlet configuration is foreseen, in presence of strong nonlinear acoustic waves which can be resolved by a CFD code and that are expected to be correctly attenuated by the broadband ODR impedance model.

Acknowledgments

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Appendix

A. Poles and weights of the ODR

<table>
<thead>
<tr>
<th>$\xi_k, s_n \text{ [rad.s}^{-1}]$</th>
<th>$\mu_{1,k}, r_{1,n} \text{ [rad.s}^{-1}]$</th>
<th>$\mu_{2,k}, r_{2,n} \text{ [rad.s}^{-1}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Mach } = 0$</td>
<td>$\beta_{\infty} = 0.5$</td>
<td>$\tau = 4.79939\text{e-04 [s]}$</td>
</tr>
<tr>
<td>$-3.81651\text{e+03} \pm 4.73456\text{e+03i}$</td>
<td>$-7.19423\text{e+02} \pm -5.44790\text{e+02i}$</td>
<td>$1.62580\text{e+03} \pm 9.99658\text{e+01i}$</td>
</tr>
<tr>
<td>$-2.76574\text{e+04} \pm 2.00029\text{e+04i}$</td>
<td>$-7.17970\text{e+03} \pm -7.33668\text{e+03i}$</td>
<td>$1.61468\text{e+04} \pm 1.93062\text{e+04i}$</td>
</tr>
</tbody>
</table>

| $\text{Mach } = 0.255$         | $\beta_{\infty} = 0.7$        | $\tau = 5.09918\text{e-04 [s]}$ |
| $-9.42490\text{e+04} \pm 3.64178\text{e+04i}$ | $-7.93287\text{e+04} \pm -1.49588\text{e+05i}$ | $5.73773\text{e+04} \pm 8.85381\text{e+04i}$ |

Table 1  Poles of the ODR of the GIT-CT57 liner - cf. $\beta_E$ in Table 2 of [8].
$$\xi_k, s_n [\text{rad.s}^{-1}] \quad \mu_{1,k}, r_{1,n} \quad \mu_{2,k}, r_{2,n}$$

<table>
<thead>
<tr>
<th>Mach=0</th>
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<td>-9.42477e+03</td>
<td>-7.29352e+03</td>
<td>7.32650e+03</td>
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<td>-7.98017e+03</td>
<td>5.73296e+03</td>
</tr>
<tr>
<td>-6.29549e+03 + 9.62600e+04i</td>
<td>-4.81024e+03 + 3.41226e+03i</td>
<td>4.22462e+03 + 3.50229e+03i</td>
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<tr>
<td>-3.89108e+03 + 1.878230e+05i</td>
<td>-6.78034e+03 + 6.41068e+03i</td>
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<td>-6.16078e+03 + 7.74032e+03i</td>
<td>5.31471e+03 + 7.12841e+03i</td>
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Table 2  Poles of the ODR of the CANNELLE Airbus SDOF wiremesh liner for the inlet case.

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<td>-5.46570e+03</td>
</tr>
<tr>
<td>-9.49996e+03 + 1.88108 e+04i</td>
<td>-6.22930e+03 + 6.02607e+03i</td>
<td>6.30385e+03 + 6.64957e+03i</td>
</tr>
<tr>
<td>-2.01231e+03 + 8.93580e+04i</td>
<td>-4.74709e+03 + 6.4691e+03 + 6.53973e+03i</td>
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<table>
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<td>-6.28318e+04</td>
<td>6.44417e+03</td>
<td>-5.63542e+03</td>
</tr>
<tr>
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<td>-7.4709e+03 + 6.67809e+03i</td>
<td>4.4691e+03 + 6.55873e+03i</td>
</tr>
<tr>
<td>-5.37529e+03 + 9.82738e+04i</td>
<td>-4.36216e+03 + 5.11727e+03i</td>
<td>3.54522e+03 + 5.03670e+03i</td>
</tr>
</tbody>
</table>

Table 3  Poles of the ODR of the CANNELLE Aircelle SDOF liner for the exhaust case.

B. Bessel function for acoustic mode injection

In an annular duct of external radius $R_{\text{ext}}$ and internal radius $R_{\text{int}}$, the pressure fluctuation for a unitary modal amplitude and phase ($\phi$) is given function of the cylindrical coordinates ($r, \theta, z$) and the modal structure ($m, n, k_{r,mn}$) as:

$$p_{mn}^\pm(z, r, \phi, t) = (B_{mn} J_{mn}(k_{r,mn}r) + C_{mn} Y_{mn}(k_{r,mn}r)) e^{i(\omega t + m\phi - k_{\pm, mn} z)}$$  \hspace{1cm} (47)

with $\pm$ defining if the propagation is on the positive or negative axis, $J_m$ and $Y_m$ the Bessel functions of first and second species, $J'_m$ and $Y'_m$ their derivatives,

$$B_{mn} = \cos(\Theta_{mn}), \quad C_{mn} = \sin(\Theta_{mn}), \quad \Theta_{mn} = \arctan \left( \frac{J'_m(k_{r,mn} R_{\text{ext}})}{Y'_m(k_{r,mn} R_{\text{ext}})} \right)$$  \hspace{1cm} (48)
with $k_{r,mn}$ the radial wavenumber (real positive), solution of (in the case of hard-walled duct):

$$\{J_m'(k_{r,mn}R_{int})\} \times \{Y_n'(k_{r,mn}R_{ext})\} - \{Y_n'(k_{r,mn}R_{int})\} \times \{J_m'(k_{r,mn}R_{ext})\} = 0 \quad (49)$$

with $k_{z,mn}$ the axial wavenumber, which is real for propagative modes and imaginary for cut-off (evanescent) modes:

$$k_{z,mn}^{\pm} = \frac{k}{1-M^2} \left( -M \pm \sqrt{1 - (1-M^2) \left( \frac{k_{r,mn}}{k} \right)^2} \right) \quad (50)$$

The coefficient $N_{mn}$ used to calculate the modal intensity in (43) is given by:

$$N_{mn} = \int_{R_{int}}^{R_{ext}} r \left[ B_{mn}J_{mn}(k_{r,mn}r) + C_{mn}Y_{mn}(k_{r,mn}r) \right]^2 dr = \frac{1}{2} \left[ \left( \frac{r^2}{k_{z,mn}^2} \right)^2 \left[ B_{mn}J_{mn}(k_{r,mn}r) + C_{mn}Y_{mn}(k_{r,mn}r) \right]^2 \right]_{r=R_{int}}^{r=R_{ext}} \quad (51)$$

and the modal shape $\Psi_{mn}$ is the real part of:

$$\Psi_{mn} = \Re \{ M k_0 + k (k_{z,mn}^{\pm} - M^2 k_{z,mn}^{\pm}) \} \quad (53)$$

**C. Characteristic convective wave in generalized coordinates**

In reference to the definition of characteristic convective waves in (18), its formulation for a boundary normal to the generalized coordinate $\chi$ and in terms of spatial derivatives of primitive variables $Q$ is expressed by:

$$\mathcal{L} = \hat{\mathbf{p}}^{-1} \frac{\partial \mathbf{Q}}{\partial \chi} = \begin{cases} \mathcal{L}_1 = \hat{u}_n \begin{pmatrix} n_x \frac{\partial \rho}{\partial \chi} + n_z \frac{\partial v}{\partial \chi} - n_y \frac{\partial w}{\partial \chi} - n_x \frac{\partial p}{\partial \chi} \\ n_x \frac{\partial v}{\partial \chi} + n_y \frac{\partial w}{\partial \chi} - n_z \frac{\partial u}{\partial \chi} - n_y \frac{\partial p}{\partial \chi} \\ n_z \frac{\partial u}{\partial \chi} + n_y \frac{\partial v}{\partial \chi} - n_x \frac{\partial w}{\partial \chi} - n_z \frac{\partial p}{\partial \chi} \end{pmatrix} \\ \mathcal{L}_2 = \hat{v}_n \begin{pmatrix} \frac{\partial \rho}{\partial \chi} \\ \frac{\partial \rho}{\partial \chi} \end{pmatrix} + n_y \frac{\partial w}{\partial \chi} - n_z \frac{\partial u}{\partial \chi} - n_y \frac{\partial p}{\partial \chi} \end{cases} \quad (54)$$

where $\hat{u}_n$ is the velocity normal to the surface $\chi$ and $\hat{c}$ the local speed of sound, both in the generalized coordinates system. In the case of a boundary normal to the $x$ direction in Cartesian coordinates, system (18) is obtained.
D. CANNELLE grid details

Tables 4 and 5 provide the geometrical details of the CANNELLE grids in terms of axial, azimuthal and radial directions for the different liner and frequency combinations.

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Table 4 Number of elements in the CANNELLE meshes in the three cylindrical directions without boundary layer refinement.

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Table 5 Number of elements in the CANNELLE meshes in the three cylindrical directions with boundary layer refinement.

References


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[37] Crandall, I. B., *Theory of Vibrating Systems and Sound*, D. Van Nostrand Company, 1926. [https://doi.org/10.1038/120544b0](https://doi.org/10.1038/120544b0)


