



HAL
open science

Stability analysis of the JCAPL equivalent fluid model equations for porous media

Ilyes Moufid, Denis Matignon, Rémi Roncen, Estelle Piot

► To cite this version:

Ilyes Moufid, Denis Matignon, Rémi Roncen, Estelle Piot. Stability analysis of the JCAPL equivalent fluid model equations for porous media. Waves 15th International Conference on Mathematical and Numerical Aspects of Wave Propagation 2022, Jul 2022, Palaiseau, France. pp. 246-247. <hal-03769040>

HAL Id: hal-03769040

<https://hal.science/hal-03769040v1>

Submitted on 13 Sep 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Stability analysis of the JCAPL equivalent fluid model equations for porous media

Ilyes Moufid^{1,2,*}, Denis Matignon¹, Rémi Roncen², Estelle Piot²

¹ISAE-SUPAERO - Université de Toulouse, 31055 Toulouse, France

²ONERA, The French Aerospace Laboratory - Université de Toulouse, 31055 Toulouse, France

*Email: ilyes.moufid@onera.fr

Abstract

The equivalent fluid model (EFM) describes the acoustic properties of rigid porous media by defining it as a fluid with an effective density and an effective compressibility. Their definition are based on the dynamic tortuosity α and the dynamic compressibility β , known to be complex-valued functions depending on frequency. Among the different models describing α and β , this paper focuses on the Johnson-Champoux-Allard-Pride-Lafarge (JCAPL) model [1] where these parameters are defined as irrational functions, behaving like fractional derivatives in the time domain. Here, we present the proof of stability of the time-domain EFM using the JCAPL model thanks to their oscillatory-diffusive (OD) representations.

Keywords: porous media, equivalent fluid model, JCAPL model, OD representation, stability.

1 Introduction

The EFM equations for rigid porous materials are recalled below in the Laplace domain:

$$\begin{cases} \rho_0 \alpha(s) s \hat{\mathbf{u}} + \nabla \hat{p} = 0, \\ \chi_0 \beta(s) s \hat{p} + \nabla \cdot \hat{\mathbf{u}} = 0, \end{cases} \quad (1)$$

where ρ_0 is the ambient fluid density, χ_0 the ambient adiabatic compressibility; velocity \mathbf{u} and pressure p are defined on $(0, \infty) \times \Omega$, with $\Omega \subset \mathbb{R}^n$, \hat{f} denotes the Laplace transform of f and s is the complex variable.

The JCAPL model defines α and β as

$$\alpha(s) := \alpha_\infty \left[1 + \frac{M}{s} + N \frac{\sqrt{1 + \frac{s}{L}} - 1}{s} \right], \quad (2)$$

$$\beta(s) := \gamma - (\gamma - 1) \left[1 + \frac{M'}{s} + N' \frac{\sqrt{1 + \frac{s}{L'}} - 1}{s} \right]^{-1}. \quad (3)$$

with parameters detailed in [2]. These expressions are based on the exact description of α and β at the high and low frequency limits, connected by a function whose singularities lie on the negative real axis of the complex plane [3].

In order to study the stability of the whole JCAPL - EFM system, a methodology based on

a *poles and cuts* technique [4] is used. It enables to recast a complex function f in a formulation built from an OD representation (4), containing an infinite number of real or complex weights and poles, but no \sqrt{s} -type terms.

$$\hat{f}(s) = \underbrace{\sum_{k \in \mathbb{Z}} \frac{r_k}{s - s_k}}_{\text{oscillatory}} + \underbrace{\int_0^\infty \frac{\mu(\xi)}{s + \xi} d\xi}_{\text{diffusive}}, \quad (4)$$

with the *well-posedness* condition $\int_0^\infty \frac{|\mu(\xi)|}{1 + \xi} d\xi < \infty$ for \hat{f} to admit an OD representation [2].

2 Oscillatory-diffusive representation

The irrational parts of α and β are first studied in order to find their OD representation. The focus is therefore on the two following transfer functions:

$$\hat{g}(s) := \frac{1}{N} \left[\frac{\alpha(s)}{\alpha_\infty} - 1 - \frac{M}{s} \right] = \frac{\sqrt{1 + \frac{s}{L}} - 1}{s},$$

$$\hat{h}(s) := \frac{\beta(s) - 1}{\gamma - 1} = \frac{N' \sqrt{1 + \frac{s}{L'}} + M' - N'}{s + N' \sqrt{1 + \frac{s}{L'}} + M' - N'}.$$

Function \hat{g} admits a diffusive representation with $\mu(\xi) \propto \xi^{-1} (\xi/L - 1)^{1/2}$, a positive diffusive weight defined for $\xi \in (L, \infty)$ and verifying the well-posedness condition.

Function \hat{h} has an oscillatory-diffusive representation containing a diffusive part and an additional isolated term, which is null for certain values of the parameters M' , N' and L' [2].

$$\hat{h}(s) = \frac{r_0}{s - s_0} + \int_{L'}^\infty \frac{\nu(\xi)}{s + \xi} d\xi, \quad (5)$$

with $s_0 < 0$, $r_0 \geq 0$ and $\nu(\xi) \propto \xi (\xi/L' - 1)^{1/2} \left((\xi - M' - N')^2 + N'^2 (\xi/L' - 1) \right)^{-1}$, which is positive and verifies the well-posedness condition.

3 Extended diffusive realization

Based on the OD representations of \hat{g} and \hat{h} , system (1) in the time domain reads:

$$\begin{cases} \partial_t \mathbf{u} + M \mathbf{u} + N (g \star \partial_t \mathbf{u}) + \frac{1}{\rho_0 \alpha_\infty} \nabla p = 0, \\ \partial_t p + (\gamma - 1) (h \star \partial_t p) + \frac{1}{\chi_0} \nabla \cdot \mathbf{u} = 0. \end{cases} \quad (6)$$

The interest of the OD representation is in the associated diffusive realization, which gives a time-local formulation of the convolution products present in (6). In this work, *extended* diffusive realizations, as

$$\begin{cases} \mathbf{z}_{\mathbf{u}}(t, \mathbf{x}) := \int_L^\infty \mu(\xi) \partial_t \boldsymbol{\phi}(\xi; t, \mathbf{x}) d\xi, \\ \partial_t \boldsymbol{\phi}(\xi; t, \mathbf{x}) = -\xi \boldsymbol{\phi}(\xi; t, \mathbf{x}) + \mathbf{u}(t, \mathbf{x}), \\ \boldsymbol{\phi}(\xi; 0, \mathbf{x}) = \mathbf{u}(0, \mathbf{x})/\xi, \end{cases} \quad (7)$$

for $\mathbf{z}_{\mathbf{u}} = (g \star \partial_t \mathbf{u})$ with $\boldsymbol{\phi}$ the diffusive variable, are used and differ from *usual* diffusive realizations by the presence of a time derivative in the convolution product. Non-null initial conditions are set for the diffusive variable to have a finite value for $\mathbf{z}_{\mathbf{u}}$ at $t = 0$ [5]. The energy functional

$$E_{\boldsymbol{\phi}}(t) := \frac{1}{2} \int_{\Omega} \int_L^\infty \mu(\xi) \xi \|\boldsymbol{\phi}(\xi; t, \mathbf{x})\|^2 d\xi d\mathbf{x},$$

the derivative of which is

$$\begin{aligned} \frac{d}{dt} E_{\boldsymbol{\phi}}(t) &= \int_{\Omega} \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{z}_{\mathbf{u}}(t, \mathbf{x}) d\mathbf{x} - \\ &\int_{\Omega} \int_L^\infty \mu(\xi) \|\partial_t \boldsymbol{\phi}(\xi; t, \mathbf{x})\|^2 d\xi d\mathbf{x}, \end{aligned} \quad (8)$$

can be defined for the extended diffusive realization (7). A realization analogous to (7) is used for $z_p := (h \star \partial_t p)$ with ψ denoting its associated diffusive variable and E_{ψ} its associated energy. The additional first-order system $r_0/(s - s_0)$ in (5) is handled by the same diffusive variable ψ and its associated energy, included in E_{ψ} , is

$$E_{\psi_0}(t) := \frac{1}{2} \int_{\Omega} r_0 (-s_0) |\psi(-s_0; t, \mathbf{x})|^2 d\mathbf{x},$$

Using the realization of the convolution products in (6) leads to the augmented system:

$$\begin{cases} \partial_t \mathbf{u} + M \mathbf{u} + N \mathbf{z}_{\mathbf{u}} + \frac{1}{\rho_0 \alpha_{\infty}} \nabla p = 0, \\ \partial_t p + (\gamma - 1) z_p + \frac{1}{\chi_0} \nabla \cdot \mathbf{u} = 0, \\ \partial_t \boldsymbol{\phi}(\xi; t, \mathbf{x}) = -\xi \boldsymbol{\phi}(\xi; t, \mathbf{x}) + \mathbf{u}(t, \mathbf{x}), \\ \partial_t \psi(\xi; t, \mathbf{x}) = -\xi \psi(\xi; t, \mathbf{x}) + p(t, \mathbf{x}), \\ \boldsymbol{\phi}(\xi; 0, \mathbf{x}) = \mathbf{u}(0, \mathbf{x})/\xi, \\ \psi(\xi; 0, \mathbf{x}) = p(0, \mathbf{x})/\xi. \end{cases} \quad (9)$$

4 Stability analysis

The stability analysis of system (9) is performed thanks to the *augmented* energy functional

$$\mathcal{E}(t) := E_m(t) + E_{\text{diff}}(t),$$

where the classical mechanical energy is

$$E_m(t) := \frac{\rho_0 \alpha_{\infty}}{2} \int_{\Omega} \|\mathbf{u}\|^2 d\mathbf{x} + \frac{\chi_0}{2} \int_{\Omega} |p|^2 d\mathbf{x},$$

and a *diffusive* energy is defined as

$$E_{\text{diff}} := \rho_0 \alpha_{\infty} N E_{\boldsymbol{\phi}}(t) + \chi_0 (\gamma - 1) E_{\psi}(t).$$

The positivity of the JCAPL diffusive weights, and the known sign of r_0 and s_0 , enables to prove the following proposition.

Theorem 4.1 *In a bounded domain Ω with no contribution at the boundary (either $p = 0$, or $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$), the augmented energy \mathcal{E} of the JCAPL-EFM satisfies $\frac{d}{dt} \mathcal{E}(t) \leq 0$.*

The key point of the proof lies in noticing that certain terms in

$$\begin{aligned} \frac{d}{dt} E_m(t) &= -\rho_0 \alpha_{\infty} \left(M \int_{\Omega} \|\mathbf{u}\|^2 d\mathbf{x} + N \int_{\Omega} \mathbf{u} \cdot \mathbf{z}_{\mathbf{u}} d\mathbf{x} \right) \\ &\quad - \rho_0 \alpha_{\infty} + \chi_0 \int_{\Omega} p \partial_t p d\mathbf{x}, \end{aligned}$$

have an opposite sign of those in the time derivative of the diffusive energy (see the first term of (8)) and can compensate exactly with $\frac{d}{dt} E_{\text{diff}}$. Moreover, following [6], Prop. 4.1 can be proved.

Proposition 4.1 *The dynamical system (9) is asymptotically stable, i.e. $(\mathbf{u}, p, \boldsymbol{\phi}, \psi) \rightarrow (\mathbf{0}, 0, \mathbf{0}, 0)$ as $t \rightarrow \infty$ in the appropriate energy space.*

Hence, \mathcal{E} describes an energy functional for (9), which enables to ensure the stability of the system without external inputs.

Acknowledgment This research has been financially supported by ONERA and by ISAE-SUPAERO, through the EUR TSAE under grant ANR-17-EURE-0005.

References

- [1] J-F. Allard, N. Atalla "Propagation of Sound in Porous Media: Modelling Sound Absorbing Materials", 2nd ed. *John Wiley & Sons, Ltd* (2009).
- [2] I. Moufid, D. Matignon, R. Roncen, E. Piot, "Energy analysis and discretization of the time-domain equivalent fluid model for wave propagation in rigid porous media", *Journal of Computational Physics* **451** (2022), pp.110888.
- [3] S.R. Pride, F.D. Morgan, A.F. Gangi "Drag forces of porous-medium acoustics", *Physical Review B* **47.9** (1993), pp. 4964-4978.
- [4] T. Hélie, D. Matignon "Representations with poles and cuts for the time-domain simulation of fractional systems and irrational transfer functions", *Signal Processing* **86.10** (2006), pp. 2516-2528.
- [5] B. Lombard, D. Matignon "Diffusive Approximation of a Time-Fractional Burger's Equation in Nonlinear Acoustics", *SIAM J. Appl. Math.* **76** (2016), pp. 1765-1791.
- [6] F. Monteghetti, G. Haine, D. Matignon "Asymptotic stability of the multidimensional wave equation coupled with classes of positive-real impedance boundary conditions", *Mathematical Control & Related Fields* **9.4** (2019), pp. 759-791.