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► **To cite this version:**

Hoang Nam Nguyen, Abdel Lisser, Vikas Vikram Singh. Random games under normal mean-variance mixture distributed independent linear joint chance constraints. 2022. hal-03768823

HAL Id: hal-03768823

<https://hal.science/hal-03768823>

Preprint submitted on 5 Sep 2022

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Random games under normal mean-variance mixture distributed independent linear joint chance constraints

Hoang Nam Nguyen · Abdel Lisser · Vikas Vikram Singh

Received: date / Accepted: date

Abstract We study an n -player random game with random payoffs and continuous strategy profiles sets. The payoff function of each player is defined by its expected value and the strategy set of each player is defined by a linear joint chance constraint. The random constraint vectors defining the joint chance constraint are independent and follow normal mean-variance mixture distributions. We propose a reformulation of the joint chance constraint of each player. We prove the existence of Nash equilibrium of this game by using the Kakutani fixed-point theorem under mild assumptions.

Keywords Chance-constrained game · Normal mean-variance mixture · Nash equilibrium.

1 Introduction

We consider an n -player CCG, where $H = \{1, 2, \dots, n\}$ is the set of players. Let $S^i \subset \mathbb{R}^{d_i}$ be the strategy set of player i which is a non-empty, convex and compact set. The product set $S = \prod_{i=1}^n S^i$ denotes the set of strategy profiles of all the players. For each $i \in H$, $S^{-i} = \prod_{j=1; j \neq i}^n S^j$ denotes the set of strategy vectors of all players j , $j \neq i$. A strategy profile $x = (x^1, x^2, \dots, x^n) \in S$ is represented as (x^i, x^{-i}) where x^i denotes the strategy of player i and x^{-i} denotes the vector of strategies of the players other than player i . In many real life applications, the strategy sets are restricted by random linear constraints, e.g., i) the constraints on total random electricity loss, energy generation and reserve arising in electricity market [20, 25], ii) risk constraints in financial market [22]. In this paper, we consider the case where the random linear constraints are formulated as a joint chance constraint. The strategy set of player i , $i \in H$, is further restricted by the following joint

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chance constraint

$$\mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i, \quad (1)$$

where $\alpha_i \in [0, 1]$ is a given probability level, $D^i = (D^{i,1}, \dots, D^{i,K_i})^T \in \mathbb{R}^{K_i}$ is a deterministic vector and $V^i = [V^{i,1}, \dots, V^{i,K_i}]^T$ is a $K_i \times d_i$ random matrix, where $V^{i,k}$ denotes the k^{th} row of matrix V^i and T denotes the transposition. Let $J^i = \{1, 2, \dots, K_i\}$ denotes the index set of i^{th} player's constraints. The feasible strategy set of player i is defined as

$$S_{\alpha_i}^i = \{x^i \in S^i \mid \mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i\}.$$

We assume that for each $i \in H$, $S_{\alpha_i}^i$ is a non-empty set. Let $\alpha = (\alpha_i)_{i \in H}$ be the confidence level vector and $S_\alpha = \prod_{i=1}^n S_{\alpha_i}^i$ be the set of all feasible strategy profiles. The payoff function of each player is defined using random variables. For each $x \in S_\alpha$, the payoff of player i is given by $f_i(x, \zeta)$, where ζ is an m -dimensional random vector. We use expected value approach to model the payoff function of each player. Therefore, the payoff function of player i is given by

$$p_i(x) = \mathbb{E}[f_i(x, \zeta)], \quad \forall x \in S_\alpha.$$

We assume that the CCG is of complete information, i.e., the payoff function, the strategy set of each player, and the confidence level vector α are known to all the players.

Definition 1 A strategy profile y^* is a Nash equilibrium of the CCG at confidence level vector α if for each $i \in H$

$$p_i(y^{i*}, y^{-i*}) \geq p_i(x^i, y^{-i*}), \quad \forall x^i \in S_{\alpha_i}^i.$$

The existence of a Nash equilibrium for a non-cooperative game in various setup has been extensively studied in the literature. It is mainly based on fixed point theorems which require the payoff function of a player i to be a continuous function of the strategies of all the players and a concave function of the strategies of player i for every fixed strategy profiles of all other players. Under Assumption 1 given below, the abovementioned continuity and concavity properties hold [22].

Assumption 1 For each player i , $i \in H$, the following conditions hold:

1. $f_i(\cdot, x^{-i}, \zeta)$ is a concave function of x^i for every $(x^{-i}, \zeta) \in S^{-i} \times \mathbb{R}^m$.
2. $f_i(\cdot)$ is a continuous function.
3. $p_i(x)$ is finite valued for every $x \in S$.

The games with the strategy sets containing chance constraints play an important role in various applications, e.g in portfolio optimization [14]. Recently, the games with strategy sets defined by chance constraints have been studied in the literature [20–22, 31, 32]. Singh and Lisser [31] considered a 2-player zero-sum game with individual chance constraints. When the row constraint vectors follow elliptical distributions, they showed the equivalence between a saddle point equilibrium problem and a primal-dual pair of second order cone programs. Singh et al. [32] considered an n -player general-sum game with individual chance constraints. Under elliptically symmetric distributions, they showed the equivalent between the a Nash equilibrium problem and the global optimization of a nonlinear optimization problem. Peng et al. [21] generalized to joint chance constraints n -player games and showed that there exists a Nash equilibrium if the row constraints vectors follow a mixture of elliptical distributions. To the best of our knowledge, the most chance-constrained games have been studied in the literature under elliptically symmetrically distributed assumption on the random constraints

vectors. However, in some situation, it is interesting to consider the case when the distribution of the row constraints vectors are not symmetric since symmetrical distributions are generally not suitable for many practical distributions. In power system scheduling problems, both wind power forecast errors and load forecasting errors are generally not normally distributed [13], and can be better fitted by generalized hyperbolic (GH) distributions. Many financial problems applications are also generally modelled by GH distributions [1, 7, 23, 24].

In this paper, we study the CCG when the strategy profile set of each player is defined by normal mean-variance mixture (a generalization of GH distributions) independent joint chance constraints. We derive a new reformulation of the joint chance constraints and show that there exist a Nash equilibrium under mild assumptions. The remainder of this paper is organized as follows. In Section 2, we introduce some basic concepts used in the paper. In Section 3, we show the existence of Nash equilibrium under assumptions on the payoff function and the threshold defined in the constraint of each player. In Sections 4 and 5, we propose some algorithm for finding Nash equilibrium and show numerical results. Finally, some conclusions can be found in Section 6.

2 Basic Concepts and Known Results

Definition 2 (Definition 2.2, [11]) A real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is r -decreasing for some real number $r \in \mathbb{R}$, if f is continuous on $(0, +\infty)$ and there exists some strictly positive real number t^* such that the function $t \mapsto t^r f(t)$ is strictly decreasing on $(t^*, +\infty)$.

Definition 3 A function $f : Q \rightarrow (0, +\infty)$ is r -concave on a convex set $Q \subset \mathbb{R}^s$ for a given $r \in (-\infty, +\infty)$ if for any $x, y \in Q$ and $\alpha \in [0, 1]$,

$$\begin{aligned} f(\alpha x + (1 - \alpha)y) &\geq [\alpha f(x)^r + (1 - \alpha)f(y)^r]^{\frac{1}{r}}, \text{ when } r \neq 0, \\ f(\alpha x + (1 - \alpha)y) &\geq f(x)^\alpha f(y)^{1-\alpha}, \text{ otherwise.} \end{aligned}$$

Definition 4 An n -dimensional random vector U follows a normal mean-variance mixture distribution with parameters (μ, γ, Σ, W) , i.e., $U \sim \text{NMVM}(\mu, \gamma, \Sigma, W)$, if we have the following representation:

$$U \stackrel{\text{d}}{=} \mu + \gamma W + \sqrt{W} \Sigma^{\frac{1}{2}} Z,$$

where:

- (i) Z is an n -dimension standard Gaussian distribution $N_n(0, \mathbb{I}_n)$.
- (ii) W a positive random variable with a density function independent of Z .
- (iii) $\Sigma \in \mathbb{R}^{n \times n}$ is a positive definite matrix and $\Sigma^{\frac{1}{2}} \in \mathbb{R}^{n \times n}$ such that $\Sigma^{\frac{1}{2}} (\Sigma^{\frac{1}{2}})^T = \Sigma$.
- (iv) μ and γ are n -dimensional real vectors.

Here, $\stackrel{\text{d}}{=}$ implies that the both sides have the same distribution.

Normal mean-variance mixture distributions are generally not symmetric. They are symmetric if and only if $\gamma = 0$ and the distribution is elliptical in this case. The family of normal mean-variance mixture distributions has an important application in finance. For a good overview, we refer to [19]. An important subset of the family of normal mean-variance mixture distributions is generalized hyperbolic (GH) distributions.

Definition 5 An n -dimensional random vector U follows a GH distribution with parameters $(\mu, \gamma, \Sigma, \chi, \psi, \lambda)$, i.e., $U \sim \text{GH}(\mu, \gamma, \Sigma, \chi, \psi, \lambda)$ if $U \sim \text{NMVM}(\mu, \gamma, \Sigma, W)$ where $W \sim N^-(\lambda, \chi, \psi)$ follows a generalized inverse Gaussian (GIG) distribution whose density function with respect to the measure of Lebesgue is:

$$f_U(w) = C_U \cdot w^{\lambda-1} \cdot \exp\left(-\frac{1}{2}(\chi w^{-1} + \psi w)\right) \cdot \mathbf{1}_{[0,+\infty)}(w),$$

where C_U is a constant, $\mathbf{1}$ denotes the indicator function and

$$\chi > 0, \psi \geq 0 \quad \text{if } \lambda < 0.$$

$$\chi > 0, \psi > 0 \quad \text{if } \lambda = 0.$$

$$\chi \geq 0, \psi > 0 \quad \text{if } \lambda > 0.$$

The family of GH distributions is a generalization of many elliptical distributions. For example, the t-multivariate distribution with parameters (Σ, μ, ν) is a particular case of hyperbolic distributions when $\lambda = \frac{-\nu}{2}$; $\chi = \nu$; $\psi = 0$; $\gamma = 0$. We summarize some selected p -dimensional elliptical distributions in Table 1.

Table 1 Representation of selected p -dimensional elliptical distributions in GH form

Distribution	μ	Σ	γ	λ	χ	ψ
Normal	μ	Σ	0	$+\infty$	$+\infty$	0
t-distribution	μ	Σ	0	$\frac{-\nu}{2}$	ν	0
Cauchy	μ	Σ	0	$\frac{-1}{2}$	1	0
Laplace	μ	Σ	0	1	0	-2
Pearson VII	μ	Σ	0	$\frac{p}{2} - N$	m	0

3 Existence of Nash equilibrium

For each $i \in H$, we assume that:

$$V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k}),$$

where $V^{i,k}$ are mutually independent, $k \in J^i$. Moreover, the support domain of $W^{i,k}$ is an open interval $(w_l^{i,k}, w_u^{i,k})$, where $0 < w_l^{i,k} \leq w_u^{i,k} < +\infty$ (l and u represent "lower" and "upper" in short, respectively). Let $\lambda_{i,k,\min}$ be the smallest eigenvalue of the positive definite matrix $\Sigma^{i,k}$. Define, $\tilde{S}_{\alpha_i}^i := S_{\alpha_i}^i \setminus \{0\}$. For $x^i \in \tilde{S}_{\alpha_i}^i$, let:

$$\begin{aligned} g^{i,k}(x^i, W^{i,k}) &:= \frac{-(x^i)^T \gamma^{i,k}}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\sqrt{W^{i,k}} \|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2}, \\ \xi^{i,k}(x^i) &:= \frac{Z^T ((\Sigma^{i,k})^{\frac{1}{2}})^T x^i}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2}, \end{aligned} \quad (2)$$

where Z is defined in Definition 4, $\|\cdot\|_2$ denotes the Euclidean norm. Let $\Omega^{i,k}$ be the set of $x^i \in S^i$ such that:

$$\begin{aligned} & - (x^i)^T \gamma^{i,k} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\sqrt{W^{i,k}}} \\ & > 4 \|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2 \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_1^{i,k}}} \right), \quad \text{for any } W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \end{aligned} \quad (3)$$

It is well known that $\xi^{i,k}(x^i)$ follows 1-dimensional standard Gaussian distribution [9]. We rewrite the constraint (1) as follows:

$$\begin{aligned} & \mathbb{P}\{V^i x^i \leq D^i\} \geq \alpha_i, \\ & \Leftrightarrow \prod_{k \in J^i} \mathbb{P}\{(V^{i,k})^T x^i \leq D^{i,k}\} \geq \alpha_i \quad (\text{using the independence of the row vectors}). \end{aligned} \quad (4)$$

Since $V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$, we deduce that (4) is equivalent to:

$$\begin{aligned} & \prod_{k \in J^i} \mathbb{P}\left(\left((\mu^{i,k})^T + W^{i,k}(\gamma^{i,k})^T + \sqrt{W^{i,k}} Z^T ((\Sigma^{i,k})^{\frac{1}{2}})^T\right) x^i \leq D^{i,k}\right) \geq \alpha_i, \\ & \Leftrightarrow \prod_{k \in J^i} \mathbb{P}\left(\frac{W^{i,k}}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2} (x^i)^T \gamma^{i,k} + \sqrt{W^{i,k}} \frac{Z^T ((\Sigma^{i,k})^{\frac{1}{2}})^T x^i}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2} \leq \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2}\right) \geq \alpha_i. \end{aligned} \quad (5)$$

Using the notations in (2), (5) can be rewritten as:

$$\begin{aligned} & \prod_{k \in J^i} \mathbb{P}\left(\frac{W^{i,k}}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2} (x^i)^T \gamma^{i,k} + \sqrt{W^{i,k}} \xi^{i,k}(x^i) \leq \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2}\right) \geq \alpha_i \\ & \Leftrightarrow \prod_{k \in J^i} \mathbb{P}\left(\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\right) \geq \alpha_i. \end{aligned} \quad (6)$$

Using the law of expectation in probability theory, we rewrite (6) as follows:

$$\begin{aligned} & \prod_{k \in J^i} \mathbb{E}\left(\mathbf{1}_{\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}}\right) \geq \alpha_i \\ & \Leftrightarrow \prod_{k \in J^i} \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}} \mid g^{i,k}(x^i, W^{i,k})\right)\right) \geq \alpha_i. \end{aligned} \quad (7)$$

Note that Z and $W^{i,k}$ are independent. Since $\xi^{i,k}(x^i)$ depends only on Z and $g^{i,k}(x^i, W^{i,k})$ depends only on $W^{i,k}$, we deduce that $\xi^{i,k}(x^i)$ and $g^{i,k}(x^i, W^{i,k})$ are independent. Then, the inner expectation can be rewritten as follows:

$$\mathbb{E} \left(\mathbf{1}_{\{g^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k})\}} \mid g^{i,k}(x^i, W^{i,k}) \right) = \Phi \left(g^{i,k}(x^i, W^{i,k}) \right),$$

where Φ is the cumulative distribution function of 1-dimensional standard Gaussian distribution.

Therefore, we rewrite (7) as follows:

$$\begin{aligned} \prod_{k \in J^i} \mathbb{E}_{W^{i,k}} \left(\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right) &\geq \alpha_i \\ \Leftrightarrow \sum_{k \in J^i} \log \left(\mathbb{E}_{W^{i,k}} \left(\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right) \right) &\geq \log(\alpha_i). \end{aligned} \quad (8)$$

The convexity of the feasible strategy set $S_{\alpha_i}^i$ plays a very important role in showing the existence of Nash equilibrium. We will show that $S_{\alpha_i}^i$ is a convex set for all $\alpha_i \in (\alpha_i^*, 1]$, where α_i^* is a threshold defined in the following assumption.

Assumption 2 For all $i \in H$, let:

$$\begin{aligned} \alpha_i^{(1)} &:= \max_{k \in J^i} \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right). \\ \alpha_i^{(2)} &:= \max_{k \in J^i} \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right]. \\ \alpha_i^{(3)} &= \max_{k \in J^i} \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_1^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \end{aligned}$$

Let $\alpha_i^* := \max(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)})$. Assume that $\alpha_i \in (\alpha_i^*, 1]$.

Lemma 1 Let Assumption 2 holds, for all $i \in H$. Then, $S_{\alpha_i}^i$ is a convex set for all $\alpha_i \in (\alpha_i^*, 1]$.

In order to prove Lemma 1, we need four following lemmas.

Lemma 2 Let Assumption 2 holds and $x^i \in \tilde{S}_{\alpha_i}^i$, $\forall i \in H$. Then $D^{i,k} > (\mu^{i,k})^T x^i$ for all $k \in J^i$.

Proof The proof is given in Appendix A.

Lemma 3 Let $x^i \in \tilde{S}_{\alpha_i}^i$, $\forall i \in H$. Then,

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] \geq \alpha_i, \quad \forall z \in [w_l^{i,k}, w_u^{i,k}], k \in J^i.$$

Proof The proof is given in Appendix B.

Lemma 4 *Let Assumption 2 holds. Then, for all $i \in H$:*

$$\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in J^i} \Omega^{i,k},$$

where Conv represents the convex hull.

Proof The proof is given in Appendix C.

Lemma 5 *Let Assumption 2 holds, then for any convex subset $Q^{i,k}$ of $\bigcap_{k \in J^i} \Omega^{i,k}$ such that $0 \notin Q^{i,k}$, $g^{i,k}(x^i, W^{i,k})$ is defined and (-2) -concave according to x^i on $Q^{i,k}$ for all $i \in H$, $k \in J^i$ and $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$.*

Proof The proof is given in Appendix D.

Using Lemmas 2, 4 and 5, we prove Lemma 1.

Proof (of Lemma 1) Let $\alpha_i \in (\alpha_i^*, 1]$, $\lambda \in [0, 1]$ and $z_1, z_2 \in S_{\alpha_i}^i$. We need to show that $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$.

Case 1: Let $z_1 = 0$ or $z_2 = 0$. Without loss of generality, we assume that $z_2 = 0$. This gives $D^{i,k} \geq 0$ for all $k \in J^i$, which in turn implies that

$$\mathbb{P}(V^i \lambda z_1 \leq D^i) \geq \mathbb{P}(V^i z_1 \leq D^i) \geq \alpha_i.$$

Hence, $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$.

Case 2: Let $z_1 \neq 0$, $z_2 \neq 0$ and $\lambda z_1 + (1 - \lambda)z_2 = 0$. In this case, $z_2 = \frac{-\lambda}{1-\lambda}z_1 \in \tilde{S}_{\alpha_i}^i$ and $z_1 \in \tilde{S}_{\alpha_i}^i$. It follows from Lemma 2 that

$$(\mu^{i,k})^T z_1 > \frac{\lambda - 1}{\lambda} D^{i,k}, \quad (\mu^{i,k})^T z_1 < D^{i,k}, \quad \forall k \in J^i.$$

This implies that $D^{i,k} \geq 0$ for all $k \in J^i$. Therefore, $\lambda z_1 + (1 - \lambda)z_2 = 0 \in S_{\alpha_i}^i$.

Case 3: Let $z_1 \neq 0$, $z_2 \neq 0$ and $0 \in \text{Seg}(z_1, z_2)$, where $\text{Seg}(z_1, z_2) =$

$\{z_1 + l(z_2 - z_1), 0 \leq l \leq 1\}$. Then, the points on the line segment $\text{Seg}(z_1, z_2)$ are either belong to $\text{Seg}(z_1, 0)$ or $\text{Seg}(0, z_2)$. It follows from Case 1 that $\text{Seg}(z_1, 0)$ and $\text{Seg}(0, z_2)$ are subset of $S_{\alpha_i}^i$. Therefore, $\lambda z_1 + (1 - \lambda)z_2 \in S_{\alpha_i}^i$ for all $\lambda \in [0, 1]$.

Case 4: Let $z_1 \neq 0$, $z_2 \neq 0$ such that $0 \notin \text{Seg}(z_1, z_2)$. It is clear that $\text{Seg}(z_1, z_2) \subset \text{Conv}(\tilde{S}_{\alpha_i}^i)$. From Lemmas 4 and 5, $g^{i,k}(\cdot, W^{i,k})$ is defined and (-2) -concave on $\text{Seg}(z_1, z_2)$, for all $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$. Therefore,

$$\begin{aligned} & g^{i,k}(\lambda z_1 + (1 - \lambda)z_2, W^{i,k}) \\ & \geq \left(\lambda (g^{i,k}(z_1, W^{i,k})^{-2} + (1 - \lambda)(g^{i,k}(z_2, W^{i,k})^{-2}) \right)^{-\frac{1}{2}}, \text{ for any } W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]. \end{aligned} \quad (9)$$

Since, $z_1 \in \tilde{S}_{\alpha_i}^i$ and $\alpha_i > \alpha_i^{(3)}$, using Lemma 3, for any $t \in [w_l^{i,k}, w_u^{i,k}]$ we have:

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{t}}{\sqrt{W^{i,k}}} g^{i,k}(z_1, t) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] > \alpha_i^{(3)}. \quad (10)$$

By definition of $\alpha_i^{(3)}$, we have:

$$\alpha_i^{(3)} \geq \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right). \quad (11)$$

Using (10) and (11), we deduce that:

$$\begin{aligned} & \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{t}}{\sqrt{W^{i,k}}} g^{i,k}(z_1, t) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] \\ & > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right). \end{aligned} \quad (12)$$

From (12), it is clear that $g^{i,k}(x^i, z) > 0$. Then, we have:

$$\frac{\sqrt{t}}{\sqrt{W^{i,k}}} g^{i,k}(z_1, t) \leq \frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} g^{i,k}(z_1, t), \quad \text{for any } W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]. \quad (13)$$

We deduce from (12) and (13) that:

$$\begin{aligned} & \Phi \left(\frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} g^{i,k}(z_1, t) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \\ & > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right), \text{ for any } t \in [w_l^{i,k}, w_u^{i,k}]. \end{aligned}$$

This implies that

$$0 < g^{i,k}(z_1, W^{i,k})^{-2} < \frac{1}{3}, \text{ for any } W^{i,k} \in [w_l^{i,k}, w_u^{i,k}].$$

Similarly,

$$0 < g^{i,k}(z_2, W^{i,k})^{-2} < \frac{1}{3}, \text{ for any } W^{i,k} \in [w_l^{i,k}, w_u^{i,k}].$$

By applying the non-decreasing function $\Phi(\cdot)$ on both side of (9), we can write

$$\begin{aligned} & \Phi \left(g^{i,k} \left(\lambda z_1 + (1 - \lambda) z_2, W^{i,k} \right) \right) \geq \\ & \Phi \left(\left(\lambda (g^{i,k}(z_1, W^{i,k}))^{-2} + (1 - \lambda) (g^{i,k}(z_2, W^{i,k}))^{-2} \right)^{-\frac{1}{2}} \right), \text{ for any } W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]. \end{aligned} \quad (14)$$

Since, $\Phi(\cdot)$ has (3)-decreasing density, from Lemma 3.1 of [11], the function $t \mapsto \Phi\left(t^{-\frac{1}{2}}\right)$ is concave on $(0, \frac{1}{3})$. Therefore, we can write

$$\begin{aligned} & \Phi\left(\left(\lambda(g^{i,k}(z_1, W^{i,k}))^{-2} + (1-\lambda)(g^{i,k}(z_2, W^{i,k}))^{-2}\right)^{-\frac{1}{2}}\right) \\ & \geq \lambda\left(\Phi\left(g^{i,k}(z_1, W^{i,k})\right)\right) + (1-\lambda)\left(\Phi\left(g^{i,k}(z_2, W^{i,k})\right)\right), \text{ for any } W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \end{aligned} \quad (15)$$

From (14) and (15), we have

$$\begin{aligned} & \Phi\left(g^{i,k}(\lambda z_1 + (1-\lambda)z_2, W^{i,k})\right) \geq \\ & \lambda\left(\Phi\left(g^{i,k}(z_1, W^{i,k})\right)\right) + (1-\lambda)\left(\Phi\left(g^{i,k}(z_2, W^{i,k})\right)\right), \text{ for any } W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \end{aligned} \quad (16)$$

By taking $\mathbb{E}_{W^{i,k}}$ on both sides of (16), we deduce that:

$$\begin{aligned} & \mathbb{E}_{W^{i,k}}\left[\Phi\left(g^{i,k}(\lambda z_1 + (1-\lambda)z_2, W^{i,k})\right)\right] \geq \\ & \lambda\left(\mathbb{E}_{W^{i,k}}\left[\Phi\left(g^{i,k}(z_1, W^{i,k})\right)\right]\right) + (1-\lambda)\left(\mathbb{E}_{W^{i,k}}\left[\Phi\left(g^{i,k}(z_2, W^{i,k})\right)\right]\right). \end{aligned} \quad (17)$$

This implies that the composition function $\mathbb{E}_{W^{i,k}}\left[\Phi\left(g^{i,k}(\cdot)\right)\right]$ is a concave function over $\text{Seg}(z_1, z_2)$, then also a log-concave function, i.e the logarithm function is concave (see [11]). It follows from the constraint (8) that $\lambda z_1 + (1-\lambda)z_2 \in S_{\alpha_i}^i$.

Next, we prove that $S_{\alpha_i}^i$ is a closed set. In fact, the closeness of $S_{\alpha_i}^i$ follows directly from the upper semi continuity of the probability function $x^i \mapsto \mathbb{P}(V^i x^i \leq D^i)$ which is a corollary of Proposition 3.1, [26]. Therefore, the feasible strategy set $S_{\alpha_i}^i$ is a compact set because, it is a closed subset of the compact set S^i . Finally, we show that there exists a Nash equilibrium of the CCG.

Theorem 1 Consider an n -player CCG defined in Section 1, where

1. Assumptions 1 and 2 hold.
2. For each $i \in H$, we assume that $V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$ and the vectors $V^{i,k}$ are mutually independent, $k \in J^i$.

Then, there exists a Nash equilibrium of the CCG for any $\alpha \in (\alpha_1^*, 1] \times \dots \times (\alpha_n^*, 1]$, where α_i^* , $i \in H$, is defined in Assumption 2.

Proof Let $\alpha \in (\alpha_1^*, 1] \times \dots \times (\alpha_n^*, 1]$. Under Assumption 1, the payoff function $p_i(x^i, x^{-i})$ is a concave function of x^i , for every $x^{-i} \in S^{-i}$, and a continuous function of x . It follows from Lemma 1 that the feasible strategy set $S_{\alpha_i}^i$, $i \in H$, is a convex set for all $\alpha_i > \alpha_i^*$. For each $i \in H$, $S_{\alpha_i}^i$ is a compact set. Then, the existence of a Nash equilibrium of the CCG follows from Theorem 4 of [8].

4 Algorithm for Finding Nash Equilibrium

For simplification of notations, we consider a 2– player CCG ($n = 2$) (the same algorithm for a general n –player game). For a fixed $x^2 \in S_{\alpha_2}^2$, the player 1 solve the following optimization problem:

$$\begin{aligned} [P1] \quad & \max \quad p_1(x^1, x^2) \\ & \text{subject to} \quad x^1 \in S_{\alpha_1}^1. \end{aligned} \quad (18)$$

The set of optimal solutions of [P1] which is also called the best response set of player 1, is given by:

$$BR_1(x^2) = \{\bar{x}^1 \mid p_1(\bar{x}^1, x^2) \geq p_1(x^1, x^2), \forall x^1 \in S_{\alpha_1}^1\} \quad (19)$$

Similarly, for a fixed $x^1 \in S_{\alpha_1}^1$, the player 2 solve the following optimization problem:

$$\begin{aligned} [P2] \quad & \max \quad p_2(x^1, x^2) \\ & \text{subject to} \quad x^2 \in S_{\alpha_2}^2. \end{aligned} \quad (20)$$

The best response set of player 2, is given by:

$$BR_2(x^1) = \{\bar{x}^2 \mid p_2(x^1, \bar{x}^2) \geq p_2(x^1, x^2), \forall x^2 \in S_{\alpha_2}^2\} \quad (21)$$

It is clear that if $x^{1*} \in BR_1(x^{2*})$ and $x^{2*} \in BR_2(x^{1*})$, then (x^{1*}, x^{2*}) is a Nash equilibrium of the CCG. We propose the best response algorithm as follows:

Algorithm 1 Best response algorithm

Step1 : Select initial feasible point $x^{2(0)} \in S_{\alpha_2}^2$ for player 2, set $k := 0$ and choose a tolerance parameter $\epsilon > 0$.

Step2 : Solve optimization problem [P1] with $x^2 = x^{2(k)}$ and find a point $x^{1(k)} \in BR_1(x^{2(k)})$.

Step3 : Solve optimization problem [P2] with $x^1 = x^{1(k)}$ and find a point $x^{2(k+1)} \in BR_2(x^{1(k)})$. If $|p_2(x^{1(k)}, x^{2(k)}) - p_2(x^{1(k)}, x^{2(k+1)})| < \epsilon$, stop and set $(x^{1*}, x^{2*}) = (x^{1(k)}, x^{2(k)})$. Otherwise, set $k := k + 1$ and go to **Step2**.

If Algorithm 1 stops, (x^{1*}, x^{2*}) is a Nash equilibrium of the CCG. By the way, the question when Algorithm 1 cycles is still open.

Using Algorithm 1, at each iteration, the two players have to solve their optimization problems [P1] and [P2]. For each $x^2 \in S_{\alpha_2}^2$, we reformulate [P1] as follows:

$$\begin{aligned} [P1] \quad & \max \quad [\max_{x^1 \neq 0} p_1(x^1, x^2), p(0, x^2)], \\ & \text{subject to} \quad \sum_{k \in J^1} \log \left(\mathbb{E}_{W^{1,k}} \left[\Phi \left(g^{1,k}(x^1, W^{1,k}) \right) \right] \right) \geq \log(\alpha_1). \end{aligned} \quad (22)$$

By introducing the auxiliary variables $z^{1,k}$, (23) can be reformulated as follows:

$$\begin{aligned}
[P1] \quad & \max_{x^1 \neq 0} [p_1(x^1, x^2), p(0, x^2)], \\
\text{subject to} \quad & i) \quad \mathbb{E}_{W^{1,k}} \left[\Phi \left(g^{1,k}(x^1, W^{1,k}) \right) \right] \geq e^{z^{1,k}}, \\
& ii) \quad \sum_{k \in J^1} z^{1,k} \geq \log(\alpha_1), \\
& iii) \quad z^{1,k} \leq 0, \forall k \in J^1.
\end{aligned} \tag{23}$$

In order to solve (23), we use the partial sampling technique to approximate the expectation $\mathbb{E}_{W^{1,k}}$. Here, we draw $N^{1,k}$ i.i.d samples, $w_1^{1,k}, \dots, w_{N^{1,k}}^{1,k}$ of the univariate distribution of $W^{1,k}$. Thanks to these samples, we get an approximation of the expected value by using Monte-Carlo method:

$$\begin{aligned}
[\bar{P}1] \quad & \max_{x^1 \neq 0} [p_1(x^1, x^2), p(0, x^2)], \\
\text{subject to} \quad & i) \quad \frac{1}{M^{1,k}} \sum_{t=1}^{N^{1,k}} \left[\Phi \left(g^{1,k}(x^1, w_t^{1,k}) \right) \right] \geq e^{z^{1,k}}, \\
& ii) \quad \sum_{k \in J^1} z^{1,k} \geq \log(\alpha_1), \\
& iii) \quad z^{1,k} \leq 0, \forall k \in J^1.
\end{aligned} \tag{24}$$

Let $C^k(x^1, W^{1,k}) := \Phi(g^{1,k}(x^1, w_t^{1,k}))$. It is clear that C^k is a continuous function. Moreover, C^k is dominated by the identity function $\mathbf{1}_{(w_t^{1,k}, w_u^{1,k})}$ which is an integrable function. Using Theorem 7.48, [27], $\mathbb{E}_{W^{1,k}} [C^k(\cdot, W^{1,k})]$ is continuous and the average sample converges to $\mathbb{E}_{W^{1,k}} [C^k(\cdot, W^{1,k})]$ w.p.1 uniformly on S^1 . Using Theorem 5.3, [27], we deduce that the optimal value of (24) converges to the optimal value of (23) when the number of samples goes to infinity. By introducing new auxiliary variables $y_t^{1,k}$, (24) can be reformulated as follows:

$$\begin{aligned}
[\bar{P}1] \quad & \max_{x^1 \neq 0} [p_1(x^1, x^2), p(0, x^2)], \\
\text{subject to} \quad & i) \quad \Phi \left(g^{1,k}(x^1, w_t^{1,k}) \right) \geq y_t^{1,k}, \\
& ii) \quad \sum_{t=1}^{N^{1,k}} y_t^{1,k} \geq N^{1,k} \cdot e^{z^{1,k}}, \\
& iii) \quad \sum_{k \in J^1} z^{1,k} \geq \log(\alpha_1), \\
& iv) \quad z^{1,k} \leq 0, \forall k \in J^1, \\
& v) \quad y_t^{1,k} \geq 0, \forall k \in J^1, 1 \leq t \leq N^{1,k}.
\end{aligned} \tag{25}$$

Using the definition of $g^{1,k}$ and the increasing monotonicity of Φ , we rewrite (25) as follows:

$$\begin{aligned}
[\bar{P}1] \quad & \max_{x^1 \neq 0} [\max_{x^1, x^2} p_1(x^1, x^2), p(0, x^2)], \\
\text{subject to} \quad & i) \quad -(x^1)^T \gamma^{1,k} \sqrt{w_t^{1,k}} + D^{1,k} - (x^1)^T \mu^{1,k} \geq \Phi^{(-1)}(y_t^{1,k}) \cdot \|(x^1)^T (\Sigma^{1,k})^{\frac{1}{2}}\|_2, \\
& ii) \quad \sum_{t=1}^{M^{1,k}} y_t^{1,k} \geq N^{1,k} \cdot e^{z^{1,k}}, \\
& iii) \quad \sum_{k \in J^1} z^{1,k} \geq \log(\alpha_1), \\
& iv) \quad z^{1,k} \leq 0, \forall k \in J^1, \\
& v) \quad y_t^{1,k} \geq 0, \forall k \in J^1, 1 \leq t \leq N^{1,k}. \tag{26}
\end{aligned}$$

(26) is a convex problem if $\Phi^{(-1)}(\cdot)$ is a log-convex function. In the following, we study necessary conditions for that.

Lemma 6 $\Phi^{(-1)}(\cdot)$ is monotonically increasing, nonnegative and convex on $[0.5, 1)$. Moreover, $\log(\Phi^{(-1)}(\cdot))$ is monotonically increasing on $[0.5, 1)$ and convex on $[\Phi(1), 1)$.

Proof The increasing monotonicity and the nonnegative property are trivial. First, we prove that $\Phi^{(-1)}(\cdot)$ is convex on $[0.5, 1)$. Thanks to the formula of derivatives of inverse functions, the first order and the second order derivatives of $\Phi^{(-1)}(\cdot)$ are:

$$\begin{aligned}
\frac{d(\Phi^{(-1)}(x))}{dx} &= \frac{1}{\Phi'(\Phi^{(-1)}(x))}. \\
\frac{d^2(\Phi^{(-1)}(x))}{dx^2} &= \frac{-1}{[\Phi'(\Phi^{(-1)}(x))]^2} \cdot \Phi''(\Phi^{(-1)}(x)) \cdot \frac{1}{\Phi'(\Phi^{(-1)}(x))}. \\
&= \frac{-\Phi''(\Phi^{(-1)}(x))}{[\Phi'(\Phi^{(-1)}(x))]^3}, \forall x \in [0.5, 1). \tag{27}
\end{aligned}$$

For all $t \in \mathbb{R}$, we have:

$$\begin{aligned}
\Phi'(t) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}. \\
\Phi''(t) &= \frac{-t}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}. \tag{28}
\end{aligned}$$

Since $\Phi^{(-1)}(t)$ is nonnegative on $[0.5, 1)$, we deduce that $\Phi'(\Phi^{(-1)}(x)) > 0$ and $\Phi''(\Phi^{(-1)}(x)) \leq 0, \forall x \in [0.5, 1)$. Then, the second derivative of $\Phi^{(-1)}(\cdot)$ is nonnegative on $[0.5, 1)$. Therefore, $\Phi^{(-1)}(\cdot)$ is convex on $[0.5, 1)$.

Next, we prove that $\log(\Phi^{(-1)}(\cdot))$ is convex on $[\Phi(1), 1)$.

Let $\varphi(\cdot) := \Phi^{(-1)}(\cdot)$. The first order and the second order derivatives of $\log(\Phi^{(-1)}(\cdot))$ are:

$$\begin{aligned}
\frac{d(\log(\Phi^{(-1)}(x)))}{dx} &= \frac{\varphi'(x)}{\varphi(x)}. \\
\frac{d^2(\log(\Phi^{(-1)}(x)))}{dx^2} &= \frac{\varphi''(x) \cdot \varphi(x) - (\varphi'(x))^2}{\varphi^2(x)}, \forall x \in [0.5, 1).
\end{aligned}$$

Using (27) and (28), we deduce that:

$$\begin{aligned}\varphi'(\cdot) &= \sqrt{2\pi} \cdot e^{\frac{\varphi^2(\cdot)}{2}} \\ \varphi''(\cdot) &= \varphi(\cdot) \cdot 2\pi \cdot e^{\varphi^2(\cdot)}\end{aligned}$$

Therefore, we have:

$$\varphi''(\cdot) \cdot \varphi(\cdot) - (\varphi'(\cdot))^2 = (\varphi^2(\cdot) - 1) \cdot 2\pi \cdot e^{\varphi^2(\cdot)}.$$

It is easy to see that $\varphi \geq 1$ on $[\Phi(1), 1)$. Hence, $\log(\Phi^{(-1)}(\cdot))$ is convex on $[\Phi(1), 1)$.

Lemma 7 *Let x^1 be a feasible solution of (24). If Assumption 2 holds, then $g^{1,k}(x^1, w_t^{1,k}) > 4 \cdot \left(\frac{\|\gamma^{1,k}\|}{\sqrt{\lambda_{1,k,\min}}} \sqrt{w_u^{1,k}} + \frac{\|\mu^{1,k}\|}{\sqrt{\lambda_{1,k,\min}}} \frac{1}{\sqrt{w_t^{1,k}}} \right)$.*

Proof The proof follows the same arguments as the proof of Lemma 4 except that we change the expectation $\mathbb{E}_{W^{1,k}}$ by the mean of its partial samples on $w_t^{1,k}$.

Lemma 8 $\|(x^1)^T(\Sigma^{1,k})^{\frac{1}{2}}\|_2$ is log-convex according to x^1 .

Proof It suffice to show that $m(x^1) := (x^1)^T(\Sigma^{1,k})x^1$ is a log-convex function which is equivalent to prove that $m''(x^1)m(x^1) \geq m'(x^1) \cdot [m'(x^1)]^T$, where $A \geq B$ is equivalent to $A - B$ is positive semidefinite. We have $m'(x^1) = 2\Sigma^{1,k}x^1$ and $m''(x^1) = 2\Sigma^{1,k}$. Then, $m''(x^1)m(x^1) - m'(x^1) \cdot [m'(x^1)]^T = 4((x^1)^T(\Sigma^{1,k})x^1) \cdot \Sigma^{1,k} - 4\Sigma^{1,k} \cdot x^1 \cdot (x^1)^T \cdot \Sigma^{1,k}$. The positive semidefiniteness of $m''(x^1)m(x^1) - m'(x^1) \cdot [m'(x^1)]^T$ is equivalent to the positivity of its quadratic form at z , for all $z \in \mathbb{R}^{d_1}$, i.e the positivity of the following term:

$$(z^T \Sigma^{1,k} z) [(x^1)^T(\Sigma^{1,k})x^1] - (z^T \Sigma^{1,k} x^1)^2.$$

However, from linear algebra, the above term is always positive. Then, we deduce the proof.

Using Lemmas 6 and (8), we prove the following theorem of the convexity of (26).

Theorem 2 *If the following condition holds:*

$$4 \left(\frac{\|\gamma^{1,k}\|}{\sqrt{\lambda_{1,k,\min}}} \sqrt{w_u^{1,k}} + \frac{\|\mu^{1,k}\|}{\sqrt{\lambda_{1,k,\min}}} \frac{1}{\sqrt{w_t^{1,k}}} \right) \geq 1,$$

then (26) is a convex problem.

Proof Let $(x^1, y_t^{1,k}, z^{1,k})$ be a feasible solution of (26). Note that we can choose $y_t^{1,k} = \Phi(g^{1,k}(x^1, w_t^{1,k}))$ without changing the optimal value of (26). Using Lemma 7, we have

$g^{1,k}(x^1, w_t^{1,k}) > 4 \cdot \left(\frac{\|\gamma^{1,k}\|}{\sqrt{\lambda_{1,k,\min}}} \sqrt{w_u^{1,k}} + \frac{\|\mu^{1,k}\|}{\sqrt{\lambda_{1,k,\min}}} \frac{1}{\sqrt{w_t^{1,k}}} \right) \geq 1$. We deduce that $y_t^{1,k} > \Phi(1)$

by using the monotonicity of Φ . Therefore, we can add the constraint $y_t^{1,k} > \Phi(1)$ without changing the optimal value of (26). Using Lemma 6, $\log(\Phi^{(-1)}(y_t^{1,k}))$ is convex according to $y_t^{1,k}$. On the other hand, using Lemma 8, $\|(x^1)^T(\Sigma^{1,k})^{\frac{1}{2}}\|_2$ is a log-convex function according to x^1 . Since the product of two log-convex functions is also a log-convex function, we deduce that $\Phi^{(-1)}(y_t^{1,k}) \cdot \|(x^1)^T(\Sigma^{1,k})^{\frac{1}{2}}\|_2$ is a log-convex function, then also a convex function. Therefore, we deduce that (26) is a convex problem.

Even when the problem (26) is a convex problem, it might has a computational difficulties due to the term $\log(\Phi^{(-1)}(y_t^{1,k}))$. The piecewise linear approximations can be used to simplify the calculations and provides lower bound and upper bound of (26). By choosing T different increasing points ξ_1, \dots, ξ_T in $(\Phi(1), 1)$, we use the piecewise linear function

$$G_L^{1,k}(x) := \max_{s=1, \dots, T} \left\{ \frac{1}{\Phi^{(-1)}(\xi_s) \Phi'(\Phi^{(-1)}(\xi_s))} (x - \xi_s) + \log(\Phi^{(-1)}(\xi_s)) \right\}$$

composed by the tangent lines at those points to approximate $\log(\Phi^{(-1)}(x))$ and get a lower bound. Moreover, if $x \in [\xi_s, \xi_{s+1}]$, we use the function

$$G_U^{1,k}(x) := \frac{\log(\Phi^{(-1)}(\xi_{s+1})) - \log(\Phi^{(-1)}(\xi_s))}{\xi_{s+1} - \xi_s} (x - \xi_s) + \log(\Phi^{(-1)}(\xi_s))$$

composed by the segments containing those points to get an upper bound.

5 Numerical Tests

All the numerical results are performed using Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD. We illustrate our theoretical results using randomly generated examples.

5.1 Piecewise Linear Approximation Results

In this subsection, we study the convergence of piecewise linear approximation method applied in (26) for $d_1 = 2$, $K^1 = 20$ and $N^{1,k} = 5$, $\forall k = 1, 2, \dots, K^1$, where d_1 is the dimension of x^1 , K^1 is the number of constraints and $N^{1,k}$ is the number of samples in the Monte-Carlo method. The deterministic vector $D^1 = (D^{1,1}, \dots, D^{1,K^1})^T$ is randomly generated in \mathbb{R}^{K^1} . For each $k = 1, 2, \dots, K^1$, $\mu^{1,k}$ and $\gamma^{1,k}$ are vectors uniformly generated in $[1, 1.1]^{d_1}$ using command "random.uniform(1,1.1)". $\Sigma^{1,k}$ is a d_1 -dimensional diagonal matrix with all elements on the main diagonal equal to $\lambda_{1,k}$ and $\lambda_{1,k}$ is uniformly generated in $[63 \times K^1, 64 \times K^1]$. $w_u^{1,k}$ is uniformly generated in $[1, 1.1]$ and $w_l^{1,k}$ is uniformly generated in $[0.9, 1]$. The objective function $p_1(x^1, x^2)$ is chosen to be linear with respect to x^1 , i.e., $p^1(x^1, x^2) = c^T x^1$, where c is a d_1 -dimensional real vector. In this setting, we choose $c = (1, 1)$. Choose α_1 such that the assumption 2 is satisfied. Such a choice of parameters verifies the condition in Theorem 2, i.e., $4 \cdot \left(\frac{\|\gamma^{1,k}\|}{\sqrt{\lambda_{1,k, \min}}} \sqrt{w_u^{1,k}} + \frac{\|\mu^{1,k}\|}{\sqrt{\lambda_{1,k, \min}}} \frac{1}{\sqrt{w_l^{1,k}}} \right) \geq 1$, $\forall k = 1, 2, \dots, K^1$ which makes

Table 2 Bounds and CPU time (s) of piecewise linear approximation method

T	Upper bound	CPU time (s)	Lower bound	CPU time (s)	Gap (%)
5	41.346	4.345	38.122	4.126	7.8%
10	40.947	20.4	39.011	16.475	4.73%
20	40.679	30.126	40.274	35.6	0.99%
50	40.589	90.849	40.571	120.4	0.04%
100	40.589	301.327	40.589	478.127	0%
500	40.589	1578.361	40.589	2016.6	0%

the problem (26) to be convex. We choose $T = 5, 10, 20, 50, 100, 500$. For each choice of T , we obtain a lower bound and an upper bound of the optimal value of (26).

The first column of Table 2 gives the value of S . The second and third columns give upper bound of the optimal value of (26) and the CPU time in seconds. The fourth and fifth columns give lower bound of the optimal value of (26) and the CPU time in seconds. The final column shows a relative difference between the upper bound and the lower bound, i.e.:

$$\text{Gap} = \frac{\text{Upper bound} - \text{Lower bound}}{\max(|\text{Upper bound}|, |\text{Lower bound}|)} \times 100\%.$$

Our method is also robust in term of K^1 , i.e., when we increase the number of constraints. Table 3 and Figure 1 shows the CPU time in second with different number of constraints $K^1 = 5, 10, 20, 100, 500$ while $T = 100$ and the other parameters are fixed.

Table 3 CPU time (s) with increasing number of constraints

K^1	CPU time (s)
5	40.126
10	100.475
20	416.4
100	1478.127
500	3016.6

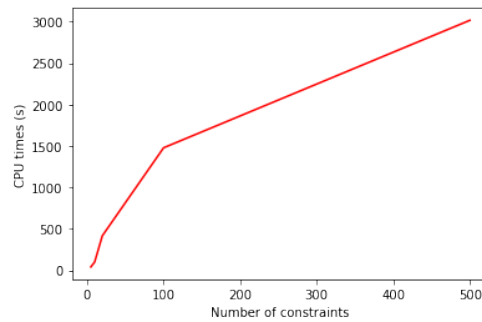


Figure 1 : CPU time (s) with different number of constraints

5.2 Nash Equilibrium of a 2-player CCG

In this subsection, we verify the performance of the best-response algorithm. We consider the following setting. For $n = 2$, let $d_1 = d_2 = 2$ and $p_1(x^1, x^2) = p_2(x^1, x^2) = (x^1)^T \cdot x^2$ (we assume that the payoff functions of both players are linear with respect to strategies of each player). Let $K^1 = K^2 = 20$ and $N^{1,k} = N^{2,k} = 5, \forall k = 1, 2, \dots, K^1$. The deterministic vector $D^1 = (D^{1,1}, \dots, D^{1,K^1})^T$ is randomly generated in $[0, 50]^{K^1}$. For each $k = 1, 2, \dots, K^1$, $\mu^{1,k}$ and $\gamma^{1,k}$ are vectors uniformly generated in $[1, 1.1]^2$. $\Sigma^{1,k}$ is a 2-dimensional diagonal matrix with all elements on the main diagonal equal to $\lambda_{1,k}$ and $\lambda_{1,k}$ is uniformly generated in $[63 \times K^1, 64 \times K^1]$. $w_u^{1,k}$ is uniformly generated in $[1, 1.1]$ and $w_l^{1,k}$ is uniformly generated in $[0.9, 1]$. We consider similar parameters for player 2. Set the tolerance parameter $\epsilon = 0.001$ for the stopping criterion and the number of iterations is 5000. Using the Best-Response algorithm in Section 4, we start with initial point $x_0^2 = (1, 1)^T$, solve alternatively the optimization problem of each player and obtain a sequence of strategy vectors for each player. We present the strategy vector of each player in \mathbb{R}^2 . Figure 2 shows the sequence of best responses of each player which converges to a Nash equilibrium point $x^{1*} := (0, 0)^T$ and $x^{2*} := (0, 0)^T$.

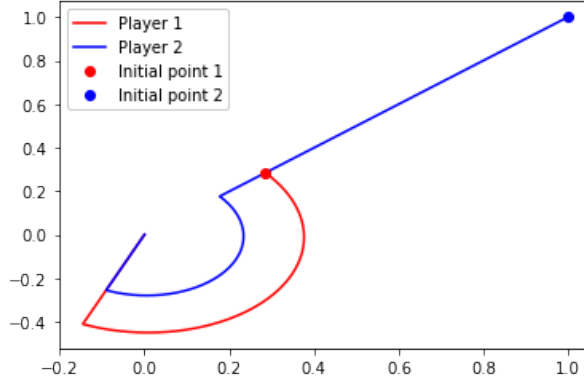


Figure 2 : Nash equilibrium of a 2-player CCG

6 Conclusion

In this paper, we studied the n -player non-cooperative chance-constrained game. We assume that the strategy profiles set of each player is given by an independent joint chance constraint and the row vectors of the stochastic matrix are normal mean-variance mixture distributed. Under mild conditions, we prove the existence of Nash equilibrium for this chance-constrained game.

Acknowledgments

This research was supported by DST/CEFIPRA Project No. IFC/4117/DST-CNRS-5th call/2017-18/2 and CNRS Project No. AR/SB:2018-07-440.

Appendices

Appendix A: Proof of Lemma 2

Let $x^i \in \tilde{S}_{\alpha_i}^i$. Using (8), we have:

$$\prod_{k \in J^i} \mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \geq \alpha_i.$$

Since, $\mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \in [0, 1]$ and $\alpha_i > \alpha_i^{(1)}$, $\forall i \in H, k \in J^i$, we deduce that:

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] > \alpha_i^{(1)}. \quad (29)$$

We have the following inequalities:

$$\begin{aligned} |(x^i)^T \gamma^{i,k}| &\leq \|x^i\|_2 \|\gamma^{i,k}\|_2, \quad \text{the Cauchy-Schwarz inequality,} \\ \sqrt{(x^i)^T \Sigma^{i,k} x^i} &\geq \sqrt{\lambda_{i,k,\min}} \|x^i\|_2, \\ \sqrt{W^{i,k}} &\leq \sqrt{w_u^{i,k}}. \end{aligned} \quad (30)$$

From (30), we deduce that:

$$\left| \frac{-(x^i)^T \gamma^{i,k}}{\sqrt{(x^i)^T \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} \right| \leq \frac{\|x^i\|_2 \|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \|x^i\|_2} \sqrt{w_u^{i,k}} = \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}}. \quad (31)$$

Hence, using the increasing monotonicity of Φ and the definition of $g^{i,k}$ in (2), we deduce from (29) and (31) that:

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\sqrt{W^{i,k}} \|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2} \right) \right] > \alpha_i^{(1)}. \quad (32)$$

By definition of $\alpha_i^{(1)}$, we have:

$$\alpha_i^{(1)} \geq \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right) \quad (33)$$

Hence, we deduce from (32) and (33) that:

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{D^{i,k} - (x^i)^T \mu^{i,k}}{\sqrt{W^{i,k}} \|(x^i)^T (\Sigma^{i,k})^{\frac{1}{2}}\|_2} \right) \right] > \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right).$$

Once again, using the increasing monotonicity of Φ , we deduce that $D^{i,k} - (x^i)^T \mu^{i,k} > 0$.

Appendix B: Proof of Lemma 3

Let $x^i \in \tilde{S}_{\alpha_i}^i$. Using Lemma 2, $D^{i,k} > (\mu^{i,k})^T x^i$ for all $k \in J^i$.

Let $R := \frac{(x^i)^T \gamma^{i,k}}{\sqrt{(x^i)^T \Sigma^{i,k} x^i}}$ and $z \in [w_1^{i,k}, w_u^{i,k}]$. Using the definition of $g^{i,k}(x^i, W^{i,k})$ in (2), we can verify the following equation:

$$g^{i,k}(x^i, W^{i,k}) = \frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{Rz - RW^{i,k}}{\sqrt{W^{i,k}}}, \quad \text{for any } W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \quad (34)$$

We have the following inequalities:

$$\begin{aligned} |(x^i)^T \gamma^{i,k}| &\leq \|x^i\|_2 \|\gamma^{i,k}\|_2, \quad \text{the Cauchy-Schwarz inequality,} \\ \sqrt{(x^i)^T \Sigma^{i,k} x^i} &\geq \sqrt{\lambda_{i,k,\min}} \|x^i\|_2, \\ \sqrt{W^{i,k}} &\leq \sqrt{w_u^{i,k}}, \\ |z - W^{i,k}| &\leq w_u^{i,k} - w_1^{i,k}. \end{aligned} \quad (35)$$

We deduce from (35) that:

$$\frac{Rz - RW^{i,k}}{\sqrt{W^{i,k}}} = \frac{(x^i)^T \gamma^{i,k}}{\sqrt{(x^i)^T \Sigma^{i,k} x^i}} \left(\frac{z - W^{i,k}}{\sqrt{W^{i,k}}} \right) \leq \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \quad (36)$$

Using the upper bound of $\frac{Rz - RW^{i,k}}{\sqrt{W^{i,k}}}$ in (36), we have:

$$g^{i,k}(x^i, W^{i,k}) \leq \frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}}, \quad \text{for any } W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \quad (37)$$

Taking $\mathbb{E}_{W^{i,k}}(\Phi(\cdot))$ on both sides of (37), we deduce that:

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \leq \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right].$$

On the other hand, since $x^i \in \tilde{S}_{\alpha_i}^i$, from (8), we have:

$$\prod_{k \in J^i} \mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \geq \alpha_i.$$

Since, $\mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \in [0, 1]$, we deduce the proof.

Appendix C: Proof of Lemma 4

Let $x^i \in S_{\alpha_i}^i$. Using Lemma 3, since $\alpha_i > \alpha_i^{(2)}$, $\forall i \in H$, then for any $z \in [w_1^{i,k}, w_u^{i,k}]$, we have:

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] > \alpha_i^{(2)}. \quad (38)$$

By definition of $\alpha_i^{(2)}$, we have:

$$\alpha_i^{(2)} \geq \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} \left(\frac{\|\gamma^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right]. \quad (39)$$

Using (38) and (39), we deduce that:

$$\begin{aligned} & \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] \\ & > \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} \left(\frac{\|\gamma^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right]. \end{aligned} \quad (40)$$

From (40), it is clear that $g^{i,k}(x^i, z) > 0$. Then, we have:

$$\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) \leq \frac{\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} g^{i,k}(x^i, z), \quad \text{for any } W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \quad (41)$$

We deduce from (40) and (41) that:

$$\begin{aligned} & \Phi \left(\frac{\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \\ & > \Phi \left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} \left(\frac{\|\gamma^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) + \frac{\|\gamma^{i,k}\| (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right]. \end{aligned} \quad (42)$$

Using the increasing monotonicity of Φ , (42) is equivalent to:

$$g^{i,k}(x^i, z) > 4 \left(\frac{\|\gamma^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right).$$

Since z is an arbitrary point in $[w_1^{i,k}, w_u^{i,k}]$, by definition of $\Omega^{i,k}$, we obtain:

$$S_{\alpha_i}^{\tilde{}} \subset \bigcap_{k \in J^i} \Omega^{i,k}.$$

Note that $\Omega^{i,k}$ is convex. Then, we deduce the proof.

Appendix C: Proof of Lemma 5

For $x^i \in Q^{i,k}$, let:

$$\begin{aligned} f^{i,k}(x^i, W^{i,k}) &:= \frac{1}{(g^{i,k}(x^i, W^{i,k}))^2} \\ &= \frac{(x^i)^T \Sigma^{i,k} x^i}{W^{i,k} [(x^i)^T \gamma^{i,k}]^2 + \frac{1}{W^{i,k}} [D^{i,k} - (x^i)^T \mu^{i,k}]^2 + 2[(x^i)^T \gamma^{i,k}][(x^i)^T \mu^{i,k} - D^{i,k}]} \\ &= ((x^i)^T \Sigma^{i,k} x^i) \cdot M_{i,k}^{-1}. \end{aligned}$$

where $M_{i,k} := W^{i,k} [(x^i)^T \gamma^{i,k}]^2 + \frac{1}{W^{i,k}} [D^{i,k} - (x^i)^T \mu^{i,k}]^2 + 2[(x^i)^T \gamma^{i,k}][(x^i)^T \mu^{i,k} - D^{i,k}]$.

It is clear that the (-2) -concavity of $g^{i,k}(x^i, W^{i,k})$ is equivalent to the convexity of $f^{i,k}(x^i, W^{i,k})$ on the domain $Q^{i,k}$. In order to prove the convexity of $f^{i,k}(x^i, W^{i,k})$ w.r.t x^i it suffices to prove that the Hessian matrix of $f^{i,k}(x^i, W^{i,k})$ w.r.t x^i is positive semi-definite on $Q^{i,k}$, $\forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$. Let $\nabla_{x^i} f^{i,k}(x^i, W^{i,k})$ be the gradient vector of $f^{i,k}(x^i, W^{i,k})$ w.r.t x^i and $H_{x^i} f^{i,k}(x^i, W^{i,k})$ be the Hessian matrix of $f^{i,k}(x^i, W^{i,k})$ w.r.t x^i . Using differential calculations in linear algebra, we can verify the following formulation of $\nabla_{x^i} f^{i,k}(x^i, W^{i,k})$:

$$\begin{aligned} \nabla_{x^i} f^{i,k}(x^i, W^{i,k}) &= 2M_{i,k}^{-1} \Sigma^{i,k} x^i \\ &\quad - 2M_{i,k}^{-2} [(x^i)^T \Sigma^{i,k} x^i] \times [W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}] \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right). \\ \forall (x^i, W^{i,k}) &\in Q^{i,k} \times [w_1^{i,k}, w_u^{i,k}]. \end{aligned} \quad (43)$$

Note that the gradient vector of $(x^i)^T \Sigma^{i,k} x^i$ is $2\Sigma^{i,k} x^i$ and the gradient vector of $M_{i,k}$ w.r.t x^i is $2[W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}] \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)$. To obtain (43), it suffices to apply the product rule of differentiation.

Similarly, by differentiating both side of (43), we have:

$$\begin{aligned} \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) &= M_{i,k} \Sigma^{i,k} + 3[(x^i)^T \Sigma^{i,k} x^i] (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^T \\ &\quad - 2[W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}] [\Sigma^{i,k} x^i (\gamma^{i,k})^T + \gamma^{i,k} (x^i)^T \Sigma^{i,k}] \\ &\quad - 2[W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}] \frac{1}{W^{i,k}} [\Sigma^{i,k} x^i (\mu^{i,k})^T + \mu^{i,k} (x^i)^T \Sigma^{i,k}]. \end{aligned} \quad (44)$$

In order to prove that $H_{x^i} f^{i,k}(x^i, W^{i,k})$ is positive semi-definite, it suffices to prove that the quadratic form of $H_{x^i} f^{i,k}(x^i, W^{i,k})$ at $z \in \mathcal{Q}^{i,k}$ is positive, i.e., $z^T H_{x^i} f^{i,k}(x^i, W^{i,k}) z \geq 0$, $\forall z \in \mathbb{R}^{d_i}$, $\forall (x^i, W^{i,k}) \in \mathcal{Q}^{i,k} \times [w_1^{i,k}, w_u^{i,k}]$.

Using (44), we have:

$$\begin{aligned}
& z^T \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) z \\
&= M_{i,k} z^T \Sigma^{i,k} z + 3[(x^i)^T \Sigma^{i,k} x^i] \left(\sqrt{W^{i,k}} (z^T \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} (z^T \mu^{i,k}) \right)^2 \\
&\quad - 4[W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}] \\
&\quad \times \left((z^T \Sigma^{i,k} x^i) [(\gamma^{i,k})^T z] + \frac{1}{W^{i,k}} (z^T \Sigma^{i,k} x^i) [(\mu^{i,k})^T z] \right) \tag{45}
\end{aligned}$$

From linear algebra, we have the following inequality:

$$|z^T \Sigma^{i,k} x^i| \leq \sqrt{z^T \Sigma^{i,k} z} \sqrt{(x^i)^T \Sigma^{i,k} x^i}. \tag{46}$$

Using (46) in (45), we deduce that:

$$\begin{aligned}
& z^T \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) z \\
&\geq M_{i,k} z^T \Sigma^{i,k} z + 3[(x^i)^T \Sigma^{i,k} x^i] \left(\sqrt{W^{i,k}} (z^T \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} (z^T \mu^{i,k}) \right)^2 \\
&\quad - 4|W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}| \times \sqrt{z^T \Sigma^{i,k} z} \sqrt{(x^i)^T \Sigma^{i,k} x^i} \\
&\quad \times \left(|(\gamma^{i,k})^T z| + \frac{1}{W^{i,k}} |(\mu^{i,k})^T z| \right). \tag{47}
\end{aligned}$$

Note that $|W^{i,k} (x^i)^T \gamma^{i,k} + (x^i)^T \mu^{i,k} - D^{i,k}| = \sqrt{W^{i,k} M_{i,k}}$ and $3[(x^i)^T \Sigma^{i,k} x^i] \left(\sqrt{W^{i,k}} (z^T \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} (z^T \mu^{i,k}) \right)^2 \geq 0$.

Then, from (47), we deduce that:

$$\begin{aligned}
& z^T \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) z \\
&\geq M_{i,k} z^T \Sigma^{i,k} z - 4\sqrt{M_{i,k}} \sqrt{z^T \Sigma^{i,k} z} \sqrt{(x^i)^T \Sigma^{i,k} x^i} \left(\sqrt{W^{i,k}} |(\gamma^{i,k})^T z| + \frac{1}{\sqrt{W^{i,k}}} |(\mu^{i,k})^T z| \right). \tag{48}
\end{aligned}$$

The case $z = 0$ is trivial. Otherwise, we have

$$\begin{aligned}
& \frac{z^T \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) z}{(z^T \Sigma^{i,k} z) ((x^i)^T \Sigma^{i,k} x^i)} \\
&\geq \frac{M_{i,k}}{(x^i)^T \Sigma^{i,k} x^i} - 4\sqrt{\frac{M_{i,k}}{(x^i)^T \Sigma^{i,k} x^i}} \left(\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^T z}{z^T \Sigma^{i,k} z} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^T z}{z^T \Sigma^{i,k} z} \right| \right). \tag{49}
\end{aligned}$$

Note that $\frac{M_{i,k}}{(x^i)^T \Sigma^{i,k} x^i} = [g^{i,k}(x^i, W^{i,k})]^2$. Since $x^i \in Q^{i,k}$, then $x^i \in \text{Conv}(\tilde{S}_{\alpha_i}^i)$. Using Lemma 4, $x^i \in \Omega^{i,k}$, i.e., $g^{i,k}(x^i, W^{i,k}) > 0$, $\forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$. Then,

$$\sqrt{\frac{M_{i,k}}{(x^i)^T \Sigma^{i,k} x^i}} = g^{i,k}(x^i, W^{i,k}).$$

Therefore, (49) is equivalent to:

$$\begin{aligned} & \frac{z^T \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) z}{(z^T \Sigma^{i,k} z)((x^i)^T \Sigma^{i,k} x^i)} \\ & \geq [g^{i,k}(x^i, W^{i,k})]^2 - 4g^{i,k}(x^i, W^{i,k}) \left(\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^T z}{z^T \Sigma^{i,k} z} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^T z}{z^T \Sigma^{i,k} z} \right| \right). \end{aligned} \quad (50)$$

We have the following inequalities:

$$\begin{aligned} & \left| \frac{(\gamma^{i,k})^T z}{z^T \Sigma^{i,k} z} \right| \leq \frac{\|\gamma^{i,k}\| \|z\|}{\sqrt{\lambda_{i,k,\min}} \|z\|} = \frac{\|\gamma^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}}, \\ & \left| \frac{(\mu^{i,k})^T z}{z^T \Sigma^{i,k} z} \right| \leq \frac{\|\mu^{i,k}\| \|z\|}{\sqrt{\lambda_{i,k,\min}} \|z\|} = \frac{\|\mu^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}}, \\ & \sqrt{W^{i,k}} \leq \sqrt{w_u^{i,k}}, \\ & \frac{1}{\sqrt{W^{i,k}}} \leq \frac{1}{\sqrt{w_1^{i,k}}}. \end{aligned} \quad (51)$$

Hence, by using (51), we deduce from (49) that:

$$\begin{aligned} & \frac{z^T \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k}(x^i, W^{i,k}) z}{(z^T \Sigma^{i,k} z)((x^i)^T \Sigma^{i,k} x^i)} \\ & \geq [g^{i,k}(x^i, W^{i,k})]^2 - 4|g^{i,k}(x^i, W^{i,k})| \left(\frac{\|\gamma^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_1^{i,k}}} \right). \end{aligned} \quad (52)$$

Since $x^i \in \Omega^{i,k}$, we deduce that the right-hand side of (52) is positive which finishes the proof.

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